Demand Learning in Two Operations Models

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ABSTRACT

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The rapid advance of information technologies largely facilitated firms' data-driven decision making. Particularly, in operations management practices, firms could continuously collect information to refine their demand knowledge, and integrate this process into their relevant operational decisions, e.g. pricing, inventory, and market entry, known as demand learning. Demand learning in complex business systems is often tangled with complex strategic interactions, thus requiring a deep understanding of how it affects the strategic relationship among players in various business setups. This thesis aims to contribute to the demand-learning literature by studying the strategic interactions in two different business relationships, one vertical and the other horizontal.

First, I consider the interactions between a retailer and a supplier in a supply chain subject to demand censorship (i.e. unobservable lost sales) when the retailer is engaging in demand learning through dynamic inventory experimentation. I study the supplier's optimal wholesale prices when the retailer is in three different situations, and find that the retailer and the supply chain may actually benefit from either myopia or censorship in contrast to the existing results, due to the supplier's different collaborative or exploitative responses to the retailer's "willingness to learn". I also identify that, with demand censorship, the collaborative behavior between the players for information acquisition may improve the system's performance.

Second, I study an online retail platform's learning process and entry policies as well as the independent seller's pricing distortion behavior to slow down this process, motivated by Amazon.com's unique dual role as both a marketplace and a merchant that allows it to use the transaction data generated by its third-party sellers to decide if to sell the same product itself. I developed a Bayesian statistical model for the platform's demand learning, proposed two types of heuristic entry policies

for the platform owner. The model predicts a pattern of price distortion, and describes the product offering choices made by the independent seller. These could potentially serve as testable results for empirical studies.

Table of Contents

List of Figures

List of Tables

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To My Family

Chapter 1

Preamble

The rapid advance of information technologies has provided firms with a tremendous amount of data, which can be used to improve decision making. Data-driven decision making has recently attracted great interests from both academia and industry. An important stream of research in Operations Management has focused on the issue of demand learning, i.e., collecting demand observations to continuously fine tune a demand model. And this constantly improving demand model is then integrated with an optimization algorithm to drive improvements in a wide range of operational decisions such as pricing, inventory, and market entry decisions.

Demand learning in complex business systems is often tangled with complex strategic interactions. For example, if a retailer invests to improve its understanding of the market demand, the supplier may choose to join the effort by e.g. lowering the wholesale price, or to exploit the retailer's incentive to demand-learn by raising the wholesale price. Strategic interactions in demand learning do not just occur in supply chains, they also happen in horizontal relationships. For example, the independent sellers doing business on an online platform may want to disrupt the platform owner's effort to discover the hot-selling items (and start selling these items itself). Therefore, a thorough understanding of demand learning in a business system is impossible without a careful study of the strategic interactions among the business players in the system.

This thesis aims to contribute to the demand-learning literature by studying the strategic interactions in two different business relationships, one vertical and the other horizontal.

1.1 A supply chain subject to demand censorship

In the first project, I study the interactions between a retailer and a supplier in a supply chain subject to demand censorship when the retailer is engaged in demand learning through dynamic inventory experimentation. Unobserved lost sales due to retail stockouts, known as the demand censorship effect, could cause self-sustaining bias in demand forecasts. Researchers proposed inventory experimentation to "learn" the real demand at the retail level, yet there is little understanding of the strategic effects of this type of demand learning in a supply chain context despite the growing trends to involve the entire supply chain in information acquisition and business analytics.

I develop a stylized model of a Stackelberg game over two periods to study the supplier's optimal wholesale price decision when the retailer acts as a newsvendor in three different situations: fully observed demand ("O"), a myopic retailer under demand censorship ("M"), and a forward-looking retailer under demand censorship ("F").

In the existing literature without considering the supplier's decision (i.e., a single-location model focusing on the retailer), the main observation has been that the retailer's performance is the best under "O", the second best under "F", and the worst under "M". But when we consider the supplier's wholesale pricing decision, the picture starts to change. Specifically, the retailer may actually benefit from either myopia or censorship. This happens either because 1) the supplier increases the wholesale price to exploit the forward-looking retailer's willingness to learn, making the myopic retailer relatively better off; or 2) the supplier decreases the wholesale price to help the retailer overcome the censorship effect (for both the "M' and "F' cases), in which case the retailer benefits from censorship and sometimes achieves a performance that is better than that under "O". On the other hand, the supply chain's performance may also benefit from either myopia or censorship, because the double-marginalization effect (i.e., decentralization inefficiency) can sometimes be mitigated due to the presence of myopia or censorship. If the supplier is allowed to modify the wholesale price in response to the updated demand information, it is possible that sometimes the retailer wants to "create censorship" so as to prevent the supplier from choosing an unfavorable wholesale price. It is also demonstrated that buy-back contracts can still coordinate the supply chain with demand censorship.

These results contribute to the literature on demand censorship, supply chain coordination, and strategic information acquisition. Our findings shed light on the retailer-supplier interactions for information acquisition in more general settings. For example, when the supplier has a stronger bargaining power and enjoys a higher share of the total surplus, she is more willing to contribute to information acquisition, whereas when the supplier is the weaker party in the relationship, she tends to take advantage of the retailer's effort for information acquisition. Interestingly, with demand censorship, the collaborative behavior between the players for information acquisition may improve the system's performance. In other words, sometimes the lack of information could provide an incentive for supply chain integration.

1.2 An e-commerce platform

The second project is motivated by Amazon.com, a unique e-commerce platform serving both as a marketplace and a merchant. There are a large number of third-party sellers doing business on Amazon.com, and their transaction data (e.g., demand and price) is readily available to the platform owner. This information enables the platform owner to decide which market to enter, which product to sell, etc. In other words, the platform owner benefits both from the volume of business of the third-party sellers (commission income) as well as from the information these businesses have generated. How could the platform deploy its analytical and computing capabilities to manage the "learning and entry" process in a massive scale? How would the third-party sellers respond to the threat of market entry from the platform owner? How does this response affect the platform owner's entry decision?

To address the above questions, I consider a simple model of an online retail platform. There is one independent seller (IS), selling a single product on the platform. The IS knows the parameters of the demand function for the product, but the platform owner (PO) only has a prior probability distribution over these parameters. The study focuses on the strategic interactions between the PO and the IS. The PO announces an entry policy. Under this entry threat, the IS chooses a retail price. For each period, the PO observes the realized demand and updates her distribution over the demand parameters. As soon as the entry condition is met, the PO enters the market, driving out the IS.

I developed a Bayesian statistical model for the PO's demand learning, proposed two types of heuristic entry policies for the PO, and studied the IS's optimal pricing decision to manipulate the PO's learning process and thus postponing the PO's entry time.

One of the key results is the characterization of the IS's optimal pricing decision. Using the optimal price when there is no entry threat from the PO (i.e., the PO commits to no entry) as a benchmark, we find that the IS sets a higher price (than the benchmark price) when the price -sensitivity parameter of the demand function is large, and sets a lower price if the price-sensitivity parameter is small. The purpose of this is to slow down the PO's learning process and delay the entry. I have also characterized what the "best' products should look like for the IS: the maximum profits from these products under the hypothetical scenario of no entry threat from the PO should be in the mid-range. In other words, the IS should avoid the extreme product choices. I have also found that the PO might be better off forgoing the entry option, and this happens only when the PO's prior distribution has small variances (and the prior means alone do not support entry). On the other hand, if the prior variances are high, signaling the PO's "openness to learn," the PO tends to want to retain the entry option. Finally, numerical examples show that the PO's profit loss can reach 10% due to the IS's pricing distortion behavior (relative to the optimal price without entry threat).

The model provides a characterization of the PO's learning ad entry process, predicts a pattern of price distortion, and describes the product offering choices made by the independent seller. These could potentially serve as testable results for empirical studies.

Chapter 2

On the Value of Demand Learning in a Supply Chain with Demand Censorship

2.1 Introduction

Unobserved lost sales due to retail stockouts, known as the demand censorship effect, has attracted considerable attention from retail practitioners and academics, for it could cause self-sustaining bias in demand forecasts for both the retailer and the supplier. In view of the difficulty to accurately account for this nearly undetectable informational loss, researchers proposed inventory experimentation to "learn" the real demand at the retail level. However, there has been little understanding of the strategic effects of demand censorship and demand learning in retail supply chains. The growing trends to involve the entire supply chain in information acquisition and business analytics driven by customer demand require concrete managerial insights into the strategic interactions between the supply chain players for this matter.

In this paper we studied the strategic implications of demand learning under demand censorship through dynamic inventory experimentation in a supply chain. We developed a stylized supply chain model with one supplier and one retailer subject to demand censorship, interacting under a constant wholesale price contract set by the supplier. We analyze the supplier's optimal wholesale price and the supply chain performances in three situations with different information structures and retailer's inventory policies: with fully observed demand ("O"), with myopic retailer under demand censorship ("M"), and with forward-looking retailer under demand censorship ("F").

Our key findings are summarized as two phenomena on the supply chain efficiency and the retailer's profits across the three situations, in contrast to the existing results without considering the supplier's strategic decisions. First, under demand censorship, the supply chain and the retailer could benefit from having or being a "myopic" retailer as opposed to the "forward-looking" retailer, due to the supplier's voluntary offer of lower wholesale price to induce "learning" from the myopic retailer. Second, with demand censorship, the supply chain and the retailer could be better off than in the situation with fully observed demand, for the supplier and the retailer's would invest to overcome censorship through lower wholesale price or higher orders. Extensive numerical studies show that these phenomena arising in the decentralized systems have significant impact that the current studies fail to account for by simply resorting to the two informational inefficiencies of censorship and the retailer's myopia. These observations are also valid in general setup with continuous demand distributions.

By including the decentralization inefficiency in supply chains, we attribute the two phenomena above to the interplay between double-marginalization and the informational inefficiencies. Namely, the presence of information inefficiencies (censorship and myopia) provides incentive for the supply chain players to invest in information acquisition, which alleviates their strategic conflicts despite of the simple contract structure. The analysis on conditions for these phenomena and their quantitative magnitude identifies the opportunity to improve supply chain efficiency and the supply chain players' profits in practice with certain environment. Moreover, it also sheds light on the strategic concerns for the retailer to adopt lost sales tracking system and inventory experimentation policy in practice, as well as the supplier counter-actions to such behaviors.

Stockouts are nearly unavoidable in retail practice for almost all product categories (grocery, electronics, and fashion etc.), and both online and offline channels. Persistent stockouts have multi-fold effects on retailers' bottom line. Most directly, it signifies failure to fulfill the potential demand and lost sales. According to Gruen & Corsten (2007)[1] , "the worldwide average level of OOS (out-of-stock) in the FMCG industry amounts to about 8 percent", which "translates to a 4 percent reduction in the average retailer's earnings per share", and "put $7 to $12 billion of sales at play in the supermarket industry"[2]. Given the generally thin operating margin for supermarkets and alike[1], such losses are substantial for retailer's profitability. The impact of retail stockouts also involve other operational and strategic concerns for retailers and suppliers, such as jeopardizing brand and store image, hampering sales and promotional effort, and encouraging trial or purchases from competitors. Yet it is often less recognized that stockouts "create a ripple effect by distorting demand and leading to inaccurate forecasts"[1], for the lost sales are usually neither reported by the customers nor recorded by the personnel, resulting in significant information loss for demand forecasts, known as "demand censorship".

The general approach to mitigate the demand censorship effect is to estimate and count in the potential lost sales during stockouts, based on historical sales data. This means is supported by research on various estimation techniques, and the retailer's rich data storage and powerful statistical toolkits enhance its viability. However, for **new, seasonal, temporary, and fashionable** products etc., adequate data accumulation is oftentimes lacked. In these cases, practitioners admit that "currently, no techniques exist to reliably and validly estimate demand under unobserved lost sales"[1], and they adopt "an iterative process that increases order sizes by a factor determined by the item's sales velocity and volatility", infused with "trial-and-error" spirit. Such inventory experimentation spurs the research of demand learning, focusing on tactical characterization of the "exploitation vs. exploration" trade-off therein. As most such practices take place at the retail level, these studies solely take the retailer's perspective, without considering its impact on the suppliers and their response.

Arguably, upon stockouts of the product in need, customers more often opt for alternative brands

[1] The operating margin for large-scale supermarkets are generally no higher than 5%.

(or products) rather than different stores, which somewhat compensates the retailer's loss. Therefore, demand censorship leading to long-term distorted orders from the suppliers could have important implications on their performance. This is partly reflected by the research sponsored by some suppliers on retail stockouts, such as Coca-Cola and P&G. These concerns are further stressed by two industry trends highlighting the supplier's involvement in information acquisition. The first is demand-driven supply chain integration to capture or even create the ever-changing customer needs, with the prominent example of fast fashion industry[3]. The second is more diverse formats of business analytics, inter- or intra-organization, with external or internal data sources, and centralized or decentralized institutions[5]. These observations raise a series of research questions on the strategic effects of demand learning under the aforementioned demand censorship in retail supply chains. How would the supplier respond to censorship effect and learning opportunities? Would she collaborate with the retailer to facilitate learning or exploiting the refined demand information owing to the retailer's effort? How would the retailer counter-act to the supplier's policies? Which party among the two has a stronger incentive to invest in demand learning in various business environments? These important issues motivate and are addressed in this paper.

The rest of the paper would be organized as follows. We first present a literature review in the rest of this section. Then we introduce the model setup to formulate and solve the two players' optimal policies, based on which we will present our major findings of the two counter-intuitive phenomena in the decentralized systems in contrast to the existing results without taking account of the supplier's strategic decisions. We also conduct extensive numerical studies to support and complement the analytical results, and discuss how other contractual forms affect the results.

2.1.1 Literature Review

This paper contributes to and is related to the literatures on demand censorship, supply chain coordination, information acquisition in distribution channels, and strategic experimentation. The above research topics have developed mostly independent of each other, so we briefly review each strand and discuss how they are integrated in this paper.

First and foremost, our work stems from the literature of demand censorship at the retail level by modeling inventory experimentation. Here we refer to Chen & Mersereau (2013) [13]for a thorough and up-to-date survey for this literature as well as the related empirical and computational work. Based on a few early analytical work to recognize and understand demand censorship effect, Lariviere & Porteus (1999)[8] developed a tractable Bayesian inventory control model to account for inventory experimentation, presenting the key insight of "ordering more than 'myopic' to learn the demand". Subsequent studies further generalize this result in different setups, with general continuos [10] and discrete demand distribution[11], perishable and non-perishable product [11], single and multiple products [11], and under Bayesian and non-Bayesian framework [12] etc.. Notably, among these studies, Besbes and Muharremoglu (2013)[12] characterized and compared the impact of demand censorship with continuous and discrete demand distribution, and pointed out that the granularity of demand information and forecast plays a critical role on the magnitude of censorship effect. Though conducted in a non-Bayesian framework that is different from ours, it directed us to adopt the Bernoulli demand system to emphasize the censorship effect.

The literature above unexceptionally takes the retailer's perspective to study the tactical effect of censorship and learning. So far, the only attempt to explore demand censorship and learning problem in a supply chain setting was from Bisi et al. (2004)[14], comparing the inventory decisions and supply chain efficiency of the centralized systems (with or without information pooling from two retailers) and the decentralized system. This manuscript didn't include an equilibrium analysis for the self-interested players, though it also touched on the implementation of the desired outcomes in decentralized systems using contracts, leaving out the critical incentive issues for the proposed dynamic contract from a game theoretical perspective. Therefore, we consider our paper as the first endeavor to investigate the strategic interactions in supply chain with the demand censorship and demand learning through inventory experimentation.

Secondly, from the perspective of supply chain management, there exists a vast literature on contracting and information sharing for supply chain coordination. We refer to Cachon (2003)[20] and Chen (2003)[19] for comprehensive surveys of this literature. Based the notion established by these research, our paper well recognizes the value to "integrate the evolving demand forecast into the planning decisions" and to resolve the incentive conflicts arising from various information structure and contract settings across supply chains. In particular, among this literature, Lariviere & Porteus (2001) set groundwork for our study on the relationship between the optimal wholesale price selling to the newsvendor, in particular, the associated double-marginalization effect, and the customer demand distribution. However, very few (if not none) of these papers explicitly consider endogenous information acquisition of supply chain players and how it is at a play in the operational decisions and strategic interactions.

Besides the papers discussed in the two surveys [20][19], an ensuing branch of literature on *collaborative planning, forecasting, and replenishment* (CPFR) ([21]) further delved into improving and sharing the evolving demand forecast between the supply chain players, and we refer to Aviv (2004) [22] for a brief survey of this literature.[24] Most notably, a recent paper by Kurtulus et al. (2012)[23] studied the supply chain players' investment in demand forecasting resulting from the strategic interactions under certain contracts. Taking such investment as a means of "costly information acquisition", it addresses a similar strategic information acquisition problem. However in their model, the informative demand signals are collected independent of the operational decisions, as opposed to via experimentation with operational instruments. This leads to the fundamental difference from the demand learning literature using "Bandit problem" framework with the "exploitation vs. exploration" trade-off attached to the operations decisions. We believe that the latter is close to the real data collection practice and helps to examine integrated business analytics policy.

In the marketing literature, there is also remarkable interest in information transmission in distribution channels. Among other work on this topic, Guo (2009)[25] considered a static model with explicit information acquisition cost in a vertical relationship with a supplier, and presents results on retailer's incentive information acquisition and sharing with the supplier resonating with our results.

Finally, we like to refer to a growing body of literature in microeconomics on dynamic contracts with information acquisition. A number of these work adopt the "Bandit problem" theoretical framework, and we refer to Bergemann & Välimäki (2006)[27] for a brief survey of its development and applications in economic literature. This literature generally consider a principal-agent setup with risky projects, and study the principal's optimal contract to induce favorable level of experimentation from the agent on payoff-related information on the project. Potential incentive conflicts pertain to the agent's effort on experimentation per se and the desirability of information. Examples include but not limit to the relationship between venture capitalist and entrepreneur on a start-up[28], pharmaceutical company and scientists on drug tests[29], and the regulator and bank management on the distressed assets[51].

To a certain extent, we view the current paper as an application of the economic model of Bandit problem in an operations management context. We start with a simple setup without asymmetric information or hidden action from the agent (retailer), and a static contract framework to catch the first order effect of strategic experimentation. We would study the corresponding dynamic contract problem in a following project, and discover interesting results on dynamic incentives. Also to catch the strategic effect purely due to experimentation, we consider a simple model without general operational concerns, such as capacity constraint, production lead time, inventory carry-over, mid-period replenishment chance, or economic concerns of complicated contract forms and asymmetric information as other study generally do. This allows us to develop a tractable model and provide basic insights into this issue.

2.2 The Model

We first present the model primitives needed to formulate the retailer's inventory problem and the supplier's optimal wholesale price problem, including the demand system, the cost structure, and the time line of events. Then we describe the information structures of different situations.

2.2.1 Model Primitives

We consider a stylized model of a supply chain with one supplier (she) and one retailer (he), selling a single perishable product over two periods. We model the interactions between the supplier and the retailer as a Stackelberg game of symmetric information, with the supplier as a Stackelberg leader to set a constant wholesale price and the retailer follows as a newsvendor.

The demand system

For tractability, we assume that the customer demand D follows a Bernoulli distribution of two possible values: $\left\{D^L, D^H\right\}$ $(0 \leq D^L < D^H)$. The probability of having a low demand realization at period $t \geq 0$ is a random variable $P_t := P\left(D_t = D^L\right)$ that follows a Beta prior known by the two players[2]:

$$P_t \sim Beta(\alpha_t, \beta_t), \alpha_t > 0, \beta_t > 0.$$

At period t, the demand follows a compound Bernoulli distribution with the expected success probability

$$P\left(D_t = D^L|\alpha_t, \beta_t\right) = \frac{\beta_t}{\alpha_t + \beta_t} := p_t.$$

And the demand uncertainty of period t can be measured by the coefficient of variation of D_t,

$$CV(D_t|\alpha_t, \beta_t) = \frac{\sqrt{p_t(1-p_t)}}{\frac{1}{1-\rho} - p_t}, \rho := \frac{D^L}{D^H} \in [0, 1),$$

which is higher with a lower ρ, representing a high demand variability, and with p_t close to $\frac{1}{2}$, representing a high information ambiguity.

The cost structure

The supply chain has a classic setup with the retailer as the newsvendor. The unit production cost c, salvage value v, retail price r are exogenous and satisfy the condition $0 \leq v \leq c \leq r$. The unit wholesale price $w \in [c, r]$ is set by the supplier. Therefore, the supply chain's critical ratio is:

$$s := \frac{r - c}{r - v} \in [0, 1],$$

[2] Beta distribution is the Bayesian conjugate distribution of Bernoulli distribution.

and the retailer's critical ratio is:

$$k(w) := \frac{r-w}{r-v} \in [0, s]. \tag{2.1}$$

The supply chain's critical ratio s (also known as the service level) determines the supply chain's performances. Similarly, $k(w)$ captures the retailer's cost structure, so we use $k\,(w)$ as the "price index" based on the one-to-one mapping between $k\,(w)$ and the wholesale price w.

Event Timeline

Through the selling season of two periods, the supply chain players are firstly informed of the prior demand information $\{\alpha_1, \beta_1\}$, then the supplier sets the constant wholesale price w accordingly for the entire selling season. At each ensuing period $t \in \{1, 2\}$, the interactions between the retailer and the supplier unfold as follows:

1. The retailer sets the inventory level y_t based on $\{\alpha_t, \beta_t\}$ and orders from the supplier, and the production is instantly completed and delivered by the supplier at a unit cost c, then the supplier collects a revenue of $w \cdot y_t$ from the retailer;

2. The customer demand D_t and the sales $x_t = \min\{y_t, D_t\}$ are realized, and the retailer collects a newsvendor revenue $r \cdot x_1 + v \cdot \{y_t - D_t\}^+$;

3. The demand information updates to $\left\{\alpha_{t+1}, \beta_{t+1}\right\}$.

The information availability and belief updating process depend on the supply chain's information structure that we describe next.

2.2.2 The Information Structures

Following the existing literature on demand censorship, we consider two information structures of the supply chain: "fully observed demand" (O) and with demand censorship.

Fully observed demand

With fully observed demand, the supplier and the retailer observe the demand realization D_1 at the first period. With the Bernoulli-Beta conjugate demand distribution, the posterior updates in the second period in a Bayesian fashion:

$$\begin{cases} \alpha_2 = \alpha_1 + 1\{D_1 = D^H\} \\ \beta_2 = \beta_1 + 1\{D_1 = D^L\} \end{cases}.$$

Demand censorship

With demand censorship, the two supply chain players only observe the sales realization as opposed to the demand realization. This means that, if the demand D_1 exceeds the inventory level y_1 at the first period ($D_1 \geq y_1$), the players observe the "stock-out" event $1\{D_1 \geq y_1\}$ and the sales $x_1 = \min\{y_1, D_1\}$ instead of D_1. In this case the belief updating in the second period is affected by the first period's inventory level with the two possible cases:

1. if $y_1 \leq D^L$, $\begin{cases} \alpha_2 = \alpha_1 \\ \beta_2 = \beta_1 \end{cases}$;
2. if $y_1 > D^L$, $\begin{cases} \alpha_2 = \alpha_1 + 1\{D_1 \geq y_1\} \\ \beta_2 = \beta_1 + 1\{D_1 = D^L\} \end{cases}$.

Under this belief updating rule, the first period's inventory level y_1 has a discontinuous effect on the posterior belief $\{\alpha_2, \beta_2\}$. When $y_1 \leq D^L$, a "stock-out" always occurs regardless of the demand realization D_1, so there is no new information becoming available, and no updating. While if $y_1 > D^L$, a "stock-out" $\{D_1 \geq y_1 (> D^L)\}$ indicates that $D_1 = D^H$ for sure as if there's no censorship effect. Therefore any $y_1 > D^L$ reveals "full information" while any $y_1 \leq D^L$ reveals "no information", so the retailer could achieve "nearly costless information acquisition" if he sets $y_1 = D^L + \varepsilon$ with an infinitesimal ε.

To prevent this unrealistic result and rationalize the retailer's decision in this case, we set a lowest inventory level for the retailer to uncover the high demand realization $D_1 = D^H$, defined as $D^L + \varepsilon, \varepsilon := \eta \cdot \left(D^H - D^L\right), \eta \in (0,1]$. η is the "cost of learning" and reflects the reality that in practice orders are placed in batch size instead of an arbitrary amount. In the analysis hereafter, we focus on the case of $\eta = 1$, meaning that the retailer has to order up to D^H to "learn" the demand information[3]. In this case the belief updating process boils down to the following two cases:

1. if $y_1 < D^H$, $\begin{cases} \alpha_2 = \alpha_1 \\ \beta_2 = \beta_1 \end{cases}$;

2. if $y_1 \geq D^H$, $\begin{cases} \alpha_2 = \alpha_1 + 1\{D_1 \geq y_1\} \\ \beta_2 = \beta_1 + 1\{D_1 = D^L\} \end{cases}$.

In this case, the expected success probability of the second period $p_2 = E(P|\alpha_2, \beta_2)$ could only take three values, corresponding to the cases of "no updating", "high demand realization", and "low demand realization" respectively:

$$\begin{cases} \text{if } y_1 < D^H, & p_1 := \frac{\beta_1}{\alpha_1+\beta_1}, \\ \text{if } y_1 = D^H, D_1 = D^H, & \underline{p}_2 := \frac{\beta_1}{\alpha_1+\beta_1+1}, \\ \text{if } y_1 = D^H, D_1 = D^L, & \bar{p}_2 := \frac{\beta_1+1}{\alpha_1+\beta_1+1}, \end{cases} .$$

We can define $I := \alpha_1 + \beta_1$ and re-write $\left\{\underline{p}_2, \bar{p}_2\right\}$ as follows:

$$\begin{aligned} \underline{p}_2 &= p_1 \cdot \left(1 - \frac{1}{I+1}\right) = p_1 - \frac{p_1}{I+1} \\ \bar{p}_2 &= p_1 \cdot \left(1 - \frac{1}{I+1}\right) + \frac{1}{I+1} = p_1 + (1-p_1) \cdot \frac{1}{I+1} \end{aligned} \tag{2.2}$$

The Beta prior $\{\alpha_1, \beta_1\}'s$ information effect can be fully captured by $\{p_1, I\}$,where the expected success rate p_1 represents the prior pessimism, and the number of accurate observations I characterizes

[3] Clearly there's no benefit from ordering higher than D^H for the purpose of learning.

the prior information richness. With a fixed I, $\left\{\underline{p}_2, \bar{p}_2\right\}$ both increase with p_1; and with fixed p_1, the difference between $\underline{p}_2$or $\bar{p}_2$ and p_1 decrease with I. So a lower information richness I could lead to a higher belief change over time and adds to the overall demand uncertainty besides the static demand uncertainty captured by the coefficient of variation of each period. Hereafter, for dispositional simplicity, we omit the subscripts in $\{\alpha_1, \beta_1\}$, and denote it as $\{\alpha, \beta\}$.

2.3 Optimization

In this section, we formulate and analyze the retailer's inventory problem and the supplier's wholesale price problem to solve the Stackelberg game in a backward induction. The simple Bernoulli demand distribution allows us to analytically solve the optimal policies and characterize the solutions using a "case" solution system. Using this "case" solution system, we can also conduct comparative statics studies and compare the optimal policies in three situations of different information structures and decision profiles to examine the impact of demand censorship and learning in a decentralized supply chain.

2.3.1 The Retailer's Inventory Problem

As the first step of the backward induction, we solve the retailer's optimal inventory policies for any given wholesale price w. We consider three different decision profiles of the retailer: "fully observed demand" (O), "myopic under demand censorship" (M), and "forward-looking under demand censorship" (F).

The retailer's inventory problem with a given w is a variation of the known inventory experimentation problem studied in the existing literature ([8][10][11]). It is shown by [12]in a non-parameter setting that demand censorship effect is more pronounced with discrete demand distribution or coarser information. So the Bernoulli-Beta setup also stresses the censorship effect besides providing tractability.

Fully Observed Demand

With fully observed demand, the inventory level of earlier periods has no information value for future decisions. As there is no inventory carry-over for the perishable product, the retailer's inventory problems are decoupled for each period and could be solved independently.

At $t \in \{1, 2\}$, based on the posterior demand information $\{\alpha_t, \beta_t\}$, the retailer sets the inventory level y_t^O to optimize his newsvendor profit of that period:

$$V_{R,t}(y_t|\alpha_t, \beta_t) = E[r \cdot \min\{D_t, y_t\} + v \cdot \{y_t - D_t\}^+ - w \cdot y_t|\alpha_t, \beta_t].$$

This classic newsvendor problem has the following solution:

$$y_t^O(k(w)|\alpha_t, \beta_t) = \begin{cases} D^L & p_t \geq k\,(w) \\ D^H & p_t < k\,(w) \end{cases}. \tag{2.3}$$

The retailer's optimal inventory decision of each period could take two possible values coinciding the Bernoulli demand distribution's discrete support. The policy is determined by the relation between the retailer's cost structure denoted by the price index $k\,(w)$ and the demand level denoted by the posterior information $\{\alpha_t, \beta_t\}$.

The supplier's optimal wholesale price decision is made up-front before the belief updates, so from the supplier's perspective, the retailer's inventory policy of two periods is aggregated into four cases:

Solution 1 (Retailer, "O") *With fully observed demand, the retailer's optimal inventory policy* $\left\{y_1^O, y_2^O|k(w), \alpha, \beta\right\}$ *is as follows:*

Case 1 $\bar{p}_2 < k\,(w)\,(< 1)$, $y_1^O = D^H, y_2^O = D^H$

Case 2 $p_1 < k\,(w) \leq \bar{p}_2$, $y_1^O = D^H, y_2^O = \begin{cases} D^H & D_1 = D^H \\ D^L & D_1 = D^L \end{cases}$

Case 3 $\underline{p}_2 < k(w) \leq p_1$, $y_1^O = D^L, y_2^O = \begin{cases} D^H & D_1 = D^H \\ D^L & D_1 = D^L \end{cases}$

Case 4 $(0 <)k(w) \leq \underline{p}_2$, $y_1^O = D^L, y_2^O = D^L$

This "case system" serves to characterize the retailer's inventory policies and the supplier's optimal wholesale price policies in all three situations due to their common discrete structure.

Myopic Retailer with Censored Demand

We pointed out in 2.2.2 that with demand censorship, the first period inventory level y_1 affects the ensuing belief updating process, therefore has an "information value". Depending on the retailer's approach to this information value, his inventory policy takes two forms. The inventory policy is "forward-looking"(F)[4] if the retailer takes account of y_1's information value to make the first period's inventory decision; and conversely the policy is "myopic"(M) if he ignores this information value and only optimizes the payoff of the first period[5].

For the myopic retailer, his first period inventory decision is the same as in "O", i.e. with fully observed demand, given the same prior information and wholesale price, since he only optimizes the profit of the first period:

$$y_1^M = y_1^O, \forall k(w), \{\alpha, \beta\}.$$

According to the information updating processes under censorship defined in 2.2.2, when $y_1 = D^L$ there is no new information becoming available, i.e. no updating. In this case, at the second period, the retailer is faced with the same demand information $\{\alpha_2, \beta_2\} = \{\alpha_1, \beta_1\}$ and the same wholesale price w as in the first period. This implies that for the myopic retailer, the following case is not feasible:

[4] We replace the term "learning" often used in the literature with "forward looking" to characterize the retailer's behavior in this situation. In general, we consider "learning" as the action of exerting effort for information acquisition, so it could occur in the supplier behavior in a supply chain setting, even when the retailer is "myopic".

[5] The "myopic" retailer also "passively" updates his belief and correctly reckons in the censorship effect.

Case 3 $\underline{p}_2 < k \leq p_1$, $y_1 = D^L, y_2 = \begin{cases} D^H & D_1 = D^H \\ D^L & D_1 = D^L \end{cases}$,

as this case is merged with

Case 4 $(0 <)k \leq \underline{p}_2$, $y_1 = D^L, y_2 = D^L$;

to become a new case:

Case 4' $(0 <)k \leq p_1$, $y_1^M = D^L, y_2^M = D^L$.

This argument is formalized as the myopic retailer's optimal inventory policy :

Solution 2 (Retailer, "M") *Under demand censorship, the myopic retailer's optimal inventory policy* $\left\{y_1^M, y_2^M | k(w), \alpha, \beta\right\}$ *is:*

Case 1 $\bar{p}_2 < k\,(w)\,(< 1)$, $y_1^M = D^H, y_2^M = D^H$

Case 2 $p_1 < k\,(w) \leq \bar{p}_2$, $y_1^M = D^H, y_2^M = \begin{cases} D^H & D_1 = D^H \\ D^L & D_1 = D^L \end{cases}$

Case 4' $(0 <)k\,(w) \leq p_1$, $y_1^M = D^L, y_2^M = D^L$

With discrete demand distribution, the retailer's optimal inventory policy in "M" is only different from the policy in "O" under the conditions of Case 3, i.e. $\underline{p}_2 < k\,(w) \leq p_1$. And in this case the myopic retailer makes lower orders than the retailer with fully observed demand, a result consistent with the existing results. For ignoring $y_1's$ information value under censorship, the myopic retailer incurs a loss of:

$$\begin{aligned} \Delta V_{R,2} &= \delta \left\{ E_{D_1|\alpha_1,\beta_1}[V_{R,2}(y_2^O|\alpha_1,\beta_1,y_1^O,D_1)] - E_{D_1|\alpha_1,\beta_1}[V_{R,2}(y_2^M|\alpha_1,\beta_1,y_1^M,D_1)] \right\} \\ &= \delta C_\Pi \cdot (1-\rho)\,(k(w) - \underline{p}_2)(1-p_1) > 0 \\ & \quad \left(\delta \in (0,1] \text{ is the discount factor, } C_\Pi := (r-v) \cdot D^H\right) \end{aligned}$$

known as $y_1's$ "information value" under censorship.

Forward-Looking Retailer with Censored Demand

To optimize his total discounted profit over two periods, the forward-looking retailer takes account of $y_1's$ information value and solves the following dynamic program for the first period's inventory level y_1^F [6]:

Problem 1 (Retailer, "F")

$$y_1^F(k(w)|\alpha_1,\beta_1) = \underset{y_1\in[0,D^L]\cup[D^L+\varepsilon,D^H]}{\arg\max} V_R^F(y_1|\alpha_1,\beta_1)$$

$$V_R^F(y_1|\alpha_1,\beta_1) = V_{R,1}(y_1|\alpha_1,\beta_1) + \delta E_{D_1|y_1}[V_{R,2}(y_2^O|\alpha_2,\beta_2)]$$

$$(\underline{p}_2 < k(w) \le p_1)$$

In Problem 1 the feasible set for y_1 is modified to be $[0, D^L] \cup \left[D^L+\varepsilon, D^H\right]$ as discussed in 2.2.2, with each interval corresponding to one of the two options, "no-learning" and "learning". When $\eta := \frac{\varepsilon}{D^H-D^L} = 1$, the second interval collapse to $\left\{D^H\right\}$, so the feasible set can be further modified as $\left\{D^L, D^H\right\}$ as there is no benefit from ordering lower than D^L. In this case the forward-looking retailer chooses from the following two options:

Case 2' $y_1^F = D^H, y_2^F = \begin{cases} D^H & D_1 = D^H \\ D^L & D_1 = D^L \end{cases}$ (learning);

Case 4 $y_1^F = D^L, y_2^F = D^L$(non-learning).

The choice criterion between these two options boils down to the following threshold[7]:

$$p_2^F := \frac{p_1+\delta(1-p_1)\cdot \underline{p}_2}{1+\delta(1-p_1)} \in (\underline{p}_2, p_1)$$

[6] When $k(w) \notin (\underline{p}_2, p_1]$, the retailer's inventory problem is not influenced by the censorship effect, so the solution is the same in all three situations. In addition, the retailer's optimal inventory decision at the second period is $y_2^O(\alpha_2, \beta_2, k(w))$ as shown by backward induction.

[7] Under censorship, Case 3 and Case 4 merge as Case 4', so there is no "Case 3" in this solution.

Solution 3 (Retailer, "F") *Under demand censorship, the "forward-looking" retailer's inventory policy* $\{y_1^F, y_2^F | k(w), \alpha, \beta\}$ *is as follows:*

Proposition 1 •

Case 1 $\bar{p}_2 < k\left(w\right)(< 1)$, $y_1^F = D^H, y_2^F = D^H$

Case 2' $p_2^F < k\left(w\right) \leq \bar{p}_2$, $y_1^F = D^H, y_2^F = \begin{cases} D^H & D_1 = D^H \\ D^L & D_1 = D^L \end{cases}$

Case 4' $(0 <)k\left(w\right) \leq p_2^F$, $y_1^F = D^L, y_2^F = D^L$.

Solution 3 shows that under censorship, the forward-looking retailer's trade-off on the costly information acquisition is represented by the price index $p_2^F \in \left(\underline{p}_2, p_1\right)$. Given the prior information $\{\alpha, \beta\}$ and for $k\left(w\right) \in \left(\underline{p}_2, p_1\right)$, "learning" only takes place when the wholesale price w is low enough so that $k\left(w\right) \in \left(p_2^F, p_1\right)$, otherwise the censorship effect cannot be prevented if the wholesale price w is so high that $k\left(w\right) \in \left(\underline{p}_2, p_2^F\right)$.

From another perspective, for a given wholesale price w, "learning" takes place if the interval $\left(p_2^F, p_1\right]$ includes the price index $k\left(w\right)$. Since the forward-looking retailer's order quantities are already fixed as in Case 2', we measure her "willingness to learn" by the width of this "learning interval":

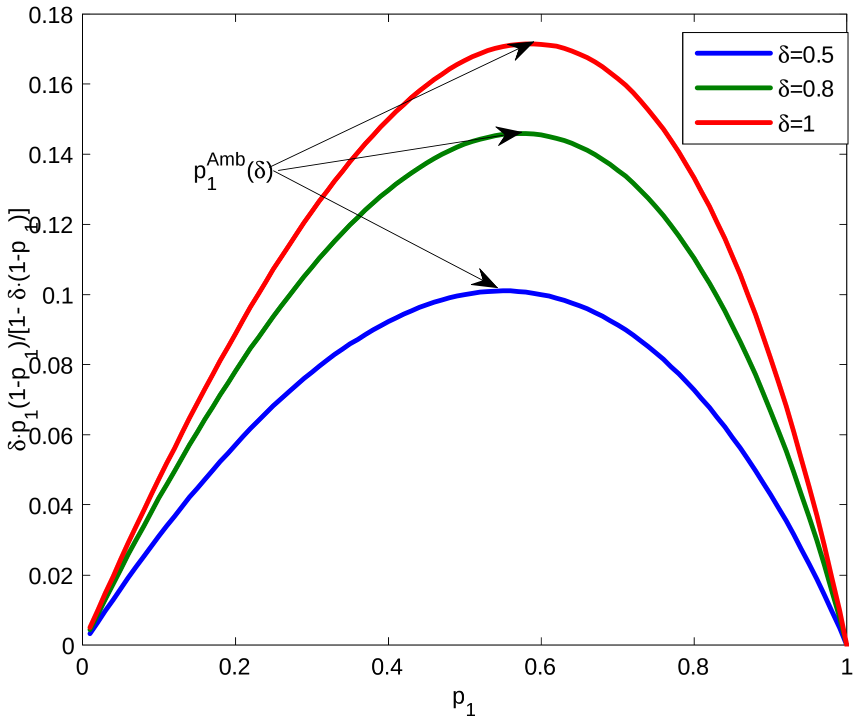
$$p_1 - p_2^F = \frac{1}{I+1} \cdot \frac{\delta\left(1-p_1\right)p_1}{1+\delta\left(1-p_1\right)}.$$

Clearly $\frac{1}{I+1}$ decreases with the prior information richness I and $\frac{\delta(1-p_1)p_1}{1+\delta(1-p_1)}$ increases with δ and is concave with p_1 as shown in Figure 2.1.

Therefore the retailer has a higher willingness to learn when there is scarce prior information (i.e. $I = \alpha + \beta$ is low), the prior information is vague (i.e. p_1 is in a medium region around $p_1^{Amb}\left(\delta\right) = \frac{1+\delta-\sqrt{1+\delta}}{\delta}$), or the retailer is more patient (i.e. δ is high).

So far we solved the retailer's inventory problem with the Bernoulli-Beta demand system in three situations ("O", "M", and "F"). The results are consistent with the existing literature's ([10][8][11]) conclusion that "the forward-looking retailer orders more than the myopic retailer to learn". Yet with

Figure 2.1: The Prior Information Ambiguity



discrete demand distribution, the decision on information acquisition under censorship amounts to the binary choice of "learning or not" as opposed to the continuous decision of "learning how much".

2.3.2 The Supplier's Wholesale Price Problem

Based on the retailer's optimal inventory policy, we proceed to examine the supplier's optimal wholesale policy. We modify the "case" solution system introduced in 2.3.1 to characterize the wholesale price problem and policies, in particular through partitions of the parameter space induced by the optimal policies.

The Case Solution System

Based on the retailer's optimal inventory policies in three situations, we formulate the supplier's optimal wholesale price problem as follows: $(* \in \{O, M, F\})$

Problem 2

$$\begin{aligned} w^*(\alpha,\beta) &= \underset{w\in[c,r)}{\arg\max} V_S^*(k(w)|\alpha,\beta) \\ V_S^*(k(w)|\alpha,\beta) &= (w-c)[y_1^*(k(w)|\alpha,\beta) + \delta E_{D_1|\alpha,\beta}[y_2^*(k(w)|\alpha,\beta,y_1^*,D_1)] \end{aligned}.$$

Since the retailer's inventory policies are discrete with respect to the price index $k\,(w)$ in all three situations (in particular, $\{y_1^*, y_2^*\}$ are step functions w.r.t. $k\,(w)$), the supplier's payoff functions V_S^* are also discontinuous with respect to $k\,(w)$. For example, with fully observed demand ("O"), $\forall k(w) \in (p_1, \bar{p}_2]$ results in the same retailer's decision:

$$\{y_1^O = D^H, y_2^O = \begin{cases} D^L & D_1 = D^L \\ D^H & D_1 = D^H \end{cases}\}.$$

So the supplier could set any $w \in [w(\bar{p}_2), w\,(p_1))$ to induce the same order quantities, and clearly she prefers the highest possible w. With slightly abuse of notations, we define the "highest" wholesale price in this interval as $w\left(p_1\right)^-$ corresponding to a price index $k\left(w\right)=p_1^+$ though it is not achievable. Therefore in "O", the supplier's optimal wholesale price decision is limited to the following four cases, each consisting of one "desirable" wholesale price w_i, the induced order quantities $\{y_{1,i}^O, y_{2,i}^O\}$, and the supplier's profit $V_{S,i}^O$:

Case 1 $k(w_1)=\bar{p}_2^+$, $y_{1,1}^O=D^H, y_{2,1}^O=D^H$,

$$V_{S,1}^O=C_\Pi\cdot(1+\delta)(s-\bar{p}_2)$$

Case 2 $k(w_2)=p_1^+$, $y_{1,2}^O=D^H, y_{2,2}^O=\begin{cases} D^H & D_1=D^H \\ D^L & D_1=D^L \end{cases}$

$$V_{S,2}^O=C_\Pi\cdot\left[(1+\delta)-\delta p_1\left(1-\rho\right)\right]\left(s-p_1\right)$$

Case 3 $k(w_3)=\underline{p}_2^+$, $y_{1,3}^O=D^L, y_{2,3}^O=\begin{cases} D^H & D_1=D^H \\ D^L & D_1=D^L \end{cases}$

$$V_{S,3}^O=C_\Pi\cdot\left[\rho(1+\delta)+\delta(1-p_1)\left(1-\rho\right)\right]\bullet\left(s-\underline{p}_2\right)$$

Case 4 $k(w_4)=0^+$, $y_{1,4}^O=D^L, y_{2,4}^O=D^L$

$$V_{S,4}^O=C_\Pi\cdot s\rho(1+\delta)$$

The supplier's optimal wholesale price decision in "O" boils down to the choice from these four cases, and it could be characterized by a partition of the primitive parameter space[8]

$$\boldsymbol{\Theta}:=\{s\in(0,1],\alpha>0,\beta>0,\delta\in(0,1],\rho\in[0,1)\}$$

as shown in Solution 4:

Solution 4 *With fully observed demand, the supplier sets the wholesale price w^O as follows:*

[8] For expositional simplicity, we aggregate all the cost parameters c, v, r in s, for s fully represents their impact on the retailer and supplier's policies. The relationship between $\left\{D^L, D^H\right\}$ is summarized by ρ too.

Case 1 when $\boldsymbol{\theta} \in \Theta_1^O$, $k(w^O) = \bar{p}_2^+$, $y_1^O = D^H, y_2^O = D^H$

Case 2 when $\boldsymbol{\theta} \in \Theta_2^O$, $k(w^O) = p_1^+$, $y_1^O = D^H, y_2^O = \begin{cases} D^H & D_1 = D^H \\ D^L & D_1 = D^L \end{cases}$

Case 3 when $\boldsymbol{\theta} \in \Theta_3^O$, $k(w^O) = \underline{p}_2^+$, $y_1^O = D^L, y_2^O = \begin{cases} D^H & D_1 = D^H \\ D^L & D_1 = D^L \end{cases}$

Case 4 when $\boldsymbol{\theta} \in \Theta_4^O$, $k(w^O) = 0^+$, $y_1^O = D^L, y_2^O = D^L$

$$\Theta_i^O := \{\boldsymbol{\theta} \in \Theta | V_{S,i}^O(\boldsymbol{\theta}) \geq V_{S,j}^O(\boldsymbol{\theta}), j \neq i\}, i = 1, ...4.$$ [9]

Similarly, we apply this approach to all three situations and express the supplier's optimal wholesale price decision with the corresponding partition of the parameter space■$\{\Theta_i^M\}_{i=1,2,4'}$ and $\{\Theta_i^F\}_{i=1,2',4'}$.

Solution 5 *With a myopic retailer under demand censorship, the supplier sets the wholesale price* w^M *as follows:*

Case 1 when $\boldsymbol{\theta} \in \Theta_1^M$, $k(w^M) = \bar{p}_2^+$, $y_1^M = D^H, y_2^M = D^H$

Case 2 when $\boldsymbol{\theta} \in \Theta_2^M$, $k(w^M) = p_1^+$, $y_1^M = D^H, y_2^M = \begin{cases} D^H & D_1 = D^H \\ D^L & D_1 = D^L \end{cases}$

Case 4' when $\boldsymbol{\theta} \in \Theta_{4'}^M$, $k(w^M) = 0^+$, $y_1^M = D^L, y_2^M = D^L$

Solution 6 *With a forward-looking retailer under demand censorship, the supplier sets the wholesale price* w^F *as follows:*

Case 1 when $\boldsymbol{\theta} \in \Theta_1^F$, $k(w^F) = \bar{p}_2^+$, $y_1^F = D^H, y_2^F = D^H$

Case 2' when $\boldsymbol{\theta} \in \Theta_2^F$, $k(w^F) = p_2^{F+}$, $y_1^F = D^H, y_2^F = \begin{cases} D^H & D_1 = D^H \\ D^L & D_1 = D^L \end{cases}$

Case 4' when $\boldsymbol{\theta} \in \Theta_{4'}^F$, $k(w^F) = 0^+$, $y_1^F = D^L, y_2^F = D^L$

[9] When there is a tie between multiple "cases", we assign θ to the set Θ_i^O with the smallest $i \in \{1, ..., 4\}$.

Comparative Statics and Characterization of the Wholesale Price Policies

We identified the partitions of Θ derived from the optimal wholesale prices in three situations, yet these expressions can not elucidate the optimal policies' properties. To describe these subsets, we first characterize the optimal wholesale prices' relationship with two critical primitive parameters $\{s, \rho\}$, representing the supply chain's cost structure and demand variability respectively.

Lemma 1 $\forall * \in \{O, M, F\}, w^*(\boldsymbol{\theta})$ *decreases with* s *and increases with* ρ.

Similar to Lemma 1, [15] also studied the relationship between the supplier's optimal wholesale prices and the supply chain's properties, with a stationary and exogenous demand distribution. It argues that within a linear distribution family, the optimal wholesale price to sell to a newsvendor decreases with the demand's coefficient of variation. Despite the evolving and endogenous demand information in our setting, Lemma 1 presents a consistent result, as the coefficient of variation of any period decrease with ρ regardless of the retailer's decision profile.

w's monotonicity w.r.t. $\{s, \rho\}$ enables us to project $\boldsymbol{\Theta}$ onto the $\{s, \rho\}$ $-$space, and generate the associated partitions of $s-$interval and $\rho-$interval to represent the corresponding partitions of Θ :

Definition 2 *Define*

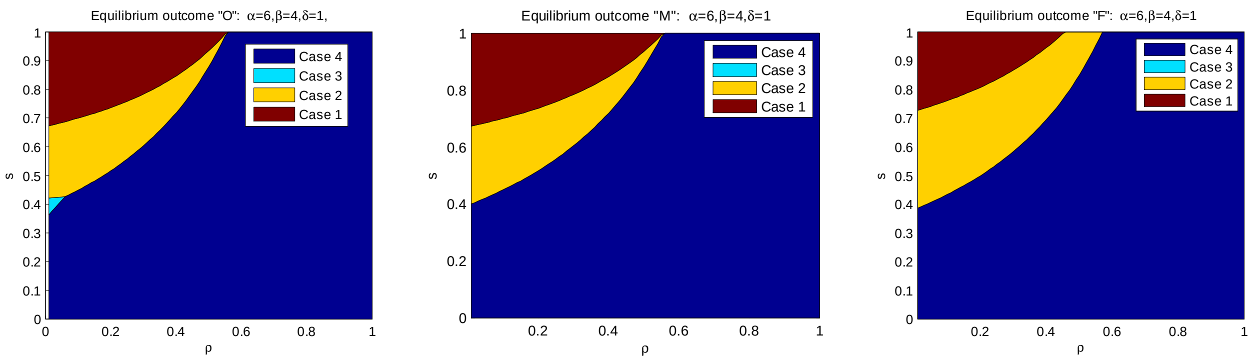
$$
\begin{aligned}
S_i^*(\boldsymbol{\theta}_{-s}) \quad &: \quad = \{s \in (0,1] | (s, \alpha, \beta, \delta, \rho) \in \Theta_i^*\}, \boldsymbol{\theta}_{-s} := \{\alpha, \beta, \delta, \rho\} \\
\varrho_i^*(\boldsymbol{\theta}_{-\rho}) \quad &: \quad = \{\rho \in [0,1) | (s, \alpha, \beta, \delta, \rho) \in \Theta_i^*\}, \boldsymbol{\theta}_{-\rho} := \{s, \alpha, \beta, \delta\}
\end{aligned}
$$

$$
s.t.\ s \in S_i^*(\boldsymbol{\theta}_{-s}) \iff \boldsymbol{\theta} \in \Theta_i^* \iff \rho \in \varrho_i^*(\boldsymbol{\theta}_{-\rho}).
$$

For more detailed characterization of $\{S_i^*\}_{i=1,\dots 4}$and $\{\varrho_i^*\}_{i=1,\dots 4}$, we refer to 1.1. We demonstrate the induced partitions of $\{s, \rho\}$ $-$space with $\{\alpha = 6, \beta = 4, \delta = 1\}$ in Figure 2.2:

By endogenizing the supplier's optimal wholesale prices, we introduced double-marginalization to the supply chain. In general, double-marginalization occurs when a wholesale price w is higher than the unit production cost c and induces a lower inventory level from the retailer than the system

Figure 2.2: The "Cases" of Supplier's Optimal Wholesale Price Decisions



optimum. In 1.1, we verify the understocking in the three situations $\{O, M, F\}$ caused by double-marginalization by comparing to the optimal inventory levels of their respective centralized systems $\{OC, MC, FC\}$[10].

2.4 Myopia and Censorship

With the retailer and the supplier's optimal policies, we could explicitly measure and compare the profits of the retailer, the supplier, and the entire supply chain in the three situations. The existing literature on demand censorship and learning ([8][10][12]) compared the retailer's profits in the three situations under exogenous wholesale prices and identified the following inequalities

$$V_R^O \geq V_R^F \geq V_R^M, \tag{2.4}$$

which are attributed to two types of informational inefficiencies for the retailer:

1. Censorship: $V_R^O \geq \max\left\{V_R^F, V_R^M\right\}$;

[10]The centralized systems' inventory problems are similar to the retailers inventory problems in their respective decentralized system with exogeneously fixed wholesale prices.

2. Myopia: $V_R^F \geq V_R^M$.

In contrast, by incorporating the supplier's strategic wholesale price decisions, we observe that demand censorship and the retailer's myopia do not necessarily cause losses to the retailer or the supply chain. We first present the conditions causing these counter-intuitive phenomena. Then we provide explanations and discuss their implications through descriptions of these conditions.

2.4.1 Impact on the Retailer's Profit

By endogenizing the supplier's wholesale price decision, we allow the supplier to respond to the two informational inefficiencies by leveraging the wholesale price. We identify the supplier's "exploitative" and "collaborative" behavior in response to the retailer's willingness to learn, that lead to the retailer's benefit from "myopia" and "censorship".

Two Counter-Intuitive Phenomena for the Retailer's Profits

In contrast to the inequalities in (2.4), we find that the retailer's profit could be higher under demand censorship or his own myopia under certain conditions, shown in the following inequalities:

$$\begin{aligned} &\text{Benefit from "censorship": } \max\left\{V_R^F, V_R^M\right\} > V_R^O \\ &\text{Benefit from "myopia": } V_R^M > \ V_R^F. \end{aligned} \tag{2.5}$$

Proposition 3 identifies the conditions for these two counter-intuitive phenomena through the partitions induced by the supplier's optimal wholesale prices in the three situations (see 1.1):

Proposition 3 *The necessary and sufficient conditions for the two counter-intuitive phenomena are:*

Observation	*Condition*
$V_R^M > V_R^F$	$\left(\Theta_2^M \cup \Theta_1^M\right) \cap \Theta_2^F := \Theta_R^{MF}$
$\max\left\{V_R^M, V_R^F\right\} := V_R^{Censor} > V_R^O$	$\left(\Theta_2^F \cup \Theta_1^F\right) \cap \left(\Theta_3^O \cup \Theta_4^O\right) := \Theta_R^{Censor,O}$

We refer to 1.2 for the proof of Proposition 3. In a nutshell, the retailer's profits are closely tied to his order quantities, so we prove it by comparing order quantities from different situations under the same parameter setting θ using the "case" solution system.

The supplier's Information Acquisition and the Retailer's Price Sensitivity

The two informational inefficiencies are associated with the supply chain's informational structures, so they not only influence the retailer but also the entire supply chain's performance. In their presence, the supplier leverages the wholesale price to respond, specifically to the retailer's different price sensitivities in the three situations.

With discrete demand distribution and discontinuous inventory policies, the retailer's price elasticities are not well-defined. With slight abuse of definition, we consider the retailer's one-sided "price sensitivity" at the demand curves' discontinuous points $k(w) = \left\{\underline{p}_2, p_2^F, p_1\right\}$ (see Figure 2.3) and observe that:

1. For $k(w) = \underline{p}_2$, the retailer in "M" and "F" has a lower sensitivity to price drop (or k's increase) than in "O"[11];

2. For $k(w) = p_2^F$, the retailer in "F" has a higher sensitivity to price drop (or k's increase) than in "O" (and "M");

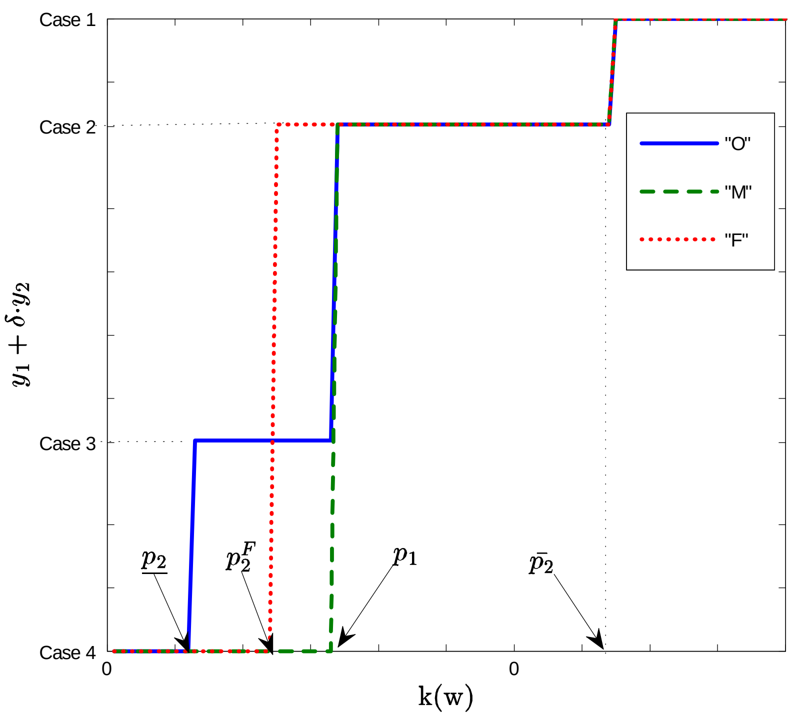
3. For $k(w) = p_1$, the retailer in "F" has a lower sensitivity to price increase (or k's drop) than in "O" (and "M"), and the retailer in "M" has a higher sensitivity to price drop (or k's increase) than in "O" (and "F").

The analysis above shows that there is no uniform ordering of the retailer's price sensitivities across the three situations. Yet we notice that the forward-looking retailer presents a "willingness to learn" by his lower sensitivity to price increase at $k(w) = p_1$ and higher sensitivity to price drop at $k(w) = p_2^F$[12]. Consequently, when $\theta \in \Theta_R^{MF}$, to exploit the forward-looking retailer's lower

[11] This also means that the retailer in "O" has a higher sensitivity to price increase. Similar equivalence apply to the other observations.

[12] And similarly the myopic retailer presents a "information stickiness" through his lower sensitivity to price drop at $k(w) = \underline{p}_2$ and higher sensitivity to price drop and $k(w) = p_1$.

Figure 2.3: The Retailer's Price (index)-Demand Curves in Three Situations



sensitivity to price increase, the supplier sets higher wholesale price $w^F(> w^M$ for $k\left(w^M\right) > p_1^+$) in "F" without causing censorship. This results in the forward-looking retailer's relative loss compared to the myopic retailer. Similarly, when $\theta \in \Theta_R^{Censor,O}$, to accommodate the forward-looking (and the myopic) retailer's higher sensitivity to price drop, the supplier sets lower price $w^F(< w^O$ when $k\left(w^F\right) > p_2^F))$ and $w^M(< w^O$ when $k\left(w^O\right) > p_1)$ to induce higher order quantities to overcome censorship and thus higher retailer profits in "F" and "M" than in "O".

These two phenomena capture the supplier's different responses to the retailer's "willingness to learn"[13]. We refer to the first as the supplier's "exploitative" behavior, when the forward-looking retailer's "learning" cost is relatively low and the supplier takes advantage of his willingness to learn; and the latter as the supplier's "collaborative" behavior, when the censorship effect is too costly for the retailer to overcome independently (in either "M" or "F"), and the supplier offers lower wholesale price to assist the retailer as an information acquisition behavior. These two behaviors result in the retailer's benefits from "myopia" and "censorship" respectively.

Characterizations of the Two Phenomena's Conditions

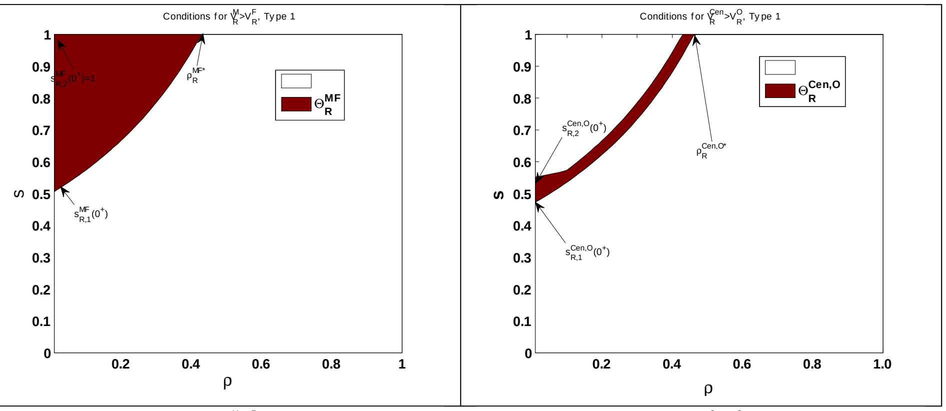
To characterize the conditions of the two phenomena, we project Θ_R^{MF} and $\Theta_R^{Censor,O}$ onto the $\{s,\rho\}$ −space. Figure 2.4 demonstrates the conditions for these two phenomena under different primitive parameter settings in the $\{s,\rho\}$ −space. We argue that the two phenomena only occur with high demand variability and service level. We also observe that the supplier is more powerful in a supply chain of lower service level s, thus has the incentive and capability to collaborate in information acquisition; while being less powerful in a supply chain of higher service level s, the supplier is incentivized to exploit the retailer's higher willingness to learn.

We define three thresholds $\left\{s^*, \rho_R^{MF*}, \rho_R^{Cen,O*}\right\}$[14] as follows to illustrate the conditions for the two

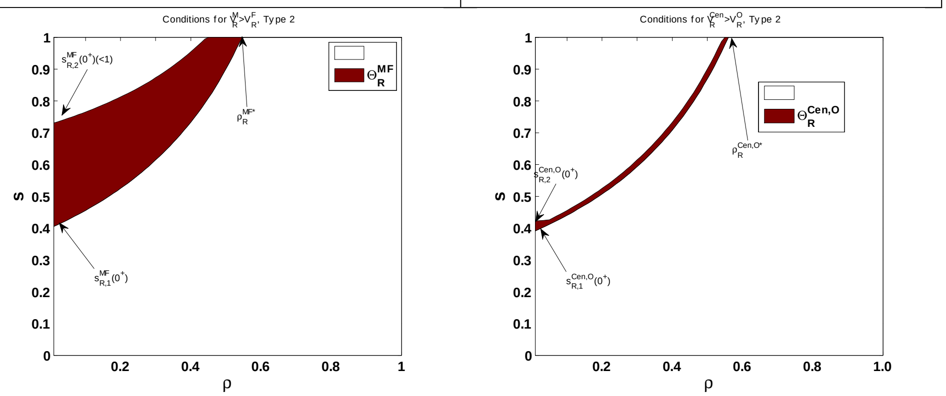
[13] The literature in CPFR and strategic experimentation[?] also discussed similar "substitutive" and "complementary" relationships between different players' investment in information acquisition.

[14] s^* is the highest possible s for the two phenomena to occur.

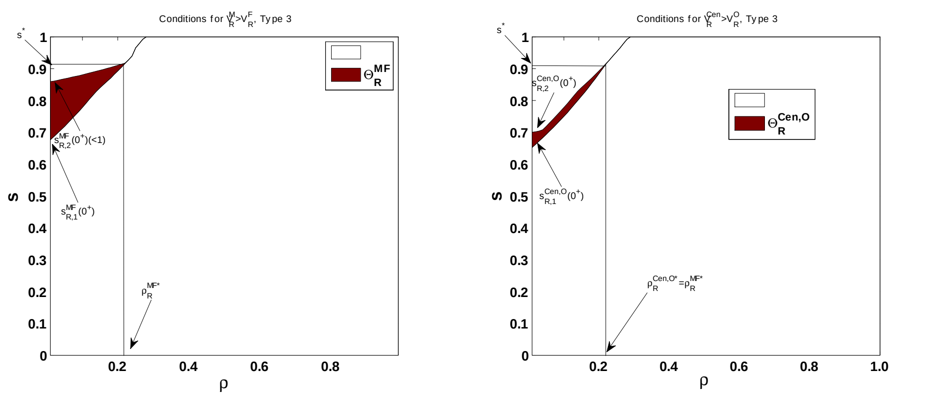
Figure 2.4: Condition for the Retailer's Benefit from Myopia or Censorship



$I \leq (1+\delta)/\delta p_1$



$I > (1+\delta)/\delta p_1$



$I > (1+\delta)/\delta p_1, s^* < 1$

phenomena not to occur.

$$
\begin{aligned}
s^* &:= \min\left\{\frac{\delta p_1 \bar{p}_2 p_2^F}{\delta p_1 \bar{p}_2 - \left(\bar{p}_2 - p_2^F\right)(1+\delta)}, 1\right\} \\
\rho_R^{MF*} &:= 1 - \max\left\{\frac{1+\delta}{\delta p_1 + \frac{s^*[1+\delta(1-p_1)]}{p_2^F}}, \min\left\{\frac{1+\delta}{\delta p_1 + \frac{s^*[1+\delta(1-p_1)]}{p_1}}, \frac{\bar{p}_2}{s^*}, 1\right\}\right\} \\
\rho_R^{Censor,O*} &:= 1 - \min\left\{\frac{1+\delta}{\delta p_1 + \frac{s^*[1+\delta(1-p_1)]}{p_2^F}}, \frac{\bar{p}_2}{s^*}, 1\right\}
\end{aligned}
$$

Lemma 2 *1. When $s \leq p_1$ or $\rho \geq \rho_R^{MF*}, V_R^M \leq V_R^F$*

2. When $s \leq p_2^F$ or $\rho \geq \rho_R^{Censor,O*}, V_R^F \leq V_R^O$

Lemma 2 shows that the two phenomena never occur when s is too low or ρ is too high, and $\left\{p_1, p_2^F\right\}$ are the lowest points of the $s-$interval while$\left\{\rho_R^{MF*}, \rho_R^{Censor,O*}\right\}$ are the right-most points of the $\rho-$interval for the two phenomena to occur. One could observe these boundaries in Figure 2.4 for the highlighted regions only lie at the upper-left corners of the $\{s, \rho\}$ space. We refer to **??** for the proof of Lemma 2. In a nutshell, we express the lower-right boundaries of Θ_R^{MF} and $\Theta_R^{Censor,O}$ in the $\{s, \rho\}$ space as ρ functions of s: $\left\{\rho_{R,2}^{MF}(s), \rho_{R,2}^{Censor,O}(s)\right\}$ [15], and locate the two ends of these two boundaries as the lowest/left-most and highest/right-most points of the regions.

Intuitively, the retailer has low willingness to learn when the supply chain's demand variability is low (corresponding to high ρ), and when the supply chain's service level is low (corresponding to low s), as in the latter case the supply is intrinsically limited and the retailer has very low bargaining power. Therefore there is little incentive for the supplier to leverage the wholesale price for higher profits either.

In practice, we observe that there is little attention to censorship effect of functional products ([18]) with stable demand corresponding to a high ρ. Also we notice that many flash sale websites for luxury fashion (e.g. Gilt and Rue La La) are often faced with significant censorship effect due to their

[15] Similarly, $\left\{\rho_{R,1}^{MF}(s), \rho_{R,1}^{Censor,O}(s)\right\}$ refer to the left boundaries of Θ_R^{MF} and $\Theta_R^{Censor,O}$, as functions of s.

short selling period (see [4]), however the stringent supply from luxury fashion suppliers corresponding to low s limits the retailer's ability and the supplier's incentive to overcome censorship.

From Figure 2.4, one can notice a "lower-left to upper-right" trend in the highlighted regions. To analytically characterize this pattern, we define $\left(s_{R,1}^{MF}(\rho), s_{R,2}^{MF}(\rho)\right]$ and $\left(s_{R,1}^{Cen,O}(\rho), s_{R,2}^{Censor,O}(\rho)\right]$ as follows:

$$\begin{aligned} \text{For } \forall\rho &\in \left(0, \rho_R^{MF*}\right], s \in \left(s_{R,1}^{MF}(\rho), s_{R,2}^{MF}(\rho)\right] \iff w^M < w^F \iff V_R^M > V_R^F \\ \text{For } \forall\rho &\in \left(0, \rho_R^{Censor,O*}\right], s \in \left(s_{R,1}^{Censor,O}(\rho), s_{R,2}^{Censor,O}(\rho)\right] \iff w^F < w^O \iff V_R^F > V_R^O \end{aligned}$$

Corollary 1 $\left\{s_{R,1}^{MF}(\rho), s_{R,2}^{MF}(\rho), s_{R,1}^{Censor,O}(\rho), s_{R,2}^{Censor,O}(\rho)\right\}$ *increase with* ρ.

Corollary 1 shows that, as the demand variability decreases, i.e. ρ increases, it takes higher service levels s for the two phenomena to take place[16]. [15]'s results show that demand uncertainty and the supply chain's service level have substitutive effects on the supplier's profit split. For instance, the supplier get a higher proportion with a lower demand uncertainty, corresponding to higher ρ, or with a lower service level s. So a higher service level offsets the increasing bargaining power of the supplier induced by a higher ρ, to provide sufficient incentive for the supplier to leverage the wholesale price for information acquisition.

Extreme Value Study for Prior Information's Impact

By projecting Θ_R^{MF} and $\Theta_R^{Censor,O}$ onto the $\{s, \rho\}$-space, we studied how the two counter-intuitive phenomena of the retailer's profit depend on the supply chain's intrinsic properties. To further delineate the prior information$\{\alpha, \beta\}$'s impact, we examine the conditions for the two phenomena in an extreme case of high demand variability, i.e.$\rho \to 0^+$[17]. We first explicate these conditions as follows:

Proposition 4 *When* $\rho \to 0^+$

[16] We also notice that the width of $S_R^{MF}(\rho) := s_{R,2}^{MF}(\rho) - s_{R,1}^{MF}(\rho)$ and $S_R^{Cen,O}(\rho) := s_{R,2}^{Cen,O}(\rho) - s_{R,1}^{Cen,O}(\rho)$ decrease with ρ,so the severity of the two phenomena of the retailer's profits decrease with ρ.

[17] Intuitively, a higher discount factor δ amplies the forward-looking behaviors of the supplier and the retailer, and the two phenomena. So we simply assumes δ is high enough in the analysis hereafter.

1.

$$\left(s_{R,1}^{MF}(\rho), s_{R,2}^{MF}(\rho)\right] := S_R^{MF}\left(0^+\right) = \left(p_1, \min\left\{\frac{\left(\bar{p}_2 - p_2^F\right)(1+\delta)}{\delta p_1} + p_2^F, 1\right\}\right]$$

2.

$$\left(s_{R,1}^{Censor,O}(\rho), s_{R,2}^{Censor,O}(\rho)\right] := S_R^{Censor,O}\left(0^+\right) = \left(p_2^F, p_1\left[1 + \delta\left(\bar{p}_2 - p_1\right)\right]\right]$$

Since the retailer's benefits from censorship and myopia are proportionate to ρ thus negligible in magnitude in this case, we measure the two phenomena's impact by the width of their conditions in the $s-$interval $S_R^{MF}(0^+)$ and $S_R^{Censor,O}(0^+)$, and study the prior information's effect on them through the information richness I and the optimism (pessimism) p_1[18]:

Corollary 2 *1. The width of $S_R^{MF}(0^+)$ decreases in I and p_1;*

2. The width of $S_R^{Cen,O}(0^+)$ decreases I; increases with $p_1 \in \left(0, p^{Censor,O}(\delta)\right]$ and decreases with $p_1 \in \left(p^{Censor,O}(\delta), 1\right]$[19];

Corollary 2 shows that both exploitative and collaborative behaviors diminish as the prior information gets richer, i.e. I increases, for the information value reduces in this case and the retailer and the supplier's incentive for information acquisition also decrease. Yet the prior information pessimism p_1 has different effects on the two types of behaviors. The supplier's exploitative behavior diminish as p_1 grows, i.e. the prior information gets more pessimistic, since in this case the retailer's willingness to learn decreases. In contrast, the supplier's collaborative behavior is more pronounced with ambiguous prior information, and is maximized at $p_1 = p_1^{Censor,O} \sim 0.53$. Intuitively, the retailer's willingness to learn is high with a more optimistic demand information and so does the supplier's; while the collaborative behavior occurs when both the retailer and the supplier have high willingness to learn with ambiguous prior information corresponding to a medium value of p_1.

We also have the following observations regarding the locations of $S_R^{MF}(0^+)$ and $S_R^{Censor,O}(0^+)$:

[18] The difference in the retailer's profits concerning the two comparisons are less measurable.

[19] $p^{Censor,O}(\delta)$ is the unique solution in $(0,1)$ for the following equation

$$-2\delta^2 p^3 + 5\delta(\delta+1)p^2 - 4(\delta+1)^2 p + (\delta+1)(\delta+2) = 0.$$

and $p^{Censor,O}(\delta) \in (0.532, 0.535)$ for $\delta \in (0.9, 0.999)$.

1. $\begin{cases} p_2^F < p_1 \\ p_1\left[1+\delta\left(\bar{p}_2-p_1\right)\right] < \min\left\{\frac{\left(\bar{p}_2-p_2^F\right)(1+\delta)}{\delta p_1}+p_2^F, 1\right\} \end{cases}$

2. $S_R^{MF}\left(0^+\right)=\left(p_1,1\right] \iff \frac{\left(\bar{p}_2-p_2^F\right)(1+\delta)}{\delta p_1}+p_2^F \geq 1 \iff I \leq \frac{1+\delta}{\delta p_1}$

The observation above first show the supplier's exploitative behavior requires a higher service level s than the collaborative behavior, as the two ends of the interval $S_R^{MF}\left(0^+\right)$ are higher than those of $S_R^{Censor,O}\left(0^+\right)$ respectively. A higher service level s caused by a low overage cost indicates a lower learning cost for the retailer and thus a higher willingness to learn of his. And in this case, the supplier is less powerful and thus more incentivized to exploit the retailer's high willingness to learn. In contrast, a lower service level s implies a high learning cost for the retailer, and induces a censorship effect entailing a loss for the entire supply chain. In this case the supplier is more powerful, thus and incentivized and capable to collaborate in learning, in which case she is essentially taking the responsibility to integrate the information acquisition effort of the supply chain. Moreover, even with very low ρ, the two phenomena (in particular the benefit from myopia) does not necessarily occur with very high s. The exception occurs with $I > \frac{1+\delta}{\delta p_1}$, and $s \in \left(\frac{\left(\bar{p}_2-p_2^F\right)(1+\delta)}{\delta p_1}+p_2^F, 1\right)$ when $\rho \to 0^+$. Intuitively, when prior information is rich and s is very high, the retailer confidently places high orders even under high wholesale price, so the supplier has little power to create the concern of censorship effect.

2.4.2 Impact on the Supplier's Profit

Similarly, we could compare the supplier's profits in the three situations based on the optimal wholesale price and order policy solutions, and make the following conclusion.

Lemma 3 *1.* $V_s^M \leq V_s^O, V_s^M \leq V_s^F$

2. $V_s^F > V_s^O \iff s \in S_S^{FO} := S_{2'}^F \cap \left(S_3^O \cap \left(0, \min\left\{p_1 + \frac{(1+\delta)\rho\left(p_1-\underline{p}_2\right)}{[1+\delta(1-p_1)](1-\rho)}, 1\right\}\right]\right)$

Lemma 3 shows that, the supplier is always worse off with a myopic retailer. However, she may benefit from the forward-looking retailer under censorship compared to with fully observed demand.

Then taking account of the result on the retailer's profit in Proposition 3, censorship can be Pareto-improving for the supply chain with a forward-looking retailer when $s \in S_S^{FO} \cap S_R^{Cen,O}$.In particular we could specify the region with high demand variability, which is exactly the condition for the supply chain to benefit from censorship:

Proposition 5 *When $\rho \to 0^+$, Censorship could be pareto-improving with a forward-looking retailer when*

$$s \in (p_1, p_1 [1 + \delta (\bar{p}_2 - p_1)]] = S^{Cen,O},$$

whose width decreases with I,increases with $p_1 \in \left(0, \frac{1}{2}\right]$ and decreases with $p_1 \in \left(\frac{1}{2}, 1\right)$.

This result of Proposition 5 shows that the presence of censorship could pareto-improve the two player's profit (and thus the supply chain profit) with a forward-looking retailer. In this situation, the retailer's willingness to learn triggers the supplier's collaborative behavior for information acquisition, leading to reduction of double-marginalization that offsets the loss due to censorship, leading to higher profits for both players. This largely contradicts with our previous understanding of censorship as an information inefficiency. We also observe that this phenomena is mostly pronounced with scarce and ambiguous prior information, corresponding to very high demand uncertainty.

2.4.3 Impact on the Supply Chain's Profit

By incorporating the supplier's strategic decision in response to the retailer's inventory policies, we also introduced decentralization inefficiency of the supply chain, which takes the form of double-marginalization under a constant wholesale price contract. In the presence of the two informational inefficiencies, the severity of double-marginalization varies across the three situations and from different parameter settings. We attribute the differences in the supply chain's profits in the three situations to such interplay between double-marginalization and the two informational inefficiencies.

Two Counter-Intuitive Phenomena for the Supply Chain's Profits

Similar to studies on the retailer's profits, we identify two phenomena for the supply chain's profit summarized by the following inequalities[20]:

$$\text{Benefit from "censorship": } \max\left\{V^F, V^M\right\} > V^O$$
$$\text{Benefit from "myopia": } V^M > \ V^F.$$

Proposition 6 characterizes the conditions for these two counter-intuitive phenomena, by referring to partitions of Θ induced by the supplier's optimal wholesale prices in the three situations and the optimal order quantities in their respective centralized systems $\{OC, MC, FC\}$:

Proposition 6 *The necessary and sufficient conditions for the two counter-intuitive phenomena are:*

Observation	*Condition*
$V^M > V^F$	$\Theta_1^M \cap \Theta_2^F := \Theta^{MF}$
$\max\left\{V^M, V^F\right\} > V^O$	$\left(\Theta_1^F \cup \Theta_2^F\right) \cap \left(\Theta_3^O \cup \Theta_4^O\right) \setminus \left(\Theta_2^F \cap \Theta_3^O \cap \Theta_3^{OC}\right) := \Theta^{Censro,O}$

By comparing the results of Proposition 3 and Proposition 6, one may immediately notice that

$$\Theta^{MF} \subset \Theta_R^{MF}, \Theta^{Censro,O} \subset \Theta_R^{Censro,O}.$$

In a nutshell, the retailer's benefit from the two informational inefficiencies need to high enough to offset the supplier's losses to suffice a net benefit for the entire supply chain, thus it requires stronger conditions for the supply chain's two phenomena.

The supplier's information acquisition and double-marginalization

We argue that the two informational inefficiencies not only affect the retailer's profits but also the entire supply chain's performance. So we can adapt the inequalities in **??** to reflect the centralized

[20] Though the inequalities in 2.4 are concerning the retailer's profit.

Table 2.1: Three Inefficiencies in Supply Chains

$$\begin{array}{ccccccl} V^{OC} & \overbrace{\geq}^{\text{censorship}} & V^{FC} & \overbrace{\geq}^{\text{myopia}} & V^{MC} & & \\ \updownarrow & & \updownarrow & & \updownarrow & \} & \text{double-marginalization} \\ V^{O} & ? & V^{F} & ? & V^{M} & & \end{array}$$

supply chains' profits affected by the two informational inefficiencies:

$$V^{OC} \geq V^{FC} \geq V^{MC}. \tag{2.6}$$

We identified the decentralization inefficiency's existence in the form of double-marginalization as shown by 1.1. We lay out the structure of the supply chains' three inefficiencies accordingly in Table 2.1:

The different ordering between $\left\{V^{O}, V^{M}, V^{F}\right\}$ under various parameter settings is driven by different double-marginalization levels in the three situations. If we represent double-marginalization's severity in the three situations by the ratio $\frac{V^{*}}{V^{*C}}, * \in \{O, M, F\}$, the two phenomena's occurrence implies the following inequalities:

$$\begin{aligned} V^{M}(\boldsymbol{\theta}) &> V^{F}(\boldsymbol{\theta}) \Rightarrow \frac{V^{M}(\boldsymbol{\theta})}{V^{MC}(\boldsymbol{\theta})} > \frac{V^{F}(\boldsymbol{\theta})}{V^{FC}(\boldsymbol{\theta})} \\ \max\left\{V^{M}(\boldsymbol{\theta}), V^{F}(\boldsymbol{\theta})\right\} &> V^{O}(\boldsymbol{\theta}) \Rightarrow \max\left\{\frac{V^{M}(\boldsymbol{\theta})}{V^{MC}(\boldsymbol{\theta})}, \frac{V^{F}(\boldsymbol{\theta})}{V^{FC}(\boldsymbol{\theta})}\right\} > \frac{V^{O}(\boldsymbol{\theta})}{V^{OC}(\boldsymbol{\theta})}. \end{aligned} \tag{2.7}$$

The inequalities in (2.7) show that upon the two phenomena's occurrence, the mitigated double-marginalization effect offsets the two informational inefficiencies and results in higher supply chain profits in the presence of censorship and myopia.

The severity of double-marginalization is governed by the retailer's order quantities, so indirectly determined by the supplier's wholesale price decision, which also serves as her instrument for information acquisition[21]. When the supplier lowers (or raises) the wholesale price in response to the

[21] The CPFR literature[23] also aknowledged similar effects: "Our findings underline that under the non-coordination contract, improved information as as result of CF has the added benefit of countering the adverse effects of double-

informational inefficiencies, and induces higher (or lower) order quantities from the retailer, the mitigation (or exacerbation) of double-marginalization offsets the informational inefficiencies (or the lack of them) and leads to a higher (or lower) supply chain profit than in other situations under the same parameter setting.

Characterization of the Two Phenomena on the Supply Chain Profits

We first characterize the conditions for the two phenomena of the supply chain by projecting Θ^{MF} and $\Theta^{Censor,O}$ onto the $\{s,\rho\}-$space. Figure 2.5[22] demonstrates these observations with different prior information:

Similarly, we again observe from Figure 2.5 that the two phenomena for thes supply chain's profit only occur (i.e. the shaded regions) with low ρ and high s. We formalize this observation by characterizing the conditions for the two phenomena not to take place, and define

$$\begin{aligned}\rho^{MF*} \quad &: \quad = 1 - \max\left\{\frac{(1+\delta)(\bar{p}_2 - p_1)}{\delta p_1 (s^* - p_1)}, \frac{\bar{p}_2}{s^*}, \frac{1+\delta}{\delta p_1 + \frac{s^*[1+\delta(1-p_1)]}{p_1}}\right\} \\ \varrho^{Censor,O*} \quad &: \quad = 1 - \min\left\{\max\left\{\min\left\{\frac{1+\delta}{\delta p_1 + \frac{s[\delta(1-p_1)+1]}{p_2^F}}, \frac{\bar{p}_2}{s^*}\right\}, \frac{p_1}{s^*}\right\}, 1\right\},\end{aligned}$$

and have the following results:

Lemma 4 *1. When $I \leq \frac{1+\delta}{\delta p_1} - 1$, or $s \leq \frac{(\bar{p}_2 - p_1)(1+\delta)}{\delta p_1} + p_1$, or $\rho \geq \rho^{MF*}, V^M \leq V^F$*

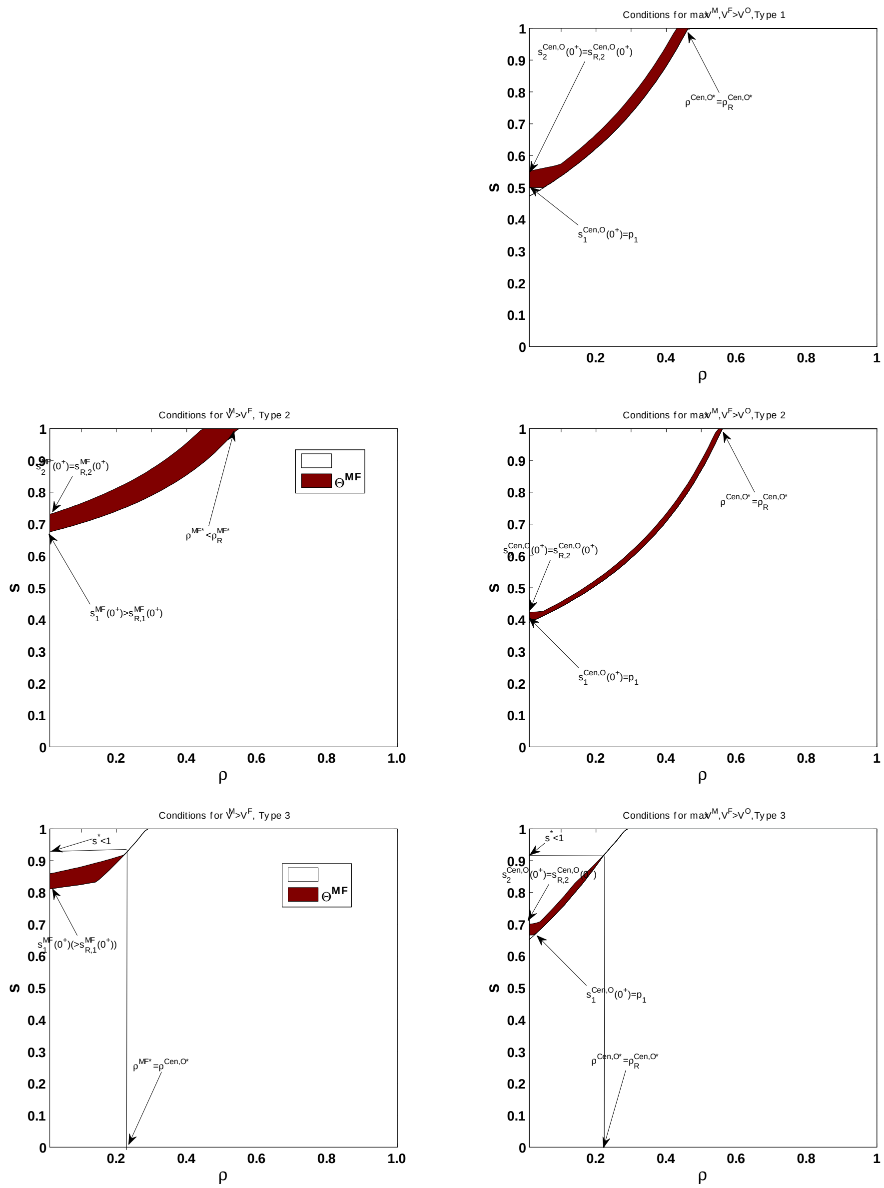
2. When $s \leq p_1$, or $\rho \geq \rho^{Censor,O*}, V^F \leq V^O$

Similar to Lemma 2, Lemma 4 shows that the two phenomena do not occur with too low demand variability, i.e. high ρ, or sufficiently low service level s. Moreover, the supply chain's benefit from myopia, aka $V^M > V^F$, does not occur with too scarce or optimistic prior information, i.e. $I \leq \frac{1+\delta}{\delta p_1} - 1$. In this case, the supplier's lower wholesale price in "M" does not suffice to induce strictly higher order

marginalization in addition to reducing the cost of supply-demand mismatch. Hence, when the inefficiency arising from double-marginalization is high, collaborative forecasting can be highly effective in countering it and delivering value for both parties."

[22] The upper-left corner is vacant as we show later when $I \leq \frac{1+\delta}{\delta p_1}, V^M \leq V^F$,so the condition is empty.

Figure 2.5: Condition for Supply Chain's Benefit from Myopia or Censorship



$I \leq (1+\delta)/\delta p_1$

$I > (1+\delta)/\delta p_1$

$I > (1+\delta)/\delta p_1, s^* < 1$

quantities from the myopic retailer than from the foreward-looking retailer (though censorship is overcome), so the myopic retailer's benefit is purely offset by the supplier's loss and the supply chain has no net benefit.

We define $\left(s_1^{MF}(\rho), s_2^{MF}(\rho)\right]$ and$\left(s_1^{Cen,O}(\rho), s_2^{Cen,O}(\rho)\right]$ as follows:

$$\begin{aligned} \text{For } \forall\rho &\in \left(0, \rho^{MF*}\right], s \in \left(s_1^{MF}(\rho), s_2^{MF}(\rho)\right] \iff V^M > V^F, \\ \text{For } \forall\rho &\in \left(0, \rho^{Censor,O*}\right], s \in \left(s_1^{Censor,O}(\rho), s_2^{Censor,O}(\rho)\right] \iff V^F > V^O, \end{aligned}$$

and use them to characterize the trend and areas of Θ^{MF} and $\Theta^{Censor,O}$:

Corollary 3 $\left\{s_1^{MF}(\rho), s_2^{MF}(\rho), s_1^{Cen,O}(\rho), s_2^{Cen,O}(\rho)\right\}$ *increase with* ρ.

Corollary 3 shows that Θ^{MF} and $\Theta^{Censor,O}$ also follow a lower-left to upper-right trend on the $\{s, \rho\}$ −space, a result similar to Corollary 1.

Extreme Value Study of Double Marginalization and Two Phenomena

Double-marginalization in supply chains takes effect when understocking occurs in the decentralized system. To examine the prior information's impact on double-marginalization in the three situations, we analyze the retailer's understocking levels in the extreme case of $\rho \to 0^+$, and decompose it across the two periods. We attribute difference in double-marginalization between the three situations to the mitigation of first period understocking in the presence of censorship and the exacerbation of second period understocking with the forward-looking retailer's willingness to learn.

With $\rho \to 0^+$, we first study conditions for double-marginalization to occur in each of the two periods with fully observed demand (i.e. "O") by comparing the order quantities in the decentralized and centralized systems. When $\rho \to 0^+$, the conditions boil down to where s falls. Corresponding to understocking in each period, we identify one $s-$interval marked by the prior and denote them as Region 1 and 2 as shown in Table 2.2 [23]:

[23] We ignore the double-marginalization effect that does not result in loss of efficiency, when the wholesale prices are higher than the production cost (defined as double-marginalization) but not causing understocking in the decentralized

Table 2.2: Understocking in "O" with High Demand Variability

Understocking	Region 1: Second Period	Region 2: First Period
$s-$ interval	$\left[\bar{p}_2, \min\{\frac{(\bar{p}_2-p_1)(1+\delta)}{\delta p_1}+p_1,1\}\right)$	$[p_1, p_1\left[1+\delta\left(\bar{p}_2-p_1\right)\right])$
$\left\{y_{1,2}^{OC}\right\}$	$\left\{D^H, D^H\right\}$	$\left\{D^H, \begin{array}{c} D^H \\ D^L \end{array}\right\}$
$\left\{y_{1,2}^{O}\right\}$	$\left\{D^H, \begin{array}{c} D^H \\ D^L \end{array}\right\}$	$\left\{D^L, \begin{array}{c} D^H \\ D^L \end{array}\right\}$
Width	$\min\left\{\frac{(\bar{p}_2-p_1)[1+\delta(1-p_1)]}{\delta p_1}, 1-\bar{p}_2\right\}$	$\delta p_1\left(\bar{p}_2-p_1\right)$
Loss of Efficiency	$\frac{\delta p_1(s-\bar{p}_2)}{(1+\delta)(s-p_1)}$	$\frac{s-p_1}{(1+\delta)(s-p_1)-\delta p_1(s-\bar{p}_2)}$

The two intervals represent conditions for understocking of the first or the second period. Similarly we could identify the understocking regions in the situations of "M" and "F" as in Table 2.3.

Table 2.3: Understocking in "M" and "F" with High Demand Variability

Region 1: Second Period	"M"	"F"
$s-$ interval	$\left[\bar{p}_2, \min\{\frac{(\bar{p}_2-p_1)(1+\delta)}{\delta p_1}+p_1,1\}\right)$	$\left[\bar{p}_2, \min\{\frac{(\bar{p}_2-p_2^F)(1+\delta)}{\delta p_1}+p_2^F,1\}\right)$
$\left\{y_{1,2}^{M/F,C}\right\}$	$\left\{D^H, D^H\right\}$	$\left\{D^H, D^H\right\}$
$\left\{y_{1,2}^{M/F}\right\}$	$\left\{D^H, \begin{array}{c} D^H \\ D^L \end{array}\right\}$	$\left\{D^H, \begin{array}{c} D^H \\ D^L \end{array}\right\}$
Width	$\min\left\{\frac{(\bar{p}_2-p_1)[1+\delta(1-p_1)]}{\delta p_1}, 1-\bar{p}_2\right\}$	$\min\left\{\frac{(\bar{p}_2-p_2^F)[1+\delta(1-p_1)]}{\delta p_1}, 1-\bar{p}_2\right\}$
Loss of Efficiency	$\frac{\delta p_1(s-\bar{p}_2)}{(1+\delta)(s-p_1)}$	$\frac{\delta p_1(s-\bar{p}_2)}{(1+\delta)(s-p_1)}$

In contrast, in the two situations with censorship, understocking only occurs at the second period. Due to the censorship effect, understocking of the first period would have impact on the posterior demand information and consequently cause significant loss to the entire supply chain. Therefore he supplier works to avoid it by leveraging the wholesale price. This difference between the two situations under censorship and "O" explains the supply chain's higher profits under censorship. We also notice the different $s-$intervals for understocking of the second period in "F" compared to those in "O" and "M' that contributes to the higher profit in "M" than in "F".

Based on comparison of Table 2.2 and Table 2.3, we identify the conditions for two phenomena of

systems compared to the centralized system optimum. This occurs when $\theta \in \Theta_i^* \cap \Theta_i^{*C}$, $* \in \{O, M, F\}, i \in \{1, ..., 4\}$.

the supply chain's profits when $\rho \to 0^+$, namely $S^{MF}(0^+)$ and $S^{Censor,O}(0^+)$:

Proposition 7 *When $\rho \to 0^+$,*

1.

$$\begin{aligned} V^M &> V^F \iff \\ I &> \frac{1+\delta}{\delta p_1} - 1 \text{ and } s \in \left(\frac{(\bar{p}_2 - p_1)(1+\delta)}{\delta p_1} + p_1, \min\left\{ \frac{\left(\bar{p}_2 - p_2^F\right)(1+\delta)}{\delta p_1} + p_2^F, 1 \right\} \right] := S^{MF}\left(0^+\right) \end{aligned}$$

2.

$$\begin{aligned} \max\left\{V^F, V^M\right\} &> V^O \iff \\ s &\in \left(p_1, p_1\left[1 + \delta\left(\bar{p}_2 - p_1\right)\right]\right] := S^{Censor,O}\left(0^+\right) \end{aligned}$$

To examine the prior information's impact on these two phenomena, we study their impact on the two types of understocking in the three situations in Lemma 5. We measure the severity of the two types of understocking by the width of the $s-$intervals[24]:

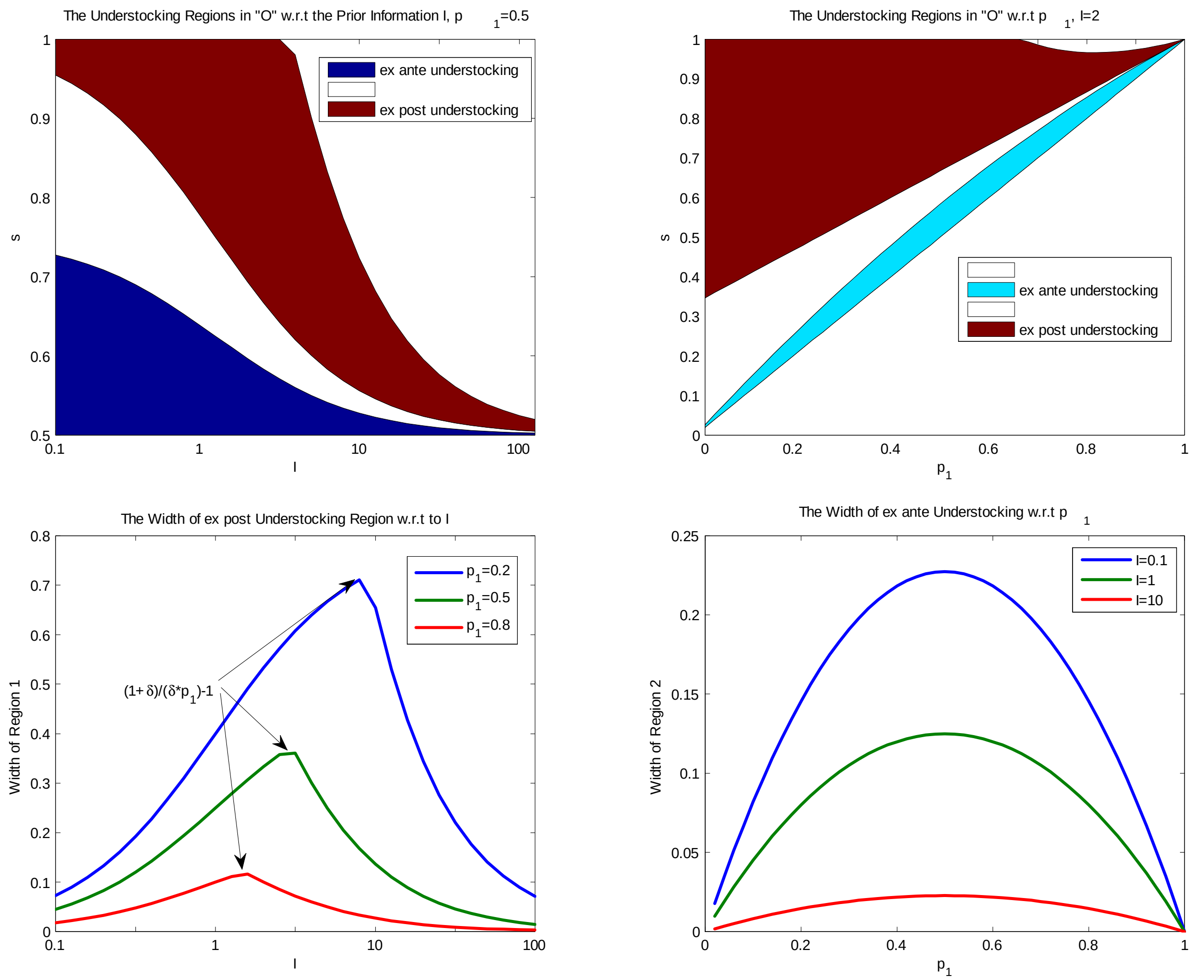
Lemma 5 *1. The width of Region 1 (second period) decreases with I (with fixed p_1 and s); increases with $p_1 \in \left(0, \frac{1}{2}\right]$ and decreases with $p_1 \in \left(\frac{1}{2}, 1\right)$ (with fixed I and s);*

2. The width of Region 2 (first period) decreases with p_1(with fixed I and s); increases when $I \in \left(0, \frac{1+\delta}{\delta p_1} - 1\right]$ in "O" and "M" and decreases with $I \in \left(\frac{1+\delta}{\delta p_1} - 1, +\infty\right)$, and increases with $I \in \left(0, \frac{1+\delta}{\delta p_1}\right]$ and decreases with $I > \frac{1+\delta}{\delta p_1}$.

As an example, we illustrate how the range of the two regions and their widths change with $\{p_1, I\}$ in the situation of "O" in Figure 2.6 ($\delta = 1$):

[24]With fixed p_1(and s), the efficiency loss in these two regions $1 - \frac{V^O}{V^{OC}}|_\theta$ increases with I; with fixed I (and s), they increase with $p_1 \in \left(0, \left(s - \sqrt{\frac{s(1-s)}{I}}\right)^+\right]$ and decreases with $p_1 \in \left(\left(s - \sqrt{\frac{s(1-s)}{I}}\right)^+, s\right]$.

Figure 2.6: The Understocking Regions and the Prior Information



Lemma 5 and Figure 2.6 indicate that the two types of understocking have different relationships with the prior information. Understocking of the first period is more pronounced with scarce prior information corresponding to low I, and an ambiguous prior p_1 around $\frac{1}{2}$. In contrast, understocking of the second period only occurs when the first period's order is high, so it diminishes when the demand information grows more pessimistic corresponding to higher p_1. Moreover, it reaches a maximal width with a medium level information richness, i.e. $I = \frac{1+\delta}{\delta p_1} - 1$ in "O" and "M" and $I = \frac{1+\delta}{\delta p_1}$

in "F". When prior information is scarce and below this level, the retailer's order quantity of the second period is largely influenced by information updating so is prevalent; also understocking of the second period gets more common as the prior information gets richer in this range. Yet when the prior information richness exceeds this level, understocking of the second period is mitigated as posterior information uncertainty is reduced. Therefore we could roughly attribute the first period understocking to information scarcity and ambiguity and the second period understocking to prior optimism and modest uncertainty. Due to the forward-looking retailer's extra precaution, he accepts a higher wholesale price to avoid censorship of the first period and consequently a more severe understocking of the second period compared to in "O" and "M".

Corollary 4 *The width of* $S^{MF}(0^+)$ *increases with* $I \in \left(\frac{1+\delta}{\delta p_1} - 1, \frac{1+\delta}{\delta p_1}\right]$, *decreases with* $I \in \left(\frac{1+\delta}{\delta p_1}, \infty\right)$;*increases with* $p_1 \in \left(\frac{1+\delta}{\delta(I+1)}, \min\left\{\sqrt{\frac{1+\delta}{\delta(I+1)}}, \frac{1+\delta}{\delta I}\right\}\right]$, *decreases with* $p_1 > \min\left\{\sqrt{\frac{1+\delta}{\delta(I+1)}}, \frac{1+\delta}{\delta I}\right\}$.

2.5 Numerical Examples and Results

In this section, we present a series of numerical studies to support and complement the analytical results. We first examine the current model, and quantify the two counter-intuitive phenomena by characterizing the scope and scale of their occurrences. We also break down the three inefficiencies, and numerically measure their impact. Then we study a model with continuous demand distribution and verify that the two phenomena for the retailer and the supply chain's profits still exist in these more general settings.

2.5.1 Numerical Analysis of the Basic Model

To measure the scope and scale of the two counter-intuitive phenomena's impact in the basic model, we compute their occurrence frequencies and the magnitude of benefits over a grid of parameter settings and. By referring to structure of inefficiencies in Table 2.1, we can attribute the differences between the three centralized systems and the differences between the centralized systems and their decentralized counterparts to the two informational inefficiencies and double-marginalization respectively, therefore also identify and quantify the impact of each of the three inefficiencies.

We first design a sample grid to represent a wide range of settings from the primitive parameters space

$$\Theta := \{(s, \alpha, \beta, \rho, \delta) | s \in (0,1], \alpha > 0, \beta > 0, \rho \in (0,1)^{25}, \delta \in (0,1]\}.$$

Among its 5 dimensions, the discount factor $\delta \in (0,1]$ captures the dynamic effect of information acquisition and wealth accumulation. A higher δ amplifies the supplier and the retailer's forward-looking behaviors, so we fix $\delta = 1$ in the numerical analysis hereafter to stress this component. Based on the optimal wholesale prices' monotonic relationship with $\{s, \rho\}$, we pick samples covering their respective supports as follows:

$$\begin{aligned} s &\in \{0.01, 0.02, ..., 1\} \\ \rho &\in \{0.1, 0.3, 0.5, 0.7, 0.9\}. \end{aligned}$$

We pick $\{\alpha, \beta\}$s to represent various levels of information richness and optimism as follows:

$$p_1 \in \{0.2, 0.4, 0.6, 0.8\}, I \in \{1, 5, 10, 50\}$$

We denote this sample grid Θ_N with $N = 8000$ different θs. To measure the frequencies of the two phenomena's occurrences and the induced benefits' relative magnitudes.

[25] For expositional simplicity, we redefine $\rho = \frac{D^L}{D^H} \in (0,1)$.

Table 2.4: Scope and Scale of the Two Phenomena's Impact

	Retailer		Supply Chain		Supplier
	$V_R^M > V_R^F$	$\max\left\{V_R^M, V_R^F\right\} > V_R^O$	$V^M > V^F$	$\max\left\{V^M, V^F\right\} > V^O$	$\max\left\{V_S^M, V_S^F\right\} > V_S^O$
σ	13.95%	2.61%	0.71%	2.53%	15.63%
ζ	31.15%	72.25%	10.62%	26.24%	7.32%

The Two Phenomena's Impact

We could derive the optimal wholesale prices and order quantities in the three situations for each $\theta \in \Theta_N$, and compute the associated payoffs for each party. Regarding each of the following inequalities

$$
\begin{aligned}
V_R^F &< V_R^M, \max\left\{V_R^F, V_R^M\right\} > V_R^O \\
V^F &< V^M, \max\left\{V^F, V^M\right\} > V^O
\end{aligned}
$$

we measure their occurrence frequency and the average relative difference between the two sides over the sample grid Θ_N. Taking the case of "$V^F < V^M$" for example, we compute the following two metrics:

$$
\begin{aligned}
&\sigma^{MF} = \tfrac{N^{MF}}{N} = \tfrac{1}{N}\textstyle\sum_{ijkn} \mathbf{1}\{V^F(s_i, \rho_j, p_{1,k}, I_n) < V^M(s_i, \rho_j, p_{1,k}, I_n)\} \\
&\zeta^{MF} = \tfrac{1}{N^{MF}}\textstyle\sum_{ijkn} \mathbf{1}\{V^F(s_i, \rho_j, p_{1,k}, I_n) < V^M(s_i, \rho_j, p_{1,k}, I_n)\}\left[1 - \tfrac{V^F(s_i, \rho_j, p_{1,k}, I_n)}{V^M(s_i, \rho_j, p_{1,k}, I_n)}\right] \\
&(\theta_{ijkn} = \left(s_i, \rho_j, p_{1,k}, I_n\right) \in \Theta_N)
\end{aligned}
$$

We present a summary of the numerical results in Table 2.4:

Table 2.4 shows that for the retailer, the benefit from myopia happens in a wider range of conditions than the benefit from censorship, yet in a lower magnitude. While for the supply chain, the benefit from censorship presents an impact of both larger scope and scale than that from myopia. We also observe that the supply chain's benefits from both myopia and censorship are of lower impact than the retailer's benefits.

In 2.4, we conducted an analytical study on the prior information's impact on conditions for the two phenomena in the extreme case of $\rho \to 0^+$ by measuring the corresponding $s-$intervals' width. By aggregating their occurrence frequency w.r.t. certain values of $\{p_1, I\}$ and ρ, we could numerically examine the prior information's impact with more general value of ρ through the width of s-intervals. We obtain the following numerical observations that are fully consistent with the analytical results in the case of $\rho \to 0^+$:

1. $\sigma_R^{MF}(\rho)$ decrease with I, decrease with p_1;
2. $\sigma_R^{Censor,O}(\rho)$ decrease with I; increase and decrease with p_1;
3. $\sigma^{MF}(\rho)$ increase and decrease with I; increase and decrease with p_1;
4. $\sigma^{Censor,O}(\rho)$ decrease with I;increase and decrease with p_1

Similarly, we could also examine the magnitude of the benefit from censorship and myopia as shown in Table 6 to Table 9 in 1.5.

The Three Inefficiencies' Impact and Interplay

In Table 2.1, we attribute the differences between the supply chain's profits in the centralized and decentralized systems to the three inefficiencies as follows:

- Myopia: $V^{MC} < V^{FC}$
- Censorship: $V^{FC} < V^{OC}$
- Double-Marginalization: $V^* < V^{*C}, * \in \{O, M, F\}$

To characterize the scope and scale of each inefficiency, we measure the occurrence frequency of these inequalities and relative difference between the two sides of them. Specifically we consider the following three metrics:

Table 2.5: Comparison of Three Sources of Inefficiency

	Demand Censorship	Myopia	Double-Marginalization		
	V^{FC} vs. V^{OC}	V^{MC} vs. V^{FC}	V^{O} vs. V^{OC}	V^{F} vs. V^{FC}	V^{M} vs. V^{MC}
$1-\xi$	0.57%	0.23%	5.78%	4.84%	5.29%
σ	9.01%	2.46%	51.86%	44.70%	42.4%
$1-\zeta$	6.31%	9.15%	11.14%	10.83%	12.48%

- ζ : the average efficiency level
- σ : the average occurrence frequency[26]
- ξ : the average efficiency level upon occurrence.

For example, for the impact of demand censorship, we calculate the three metrics by comparing V^{FC} to V^{OC} over the sample grid of $\Theta_N := \big\{(s_i, \rho_j, p_{1,k}, I_n)_{ijkn}\big\}$:

$$\begin{aligned}
\xi^{OF,C} &= \tfrac{1}{N}\textstyle\sum_{ijkn} \tfrac{V^{FC}(s_i,\rho_j,p_{1,k},I_n)}{V^{OC}(s_i,\rho_j,p_{1,k},I_n)}\\
\sigma^{OF,C} &= \tfrac{1}{N}\textstyle\sum_{ijkn} \mathbf{1}\{V^{FC}(s_i,\rho_j,p_{1,k},I_n) < V^{OC}(s_i,\rho_j,p_{1,k},I_n)\}\\
\zeta^{OF,C} &= \tfrac{1}{N\sigma^{OF,C}}\textstyle\sum_{ijkn} \mathbf{1}\{V^{FC}(s_i,\rho_j,p_{1,k},I_n) < V^{OC}(s_i,\rho_j,p_{1,k},I_n)\}\tfrac{V^{FC}(s_i,\rho_j,p_{1,k},I_n)}{V^{OC}(s_i,\rho_j,p_{1,k},I_n)}
\end{aligned}$$

We argue that, for any set of comparison, there exists the following relationship between these three metrics:

$$1-\xi = \sigma\cdot(1-\zeta).$$

Intuitively, $1-\xi$ represents the average efficiency loss over the entire sample grid, while $1-\zeta$ only captures the average loss upon strict inefficiency occurrence as it does not occur to certain samples, therefore $\sigma < 1$.

We apply similar formulas to other sets of comparisons and present the numerical results regarding these comparisons in Table 2.5:

Table 2.5 shows that the two informational inefficiencies have a much lower impact on the supply chain performances than double-marginalization, mainly due to their lower occurrence frequencies. We

[26] we only take account of the occurence of the "strict inequality".

	Myopia		Censorship	
	$\frac{V^F}{V^{FC}} > \frac{V^M}{V^{MC}}$	$\frac{V^F}{V^{FC}} < \frac{V^M}{V^{MC}}$	$\frac{V^O}{V^{OC}} < \max\left\{\frac{V^M}{V^{MC}}, \frac{V^F}{V^{FC}}\right\}$	$\frac{V^O}{V^{OC}} > \max\left\{\frac{V^M}{V^{MC}}, \frac{V^F}{V^{FC}}\right\}$
σ	1.81%	3.09%	11.38%	0.0625%
ζ	38.71%	8.18%	10.59%	28.56%

Table 2.6: Double-Marginalization Levels w/o the Informational Inefficiencies

also notice a considerable difference in the double-marginalization levels across the three situations. These two observations serve to explain why the two informational inefficiencies can not fully account for the three decentralized systems' different performances, and the necessity to consider double-marginalization and its interplay with the informational inefficiencies.

We also notice that there is no the uniform ordering of these double-marginalization levels in the three situations across all parameter settings. Namely, for each of the two information inefficiency, we observe conditions for the double-marginalization to be either higher or lower with or without them. To illustrate this observation, we measure the frequencies for the double-marginalization level of each side to be higher and the differences in the double-marginalization levels. For example, for censorship, we compare the double-marginalization levels in "F" and "M", and compute the following two metrics for each direction:

$$\begin{array}{l} \sigma_{DM}^{MF\pm} = \frac{1}{N}\sum_{ijn} \mathbf{1}\{\frac{V^F}{V^{FC}}(s_i, \rho_j, p_{1,k}, I_n) \gtrless \frac{V^M}{V^{MC}}(s_i, \rho_j, p_{1,k}, I_n)\} \\ \zeta_{DM}^{MF\pm} = \frac{1}{N\sigma_{DM}^{MF\pm}}\sum_{ijn} \mathbf{1}\{\frac{V^F}{V^{FC}}(s_i, \rho_j, p_{1,k}, I_n) \gtrless \frac{V^M}{V^{MC}}(s_i, \rho_j, p_{1,k}, I_n)\} \cdot |\frac{V^F}{V^{FC}} - \frac{V^M}{V^{MC}}|(s_i, \rho_j, p_{1,k}, I_n) \end{array}.$$

We present the results summarized over Θ_N in Table 2.6:

Table 2.6 shows that for the two situations under censorship, for a wider range of conditions, the double-marginalization is more severe in "F", yet the difference is of much higher magnitude once it is more severe in "M". Similarly, for a much wider range of conditions, the double-marginalization level is more severe with fully observed demand; yet once it is more severe under censorship, the difference between the two levels is higher.

2.5.2 Numerical Analysis for an Extension with Continuous Demand Distribution

[?] studied the retailer's demand learning through inventory experimentation under censorship in a multi-period model with a continuous "newsvendor" distribution with Bayesian conjugate. However, we were not able to solve for the supplier's optimal wholesale prices analytically that prevent us from obtaining more structural results. We resort to numerical results instead to examine this more general setting, in particular searching for evidence of the two counter-intuitive phenomena for the retailer and the supply chain's profits therein. We found that the two counter-intuitive phenomena still exist in this setting (Table 6 and Table 7), though of a smaller scale.

We consider [8]'s model of two periods with an Weibull demand distribution with the density function $\psi(\cdot)$, whose shape parameter l is known and scale parameter ω follows a gamma distribution with hyperparameters $\{a, S\}$, and a compound density $\phi(\cdot)$:

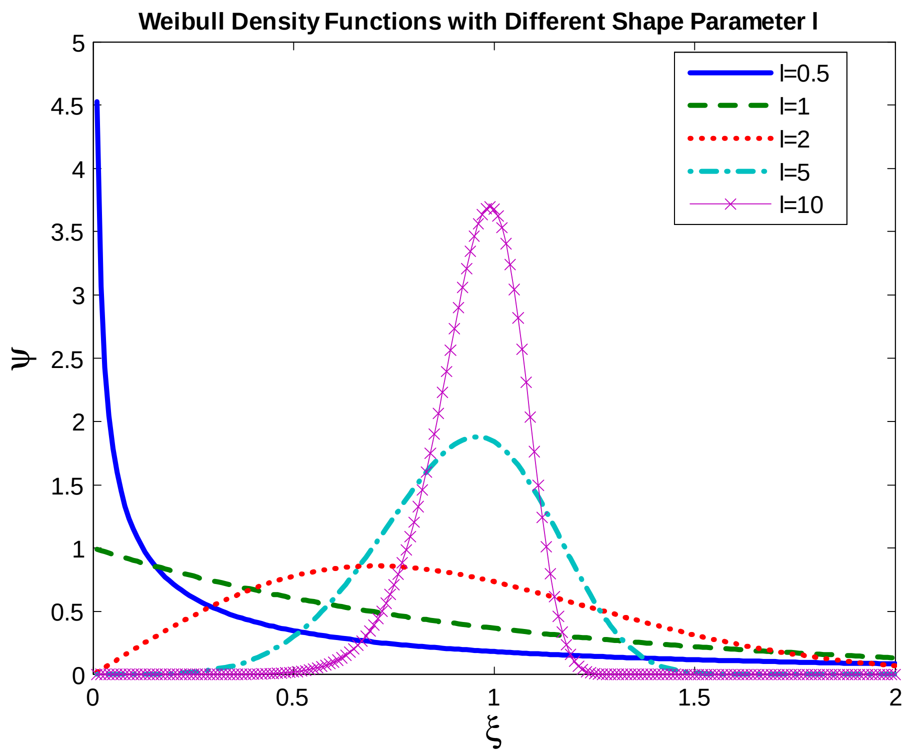
$$\begin{aligned}\psi(\xi|\omega) &= 1 - e^{-\omega d(\xi)}, g(\omega|a, S) = \frac{S^a \omega^{a-1} e^{-S\omega}}{\Gamma(a)} \\ \phi(\xi|a, S) &= \frac{aS^a d'(\zeta)}{[S + d(\xi)]^{a+1}}.\end{aligned}$$

With its scale parameter being normalized as $S = 1$, the compound demand distribution has a density function $\phi(\cdot)$:

$$\begin{aligned}\psi(\xi|\omega) &= 1 - e^{-\omega \xi^l}, g(\omega|a) = \frac{\omega^{a-1} e^{-\omega}}{\Gamma(a)} \\ \phi(\xi|a) &= \frac{a l \xi^{l-1}}{\left(1 + \xi^l\right)^{a+1}}\end{aligned}$$

Figure 2.7 shows the Weibull distribution's shape of different l with its scale parameter λ being normalized as 1. In the basic model, the demand distribution's shape is jointly determined by $\{\rho, p_1\}$. Comparably, a lower l indicates a more pessimistic demand distribution, to a certain extent corresponding to a lower ρ and higher p_1.The Gamma prior's shape parameter a captures the prior

Figure 2.7: Probability Density Function of Weibull Distribution, l: shape parameter



information richness similar to I in the basic model. Besides, $s \in (0, 1]$ represents the supply chain cost structure as in the basic model.

Based on the analytical results for the retailer's optimal inventory policies, we could solve for the supplier's optimal wholesale prices in the three situations, and derive the payoffs for each party for comparison across the three situations. In the numerical study, we consider a sample grid consisting of the following combinations of $\{s, l, a\}$ with 1000 samples of each parameter combination:

$$s \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$$

l	$a\left(> \frac{1}{l}\right)$
0.5	$\{2.01, 2.1, 3, 5, 10, 50, 100, 500, 1000, 5000\}$
1	$\{1.01, 1.1, 1.3, 2, 5, 10, 50, 100, 500, 1000\}$
2	$\{0.501, 0.6, 1, 2, 5, 10, 50, 100, 500, 1000\}$
5	$\{0.201, 0.3, 1, 3, 5, 10, 50, 100, 500, 1000\}$
10	$\{0.101, 0.2, 1, 2, 5, 10, 50, 100, 500, 1000\}$

We present the numerical evidence of the two counter-intuitive phenomena over the entire sample grid in Table 2.7:

Table 2.7: Retailer's Profits Comparison with Weibull Distribution

Retailer	$\max\left\{V_R^M, V_R^F\right\} > V_R^O$		$V_R^M > V_R^F$	
l	σ	$1-\zeta$	σ	$1-\zeta$
0.5	20%	4.90×10^{-7}	22%	6.67×10^{-6}
1	16%	1.26×10^{-3}	12%	6.61×10^{-7}
2	42%	4.50×10^{-3}	6%	4.11×10^{-7}
5	52%	1.47×10^{-2}	10%	7.97×10^{-7}
10	58%	1.82×10^{-2}	2%	1.69×10^{-6}

Table 2.8: Supply Chain's Profits Comparison with Weibull Distribution

Supply Chain	$\max\left\{V^M, V^F\right\} > V^O$		$V^M > V^F$	
l	σ	$1-\zeta$	σ	$1-\zeta$
0.5	12%	2.16×10^{-7}	12%	2.11×10^{-7}
1	8%	9.03×10^{-7}	8%	2.80×10^{-7}
2	6%	2.30×10^{-7}	6%	8.23×10^{-8}
5	4%	4.97×10^{-8}	8%	1.39×10^{-7}
10	10%	7.28×10^{-8}	2%	1.41×10^{-7}

Table 2.7 and Table 2.8 show that the two phenomena for the retailer and the supply chain's

profits do occur with the continuous demand distribution, though the differences in profits are of a smaller scale. In particular we observe the retailer's benefit from censorship in a relatively wider range of conditions and of a large magnitude, a result slightly different from our observation of the basic model. And the scope and scale of this phenomena's impact increases with a. We notice that $V_R^M > V_R^F$ only occurs with very large a; $\max\left\{V_R^M, V_R^F\right\} > V_R^O$ occurs with relatively large s and large a;while $V^M > V^F$ and $\max\left\{V^M, V^F\right\} > V^O$only occur with very large a.

We also noticed in our numerical studies that the observations of $V_R^M > V_R^F$ coincide with $w^M < w^F$; however we always observe $w^M < w^O, w^F < w^O$. This may lead to future study to further investigate this theoretical framework.

2.6 Discussions and Concluding Remarks

Our analysis of the censorship and demand learning in a decentralized supply chain is in the context of constant wholesale price contract. The presence of censorship and learning opportunity affects the strategic interactions in a supply chain by changing the decision power allocation and incentive alignment. The strategic interactions also complicate the implications of demand censorship and the retailer's myopia, which might not always cause inefficiencies to the system or certain players.

The constant wholesale price contract is simple for our study and widely used in practice. Yet we could take this approach further to examine various forms of dynamic interactions in supply chains with information acquisition opportunities, to extend our understanding on how the information acquisition behaviors change with strategic concerns, and how the incentive and coordination of different players change with this decision being endogenized. In particular, to complement and contrast with our study on constant wholesale price contract, we discuss two other contractual forms. One is dynamic wholesale price contract, and another one buy-back contract that could potentially coordinate the supply chain and eliminate the decentralization inefficiency.

Allowing the supplier to update the wholesale price based on the demand information provides her with a stronger decision power and strategic tension with the retailer. In the situations of "O" and

"M", this better integrates the supply chain and reduces the decentralization inefficiency. While in "F", it gives rise to an incentive for the forward-looking retailer to create censorship and manipulate the posterior wholesale price, and the need for the supplier to prevent it.

We argue that a constant buy-back contract could coordinate the supply chain in the presence of censorship and the forward-looking retailer's learning. In despite of the simple contract structure, this result is due to the fact the benefit from learning is proportional to the supply chain's future profit and could be allocated to the two players in a similar fashion. Therefore the incentive for information acquisition could be coordinated through the same contract that coordinates the inventory levels.

2.6.1 Dynamic Wholesale Price Contract

In the basic model we assume that the supplier's wholesale price is determined up-front and remains constant throughout the two periods. The supplier has to take account of the potential information change in the future when deciding on the wholesale price. Realistically, the supplier could update the wholesale price over time according to the demand information. In this case, the posterior information not only affects the retailer's inventory decision but also the supplier's wholesale price. The supplier's increasing pricing power and flexibility could affect the supply chain's performances and strategic information acquisition behaviors of the two players.

We continue with the cost structure, demand distribution and belief updating rules, and the three decision profiles of the retailer from the basic model. With the dynamic wholesale price policy, at each period $t \in \{1, 2\}$, the interactions between the retailer and the supplier unfold as follows:

1. Based on the prior demand information $\{\alpha_t, \beta_t\}$, the supplier sets the constant wholesale price $\widetilde{w}_t$

2. The retailer sets the inventory level $\widetilde{y}_t$ and orders from the supplier; the production is instantly completed and delivered by the supplier at a unit cost c, then the supplier collects a revenue of $\widetilde{w}_t \cdot \widetilde{y}_t$ from the retailer;

3. The customer demand D_t and the sales $x_t = \min\{\widetilde{y}_t, D_t\}$ are realized, and the retailer collects

a newsvendor revenue $r \cdot x_1 + v \cdot \{\widetilde{y}_t - D_t\}^+$;

4. The demand information updates to $\{\alpha_{t+1}, \beta_{t+1}\}$.

In the situations of "O" and "M", the retailer's inventory decisions are still decoupled over time. The retailer takes the wholesale price of each period as exogenous and optimizes his newsvendor profit of the that period, so he still follows the inventory policy in 2.3. We could solve for the supplier's optimal wholesale price policies $\{\widetilde{w}_t^O\}_{t=1,2}$ and $\{\widetilde{w}_t^M\}_{t=1,2}$ in backward inductions.

Lemma 6 *1.* $\widetilde{k}_t^O = k\left(\widetilde{w}_t^O\right) = \begin{cases} p_t^+ & 0 < p_t < s(1-\rho) \\ 0 & o.w. \end{cases}, t \in \{1,2\}.$

2. $\widetilde{k}_1^M = k\left(\widetilde{w}_1^M\right) = \begin{cases} p_1^+ & 0 < \tilde{p}_2^M := \frac{p_1 + \delta(1-p_1)\underline{p}_2}{1+\delta(1-p_1)} < s(1-\rho) \\ 0 & o.w. \end{cases},$

$\widetilde{k}_2^M = k\left(\widetilde{w}_2^M\right) = \begin{cases} p_2^+ & 0 < p_2 < s(1-\rho) \\ 0 & o.w. \end{cases}.$

Based on the results of Lemma 6, we could define the following partitions of Θ, which yield the same order expected quantities with the corresponding subset of Θ under constant wholesale price contract[27].

$$\begin{cases} \widetilde{\Theta}_1^O := & \left\{s \in \left(\frac{\bar{p}_2}{1-\rho}, 1\right]\right\} \\ \widetilde{\Theta}_2^O := & \left\{s \in \left(\frac{p_1}{1-\rho}, \frac{\bar{p}_2}{1-\rho}\right]\right\} \\ \widetilde{\Theta}_3^O := & \left\{s \in \left(\frac{\underline{p}_2}{1-\rho}, \frac{p_1}{1-\rho}\right]\right\} \\ \widetilde{\Theta}_4^O := & \left\{s \in \left(0, \frac{\underline{p}_2}{1-\rho}\right]\right\} \end{cases}, \begin{cases} \widetilde{\Theta}_1^M := & \left\{s \in \left(\frac{\bar{p}_2}{1-\rho}, 1\right]\right\} \\ \widetilde{\Theta}_{2'}^M := & \left\{s \in \left(\frac{\tilde{p}_2^M}{1-\rho}, \frac{\bar{p}_2}{1-\rho}\right]\right\} \\ \widetilde{\Theta}_4^M := & \left\{s \in \left(0, \frac{\tilde{p}_2^M}{1-\rho}\right]\right\} \end{cases},$$

By comparing the supply chain's performances under constant wholesale price and dynamic wholesale prices with the same parameter setting $\theta \in \Theta$ and the same demand realization of the first period, we have the following observations:

Corollary 5 *1.* $\{\widetilde{y}_t^*\} \geq \{y_t^*\}, \widetilde{V}^* \geq V^*, \widetilde{V}_R^* \geq V_R^*, * \in \{O, M\}$

[27] Note that the wholesale prices and order quantities with dynamic wholesale price contract is dependent on D_1.

2. $\widetilde{y}_t^*(\theta, D_1) > y_t^*(\theta, D_1) \iff \widetilde{w}_t^*(\theta, D_1) < w_t^*(\theta, D_1)$

3. $\widetilde{V}_S^O > V_S^O \iff \theta \in \left\{\Theta_4^O \cap \left\{\widetilde{\Theta}_2^O \cup \widetilde{\Theta}_3^O\right\}\right\} \cup \left\{\Theta_3^O \cap \left\{\widetilde{\Theta}_1^O \cup \widetilde{\Theta}_2^O\right\}\right\}$

$\widetilde{V}_S^M > V_S^M \iff \theta \in \left\{\Theta_4^M \cap \left\{\widetilde{\Theta}_2^M \cup \widetilde{\Theta}_1^M\right\}\right\}$

Corollary 5 shows that the supply chain and the retailer both have higher profits with dynamic wholesale price contract due to higher order quantities. When the supplier does not need to commit to a price for the entire two periods, she lowers the price when it could induce a higher order quantity, therefore mitigates the understocking caused by double-marginalization. In particular, the higher of the first period could also mitigates the censorship effect in "M". So the price flexibility increase the supplier's decision power, reduces the demand uncertainty and information acquisition cost she is exposed to, and may be pareto-improving for the supply chain.

In contrast, for the forward-looking retailer, the updated demand information could lead to potentially more or less favorable wholesale price of the second period, so there arises an incentive for him to manipulate the posterior information and consequently the second period's wholesale price through the first period's inventory decision. Therefore the information acquisition opportunity has another implication for the retailer than its information value for his inventory policy. Taking this into account, we identify conditions for the forward-looking retailer to create censorship and the supplier to adjust the first period's wholesale price in response to such a strategic behavior. In this case, we consider the two player's optimal policies $\left\{w_1^F, y_1^F, w_2^F|_{D_1}, y_2^F|_{D_1}\right\}$ to constitute a sequential equilibrium/perfect Bayesian equilibrium.

We define

$$\begin{aligned}
\widetilde{k}_1^F \quad &: \quad = p_1\left(1 + \frac{\delta\rho\overline{p}_2}{1-\rho}\right)^+, \widetilde{s}_1^F := \frac{p_1}{1-\rho} \cdot \frac{1 + \frac{\delta\overline{p}_2(2\rho-1)}{1-\rho}}{1-\delta p_1} \\
\widetilde{k}_2^F \quad &: \quad = p_1\left\{1 - \frac{\delta\rho\left(1-\overline{p}_2\right)}{1-\rho}\right\}^+, \widetilde{s}_2^F := \frac{p_1}{1-\rho} \cdot \frac{1 + \frac{\delta(1-\overline{p}_2)(1-2\rho)}{1-\rho}}{1+\delta\left(1-p_1\right)}
\end{aligned}$$

and the supplier and the forward-looking retailer's optimal policies are as follows:

Proposition 8 *1. When* $s \in \left(\frac{\overline{p}_2}{1-\rho}, 1\right], k\left(w_t^F\right) = p_t^+, y_t^F = D^H, t \in \{1, 2\}$

2. *When* $s \in \left(\max\left\{\widetilde{s}_1^F, \frac{p_1}{1-\rho}\right\}, \frac{\overline{p}_2}{1-\rho}\right]$, $\begin{cases} k\left(w_1^F\right) = \widetilde{k}_1^{F+}, \\ y_1^F = D^H \end{cases}$,

$$\begin{cases} k\left(w_2^F\right)|_{D_1=D_H} = \underline{p}_2^+, \\ y_2^F|_{D_1=D_H} = D^H \end{cases}, \begin{cases} k\left(w_2^F\right)|_{D_1=D_L} = 0^+, \\ y_2^F|_{D_1=D_L} = D^L \end{cases};$$

3. *When* $s \in \left(\frac{p_1}{1-\rho}, \min\left\{\widetilde{s}_1^F, \frac{\overline{p}_2}{1-\rho}\right\}\right]$, $\begin{cases} k\left(w_1^F\right) = 0^+, \\ y_1^F = D^L \end{cases}$, $\begin{cases} k\left(w_2^F\right) = p_1^+, \\ y_2^F = D^H \end{cases}$,

4. *When* $s \in \left(\max\left\{\widetilde{s}_2^F, \frac{\underline{p}_2}{1-\rho}\right\}, \frac{p_1}{1-\rho}\right]$, $\begin{cases} k\left(w_1^F\right) = \widetilde{k}_2^{F+}, \\ y_1^F = D^H \end{cases}$,

$$\begin{cases} k\left(w_2^F\right)|_{D_1=D^H} = \underline{p}_2^+, \\ y_2^F|_{D_1=D^H} = D^H \end{cases}, \begin{cases} k\left(w_2^F\right)|_{D_1=D_L} = 0^+, \\ y_2^F|_{D_1=D_L} = D^L \end{cases}$$

5. *When* $s \in \left(0, \min\left\{\widetilde{s}_2^F, \frac{p_1}{1-\rho}\right\}\right]$, $\begin{cases} k\left(w_1^F\right) = 0^+, \\ y_1^F = D^L \end{cases}$, $\begin{cases} k\left(w_2^F\right) = 0^+, \\ y_2^F = D^L \end{cases}$

We refer to 1.4 for the proof, by layout the game tree and work backward for the sequential equilibrium. We notice that when $s \in \left(\frac{p_1}{1-\rho}, \min\left\{\widetilde{s}_1^F, \frac{\overline{p}_2}{1-\rho}\right\}\right]$, the forward-looking retailer deliberately deviates from the optimal order of the current period D^H to create censorship, so that in the second period the supplier continues to offer a favorable wholesale price. And when $s \in \left(\max\left\{\widetilde{s}_1^F, \frac{p_1}{1-\rho}\right\}, \frac{\overline{p}_2}{1-\rho}\right]$, the supplier has to offer a lower wholesale price at the first period[28] to prevent the retailer from creating censorship at the first period. This result shows that the though the supplier has a higher decision power with the dynamic wholesale price, the retailer is still directly taking control the inventory thus the information acquisition scheme, so the incentive misalignment could create more intensive strategic tension.

[28] $\widetilde{k}_1^{F+} > p_1$, so $w\left(\widetilde{k}_1^{F+}\right) = \widetilde{w}_1^{F+} < w\left(p_1\right)$

2.6.2 Supply Chain Coordination with Contracts

In our basic model, we consider a simple constant wholesale price that causes double-marginalization effect to interplay with the two informational inefficiencies. A large body of literature discusses various contract forms that coordinate the supply chain. A natural question is how do these contract perform in the presence of demand censorship and the retailer's learning through inventory experimentation. We argue that constant buy-back contracts could coordinate the supply chain being subject to censorship. In other words, such contracts not only coordinate the profit split but also the information acquisition effort of the two players.

We consider a general continuous demand distribution with the cdf $\phi(\cdot)$, hyperparameters θ, and prior belief $\pi(\cdot|\cdot)$, and continue using the cost structure parameters $\{c, v, r, w, b\}$.Then the integrated supply chain's profit of period n with order quantity y_n is ω_n:

$$\omega_n(\pi'_n, y_n) = (r-c)\left[\int_0^{y_n} x\phi'_n(x)\,dx + \int_{y_n}^{+\infty} y_n\phi'_n(x)\,dx\right] - (c-v)\int_0^{y_n}(y_n - x)\,\phi'_n(x)\,dx, \quad (2.8)$$

where $\phi'_n(x) = \int_\Theta f(x|\theta)\,\pi'_n(\theta|x_{n-1})\,dx, \pi'_n(\theta|x_{n-1}) = \begin{cases}\pi_n(\theta|x_{n-1}), \text{if } x_{n-1} < y_{n-1} \\ \pi^c_n(\theta|y_{n-1}), \text{if } x_{n-1} \geq y_{n-1}\end{cases}$. And the belief updating rule under censorship is as follows:

$$\pi_n(\theta|x_{n-1}) = \frac{f(x_{n-1}|\theta)\,\pi'_{n-1}(\theta)}{\int_\Theta f(x|\theta')\,\pi'_{n-1}(\theta')\,d\theta'}, \pi^c_n(\theta|y_{n-1}) = \frac{\int_{y_{n-1}}^{+\infty} f(x|\theta)\,\pi'_{n-1}(\theta)\,dx}{\int_\Theta\int_{y_{n-1}}^{+\infty} f(x|\theta')\,\pi'_{n-1}(\theta')\,dx d\theta'}$$

The retailer's profit with the same posterior belief could be derived by replacing c with w, v with b in (2.8).Note that the demand information $\phi'_n(x)$ is common knowledge for all parties and is independent of their cost structure. So if we set

$$\frac{r-w}{r-c} = \frac{w-b}{c-v} := \zeta, \quad (2.9)$$

the retailer's profit of one period is proportionate to the supply chain's with the same order quantity:

$$\omega_n^R(\pi'_n, y_n) = \zeta\omega_n(\pi'_n, y_n).$$

Over time, the supply chain (or the retailer)'s NPV from period n on is:

$$\begin{aligned}
\rho_n(\pi'_n) &= \max_{y_n \in R^+} \rho_n(\pi'_n, y_n), \ y_n(\pi'_n) = \arg\max_{y_n \in R^+} \rho_n(\pi'_n, y_n) \\
\rho_n(\pi'_n, y_n) &= \omega_n(\pi'_n, y_n) + \delta\left[\int_0^{y_n} \rho_{n+1}(\pi_{n+1}(\cdot|x))\,\phi'_n(x)\,dx + \rho_{n+1}\left(\pi^c_{n+1}(\cdot|y_n)\right)\overline{\Phi}'_n(y_n)\right]
\end{aligned}$$

Proposition 9 *If a buy-back contract* (w, b) *satisfies (2.9), then for any period* $n \in \{1, 2, ..., N\}$, *the optimal order quantities* $y_n(\pi'_n) = y_n^R(\pi'_n)$.

This coordination result relies on the fact that the additional information cost due to censorship is proportional to future profit, therefore if the future profit is split between the retailer and the supplier proportionately, their exposure to such risk is also allocated proportionately.

Chapter 3

Strategic Interactions on an E-commerce Platform: Pricing, Demand Learning, and the Threat of Entry

3.1 Introduction

Amazon.com, known as the biggest online retailer in the US, has hybrid revenue streams. It serves as a online retail platform (Amazon Marketplace) for "third-party" merchants to list and sell their products on the website by charging a mix of commissions and fees. Meanwhile Amazon offers its own products as a reseller on its own marketplace to compete with other sellers. As the platform owner, Amazon has access to all transaction records, customer browsing data, as well as services requested by sellers, including advertising, fulfillment, and web service etc. that provide a superior advantage against other sellers having created the information by using the platform.

This situation was already recognized by many third-party sellers[32] until an article from the Wall Street Journal[31] attracted wider public attention. The story captures an instance of Amazon discovering a niche hot-selling item (pillow pets of NFL mascots) and taking it over from the independent seller by offering the same product with a more competitive price and prominent advertising spot. It soon sparked discussions among business commentators, blog writers, sellers, and Amazon employee[36][33][34][35] around Amazon's strategic trade-off, how sellers should react, and how different business functions of Amazon practically operate in such situations etc.

Amazon started as an online book seller in 1995, reached a $2.8 billion revenue within the first 5 years, and expanded from 1 category to 16 main categories in 15 years, known as "one of the fastest growths in the Internet's history"[30]. Amazon's marketplace was launched at 2000, initially considered by experts as a mistake for the sake of cannibalization. Yet the marketplace's growth soon surpassed Amazon's own sales and prompted it to offer advertising, fulfilment, computation, and cloud services through the platform. In 2012, Amazon reported a sales of over $60 billion[1], of which about $10 billion is service sales, i.e. the commission and fees from third-party sellers. This shows that roughly 40% of the total products sold on Amazon.com are from third-party sellers[37].

As some observers point out, Jeff Bezos, the founder and CEO of Amazon, has an ideal to build an ultimate "one-stop shop" with exhaustive selection, unbeatable prices, and great convenience to lock in all consumer needs. Having pioneered on product offering, tested pricing and inventory, and built customer awareness, the third-party sellers very much assisted Amazon with the impressive growth and expansion. Amazon could leverage its enormous bargaining power against suppliers, logistic facilities, analytical capability, and customer traffic to maintain favorable price and service to further attract customers and sellers to its platform. Therefore it is critical for Amazon to centralize and capitalize on the crowd-sourced knowledge from third-party sellers while retaining them as an indispensable revenue source and users.

While most online retailers take on only one role among the two most common business models, i.e. a merchant (e.g. most brick-and-mortar stores' online channel) or a platform (e.g. eBay.com and

[1] Amazon.com, Inc. (2012). Form 10-K 2012,
http://edgar.secdatabase.com/1562/119312513028520/filing-main.htm

Taobao.com) [62], Amazon is not the only one that seeks to adopt a hybrid of the two. Traditional retailers Sears, Walmart[40], and Staples[41] among others launched their online marketplaces to third party sellers after opening their online channel. jd.com, the second largest e-commerce website in China that started with its proprietary business also opened its website and logistic facilities to other B2C sellers for a commission. Yet few achieved a comparable dual-success through this path because of lack of participation, low sales volume, and inadequate quality and service from third -party sellers.

This contrast triggers a series of strategic questions around an online retailer's business model choice and execution. Under what conditions should an online retailer choose such a hybrid model, in particular when it starts as a merchant? How could they successfully implement this strategy by incorporating demand learning into its product offering decisions? Moreover, what strategic reactions from the third party sellers should be accounted for in designing this learning and entry policy? As two-sided marketplaces penetrate into more business sectors along with the rising of "sharing economy", the competitive yet collaborative relationship between the owner and the users of a platform requires a deeper understanding. In particular, how could the platform owners leverage the vast amount of information generated by users to improve operational efficiency and consumer welfare lies at the centre of the debates.

Jiang, Jerath, and Srinivasan's paper in 2011[47] looked into a similar problem motivated by Amazon's competitive and collaborative relationship with the third party sellers, and highlighted the platform owner's preference over large sales volume[2] and the trade-off around products with medium demand level, namely the "mid-tail". In this study, the authors assume that the independent seller leverages the hidden service effort level to prevent the platform owner from learning the product's true demand parameters, and the platform owner regulates the commission rate to elicit demand information. Based on these, the authors establish a two-period signaling game theoretical model to characterize the equilibrium behaviors of two players and argue that the platform owner sets a higher

[2] This might be different for merchants already selling products of high volume across many categories offline. The incentive for Walmart to open a marketplace might be only to enrich its online product offering and increse customer exposure.

commission rate with prospect of higher demand to separate the product with higher demand from those with lower demand.

Nonetheless, we notice two features in behaviors of the platform owner and the independent seller that need to be addressed more practically. First, when faced with a large number of products and sellers, the platform owner has a low strategic awareness and flexibility in interactions with each individual seller. Lacking the ability to examine each seller's motivation and set a unique commission rate accordingly, the platform owner demands a more homogeneous analytical process and operational instrument for her learning and entry decisions. For instance, Amazon deploys an extensive scheme to monitor the sales records of all newly-listed product items for a period of time before selecting the "promising" ones for buyer teams to review. Second, though technically an independent seller may manipulate the service level to influence the sales observation of the platform owner, it could effectively damage his reputation and get banned by the platform owner for quality purpose. Instead, we believe the seller's price decision influences sales level to a larger extent.

With these observations, we are further concerned with the following research questions. What is a scalable demand learning model with little assumptions on independent sellers to support the platform owner's ("PO" hereafter) entry decision? How would the independent seller ("IS") set his price in response to the PO's learning and entry threat, and can the PO's scheme account for it reasonably? Can we explain or predict some collective features of the marketplace based on the PO and the IS's behaviors?

To address these questions, we establish a Bayesian statistical model of the PO's learning for the IS's private demand information. In presence of other private information of the IS (e.g. cost parameter), we argue that the PO can hardly infer the demand information immediately upon observing the IS's pricing decision as a signal, but keep updating her belief also based on sales observations until her posterior belief meets the entry threshold. We adopt a bivariate demand function and a continuous prior distribution for the PO's belief, and discuss how it effectively supports the PO's entry decision.

With this model, we discover that in general, the IS sets a higher price when the price sensitivity is higher and lower price if the price sensitivity is lower to slow down the PO's learning and entry.

Consistent with the "mid-tail" observation in [47],the IS aims to avoid an exceedingly high sales volume compared to the PO's prior belief, so the price is adjusted based on the actual price sensitivity level relative to the prior. We also identify the set of product characteristics that induce the highest payoff for the IS under the PO's entry threat, whose market size range from low to high but maintains a medium profitability level to avoid the PO's early entry. This provides an interpretation and characterization of the IS's product offering choices on the marketplace, and is potentially testable for empirical studies.

Finally, we propose two types of heuristic entry policies for the PO marked as "profit driven" and "revenue driven" respectively. We evaluate the PO's performance under various entry thresholds based on the true demand parameters and a fixed price observation, and identify the existence of optimal entry threshold in each situation. In particular, we discuss conditions under which the PO is better off by forgoing the entry option, and highlight the prior variance's impact on this issue among other model parameters. A higher variance represents the PO's "openness to learn" for a highly unknown market, and leads to retaining the entry option even when the demand is low according to the prior. We also identify the PO's loss of up to 10% due to the IS's pricing distortion behavior, and establish the equivalency between the two types of entry policies based on the IS's fixed price.

The rest of this chapter is organized as follows. We first discuss the relevant literature in the ensuing subsection, and establish the model setup in Section 3.2. We devote Section 3.3 to discussion of model choice as well as the challenges for the PO's learning and entry process. Then we establish the IS's pricing problem and present its properties in Section 3.4, and discuss the impact of different entry threshold under the revenue-driven entry policy in Section 3.5. We present the observations arising from the profit-driven entry policy and the comparison between the two types of policies in Section 3.6. In Section 3.7, we justify the choice of a bivariate demand model and generalize the results from an uncorrelated bi-variate distribution to other settings. Finally we conclude with complementary discussions of several interesting open questions.

3.1.1 Literature Review

The literature to our study are from four major areas encompassing research in marketing, economics, operations management, and statistics. In particular, our research motivation is related to retail channel structure and two-sides markets, while the methodology relates to pricing with demand learning and strategic experimentation.

Retail Channel Structure Similar competitive and collaborative relationship also exists in traditional retail channel and supply chains. The literature on "store- within-a-store"[42] and consignment stores[43] studied retailers renting out physical space to suppliers or independent resellers, focusing on the trade-off between loss of control and increasing customer traffic and competition. Studies of private labels[44][45] focus on retailers taking advantage of market contact to compete with national brand suppliers. Yet online platforms differ from traditional channel in a few aspects. First of all, brick-and -mortar retailers are highly limited by physical space and resources to open marketplaces merely for the information benefit. Second, the information brick-and-mortar retailers obtain can hardly compete in volume and granularity with that from online platforms to facilitate vertical expansion. Last, traditional retailers often encounter product exclusivity when competing with suppliers so need to create new brands to complement the value proposition or to capitalize on the supplier's brand equity. While in e-commerce, third-party sellers rarely enjoys an exclusive relationship with the supplier and suppliers tend to deem the relationship with marketplaces as open and collaborative.

There is also a growing literature studying channel structures in e-commerce. Abhishek et al. (2013)[46] examined the e-tailer's choice between agency-selling and reselling by measuring their impact on sales in the traditional off-line channel. We highlighted Jiang et al. (2011)[47]'s study of a similar problem on Amazon's entry based on demand learning through third-party sellers. We share a view with Jiang et al. (2011) on the platform owner's preference on offering products with higher demand volume, yet we adopt a different modeling approach by acknowledging the platform owner's lack of strategic awareness and flexibility when interacting with each independent seller, the independent seller's pricing decision as the main operational lever, and the entry timing's importance. This directs us to resort to dynamic demand learning as the main research methodology instead of classical signaling game theory.

Two-sided Markets Two-sided markets in the economics literature refer to platforms that attract "buyers" and "sellers" to meet and strike deals, in particular to make profits from facilitating such interactions[64]. Examples of two-sided markets are abundant, including credit cards, playstations, ad agency, and appstore for smart devices as the more traditional ones and Airbnb, Uber to represent online two-sided markets that drive the rapid-growing "sharing economy". Rochet & Tirole (2006)[63], Armstrong(2006)[66] and their citations constitute a comprehensive introduction to this strand of literature, focusing on two-sides markets' network effects and growth dynamics, user behavior and interaction mechanism design, and pricing structures. Notably, Rysman (2009)[61] and Hagiu (2007)[62] pointed out Amazon's hybrid nature of "two-sidednes" and "one-sidedness". Hagiu (2007) discussed many issues that potentially affect the choice between these two modes and the conditions to support each as a better choice. For example, it listed high demand uncertainty, asymmetric information from the seller, the need for seller's ongoing investment, and consumer demand for variety as market features that favor the platform mode, and economies of scale, strong product complementarity to favor the merchant mode. In contrast, we focus on modeling the operational decisions supporting system and the trade-off facing a platform owner in retail practices on a product level.

Pricing with Demand Learning The Bandit problem on price experimentation originates from stylized model by Rothschild(1974)[58], on learning from two possible demand models. Later, McLennan(1984)[59] identified an "incomplete learning" issue for this problem, and Aghion et al.(1991)[60] further elaborate on it and discussed the conditions to successfully "learn" the true parameters in a long run. There is an operations management literature on dynamic pricing with demand learning[53][56][55][54]in revenue management context. Given that the basic dynamic pricing problem is hard to solve explicitly, an extra uncertainty further complicates it. So most of these studies resorts to asymptotic analysis or approximation for near-optimal policies. Moreover, these work mostly place the uncertainty on traffic intensity or market size related parameters instead of price-sensitivity, the learning for which depends largely on the existence of a time or inventory constraint.

Strategic Experimentation In addition to the literature on Bandit Problems we discuss in the previous sections[27], we notice a body of work using similar continuous time setup as in our model. Most Bandit problems are known to have no explicit solutions, which prohibits the understanding and applications of them. Karatzas(1984)[48] solved a class of continuous time problems with Brownian motion driven uncertainty and showed hope on this direction. Thereafter, many economic papers adopt this setup to study complicated strategic issues with learning. Bolton & Harris (1999)[49] demonstrated an example by solving the symmetric oligopoly strategies on a two-armed bandit problem. Harrison & Sunar (2013)[52] applied the similar framework to an investment problem from the operations management perspective. Keller et al. (2005)[50] solved an example with a risky arm yielding payoffs after an exponential distributed time. Garfagnini (2011)[51] applied this framework in a principal-agent setup on bank management. These work all assume a linear (or constant) learning cost and linear (or constant) signal structure for tractability.

3.2 Model Setup

We consider a discrete time model of infinite horizon on the dynamic interactions between the platform owner (PO) and an independent seller (IS) on the marketplace.

At the start, the IS is present on the market to sell one product, having private information over its demand characteristics as well as other profit-related factors (such as his unit variable cost). The PO only has a prior belief over the product's demand information, based on which she announces an entry threshold as a function of her posterior belief (and other public information such as the IS's potential selling price), and commits to offering the product, i.e. entering the market, once the pre-specified function of her posterior hits the threshold.

According to the PO's entry policy, the IS sets the product's price and starts selling the product while paying a commission to the PO proportional to realized sales of each period. As time progresses, the PO engages in a dynamic statistical learning for the product's demand parameters based on observations of the IS's price and demand realizations. Once the PO enters the market, she pays a one-time setup cost, drives the IS out of the market, and retains all revenue from selling the product thereafter.

We adopt a bi-variate linear demand function with noise, whose parameters follow a bi-variate normal distribution to represent a wide product space and to enrich the two player's strategy space. We allow for other type of private information from the PO besides the demand information, e.g. the IS's cost structure.

3.2.1 Timeline

The PO and the IS's interactions unfold over time as follows:

1. IS is at the market at $t = 0$ with the market condition $\boldsymbol{\theta}^*$;
2. Based on the profit related parameters $\boldsymbol{\psi}$, PO sets the entry policy $f\left(\boldsymbol{\theta}_t|p^{IS}, \boldsymbol{\theta}_0, \boldsymbol{\psi}\right)$ and commits to it;
3. IS sets his price p^{IS};PO collects a $\gamma-$revenue commission from the IS's sales, and updates her belief $\boldsymbol{\theta}_t = \boldsymbol{g}\left(\boldsymbol{\theta}_{t-1}|D_t, p^{IS}\right)$;
4. PO enters the market at the end of $\tau = \min\left\{t \geq 0|f\left(\boldsymbol{\theta}_t|p^{IS}, \boldsymbol{\theta}_0\right) \geq 0\right\}$ at an entry cost C,and sets the product price as p^{PO} from $t > \tau$ based on $\boldsymbol{\theta}_t$ to sell it thereafter;
5. IS exists the market and the PO collects all revenue from $\tau + 1$.

3.2.2 Demand Function and Information Structure

We adopt a linear price-demand function with two parameters following a bivariate normal distribution. The demand D_t at t is determined by the price p^{IS}, the demand parameters $\boldsymbol{\theta}^* := \{a, b\}$, and a white noise ε_t :

$$D_t = a - bp^{IS} + \varepsilon_t, \varepsilon_t \sim N\left(0, 1\right), p^{IS} \geq 0.$$

The true demand parameters $\boldsymbol{\theta}^*$ is only known to the IS, while the PO has prior information $\boldsymbol{\theta}_0 := (\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$ over its distribution at $t = 0$:

$$\begin{aligned}
(a,b)^T &\sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0), \\
\boldsymbol{\mu}_0 &= (\mu_a, \mu_b)^T \\
\boldsymbol{\Sigma}_0 &= \begin{pmatrix} \sigma_a^2 & \rho\sigma_a\sigma_b \\ \rho\sigma_a\sigma_b & \sigma_b^2 \end{pmatrix} \\
\rho &\in (-1,1), \sigma_a, \sigma_b > 0
\end{aligned}$$

Afterwards, the PO updates the posterior belief over $(a,b)^T$ in a Bayesian fashion based on the observations of $\left\{p^{IS}, D_t\right\}_{t>0}$.

Linear demand models are widely used in the existing literature. In particular in the demand learning context, for simplicity they often take an uni-variate form, which is equivalent to making the assumption of

$$a = b \cdot K$$

with a fixed K. In bi-variate models, there is often a discrete support for the parameters such as

$$(a,b) \in \{(a_1, b_1), (a_2, b_2)\}.$$

Here we consider a more general bi-variate model with a continuum support to enrich the product space. It allows for a more general characterization of the IS's pricing behavior to influence the PO's learning and entry decision. The normal prior for the demand parameters combined with a normal-distributed noise term renders simple belief updating rule for the PO.

3.2.3 Profit Functions and Cost Structure

Besides the demand information, the two players' profits are jointly determined by another set of parameters $\boldsymbol{\psi} := \left\{\delta, C, \gamma, c^{PO}, c^{IS}\right\}$. Here δ is the discount factor for the two players, C is the PO's entry cost, γ is the revenue commission rate, and $\left\{c^{PO}, c^{IS}\right\}$ are the unit variable costs for the two players. We assume that $\left\{\delta, C, \gamma, c^{PO}\right\}$ are public information while c^{IS} is private information for the IS.

The two players' expected discounted payoffs over the entire time horizon are

$$\begin{aligned} V^{PO}\left(f, p^{PO}|\boldsymbol{\theta}_0, p^{IS}\right) &= E\left[\begin{array}{cc} \Sigma_{t=1}^{\tau}\gamma p^{IS}\left(a - bp^{IS}\right)\delta^t - \delta^\tau C & \\ +\Sigma_{t=\tau+1}^{+\infty}\left(p^{PO} - c^{PO}\right)\left(a - bp^{PO}\right)\delta^t & |\boldsymbol{\theta}_0 \end{array}\right] \\ V^{IS}\left(p^{IS}|f, \boldsymbol{\theta}_0, \boldsymbol{\theta}^*\right) &= E\left[\Sigma_{t=1}^{\tau}\left[(1-\gamma)p^{IS} - c^{IS}\right]\left(a - bp^{IS}\right)\delta^t\right] \\ s.t.\ \tau &= \min\left\{t \geq 0 | f\left(\boldsymbol{\theta}_t|p^{IS}, \boldsymbol{\theta}_0, \boldsymbol{\psi}\right) \geq 0\right\}, \boldsymbol{\theta}_t = \boldsymbol{g}\left(\boldsymbol{\theta}_{t-1}|D_t, p^{IS}\right) \end{aligned}$$

For dispositional simplicity, we assume that $c^{IS} > c^{PO}$ for most of this paper and normalize c^{PO} as 0[3].

3.3 The PO's Belief Updating and Entry Policy

The interactions between the IS and the PO take place sequentially and following a backward induction, we first examine and characterize the PO's learning behavior and entry policies for given price set by the IS. In particular, we discuss the differences between "strategic learning" and "statistical learn-

[3] We could make the transformation of

$$p - c^{PO} = p', a - bc^{PO} = a', c' = c^{IS} - c^{PO}$$

to ensure

$$\begin{aligned} p'\left(a' - bp'\right) &= \left(p - c^{PO}\right)(a - bp). \\ \left(p - c'\right)\left(a' - bp'\right) &= \left(p - c^{IS}\right)(a - bp) \end{aligned}$$

So this normalization does not affect the profit parts of the two players' payoff functions. However the PO's revenue-based commission rate would be affected since

$$\gamma p(a - bp) - \gamma p'\left(a' - bp'\right) = \gamma c^{PO}\left(a' - bp'\right) \neq 0.$$

ing" for the PO, and explain the reason that we adopt the latter in our model of the PO's information acquisition behavior. We discuss the PO's entry policy based on her statistical learning scheme, and identify two types of heuristic policies given the difficulty in searching for the optimum. Namely we focus on the revenue-driven ("RD") threshold policy and the profit-driven ("PD") threshold policy.

3.3.1 Strategic Learning and Statistical Learning

In the classic principal-agent model with asymmetric information, the observable action of the agent serves as a channel for information revelation (i.e. "signaling") and/or information solicitation (i.e. "screening"). These models involve heavy assumptions on the two players' strategic behaviors such as forming consistent beliefs over the agent's action and the principal's interpretation, and we name it as "strategic learning".

We argue that as the agent's private information structure gets more complex, e.g. having a continuous or multi-dimensional support, the information the principal obtains from such strategic learning diminishes in clarity. For example, it requires very restrictive belief to support a more informative separating equilibrium compared to the pooling or hybrid equilibrium. In this case, the principal essentially takes the agent's actions as exogenous and uses it in combination with other signals to infer the agent's private information of interest, so the principal's information acquisition behavior could be approximated by a statistical learning process. In the appendix, we provide a few signaling game examples with two dimensional private information, among which the principal is interested in learning only one. These examples demonstrate the limitation of current strategic learning models and how it could be represented by statistical learning process.

In our model, the IS's price decision is not only affected by her private demand information $\boldsymbol{\theta}^* := \{a, b\}$ but also her private unit variable cost c^{IS}. While the latter does not influence the PO's profit directly and is of no interest for her to learn. So the IS's price p^{IS} by itself only serves as an ambiguous signal for the demand information, and the PO's inference of $\boldsymbol{\theta}^*$ is mainly driven by the statistical learning based on observations of $\left\{p^{IS}, D_t\right\}_{t>0}$. In particular, in the PO's interactions with a large group of IS, each with distinctive business objective and decision criteria, it is practical for the PO to adopt an effective learning mechanism built on little strategic assumption and knowledge on each individual seller's motivation. With these observations, we establish the PO's information acquisition process through Bayesian statistical learning and deem the strategic learning component as negligible.

3.3.2 Belief Updating Rule

We develop the PO's belief updating rule of $\boldsymbol{\theta}_t$ based on statistical learning using observations of $\left\{p_t^{IS}, D_t\right\}_{t>0}$ as exogenous signals. We first define that:

$$\begin{aligned} \boldsymbol{X}_t &= \begin{pmatrix} 1 & 1... & 1 \\ -p_1^{IS} & -p_2^{IS}... & -p_t^{IS} \end{pmatrix}^T, \boldsymbol{y}_t = (d_1, d_2, ...d_t)^T \\ d_i &= a - bp_i^{IS} + \varepsilon_i, i = 1, ..., t. \end{aligned}$$

Upon observing $\{\boldsymbol{X}_t, \boldsymbol{y}_t\}$, the PO's posterior belief on $\{a, b\}$ still follows a bi-variate normal distribution, whose parameters $\{\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t\}$ are:

$$\begin{aligned} \boldsymbol{\mu}_t &= \boldsymbol{\Sigma}_t \left(\boldsymbol{X}_t^T \boldsymbol{y}_t + \boldsymbol{\Sigma}_0^{-1} \boldsymbol{\mu}_0\right). \\ &= \boldsymbol{\Sigma}_t \begin{pmatrix} \Sigma d_i + \frac{\frac{\mu_a}{\sigma_a^2} - \frac{\rho\mu_b}{\sigma_a\sigma_b}}{1-\rho^2} \\ -\Sigma p_i^{IS} d_i + \frac{\frac{\mu_b}{\sigma_b^2} - \frac{\rho\mu_a}{\sigma_a\sigma_b}}{1-\rho^2} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \boldsymbol{\Sigma}_t &= \left(\boldsymbol{X}_t^T \boldsymbol{X}_t + \boldsymbol{\Sigma}_0^{-1}\right)^{-1} \\ &= \frac{\begin{pmatrix} \frac{1}{(1-\rho^2)\sigma_b^2} + \Sigma\left(p_i^{IS}\right)^2 & \frac{\rho}{(1-\rho^2)\sigma_a\sigma_b} + \Sigma p_i^{IS} \\ \frac{\rho}{(1-\rho^2)\sigma_a\sigma_b} + \Sigma p_i^{IS} & \frac{1}{(1-\rho^2)\sigma_a^2} + t \end{pmatrix}}{\frac{1}{(1-\rho^2)\sigma_a^2\sigma_b^2} + t\Sigma\left(p_i^{IS}\right)^2 - \left(\Sigma p_i^{IS}\right)^2 + \frac{t}{(1-\rho^2)\sigma_b^2} + \frac{\Sigma\left(p_i^{IS}\right)^2}{(1-\rho^2)\sigma_a^2} - \frac{2\rho\Sigma p_i^{IS}}{(1-\rho^2)\sigma_a\sigma_b}} \end{aligned}$$

In this study, we assume that the IS's price stays constant, i.e. $\forall i \geq 1$, $p_i^{IS} = p^{IS}$, then

$$
\begin{aligned}
\boldsymbol{\mu}_t &= \boldsymbol{\Sigma}_t \left(\boldsymbol{X}_t^T \boldsymbol{y}_t + \boldsymbol{\Sigma}_0^{-1} \boldsymbol{\mu}_0 \right). \\
&= \boldsymbol{\Sigma}_t \begin{pmatrix} \Sigma d_i + \frac{\frac{\mu_a}{\sigma_a^2} - \frac{\rho \mu_b}{\sigma_a \sigma_b}}{1-\rho^2} \\ -p^{IS} \Sigma d_i + \frac{\frac{\mu_b}{\sigma_b^2} - \frac{\rho \mu_a}{\sigma_a \sigma_b}}{1-\rho^2} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\Sigma}_t &= \left(\boldsymbol{X}_t^T \boldsymbol{X}_t + \boldsymbol{\Sigma}_0^{-1} \right)^{-1} \\
&= \frac{\begin{pmatrix} \frac{1}{(1-\rho^2)\sigma_b^2} + t\left(p^{IS}\right)^2 & \frac{\rho}{(1-\rho^2)\sigma_a \sigma_b} + t p^{IS} \\ \frac{\rho}{(1-\rho^2)\sigma_a \sigma_b} + t p^{IS} & \frac{1}{(1-\rho^2)\sigma_a^2} + t \end{pmatrix}}{\frac{1}{(1-\rho^2)\sigma_a^2 \sigma_b^2} + t(t-1)\left(p^{IS}\right)^2 + \frac{t}{(1-\rho^2)\sigma_b^2} + \frac{t(p^{IS})^2}{(1-\rho^2)\sigma_a^2} - \frac{2\rho t p^{IS}}{(1-\rho^2)\sigma_a \sigma_b}}
\end{aligned}
$$

For notational simplicity we represent these expressions as $\boldsymbol{\theta}_t = g\left(\boldsymbol{\theta}_{t-1} | D_t, p^{IS}\right)$.

3.3.3 Entry Policy

The PO's entry policy decision serves as the first strategic move of the two players' interactions, and directly determine the IS's best pricing response. We first formulate and discuss the PO's optimal entry policy problem, and due to the technical difficulty, we propose two types of heuristic threshold entry policies.

The PO's Optimal Entry Policy

We argue that, due to the IS's multi-dimensional private information, the PO's information acquisition mainly comprises statistical learning instead of strategic learning, and assume she takes the IS's price decision as exogenous for the purpose of learning. Similarly, due to the complex private information structure, the PO can hardly anticipate the IS's pricing decision in response to her entry policy. So we only consider the entry policies that involve in the IS's price as an exogenous parameter

$f\left(\boldsymbol{\theta}_t|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right)$.

In this case, the optimal entry policy $f^*\left(\boldsymbol{\theta}_t|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right)$ is derived from the optimal solution of the following problem dynamic programing at the end of each time period t:

$$V^{PO*}\left(\boldsymbol{\theta}_t|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right)=\max\left\{\frac{\delta}{1-\delta}\frac{\mu_{a,t}^2}{4\mu_{b,t}}-C,\delta\left[\gamma p^{IS}\left(\mu_{a,t}-\mu_{b,t}p^{IS}\right)+EV^{PO*}\left(\boldsymbol{\theta}_{t+1}|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right)\right]\right\} \tag{3.1}$$

and the state variable follows the transition rule $\boldsymbol{\theta}_t=g\left(\boldsymbol{\theta}_{t-1}|D_t,p^{IS}\right)$. Intuitively,

$$\frac{\delta}{1-\delta}\frac{\mu_{a,t}^2}{4\mu_{b,t}}-C=E\left[\Sigma_{i=t+1}^{\infty}p^{PO}\left(a-bp^{PO}\right)\delta^{i-t}|\boldsymbol{\theta}_t\right]-C$$

represents the expected continuation value when the PO decides to enter at the end of time t and sets the optimal price to be

$$p^{PO}=\frac{\mu_{a,t}}{2\mu_{b,t}}$$

based on the posterior information of that moment;while

$$\delta\left[\gamma p^{IS}\left(\mu_{a,t}-\mu_{b,t}p^{IS}\right)+EV^{PO*}\left(\boldsymbol{\theta}_{t+1}|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right)\right]$$

represents the expected continuation value when the PO decides to not enter at the end of time t and wait until the end of period $t+1$ to collect the revenue commission $\delta\gamma p^{IS}\left(\mu_{a,t}-\mu_{b,t}p^{IS}\right)^+$ and the expected continuation value from then on. By comparing these two options of "entering now" and "waiting till next time", the PO's entry policy $f^*\left(\boldsymbol{\theta}_t|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right)$ entails an immediate entry when

$$\begin{aligned}V^{PO*}\left(\boldsymbol{\theta}_t|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right) &= \frac{\delta}{1-\delta}\frac{\mu_{a,t}^2}{4\mu_{b,t}}-C\\ &\geq \delta\left[\gamma p^{IS}\left(\mu_{a,t}-\mu_{b,t}p^{IS}\right)+EV^{PO*}\left(\boldsymbol{\theta}_{t+1}|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right)\right],\\ &\iff f^*\left(\boldsymbol{\theta}_t|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right)\geq 0.\end{aligned}$$

Two Types of Heuristic Entry Policies

Notably, the problem in (3.1) is intractible with a continuum and multi-dimensional state space, i.e. suffering from "curse of dimensionality". We consider a simplification[4] by restricting the PO's choice to be between "entering now" and "never entering" for each period t. In this case, the PO's entry policy is derived as follows[5]:

[4] In general, we assume a minimal rationality for the PO in the interactions with the specific individual IS for the large total number of independent sellers she deals with and the associated information asymmetry. While we assume that the IS acts rationally, i.e. taking best response based on his privated information and the pricipal's policy commitment.

[5] For a generic entry policy $f\left(\boldsymbol{\theta}_t|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right)$, the PO's expected payoff following this policy is:

$$\begin{aligned} V^{PO}\left(f|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right) &= E\left[\Sigma_{t=1}^{\tau}\gamma p^{IS}\left(a-bp^{IS}\right)^{+}\delta^t-\delta^\tau C+\Sigma_{t=\tau+1}^{+\infty}p^{PO}\left(a-bp^{PO}\right)\delta^t|\boldsymbol{\theta}_0\right] \\ s.t.\ \tau &= \min\left\{t\geq 0|f\left(\boldsymbol{\theta}_t|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right)\geq 0\right\}, \\ \boldsymbol{\theta}_t &= \boldsymbol{g}\left(\boldsymbol{\theta}_{t-1}|D_t,p^{IS}\right). \end{aligned}$$

We express the expected payoff as $V^{PO}\left(f|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right)=E\left[V^{PO}\left(\tau|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right)\right]$, and first study the payoff conditional on the entry time τ :

$$\begin{aligned} V^{PO}\left(\tau|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right) &= E\left[\Sigma_{t=1}^{\tau}\gamma p^{IS}\left(a-bp^{IS}\right)^{+}\delta^t-\delta^\tau C+\Sigma_{t=\tau+1}^{+\infty}p^{PO}\left(a-bp^{PO}\right)\delta^t|\tau\right] \\ &= \Sigma_{t=1}^{+\infty}\gamma p^{IS}\left(\mu_a-\mu_b p^{IS}\right)^{+}\delta^t \\ &\quad -E\left[\delta^\tau C-\Sigma_{t=\tau+1}^{+\infty}\left[p^{PO}\left(a-bp^{PO}\right)-\gamma p^{IS}\left(a-bp^{IS}\right)^{+}\right]\delta^t|\tau\right]. \end{aligned}$$

The PO's optimal price is set right after the entry decision is make and the information available at the moment is $\boldsymbol{\theta}_\tau$, therefore

$$\begin{aligned} & V^{PO}\left(f|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right)-\frac{\gamma p^{IS}\left(\mu_a-\mu_b p^{IS}\right)^{+}\delta}{1-\delta} \\ = & E\left[V^{PO}\left(\tau|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right)-\frac{\gamma p^{IS}\left(\mu_a-\mu_b p^{IS}\right)^{+}\delta}{1-\delta}\right] \\ = & \frac{\delta}{1-\delta}E\left\{\delta^\tau\left[\frac{\mu_{a,\tau}^2}{4\mu_{b,\tau}}-\gamma p^{IS}\left(\mu_{a,\tau}-\mu_{b,\tau}p^{IS}\right)^{+}-\frac{C\left(1-\delta\right)}{\delta}\right]\right\} \end{aligned}$$

An optimal entry policy f^* needs to balance the trade-off between a higher incremental profit $\frac{\mu_{a,\tau}^2}{4\mu_{b,\tau}}-\gamma p^{IS}\left(\mu_{a,\tau}-\mu_{b,\tau}p^{IS}\right)^{+}-\frac{C(1-\delta)}{\delta}$ and an earlier entry time index τ.If the PO discards the concerns for an optimal entry time but enters as soon as the incremental profits is positive, she essentially adopts the strategy of "entering now or never".

$$
\begin{aligned}
f^{*}\left(\boldsymbol{\theta}_t|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right) &\geq 0 \iff \\
V^{PO*}\left(\boldsymbol{\theta}_t|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right) &= \frac{\delta}{1-\delta}\frac{\mu_{a,t}^2}{4\mu_{b,t}} - C \\
&\geq \frac{\delta}{1-\delta}\gamma p^{IS}\left(\mu_{a,t}-\mu_{b,t}p^{IS}\right),
\end{aligned}
$$

based on which we define

$$
f_{C_\gamma^*} := \frac{\mu_{a,\tau}^2}{4\mu_{b,\tau}} - \gamma p^{IS}\left(\mu_{a,\tau}-\mu_{b,\tau}p^{IS}\right)^{+} - \frac{C\left(1-\delta\right)}{\delta}.
$$

This policy essentially entails an entry as soon as the PO's incremental profit after entry is positive, therefore we consider it "profit-driven" heuristic. Similarly, we consider a "revenue-driven" heuristic under which the PO enters as soon as her incremental revenue[6] after entry is positive:

$$
f_{C^*} = \frac{\mu_{a,\tau}^2}{4\mu_{b,\tau}} - C^*.
$$

Namely, C_γ^* and C^* are thresholds for the PO's minimum incremental profit or post entry revenue respectively.

We argue that the optimal entry policy involves in three layers of trade-off as follows:

1. entry cost index $\frac{C(1-\delta)}{\delta}$ and post-entry revenue $\frac{\mu_{a,\tau}^2}{4\mu_{b,\tau}}$;
2. post-entry net profit $\frac{\mu_{a,\tau}^2}{4\mu_{b,\tau}} - \frac{C(1-\delta)}{\delta}$ and pre-entry net profit $\gamma p^{IS}\left(\mu_{a,\tau}-\mu_{b,\tau}p^{IS}\right)^{+}$;
3. incremental profit $\frac{\mu_{a,\tau}^2}{4\mu_{b,\tau}} - \gamma p^{IS}\left(\mu_{a,\tau}-\mu_{b,\tau}p^{IS}\right)^{+} - \frac{C(1-\delta)}{\delta}$ and entry time index τ.

While $f_{C_\gamma^*}$ looks at the first two, and f_{C^*} focuses on the first one. For tractability, we assume that the PO adopts f_{C^*} and study the IS and the PO's behaviors and performances in this case for most of

[6] Here we use revenue-driven for f_{C^*} to contrast with the "net profit", though $\frac{\mu_{a,\tau}^2}{4\mu_{b,\tau}}$ is actually the PO's profit post entry as we normalize the PO's varaible cost as 0.

this study. We discuss the results when the PO adopts $f_{C^*_\gamma}$ and compare the two systems in Section 3.6.

3.4 The IS's Optimal Pricing Policy and Performance

In this section, we study the IS's optimal pricing decision as well as the associated payoff functions of the IS. Given the PO's belief updating rule based on statistical learning and entry policy f_{C^*}, the IS could anticipate the PO's belief updating trajectory and her entry time based on the true demand parameters $\{a, b\}$. Then we could formulate the IS's optimal pricing problem.

3.4.1 The IS's Anticipation of the PO's Entry

The IS's Anticipation of the PO's Belief Evolution

Knowing the true value of $\{a, b\}^T := \boldsymbol{\mu}^*$, the PO's posterior mean $\boldsymbol{\mu}_t$ follows a two-dimensional random walk from the IS's perspective:

$$\begin{aligned} \boldsymbol{\mu}_t &= \boldsymbol{\Sigma}_t \left(\boldsymbol{X}_t^T \boldsymbol{X}_t \boldsymbol{\mu}^* + \boldsymbol{\Sigma}_0^{-1} \boldsymbol{\mu}_0 \right) + \boldsymbol{\Sigma}_t \boldsymbol{X}_t^T \boldsymbol{\varepsilon}_t \\ &= \boldsymbol{\mu}_t^D + \boldsymbol{\mu}_t^\epsilon, \end{aligned}$$

with a deterministic trajectory $\boldsymbol{\mu}_t^D = \boldsymbol{\Sigma}_t \left(\boldsymbol{X}_t^T \boldsymbol{X}_t \boldsymbol{\mu}^* + \boldsymbol{\Sigma}_0^{-1} \boldsymbol{\mu}_0 \right)$ and a stochastic trajectory $\boldsymbol{\mu}_t^\epsilon = \boldsymbol{\Sigma}_t \boldsymbol{X}_t^T \boldsymbol{\varepsilon}_t$.

With the assumption $p_i^{IS} = p$, we define

$$\Delta \boldsymbol{\mu}(p) := \begin{pmatrix} \frac{1}{\sigma_b^2} - \frac{\rho p}{\sigma_a \sigma_b} \\ \frac{\rho}{\sigma_a \sigma_b} - \frac{p}{\sigma_a^2} \end{pmatrix},$$

and get:[7]

$$
\begin{aligned}
\boldsymbol{\mu}_t^D - \boldsymbol{\mu}_0 &= \frac{t\left[(a-\mu_a) - p\,(b-\mu_b)\right]}{\frac{1}{\sigma_a^2\sigma_b^2} + t\left(\frac{1}{\sigma_b^2} + \frac{p^2}{\sigma_a^2} - \frac{2\rho p}{\sigma_a\sigma_b}\right)} \boldsymbol{\Delta\mu}\,(p) \\
\boldsymbol{\mu}_t^{\epsilon} &= \frac{\Sigma\varepsilon_i}{\frac{1}{\sigma_a^2\sigma_b^2} + t\left(\frac{1}{\sigma_b^2} + \frac{p^2}{\sigma_a^2} - \frac{2\rho p}{\sigma_a\sigma_b}\right)} \boldsymbol{\Delta\mu}\,(p)\,.
\end{aligned}
$$

We have the following observations on the PO's belief evolution trajectory from the IS's perspective (see Figure 3.1):

7

$$
\begin{aligned}
\boldsymbol{\mu}_t^D &= \\
&\frac{\left[\begin{pmatrix} \frac{\mu_a}{\sigma_a^2\sigma_b^2} \\ \frac{\mu_b}{\sigma_a^2\sigma_b^2} \end{pmatrix} + t\begin{pmatrix} \frac{a}{\sigma_b^2} + p\left[\frac{\mu_b-b}{\sigma_b^2} - \frac{\rho(\mu_a+a)}{\sigma_a\sigma_b}\right] + p^2\left[\frac{\mu_a}{\sigma_a^2} - \frac{\rho(\mu_b-b)}{\sigma_a\sigma_b}\right] \\ \frac{\mu_b}{\sigma_b^2} - \frac{\rho(\mu_a-a)}{\sigma_a\sigma_b} + p\left[\frac{\mu_a-a}{\sigma_a^2} - \frac{\rho(\mu_b+b)}{\sigma_a\sigma_b}\right] + \frac{bp^2}{\sigma_a^2} \end{pmatrix}\right]}{\frac{1}{\sigma_a^2\sigma_b^2} + t\left(\frac{1}{\sigma_b^2} + \frac{p^2}{\sigma_a^2} - \frac{2\rho p}{\sigma_a\sigma_b}\right)} \\
\boldsymbol{\mu}_t^{\epsilon} &= \frac{\Sigma\varepsilon_i \begin{pmatrix} \frac{1}{\sigma_b^2} - \frac{\rho p}{\sigma_a\sigma_b} \\ \frac{\rho}{\sigma_a\sigma_b} - \frac{p}{\sigma_a^2} \end{pmatrix}}{\frac{1}{\sigma_a^2\sigma_b^2} + t\left(\frac{1}{\sigma_b^2} + \frac{p^2}{\sigma_a^2} - \frac{2\rho p}{\sigma_a\sigma_b}\right)} \xrightarrow{} t \to +\infty 0.
\end{aligned}
$$

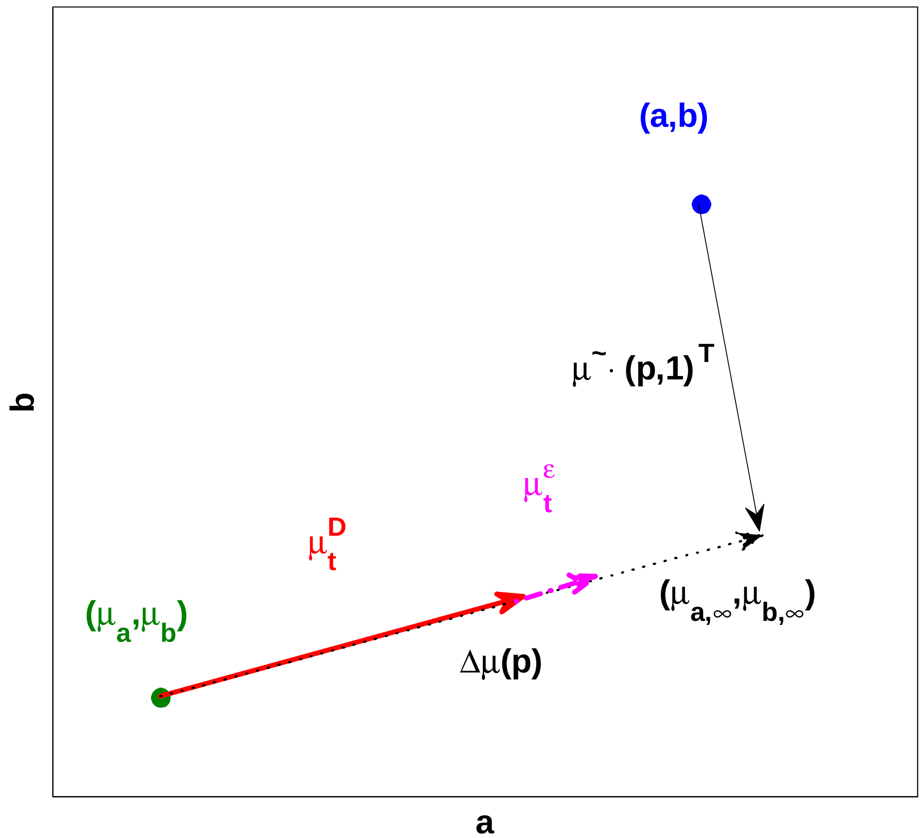


Figure 3.1: The IS's Anticipation of PO's Belief Evolution

1. **Stability**

   The stochastic trajectory $\boldsymbol{\mu}_t^\epsilon$ converges to 0 (in probability) as t grows, so asymptotically $\boldsymbol{\mu}_t^D$ dominates the belief evolution. Note that when $p = \frac{a-\mu}{b-\mu}$, the deterministic component of the PO learning is eliminated, i.e. $\boldsymbol{\mu}_t^D - \boldsymbol{\mu}_0 = 0$, so her learning is fully driven by the random component $\boldsymbol{\mu}_t^\epsilon$. This is an issue well-recognized in the existing demand learning literature[56][59] with linear, bivariate demand models, caused by an identical expected demand level under the prior and true demand information based on a "stalling price", here $p = \frac{a-\mu}{b-\mu}$.

2. **Direction**

   $\boldsymbol{\mu}_t^D$ is moving along the direction of $\Delta\boldsymbol{\mu}(p)$ starting from $\boldsymbol{\mu}_0$; $\Delta\boldsymbol{\mu}(p)$ is non-zero as long as the PO's bi-nomial prior on the demand parameters is indegenerate $\rho \in (-1, 1)$.

3. **Asymptoticity**

   As $t \to +\infty$,

$$\begin{aligned}
\boldsymbol{\mu}_t &\to \boldsymbol{\mu}_0 + \frac{(a-\mu_a) - p(b-\mu_b)}{\frac{1}{\sigma_b^2} + \frac{p^2}{\sigma_a^2} - \frac{2\rho p}{\sigma_a\sigma_b}} \boldsymbol{\Delta\mu}(p) := \boldsymbol{\mu}_{+\infty} \\
&= \begin{pmatrix} a \\ b \end{pmatrix} + \widetilde{\mu}(p) \cdot \begin{pmatrix} p \\ 1 \end{pmatrix} \\
\widetilde{\mu}(p) &= \frac{\left[\frac{\mu_b - b}{\sigma_b^2} - \frac{\rho(\mu_a - a)}{\sigma_a\sigma_b}\right] + p\left[\frac{\mu_a - a}{\sigma_a^2} - \frac{\rho(\mu_b - b)}{\sigma_a\sigma_b}\right]}{\frac{1}{\sigma_b^2} + \frac{p^2}{\sigma_a^2} - \frac{2\rho p}{\sigma_a\sigma_b}}
\end{aligned}$$

Note that $\widetilde{\mu}(p)$ is a scaler, and when $\widetilde{\mu}(p) \neq 0$, the PO's learning is inconsistent. This information loss is due to the insufficient observation (i.e. one-dimensional variation from $\{D_t\}_t$) for a two-dimensional inference. Since

$$\frac{|\mu_{a,t}^D - a|}{\left\|\boldsymbol{\mu}_t^D - (a,b)\right\|} = \frac{p}{\sqrt{p^2+1}}, \frac{|\mu_{b,t}^D - b|}{\left\|\boldsymbol{\mu}_t^D - (a,b)\right\|} = \frac{1}{\sqrt{p^2+1}},$$

IS's price p regulates the information acquisition effort (and accuracy) devoted to a and b. The higher p is, the more effort and accuracy is allocated to b while a's evolution is proportional to the change in b based on the relative error; and vice versa.

4. and it governs $a's$ information updating, and vice versa:

$$\boldsymbol{\mu}_t^D \xrightarrow{t \to +\infty} \begin{cases} \xrightarrow{p \to 0} \left(a, \mu_b + \frac{\rho\sigma_b(a-\mu_a)}{\sigma_a}\right) \\ \xrightarrow{p \to +\infty} \left(\mu_a + \frac{\rho\sigma_a(b-\mu_b)}{\sigma_b}, b\right) \end{cases}$$

$\boldsymbol{\mu}_t^D$ converge to $(a,b)^T$,i.e. $\widetilde{\mu}(p) = 0$, only when

$$p = \frac{\sigma_a}{\sigma_b} \frac{\frac{\mu_b - b}{\sigma_b} - \frac{\rho(\mu_a - a)}{\sigma_a}}{\frac{\rho(\mu_b - b)}{\sigma_b} - \frac{\mu_a - a}{\sigma_a}} := p^{Consistent}$$

or

(a) $\frac{b-\mu_b}{\sigma_b} = \pm\frac{a-\mu_a}{\sigma_a}, \rho = \pm 1$ (learning along the right direction, regardless of p)

(b) $b = \mu_b, a = \mu_a$

5. **Consistence**

From the PO's perspective, the demand D_t follows a normal distribution with known variance and unknown mean $a - bp$. Based on her prior and the price observation p, the PO's posterior of $a - bp$ follows $N\left(\boldsymbol{\mu}_t \cdot \overrightarrow{p}, \overrightarrow{p}^T \boldsymbol{\Sigma}_t \overrightarrow{p}\right)$, where $\overrightarrow{p} = \begin{pmatrix} 1 \\ -p \end{pmatrix}$, and her learning of the demand level is consistent.

The PO's Entry Time under f_{C^*}

To find the stopping time $\tau := \min\left\{t \geq 0 | \frac{\mu_{a,t}^2}{4\mu_{b,t}} \geq C^*\right\}$ from the PO's perspective, we write $p^{IS} = p$ for simplicity and consider the change of coordinates $\boldsymbol{\mu}_t = \boldsymbol{\mu}_0 + T^{-1}(p)\,\boldsymbol{\mu}_t'(p)$ defined by

$$\begin{aligned} T(p) &= \iota(p)^{-1} \begin{pmatrix} \frac{1}{\sigma_b^2} - \frac{\rho p}{\sigma_a\sigma_b} & \frac{\rho}{\sigma_a\sigma_b} - \frac{p}{\sigma_a^2} \\ -\left(\frac{\rho}{\sigma_a\sigma_b} - \frac{p}{\sigma_a^2}\right) & \frac{1}{\sigma_b^2} - \frac{\rho p}{\sigma_a\sigma_b} \end{pmatrix} \\ \iota(p) &= \sqrt{\left(\frac{1}{\sigma_b^2} - \frac{\rho p}{\sigma_a\sigma_b}\right)^2 + \left(\frac{\rho}{\sigma_a\sigma_b} - \frac{p}{\sigma_a^2}\right)^2} \end{aligned}$$

In this new coordinate frame, the deterministic and random components of the PO's learning are

$$\begin{aligned}
\boldsymbol{\mu}_t'^D &= \frac{t\left[(a-\mu_a)-p\,(b-\mu_b)\right]\iota\,(p)}{\frac{1}{\sigma_a^2\sigma_b^2}+t\left(\frac{1}{\sigma_b^2}+\frac{p^2}{\sigma_a^2}-\frac{2\rho p}{\sigma_a\sigma_b}\right)}\begin{pmatrix}1\\0\end{pmatrix}\\
\boldsymbol{\mu}_t'^\epsilon &= \frac{\iota\,(p)\,\Sigma\varepsilon_i}{\frac{1}{\sigma_a^2\sigma_b^2}+t\left(\frac{1}{\sigma_b^2}+\frac{p^2}{\sigma_a^2}-\frac{2\rho p}{\sigma_a\sigma_b}\right)}\begin{pmatrix}1\\0\end{pmatrix},
\end{aligned}$$

which help to solve for the PO's entry time τ through the following process:

1. Define $\omega=\arctan\frac{\frac{\rho}{\sigma_a\sigma_b}-\frac{p}{\sigma_a^2}}{\frac{1}{\sigma_b^2}-\frac{\rho p}{\sigma_a\sigma_b}}$[8],then ($\mu_t'$ is a scalar to solve, marking the distance the PO's belief moves along $\Delta\boldsymbol{\mu}\,(p)$)

$$\boldsymbol{\mu}_t=\begin{pmatrix}\mu_a\\\mu_b\end{pmatrix}+\begin{pmatrix}\cos\omega\\\sin\omega\end{pmatrix}\mu_t' \tag{3.2}$$

2. Insert (3.2) into the entry condition boundary $\frac{\mu_{a,\tau}^2}{4\mu_{b,\tau}}=C^*$, and obtain[9]

$$\mu_\tau'=\begin{cases}\frac{4\sqrt{C^*\cos\omega(\mu_b\cos\omega-\mu_a\sin\omega)+C^{*2}\sin^2\omega}-(2\mu_a\cos\omega-4C^*\sin\omega)}{2\cos^2\omega} & \text{if }\cos\omega\neq0\\ \mu_b-\frac{\mu_a^2}{4C^*} & \text{if }\cos\omega=0\text{ and }\sin\omega=-1\\ \text{no solution} & o.w.\end{cases} \tag{3.3}$$

3. The stopping time for $\boldsymbol{\mu}_t'^D+\boldsymbol{\mu}_t'^\varepsilon$ to hit the entry threshold at $\begin{pmatrix}\mu_\tau'\\0\end{pmatrix}$ is equal to the stopping

[8] $\begin{pmatrix}\frac{1}{\sigma_b^2}-\frac{\rho p}{\sigma_a\sigma_b}\\\frac{\rho}{\sigma_a\sigma_b}-\frac{p}{\sigma_a^2}\end{pmatrix}$ is the belief evolution direction

[9]

$$\begin{aligned}
\alpha &= \frac{4\sqrt{C^*\mu\,(1-\rho p)\,(1+p)\,(1-\rho)+C^{*2}\,(\rho-p)^2}-\left[2\mu\,(1-\rho p)-4C^*\,(\rho-p)\right]}{2\sigma^2\,(1-\rho p)^2}\\
\beta &= \alpha\left(1+p^2-2\rho p\right)\sigma^2-(a-\mu)+p\,(b-\mu)
\end{aligned}$$

time for the random walk $\Sigma_{i=1}^{t}\varepsilon_i$ to hit the line $\alpha(p)+\beta(p)t$

$$\begin{cases} \alpha(p) = \frac{\mu'_{\tau}}{\sigma_a^2\sigma_b^2\iota(p)} \\ \beta(p) = \alpha(p)\sigma_a^2\sigma_b^2\left(\frac{1}{\sigma_b^2}+\frac{p^2}{\sigma_a^2}-\frac{2\rho p}{\sigma_a\sigma_b}\right)-(a-\mu_a)+p(b-\mu_b) \end{cases} \tag{3.4}$$

4. To calculate $E\delta^{\tau}$, we approximate the random walk $\Sigma_{i=1}^{t}\varepsilon_i$ with a standard Brownian motion $\{B(t)\}_{t\geq 0}$ .Given the stopping time for $\{B(t)\}_{t\geq 0}$ to hit the line $\alpha(p)+\beta(p)t$ is $T_{\alpha\beta}$ (please refer to (5.5) on Page 362 of [65]), for $\forall\theta>0$,

$$E\left[e^{-\theta T_{\alpha\beta}}\right] = e^{-\alpha\left[\beta+\sqrt{\beta^2+2\theta}\right]}.$$

By replacing $-\ln\delta=\theta, \tau\sim T_{\alpha\beta}$, we have

$$\begin{aligned} E\delta^{\tau} &= Ee^{\tau\ln\delta}\sim e^{-\widehat{\tau}(p)}, \\ \widehat{\tau}(p) &= \alpha\left[\beta+\sqrt{\beta^2-2\ln\delta}\right]. \end{aligned} \tag{3.5}$$

3.4.2 The IS's Optimal Pricing Problem

Given the PO's belief updating rule based on statistical learning and entry policy f_{C^*}, the IS is maximizing his expected NPV V^{IS} through the decision of price p^{IS}:

$$\begin{aligned} p^* &= \underset{p\in(0,\infty)}{\arg\max} V^{IS} \qquad (3.6) \\ V^{IS} &= \frac{\delta}{1-\delta}\left[(1-\gamma)p-c^{IS}\right](a-bp)^{+}(1-E\delta^{\tau}) \\ \frac{V^{IS}(1-\delta)}{\delta(1-\gamma)} &\simeq (p-c)(a-bp)\left[1-e^{-\widehat{\tau}(p)}\right], \\ c &:= \frac{c^{IS}}{1-\gamma}<\frac{a}{b} \end{aligned}$$

where $\{\alpha(p), \beta(p)\}$ are defined as in (3.4). WLOG[10], we can normalize $\mu_a = \mu_b := \mu$, and define $\widehat{V}^{IS} = \frac{V^{IS}(1-\delta)}{\delta}$ for dispositional simplicity.

V^{IS} is non-negative in the compact set $p \in \left[c, \frac{a}{b}\right]$, so the optimal solution(s) p^* to the IS's problem (3.6) do exist. Yet the IS's complex payoff function V^{IS} makes it impossible to get the close form solution of the optimal price p^*. So we resort to a combination of analytical and numerical studies to investigate the optimal solution(s) $\{p^*(\boldsymbol{\theta}^*, \boldsymbol{\theta}_0, C^*)\}'$s properties.

In particular, without the PO's entry threat, the IS's optimal price is $p^M := \frac{a+bc}{2b}$ that serves as a benchmark for p^*. We focus on the comparison between p^* and p^M or the IS's pricing distortion behavior, namely if the IS sets a higher or lower price to postpone the PO's entry, and identify the condition for each case. Moreover, based on uni-modality assumption of some special cases, we discover the IS's optimal product selections $\{a^*(b), b\}$ within the product space that helps to explain the seller's competitive behavior on the market place.

[10] When $\mu_a \neq \mu_b$, we define

$$\begin{aligned} \mu_b' &= \mu_a,\ b' = b \cdot \frac{\mu_a}{\mu_b}, \mu_{b,t}' = \mu_{b,t} \cdot \frac{\mu_a}{\mu_b}, \sigma_b' = \sigma_b \cdot \frac{\mu_a}{\mu_b} \\ p' &= p \cdot \frac{\mu_b}{\mu_a}, c' = c \cdot \frac{\mu_b}{\mu_a}, C^{*\prime} = C^* \cdot \frac{\mu_b}{\mu_a} \end{aligned}$$

then the PO's prior is

$$\begin{aligned} \{a, b'\} &= \left\{a, b \cdot \frac{\mu_a}{\mu_b}\right\} \sim N\left(\boldsymbol{\mu}_0', \boldsymbol{\Sigma}_0'\right), \\ \boldsymbol{\mu}_0' &= (\mu_a, \mu_b') \\ \boldsymbol{\Sigma}_0' &= \begin{pmatrix} \sigma_a^2 & \rho\sigma_a\sigma_b' \\ \rho\sigma_a\sigma_b' & \sigma_b'^2 \end{pmatrix}. \end{aligned}$$

and the belief updating rule follows.

The demand level and profit level are

$$\begin{aligned} \mu_{a,t} - \mu_{b,t}' p' &= \mu_{a,t} - \mu_{b,t} p, \\ (p' - c')\left(\mu_{a,t} - \mu_{b,t}' p'\right) &= (p - c)\left(\mu_{a,t} - \mu_{b,t} p\right) \cdot \frac{\mu_b}{\mu_a}, \end{aligned}$$

and the entry threshold is

$$\begin{aligned} \frac{\mu_{a,t}^2}{4\mu_{b,t}} \geq C^* &\iff \frac{\mu_{a,t}^2}{4\mu_{b',t}} \cdot \frac{\mu_b}{\mu_a} \geq C'^* \cdot \frac{\mu_b}{\mu_a} \\ &\iff \frac{\mu_{a,t}^2}{4\mu_{b',t}} \geq C'^*. \end{aligned}$$

Multi-Modality and Ineffective Learning

We argue that, despite the existence of p^*, $V^{IS}(p)$'s multi-modality does not allow for straightforward comparative statics of $p^*(\boldsymbol{\theta}^*, \boldsymbol{\theta}_0, C^*)$ based on FOC. We demonstrate the example of

$$\left\{ \begin{array}{c} \sigma_a = \sigma_b = 1, \mu = 3, a = 4, b = 2, \rho \in [0.6, 1) \\ c = 0, \delta = 0.99, C^* = \frac{5}{4} \end{array} \right\}$$

in Figure 3.2.

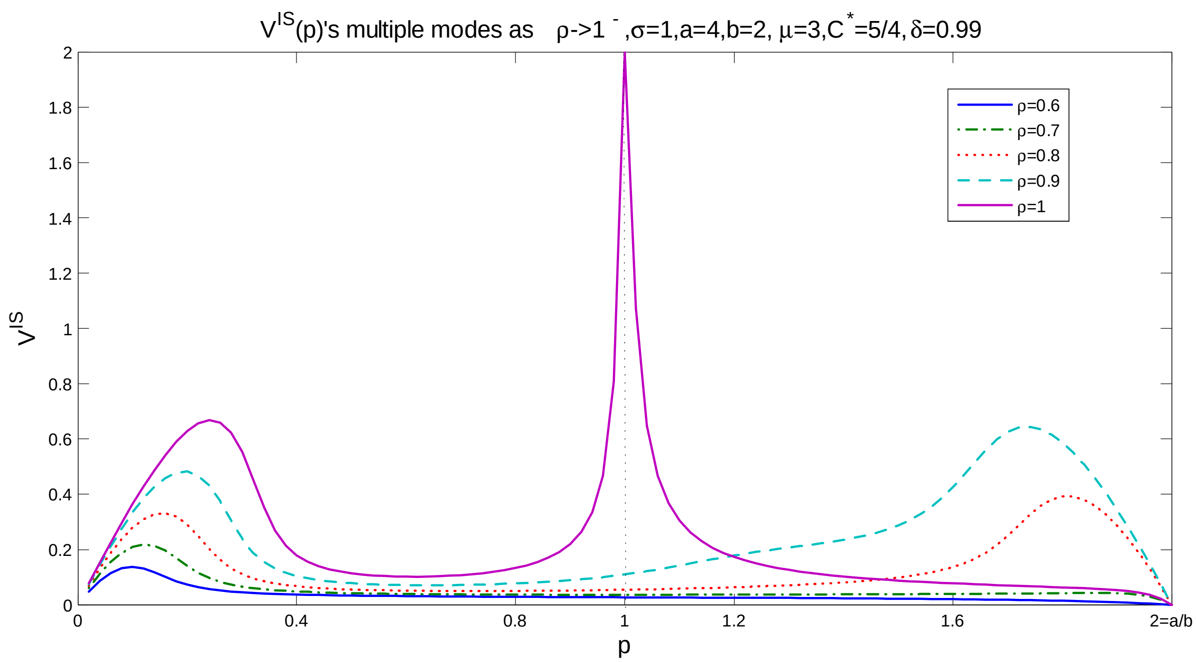


Figure 3.2: Multi-Modality in the IS's Obejective Functions

It shows that in this case $V^{IS}(p, \rho)$ has more than one modes when $\rho \geq 0.8$, $p^*(\rho)$ is discontinuous. In this example, $p^* \in (0, 0.4)$ when $\rho < 0.8$; $p^* \in (1.6, 2)$, when $\rho \in \{0.8, 0.9\}$; and $p^* = 1$,when $\rho \to 1^-$.

In general, when $\frac{\sigma_a}{\sigma_b} \in \left[c, \frac{a}{b}\right], \rho \to 1^-, p \to \frac{\sigma_a}{\sigma_b}$, the PO's belief evolution vector (and the bi-nomial prior belief) becomes degenerate

$$\Delta \boldsymbol{\mu}(p) = \begin{pmatrix} \frac{1}{\sigma_b^2} - \frac{\rho p}{\sigma_a \sigma_b} \\ \frac{\rho}{\sigma_a \sigma_b} - \frac{p}{\sigma_a^2} \end{pmatrix} \to 0,$$

in which case the PO's learning is ineffective, so $\mu_t \equiv \mu_0$, $\tau = \infty, V^{IS}(p) = p(a - bp)$, causing a spike in $V^{IS}(p)$. In the special case of $\sigma_a = \sigma_b$, this amounts to $a \geq b$, $p = 1$.

Remark 1 *Though the IS could effectively deter the PO's entry with this "ineffective learning price", it is not necessarily the IS's optimal price. We demonstrate this observation of in Figure 3.3 of the example of* $\left\{\sigma = 0.3, \mu = 3, C^* = \frac{5}{4}, a = 3.75, b = 2.6, \rho \in [0.6, 1), c = 0, \delta = 0.99\right\}$. *This set of* $V^{IS}(p)$ *curves show that though there is a (right) mode in* $V^{IS}(p)$ *value around "ineffective learning price", the optimum lies at the left mode.*

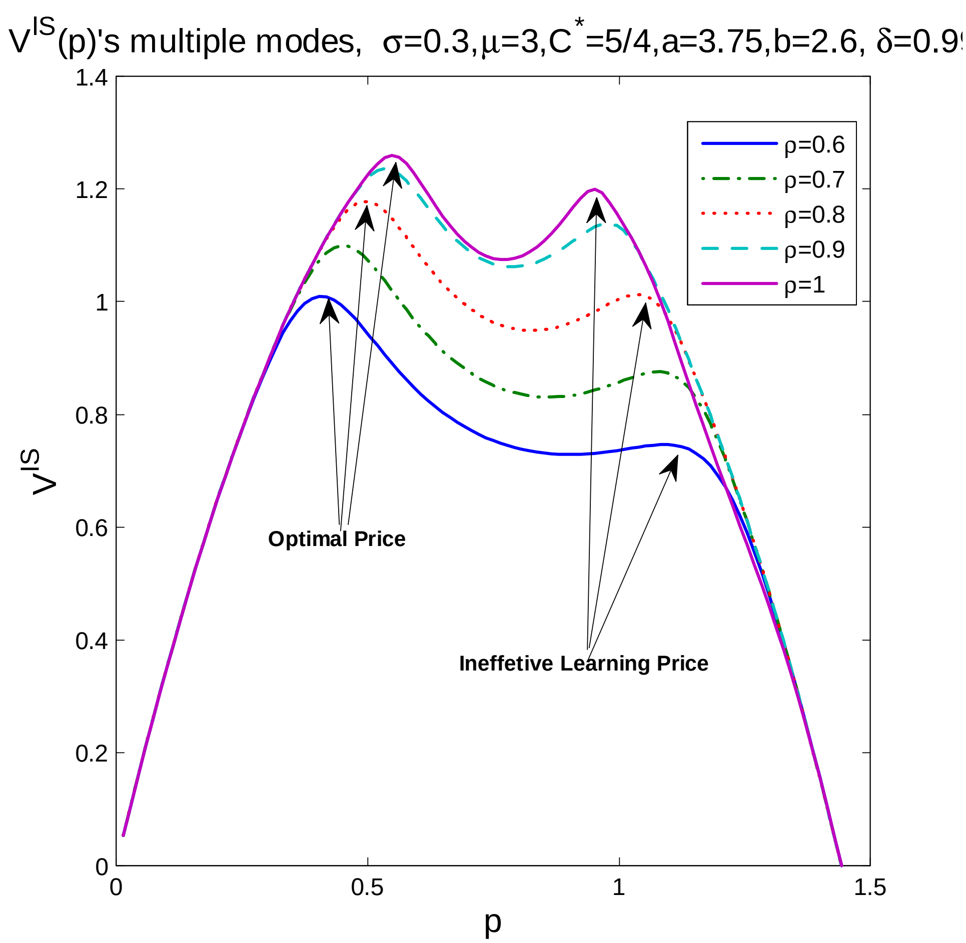


Figure 3.3: The IS's Optimal Price and the "Ineffetive Learning" Price

Lacking $V^{IS}(p)'s$ uni-modality property in general, we start with a special case of $\rho = 0$ for which we observe uni-modality from extensive numerical and analytical studies, and for simplicity we assume that $\sigma_a = \sigma_b := \sigma$[11]. We discuss how the observations from this special case apply to a general ρ in Section 3.7.

[11] This assumption is for simplicity and may cause the loss of generality.

3.4.3 The IS's Optimal Price Policies with $\rho = 0$

In this special case of $\rho = 0$, we obtain analytical results regarding p^*'s properties and complement them with numerical results. In particular, we examine the comparison between the optimal price and the myopic price, and the optimal price's implication on the PO's profits.

Analytical Results on Trends of Entry time and the Optimal Price Comparison

A focal question around the optimal price p^* is how it is compared to p^M, the optimal price without entry threat. The metric $\frac{p^*}{p^M} - 1$ characterizes the IS's price distortion behavior subject to the PO's entry threat, and relates to how he practically postpones the PO's entrance by leveraging the price.

We argue that the entry time τ's monotonicity w.r.t. p is sufficient to determine $\left(\frac{p^*}{p^M} - 1\right)' s$ sign, and find sufficient conditions on $\{a, b\}$ for τ's monotonicity. We show that the products with price sensitivity b low (high) enough is sold at price p^* lower (higher) than p^M by the IS, a result consistent with statistical learning theory.

Lemma 7 *$p^* \geq (\leq)\ p^M = \frac{a+bc}{2b}$ if $\widehat{\tau}(p)$ increases (decreases) in $p \in \left[c, \frac{a}{b}\right]$.*

Please refer to 2.2 for the proof of Lemma 7. The result of Lemma 7 directs us to search for conditions supporting $\tau' s$ monotonicity. Note that

$$\begin{aligned} \widehat{\tau} \quad &: \quad = \alpha \left[\beta + \sqrt{\beta^2 - 2\ln\delta}\right] \\ \frac{\widehat{\tau}'}{\widehat{\tau}} \quad &= \quad \frac{\alpha'}{\alpha} + \frac{\beta'}{\sqrt{\beta^2 - 2\ln\delta}}, \end{aligned}$$

so the sign of $\widehat{\tau}'$ could be determined by the signs of $\frac{\alpha'}{\alpha}$ and $\frac{\beta'}{\sqrt{\beta^2 - 2\ln\delta}}$. When $\rho = 0$, we know that

$$\frac{\alpha'}{\alpha} = \frac{-C^*}{\sqrt{C^*\mu(1+p) + C^{*2}p^2}} < 0,$$

so what is left is to determine the sign of $\frac{\beta'}{\sqrt{\beta^2 - 2\ln\delta}}$. The following result helps to characterize the sign of β' and consequently $\widehat{\tau}' s$ monotonicity.

Lemma 8 *$\beta(p)$ is convex w.r.t. $p \in [0,\infty)$ when $\rho = 0$.*

Please refer to 2.2 for the proof of Lemma 8. With Lemma 8's result, it suffices to show $\beta' < 0$ and $\widehat{\tau}' < 0$ for $\forall p \in \left[c, \frac{a}{b}\right]$, if $\beta'\left(\frac{a}{b}\right) < 0$; similarly, $\beta' > 0$ for $\forall p \in \left[0, \frac{a}{b}\right]$, if $\beta'(0) < 0$, which helps us to further determine the sufficient conditions for $\widehat{\tau}' < 0$. We present the sufficient conditions for τ to be increasing or decreasing, and consequently p^* being higher or lower than p^M based on these two lemma.

Proposition 10 *1. If $b \geq \bar{b}(a) := \mu + \left(2C^* - \sqrt{C^*\mu} + \sqrt{\frac{C^*}{\mu}\left[\left(2\sqrt{C^*\mu} - a\right)^2 - 2\ln\delta\right]}\right)$, then $p^* \geq p^M$.*

2. If $b \leq \underline{b} := \frac{\mu^2}{4C^*}$, then $p^* \leq p^M$.

Please refer to 2.2 for the proof of Proposition 10. The rough idea is that for the IS to postpone the PO's entry, he needs to avoid the occurrence of exceedingly high sale observations, by minimizing

$$
\begin{aligned}
\Delta D_t &= E\left[D_t|a,b\right] - E\left[D_t|\mu_a,\mu_b\right] \\
&= (a-\mu) - (b-\mu)\,p.
\end{aligned}
$$

Therefore if $b-\mu$ is positive and very large, he will set a higher p; and vice versa. In this case, when the actual market demand is highly price-sensitive, the IS would distort the price upward compared to the situation without the PO's entry threat, as a "reverse" attempt.

We established sufficient conditions for $\frac{p^*}{p^M}$ to be higher or lower than 1 by defining two thresholds of b, $\left\{\underline{b}, \bar{b}(a)\right\}$. An example of $\left\{\underline{b}, \bar{b}(a)\right\}$ for

$$\left\{\mu = 3, C^* = \frac{5}{4}, \sigma = 0.5, \delta = 0.99, c = 0\right\}$$

is shown in Figure 3.4.

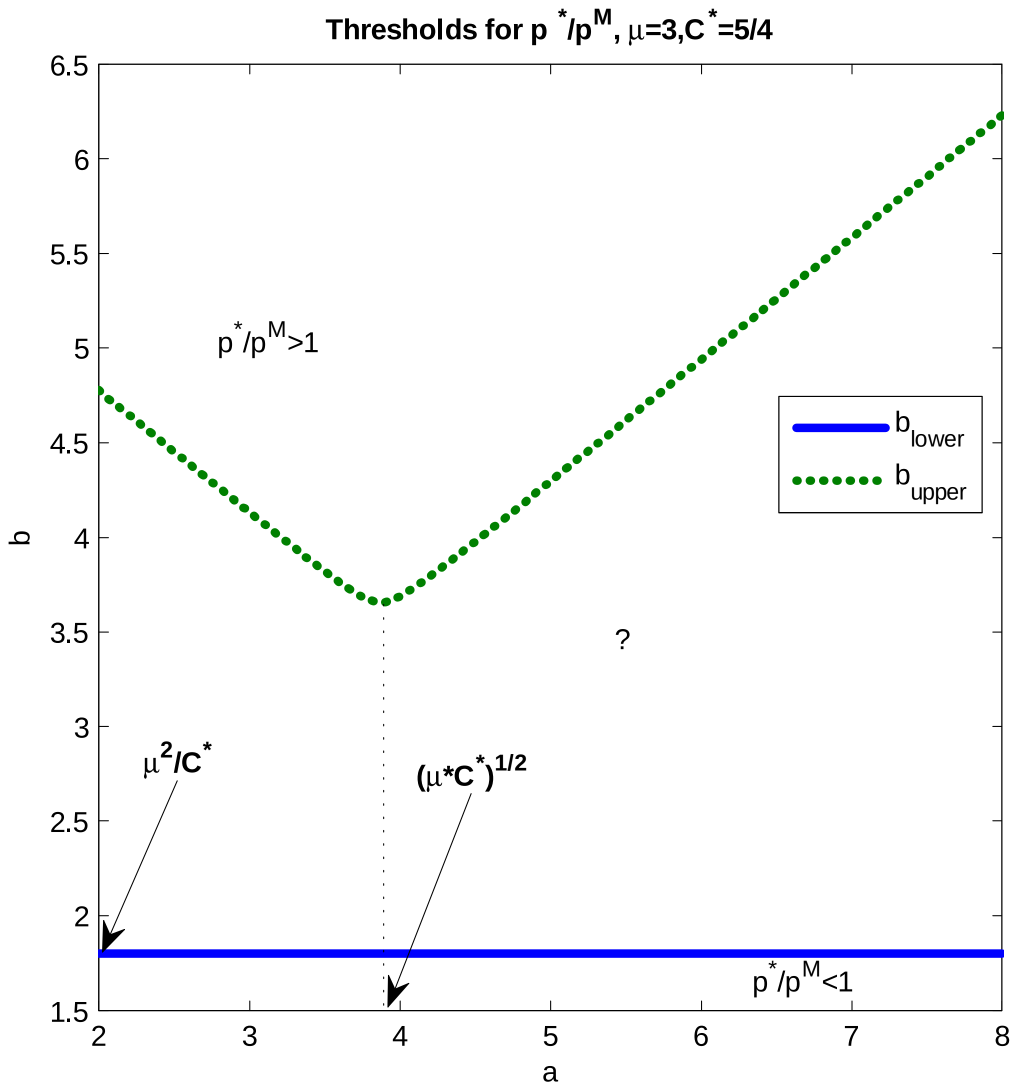


Figure 3.4: Two Thresholds of b for Increasing and Decreasing Entry Time

The following questions remain unanswered regarding $\frac{p^{IS}}{p^M}{}'s$ value, and to address them we need to resort to numerical studies:

1. In the gap between $\left\{\underline{b}, \bar{b}\,(a)\right\}$, what is the sign of $\frac{p^*}{p^M} - 1$?

2. Within (and between) the two regions bounded by $\left\{\underline{b}, \bar{b}\,(a)\right\}$, how does $\frac{p^*}{p^M}$ change with $\{a, b\}$?

Preliminary numerical results show that the IS's objective function is unimodal with $\rho = 0$, so we could rely on numerical solution of the unique p^* in the analysis hereafter.

Numerical Study for $\frac{p^*}{p^M}$

A Boundary for the Sign of $\frac{p^*}{p^M} - 1$ in $\{a, b\}$ Space We numerically examine the sign of $\frac{p^*}{p^M} - 1$ in the area between $\left\{\underline{b}, \bar{b}\,(a)\right\}$, and find in a broad range of parameter settings:

$$\begin{aligned} \exists \widehat{b}\,(a) &\in \left(\underline{b}, \bar{b}\,(a)\right), s.t. \\ \forall b &> \widehat{b}\,(a), \frac{p^*}{p^M} - 1 > 0; \forall b < \widehat{b}\,(a), \frac{p^*}{p^M} - 1 < 0. \end{aligned}$$

We demonstrate $\widehat{b}\,(a)$ (labeled as b_{num} in the graphics) for the same example for Figure 3.4 in Figure 3.5.

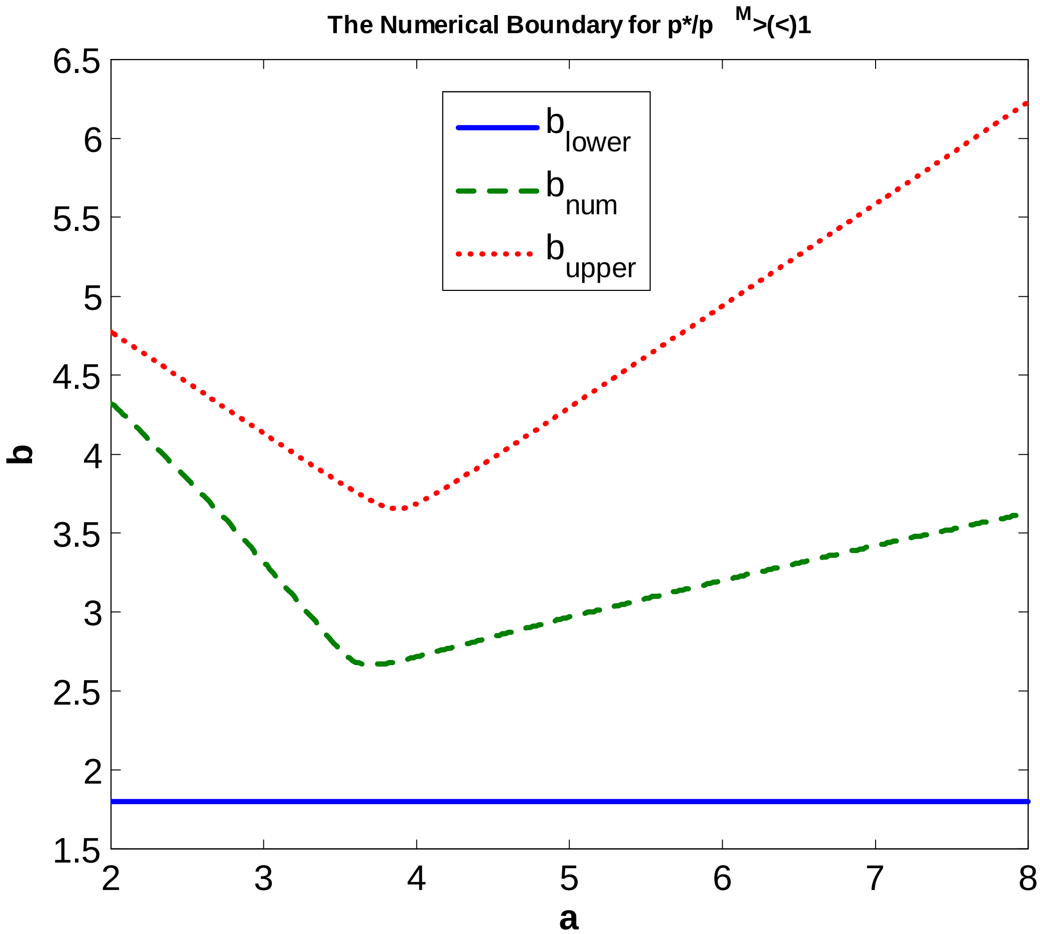


Figure 3.5: The numerical threshold of b for the price distortion's direction

Numerical comparative statics of $\frac{p^*}{p^M}$ Besides the sign of $\frac{p^*}{p^M} - 1$, we examine how $\frac{p^*}{p^M}$'s value change with $\{a, b\}$, to represent how the price distortion level being influenced by product characteristics.

Fixing b and changing a By fixing b and changing a, we observe two product types of low or high price sensitivity presenting the opposite trends of $\frac{p^*}{p^M}-1$[12], which could be unified under the same framework by measuring $|\frac{p^*}{p^M}-1|$. In particular, when σ is large enough[13] and $b<(>)\,\widehat{b}\,(a)$, $\exists a_1\,(b,\sigma)$, s.t. $\frac{p^*}{p^M}$ minimizes (maximizes) at $a_1\,(b,\sigma)$; $\frac{p^*}{p^M}$ decreases (increases) when $a<a_1\,(b,\sigma)$, and increases (decreases) with $a>a_1\,(b,\sigma)$[14]. Therefore the price distortion level $|\frac{p^*}{p^M}-1|$ is maximized at $a_1\,(b,\sigma)$ for either type. We demonstrate the cases of $\{b\in\{1.5,2,2.5\}\,,\sigma=0.5\}$ and $\{b=\{4.5,5,5.5\}\,,\sigma=0.5\}$ in Figure 3.6. Intuitively, a product with low market size, i.e. $a<a_1\,(b,\sigma)$, is not worth the PO's entry but grows more attractive as a increases, therefore IS needs to increase the price distortion level to prevent the PO's potential entry by accident. When the market size is large enough, the IS's price distortion becomes ineffective given the PO's fast learning, so the IS reduces the effort of pricing distortion.

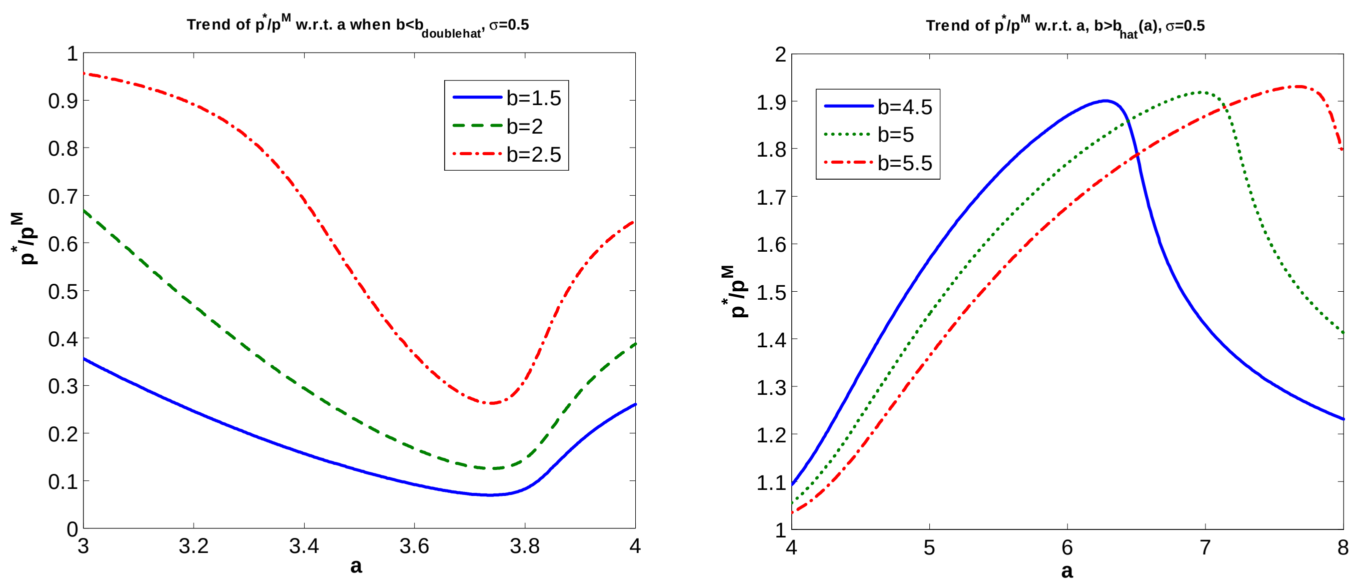


Figure 3.6: The IS's price distortion behaviors in the two regions

[12] One could combine these studies to come up with results about parameter changes across $\widehat{b}\,(a)$.

[13] when σ is very small, we observe $\frac{p^*}{p^M}$ decrease with a, therefore $a_1\,(b)$ may not exist

[14] $a_1\,(b,\sigma)$ remains at a relatively stable level while slightly decreasing in the case of $b<\widehat{b}\,(a)$.

Table 3.1: a1's relationship with b and the coefficient of variantion

Lower b						
$\sigma = 0.5$	b	1.5	1.8	2	2.2	2.5
	$a_1(b,\sigma)$	3.74	3.74	3.74	3.74	3.74
$b = 2$	σ	0.1	0.2	0.5	2	5
	$a_1(b,\sigma)$	3.855	3.69	3.74	3.74	3.74
Higher b						
$\sigma = 0.5$	b	4.5	4.8	5	5.2	5.5
	$a_1(b,\sigma)$	6.28	6.692	6.97	7.248	7.67
$b = 5$	σ	0.1	0.2	0.5	2	5
	$a_1(b,\sigma)$	6.55	6.90	6.97	6.97	6.97

We also present a summary of $a_1(b,\sigma)' s$ trends w.r.t. $\{b,\sigma\}$ in Table 3.1, and notice that when $b < \widehat{b}(a)$, $a_1(b,\sigma)$ stays relatively stable w.r.t. b; decreases and the then increases w.r.t. σ[15]; when $b > \widehat{b}(a)$, $a_1(b,\sigma)$ increases w.r.t. b, and increases w.r.t. σ.

Fixing a and changing b When fixing a and changing b, we observe that $\frac{p^*}{p^M}$ increases then decreases with b, converging to 0 and 1 with very low and high b respectively, and maximizing at $\widetilde{b}(a)$ (Figure 3.7). Intuitively, products of very low price sensitivity is very attractive for the PO's entry so the IS needs to particularly lower the price to prevent so; while products of very high price sensitivity is not attractive so the IS does not need to deviate from p^M. Based on an extensive set of numerical studies, we observe that the price distortion $\left|\frac{p^*}{p^M} - 1\right|$ is maximized at $b = 0$ as

$$\max \frac{p^*}{p^M} = \frac{p^*}{p^M}|_{b=\widetilde{b}(a)} < 2.$$
$$\min \frac{p^*}{p^M} = \frac{p^*}{p^M}|_{b\to 0^+} = 0$$

[15] minimizes at $\sigma = 0.2$

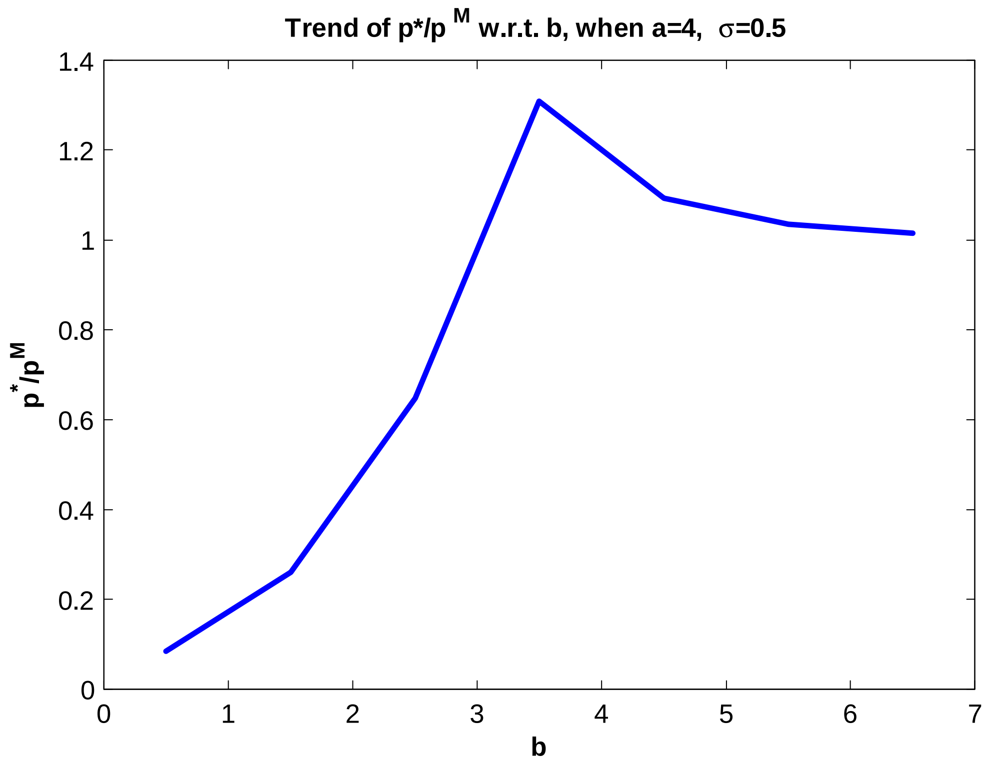


Figure 3.7: The IS's Price Distortion w.r.t. b with Fixed a

3.4.4 The IS's Product Selection and Payoffs

Based on the IS's optimal pricing decision, we could derive the IS's payoff $V^{IS*}(a,b) = V^{IS}(p^*, a, b)$. By comparing $V^{IS*}(a,b)$ across different demand characteristics $\{a,b\}$ with other parameters fixed, we identify the optimal parameter settings for the IS's payoff, as well as the comparison of its performance to other settings. In particular, we discover that the optimal settings $\{a^*(b), b\}$ locate along a line segment. Moreover, we characterize how the IS's payoff diminishes as $\{a,b\}$ moves away from the optimal product selection on the line segment.

Optimal Product Selections

Based on an uni-modality assumption[16] in the case of $\rho = 0$, we establish the following results regarding how the IS's optimal performance depend on the product true demand parameters.

Proposition 11 *When $V^{IS}(p|\rho = 0)$ is unimodal, $\exists \{p^{**}(\theta_0, \boldsymbol{\psi}), q^{**}(\theta_0, \boldsymbol{\psi})\}$, s.t. $\forall \{a^*, b^*\} > 0$ with the relationship*

$$a^* = q^{**} + b^* p^{**}$$

have the following properties:

1. $p^*(a^*, b^*) = p^{**}$;

2. $V^{IS*}(a^*, b^*) := V^{IS}(a^*, b^*, p^*(a^*, b^*)) = V^{IS**}$

3. $\forall \{a, b\} > 0$ *and* $a > bc, V^{IS*}(a, b) \leq V^{IS**}$

For example, when $\left\{\mu = 3, C^* = 5, \sigma = 0.5, c = \frac{c^{IS}}{1-\gamma} = 0.3\right\}$, based on our computation

$$\begin{aligned} p^{**} &= 0.7562 \\ q^{**} &= 1.2465 \\ V^{IS**} &= 0.4826, \end{aligned}$$

so

$$a^*(b) = 1.2465 + 0.7562b.$$

From the left panel of Figure 3.8, we notice that $a^*(b)$ presents a linear relationship w.r.t. $b, i.e.$ grows with b. And the right panel shows that the $\{b, a^*(b)\}$ line crosses the boundary marked by $\widehat{b}(a)$, thus $|\frac{p^*}{p^M} - 1|$ could take both signs along the optimal product selection $\{b, a^*(b)\}$.

[16] We verified this result with extensive numerical tests but it is exceedingly difficult to show in general, due to the complex form of the objective function involving exponential and root functions. We prove the special case of $C^* = \frac{\mu}{4}(1 + \varepsilon)$ with deterministic approximation of τ in 2.2.

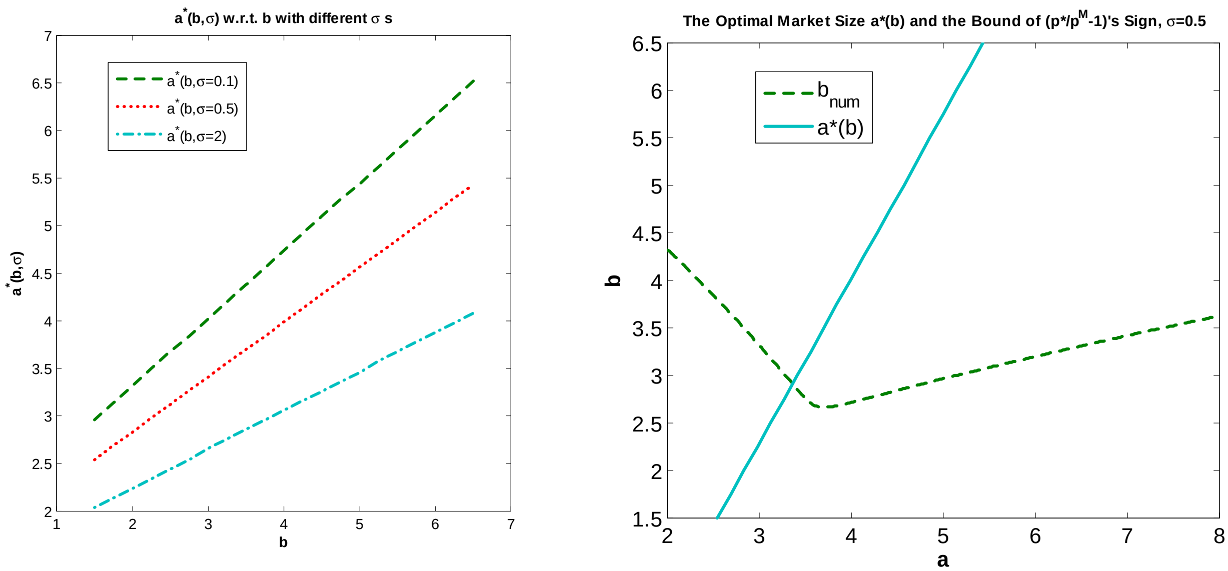


Figure 3.8: The Best Product Selections

Extreme Product Selections

Given the optimal product selection $\{a^*(b), b\}$ along the line

$$a = q^* + bp^*,$$

we observe that the IS's payoff deteriorates as $\{a, b\}$ moves away from the optimal product selections. In particular, we define

$$\Theta_\xi = \left\{\{a, b\} | V^{IS*}(a, b) := V^{IS}(a, b, p^*(a, b)) < \xi V^{IS**}, \xi \in (0, 1)\right\}$$

as the set of product characteristics that induce a lower payoff for the IS than a proportion, i.e.ξ, of the highest optimal payoff V^{IS**}. Θ_ξ include product characteristics at the upper-left and lower-right corners of the (a, b)-space and expands as ξ increases. We demonstrate $\Theta_{0.9}$ and $\Theta_{0.5}$ for the example of$\left\{\mu=3, C^{*}=\frac{5}{4}, \sigma=0.5, c=\frac{c^{IS}}{1-\gamma}=0.3, a\in[2,8], b\in[1.5,6.5]\right\}$ in Figure :

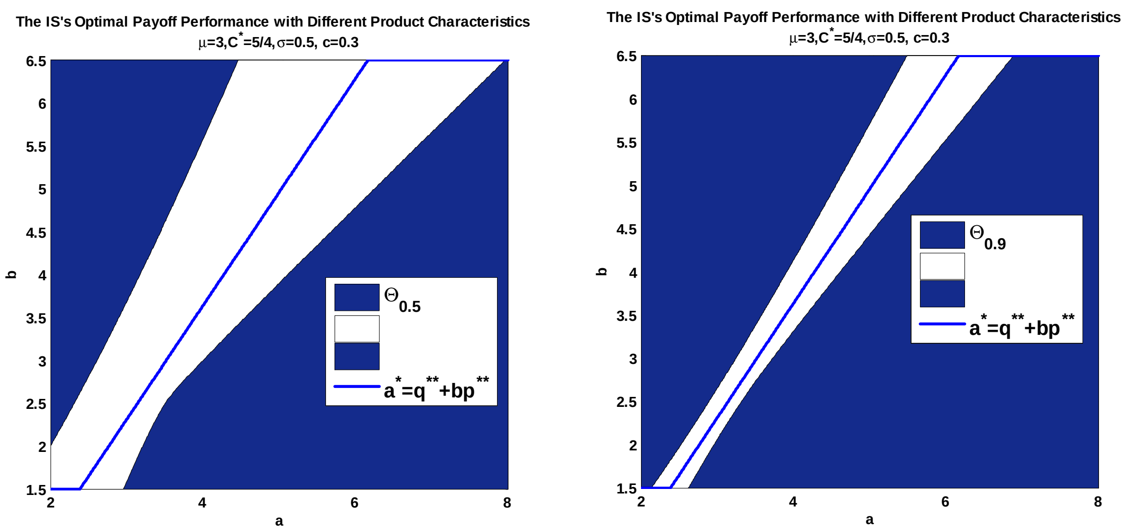


Figure 3.9: The IS's Product Selections Satisfying a Minimum Profit Level

The following Proposition 12 characterizes the IS's payoffs at these extreme cases of $\{a, b\}$, and shows the differences between the two extremes.

Proposition 12 *1. For* $\forall b>0$, $\lim_{a\to bc^{+}} V^{IS*}(a,b)=0$, $\lim_{a\to+\infty} V^{IS*}(a,b)=\frac{-\ln\delta\mu(4C^{*}-\mu)}{\left(4+\frac{\mu^{2}}{bC^{*}}\right)\sigma^{2}C^{*}}>0$

2. For $\forall a>0$, $\lim_{b\to\left(\frac{a}{c}\right)^{-}} V^{IS*}(a,b)=0$, $\lim_{b\to0^{+}} V^{IS*}(a,b)=0$.

Please refer to 2.2 for the proof of Proposition 12. Though the IS's payoff does not fully diminish as a grows very large, its limit is still very small compared to V^{IS**}. For example, with $\left\{\mu=3, C^{*}=\frac{5}{4}, \sigma=0.5, c=\frac{c^{IS}}{1-\gamma}=0.3\right\}$,

$$\frac{\frac{-\ln\delta\mu(4C^{*}-\mu)}{\left(4+\frac{\mu^{2}}{bC^{*}}\right)\sigma^{2}C^{*}}}{V^{IS**}}\le\frac{\frac{-\ln\delta\mu(4C^{*}-\mu)}{4\sigma^{2}C^{*}}}{V^{IS**}}<0.1.$$

3.5 The PO's Entry Threshold and Performances

In this section we study the PO's entry threshold choice based on the IS's best response of price. We assume that the PO is unable to fully anticipate the IS's pricing decision in response to the entry threshold she set, due to the lack of complete information. To understand how the PO's entry threshold choice affects her performance, we take the researcher's persective by first assuming the PO has some preliminary information allowing her to evaluate different entry thresold based on her prior belief. We then characterize how this addtional information affects her optimal entry threshold choice and ensued performance.

The PO needs to take account of the IS's pricing decision to compare the outcome from different entry threshold. And in this current model, the extra uncertainty preventing the PO from rationally anticipating the IS's pricing response based on her prior belief is the IS's private information of her variable cost c^{IS}. We examine two approaches by assuming the PO has preliminary information on the IS's price or has preliminary information on c^{IS} respectively, and name them as "cost-based" and "price-based" approach respectively.

Regarding the PO's entry policy, the first and foremost question is if retaining the entry option is beneficial, corresponding to the case of $C^* = +\infty$. Also one would be interested in finding the "optimal" entry threshold and its sensitivity to certain parameters such as the PO's prior belief. In both approaches, we conduct a comparison between the PO's performance under finite C^*s and $C^* = +\infty$ and identify conditions for the PO to be better off by forgoing the entry decision, and examine how the PO's performance change with her choice of C^*. We also identify the PO's loss due to the IS's pricing distortion behavior as well as the PO's loss due to demand information asymmetry.

3.5.1 Cost-Based Approach

The PO's Expected Performance based on Distributional Aggregation

We first specify the PO's expected payoff function from the researcher perspective by explicating the IS's private information $\boldsymbol{\theta}^* = \{a, b\}$ and c^{IS}

$$\widehat{V}^{PO}\left(C^* | p^*\left(\boldsymbol{\theta}^*, C^*\right), \boldsymbol{\theta}^*, \boldsymbol{\psi}\right) = \gamma p^*\left(a - bp^*\right)^+ + e^{-\widehat{\tau}}\left\{p^{PO}\left(a - bp^{PO}\right) - \gamma p^*\left(a - bp^*\right)^+ - C_0\right\},$$

where

$$C_0 := C \cdot \frac{1-\delta}{\delta}, \widehat{V}^{PO} := V^{PO} \cdot \frac{\delta}{1-\delta},$$

and

$$p^{PO} = \frac{\mu_{a,\tau}}{2\mu_{b,\tau}}$$

is the PO's optimal pricing decision based on her posterior upon her entry decision, and p^* is the IS's optimal pricing decision in response to an entry threshold choice C^*.

Based on the analysis so far, we could compute the IS's optimal price $p^*\left(\boldsymbol{\theta}^*, C^*\right)$ and the ensuing PO's payoff $\widehat{V}^{PO}\left(C^* | p^*, \boldsymbol{\theta}^*, \boldsymbol{\psi}\right)$ for a fixed C^* and any realization of $\boldsymbol{\theta}^* = \{a, b\}$ and c^{IS}.To evaluate the PO's payoff under different entry threshold on an aggregate level, we take the expectation of $\widehat{V}^{PO}\left(C^* | p^*, \boldsymbol{\theta}^*, \boldsymbol{\psi}\right)$ over the prior distribution $\boldsymbol{\theta}^* \sim N\left(\boldsymbol{\theta}_0\right)$ for given c^{IS}:

$$E\left[\widehat{V}^{PO}\left(C^* | p^*\left(\boldsymbol{\theta}^*, C^*\right), \boldsymbol{\theta}^*, \boldsymbol{\psi}\right) | N\left(\boldsymbol{\theta}_0\right)\right] := \widehat{V}^{PO}\left(C^* | p^*, N\left(\boldsymbol{\theta}_0\right), \boldsymbol{\psi}\right),$$

and compare them across different values of C^*, in particular to the case of $C^* = +\infty$,i.e. the PO never entering.

Besides c^{IS}, we also change and examine the effect of C_0, and the prior information $\{\mu, \sigma\}$ while keeping the correlation coefficient ρ fixed[17]. In Figure 3.10, we demonstrate how $\widehat{V}^{PO}\left(C^* | p^*, N\left(\boldsymbol{\theta}_0\right), \boldsymbol{\psi}\right)$ change with C^* for a selection of $\left\{c^{IS}, C_0, \mu, \sigma\right\}$ values, and for comparison we display the value of

[17] ρ affects the overall profitability to a very large extent as well, yet we leave the discussion to 3.7.

$\widehat{V}^{PO}\left(+\infty|p^*, N\left(\boldsymbol{\theta}_0\right), \boldsymbol{\psi}\right)$ towards the end of each curve.

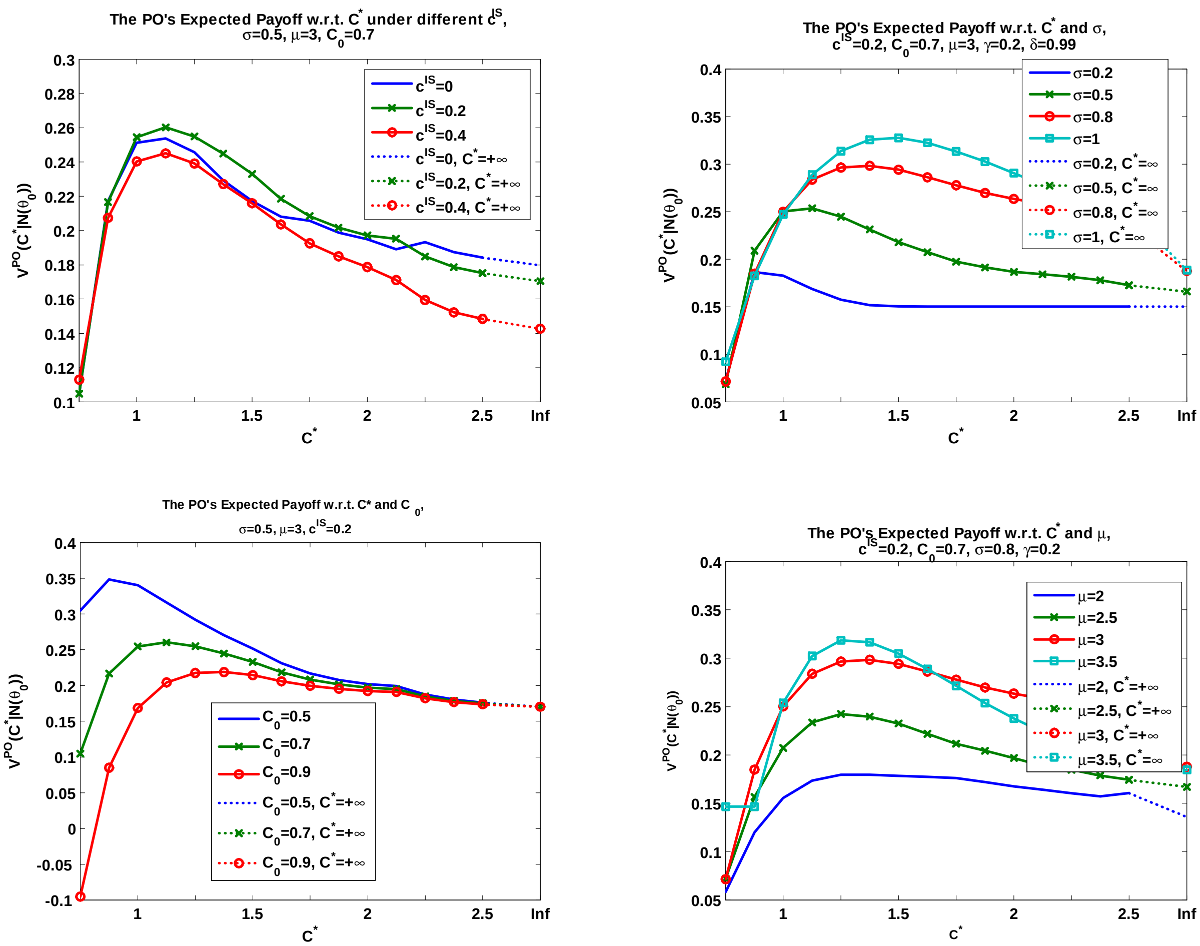


Figure 3.10: The PO's Performances by Aggregating over the Prior Distribution

Figure 3.10 demonstrate the effect of different factors on the PO's expected payoff under different entry threshold C^*. In general, the prior distribution variance σ plays a important role on the overall attractiveness of the market and the entry cost C_0 determines the entry profitability, while μ[18] and

[18] Under the assumption of $\mu_a = \mu_b := \mu$, the magnitude of μ represents the revenue size according to the PO's prior,

the IS's unit variable cost c^{IS} play relatively lesser roles. In particular, we observe that with other factors fixed, the PO is better off by forgoing the entry option only when σ is very low and C^* is very low. In this case, the PO is very confident with her prior information that does not support up-front or early entry as $\frac{\mu}{4} < C^*$.

The unknown c^{IS}'s relatively low effect on the PO's entry threshold choice shows that the PO could effectively choose a well-performing entry threshold based on the prior distribution information and other profit related parameters, as long as she knows the range of the PO's cost parameter. However, as we observe a large variance across the IS's optimal prices p^* for different c^{IS} with fixed $\{a, b\}$, p^* could not effectively serve as a signal for the PO to infer $\{a, b\}$.

The PO's Loss from Incomplete Information and Price Distortion

Based on the PO's payoff with the IS's optimal pricing decison, we could also examine the effect of the IS's pricing distortion behavior on the PO's performance and entry threshold choice. As a benchmark, we calculate the PO's payoff by replacing the IS's optimal pricing decision p^* with his optimal price without the entry threat p^M :

$$\widehat{V}^{PO}\left(C^*|p^M, N(\boldsymbol{\theta}_0), \boldsymbol{\psi}\right) := E\left[\widehat{V}^{PO}\left(C^*|p^M(\boldsymbol{\theta}^*), \boldsymbol{\theta}^*, \boldsymbol{\psi}\right)|N(\boldsymbol{\theta}_0)\right].$$

Another benchmark is the PO's average payoff when she is informed of all the $\{a, b\}$ realizations under the prior distribution, so she could choose between entering up-front or never entering for each $\{a, b\}$ realization, known as the full information benchmark:

$$\widehat{V}^{PO}(\{0, +\infty\}|N(\boldsymbol{\theta}_0), \boldsymbol{\psi}) = E\left[\max\left\{\widehat{V}^{PO}(0|\boldsymbol{\theta}^*, \boldsymbol{\psi}), \widehat{V}^{PO}(\infty|\boldsymbol{\theta}^*, \boldsymbol{\psi})\right\}|N(\boldsymbol{\theta}_0)\right].$$

We compare the PO's performance against these two benchmarks and demonstrate an example of $\left\{\sigma = 0.8, c^{IS} = 0\right\}$ in Figure 3.11 and more results for different σs in Table 3.2. These results show

though μ_a and μ_b evolve separately in the belief updating process. Generally, μ could be replaced by $\frac{\mu_a^2}{\mu_b}$ for a similar purpose.

that the IS's pricing distortion behavior leads to a significant loss (up to 9.92%) to the PO, comparable to her total loss due to demand information asymmetry (up to 13.91%). If the PO is to decide her entry threshold based on her prior information, she would choose a larger entry threshold (by about 2%) if not accounting for the IS's pricing behavior.

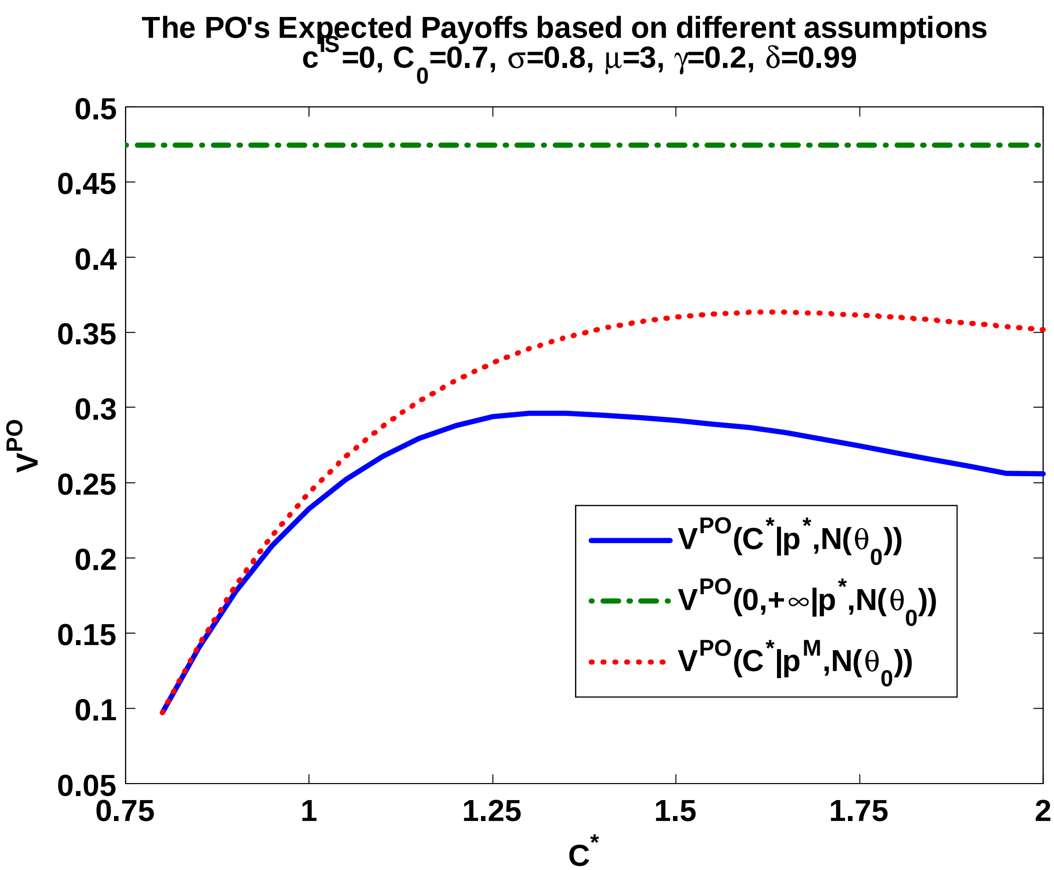


Figure 3.11: The PO's Loss due to the IS's Price Distortion and Information Asymmetry

Table 3.2: The PO's Performances with Different Variance Parameters and Assumptions

σ	0.2	0.5	0.8	1
C^{**}	0.875	1	1.0625	1.25
C^{M*}	0.875	1.0625	1.1875	1.3125
$\widehat{V}^{PO}\left(C^{**}\|p^*\right)$	0.2161	0.2567	0.3550	0.3618
$\widehat{V}^{PO}\left(C^{M*}\|p^M\right)$	0.2247	0.2750	0.3941	0.3856
$\widehat{V}^{PO}\left(0,\infty\|p^*\right)$	0.2338	0.2866	0.4123	0.4095
$1-\frac{\widehat{V}^{PO}(C^{**}\|p^*)}{\widehat{V}^{PO}(C^{M*}\|p^M)}$	3.85%	6.64%	9.92%	6.18%
$1-\frac{\widehat{V}^{PO}(C^{**}\|p^*)}{\widehat{V}^{PO}(0,\infty\|p^*)}$	7.57%	10.44%	13.91%	11.67%

Conditions for the PO to Forego the Entry Option

In the extreme case of $C^* = +\infty$,

$$\begin{aligned} \widehat{\tau} &= +\infty, p^* = p^M = \frac{a}{2b} + \frac{c^{IS}}{2\left(1-\gamma\right)} \\ \widehat{V}^{PO}\left(+\infty|p^* = p^M, \boldsymbol{\theta}^*, \boldsymbol{\psi}\right) &= \frac{\gamma}{4b}\left[a^2 - \left(\frac{bc^{IS}}{1-\gamma}\right)^2\right], \end{aligned}$$

so the PO is better off by foregoing the entry option when

$$\widehat{V}^{PO}\left(C^*|p^*\left(\boldsymbol{\theta}^*, C^*\right), \boldsymbol{\theta}^*, \boldsymbol{\psi}\right) < \widehat{V}^{PO}\left(+\infty|p^* = p^M, \boldsymbol{\theta}^*, \boldsymbol{\psi}\right).$$

In particular, when $C^* \leq \frac{\mu_{a,0}^2}{4\mu_{b,0}}$, i.e. the PO enters up-front, this amounts to

$$\begin{aligned} \widehat{V}^{PO}\left(+\infty|p^* = p^M, \boldsymbol{\theta}^*, \boldsymbol{\psi}\right) &> \widehat{V}^{PO}\left(0|p^*, \boldsymbol{\theta}^*, \boldsymbol{\psi}\right) = \frac{a^2}{4b} - C_0 \\ C_0 &> \frac{\left(1-\gamma\right)a^2 + \left(\frac{bc^{IS}}{1-\gamma}\right)^2\gamma}{4b} \end{aligned}$$

This shows that compared to entering up-front, a lower market size a, the IS's lower unit cost c^{IS}, as well as a medium level of b drives the PO to opt for never entering. For a general entry threshold C^*, conditions for the PO to forego the entry option depend more specifically on the IS's pricing decision, so we resort to an extensive set of numerical tests over the following sample grid to examine these conditions

$$\begin{aligned} a &\in [2,8], b \in [1.5, 6.5], \\ c^{IS} &\in \{0, 0.1, ..., 0.4\}, C^* \in \{0.75, 1, 1.25, ..., 2\}. \end{aligned}$$

In particular, for each $\left\{C^*, c^{IS}\right\}$ combination, we find the division of the $\{a,b\}$ −space as $\left\{\boldsymbol{\theta}^*|\widehat{V}^{PO}\left(C^*|p^*\left(\boldsymbol{\theta}^*, C^*\right), \boldsymbol{\theta}^*, \boldsymbol{\psi}\right) <\right.$ and demonstrate the boundaries between the two parts of a selection of $\left\{C^*, c^{IS}\right\}$ values in Figure 3.12.

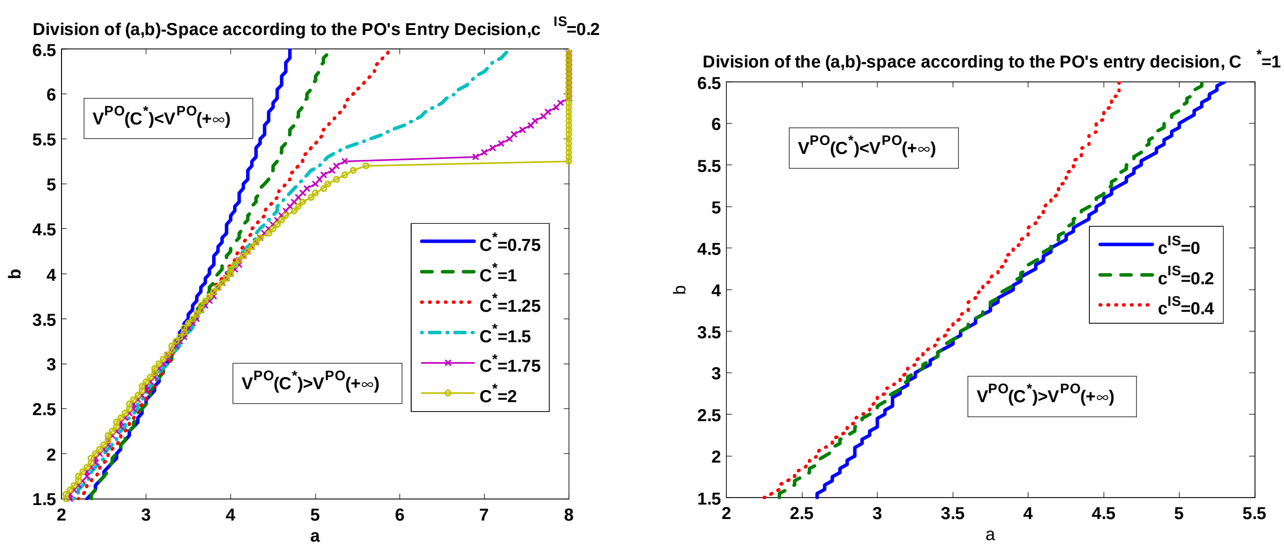


Figure 3.12: Product Characteristics for the PO's to Forgo the Entry Option

In general, the PO opts out of entering with lower a and larger b, and as the entry threshold C^*

increases, the region of $\{a, b\}$ that supports opting for never entering expands. However, the impact of the IS's variable cost c^{IS} varies across different parameter settings. In general, a higher c^{IS} makes the entry option more attractive for the PO, so the $\{a, b\}$ −region that supports "never entering" shrinks.

3.5.2 Price-Based Approach

From the PO's perspective, accessing the IS's cost information allows her to form a rational expectation of the IS's pricing decision. However if she have access to certain information on the IS's price instead of his cost parameter, she could also evaluate the outcome of her entry threshold decision. This becomes possible when the PO has accumulated price observation over a long period of time or offerings of the same/similar product.

Here we assume that the PO has access to the IS's pricing decision p^{IS}, based on which and her prior information, she could anticipate her entry time and consequently her expected payoff:

$$\widehat{V}^{PO}\left(C^*|p^{IS}, \boldsymbol{\theta}^*, \boldsymbol{\psi}\right) = \gamma p^*\left(a - bp^*\right)^+ + e^{-\widehat{\tau}}\left\{p^{PO}\left(a - bp^{PO}\right) - \gamma p^{IS}\left(a - bp^{IS}\right)^+ - C_0\right\},$$

and similarly we could take its expectation over the PO's prior belief:

$$E\left[\widehat{V}^{PO}\left(C^*|p^{IS}, \boldsymbol{\theta}^*, \boldsymbol{\psi}\right)|N\left(\boldsymbol{\theta}_0\right)\right] := \widehat{V}^{PO}\left(C^*|p^{IS}, N\left(\boldsymbol{\theta}_0\right), \boldsymbol{\psi}\right).$$

Then similar to the cost-based approach, we could compare the PO's expected payoff across different entry threshold C^* based on numerical studies.We demonstrate the results of the following example in the left panel of Figure 3.13:

$$\begin{aligned}
\mu_{a,0} &= \mu_{b,0} := \mu = 3, \sigma_a = \sigma_b = \sigma = 1, \\
C_0 &= 0.7, \gamma = 0.2, p^{IS} \in \{0.5, 1, 1.5, 2\}, \delta = 0.99.
\end{aligned}$$

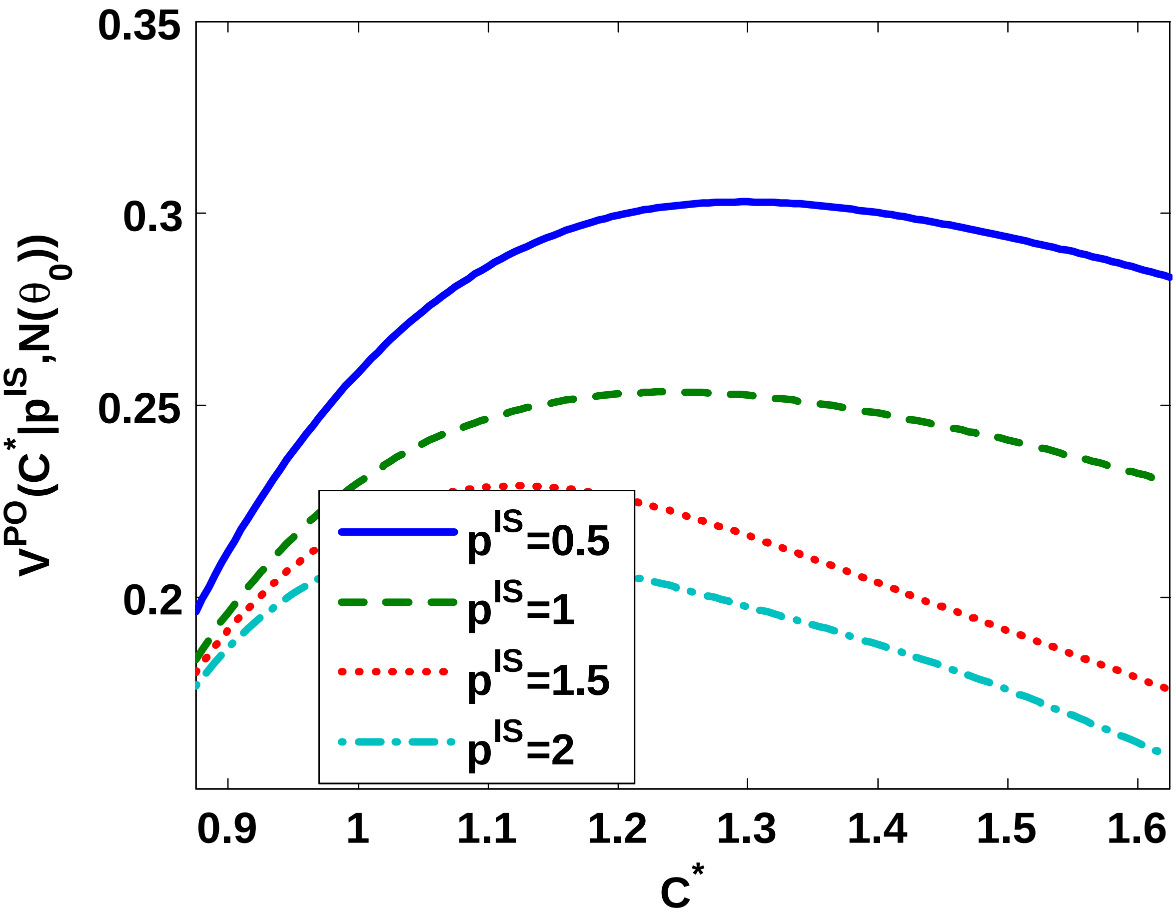


Figure 3.13: The PO's Price-Based Payoff Function w.r.t. Entry Threshold

As we notice the PO's payoff function $\widehat{V}^{PO}\left(C^*|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right)$ presents a concave shape w.r.t. C^* for any fixed p^{IS}, we could identify the optimum $C^{**}\left(p^{IS}\right)$ and the associated payoff $\widehat{V}^{PO}\left(C^{**}|p^{IS},\boldsymbol{\theta}_0,\boldsymbol{\psi}\right)$ as shown in Figure 3.14:

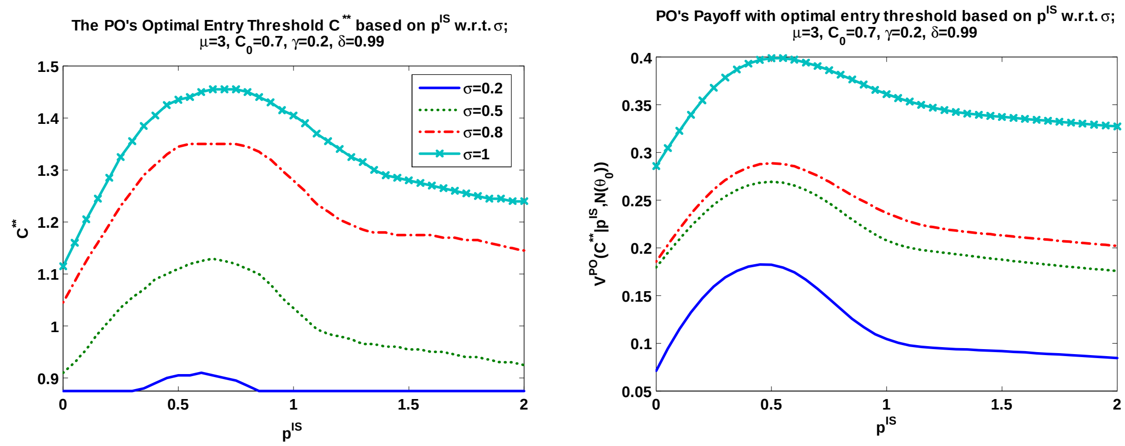


Figure 3.14: The PO's Optimal Entry Threshold and Payoff for Fixed Price

The "optimal" entry threshold C^{**} reaches its highest with a price between 0.5 and 1, and increases with σ;while the optimal payoff with this "optimal" entry threshold C^{**} reaches its highest at $p^{IS} = 0.5$.

3.6 The Alternative Profit-Driven Entry Policy

In this section we study the two players' performances when the PO adopts the profit-driven entry policy $f_{C_\gamma^*}$. We focus on the unique observations in this case and evaluate the PO's performances under the two types entry policies. In particular, we examine how the revenue commission rate γ affect the IS's optimal performances and optimal product selections, and highlight the IS's preference for a higher γ for a delayed entry.

3.6.1 Commission Rate $\gamma's$ Effect on The IS's Performance

The PO's Entry Time under the Alternative Entry Policy

To understand the two player's behaviors under $f_{C^*_\gamma}$, we first derive the PO's entry time index $\widehat{\tau}_\gamma$ that determines the differences arising from the alternative entry policy as other parts of the IS's problem remains the same.

The entry time index

$$\widehat{\tau}_\gamma := \min\left\{ t \geq 0 | \frac{\mu_{a,t}^2}{4\mu_{b,t}} - \gamma p \left(\mu_{a,t} - \mu_{b,t} p\right)^+ \geq C^*_\gamma \right\}$$

could be found following a similar procedure,by replacing [19]the equation of $\frac{\mu_{a,t}^2}{4\mu_{b,t}} = C^*$ with

$$\frac{\mu_{a,t}^2}{4\mu_{b,t}} - \gamma p \left(\mu_{a,t} - \mu_{b,t} p\right)^+ = C^*_\gamma \tag{3.7}$$

to solve for $\mu'_{\gamma,t}$, and inserting $\mu'_{\gamma,t}$ into the expressions of 3.4 and 3.5 for $\{\alpha, \beta\}$ and $\widehat{\tau}_\gamma$ respectively.

The solution for the general case of $A > 0, \Delta \geq 0, \mu'_{\tau,\gamma} \geq \frac{-\left(\mu_{a,0} - \mu_{b,0} p\right)}{\cos\omega - p\sin\omega}$ is

$$\mu'_{\tau,\gamma} = \frac{-\mathcal{B} + \sqrt{\Delta}}{2\mathcal{A}} > 0, \tag{3.8}$$

and in the special case of $\mu_{a,0} = \mu_{b,0} := \mu$:

$$\begin{aligned}
\mathcal{A} &= \cos^2\omega - 4\gamma p \sin\omega\cos\omega + 4\gamma p^2 \sin^2\omega \geq 0 \\
\mathcal{B} &= 2\mu\cos\omega - 4C^*_\gamma \sin\omega - 4\gamma p \sin\omega \mu \left(1 - p\right) - 4\gamma p \mu \left(\cos\omega - p\sin\omega\right) \\
\mathcal{C} &= \mu^2 \left[1 - 4\gamma p \left(1 - p\right)\right] - 4C^*_\gamma \mu \\
\Delta &= \mathcal{B}^2 - 4\mathcal{A}\mathcal{C}.
\end{aligned}$$

Otherwise when $\Delta < 0$ or $\mu'_{\tau,\gamma} < \frac{-\left(\mu_{a,0} - \mu_{b,0} p\right)}{\cos\omega - p\sin\omega}$, the condition 3.7 is reduced to $\frac{\mu_{a,t}^2}{4\mu_{b,t}} = C^*_\gamma$ as $\mu_{a,\tau} -$

[19] We assume that $\mu\left[1 - 4\gamma p \left(1 - p\right)\right] < 4C^*_\gamma$

$\mu_{b,\tau}p \le 0$. In this case when $\mu'_\tau < \frac{-\left(\mu_{a,0}-\mu_{b,0}p\right)}{\cos\omega - p\sin\omega}$[20], the solution is simply

$$\mu'_{\tau,\gamma} = \mu'_\tau.$$

Then similarly we could obtain $\widehat{\tau}_\gamma = \alpha\left\{\beta + \sqrt{\beta^2 - 2\ln\delta}\right\}$ through

$$\begin{cases} \alpha\left(p\right) = \frac{\mu'_{\tau,\gamma}}{\sigma_a^2\sigma_b^2\iota(p)} \\ \beta\left(p\right) = \alpha\left(p\right)\sigma_a^2\sigma_b^2\left(\frac{1}{\sigma_b^2} + \frac{p^2}{\sigma_a^2} - \frac{2\rho p}{\sigma_a\sigma_b}\right) - \left(a - \mu_{a,0}\right) + p\left(b - \mu_{b,0}\right) \end{cases}.$$

The IS's Optimal Payoff and Product Selection

Under the alternative entry policy $f_{C^*_\gamma}$, one major difference arises from the complex effect of the PO's commission rate γ. Under the f_{C^*} entry policy, the PO's entry time is independent from γ and $\gamma's$ effect on the IS's pricing decision could be subsumed in the factor $c = \frac{c^{IS}}{1-\gamma}$, so the IS's payoff always decreases in γ.

In contrast under $f_{C^*_\gamma}$, the PO's entry time index $\widehat{\tau}_\gamma$ increases with γ even with the entry threshold C^*_γ and other parameters being fixed, while the IS's revenue decreases with γ. Therefore these two contradicting effects of γ on the IS's payoff leads to a potential better performance under a higher γ. We demonstrate how $\widehat{V}^{IS}\left(p^*, \gamma|a, b\right)$ change with γ with the numerical results of

$$\begin{aligned} & \{C^* = 1, \mu = 3, \sigma = 0.5\} \\ a &\in \{2.5, 3.5\}, b \in \{2, 4\} \end{aligned}$$

.

[20] to guarantee $\mu_{a,\tau} - \mu_{b,\tau}p \le 0$

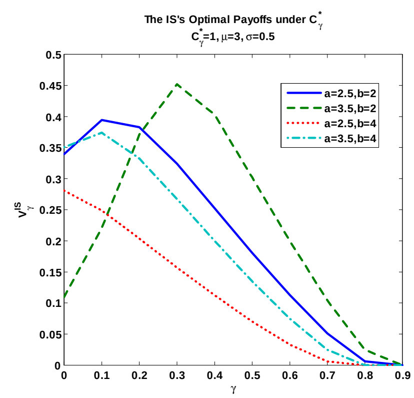


Figure 3.15: The IS's Optimal Payoffs w.r.t. Commission Rate

Figure 3.15 shows that $\widehat{V}^{IS}(p^*, \gamma|a, b)$ appears to be concave in γ so one could identify an "optimal" γ^{IS} from the IS's perspective. In particular γ^{IS} is lower when the entry opportunity is less attractive, i.e. $\frac{a^2}{b}$ is low, since its entry time effect is lessened compared to the revenue scale effect. On the contrary, γ^{IS} is higher when the entry opportunity is more attractive, i.e. $\frac{a^2}{b}$ is high.

Further more, we could also identify the IS's optimal product selection set under the alternative entry policy $f_{C^*_\gamma}$.

Corollary 6 $\exists \left\{p^{**}_\gamma(\theta_0, \boldsymbol{\psi}), q^{**}_\gamma(\theta_0, \boldsymbol{\psi})\right\}$, *s.t.* $\forall \{a^*, b^*\} > 0$ *with the relationship*

$$a^* = q^{**}_\gamma + b^* p^{**}_\gamma$$

have the following properties:

1. $p^*_\gamma(a^*, b^*) = p^{**}_\gamma$;

2. $V^{IS*}_\gamma(a^*, b^*) := V^{IS}\left(a^*, b^*, p^*_\gamma(a^*, b^*)\right) = V^{IS**}_\gamma$

3. $\forall \{a, b\} > 0$ *and* $a > bc, V^{IS*}_\gamma(a, b) \leq V^{IS**}_\gamma$

Based on Corollary 6, we could examine the effect of γ on $\left\{p^{**}_\gamma, q^{**}_\gamma, V^{IS**}_\gamma\right\}$ with the other parameters being fixed, to characterize the IS's performances as a result of the commission rate. In general, we observe that $\left\{V_\gamma^{IS**}, q_\gamma^{**}\right\}$ have an increasing and decreasing trend w.r.t. γ, while p_γ^{**} increase with γ. We demonstrate the example with the following set of the parameters:

$$\left\{C_\gamma^* = 1, \mu = 3, \sigma = 0.5, a \in [2, 8], b \in [1.5, 6.5], \gamma \in [0, 0.8]\right\}$$

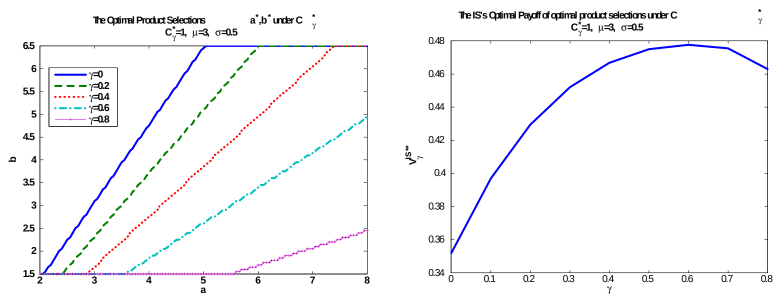


Figure 3.16: The IS's Performances of Optimal Product Selections w.r.t. Commission Rate

From Figure 3.16 we demonstrate the optimal product selection bundles $\left\{a_\gamma^*(b), b\right\}$ of different γ, and observe that the IS's optimal payoff of these optimal product selections always presents a concave pattern w.r.t. γ, therefore there exists an optimal γ^{IS*} across all $\{\gamma, a, b\}$ values, which is around 0.6 in the case examined. Intuitively, if the PO adopts $f_{C_\gamma^*}$ and the PO is to choose his product offering across categories with different commission rates, he could first settle with a category with a favorable commission rate γ^{IS*}, then choose any product among $\left\{a_\gamma^*(b), b\right\}$.

3.6.2 The PO's Payoffs under the Two Entry Policies

In this subsection, we examine the IS and the PO's performances when the PO adopts the two types of entry policies. Similar to the studies under the revenue-driven entry policies f_{C^*}, we compare the two policies based on the IS's private information or the IS's pricing decision. We observe that on an aggregate level over the prior distribution of the IS's private information, the PO's payoff is consistently higher under $f_{C_\gamma^*}$. However, if the PO could set the entry thresholds based on the IS's price, her payoffs could be higher under either entry policy.

Comparison of Cost-Based Payoffs

We first examine the PO's payoffs under the two types entry policies when we vary the entry threshold C^* and C^*_γ for give prior information and c^{IS}. We observe that $\widehat{V}^{PO}\left(C^*|p^*, N\left(\boldsymbol{\theta}_0\right), \boldsymbol{\psi}\right)$ and $\widehat{V}^{PO}\left(C^*_\gamma|p^*, N\left(\boldsymbol{\theta}_0\right), \boldsymbol{\psi}\right)$ both present a concave shape w.r.t. C^* and C^*_γ respectively. However, the highest possible payoffs under $f_{C^*_\gamma}$ is consistently higher. This means that when the PO chooses the best performing C^*_γ, she can achieve an average payoff not achievable under f_{C^*}. We present this observation in Figure 3.17, and the relative gap in the PO's performances in Table 3.3.

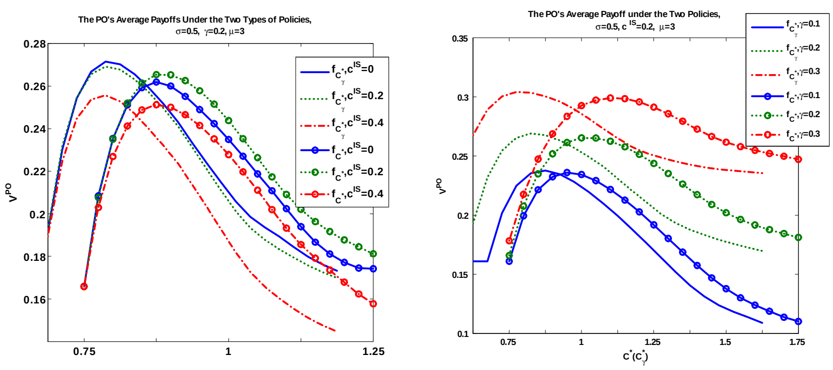


Figure 3.17: The PO's Average Payoff over Prior under Two Policies

Table 3.3: Relative Difference in The PO Average Payoffs under Two Policies

$1-\frac{\widehat{V}^{PO}(C^{**}\|p^*,N(\boldsymbol{\theta}_0),\boldsymbol{\psi})}{\widehat{V}^{PO}\left(C^{**}_\gamma\|p^*,N(\boldsymbol{\theta}_0),\boldsymbol{\psi}\right)}$	$\gamma=0.1$	$\gamma=0.2$	$\gamma=0.3$
$c^{IS}=0$	2.39%	4.77%	6.50%
$c^{IS}=0.2$	0.8%	1.65%	2.36%
$c^{IS}=0.4$	0.78%	1.63%	2.62%

It is not a surprising result that the PO's average expected profit is higher under $f_{C^*_\gamma}$ as the policy $f_{C^*_\gamma}$ is explicitly seeking a higher profit while f_{C^*} focuses more on the post entry revenue.

Comparison of Price-Based Payoffs

Similar to the study under f_{C^*}, we evaluate the PO's performance under different C^*_γ for given price p^{IS} and identify the best performing entry threshold and the payoff associated to compare with the results under f_{C^*}.We observe that the PO can achieve the same highest payoff under the two types of policies for any fixed p^{IS}, i.e. $\widehat{V}^{PO}\left(C^{**}|p^{IS}, \boldsymbol{\theta}_0, \boldsymbol{\psi}\right) = \widehat{V}^{PO}\left(C^{**}_\gamma|p^{IS}, \boldsymbol{\theta}_0, \boldsymbol{\psi}\right)$.

Table 3.4: The PO's Payoff under Optimal Entry Threshold w.r.t. IS's Price under Two Policies

$V^{PO*}\left(N(\theta)\|p^{IS}\right)$ under two policies				
σ	0.2	0.5	0.8	1
V^{PO*}_{RD}	0.181	0.212	0.287	0.279
V^{PO*}_{PD}	0.180	0.214	0.287	0.278

3.7 The Univariate Model and the Impact of ρ

In this section, we discuss the necessity to adopt the current bivariate demand model through the different results from a univariate model. In particular, we observe that the IS's optimal price is always higher than the optimal price without entry threat, as opposed to the pricing distortion behavior in two ways we observe in a general model.

We compare the IS and the PO's performances under different values of the correlation coefficient ρ, and find that while the IS is better off with higher values of ρ, the PO's performance depend on the true demand parameters and could be better with lower values of ρ. In particular, when the demand parameters presents an attractive opportunity for entry, the PO is better off with $\rho = -1$ as it assigns higher probability to realizations of higher a and lower b, thus entails a more optimistic perspective and earlier entry.

3.7.1 The Uni-variate Demand Model and the IS's Pricing Distortion Behavior

If the PO adopts a univariate model as follows

$$D = a(1-p) + \varepsilon,$$

it is essentially a special case of our current model with the assumption of

$$\rho = 1, a = b, \mu_a = \mu_b, \sigma_a = \sigma_b \tag{3.9}$$

We argue that in this case, the IS's pricing distortion behavior is always along one direction.

Proposition 13 *Under the assumptions in (3.9)* $p^* \geq p^M$.

In contrast, we already observe in the bi-variate model when $\rho = 0$, the IS's pricing distortion behavior could be in two ways, as shown by Proposition 10. We would demonstrate that with a general ρ or even in the special case of $\rho = 1$, this is still true. We believe it is very important for the PO to develop a directionally consistent understanding of the IS's pricing behavior by adopting a general bi-variate demand model.

The IS's pricing behavior with a General ρ

In Proposition 10, we identified the sufficient conditions for the IS to set $p^* < p^M$ when $\rho = 0$, and similarly we argue that when ρ is low enough, the IS still sets $p^* < p^M$ when b is low enough.

Proposition 14 *When* $\rho < \frac{\mu + C^* - \sqrt{C^*(2\mu + C^*)}}{\mu}, (0 <) b < \frac{\mu}{2C}(\mu - 2C), p^* < p^M$

Please refer to 2.3 for the proof of Proposition 14. We resort to numerical studies to demonstrate that, when ρ is low enough, there exists a boundary $\widehat{b}(a|\rho)$ that divides the $\{a, b\}$-space into two parts of "high b" and "low b", within which we observe $p^* > p^M$ and $p^* < p^M$ respectively. In the left panel of Figure 3.18, we demonstrate the boundaries $\widehat{b}(a|\rho)$ for $\rho \leq 0.6$ and observe that $\widehat{b}(a|\rho)$ decreases with ρ. We also observe that such a division of the $\{a, b\}$ $-$space simply along the $b-$axis may not exists anymore for larger ρs. For example, in the second graph of Figure 3.18, we show that when $\rho = 0.8$, the region for $p^* < p^M$ in $\{a, b\}$ $-$space is no longer defined by an upper bound of b. In other words, even when b is very low, there may still exist an $\{a, b\}$ s.t. $p^* (a, b) > p^M (a, b)$.

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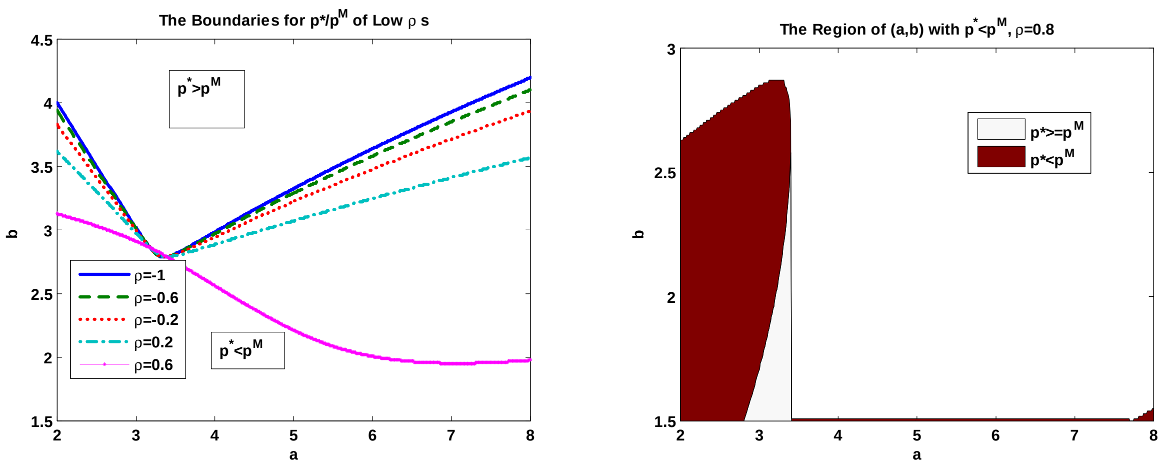

Figure 3.18: The IS price distortion behaviors with Low Correlation

With larger ρ, we develop the following result for the possibility of $p^* < p^M$.

Proposition 15 *When* $a > b, \exists \omega_0 (\boldsymbol{\theta}_0, \boldsymbol{\theta}^*, \boldsymbol{\psi}) > 0,$ *s.t.* $\forall \rho = 1 - \omega > 1 - \omega_0 (\boldsymbol{\theta}_0, \boldsymbol{\theta}^*, \boldsymbol{\psi}),$
$\tau(\rho)$ *decreases in* $\left[\max\left\{c, 1 - \left(\frac{\frac{a-b}{2} - \frac{\ln \delta}{a-b}}{4C^* - b}\right)^+ [1 + O(\omega)]\right\}, \min\left\{1 + \frac{\frac{a-b}{2} - \frac{\ln \delta}{a-b}}{b}[1 - O(\omega)], \frac{a}{b}\right\}\right].$

In Figure 3.19, we demonstrate the regions in $\{a, b\}$ $-$space inducing a lower optimal price $p^* < p^M$ under $\rho = 1$ and $\rho = 0.99$ for the example with[21]

$$\begin{aligned} \delta &= 0.99, \sigma = 0.5, \mu = 3 \\ c^{IS} &= 0.2, \gamma = 0.2, C^* = 1, C_0 = 0.7 \end{aligned}$$

[21] All the numerical studies in this section are based on this sample.

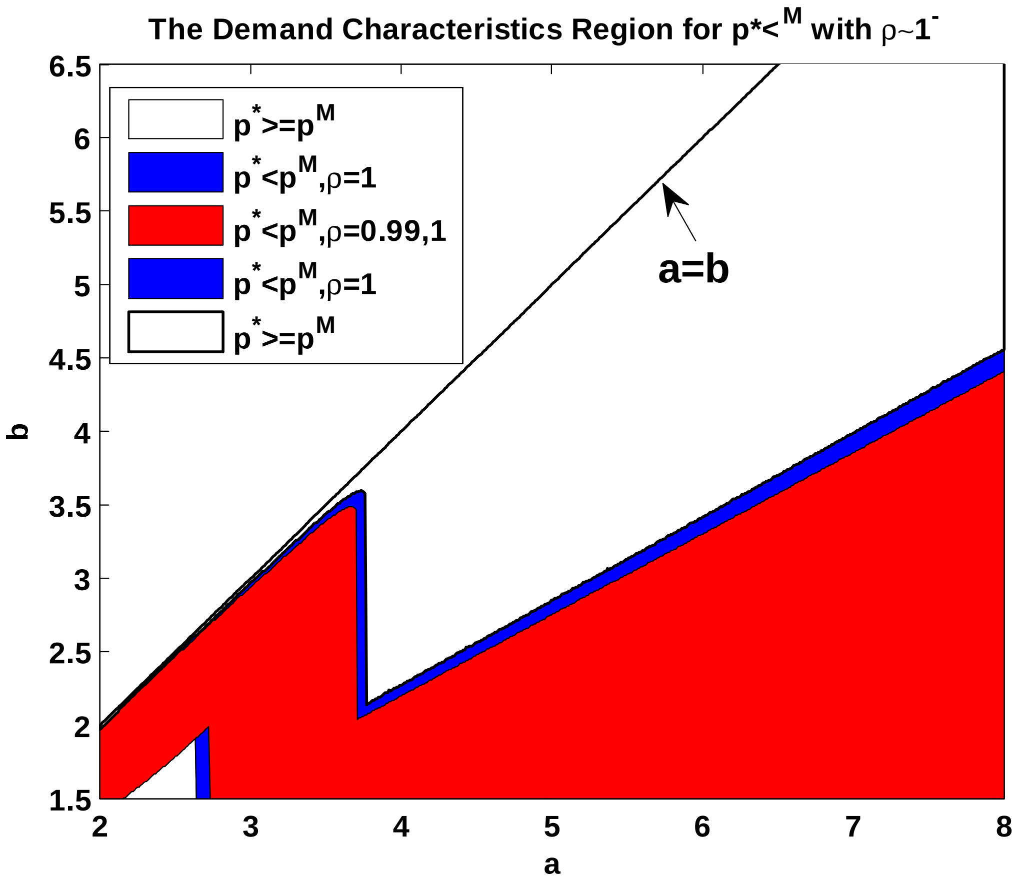

Figure 3.19: The IS's price distortion behaviors with High Correlation

3.7.2 The Two Player's Performances with General ρ

In this subsection, we further examine how in general the value of ρ affects the two player's performances as well as the model's prediction of the IS's optimal price. We first argue that under certain conditions, the IS optimal price might still be lower than p^M, to contrast with the prediction of the one-dimensional demand model. Moreover, we compare the PO and the IS's performances under different values of ρ to demonstrate that ρ could be an endogenous choice by the PO to determine her learning and entry strategy. In particular, we observe that a low ρ assigns higher probability to realizations of higher a and lower b, thus might lead to an earlier entry and higher payoff to the PO when the actual $\{a, b\}$ presents an attractive opportunity.

The Two Players' Performances

Based on the IS's optimal pricing decision, we could numerically compute the two players' performances under different values of ρ. We first fix $\{a, b\}$ and $\{\boldsymbol{\psi}, \boldsymbol{\theta}_0\}$ to examine how the IS's optimal pricing p^*, entry time index $\widehat{\tau}$, and the induced payoffs $\left\{V^{IS}(p^*), V^{PO}(p^*)\right\}$ change with ρ.

From the numerical results over a wide range of $\{a, b\}$ values, we observe that while the IS's payoff is consistently increasing with ρ, the PO's performance could be either increasing or decreasing w.r.t. the value of ρ. We consider the following two samples

$$\{a_1, b_1\} = \{5.2, 3.7\}, \{a_2, b_2\} = \{2.6, 3.9\},$$

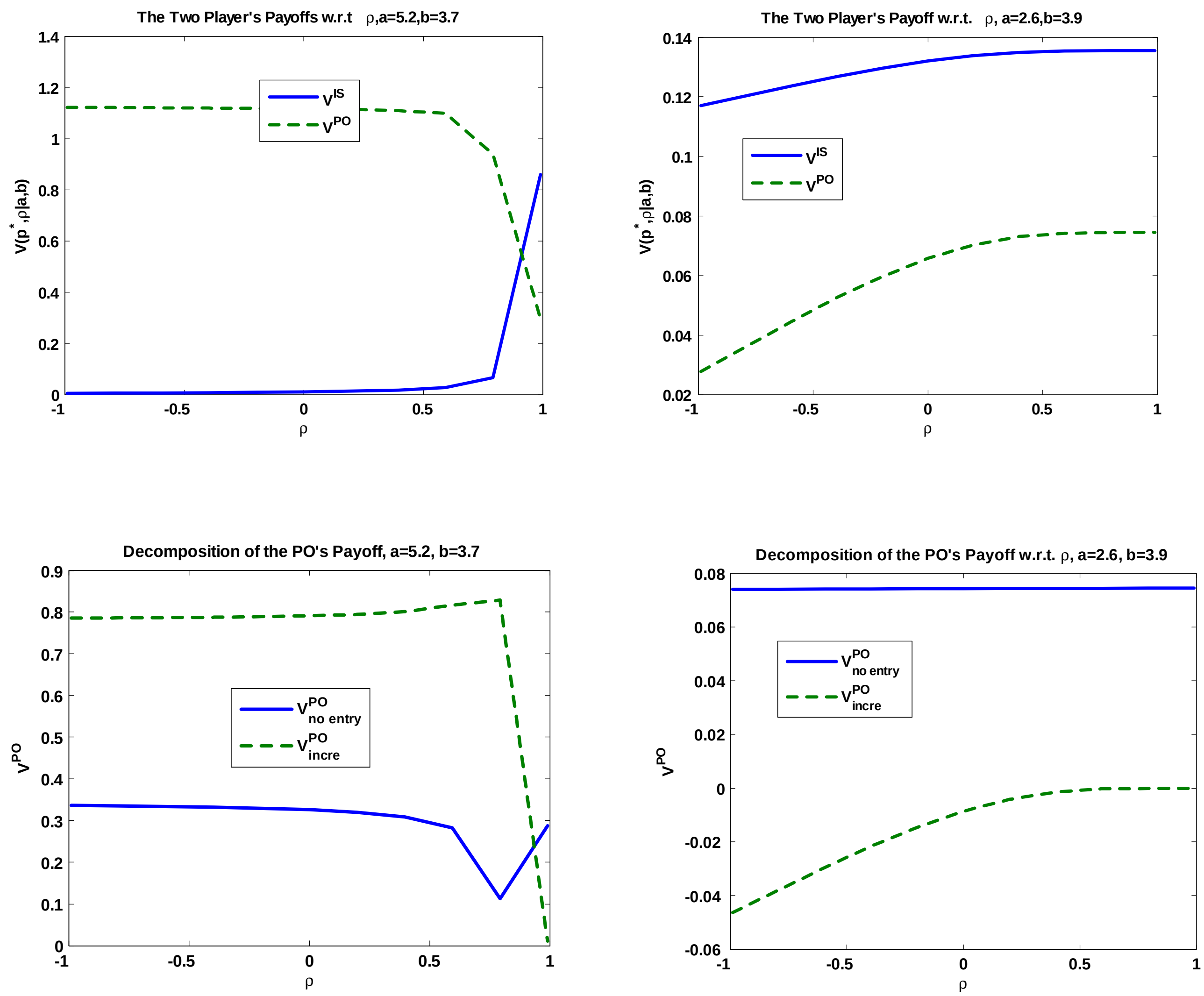

Figure 3.20: The Two Player's Payoff w.r.t r Coefficient of Variation for Two Samples

The top panels of Figure 3.20 show that the PO's payoff decreases with ρ when $\{a,b\}$ presents an attractive entry opportunity, namely when $\frac{a^2}{b}$ is high, and increases when $\{a,b\}$ presents an unattractive entry opportunity. Therefore the $\{a, b\}$ −space can be divided into two parts according to $\widehat{V}^{PO}(p^*, \rho|a, b)$'s relation with ρ, as shown in Figure 3.21.

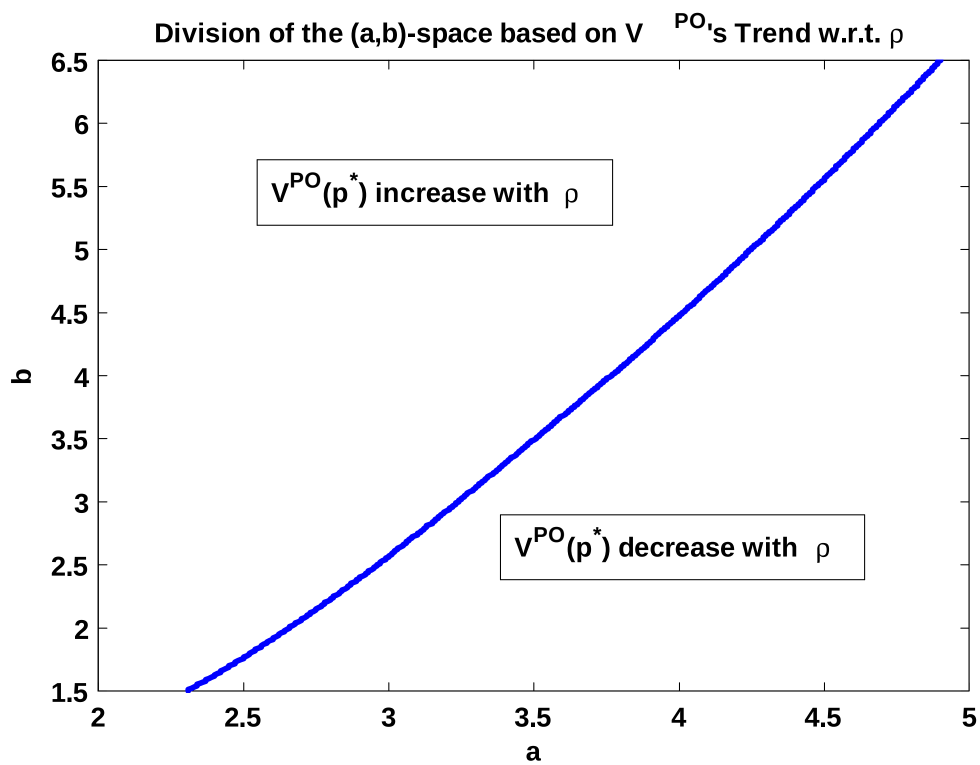

Figure 3.21: Division of the Product Space

Intuitively, with $\rho = -1$, the PO's posterior belief assigns higher probability to realizations of $\{a > \mu, b < \mu\}$ or $\{a < \mu, b > \mu\}$; while with $\rho = 1$, the PO's posterior belief assigns higher probability to realizations of $\{a > \mu, b > \mu\}$ or $\{a < \mu, b < \mu\}$. Therefore, for an more attractive $\{a, b\}$ bundle, the PO's posterior belief with $\rho = -1$ would more likely evolve to the area of $\{a > \mu, b < \mu\}$, and triggers an earlier entry compared to the case of $\rho = 1$. While for an unattractive $\{a, b\}$ bundle, with $\rho = -1$ the PO's entry is most likely triggered by an erroneous entry with a posterior belief in the area of $\{a > \mu, b < \mu\}$. To confirm this hypothesis, we decompose the PO's payoff into two components

$$\begin{aligned}\widehat{V}^{PO} &= \widehat{V}^{PO}_{noentry} + \widehat{V}^{PO}_{incre} \\ \widehat{V}^{PO}_{noentry} &: = \gamma p^* \left(\mu_{a,0} - \mu_{b,0} p^*\right)^+, \widehat{V}^{PO}_{incre} := e^{-\widehat{\tau}} \left[C^* - \gamma p^* \left(\mu_{a,\tau} - \mu_{b,\tau} p^*\right)^+ - C_0\right]\end{aligned}$$

and examine the two components' trends w.r.t. ρ. The lower panels of Figure 3.20 show that for the sample of $\{a_1, b_1\} = \{5.2, 3.7\}$, the incremental payoff post entry $\widehat{V}^{PO}_{incre}$ was much higher with lower ρ, mainly due to a lower entry time $\widehat{\tau}$ and positive post entry incremental profit, while the PO's no-entry payoff $\widehat{V}^{PO}_{noentry}$ first decreases with ρ as the IS is more pressed by the entry threat and distorts the optimal price p^* to a higher level. In contrast, the PO's no-entry payoff $\widehat{V}^{PO}_{noentry}$ holds almost constant for $\{a_2, b_2\} = \{2.6, 3.9\}$ as the IS barely manipulates the price, and her loss from an erroneous entry leading to negative $\widehat{V}^{PO}_{incre}$ is reduced with more cautious entry under $\rho = 1$.

Based on results from the case of $\rho = 0$, we examine the existence of optimal product selections $\{a^*(b|\rho), b\}$ from the IS's perspective. In a wide range of ρ, we still observe such line segments in the $\{a, b\}$-space defined by $\{p^{**}(\rho), q^{**}(\rho)\}$

$$a^*(b|\rho) = q^{**}(\rho) + bp^{**}(\rho)$$

that give rise to the highest payoffs for the IS

$$V^{IS**}(\rho) := V^{IS*}(a^*(b|\rho), b).$$

Moreover, within a valid range of ρ, we observe that $\left\{p^{**}(\rho), q^{**}(\rho), \widehat{V}^{IS**}(\rho)\right\}$ all increase with ρ, a result consistent with the observations from Figure 3.20:

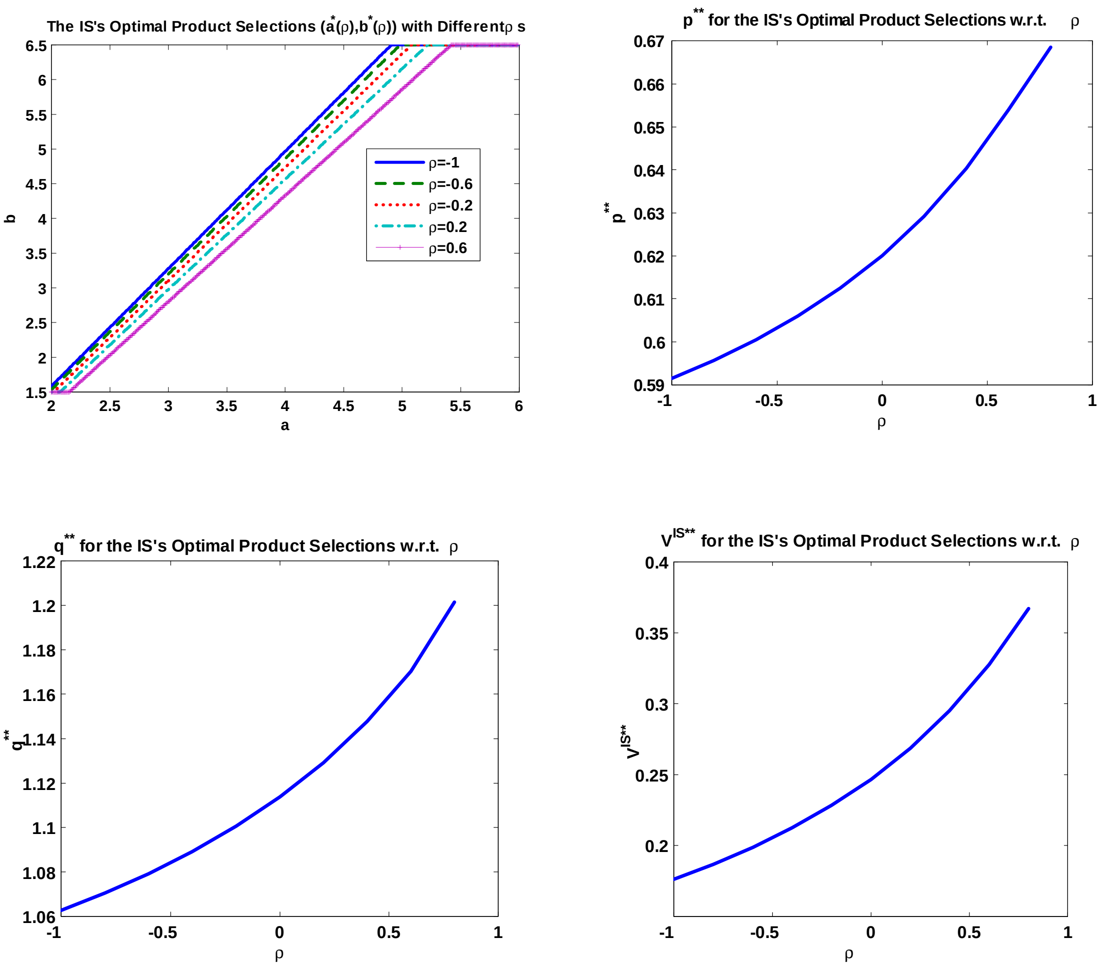

Figure 3.22: The IS's Optimal Product Selections with Different Coefficients of Variation

3.8 Conclusion and Discussion

Motivated by Amazon.com's informational advantage to fuel its rapid expansion into new product categories and accurate product offerings through a unique "merchant-platform" dual role, we conduct this study to characterize a platform owner's "learning-while-earning" process as well as the independent sellers' operational reactions when faced with such threat and opportunity from selling on a popular platform. As two-sided marketplaces become increasingly important in our economy and consumer experiences, many facets of their operations remain novel and pose challenges to practitioners' strategic moves. Our study contributes to a better understanding of the owner-and-users relationship of a retail platform, as well as the user-generated-information as one of the most valuable assets for the owner to capitalize.

We proposed a general demand learning and entry decision model for the platform owner faced with a large number of products sharing some common market conditions. We analyzed the independent seller's different pricing distortion behaviors for two types of products with "high price sensitivity" or "low price sensitivity". This helps to explain the price dispersion for similar products on the marketplace in the presence of price transparency and fierce competition. We discussed the PO's practical "revenue-driven" and "profit-driven" threshold entry policies, and identified their equivalence under the IS's fixed price.

As our model has a very rich structure, this paper only considers a few special situations and focuses on a few operational instrument as decision variables. Below we briefly discuss some potential model setups as well as the current findings' sensitivity to some parameter changes. We believe these questions are highly interesting and an in-depth study of them would largely contribute to the demand learning literature.

The IS's Dynamic Price Scheme In this paper, we assume that the IS sets a price up-front and keeps it constant for tractability. As the current revenue-driven and profit-driven entry policies of the PO do not guarantee optimality but depend on the IS's selling price as a parameter, it does not take a constant price from the IS to justify. If the IS could change his selling price p^{IS} without restriction or commitment, the PO's entry decision would update accordingly, as if the process restarts from $t = 0$ for her. In this case, the IS's changing price would add variation to the PO's observation and expedite her learning process. On the other hand, the IS could flexibly regulate his price distortion of each period according to the current "entry threat" level.

By enabling the IS to change his price each period, the problem boils down to a dynamic programming with the PO's posterior as the state variables, which again becomes intractable because of the curse of dimensionality. One potential way to incorporate this trade-off from the IS dynamic pricing scheme is to allow for a finite number of price changes from the IS, or a finite set of time points for the changes to occur. Another approach is to require the IS to announce the time-dependent pricing scheme up-front. In both cases, the PO is informed of the IS's higher degree of freedom in his action space, and should take it into account. More detailed discussion is beyond the scope of this study and remains open for future exploration.

The IS's Variable Cost c^{IS} In our major analysis, we assumed that $c^{IS} > c^{PO}$ to highlight the PO's higher operational efficiency to support her entry decision. However, this might not be necessary. Also we noticed that the PO's average performance is not monotone with c^{IS} in Figure 3.17, and is higher with a medium level c^{IS} in many cases under both types of heuristic entry policies. This indicates that a deeper understanding $c^{IS\prime}s$ effect is needed for future research, as well as a study for more general values of c^{IS}.

The PO's Optimal γ So far, our study of the IS's optimal pricing and the PO's entry threshold choice is based on the assumption of $c = \frac{c^{IS}}{1-\gamma} > \frac{a}{b}$. Yet as the PO is not informed the true demand parameters $\{a, b\}$, a too high γ may break this assumption. Namely when $c = \frac{c^{IS}}{1-\gamma} > \frac{a}{b}$, it is not profitable for the IS to be present in the marketplace. So if the commission rate γ is endogenous for the PO and applies to the entire product category that shares the prior distribution of the demand parameters, the PO needs to weigh the trade-off between low commission income and low participation to set γ. This trade-off not only applies regardless of her adopt of $f_{C^*_\gamma}$ or f_{C^*}, and is present even without her learning and entry threat.

Here we assume that if $c = \frac{c^{IS}}{1-\gamma} \geq \frac{a}{b}$, then $V^{IS}\left(c^{IS}, \gamma, a, b\right) = V^{PO}\left(c^{IS}, \gamma, a, b\right)$, and consider how the PO's payoff aggregated over the prior distribution $\widehat{V}^{PO}\left(\gamma | p^*, N\left(\boldsymbol{\theta}_0\right), \boldsymbol{c}^{IS}\right)$ change with γ. Intuitively, when the IS's unit unit cost c^{IS} is small, the "no participation" effect is much lessened for the PO so her optimal γ^{PO} would be higher, and vice versa.

We demonstrate the example of

$$\left\{ \begin{array}{c} C_{\gamma}^{*} = 1, \mu = 3, \sigma = 0.5, \gamma \in [0, 0.9] \\ c^{IS} = \{0, 0.1, ...0.4\}, C_0 = 0.7 \end{array} \right\}$$

that shows consistent results with the intuition. And we observe little influence by the IS's pricing distortion behavior in this case.

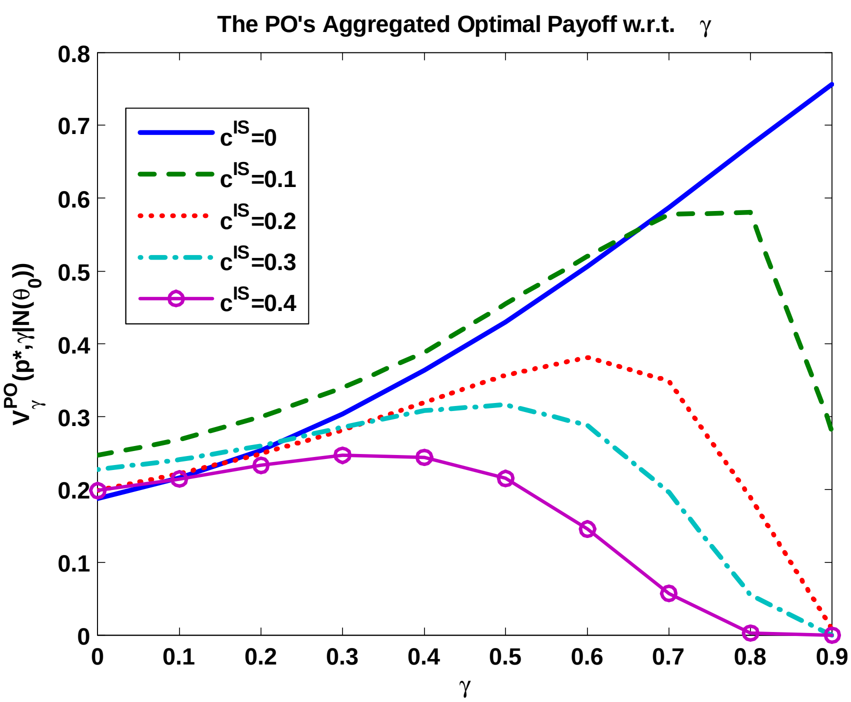

Figure 3.23: The PO's Aggregated Payoff w.r.t. Commission Rate

The PO's Strategic Choice of the Prior Belief The analysis above shows that how the PO's performance depends on values of ρ varies with the value of $\{a, b\}$. We examine the impact of ρ on the PO's potential payoff on an aggregated level by averaging over a set of $\{a, b\}$ samples. Instead of examining an evenly distributed sample grid, we consider the two player's performance w.r.t. the prior $\rho \in [-1, 1]$ on two more realistic sample sets:

1. A $\{a, b\}_{\rho,\sigma}$ sample of size 200 following the PO's prior distribution $N(\mu_0, \sigma, \rho)$, $\sigma \in \{0.2, 0.5, 0.8, 1\}$;

2. A $\{a, b\}_{0,\sigma}$ sample of size 200 following $N(\mu_0, \sigma, \rho = 0)$, $\sigma \in \{0.2, 0.5, 0.8, 1\}$.

We choose these two sets of samples mainly for two reasons. One is that the PO's prior belief might not be consistent with the generating distribution of products in one category; another is a different generating distribution fundamentally affect the sample's performance in our setting. Similar to the PO's prior belief's impact, a generating distribution with $\rho = -1$ essentially create more profitable entry opportunities, which is consistent with the results of Sample 1 shown in Figure 3.24.

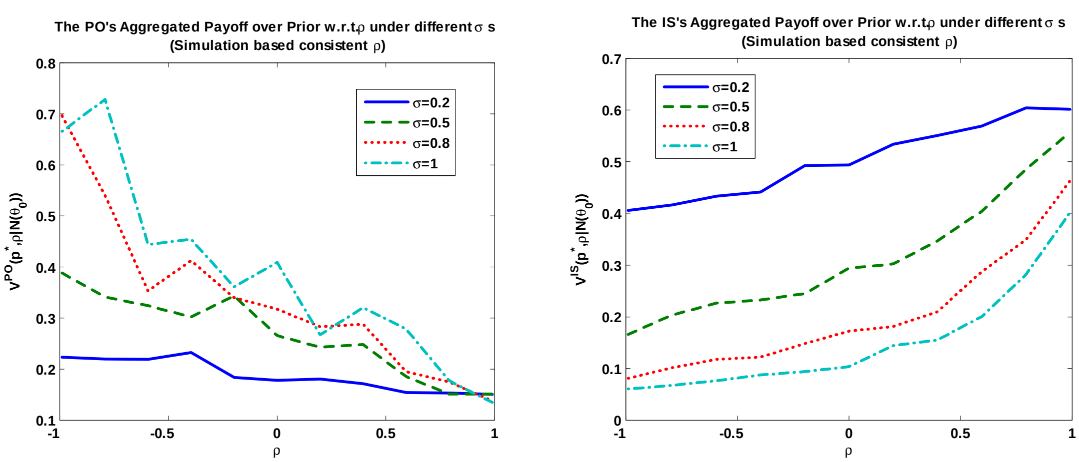

Figure 3.24: The Two Player's Aggregated Payoff w.r.t. the Actual Coefficient of Variantion

In contrast, the results from Sample 2 distinguish the impact of the PO's prior from the generating distribution. In particular, it shows that the PO's prior as a whole affects the PO's average performance in different ways. We observe that the PO's payoff maximized at $\rho = -1$ when σ is low, and as σ increases, the PO's payoff becomes increasing with ρ when it is small but all finally drops with large enough ρ.

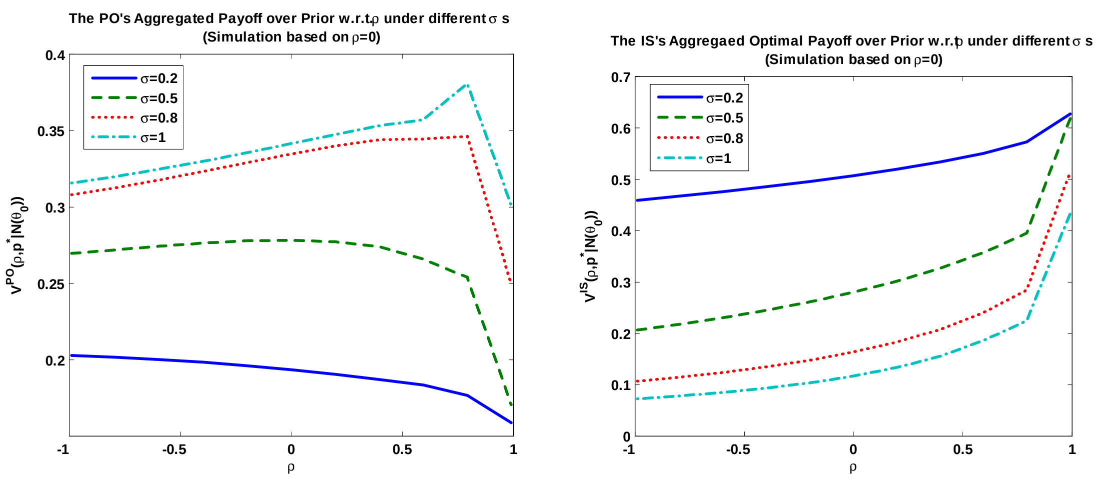

Figure 3.25: The Two Player's Aggregated Payoff w.r.t. the PO's perceived Coefficient of Variantion

The result from Sample 2 shows that the PO's average payoff is not necessarily the highest when her prior belief is consistent with the generating distribution, namely when $\rho = 0$. In this case, the variance in her performance is mainly due to her model selection. When the purpose of learning is to guide her entry decision, it is not necessary to refine her prior belief to be as close as possible to the actual distribution of demand characteristic. The PO's choice of her prior belief reflects more of her learning and entry strategy as opposed to the actual market condition. When the PO is confident about the prior mean and has lower willingness to learn, a lower ρ focuses her attention only to the exceptionally attractive entry opportunities as the prior show low attraction. While when the PO is more open to learning due to higher σ, a higher ρ leads to higher consistency.

We repeat this study for various c^{IS} and observe the same pattern, therefore even though the PO cannot fully anticipate the IS's pricing decision or infer the demand parameters upon observing the

IS's price due to the information asymmetry[22], she is informed of the effect of her prior belief choice, as a part of her learning and entry strategy.

We also notice that the IS's optimal payoff based on the optimal product selections $V^{IS**}(\mu, C^*, \sigma)$ drops sharply (i.e. exponentially) with a growing σ, showing that the IS's payoff is largely affected by the PO's learning behavior. A higher σ can be interpreted as the PO's openness to learning and leads to faster learning, but more possibly erroneous learning or high variation in profitability, so could be perceived as riskier by the both the IS and the PO regardless of the true demand. As we could tell from Table 3.2, the PO's average payoff based on complete information $\widehat{V}^{PO}(0, \infty|p^*)$ is not monotonically increasing with σ. Therefore, from the PO's perspective, there could be an "optimal prior σ" based on this trade-off.

[22] For example, when c^{IS} vary from 0 to 0.4, the IS's optimal price for a fixed $\{a, b\}$ could vary by 90%, and the IS's average optimal price over the generating distribution could vary by 50%.

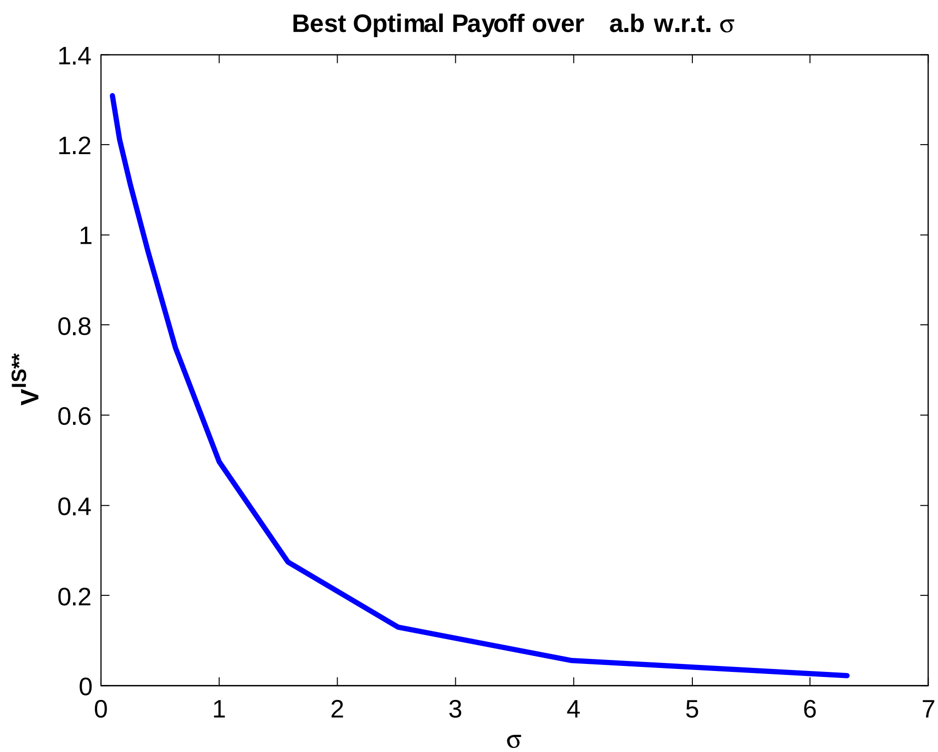

Figure 3.26: The Best Optimal Payoff's Relationship to the Prior Coefficien of Variation

Finally, we notice from Table 3.10 that the PO's average payoff over the prior distribution does not increase with the prior mean μ. This might be related to the trade-off between exploration and exploitation, for a higher μ would lead to more cautious decision on entry threshold and potentially missing many profitable entry opportunities. A more holistic sensitivity analysis of the PO's prior is of great interest but remain open for future exploration.

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Appendix

1 Chapter 2

1.1 Optimal Solutions

The Retailer's Optimal Inventory Policies

Proof of Proposition 3. The result for $k \geq p_1$ or $k < \underline{p}_2$ is straightforward for the retailer, since the second period profit is not affected by the first period's inventory decision. So we focus on the case of $k \in \left(\underline{p}_2, p_1\right]$. The retailer's loss of the first period by ordering D^H instead of the D^L, optimum of the period is $C_\Pi \cdot (k - p_1)(1 - \rho)$; and her benefit of the second period by overcoming the censorship effect is $\Delta V_{R,2} = \delta C_\Pi \cdot (1 - \rho)(k - \underline{p}_2)(1 - p_1)$. So the retailer orders D^H when

$$\begin{aligned} C_\Pi \cdot (k - p_1)(1 - \rho) + \delta C_\Pi \cdot (1 - \rho)(k(w) - \underline{p}_2)(1 - p_1) &> 0 \\ \frac{p_1 + \delta(1 - p_1) \cdot \underline{p}_2}{1 + \delta(1 - p_1)} &< k \end{aligned}$$

■

The Supplier's Optimal Wholesale Price Policies

Then the supplier's optimal wholesale price in "O" could be expressed as:

Solution 7 *With fully observed demand, the supplier's optimal wholesale prices*

$$\begin{cases} S_1^O := & \left[\max\{0, \frac{(\bar{p}_2-p_1)(1+\delta)}{\delta p_1(1-\rho)}+p_1, \frac{(\bar{p}_2-\underline{p}_2)(1+\delta)}{(1-\rho)(1+\delta p_1)}+\underline{p}_2, \frac{\bar{p}_2}{1-\rho}\}, 1\right) \\ S_2^O := & \left[\max\{0, \frac{\left(\frac{1+\delta}{1-\rho}-\delta p_1\right)p_1}{1+\delta(1-p_1)}, \left(\frac{1+\delta}{1-\rho}-\delta p_1\right)(p_1-\underline{p}_2)+\underline{p}_2\}, \min\{\frac{(\bar{p}_2-p_1)(1+\delta)}{\delta p_1(1-\rho)}+p_1, 1\}\right) \\ S_3^O := & \left[\{\frac{\left(\frac{1+\delta}{1-\rho}-\delta p_1-1\right)\underline{p}_2}{\delta(1-p_1)}\}^+, \min\{\frac{(\bar{p}_2-\underline{p}_2)(1+\delta)}{(1+\delta p_1)(1-\rho)}+\underline{p}_2, \left(\frac{1+\delta}{1-\rho}-\delta p_1\right)(p_1-\underline{p}_2)+\underline{p}_2, 1\}\right) \\ S_4^O := & \left[0, \min\{\frac{\left(\frac{1+\delta}{1-\rho}-\delta p_1-1\right)\underline{p}_2}{\delta(1-p_1)}, \frac{\left(\frac{1+\delta}{1-\rho}-\delta p_1\right)p_1}{1+\delta(1-p_1)}, \frac{\bar{p}_2}{1-\rho}\right). \end{cases}$$

And her optimal wholesale price in "M" is:

Solution 8 $$\begin{cases} S_1^M := & [\max\{0, \frac{(\bar{p}_2-p_1)(1+\delta)}{\delta p_1(1-\rho)}+p_1, \frac{\bar{p}_2}{1-\rho}\}, 1] \\ S_2^M := & [\{\frac{\left(\frac{1+\delta}{1-\rho}-\delta p_1\right)p_1}{1+\delta(1-p_1)}\}^+, \min\{\frac{(\bar{p}_2-p_1)(1+\delta)}{\delta p_1(1-\rho)}+p_1, 1\}] \\ S_4^M := & [0, \min\{\frac{\left(\frac{1+\delta}{1-\rho}-\delta p_1\right)p_1}{1+\delta(1-p_1)}, \frac{\bar{p}_2}{1-\rho}\}] \end{cases}$$

In general, with $\eta \in (0,1]$, the supplier's optimal wholesale price in "F" is:

Solution 9 $$\begin{cases} S_1^F := & \left[\max\{0, \frac{(\bar{p}_2-p_1)(1+\delta)}{\delta p_1(1-\rho)}+p_1, \frac{[\bar{p}_2-p_2^F(\eta)](1+\delta)}{(1-\rho)(1+\delta p_1-\eta)}+p_2^F(\eta), \frac{\bar{p}_2}{1-\rho}\}, 1\right) \\ S_2^F := & \left[\max\{0, \frac{\left(\frac{1+\delta}{1-\rho}-\delta p_1\right)p_1}{1+\delta(1-p_1)}, \frac{\left(\frac{1+\delta}{1-\rho}-\delta p_1\right)[p_1-p_2^F(\eta)]}{1-\eta}+p_2^F(\eta)\}, \min\{\frac{(\bar{p}_2-p_1)(1+\delta)}{\delta p_1(1-\rho)}+p_1, 1\}\right) \\ S_3^F := & \left[\{\frac{[\frac{1+\delta}{1-\rho}-\delta p_1+(\eta-1)]p_2^F(\eta)}{\delta(1-p_1)+\eta}\}^+, \min\{\frac{[\bar{p}_2-p_2^F(\eta)](1+\delta)}{(1+\delta p_1-\eta)(1-\rho)}+p_2^F(\eta), \frac{\left(\frac{1+\delta}{1-\rho}-\delta p_1\right)[p_1-p_2^F(\eta)]}{1-\eta}+p_2^F(\eta), 1\}\right) \\ S_4^F := & \left[0, \min\{\frac{[\frac{1+\delta}{1-\rho}-\delta p_1+(\eta-1)]\cdot p_2^F(\eta)}{\delta(1-p_1)+\eta}, \frac{\left(\frac{1+\delta}{1-\rho}-\delta p_1\right)p_1}{1+\delta(1-p_1)}, \frac{\bar{p}_2}{1-\rho}\right) \end{cases},$$

And when $\eta = 1$, Case 2 and Case 3 merge into Case 2' in "F", so the supplier's optimal wholesale price in "F" is:

Solution 10 $$\begin{cases} S_1^F := & \left[\max\{0, \frac{(\bar{p}_2-p_1)(1+\delta)}{\delta p_1(1-\rho)}+p_1, \frac{(\bar{p}_2-p_2^F)(1+\delta)}{\delta p_1(1-\rho)}+p_2^F, \frac{\bar{p}_2}{1-\rho}\}, 1\right) \\ S_{2'}^F := & \left[\{\frac{\left(\frac{1+\delta}{1-\rho}-\delta p_1\right)p_2^F}{\delta(1-p_1)+1}\}^+, \min\{\frac{(\bar{p}_2-p_2^F)(1+\delta)}{\delta p_1(1-\rho)}+p_2^F, 1\}\right) \\ S_4^F := & \left[0, \min\{\frac{\left(\frac{1+\delta}{1-\rho}-\delta p_1\right)p_2^F}{\delta(1-p_1)+1}, \frac{\left(\frac{1+\delta}{1-\rho}-\delta p_1\right)p_1}{1+\delta(1-p_1)}, \frac{\bar{p}_2}{1-\rho}\right) \end{cases}$$

Proof of Lemma 1. This result is straight-forward by comparing the conditions for different order quantities in the three situations, and by noting that order quantities decreases with the wholesale prices. ■

Centralized Systems and Double-Marginalization

We consider the centralized supply chains with the same demand, cost, and information structures as in the decentralized systems. In all three situations of the centralized system ("OC", "MC", and "FC"), the decision maker sets the inventories to optimize the system total profits. It is worth noting that the inventory problems of the centralized systems are structurally identical to the retailer's inventory problems in the respective decentralized systems, simply by replacing the retailer's critical ratio $k(w)$ with the supply chain's critical ratio s. So we can easily derive the optimal inventory levels of the centralized supply chains in all three situations as follows: (here we only consider $\eta = 1$)

Solution 11 *The optimal inventory policy for a centralized supply chain* $\left\{y_1^{OC}, y_2^{OC}|s, \alpha, \beta\right\}$ *is:*

Case 1 $(1 >)s > \bar{p}_2$, $y_1^{OC} = D^H, y_2^{OC} = D^H$;

Case 2 $\bar{p}_2 \geq s > p_1$, $y_1^{OC} = D^H, y_2^{OC} = \begin{cases} D^H & D_1 = D^H \\ D^L & D_1 = D^L \end{cases}$;

Case 3 $p_1 \geq s > \underline{p}_2$, $y_1^{OC} = D^L, y_2^{OC} = \begin{cases} D^H & D_1 = D^H \\ D^L & D_1 = D^L \end{cases}$

Case 4 $\underline{p}_2 \geq s(> 0)$, $y_1^{OC} = D^L, y_2^{OC} = D^L$

Solution 12 *The optimal inventory policy for a myopic supply chain under demand censorship* $\left\{y_1^{MC}, y_2^{MC}|s, \alpha, \beta\right\}$ *is:*

Case 1 $(1 >)s > \bar{p}_2$, $y_1^{MC} = D^H, y_2^{MC} = D^H$

Case 2 $\bar{p}_2 \geq s > p_1$, $y_1^{MC} = D^H, y_2^{MC} = \begin{cases} D^H & D_1 = D^H \\ D^L & D_1 = D^L \end{cases}$

Case 4' $p_1 \geq s(> 0)$, $y_1^{MC} = D^L, y_2^{MC} = D^L$.

Solution 13 *The optimal inventory policy for a forward-looking supply chain under demand censorship* $\left\{y_1^{FC}, y_2^{FC}|s, \alpha, \beta\right\}$ *is:*

Case 1 $(1 >)s > \bar{p}_2$, $y_1^{FC} = D^H, y_2^{FC} = D^H$

Case 2' $\bar{p}_2 \geq s > p_2^F$, $y_1^{FC} = D^H, y_2^{FC} = \begin{cases} D^H & D_1 = D^H \\ D^L & D_1 = D^L \end{cases}$

Case 4' $p_2^F \geq s(>0)$, $y_1^{FC} = D^L, y_2^{FC} = D^L$.

Similar to the results of the decentralized systems in **??**, we could define the partitions of Θ and the projected partitions on S induced by the optimal order quantities in the centralize systems:

$$\begin{cases} S_1^{OC} := & [\bar{p}_2, 1) \\ S_2^{OC} := & [p_1, \bar{p}_2) \\ S_3^{OC} := & \left[\underline{p}_2, p_1\right) \\ S_4^{OC} := & \left[0, \underline{p}_2\right). \end{cases} \begin{cases} S_1^{MC} := & [\bar{p}_2, 1) \\ S_2^{MC} := & [p_1, \bar{p}_2) \\ S_4^{MC} := & [0, p_1) \end{cases} \begin{cases} S_1^{FC} := & [\bar{p}_2, 1) \\ S_{2'}^{FC} := & \left[p_2^F, \bar{p}_2\right) \\ S_4^{FC} := & \left[0, p_2^F\right). \end{cases}$$

Based on these results, we could easily compare the order quantities and total profits of the three situations to justify the two informational inefficiencies:

Corollary 7

$$\begin{aligned} y_t^{OC} &\geq y_t^{MC}, y_t^{FC} \geq y_t^{MC}, t = 1, 2 \\ \Pi^{OC} &\geq \Pi^{FC} \geq \Pi^{MC}. \end{aligned}$$

Corollary 8

$$\forall \boldsymbol{\theta} \in \Theta, \forall * \in \{O, M, F\}, \{y_1^*(\boldsymbol{\theta}), y_2^*(\boldsymbol{\theta})\} \leq \{y_1^{*C}(\boldsymbol{\theta}), y_2^{*C}(\boldsymbol{\theta})\}$$

1.2 Conditions for the Two Phenomena

Proof of Proposition 2. The proof is based on the optimal wholesale price solutions and the respective partitions of Θ in the three situations as in **??**. We first summarize the retailer, the supplier, and the supply chain's total profits in Table 5 using the uniform "case" framework: $(* \in \{O, M, F\})$

Table 5: Profits of the Players and Supply Chain

Case	$\{y_1^*, y_2^*\}$	w^*	V_i^*/C_Π	$V_{S,i}^*/C_\Pi$	$V_{R,i}^*/C_\Pi$
1	D^H, D^H	$\overline{p}_2^+$	$(1+\delta)\left[s-(1-\rho)\,p_1\right]$	$(1+\delta)(s-\overline{p}_2)$	$(1+\delta)\left[\overline{p}_2-(1-\rho)\,p_1\right]$
2	$D^H, \begin{smallmatrix} D^H \\ D^L \end{smallmatrix}$	p_1^+	$(1+\delta)\left[s-(1-\rho)\,p_1\right]$ $-\delta p_1(s-\overline{p}_2)\,(1-\rho)$	$\left[(1+\delta)-\delta p_1\,(1-\rho)\right]\cdot$ $(s-p_1)$	$(1+\delta)\rho p_1+$ $\delta p_1(\overline{p}_2-p_1)\,(1-\rho)$
$2'$	$D^H, \begin{smallmatrix} D^H \\ D^L \end{smallmatrix}$	p_2^{F+}	$(1+\delta)\left[s-(1-\rho)\,p_1\right]$ $-\delta p_1(s-\overline{p}_2)\,(1-\rho)$	$\left[(1+\delta)-\delta p_1\,(1-\rho)\right]\cdot$ $\left(s-p_2^F\right)$	$\rho(1+\delta)p_2^F$
3	$D^L, \begin{smallmatrix} D^H \\ D^L \end{smallmatrix}$	$\underline{p}_2^+$	$s\rho(1+\delta)+$ $\delta(1-p_1)\,(1-\rho)\left(s-\underline{p}_2\right)$	$[\rho(1+\delta)+$ $\delta(1-p_1)\,(1-\rho)]\cdot$ $(s-\underline{p}_2)$	$\rho(1+\delta)\underline{p}_2$
4	D^L, D^L	0^+	$s\rho(1+\delta)$	$s\rho(1+\delta)$	0

From Table 5, we observe that the supplier's profits follow[23]:

$$\begin{aligned} V_{S,i}^O(\theta) &= V_{S,i}^M(\theta) = V_{S,i}^F(\theta),\, i=1,4 \\ V_{S,i}^O(\theta) &= V_{S,i}^M(\theta) < V_{S,i}^F(\theta),\, i=2, \end{aligned} \tag{10}$$

meaning that without changing the wholesale price, the supplier makes the same profit from the same order quantities across the three situations, and more profit with higher wholesale price in Case 2' in "F".

And the retailer's profits follow:

$$V_{R,1}^*(\theta) > V_{R,2}^{O,M}(\theta) > V_{R,2'}^F(\theta) > V_{R,3}^O(\theta) > V_{R,4}^*(\theta),\, * \in \{O, M, F\}, \tag{11}$$

meaning that the retailer's profit increases with order quantities across the three situations, and slightly decreases (with no changes in order quantities) with the wholesale price in Case 2'.

The supply chain profit only depends on order quantities, so

$$\forall\theta, V_i^M(\theta) = V_i^O(\theta) = V_i^F(\theta),\, i=1,...,4. \tag{12}$$

To compare the supply chain profits, we refer to the centralized system optimal outcome in 1.1. For ex-

[23]Note that Case 3 is only present in "O".

ample, $\forall\theta \in \Theta_3^{OC}, V_3^O(\theta) > V_i^O(\theta), i = 1, 2, 4$. The result of Proposition 8 about understocking/double-marginalization effect could be summarized in the set language: $\Theta_i^* \subset \cup_{j=1}^{i} \Theta_j^{*C}, * \in \{O, M, F\}$, meaning the optimal order quantities in the respective centralized system can only be higher than (or equal to) the order quantities in the decentralized system induced by the optimal wholesale price. This relationship bridges the centralized and decentralized systems and enables us to conduct comparison on supply chain profits.

We prove the results of Proposition 2 based on the pair-wise comparisons of the three situations through the following three lemmas.

Lemma 9 (Comparison between "M" and "F")

$$\Theta_1^F \subset \Theta_1^M, \Theta_{2'}^F \supset \Theta_2^M, \Theta_4^F \subset \Theta_4^M$$

■

Sketchy Proof of Lemma 9.

Based on (10), $\forall\theta \in \Theta_1^F, V_{S,1}^M(\theta) = V_{S,1}^F(\theta) \geq V_{S,2'}^F(\theta) > V_{S,2}^M(\theta), V_{S,1}^M(\theta) = V_{S,1}^F(\theta) \geq V_{S,4}^F(\theta) = V_{S,4}^M(\theta)$, thus $\theta \in \Theta_1^M$,so $\Theta_1^F \subset \Theta_1^M$.Similarly we could show that $\Theta_4^F \subset \Theta_4^M$, and $\Theta_{2'}^F \supset \Theta_2^M$.□

The result of Lemma 9 indicates that the difference between "F" and "M“ could be summarized as the expansion of Case 2' in "F". So we observe different order quantities between "M" and "F" when $\theta \in \Theta_{2'}^F \backslash \Theta_2^M = \left(\Theta_1^M \backslash \Theta_1^F\right) \cup \left(\Theta_4^M \backslash \Theta_4^F\right)$[24], and different profit allocation in $\Theta_2^M = \Theta_{2'}^F \cap \Theta_2^M$ with the same order quantities in "M" and "F".

Based on (11), the retailer's profit is higher in "M" than "F" when $\theta \in \Theta_1^M \backslash \Theta_1^F$ due to the higher order quantities (and lower w) and in Θ_2^M due to lower w. So

$$V_R^M(\theta) > V_R^F(\theta) \iff \theta \in \left(\Theta_1^M \backslash \Theta_1^F\right) \cup \Theta_2^M = \left(\Theta_2^M \cup \Theta_1^M\right) \cap \Theta_{2'}^F := \Theta_R^{MF}.$$

The supply chain profits are only different in $\left(\Theta_1^M \backslash \Theta_1^F\right) \cup \left(\Theta_4^M \backslash \Theta_4^F\right)$ with different order quantities.$\forall\theta \in \Theta_4^M \backslash \Theta_4^F = \Theta_4^M \cap \Theta_{2'}^F \subset \Theta_{2'}^{FC} \cup \Theta_1^{FC}$, if $V^M(\theta) > V^F(\theta)$, then $V_4^M(\theta) = V_4^F(\theta) > V_{2'}^F(\theta) \Rightarrow \theta \in$

[24] The order quantities are higher in "F" if $\theta \in \Theta_4^M \backslash \Theta_4^F$ and lower if $\theta \in \Theta_1^M \backslash \Theta_1^F$.

Θ_4^{FC},contradiction; so $V^M(\theta) < V^F(\theta)$[25]. Following similar logic, for $\forall\theta \in \Theta_1^M \backslash \Theta_1^F = \Theta_1^M \cap \Theta_{2'}^F \subset \Theta_1^{MC} = \Theta_1^{FC}$, so $V_1^M(\theta) > V_2^M(\theta) = V_2^F(\theta), V^M(\theta) > V^F(\theta)$. To sum up,

$$V^M(\theta) > V^F(\theta) \iff \theta \in \Theta_1^M \backslash \Theta_1^F = \Theta_1^M \cap \Theta_{2'}^F := \Theta^{MF}.$$

So far we proved the results in the first column of Proposition 2.

Note that $\Theta_{(R)}^{Cen,O} := \left\{ \theta \in \Theta | \max\left\{ V_{(R)}^M(\theta) \ , V_{(R)}^F(\theta) \right\} > V_{(R)}^O \ (\theta) \right\} = \Theta_{(R)}^{MO} \cup \Theta_{(R)}^{FO}$,where $\Theta_{(R)}^{MO} := \left\{ \theta \in \Theta | V_{(R)}^M(\theta) \ > V_{(R)}^O \ (\theta) \right\}$,

$\Theta_{(R)}^{FO} := \left\{ \theta \in \Theta | V_{(R)}^F(\theta) \ > V_{(R)}^O \ (\theta) \right\}$,so we compare "M" and "O", and "F" and "O" separately.

Lemma 10 (Comparison between "M" and "O")

$$\Theta_1^O \subset \Theta_1^M, \Theta_2^O \subset \Theta_2^M, \Theta_4^O \subset \Theta_4^M$$

■

Sketchy Proof of Lemma 10.

Based on (10), $\forall\theta \in \Theta_1^O, V_{S,1}^M(\theta) = V_{S,1}^O(\theta) \geq V_{S,2}^O(\theta) = V_{S,2}^M(\theta), V_{S,1}^M(\theta) = V_{S,1}^O(\theta) \geq V_{S,4}^O(\theta) = V_{S,4}^M(\theta)$, so $\theta \in \Theta_1^M, \Theta_1^O \subset \Theta_1^M$.Similarly we could prove that $\Theta_2^O \subset \Theta_2^M, \Theta_4^O \subset \Theta_4^M$.□

The result of Lemma 10 shows that the difference in "M" compared to "O" is simply due to the removing of Case 3. So the supplier keeps her choice of Case 1, 2, 4 in "M" if she chooses it in "O", yet has to opt for Case 1,2,and 4 if she chooses Case 3 in "O", therefore we only observe different outcomes between "M" and "O' in $\Theta_3^O = \left(\Theta_1^M \backslash \Theta_1^O\right) \cup \left(\Theta_2^M \backslash \Theta_2^O\right) \cup \left(\Theta_4^M \backslash \Theta_4^O\right)$.

Following similar logic in the comparison between "M" and "F", it is easy to verify that

$$V_R^M(\theta) > V_R^O(\theta) \iff \theta \in \left(\Theta_1^M \backslash \Theta_1^O\right) \cup \left(\Theta_2^M \backslash \Theta_2^O\right) = \left(\Theta_1^M \cup \Theta_2^M\right) \cap \Theta_3^O.$$

And for $\forall\theta \in \left(\Theta_1^M \cup \Theta_2^M\right) \cap \Theta_3^O \subset \Theta_1^{MC} \cup \Theta_2^{MC} = \Theta_1^{OC} \cup \Theta_2^{OC}$, if $V_3^O(\theta) > V_2^M(\theta) = V_2^O(\theta)$

[25] The system optimum are Case 1 or Case 2 in "FC", thus the double-marginalization (i.e. understocking) is more severe in "M" and leads to lower supply chain profit.
Also, we ignore the case of $V^M(\theta) = V^F(\theta)$,

or $V_3^O(\theta) > V_1^M(\theta) = V_1^O(\theta), \theta \in \Theta_3^{OC}$,contradiction! So $\forall\theta \in \left(\Theta_1^M \cup \Theta_2^M\right) \cap \Theta_3^O, V^M(\theta) > V^O(\theta)$.Similarly we could argue that $\forall\theta \in \Theta_4^M \backslash \Theta_4^O, V^M(\theta) < V^O(\theta)$,so

$$V^M(\theta) > V^O(\theta) \iff \theta \in \left(\Theta_1^M \cup \Theta_2^M\right) \cap \Theta_3^O.$$

To sum up,

$$\Theta^{MO} = \Theta_R^{MO} = \left(\Theta_1^M \cup \Theta_2^M\right) \cap \Theta_3^O.$$

The comparison between "F" and "O" is less straightforward, we first ague that:

Lemma 11 (Comparison between "F" and "O") *1.* $V_R^F(\theta) > V_R^O(\theta) \iff \left(\Theta_1^F \cup \Theta_{2'}^F\right) \cap \left(\Theta_3^O \cup \Theta_4^O\right)$

2. $V^F(\theta) > V^O(\theta) \iff \left(\Theta_1^F \cup \Theta_{2'}^F\right) \cap \left(\Theta_3^O \cup \Theta_4^O\right) \setminus \left(\Theta_2^F \cap \Theta_3^O \cap \Theta_3^{OC}\right)$

■

Proof of Lemma11. :

1. Following the logic of the previous comparisons, it's easy to show that $\theta \in \left(\Theta_1^F \cup \Theta_{2'}^F\right) \cap \left(\Theta_3^O \cup \Theta_4^O\right) \Rightarrow V_R^F(\theta) > V_R^O(\theta)$.It suffices to show that

$$\theta \notin \left(\Theta_1^F \cup \Theta_{2'}^F\right) \cap \left(\Theta_3^O \cup \Theta_4^O\right) \Rightarrow V_R^F(\theta) \leq V_R^O(\theta). \tag{13}$$

   If $\theta \in \left(\Theta_1^F \cup \Theta_{2'}^F\right) \cap \left(\Theta_1^O \cup \Theta_2^O\right)$,the retailer gets a higher profit in "F" than in "O" only when $\theta \in \Theta_1^F \cap \Theta_2^O$ according to 11. Yet since $\forall s \in S_1^F(\theta_{-s}), s > \frac{(\bar{p}_2 - p_1)(1+\delta)}{\delta p_1(1-\rho)} + p_1$,while $\forall s \in S_2^O(\theta_{-s})$ $s < \frac{(\bar{p}_2 - p_1)(1+\delta)}{\delta p_1(1-\rho)} + p_1$,so $\Theta_1^F \cap \Theta_2^O = \emptyset$.If $\theta \in \Theta_4^F$,again the retailer cannot get a higher profit in "F" than in "O" according to 11.

2. Based on the first result of Lemma 11, it's also easy to verify that

$$\theta \notin \left(\Theta_1^F \cup \Theta_{2'}^F\right) \cap \left(\Theta_3^O \cup \Theta_4^O\right) \Rightarrow V^F(\theta) \leq V^O(\theta),$$

and when $\theta \in \Theta_{2'}^F \cap \Theta_3^O \cap \Theta_3^{OC}, V_3^O(\theta) > V_2^O(\theta) = V_2^F(\theta)$,so $V^F(\theta) < V^O(\theta)$.On the other hand, $\left(\Theta_1^F \cup \Theta_2^F\right) \cap \left(\Theta_3^O \cup \Theta_4^O\right) \setminus \left(\Theta_{2'}^F \cap \Theta_3^O \cap \Theta_3^{OC}\right) = \left\{\Theta_1^F \cap \left(\Theta_3^O \cup \Theta_4^O\right)\right\} \cup \left\{\Theta_{2'}^F \cap \Theta_4^O\right\} \cup \left\{\Theta_{2'}^F \cap \Theta_3^O \backslash \Theta_3^{OC}\right\}$.Therefore when $\theta \in \Theta_3^{OC} \subset \Theta_1^{FC} = \Theta_1^{OC}$,so $V_1^F(\theta) = V_1^O(\theta) > V_{3,4}^O(\theta)$;when $\theta \in \Theta_{2'}^F \cap \Theta_4^O \subset \Theta_{2'}^{FC}$,so $V_{2'}^F(\theta) > V_4^F(\theta) = V_4^O(\theta)$;when $\theta \in \left(\Theta_{2'}^F \cap \Theta_3^O\right) \backslash \Theta_3^{OC} = \left(\Theta_{2'}^F \cap \Theta_3^O\right) \cap \left(\Theta_1^{OC} \cup \Theta_2^{OC}\right)$,so $V_{2'}^F(\theta) = V_2^O(\theta) > V_3^O(\theta)$; so in summary

$$\theta \in \left(\Theta_1^F \cup \Theta_{2'}^F\right) \cap \left(\Theta_3^O \cup \Theta_4^O\right) \setminus \left(\Theta_{2'}^F \cap \Theta_3^O \cap \Theta_3^{OC}\right) \iff V^F(\theta) > V^O(\theta).$$

Based on the results of Lemma 10 and Lemma 11, it is easy to verify that

$$\Theta_{(R)}^{M,O} = \left(\Theta_1^M \cup \Theta_2^M\right) \cap \Theta_3^O \subset \left(\Theta_1^F \cup \Theta_{2'}^F\right) \cap \left(\Theta_3^O \cup \Theta_4^O\right) \setminus \left(\Theta_{2'}^F \cap \Theta_3^O \cap \Theta_3^{OC}\right) = \Theta^{FO} \subset \Theta_R^{FO}$$

so

$$\begin{aligned} \Theta_R^{Censor,O} &= \Theta_R^{F,O} = \left(\Theta_1^F \cup \Theta_{2'}^F\right) \cap \left(\Theta_3^O \cup \Theta_4^O\right) \\ \Theta^{Censor,O} &= \Theta^{F,O} = \left(\Theta_1^F \cup \Theta_{2'}^F\right) \cap \left(\Theta_3^O \cup \Theta_4^O\right) \setminus \left(\Theta_{2'}^F \cap \Theta_3^O \cap \Theta_3^{OC}\right). \end{aligned}$$

■

1.3 Characterization of the Two Phenomena and Extreme Value Studies

Study for the Retailer's Two Phenomena

Proof of Proposition 4. Based on Proposition 3 and the projected partitions of S induced by the optimal wholesale prices in **??**, the conditions for the two phenomena could be written through the S partitions as follows:

$$\begin{aligned} S_R^{MF} &: = \left(S_2^M \cup S_1^M\right) \cap S_2^F \\ S_R^{Censor,O} &: = \left(S_2^F \cup S_1^F\right) \cap \left(S_3^O \cup S_4^O\right) \end{aligned}$$

We characterize these subsets through their boundaries. For example, since $S_2^M \cup S_1^M$ and S_1^M are both intervals, their intersection S_R^{MF}could be defined as $\left(S_{R,1}^{MF}\left(\theta_{-s}\right), S_{R,2}^{MF}\left(\theta_{-s}\right)\right]$,where

$$\begin{aligned}
S_{R,1}^{MF} \quad &: \quad = \max\left\{\min\{\frac{\left(\frac{1+\delta}{1-\rho}-\delta p_1\right)p_1}{1+\delta(1-p_1)}, \frac{\bar{p}_2}{1-\rho}, 1\}, \{\frac{\left(\frac{1+\delta}{1-\rho}-\delta p_1\right)p_2^F}{\delta(1-p_1)+1}\}^+\right\} \underrightarrow{\rho \to 0^+} p_1 \\
S_{R,2}^{MF} \quad &: \quad = \min\left\{\frac{(\bar{p}_2-p_1)(1+\delta)}{\delta p_1(1-\rho)}+p_1, 1\right\} \underrightarrow{\rho \to 0^+} \min\{\frac{(\bar{p}_2-p_1)(1+\delta)}{\delta p_1}+p_1, 1\}
\end{aligned}$$

Similarly,

$$\begin{aligned}
S_{R,1}^{Censor,O} \quad &: \quad = \min\{\{\frac{\left(\frac{1+\delta}{1-\rho}-\delta p_1\right)p_2^F}{\delta(1-p_1)+1}\}^+, \frac{\bar{p}_2}{1-\rho}\} \underrightarrow{\rho \to 0^+} p_2^F, \\
S_{R,2}^{Censor,O} \quad &: \quad = \max\left\{\begin{array}{c} \min\{\frac{(\bar{p}_2-\underline{p}_2)(1+\delta)}{(1+\delta p_1)(1-\rho)}+\underline{p}_2, \left(\frac{1+\delta}{1-\rho}-\delta p_1\right)(p_1-\underline{p}_2)+\underline{p}_2, 1\}, \\ \min\left\{\frac{\left(\frac{1+\delta}{1-\rho}-\delta p_1-1\right)\underline{p}_2}{\delta(1-p_1)}, \frac{\left(\frac{1+\delta}{1-\rho}-\delta p_1\right)p_1}{1+\delta(1-p_1)}, \frac{\bar{p}_2}{1-\rho}\right\} \end{array}\right\} \\
&\underrightarrow{\rho \to 0^+} \min\{p_1\left[1+\delta\left(\bar{p}_2-p_1\right)\right], 1\}.
\end{aligned}$$

In a similar approach, we could translate the above expressions into intervals of ρ :

$$\begin{aligned}
\varrho_{R,1}^{MF} \quad &: \quad = 1-\min\left\{\frac{\left(\bar{p}_2-p_2^F\right)(1+\delta)}{\left(s-p_2^F\right)\delta p_1}, 1\right\} \\
\varrho_{R,2}^{MF} \quad &: \quad = 1-\max\left\{\frac{1+\delta}{\delta p_1+\frac{s[1+\delta(1-p_1)]}{p_2^F}}, \min\left\{\frac{1+\delta}{\delta p_1+\frac{s[1+\delta(1-p_1)]}{p_1}}, \frac{\bar{p}_2}{s}, 1\right\}\right\}
\end{aligned}$$

$$\begin{aligned}
\varrho_{R,1}^{Cen,O} \quad &: \quad = 1-\max\left\{\min\left\{\frac{\bar{p}_2}{s}, \frac{1+\delta}{\frac{s\delta(1-p_1)}{\underline{p}_2}+1+\delta p_1}\right\}, \min\left\{\frac{1+\delta}{\delta p_1+\frac{s[1+\delta(1-p_1)]}{p_1}}, \frac{1+\delta}{\delta p_1+\frac{s-\underline{p}_2}{p_1-\underline{p}_2}}\right\}\right\} \\
\varrho_{R,2}^{Cen,O} \quad &: \quad = 1-\min\left\{\frac{1+\delta}{\delta p_1+\frac{s[\delta(1-p_1)+1]}{p_2^F}}, \frac{\bar{p}_2}{s}, 1\right\}
\end{aligned}$$

■

Proof of Lemma 2 and Corollary 1.

Clearly the boundaries of the $s-$intervals $\left\{S_{R,1}^{MF}\left(\theta_{-s}\right), S_{R,2}^{MF}\left(\theta_{-s}\right), S_{R,1}^{Cen,O}\left(\theta_{-s}\right), S_{R,2}^{Cen,O}\left(\theta_{-s}\right)\right\}$ all increase with ρ, for their dependence of the $1-\rho$ factor. So the lowest possible s points for the two phenomena are $S_{R,1}^{MF}\left(\rho=0^{+}\right)=p_1$ and $S_{R,1}^{Cen,O}\left(\rho=0^{+}\right)=p_2^F$ respectively. Similarly we could tell that the boundaries of the $\rho-$intervals increase with s as well, so the rightmost possible ρ points for the two phenomena are $\rho_{R,2}^{MF}\left(s_R^{MF*}\right)$ and $\rho_{R,2}^{Cen,O}\left(s_R^{Cen,O*}\right)$. So it suffices to find the highest possible s_R^{MF*} satisfying the following conditions:

$$\begin{aligned}
s^* &= \max s \in (0,1] \\
&\quad s.t. \min\left\{\frac{\left(\bar{p}_2-p_2^F\right)(1+\delta)}{\left(s-p_2^F\right)\delta p_1}, 1\right\} \geq \\
&\quad \max\left\{\frac{1+\delta}{\delta p_1+\frac{s[1+\delta(1-p_1)]}{p_2^F}}, \min\left\{\frac{1+\delta}{\delta p_1+\frac{s[1+\delta(1-p_1)]}{p_1}}, \frac{\bar{p}_2}{s}, 1\right\}\right\}
\end{aligned}$$

We could show that the inequality holds iff. $\begin{cases} \frac{\left[\bar{p}_2-p_2^F\right](1+\delta)}{\left[s-p_2^F\right]\delta p_1} > \frac{1+\delta}{\delta p_1+\frac{s[1+\delta(1-p_1)]}{p_2^F}} \iff s \geq \bar{p}_2 \\ 1 > \frac{1+\delta}{\delta p_1+\frac{s[1+\delta(1-p_1)]}{p_2^F}} \iff s \geq p_2^F \\ \frac{\left[\bar{p}_2-p_2^F\right](1+\delta)}{\left[s-p_2^F\right]\delta p_1} > \min\left\{\frac{1+\delta}{\delta p_1+\frac{s[1+\delta(1-p_1)]}{p_1}}, \frac{\bar{p}_2}{s}, 1\right\} \end{cases}$, and the last inequality holds iff anyone of the following holds:

$$\begin{aligned}
& s\left[1-\tfrac{1+\delta(1-p_1)}{\delta p_1^2}\left(\bar{p}_2-p_2^F\right)\right] \leq \bar{p}_2 \\
& s\left[\delta p_1\bar{p}_2-\left(\bar{p}_2-p_2^F\right)(1+\delta)\right] \leq \delta p_1\bar{p}_2 p_2^F \\
& s \leq \tfrac{\left[\bar{p}_2-p_2^F\right](1+\delta)}{\delta p_1}+p_2^F \\
\iff \quad & s \leq \max\left\{\frac{\bar{p}_2}{1-\frac{1+\delta(1-p_1)}{\delta p_1^2}\left(\bar{p}_2-p_2^F\right)}, \frac{\delta p_1\bar{p}_2 p_2^F}{\delta p_1\bar{p}_2-\left(\bar{p}_2-p_2^F\right)(1+\delta)}, \frac{\left[\bar{p}_2-p_2^F\right](1+\delta)}{\delta p_1}+p_2^F\right\} \\
\iff \quad & s \leq \frac{\delta p_1\bar{p}_2 p_2^F}{\delta p_1\bar{p}_2-\left(\bar{p}_2-p_2^F\right)(1+\delta)}
\end{aligned}$$

so $s_R^{MF*}=\min\left\{\frac{\delta p_1\bar{p}_2 p_2^F}{\delta p_1\bar{p}_2-\left(\bar{p}_2-p_2^F\right)(1+\delta)}, 1\right\}$.We could also show that $s_R^{Cen,O*}=s_R^{MF*}:=s^*$. ■

Proof of Corollary 2.

1. The width of $S_R^{MF}(0^+)$ is

   $\min\left\{\frac{\left(\bar{p}_2 - p_2^F\right)(1+\delta)}{\delta p_1} + p_2^F, 1\right\} - p_1 =$

   $$\begin{cases} 1 - p_1 & \frac{\left(\bar{p}_2 - p_2^F\right)(1+\delta)}{\delta p_1} + p_2^F > 1 \iff I < \frac{1+\delta}{\delta p_1} \\ \frac{1-p_1}{I+1}\left(1 + \frac{1+\delta}{\delta p_1}\right) & o.w. \end{cases}.$$

   So it decreases with $I \geq \frac{1+\delta}{\delta p_1}$ and remains constant for $I < \frac{1+\delta}{\delta p_1}$; and decreases with p_1.

2. The width of $S_R^{Cen,O}(0^+)$ is

   $p_1\left[1 + \delta\left(\bar{p}_2 - p_1\right)\right] - p_2^F = \frac{\delta p_1(1-p_1)}{I+1}\frac{2+\delta(1-p_1)}{1+\delta(1-p_1)}$; so it decreases with I.

■

Study for the Supply Chain's Two Phenomena and the Supplier's Profits

Proof of Proposition 7. Similar to the studies of the retailer, based on conditions for the two phenomena of the supply chain's profits

$$\begin{aligned} S^{MF} &: = S_1^M \cap S_2^F \\ S^{Censor,O} &: = \left(S_2^F \cup S_1^F\right) \cap \left(S_3^O \cup S_4^O\right) \setminus \left(S_2^F \cap S_3^O \cap S_3^{OC}\right), \end{aligned}$$

we could characterize these subsets through their boundaries:

$$\begin{aligned} S_1^{MF} &: = \max\left\{\frac{\left(\frac{1+\delta}{1-\rho} - \delta p_1\right)p_2^F}{\delta(1-p_1)+1}, \frac{(\bar{p}_2 - p_1)(1+\delta)}{\delta p_1(1-\rho)} + p_1, \frac{\bar{p}_2}{1-\rho}\right\} \\ &\xrightarrow{\rho \to 0^+} \min\left\{\frac{(\bar{p}_2 - p_1)(1+\delta)}{\delta p_1} + p_1, 1\right\}, \\ S_2^{MF} &: = \min\{\frac{\left(\bar{p}_2 - p_2^F\right)(1+\delta)}{\delta p_1(1-\rho)} + p_2^F, 1\}\xrightarrow{\rho \to 0^+} \min\{\frac{\left(\bar{p}_2 - p_2^F\right)(1+\delta)}{\delta p_1} + p_2^F, 1\} \end{aligned}$$

Note that $S_1^{MF} < S_2^{MF} \iff \frac{\left(\bar{p}_2 - p_2^F\right)(1+\delta)}{\delta p_1} + p_2^F < 1 \iff I > \frac{1+\delta}{\delta p_1} - 1.$

$$\begin{aligned} S_{R-,1}^{Censor,O} \quad &: \quad = \max\left\{\{\frac{\left(\frac{1+\delta}{1-\rho} - \delta p_1 - 1\right)\underline{p}_2}{\delta(1-p_1)}\}^+, \{\frac{\left(\frac{1+\delta}{1-\rho} - \delta p_1\right)p_2^F}{\delta(1-p_1)+1}\}^+\right\} \xrightarrow{\rho \to 0^+} p_2^F \\ S_{R-,2}^{Censor,O} := p_1 \quad &, \end{aligned}$$

So

$$S^{Censor,O} = \left(S_{R,1}^{Censor,O}(\theta_{-s}), S_{R,2}^{Censor,O}(\theta_{-s})\right] \setminus \left(S_{R-,1}^{Censor,O}(\theta_{-s}), S_{R-,2}^{Censor,O}(\theta_{-s})\right].$$

We conjecture that:

Conjecture 1 *when* $\left(S_{R-,1}^{Cen,O}(\theta_{-s}), S_{R-,2}^{Cen,O}(\theta_{-s})\right] \neq \emptyset, S_{R,1}^{Cen,O}(\theta_{-s}) = S_{R-,1}^{Cen,O}(\theta_{-s})$; *so*

$$\begin{aligned} S^{Cen,O} \quad &= \quad \left(\max\left\{S_{R,1}^{Cen,O}(\theta_{-s}), S_{R-,2}^{Cen,O}(\theta_{-s})\right\}, S_{R,2}^{Cen,O}(\theta_{-s})\right] \\ &= \quad \left(\max\left\{\min\{\{\frac{\left(\frac{1+\delta}{1-\rho} - \delta p_1\right)p_2^F}{\delta(1-p_1)+1}\}^+, \frac{\bar{p}_2}{1-\rho}\}, p_1\right\}, S_{R,2}^{Cen,O}(\theta_{-s})\right] \\ &: \quad = \left(S_1^{Cen,O}(\theta_{-s}), S_2^{Cen,O}(\theta_{-s})\right] \end{aligned}$$

is also an interval.

Similarly the intervals of the $\rho-$axis

$$\begin{aligned} \varrho_1^{MF} \quad &: \quad = 1 - \min\left\{\frac{\left(\bar{p}_2 - p_2^F\right)(1+\delta)}{\left(s - p_2^F\right)\delta p_1}, 1\right\}, \\ \varrho_2^{MF} \quad &: \quad = 1 - \max\left\{\frac{(1+\delta)(\bar{p}_2 - p_1)}{\delta p_1 (s - p_1)}, \frac{\bar{p}_2}{s}, \frac{1+\delta}{\delta p_1 + \frac{s[1+\delta(1-p_1)]}{p_1}}\right\} \end{aligned}$$

$$\varrho_1^{Cen,O} \quad : \quad = 1 - \max\left\{ \min\left\{ \frac{\bar{p}_2}{s}, \frac{1+\delta}{\frac{s\delta(1-p_1)}{\underline{p}_2} + 1 + \delta p_1} \right\}, \min\left\{ \frac{1+\delta}{\delta p_1 + \frac{s[1+\delta(1-p_1)]}{p_1}}, \frac{1+\delta}{\delta p_1 + \frac{s-\underline{p}_2}{p_1-\underline{p}_2}} \right\} \right\}$$

$$\varrho_2^{Cen,O} \quad : \quad = 1 - \min\left\{ \max\left\{ \min\left\{ \frac{1+\delta}{\delta p_1 + \frac{s[\delta(1-p_1)+1]}{p_2^F}}, \frac{\bar{p}_2}{s} \right\}, \frac{p_1}{s} \right\}, 1 \right\}$$

■

Proof of Lemma 4 and Corollary 3. Clearly the boundaries of the $s-$intervals $\left\{ S_1^{MF}(\theta_{-s}), S_2^{MF}(\theta_{-s}), S_1^{Cen,O}(\theta_{-s}), \right.$ all increase with ρ, for their dependence of the $1-\rho$ factor. So the lowest possible s points for the two phenomena are $S_1^{MF}(0^+) = \min\left\{ \frac{(\bar{p}_2-p_1)(1+\delta)}{\delta p_1} + p_1, 1 \right\}$ and $S_1^{Cen,O}(0^+) = p_1$ respectively. We could easily show that $s^{Cen,O*} = s^{MF*} := s^*$ as $\rho \to 0^+$. ■

Proof of Lemma 5. Note that in the expressions of the width and efficiency loss, the change of prior information $\{\alpha, \beta\}$ impact p_1 and $\bar{p}_2$.Since

$$\bar{p}_2 = \frac{1}{I+1} + p_1 \cdot \left(1 - \frac{1}{I+1}\right)$$

so we could delineate the impact of p_1 and I through the following transformations:

$$\begin{aligned}
1 - \bar{p}_2 &= (1-p_1) \cdot \left(1 - \frac{1}{I+1}\right) \\
p_1(\bar{p}_2 - p_1) &= \frac{p_1(1-p_1)}{I+1} \\
\frac{(\bar{p}_2 - p_1)[1+\delta(1-p_1)]}{p_1} &= \frac{(1-p_1)[1+\delta(1-p_1)]}{p_1(I+1)} \\
\frac{p_1(s-\bar{p}_2)}{(s-p_1)} &= p_1\left[1 - \frac{1-p_1}{(I+1)(s-p_1)}\right] \\
\frac{s-p_1}{(1+\delta)(s-p_1) - \delta p_1(s-\bar{p}_2)} &= \frac{1}{(1+\delta) - \delta p_1\left[1 - \frac{1-p_1}{(I+1)(s-p_1)}\right]}
\end{aligned}$$

1. Note that

$$1 - \bar{p}_2 < \frac{(\bar{p}_2 - p_1)[1+\delta(1-p_1)]}{\delta p_1} \iff \frac{1+\delta(1-p_1)}{\delta p_1} > I,$$

so when $I < \frac{1+\delta(1-p_1)}{\delta p_1}$, the width of Region 1 is $1-\bar{p}_2$ and increases with I, and when $I \geq$

$\frac{1+\delta(1-p_1)}{\delta p_1}$,the width of Region 1 is $\frac{(\bar{p}_2-p_1)[1+\delta(1-p_1)]}{\delta p_1}$ and decreases with I. It is easy to tell that the width of Region 2 $\delta p_1(\bar{p}_2-p_1)$ decreases with I, and the loss of efficiency in the two regions$\frac{\delta p_1(s-\bar{p}_2)}{(1+\delta)(s-p_1)}$ and $\frac{s-p_1}{(1+\delta)(s-p_1)-\delta p_1(s-\bar{p}_2)}$ increases with I.

2. It is easy to tell that $p_1(\bar{p}_2-p_1)$ is concave with p_1, and $\frac{(\bar{p}_2-p_1)[1+\delta(1-p_1)]}{p_1}$ decreases with p_1.We could also show that

$$\frac{d\left\{p_1\left[1-\frac{1-p_1}{(I+1)(s-p_1)}\right]\right\}}{dp_1}=\frac{1}{I+1}\left[I-\frac{s(1-s)}{(s-p_1)^2}\right]$$

so the efficiency loss in these two regions increases with $p_1\in\left(0,\left(s-\sqrt{\frac{s(1-s)}{I}}\right)^+\right]\cup\left(\min\left\{s+\sqrt{\frac{s(1-s)}{I}},1\right\},1\right]$ and decreases with $p_1\in\left(\left(s-\sqrt{\frac{s(1-s)}{I}}\right)^+,\min\left\{s+\sqrt{\frac{s(1-s)}{I}},1\right\}\right]$.

■

The proof of Corollary **??** could be easily derived based on the results of Lemma 5.

Proof of Lemma 3.

1. To compare the supplier's profit in "M" and "O", we first notice that the supply chain performance is different in these two situations only when $\theta\in\Theta_3^O=\left(\Theta_1^M\backslash\Theta_1^O\right)\cup\left(\Theta_2^M\backslash\Theta_2^O\right)\cup\left(\Theta_4^M\backslash\Theta_4^O\right)$ as shown by Lemma 10. $\forall\theta\in\Theta_3^O\cap\Theta_i^M, i\in\{1,2,4\}, V_S^M(\theta)=V_{S,i}^M=V_{S,i}^O<V_{S,3}^O=V_S^O(\theta)$.Note that the last inequality is due to the fact that the supplier chooses Case 3 in "O". So $\forall\theta\in\Theta, V_S^M(\theta)\leq V_S^O(\theta)$. Similarly we could verify the result on comparison between "M" and "F". In general, the supplier is worse off in "M" compared to "O" and "F" since there are fewer options due to the removing of Case 3.

2. To compare the supplier's profits in "F" and "O", we first claim that $V_S^F(\theta)>V_S^O(\theta)\Rightarrow\theta\in\Theta_{2'}^F$

   (a) We prove the claim above by contradiction. Note that $\forall i\in\{1,4\}$, $\Theta_i^F\subset\Theta_3^O\cup\Theta_i^O$, o.w. we observe that $V_{S,i}^O<V_{S,2}^O<V_{S,2'}^F\Rightarrow\theta\in\Theta_{2'}^F$, or $V_{S,i}^O<V_{S,5-i}^O=V_{S,5-i}^F<V_{S,i}^F=V_{S,i}^O$, contradiction! If $\theta\in\Theta_i^F\cap\Theta_i^O, V_S^O(\theta)=V_{S,i}^O=V_{S,i}^F=V_S^F(\theta)$; if $\theta\in\Theta_i^F\cap\Theta_3^O, V_S^O(\theta)=V_{S,3}^O>V_{S,i}^O=V_{S,i}^F=V_S^F(\theta)$, so $V_S^O(\theta)\geq V_S^F(\theta)$.

(b) Then we claim that $\forall\theta \in \Theta^F_{2'} \cap \Theta^O_i, i \in \{1,2,4\}, V^F_S(\theta) > V^O_S(\theta)$, due to the fact that $V^F_S(\theta) = V^F_{S,2'} > V^F_{S,i} = V^O_{S,i} = V^O_S(\theta)$. For $\forall\theta \in \Theta^F_{2'} \cap \Theta^O_3, V^F_S(\theta) = V^F_{S,2'} > V^O_{S,3} = V^O_S(\theta) \iff$

$$[(1+\delta) - \delta p_1(1-\rho)] \cdot \left(s - p^F_2\right) > [\rho(1+\delta) + \delta(1-p_1)(1-\rho)] \cdot (s - \underline{p}_2)$$

$$\iff s > p_1 + \frac{(1+\delta)\rho\left(p_1 - \underline{p}_2\right)}{[1+\delta(1-p_1)](1-\rho)}$$

■

1.4 Discussion and Concluding Remarks

Proof of Proposition 1. We do not need to consider the cases of $s > \frac{\overline{p}_2}{1-\rho}$ or $s < \frac{\underline{p}_2}{1-\rho}$, as the retailer's expected profit of the second period is not affected by the information updating. So we focus on the cases of $s \in \left(\frac{p_1}{1-\rho}, \frac{\overline{p}_2}{1-\rho}\right]$ and $s \in \left(\frac{\underline{p}_2}{1-\rho}, \frac{p_1}{1-\rho}\right]$, and study the two players' policies to constitute a sequential equilibrium. Note that for the second period, the two players' optimal policies remain the same as in the other two situations: $\begin{cases} s > \frac{p_2}{1-\rho}, & k\left(\widetilde{w}^F_2\right) = p^+_2, \widetilde{y}^F_2 = D^H \\ s \le \frac{p_2}{1-\rho}, & k\left(\widetilde{w}^F_2\right) = 0^+, \widetilde{y}^F_2 = D^L \end{cases}$. Then for each case, we develop the game tree of the two players' first period decisions.

1. When $s \in \left(\frac{p_1}{1-\rho}, \frac{\overline{p}_2}{1-\rho}\right]$, the two players' policies and the outcomes are

$k\left(\widetilde{w}_1^F\right)$	$\widetilde{k}_1^{F+}$		
$\widetilde{y}_1^F$	D^L	D^H	
$\widetilde{V}_{R,1}^F$	$\widetilde{k}_1^F \rho$	$\widetilde{k}_1^F - (1-\rho)\, p_1$	
$\widetilde{V}_{S,1}^F$	$\left(s - \widetilde{k}_1^F\right)\rho$	$s - \widetilde{k}_1^F$	
D_1	$\left\{D^H, D^L\right\}$	$P\left(D_1 = D^H\right) = 1 - p_1$	$P\left(D_1 = D^L\right) = p_1$
p_2	p_1	$\underline{p}_2$	$\overline{p}_2$
$k\left(\widetilde{w}_2^F\right)$	p_1^+	$\underline{p}_2^+$	0^+
$\widetilde{y}_2^F$	D^H	D^H	D^L
$\widetilde{V}_{R,2}^F$	$p_1 \rho$	$\underline{p}_2 \rho$	0
$\widetilde{V}_{S,2}^F$	$s - p_1$	$s - \underline{p}_2$	$s\rho$

So for the forward-looking retailer to order $\widetilde{y}_1^F = D^L$, it requires

$$
\begin{aligned}
\widetilde{V}_R^F\left(\widetilde{y}_1^F = D^L\right) &> \widetilde{V}_R^F\left(\widetilde{y}_1^F = D^H\right) \\
\widetilde{k}_1^F \rho + \delta p_1 \rho &> \widetilde{k}_1^F - (1-\rho)\, p_1 + \delta\,(1-p_1)\, \underline{p}_2 \rho \\
\widetilde{k}_1^F &< p_1 \left(1 + \frac{\delta \rho \overline{p}_2}{1-\rho}\right).
\end{aligned}
$$

Therefore the supplier's wholesale price choice is between the following two options

$$
\begin{array}{cc}
k\left(\widetilde{w}_2^F\right) = p_1 \left(1 + \frac{\delta\rho\overline{p}_2}{1-\rho}\right)^+, & \widetilde{y}_2^F = D^H \\
k\left(\widetilde{w}_2^F\right) = 0^+, & \widetilde{y}_2^F = D^L
\end{array},
$$

and she chooses $k\left(\widetilde{w}_2^F\right)=p_1\left(1+\frac{\delta\rho\overline{p}_2}{1-\rho}\right)$ when $\widetilde{V}_S^F\left(k\left(\widetilde{w}_2^F\right)=p_1\left(1+\frac{\delta\rho\overline{p}_2}{1-\rho}\right)\right)>\widetilde{V}_S^F\left(k\left(\widetilde{w}_2^F\right)=0^+\right)$

$$\begin{aligned} & \left(s-p_1\left(1+\frac{\delta\rho\overline{p}_2}{1-\rho}\right)\right)+\delta\left[p_1 s\rho+(1-p_1)\left(s-\underline{p}_2\right)\right] \\ > & \left(s-p_1\left(1+\frac{\delta\rho\overline{p}_2}{1-\rho}\right)\right)\rho+\delta\left(s-p_1\right) \\ s > & \widetilde{s}_1^F := \frac{p_1}{1-\rho}\cdot\frac{1+\frac{\delta\overline{p}_2(2\rho-1)}{1-\rho}}{1-\delta p_1} \end{aligned}$$

2. When $s\in\left(\frac{p_2}{1-\rho}\frac{p_1}{1-\rho}\right]$, the two players' policies and the outcomes are

$k\left(\widetilde{w}_1^F\right)$	$\widetilde{k}_1^{F+}$		
$\widetilde{y}_1^F$	D^L	D^H	
$\widetilde{V}_{R,1}^F$	$\widetilde{k}_1^F\rho$	$\widetilde{k}_1^F-(1-\rho)\,p_1$	
$\widetilde{V}_{S,1}^F$	$\left(s-\widetilde{k}_1^F\right)\rho$	$s-\widetilde{k}_1^F$	
D_1	$\left\{D^H,D^L\right\}$	$P\left(D_1=D^H\right)=1-p_1$	$P\left(D_1=D^L\right)=p_1$
p_2	p_1	$\underline{p}_2$	$\overline{p}_2$
$k\left(\widetilde{w}_2^F\right)$	0^+	$\underline{p}_2^+$	0^+
$\widetilde{y}_2^F$	D^L	D^H	D^L
$\widetilde{V}_{R,2}^F$	0	$\underline{p}_2\rho$	0
$\widetilde{V}_{S,2}^F$	$s\rho$	$s-\underline{p}_2$	$s\rho$

So for the forward-looking retailer to order $\widetilde{y}_1^F=D^L$, it requires

$$\begin{aligned} \widetilde{V}_R^F\left(\widetilde{y}_1^F=D^L\right) &> \widetilde{V}_R^F\left(\widetilde{y}_1^F=D^H\right) \\ \widetilde{k}_1^F\rho &> \widetilde{k}_1^F-(1-\rho)\,p_1+\delta\,(1-p_1)\,\underline{p}_2\rho \\ \widetilde{k}_1^F &< p_1\left\{1-\frac{\delta\rho\,(1-\overline{p}_2)}{1-\rho}\right\}^+. \end{aligned}$$

Therefore the supplier's wholesale price choice is between the following two options

$$\begin{array}{cc} k\left(\widetilde{w}_2^F\right) = p_1\left(\left\{1-\frac{\delta\rho(1-\overline{p}_2)}{1-\rho}\right\}^+\right)^+, & \widetilde{y}_2^F = D^H \\ k\left(\widetilde{w}_2^F\right) = 0^+, & \widetilde{y}_2^F = D^L \end{array},$$

and she chooses $k\left(\widetilde{w}_2^F\right) = p_1\left(\left\{1-\frac{\delta\rho(1-\overline{p}_2)}{1-\rho}\right\}^+\right)^+$ when

$$\widetilde{V}_S^F\left(k\left(\widetilde{w}_2^F\right) = p_1\left(\left\{1-\frac{\delta\rho\left(1-\overline{p}_2\right)}{1-\rho}\right\}^+\right)^+\right) > \widetilde{V}_S^F\left(k\left(\widetilde{w}_2^F\right) = 0^+\right)$$

$$\begin{aligned} & \left(s - p_1\left(\left\{1-\frac{\delta\rho\left(1-\overline{p}_2\right)}{1-\rho}\right\}^+\right)^+\right) + \delta\left[p_1 s\rho + (1-p_1)\left(s-\underline{p}_2\right)\right] \\ > \quad & \left(s - p_1\left(\left\{1-\frac{\delta\rho\left(1-\overline{p}_2\right)}{1-\rho}\right\}^+\right)^+\right)\rho + \delta s\rho \\ s \quad > \quad & \widetilde{s}_2^F := \frac{p_1}{1-\rho}\cdot\frac{1+\frac{\delta(1-\overline{p}_2)(1-2\rho)}{1-\rho}}{1+\delta\left(1-p_1\right)} \end{aligned}$$

■

Proof of Proposition 9. We prove the result by backward induction.

We first argue that when $n = N$ and with the same prior π'_N,

$$\rho_N\left(\pi'_N, y_N\right) = \omega_N\left(\pi'_N, y_N\right) = \frac{\omega_N^R\left(\pi'_N, y_N\right)}{\zeta} = \frac{\rho_N^R\left(\pi'_N, y_N\right)}{\zeta}$$

for any y_N, so the contract (w, b) could coordinate the supply chain based on known results, yielding $y_N\left(\pi'_N\right) = y_N^R\left(\pi'_N\right), \rho_N\left(\pi'_N\right) = \rho_N^R\left(\pi'_N\right).$

If for all $n > m \geq 1$, we have $y_n\left(\pi'_n\right) = y_n^R\left(\pi'_n\right)$ and thus $\rho_n\left(\pi'_n\right) = \rho_n^R\left(\pi'_n\right)$, then for $n = m$,we

Table 6: Supply Chain Efficiency Comparison: "M" vs "F"

Supply Chain Efficiency Comparison: "M" vs "F", $\left\{\sigma^{MF}, \zeta^{MF}(p_1, I)\,|\rho\right\}$

I	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
1	0	0	0	0	0
5	3.5%, 15.9%	1.25%,11.74%	0	0	0
10	2.5%, 14.2%	2.25%, 9.55%	1.5%, 6.71%	0	0
50	0.75%,8.13%	0.5%, 5.2%	0.75%, 3.65%	1.25%, 2.32%	0
p_1	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
0.2	0.5%, 7.43%	0.5%, 5.20%	0.75%, 3.65%	1.25%, 2.32%	0
0.4	3%, 13.33%	2.25%, 9.55%	1.5%, 6.71%	0	0
0.6	3%, 16.32%	1.25%, 11.74%	0	0	0
0.8	0.25%, 16.57%	0	0	0	0

know that for any y_m and π'_m, and $\forall x$:

$$\frac{\omega_m^R\left(\pi'_m, y_m\right)}{\omega_m\left(\pi'_m, y_m\right)} = \frac{\rho_{m+1}^R\left(\pi'_{m+1}\left(\cdot|x\right)\right)}{\rho_{m+1}\left(\pi'_{m+1}\left(\cdot|x\right)\right)} = \zeta,$$

therefore $\frac{\rho_m^R\left(\pi'_m, y_m\right)}{\rho_m\left(\pi'_m, y_m\right)} = \zeta$. If $y_m\left(\pi'_m\right)$ is the optimal order quantity for the supply chain, then it's also optimal for the retailer.

Note that if the retailer's and the supply chain's order quantities are the same through all past periods, they also hold the same (expected) posterior depending on the realized sales volume. Therefore we proved the proposition. ■

1.5 Complementary Numerical Results

Basic Model

2 Chapter 3

2.1 Variation of the Spence's Signaling Game and Strategic Learning

We discuss a variation of the classic "Spence's job market signaling" game[67], in which the agent faces several sources of uncertainty that jointly influence the cost of signaling, while the principal's

Table 7: Supply Chain Efficiency Comparison: Censored vs. "O"

Supply Chain Efficiency Comparison: Censored vs. "O", $\sigma^{Censor,O}(p_1, I)\,|\rho$

I	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
1	13.5%, 22.52%	12.25%, 24.83%	8.25%, 24.41%	4.75%, 19.07%	0
5	2.25%, 43.76%	2.25%, 40.56%	2%, 31.93%	1.25%, 22.01%	0
10	1%, 49.33%	0.75%, 40.7%	1.25%, 31.62%	0.75%, 21.87%	0
50	0	0	0.25%, 32.31%	0	0
p_1	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
0.2	3.5%, 25.43%	4.25%, 22.19%	5.25%, 23.46%	6.75%, 19.92%	0
0.4	5%, 27.43%	6.5%, 24.94%	6.5%, 29.18%	0	0
0.6	5.75%, 27.57%	4.5%, 37.68%	0	0	0
0.8	2.5%, 26.88%	0	0	0	0

Table 8: Retailer's Profit Comparison: "M" vs "F"

Retailer's Profit Comparison: "M" vs "F", $\sigma_R^{MF}(p_1, I)\,|\rho$

I	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
1	42.5%, 65.61%	28.5%, 43.94%	17.25%, 34.25%	7.25%, 28.36%	0
5	39.25%, 40.07%	28.75%, 19.04%	17.25%, 12.8%	7.25%, 9.98%	0
10	29.25%, 28.2%	24.25%, 13.18%	17.25%, 8.76%	7.25%, 5.51%	0
50	4.75%, 12.05%	3.75%, 5.15%	3%, 4.81%	1.5%, 9.25%	0
p_1	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
0.2	62.75%, 44.94%	57%, 24.75%	46.5%, 17.71%	23.25%, 14.27%	0
0.4	35.75%, 46.69%	26%, 25.36%	8.25%, 18.64$	0	0
0.6	15.5%, 44.59%	2.25%, 30.5%	0	0	0
0.8	1.75%, 36.01%	0	0	0	0

Table 9: Retailer's Profit Comparison: Censored vs "O"

Retailer's Profit Comparison: Censored vs "O", $\sigma_R^{Censor,O}(p_1, I)\,|\rho$

I	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
1	15.25%, 58.75%	2.25%, 61.62%	8.25%, 71.82%	4.75%, 82.58%	0
5	2.25%, 84.21%	2.25%, 100%	2%, 100%	1.25%, 100%	0
10	1%, 100%	0.75%, 100%	1.25%, 100%	0.75%, 100%	0
50	0	0	0.25%, 100%	0	0
p_1	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
0.2	4.25%, 64.33%	4.25%, 59.49%	5.25%, 68.65%	6.75%, 87.74%	0
0.4	5.75%, 68.42%	6.5%, 60.20%	6.5%, 89.55%	0	0
0.6	6%, 65.83%	4.5%, 91.27%	0	0	0
0.8	2.5%, 49.47%	0	0	0	0

payoff only depends on one dimension of the private information. This setup is related but different from "noisy signaling game"[70]. In the latter case, the signal observed by the principal besides the deterministic component affected by the agent's action, also adding complexity to the principal's inference. We demonstrate that as the information structure gets more complex, the equilibrium belief contains little information. The principal's ability to infer the only "valuable" private information amid other uncertainty is limited.

The classic Spence's job market game considers an agent (student, he) of two possible types ("quality") $t \in \{1, 2\}$. The agent's type is valuable for the principal (employer, she), who only has a prior distribution over the agent's type $P(t = 1) := p$.The agent could exert an observable effort ("education") $e \in [0, +\infty)$ as a signal of his type that is less costly for the agent of high type. And the principal offers a monetary compensation $w(\cdot) \in [0, +\infty)$ depending on the observed e.The utility functions of the players are as follows, and both have a zero reserved-value for participation.

$$\begin{aligned} U_P &= t - w(e) \\ U_A &= w(e) - \frac{e}{t}. \end{aligned}$$

Among the multiple sequential equilibria for this game, the Riley equilibrium[68] is the known as the "least costly separating equilibrium" and the unique equilibrium that survives the Intuitive Criteria [69] among other refine schemes[26].

$$\begin{aligned} \widetilde{p} &= P(t = 1|e) = \begin{cases} 0 & e = 1 \\ 1 & o.w. \end{cases} \\ e &= \begin{cases} 0 & t = 1 \\ 1 & t = 2 \end{cases}, w = \begin{cases} 1 & e = 0 \\ 2 & e = 1 \end{cases}.^{27} \end{aligned}$$

[26] Equilibrium selection/refinement is beyond the scope of this report, since we aim to argue that in our setup, there does not exist any equilibrium that fully separate the types of agents along the dimension of the valuable information for the principal.

Spence's Job Market Game, Variation 1

In this variation, we assume that the agent's private information is two dimensional, $t = (t_1, t_2)$, $t_1 \in \{1, 2\}, t_2 \in \{-0.5, 0.5\}$,both unknown to the principal yet only t_1 affecting her payoff as in the classic setting. The principal has a prior over the agent's type:

$$P(t_1 = 1) := p \in (0,1), P(t_2 = -0.5) := \frac{1}{2} \in (0,1),$$

the two dimensions are independent[28].The agent's cost for education $e \in [0, +\infty)$ is jointly determined by $\{t_1, t_2\}$. Specifically, the two players' utility functions are:

$$\begin{aligned} U_P &= t_1 - w(e) \\ U_A &= w(e) - \frac{e}{t_1 + t_2}. \end{aligned}$$

We present the equilibria that survive the Intuitive Criterion (IC) for this variation as follows, among which the second is not a separating equilibrium w.r.t. t_1. Since education is equally costly for the agent of two sub-types: $\{1, 0.5\}$ and $\{2, -0.5\}$, they may choose the same action in the equilibrium with a contract based on the observable education level and remain unseparated. In this case, the equilibrium is a "separating equilibrium" w.r.t. $t = t_1 + t_2$. So for this example, the existence of multiple equilibria and hybrid equilibrium largely affect the principal's information acquisition.

Example 1 (Equilibrium 1)
$$\begin{array}{cccc} t = (t_1, t_2) & e & w & U_a \\ \left(1, \pm\frac{1}{2}\right) & 0 & 1 & 1 \\ \left(2, \pm\frac{1}{2}\right) & \frac{3}{2} & 2 & \begin{cases} 1 & t_2 = -\frac{1}{2} \\ \frac{7}{5} & t_2 = -\frac{1}{2} \end{cases} \end{array},$$

[28] Here the independence and correlation between the two dimensions are not essential on our argument and results, as long as $E[t|t_1]$ increase with t_1. Alternatively, we could assume that $P(t_2 = -0.5|t_1 = 1) := q_1, P(t_2 = -0.5|t_1 = 2) := q_2, q_i \in (0,1)$

$$\widetilde{p} = P\left(t_1 = 1|e\right) = \begin{cases} 1 & e \in \left[0, \frac{3}{2}\right) \\ 0 & e \in \left[\frac{3}{2}, +\infty\right) \end{cases}$$

Example 2 (Equilibrium 2)

$$\begin{array}{llll} t = (t_1, t_2) & e & w & U_a \\ \left(1, -\frac{1}{2}\right) & 0 & 1 & 1 \\ \left(1, \frac{1}{2}\right), \left(2, -\frac{1}{2}\right) & \frac{1-p}{2} & 2-p & \frac{5-2p}{3} \\ \left(2, \frac{1}{2}\right) & \frac{1+2p}{2} & 2 & \frac{9-2p}{2} \end{array},$$

$$\widetilde{p} = P\left(t_1 = 1|e\right) = \begin{cases} 1 & e \in \left[0, \frac{1-p}{2}\right) \\ p & e \in \left[\frac{1-p}{2}, \frac{1+2p}{2}\right) \\ 0 & e \in \left[\frac{1+2p}{2}, +\infty\right) \end{cases}$$

Spence's Job Market Game, Variation 2

Similarly, we consider another variation in which the agent's private information is $t = (t_1, t_2), t_1 \in \{1, 2\}, t_2 \in (0, +\infty)$. And the principal's prior distribution is $P(t_1 = 1) = p, t_2|t_1 \sim Exp\{1\}$. The two players' payoffs are

$$\begin{aligned} U_P &= t_1 - w \\ U_A &= w - \frac{e}{t_1 \cdot t_2}. \end{aligned}$$

We consider the following equilibrium: $(t_a = t_1 \cdot t_2 \in (0, +\infty))$

$$\begin{aligned} P\left[t_1 = 1|t_a\left(e\right)\right] &= \frac{pe^{-t_a}}{pe^{-t_a} + \frac{1}{2}\left(1-p\right)e^{-\frac{t_a}{2}}} = \frac{2p}{2p + \left(1-p\right)e^{\frac{t_a}{2}}} \in \left(0, \frac{2p}{1+p}\right) \\ w(t_a) &= E[t_1|t_a\left(e\right)] = 1 + \frac{(1-p)}{2pe^{-\frac{t_a}{2}} + (1-p)} \\ e(t_a) &= \int_0^{t_a} w'(s)sds. \end{aligned}$$

This equilibrium is "separating" w.r.t. $t_a = t_1 \cdot t_2 \in (0, +\infty)$, yet the principal's belief over t_1

remains ambiguous. As the extra uncertainties like t_2 in these variations get more complex, the extra information obtained from such signaling quickly diminishes.

Sketch Proof of Variation 1. Given any belief updating rule of the principal, $w = E[t_1|e]$ is the best response;

1. It's easy to show that the agent's utility function satisfies single-crossing property for both $t_a = t_1 + t_2$, and each dimension $t_i, i \in \{1,2\}$; thus the equilibrium effort $e(t)$ is non-decreasing w.r.t. $t_i, i \in \{1,2,a\}$;therefore w is also non-decreasing w.r.t. e.

2. (Contradiction) Suppose in one equilibrium, $e_2 < e_3$,then $w_2 \leq w_3$;since $e_1 \leq e_2, e_3 \leq e_4$,the principal could rationally assign $\widetilde{p}(e_1) = \widetilde{p}(e_2) = 1$, $\widetilde{p}(e_3) = \widetilde{p}(e_4) = 0$,and separate the agents w.r.t. t_1 :

$$\begin{aligned} w_1 &= w_2 = 1 = E[t_1|e_1 \text{ or } e_2] \\ w_3 &= w_4 = 2 = E[t_1|e_3 \text{ or } e_4] \end{aligned}$$

Then by equilibrium definition, if $(0 \leq) e_1 < e_2$, type 2 agent for sure has an incentive to send e_1 instead of e_2, so $e_1 = e_2 (= 0)$. Similarly, if $(0 \leq) e_3 < e_4$,then type 4 agent for sure has the incentive to send e_3 as opposed to e_4, so $e_3 = e_4 := e$. For type 2 and type 3 to be indifferent between mimicking each other and maintaining the status quo, we need

$$\begin{aligned} w_3 - e_3/1.5 &= w_1 - e_1/2.5 = 1 \\ e_3 &= 1.5 \end{aligned}$$

■

2.2 The IS's Pricing Policy with $\rho = 0, \sigma_a = \sigma_b$

Solution 14 (**$\{a, \beta\}$ with $\rho = 0, \sigma_a = \sigma_b$**)

$$\begin{cases} \alpha = \frac{2\sqrt{C^*\mu(1+p)+C^{*2}p^2}-(\mu+2C^*p)}{\sigma^2} > 0 \\ \beta = \left(1+p^2\right)\left[2\sqrt{C^*\mu\left(1+p\right)+C^{*2}p^2}-\left(\mu+2C^*p\right)\right]-\left(a-\mu\right)+p\left(b-\mu\right) \\ \beta' = \left[2\sqrt{C^*\mu\left(1+p\right)+C^{*2}p^2}-\left(\mu+2C^*p\right)\right]\left[2p-\frac{C^*\left(1+p^2\right)}{\sqrt{C^*\mu(1+p)+C^{*2}p^2}}\right]+\left(b-\mu\right) \\ = \left(\mu+2C^*p\right)2p\left[\sqrt{1+\frac{\mu(4C^*-\mu)}{(\mu+2C^*p)^2}}-1\right]\left[1-\frac{\frac{\frac{1}{p}+p}{\frac{\mu}{2C^*}+p}}{4\sqrt{1+\frac{\mu(4C^*-\mu)}{(\mu+2C^*p)^2}}}\right]+\left(b-\mu\right) \end{cases},$$

Proof of Lemma 7. Note that $1-e^{-\tau(p)}$ increases(decreases) in p if $\tau\left(p\right)$ increases(decreases) in p. Also $\left(p-c\right)\left(a-bp\right)$ increases in $\left[c, \frac{a+bc}{2b}\right]$, and decreases in $\left[\frac{a+bc}{2b}, \frac{a}{b}\right]$.Therefore if $\tau\left(p\right)$ increases in p, then $V^{IS} = \left(p-c\right)\left(a-bp\right)\left(1-e^{-\tau(p)}\right)$ increases in $\left[c, \frac{a+bc}{2b}\right]$, and $p^* \in \left[\frac{a+bc}{2b}, \frac{a}{b}\right] \geq p^M$.Similarly we argue that if $\tau\left(p\right)$ decreases in $p, p^* \in \left[c, \frac{a+bc}{2b}\right] \leq p^M$. ■

Proof of Lemma 8.

We aim to show $\beta^{''} \geq 0$ in the feasible set so that β is convex.

Define $x := \sqrt{p^2+\frac{\mu(1+p)}{C^*}} \in \left(p, p+2\right)$,then $\frac{\mu}{C^*} = \frac{x^2-p^2}{1+p}$, and

$$\begin{aligned} \kappa\left(x,p\right) &= \frac{2\sqrt{C^*\mu\left(1+p\right)+C^{*2}p^2}-\left(\mu+2C^*p\right)}{x} > 0 \\ \beta^{''} &= \kappa\left(x,p\right)\widetilde{\beta^{''}}\left(x,p\right), \end{aligned}$$

so it suffices to prove

$$\widetilde{\beta^{''}}\left(x,p\right) = x+\frac{1+p^2}{2x}+\frac{p\left(p+2\right)\left(1+p^2\right)}{4\left(1+p\right)x^2}+\frac{1+p^2}{4\left(1+p\right)}-2p \geq 0. \tag{14}$$

If $\frac{1+p^2}{4(1+p)}-2p = -\frac{7p^2+8p-1}{4(1+p)} \geq 0$, then the argument is proved; otherwise it suffices to show that $\forall x \in \left(p, p+2\right)$

$$x+\frac{1+p^2}{2x} \geq \frac{7p^2+8p-1}{4\left(1+p\right)}.$$

If $\exists x \in (p, p+2)$, s.t. $x + \frac{1+p^2}{2x} < \frac{7p^2+8p-1}{4(1+p)} \iff$

$$4(1+p)x^2 - \left(7p^2+8p-1\right)x + 2\left(1+p^2\right)(1+p) < 0.$$

Note that $\Delta = \left(7p^2+8p-1\right)^2 - 32(1+p)^2\left(1+p^2\right) < 0$ for $\forall 0 < p \leq 1.3$. So $\widetilde{\beta''}(x,p) > 0$ when $p \leq 1.3$. Otherwise we aim to prove the result by showing that

- $\widetilde{\beta''}(x,p)$ minimizes at $\widetilde{x}(p) \in (p, p+2)$
- $\widetilde{\beta''}(\widetilde{x}(p), p) > 0$.

Consider

$$\frac{\partial \widetilde{\beta''}(x,p)}{\partial x} = \frac{1}{x^3}\left[x^3 - \frac{1+p^2}{2}x - \frac{p(p+2)\left(1+p^2\right)}{2(1+p)}\right] \tag{15}$$

Since $\frac{\partial \widetilde{\beta''}(x,p)}{\partial x}x^3|_{x=p} < 0, \frac{\partial \widetilde{\beta''}(x,p)}{\partial x}x^3|_{x=p+2} > 0, \exists \widetilde{x}(p) \in (p, p+2)$ $s.t. \frac{\partial \widetilde{\beta''}(\widetilde{x},p)}{\partial x} = 0$. We solve for this unique $\widetilde{x}(p)$ in Solution 15, and $\frac{\partial \widetilde{\beta''}(\widetilde{x},p)}{\partial x} < (>)\, 0 \iff x < (>)\, \widetilde{x}$. So $\widetilde{\beta''}(x,p)$ decreases in $x \in (p, \widetilde{x}(p))$, and increases in $x \in (\widetilde{x}, p+2)$, minimizing at $x = \widetilde{x}(p)$.

To show that $\widetilde{\beta''}(\widetilde{x},p) \geq 0$, we notice that $\left(\widetilde{x}^2 - \frac{1+p^2}{2}\right)\widetilde{x} = \frac{p(p+2)\left(1+p^2\right)}{2(1+p)}$, then

$\widetilde{\beta''}(\widetilde{x},p) = \frac{3\widetilde{x}}{2} + \frac{1+p^2}{4\widetilde{x}} - \frac{7p^2+8p-1}{4(1+p)} > 0 \iff 6(1+p)\widetilde{x}^2 - \left(7p^2+8p-1\right)\widetilde{x} + \left(1+p^2\right)(1+p) > 0$

$\iff \widetilde{x} > \frac{\left(7p^2+8p-1\right)+\sqrt{\left(7p^2+8p-1\right)^2-24(1+p)^2\left(1+p^2\right)}}{12(1+p)} = \widetilde{\widetilde{x}}(p)$ [29]. $\widetilde{\widetilde{x}}(p) \leq p$ when $p \leq 2.4$; so we only need to consider the case when $\widetilde{\widetilde{x}}(p) > p > 2.4$, in this case we use the base of $p = \frac{1}{q}, q < 1$; in this case

$\frac{\left(7p^2+8p-1\right)+\sqrt{\left(7p^2+8p-1\right)^2-24(1+p)^2\left(1+p^2\right)}}{12(1+p)} \sim p\left(1 + \frac{1}{5p} - \frac{76}{125p^2} + o\left(\frac{1}{p^2}\right)\right)$

and $\widetilde{x} \sim p\left(1 + \frac{1}{5p} + \frac{19}{125p^2} + o\left(\frac{1}{p^2}\right)\right)$, so $\widetilde{x} > \frac{\left(7p^2+8p-1\right)+\sqrt{\left(7p^2+8p-1\right)^2-24(1+p)^2\left(1+p^2\right)}}{12(1+p)}$. ■

[29] when $p > 1.3$, $\left(7p^2+8p-1\right)^2 - 24(1+p)^2\left(1+p^2\right) > 0$

Solution 15 *Solving* $\widetilde{x}$ *for* $\frac{\partial \widetilde{\beta''}(x,p)}{\partial x} = 0:$

$$\begin{aligned}
a &= 1, b = 0, c = -\frac{1+p^2}{2}, d = -\frac{p(p+2)\left(p^2+1\right)}{2(p+1)} \\
A &= \frac{3\left(p^2+1\right)}{2}, B = \frac{9p(p+2)\left(p^2+1\right)}{2(p+1)}, C = \frac{\left(p^2+1\right)^2}{4} \\
\Delta &= \frac{3\left(p^2+1\right)^2}{4(p+1)^2}\left[27p^2(p+2)^2 - 2\left(p^2+1\right)(p+1)^2\right] \\
\widetilde{\Delta} &: = 27p^2(p+2)^2 - 2\left(p^2+1\right)(p+1)^2
\end{aligned}$$

We argue that when $p \geq 0.2, \widetilde{\Delta} > 0,$*since*

$$\begin{aligned}
\widetilde{\Delta}(0.2) &= 2.232 > 0, \widetilde{\Delta}'(0.2) = 50.88 > 0, \widetilde{\Delta}^{(2)}(0.2) = 344.8 > 0, \\
\widetilde{\Delta}^{(3)}(0.2) &= 744 > 0, \widetilde{\Delta}^{(4)} = 600 > 0.
\end{aligned}$$

In this case $Y_{1,2} = Ab + 3a\left(\frac{-B \pm \sqrt{B^2-4AC}}{2}\right)$*,the unique real solution is* $\widetilde{x} = \frac{-b-\left(\sqrt[3]{Y_1}+\sqrt[3]{Y_2}\right)}{3a}$.

Conjecture 2 *If* $\tau'(0) \geq 0$, *then* $\tau'(p) \geq 0$ *for all* $p \in \left[0, \frac{a}{b}\right]$.

Sketch Proof of Conjecture 2. $\frac{\tau'}{\tau} = \frac{\alpha'}{\alpha} + \frac{\beta'}{\sqrt{\beta^2 - 2\ln\delta}}$, $\frac{\alpha'}{\alpha} = \frac{-C^*}{\sqrt{C^*\mu(1+p)+C^{*2}p^2}} < 0$ and increases in p. If $\tau'(0) \geq 0$, then $\beta'(0) > 0$ and $\beta'(p) \geq 0$ for all $p \in \left[c, \frac{a}{b}\right]$ as β is convex in p; also β increases in $\left[c, \frac{a}{b}\right]$ and positive.

If $\beta(0) < 0$, then $\exists \widetilde{p} \in \left(c, \frac{a}{b}\right], s.t. \forall p \leq \widetilde{p}, \beta(p) \leq 0$. Then $\beta^2 - 2\ln\delta$ decreases for $p \in [0, \widetilde{p}]$ and $\frac{\beta'}{\sqrt{\beta^2-2\ln\delta}}$ increases, so $\frac{\tau'}{\tau}$ increases in $[0, \widetilde{p}]$ and is positive.

When $\beta(0) \geq 0$ or $p \in \left(\widetilde{p}, \frac{a}{b}\right]$, $\frac{\beta'}{\sqrt{\beta^2-2\ln\delta}}$ may not increase anymore.

Taylor expansion[30] shows that

$$\frac{\tau'}{\tau} \sim \frac{2}{p^2}\left[\frac{\frac{\mu^2}{2C^*}\left(1-\frac{\mu}{4C^*}\right)+(a-\mu)}{b-\frac{\mu^2}{4C^*}} + \frac{\mu}{4C^*}\right] + O\left(\frac{1}{p^3}\right)$$

decreases in p and approaches 0^+. so $\tau' \geq 0$ when $\beta \geq 0$. ■

[30] This approach is not fully rigorous so the conjecture is not completely proved.

Proof of Proposition 10. Based on Conjecture 2, it's sufficient if $\tau'(0) \geq 0 \iff \frac{\tau'}{\tau}(0) \geq 0$

$$\begin{aligned}
\frac{\tau'}{\tau}|_{p=0} &= \frac{\alpha'}{\alpha}|_{p=0} + \frac{\beta'}{\sqrt{\beta^2 - 2\ln\delta}}|_{p=0} \\
&= -\frac{1}{2}\sqrt{\frac{C^*}{\mu}} + \frac{\frac{\sqrt{C^*\mu}}{2} - \frac{C^*}{2} + (b-\mu)}{\sqrt{\left(\sqrt{C^*\mu} - a\right)^2 - 2\ln\delta}} \geq 0
\end{aligned}$$

Then $b \geq \bar{b}(a) := \mu + \frac{1}{2}\left(C^* - \sqrt{C^*\mu} + \sqrt{\frac{C^*}{\mu}\left[\left(\sqrt{C^*\mu} - a\right)^2 - 2\ln\delta\right]}\right)$.

Since $\frac{\alpha'}{\alpha} < 0$, it suffices for τ to decrease if $\beta' < 0$. As β is convex, it suffices for $\beta'(p) \geq 0$ for $\forall p \in \left(0, \frac{a}{b}\right)$ if $\beta'\left(\frac{a}{b}\right) < 0$ or $\lim_{p\to+\infty}\beta'(p) := \beta'(\infty) < 0$.

$$\begin{aligned}
\lim_{p\to+\infty}\beta'(p) \quad &: \quad = \beta'(\infty) = b - \frac{\mu^2}{4C^*} < 0 \\
&\iff \quad b \leq \frac{\mu^2}{4C^*} := \underline{b}
\end{aligned}$$

■

Proof of $V^{IS\prime}s$ Unimodality in One Special Case. When $C^* = \frac{\mu}{4}(1+\varepsilon)$, we consider the deterministic approximation of the function $\widehat{\tau}$, where $\widehat{\tau}$ is finite only when $\beta < 0$ and $\widehat{\tau} \sim \frac{\alpha}{-\beta}$. In this case, $F(p) = \frac{[(a-bp)-\mu(1-p)]\left[2\sqrt{C^*\mu(1+p)+C^{*2}p^2}+(\mu+2C^*p)\right]}{\mu(4C^*-\mu)} - \left(1+p^2\right)$

$\sim [(a-bp)-\mu(1-p)]\frac{\left[2\sqrt{C^*\mu(1+p)+C^{*2}p^2}+(\mu+2C^*p)\right]}{\mu(4C^*-\mu)} - \left(1+p^2\right)$

$\frac{\left[2\sqrt{C^*\mu(1+p)+C^{*2}p^2}+(\mu+2C^*p)\right]}{\mu(4C^*-\mu)} \sim \frac{\sqrt{4(1+p)(1+\varepsilon)+p^2(1+\varepsilon)^2}+[2+(1+\varepsilon)p]}{2\mu\varepsilon} \sim \frac{p+2+p\varepsilon+\frac{\varepsilon}{p+2}}{\mu\varepsilon}$

$\widehat{\tau} \sim \frac{-\ln\delta\mu\varepsilon}{\sigma^2}\frac{1}{\left(p+2+p\varepsilon+\frac{\varepsilon}{p+2}\right)[(a-bp)-\mu(1-p)]-\mu(1+p^2)\varepsilon} \sim \frac{-\ln\delta\mu\varepsilon}{\sigma^2}\frac{1}{(p+2)[(a-bp)-\mu(1-p)]}$

$1-e^{-\widehat{\tau}} \sim \widehat{\tau} \sim \frac{-\ln\delta\mu\varepsilon}{\sigma^2}\frac{1}{(p+2)[(a-bp)-\mu(1-p)]}$

$V^{IS}(p) \sim \frac{(a-bp)(p-c)}{(p+2)[(a-bp)-\mu(1-p)]}\frac{-\ln\delta\mu\varepsilon}{\sigma^2}$

$A = b^2(c+2) - \mu(a+b+bc), B = \mu(4b+2ac) - 4ab - 2abc;$

$C = a^2(c+2) - \mu(2a-ac+2bc), \Delta \sim 4\mu(a-b)(a+2b)(c+2)[(a-bc)-\mu(1-c)].$

1. if $a < b$ and $(a-bc) > \mu(1-c) \iff \mu < bc + \mu(1-c) < a < b$

   $(a-bp) - \mu(1-p) = (a-\mu) - (b-\mu)p > 0 \iff c < p < \frac{a-\mu}{b-\mu} < \frac{a}{b}$

$V^{IS'}(c) \sim (c+2)(a-bc)[(a-bc)-\mu(1-c)] > 0$

$V^{IS'}\left(\frac{a}{b}\right) \sim -\mu(a-b)(a+2b)(a-bc) > 0$

$V^{IS'}\left(\frac{a-\mu}{b-\mu}\right) \sim \frac{-\mu(a-b)(a+2b-3\mu)[a-bc-\mu(1-c)]}{(b-\mu)^2} > 0$

V^{IS} increase in $\left[c, \frac{a-\mu}{b-\mu}\right]$

(a) if $\frac{a-\mu}{b-\mu} > \frac{a+bc}{2b} \iff a > \frac{b(2\mu+bc-c\mu)}{b+\mu}, p^* = \frac{a-\mu}{b-\mu} > p^M, V^{IS}(p) \sim \frac{(b-a)\mu[(a-bc)-\mu(1-c)]}{b-\mu}$

(b) if $\frac{a-\mu}{b-\mu} < \frac{a+bc}{2b} \iff a < \frac{b(2\mu+bc-c\mu)}{b+\mu}, p^* = \frac{a+bc}{2b}, \widehat{\tau} = +\infty, V^{IS}(p) = (a-bp)(p-c)$

2. $a > b$ and $(a-bc) < \mu(1-c) \iff b < \mu, b < a < bc + \mu(1-c) < \mu$

$(a-bp) - \mu(1-p) = (a-\mu) - (b-\mu)p > 0 \iff \frac{\mu-a}{\mu-b} < p < \frac{a}{b}$

$V^{IS'}\left(\frac{a-\mu}{b-\mu}\right) \sim \frac{-\mu(a-b)(a+2b-3\mu)[a-bc-\mu(1-c)]}{(b-\mu)^2} < 0$

$V^{IS'}\left(\frac{a}{b}\right) \sim -\mu(a-b)(a+2b)(a-bc) < 0$

$\Delta < 0, V^{IS}$ decrease in $\left[\frac{\mu-a}{\mu-b}, \frac{a}{b}\right]$

(a) if $\frac{\mu-a}{\mu-b} < \frac{a+bc}{2b} \iff a < \frac{b(2\mu+bc-c\mu)}{b+\mu}, p^* = \frac{\mu-a}{\mu-b} < p^M, V^{IS}(p) \sim \frac{(a-b)\mu[(a-bc)-\mu(1-c)]}{\mu-b}$

(b) if $\frac{\mu-a}{\mu-b} > \frac{a+bc}{2b} \iff a > \frac{b(2\mu+bc-c\mu)}{b+\mu}, p^* = \frac{a+bc}{2b} = p^M, V^{IS}(p) = (a-bp)(p-c)$

3. $a < b < bc + \mu(1-c) < \mu, (a-bp) - \mu(1-p) < 0, p^* = p^M, V^{IS}(p) = (a-bp)(p-c)$

as $(a-bp) - \mu(1-p) < 0, \forall p \in \left[c, \frac{a}{b}\right]$

4. $a > bc + \mu(1-c) > b, b < \mu, (a-bp) - \mu(1-p) > 0 \iff \frac{a}{b} > p > \frac{\mu-a}{\mu-b},$

if $\frac{\mu-a}{\mu-b} < \frac{a+bc}{2b} \iff b < bc + \mu(1-c) < a < \frac{b(2\mu+bc-c\mu)}{b+\mu} < \mu,$

$V^{IS'}\left(\frac{a-\mu}{b-\mu}\right) \sim \frac{-\mu(a-b)(a+2b-3\mu)[a-bc-\mu(1-c)]}{(b-\mu)^2} < 0$

since $a + 2b - 3\mu < \frac{b(2\mu+bc-c\mu)}{b+\mu} + 2b - 3\mu = \frac{1}{b+\mu}\left[(c+2)b^2 + (1-c)\mu b - 3\mu^2\right] < 0$ as $b < \mu =$

$V^{IS'}\left(\frac{a}{b}\right) \sim -\mu(a-b)(a+2b)(a-bc) < 0$

so V^{IS} decrease in $\left[\frac{\mu-a}{\mu-b}, \frac{a}{b}\right]$ and increase in $\left[c, \frac{\mu-a}{\mu-b}\right], p^* = \frac{\mu-a}{\mu-b} < p^M;$

if $\frac{\mu-a}{\mu-b} > \frac{a+bc}{2b} \iff a > \frac{b(2\mu+bc-c\mu)}{b+\mu}$

$V^{IS'}\left(\frac{a-\mu}{b-\mu}\right) \sim \frac{-\mu(a-b)(a+2b-3\mu)[a-bc-\mu(1-c)]}{(b-\mu)^2}$

$V^{IS'}\left(\frac{a}{b}\right) \sim -\mu\,(a-b)\,(a+2b)\,(a-bc) < 0$

$a > b > bc + \mu\,(1-c) > \mu,$

$a < bc + \mu\,(1-c) < b, b > \mu$

$V^{IS'}(c) \sim (c+2)\,(a-bc)\,[(a-bc) - \mu\,(1-c)] > 0$

$V^{IS'}\left(\frac{a}{b}\right) \sim -\mu\,(a-b)\,(a+2b)\,(a-bc) < 0$

so if $V^{IS'}\left(\frac{a}{b}\right) > 0$, $\Delta < 0$, $V^{IS'}\left(\frac{a}{b}\right)$ always positive

therefore at most one optimum

■

Proof of Proposition 11. We first assume that $V^{IS}(p)$ is unimodal w.r.t. $p \in \left[c, \frac{a}{b}\right]$ thus we could refer to the FOC for the optimal price $p^*(a,b)$.

$$\begin{aligned} \frac{dV^{IS}}{dp}|p &= p^* = 0 \iff \\ \frac{2bp^* - (a+bc)}{(p^*-c)(a-bp^*)} &= \frac{\tau'(p^*)}{e^{\tau(p^*)} - 1}. \end{aligned} \tag{16}$$

We consider the function $V^{IS*}(a,b) = V^{IS}(a,b,p^*(a,b))$, and take its derivative w.r.t. a.

$$\begin{aligned} \frac{dV^{IS*}(a,b)}{da} &= \frac{dV^{IS}(a,b,p^*(a,b))}{da} \\ &= \frac{\partial V^{IS}(a,b)}{\partial a} + \frac{\partial V^{IS}(a,b)}{\partial p}\frac{dp}{da}|_{p=p^*} \\ &= \frac{\partial V^{IS}(a,b)}{\partial a}|_{p=p^*}, \end{aligned}$$

therefore if $\exists a^*(b)$ s.t. $V^{IS*}(a^*(b), b) = \max_a V^{IS*}(a, b)$, then

$$\begin{aligned}
\frac{\partial V^{IS}(a,b)}{\partial a}|_{p=p^*,a=a^*} &= 0 \Rightarrow \\
\frac{1}{a^* - bp^*(a^*, b)} &= \frac{\tau(p^*, a^*)}{e^{\tau(p^*,a^*)} - 1} \frac{1}{\sqrt{\beta(p^*, a^*)^2 - 2\ln\delta}}. \qquad (17)
\end{aligned}$$

Define $q^*(b) = a^*(b) - bp^*(a^*(b), b)$, then $\{p^*(b), q^*(b)\}$ satisfy the conditions set by (17)

$$\begin{aligned}
& \frac{1}{q^*} - \frac{\tau(p^*, q^*)}{e^{\tau(p^*,q^*)} - 1} \frac{1}{\sqrt{\beta(p^*, q^*)^2 - 2\ln\delta}} \qquad (18) \\
: \quad & = F_1(p, q|\theta_0, \boldsymbol{\psi}) = 0 \qquad (19)
\end{aligned}$$

By inserting (17), (16) can be re-written as

$$\begin{aligned}
\frac{2bp^* - (a + bc)}{(p^* - c)q} &= \frac{\tau'(p^*)}{e^{\tau(p^*)} - 1} \\
b - \frac{q^*}{p^* - c} &= \frac{\frac{d\tau}{dp}(p^*)}{\tau} \sqrt{\beta(p^*, q^*)^2 - 2\ln\delta} \\
b &= \frac{q^*}{p^* - c} + \left(\frac{\alpha'}{\alpha}\sqrt{\beta(p^*, q^*)^2 - 2\ln\delta} + \beta'\right)
\end{aligned}$$

or

$$\begin{aligned}
& \frac{q^*}{p^* - c} + (\mu + 2C^*p)\, 2p\left[\sqrt{1 + \frac{\mu(4C^* - \mu)}{(\mu + 2C^*p)^2}} - 1\right]\left[1 - \frac{\frac{\frac{1}{p} + p}{\frac{\mu}{2C^*} + p}}{4\sqrt{1 + \frac{\mu(4C^* - \mu)}{(\mu + 2C^*p)^2}}}\right] \qquad (20) \\
& - \frac{C^*\sqrt{\beta(p^*, q^*)^2 - 2\ln\delta}}{\sqrt{C^*\mu(1 + p^*) + C^{*2}p^{*2}}} - \mu \\
: \quad & = F_2(p, q|\theta_0, \boldsymbol{\psi}) = 0 \qquad (21)
\end{aligned}$$

Since (18) and (20) could be written as functions of (p, q) with parameters of $(\theta_0, \boldsymbol{\psi})$, and are independent, we could solve for the unique $\{p^{**}(\theta_0,\boldsymbol{\psi}), q^{**}(\theta_0,\boldsymbol{\psi})\}$ that satisfy

$$\begin{cases} F_1(p,q|\theta_0,\boldsymbol{\psi})=0 \\ F_2(p,q|\theta_0,\boldsymbol{\psi})=0 \end{cases}.$$

This means for $\forall\{b,a^*(b)\}, \exists\{p^{**},q^{**}\}$ s.t.

$$a^*(b)=q^{**}+bp^{**},$$

therefore $\{b,a^*(b)\}$ form a line segment. Moreover, $\forall\{b,a^*(b)\}$

$$\begin{aligned} V^{IS*}(a^*(b),b) &= V^{IS}(p^{**},q^{**}) \\ &= (p^{**}-c)\,q^{**}\left[1-e^{-\tau(p^{**},q^{**})}\right] \\ : &= V^{IS**}. \end{aligned}$$

■

Proof of Proposition 12. When $a\to bc^+$,since $p^*\in\left[c,\frac{a}{b}\right]$, so $p^*\to bc^+$, $(p^*-c)(a-bp^*)\to 0^+$;

1. so $V^{IS*}(a,b)=(p^*-c)(a-bp^*)\left(1-e^{-\tau\left(a,b,p^2\right)}\right)\to 0^+$,as $1-e^{-\tau}\leq 1$.

   Similarly we could prove that $V^{IS*}(a,b)\to 0^+$ as $b\to\left(\frac{a}{c}\right)^-$.

2. For $a\to+\infty$, we show that $V^{IS*}(a,b)$ does not fully diminish but remains above a positive level. By definition

$$\begin{aligned} V^{IS*}(a,b) &= V^{IS}(a,b,p^*)\geq V^{IS}\left(a,b,p^M\right) \\ V^{IS}\left(a,b,p^M\right) &= \left(p^M-c\right)(a-bp)\left(1-e^{-\tau\left(a,b,p^M\right)}\right) \end{aligned}$$

   Note that $p^M=\frac{a+bc}{2b}\sim\frac{a}{2b}(1+o(a))$, so $\left(p^M-c\right)(a-bp)\sim\frac{a^2}{4b}\left(1+o\left(\frac{1}{a}\right)\right)$.

   Consider $1-e^{-\tau\left(a,b,p^M\right)}$, and note that $\tau=\alpha\left(\beta+\sqrt{\beta^2-2\ln\delta}\right)$

   $\alpha=\frac{\mu(4C^*-\mu)}{\sigma^2\left[2\sqrt{C^*\mu(1+p)+C^{*2}p^2}+(\mu+2C^*p)\right]}\sim\frac{\mu(4C^*-\mu)}{\sigma^2 4C^*p\left(1+o\left(\frac{1}{p}\right)\right)}\sim\frac{b\mu(4C^*-\mu)}{2\sigma^2C^*a}\left(1+o\left(\frac{1}{a}\right)\right)$

$\beta = \frac{\mu(4C^*-\mu)\left(1+p^2\right)}{2\sqrt{C^*\mu(1+p)+C^{*2}p^2}+(\mu+2C^*p)} - (a-\mu) + p\,(b-\mu)$

$\sim \left[\frac{\mu(4C^*-\mu)}{4C^*}\left(1+o\left(\frac{1}{p}\right)\right)+b-\mu\right]p-(a-\mu) \sim -\frac{a}{2b}\left(\frac{\mu^2}{4C^*}+b\right)\left(1+o\left(\frac{1}{a}\right)\right)$

$\beta+\sqrt{\beta^2-2\ln\delta} = \beta+|\beta|\sqrt{1-\frac{2\ln\delta}{\beta^2}} \sim \beta-\beta\left(1-\frac{\ln\delta}{\beta^2}+o\left(\frac{1}{\beta^2}\right)\right) \sim \frac{\ln\delta}{\beta}\left(1+o\left(\frac{1}{\beta}\right)\right)$

$\sim \frac{-2b\ln\delta}{a\left(\frac{\mu^2}{4C^*}+b\right)}\left(1+o\left(\frac{1}{a}\right)\right)$

$\tau = \alpha\left(\beta+\sqrt{\beta^2-2\ln\delta}\right) \sim \frac{-b^2\ln\delta\mu(4C^*-\mu)}{a^2\left(\frac{\mu^2}{4C^*}+b\right)\sigma^2C^*}\left(1+o\left(\frac{1}{a}\right)\right)$

So $1-e^{-\tau\left(a,b,p^M\right)} \sim 1-[1+(-\tau)+o(\tau)] \sim \tau\,(1+o(\tau)) \sim \frac{-b^2\ln\delta\mu(4C^*-\mu)}{a^2\left(\frac{\mu^2}{4C^*}+b\right)\sigma^2C^*}\left(1+o\left(\frac{1}{a}\right)\right)$

$V^{IS}\left(a,b,p^M\right) \sim \frac{-\ln\delta\mu(4C^*-\mu)}{4\left(1+\frac{\mu^2}{4bC^*}\right)\sigma^2C^*}\left(1+o\left(\frac{1}{a}\right)\right).$

So $\lim_{a\to+\infty} V^{IS*}(a,b) \geq \frac{-\ln\delta\mu(4C^*-\mu)}{4\left(1+\frac{\mu^2}{4bC^*}\right)\sigma^2C^*} > 0.$

3. Similarly we could prove that $\lim_{b\to0^+} V^{IS*}(a,b) \sim \frac{-\ln\delta(4C^*-\mu)a\varepsilon}{\mu\sigma^2}(1+o(\varepsilon)) \to 0.$

   Note that $p^M \sim \frac{a}{2b}(1+o(\varepsilon))$, so $\left(p^M-c\right)(a-bp) \sim \frac{a^2}{4b}(1+o(\varepsilon))$.

   $\alpha \sim \frac{2\varepsilon\mu(C^*-\mu)}{\sigma^2C^*}(1+o(\varepsilon)), \beta \sim -\frac{a\mu^2}{2bC^*}(1+o(\varepsilon))$

   $\beta+\sqrt{\beta^2-2\ln\delta} \sim \frac{-8\ln\delta C^*\varepsilon}{\mu^2}(1+o(\varepsilon))$

   $\tau \sim \frac{-4\ln\delta(4C^*-\mu)\varepsilon^2}{\mu\sigma^2}(1+o(\varepsilon)),\ 1-e^{-\tau\left(a,b,p^M\right)} \sim \frac{-4\ln\delta(4C^*-\mu)\varepsilon^2}{\mu\sigma^2}(1+o(\varepsilon))$

   $V^{IS}\left(a,b,p^M\right) \sim \frac{-\ln\delta(4C^*-\mu)a\varepsilon}{\mu\sigma^2}(1+o(\varepsilon))$

■

2.3 Analytical Results of Alternative Entry Policy and General ρ

Proof. Consider $\boldsymbol{\theta}_\gamma := \left\{\{a,b\}\,|\,\frac{a^2}{4b}-\gamma_1 p\,(a-bp) \geq C^*_\gamma\right\}$; then $\boldsymbol{\theta}_{\gamma_1} \subset \boldsymbol{\theta}_{\gamma_2}$.

$$\begin{aligned}\frac{\mu_{a,t}^2}{4\mu_{b,t}} - \gamma_1 p\left(\mu_{a,t}-\mu_{b,t}p\right)^+ &\leq C^*_\gamma \\ \frac{\frac{\mu_{a,t}^2}{4\mu_{b,t}}-C^*_\gamma}{p\left(\mu_{a,t}-\mu_{b,t}p\right)^+} &\leq \gamma_1<\gamma_2\end{aligned}$$

so $\tau_1<\tau_2$. ■

Proof of Proposition 13. when $a \leq b, p \leq \frac{a}{b} \leq 1$,so $\alpha = \frac{(4C^* - \mu)}{\sigma^2(1-p)}, \beta = (4C^* - b)(1-p) + (b-a)$

since $\frac{\tau'}{\tau} = \frac{\alpha'}{\alpha} + \frac{\beta'}{\sqrt{\beta^2 - 2\ln\delta}}$

$\frac{\alpha'}{\alpha} = \frac{1}{1-p} > 0, \beta' = b - 4C^*, \frac{\tau'}{\tau} = \frac{b-4C^*}{\sqrt{\beta^2 - 2\ln\delta}} + \frac{1}{1-p} > 0$

so τ increase in p^{IS}, and $p^* > p^M$. ■

Lemma 12 *When $\rho = 1$, $a > b$,*

1. *$\widehat{\tau}$ increases with p when* $p \in \left(c, 1 - \left(\frac{\frac{a-b}{2} - \frac{\ln\delta}{a-b}}{4C^* - b}\right)^+\right) \cup \left(1 + \frac{\frac{a-b}{2} - \frac{\ln\delta}{a-b}}{b}, \frac{a}{b}\right)$;

2. *$\widehat{\tau}$ decreases with p when* $p \in \left[\max\left\{c, 1 - \left(\frac{\frac{a-b}{2} - \frac{\ln\delta}{a-b}}{4C^* - b}\right)^+\right\}, \min\left\{1 + \frac{\frac{a-b}{2} - \frac{\ln\delta}{a-b}}{b}, \frac{a}{b}\right\}\right]$.

Proof of Lemma 12. Again when $p < 1$

$\alpha = \frac{(4C^* - \mu)}{\sigma^2(1-p)}, \beta = (4C^* - b)(1-p) + (b-a)$

$\frac{\alpha'}{\alpha} = \frac{1}{1-p} > 0, \beta' = b - 4C^*, \frac{\tau'}{\tau} = \frac{b-4C^*}{\sqrt{\beta^2 - 2\ln\delta}} + \frac{1}{1-p} < 0$

when $b < 4C^* - \sqrt{-2\ln\delta}, a \in 4C^* - b \pm \sqrt{(4C^* - b)^2 + 2\ln\delta}$

when $p < 1 - \frac{\frac{a-b}{2} - \frac{\ln\delta}{a-b}}{4C^* - b}$

$\frac{\tau'}{\tau} = \frac{b-4C^*}{\sqrt{\beta^2 - 2\ln\delta}} + \frac{1}{1-p} > 0$ when $p > 1 - \frac{\frac{a-b}{2} - \frac{\ln\delta}{a-b}}{4C^* - b}$

when $p > 1$

$\alpha = \frac{\mu}{\sigma^2(p-1)}, \beta = -(a - bp)$

$\frac{\alpha'}{\alpha} = -\frac{1}{p-1} < 0, \beta' = b > 0,$

$\frac{\tau'}{\tau} = \frac{\alpha'}{\alpha} + \frac{\beta'}{\sqrt{\beta^2 - 2\ln\delta}} = \frac{b}{\sqrt{\beta^2 - 2\ln\delta}} - \frac{1}{p-1}$

or $\frac{\tau'}{\tau} < 0$ when $p \leq 1 + \frac{\sqrt{-2\ln\delta}}{b} < 1 + \frac{(a-b) - \frac{2\ln\delta}{a-b}}{2b}$

$\frac{\tau'}{\tau} > 0$ when $p > 1 + \frac{\frac{a-b}{2} - \frac{\ln\delta}{a-b}}{b}$

so $\tau' < 0$ when $p \in \left(1 - \frac{\frac{a-b}{2} - \frac{\ln\delta}{a-b}}{4C^* - b}, 1 + \frac{\frac{a-b}{2} - \frac{\ln\delta}{a-b}}{b}\right)$ ■

Proof of Corollary 15. We could argue that for $\rho = 1 - \varepsilon \to 1^-$, we have the following approximation

$$F(\rho, p) \sim \begin{cases} \mu\left[(1-p) + \varepsilon(1+2p)\right] & p < 1 \\ (4C^* - \mu)(p - 1 + \varepsilon) & p > 1 \end{cases}$$

$\beta = \alpha\left(1 + p^2 - 2\rho p\right)\sigma^2 - (a - \mu) + p(b - \mu)$

$$\sim \begin{cases} (4C^* - a) - (4C^* - b)\,p + \frac{(4C^*-\mu)(1+p)(2p-1)\varepsilon}{1-p} & p < 1 \\ -(a - bp) + \frac{\mu(1+p)\varepsilon}{p-1} & p > 1 \end{cases}$$

$$\frac{\alpha'}{\alpha} \sim \begin{cases} \frac{1-\frac{3\varepsilon}{1-p}}{1-p} & p < 1 \\ \frac{1-\frac{\varepsilon}{p-1}}{1-p} & p > 1 \end{cases},\ \beta' \sim \begin{cases} b - 4C^* - \frac{2(4C^*-\mu)p(p-2)\varepsilon}{(1-p)^2} & p < 1 \\ b - \frac{2\mu\varepsilon}{(p-1)^2} & p > 1 \end{cases}$$

So the results of Claim remains true for ε small enough. ■

Proof of Proposition 14. $\alpha = \frac{4\sqrt{C_T\mu(1-\rho p)(1+p)(1-\rho)+C_T^2(\rho-p)^2}-[2\mu(1-\rho p)-4C_T(\rho-p)]}{2\sigma^2(1-\rho p)^2}$

$$= \frac{(4C^*-\mu)\mu}{\sigma^2\left\{2\sqrt{C_T\mu(1-\rho p)(1+p)(1-\rho)+C_T^2(\rho-p)^2}+[\mu(1-\rho p)-2C_T(\rho-p)]\right\}} = \frac{(4C^*-\mu)\mu}{\sigma^2}\frac{1}{F(\rho,p)}$$

$$\beta = \frac{(4C^*-\mu)\mu\left(1+p^2-2\rho p\right)}{F(\rho,p)} - (a-\mu) + p\,(b-\mu)$$

$$F(\rho,p) = 2\sqrt{C_T\mu\,(1-\rho p)\,(1+p)\,(1-\rho) + C_T^2\,(\rho-p)^2} + [\mu\,(1-\rho p) - 2C_T\,(\rho-p)] > 0$$

$$\frac{\alpha'}{\alpha} = \frac{-F'}{F}, \beta' = (4C^*-\mu)\,\mu\left[\frac{2(p-\rho)}{F(\rho,p)} - \frac{\left(1+p^2-2\rho p\right)F'(p)}{F^2(\rho,p)}\right] + (b-\mu)$$

$$\frac{\tau'}{\tau} = \frac{\alpha'}{\alpha} + \frac{\beta'}{\sqrt{\beta^2-2\ln\delta}}$$

$$= \frac{-F'}{F} + \frac{(4C^*-\mu)\mu\left[\frac{2(p-\rho)}{F(\rho,p)} - \frac{\left(1+p^2-2\rho p\right)F'(p)}{F^2(\rho,p)}\right]+(b-\mu)}{\sqrt{\beta^2-2\ln\delta}}$$

$$= \frac{-F'}{F}\left[1 + \frac{(4C^*-\mu)\mu\left(1+p^2-2\rho p\right)}{F\sqrt{\beta^2-2\ln\delta}}\right] + \frac{\frac{2\left(4C^*-\mu\right)\mu(p-\rho)}{F(\rho,p)}+(b-\mu)}{\sqrt{\beta^2-2\ln\delta}}$$

$\frac{\tau'}{\tau} < 0$ if $F' > 0$, $\frac{2(4C^*-\mu)\mu(p-\rho)}{F(\rho,p)} + (b-\mu) < 0$ for all p.

$$F' = \frac{C_T\mu(1-\rho)(1-\rho-2\rho p)-2C^2(\rho-p)}{\sqrt{C_T\mu(1-\rho p)(1+p)(1-\rho)+C_T^2(\rho-p)^2}} + (2C - \mu\rho) > 0$$

when $\rho < \frac{\mu+C^*-\sqrt{C^*(2\mu+C^*)}}{\mu} < \frac{2C}{\mu}$

and $0 < \rho < p < \frac{1}{\rho}$ or $\rho < 0$ $(p > \rho, 1 > \rho p)$

$$\frac{p-\rho}{F(\rho,p)} = \frac{p-\rho}{2\sqrt{C_T\mu(1-\rho p)(1+p)(1-\rho)+C_T^2(\rho-p)^2}+[\mu(1-\rho p)+2C_T(p-\rho)]} < \frac{1}{4C}$$

$$b < \mu - 2\,(4C^*-\mu)\,\mu\frac{p-\rho}{F(\rho,p)}$$

$$2\,(4C^*-\mu)\,\mu\frac{p-\rho}{F(\rho,p)} < \frac{2(4C^*-\mu)\mu}{4C}$$

$$\mu - 2\,(4C^*-\mu)\,\mu\frac{p-\rho}{F(\rho,p)} > \frac{\mu}{2C}\,(\mu - 2C)$$

sufficient if $b < \frac{\mu}{2C}\,(\mu-2C)$

so α decrease in p, and $\frac{-F'}{F} < 0$. ■