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BILINEAR SYSTEM IDENTIFICATION BY MINIMAL-ORDER STATE OBSERVERS

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Bilinear systems offer a promising approach for nonlinear control because a broad class of nonlinear problems can be reformulated and approximated in bilinear form. System identification is a technique to obtain such a bilinear approximation for a nonlinear system from input-output data. Recent discrete-time bilinear model identification methods rely on Input-Output-to-State Representations (IOSRs) derived via the interaction matrix technique. A new formulation of these methods is given by establishing a correspondence between interaction matrices and the gains of full-order bilinear state observers. The new interpretation of the identification methods highlights the possibility of utilizing minimal-order bilinear state observers to derive new IOSRs. The existence of such observers is discussed and shown to be guaranteed for special classes of bilinear systems. New bilinear system identification algorithms are developed and the corresponding computational advantages are illustrated via numerical examples.

INTRODUCTION

System identification has proven to be a powerful technique to develop a mathematical model of a dynamic system, to be used for example for analysis or controller design. It has found applications in many fields and in particular in aerospace engineering. A notable example is Observer/Kalman filter IDentification (OKID), originally developed for the identification of lightly-damped linear structures and distributed by NASA (see for example References 1, 2). Although well-established techniques exist for linear systems, this is not the case for nonlinear systems. Bilinear models are a particular class of nonlinear models. Some dynamical systems are inherently bilinear, e.g. those that involve the regulation of thermal energy by fluid flow (References 3, 4, 5). More importantly, by increasing the state dimension a bilinear model can be used to approximate a more general nonlinear system (References 6, 7, 8). This property, together with their relatively simple mathematical structure, makes bilinear models an appealing bridge between linear and nonlinear systems. Interest in bilinear systems has recently grown after a technique formally known as Carleman linearization was found to achieve such an approximation (References 9, 10). One can then address more general nonlinear control problems by designing simpler bilinear state estimators and controllers. System identification can also be a tool to find the bilinear approximation of a nonlinear system from measured input-output data.

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Bilinear systems can also mean systems having multiple variables, and having the property that they are linear in each variable if the remaining variables are held constant. In satellite attitude dynamics, the Euler equations for the satellite angular velocity as a function of applied torque are bilinear equations in this sense. One needs to add to these equations, additional equations that represent the kinematics of rigid body motion, which then give the satellite attitude as a function of time that results from the angular velocity history. There are various choices for these kinematic equations: one can use 9 direction cosine matrix elements, or 6 direction cosine matrix elements, or 4 parameter representations such as quaternions, Rodriguez parameters, and Euler Rodriguez parameters, or 3 parameter sets of Euler angles. The first few of these choices also produce bilinear equations in this sense, so that the satellite attitude control problem can be thought of as a set of 12 or 9 bilinear equations. The literature on bilinear systems usually makes use of a different form of bilinear equation that only contains nonlinear terms that are products of a state variable with the input. Carleman linearization will convert the satellite attitude dynamics bilinear problem into this form of bilinear equations (References 10, 11). Alternatively, bilinear system identification will provide a bilinear model directly from input-output data. One can then think of system identification, state estimators, and control methods for bilinear systems as an approach to handle the satellite attitude control problem.

Recently two novel approaches were presented for bilinear system identification. References 12, 13 proposed the identification of a bilinear system via an Equivalent Linear Model (ELM). References 14, 15 introduced an Intersection Subspace (IS) method to identify the bilinear system via the reconstruction of the bilinear state, obtained by intersecting two data matrices. At the core of both approaches are relationships expressing the bilinear state as a linear combination of input-output data. References 13, 14 formalized the concept by calling such relationships Input-Output-to-State Representations (IOSRs). Reference 12 developed IOSRs via perturbation theory based on an excitation made of a special combination of sinusoidal inputs. References 13, 14 derived IOSRs via interaction matrices with the aim of developing an identification method that can be used with arbitrary excitation input. Reference 15 relied on interaction matrices as well but in conjunction with a specialized input, making the most of the linear-time-varying properties of bilinear models. Depending on which IOSRs are used in the implementation, the ELM and IS approaches give rise to different identification algorithms for bilinear systems.

References 16, 17 formulated a full-order bilinear state observer and established a fundamental link between the observer gains and the interaction matrices at the core of the IOSRs developed in References 13, 14. Not only this link inspired a method for bilinear observer gain design based on system identification techniques, but it also provided a new interpretation of the identification methods presented in References 13, 14. In this paper we formulate such methods relying on bilinear observers instead of interaction matrices. Additionally, we develop new IOSRs by exploiting the new formulation, which can be based on full-order bilinear observers as well as minimal-order bilinear observers. Indeed, interesting results on the latter already exist in the literature (Reference 18) and are now applied for the first time to bilinear system identification. Numerical examples are provided to validate the theoretical development and illustrate the advantages of the new identification algorithms. As a last note, we focus on discrete-time methods that are particularly suitable for digital implementation although physical dynamical systems are in continuous time. Reference 11 shows how a continuous-time bilinear state-space model can be approximated by a discrete-time version without losing the simple bilinear structure. Direct identification of continuous-time bilinear models for a continuous-time bilinear or nonlinear system is another track of research that is being vigorously pursued (References 10, 19, 20, 21, 22).

PROBLEM STATEMENT

Consider an n-state, single-input, q-output bilinear system in state-space form

$$x(k+1) = Ax(k) + Nx(k)u(k) + Bu(k) \tag{1a}$$

$$y(k) = Cx(k) + Du(k) \tag{1b}$$

where $x \in \mathbb{R}^{n\times 1}$ is the state vector, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}^{q\times 1}$ is the output vector and $A, N \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{q\times n}$, and $D \in \mathbb{R}^{q\times 1}$ are the matrices governing the dynamics of the bilinear system. A single set of length l of input-output data that starts from some unknown initial state $x(0)$ is given

$$\{u(k)\} = \{u(0),\ \ u(1),\ \ u(2),\ \ \ldots,\ \ u(l-1)\} \tag{2a}$$

$$\{y(k)\} = \{y(0),\ \ y(1),\ \ y(2),\ \ \ldots,\ \ y(l-1)\} \tag{2b}$$

The objective is to identify the system of Eq. (1), i.e. the system matrices A, N, B, C, and D, with the input-output data provided in Eq. (2). The data is assumed to be of sufficient length and richness so that the system of Eq. (1) can be correctly identified. For simplicity, we focus on the single-input case in this paper. Extension to the multi-input case can be made without conceptual difficulties.

APPROACH

References 12, 13, 14, 15 proposed two new approaches to bilinear system identification, namely the Equivalent Linear Model (ELM) method and the Intersection Subspace (IS) method. At the core of both approaches is the following linear relationship between the state x of the bilinear system and a superstate z defined solely in terms of input-output data

$$x(k) = Tz(k) \tag{3}$$

where T is a constant matrix. Relations of the form of Eq. (3) are referred to as Input-Output-to-State Representations (IOSRs). Depending on the specific choice of IOSR (see next section), different algorithms of ELM and IS type can be devised. To keep the presentation as self-contained as possible, the IS method is not illustrated via examples, since it requires IOSRs presented in previous papers (References 14, 15). Nevertheless, a quick overview if the IS method is provided to emphasize how the new IOSR developed in the present work can also be applied to implement IS-based algorithms for bilinear system identification, extending the range of algorithms presented in References 14, 15. Additionally, the IS method in its linear version is used in the examples at the end of this paper to identify the ELM.

Equivalent Linear Model (ELM) Method

If a relation like Eq. (3) is available, the bilinear identification problem is dramatically simplified by rewriting the original bilinear system of Eq. (1) in the form of a linear system. Substituting Eq. (3) into Eq. (1a) we obtain the following Equivalent Linear Model (ELM)

$$x(k+1) = Ax(k) + B_{ELM}w(k) \tag{4a}$$

$$y(k) = Cx(k) + Du(k) \tag{4b}$$

where

$$B_{ELM} = \begin{bmatrix} B & NT \end{bmatrix} \tag{5a}$$

$$w(k) = \begin{bmatrix} u(k) \\ z(k)u(k) \end{bmatrix} \tag{5b}$$

The ELM is a linear state-space model with known input $\{w(k)\}$ and output $\{y(k)\}$ history. Thus it can be identified by any linear identification method and then used to recover the original bilinear system matrices. Reference 13 suggests to use the identified ELM to reconstruct the bilinear state history and then estimating N by classic least squares (Moore Penrose pseudoinversion)on Eq. (1a)

$$N = \begin{bmatrix} h(k_{ss}+1) & \dots & h(l-1) \end{bmatrix} \begin{bmatrix} x(k_{ss})u(k_{ss}) & \dots & x(l-2)u(l-2) \end{bmatrix}^{\dagger} \tag{6}$$

where

$$h(k+1) = Nx(k)u(k) = x(k+1) - Ax(k) - Bu(k) \tag{7}$$

The matrices A, B, C, and D of the bilinear system can be directly read from the identified ELM, as per Eqs. (4) and (5a). Note that the identified model is not necessarily in the original coordinate system. As usual with state-space formulation, the change in coordinate system does not affect the identified model validity.

Intersection Subspace (IS) Method

Assume two IOSRs are available for the bilinear system of Eq. (1)

$$x(k) = T_a z_a(k) \qquad\qquad x(k) = T_b z_b(k) \tag{8}$$

and define the following matrices

$$X = \begin{bmatrix} x(k_i) & x(k_i+1) & x(k_i+2) & \dots & x(k_f) \end{bmatrix} \tag{9}$$

$$Z_a = \begin{bmatrix} z_a(k_i) & z_a(k_i+1) & z_a(k_i+2) & \dots & z_a(k_f) \end{bmatrix} \tag{10}$$

$$Z_b = \begin{bmatrix} z_b(k_i) & z_b(k_i+1) & z_b(k_i+2) & \dots & z_b(k_f) \end{bmatrix} \tag{11}$$

where k_i and k_f are the initial and final time steps for which Eq. (8) holds. Then we can write

$$X = T_a Z_a \qquad\qquad X = T_b Z_b \tag{12}$$

The row space of the (minimum-dimension) state X is a subspace of the row space of Z_a and also a subspace of the row space of Z_b. The row space of X must then lie in the intersection between the row spaces of the two superspaces Z_a and Z_b. The problem of reconstructing the state history $\{x(k)\}$ is therefore reduced to finding the intersection of two vector spaces, which can be done by two singular value decompositions (SVDs), as shown in Reference 14. Having reconstructed the bilinear state history, the identification problem is dramatically simplified and from Eq. (1) we can estimate the matrices by classic least-squares (Moore-Penrose pseudo-inversion)

$$\begin{bmatrix} A & B & N \end{bmatrix} = \begin{bmatrix} x(k_i+1) & \dots & x(k_f) \end{bmatrix} \begin{bmatrix} x(k_i) & \dots & x(k_f-1) \\ u(k_i) & \dots & u(k_f-1) \\ x(k_i)u(k_i) & \dots & x(k_f-1)u(k_f-1) \end{bmatrix}^{\dagger} \tag{13a}$$

$$\begin{bmatrix} C & D \end{bmatrix} = \begin{bmatrix} y(k_i) & \dots & y(k_f) \end{bmatrix} \begin{bmatrix} x(k_i) & \dots & x(k_f) \\ u(k_i) & \dots & u(k_f) \end{bmatrix}^{\dagger} \tag{13b}$$

INPUT-OUTPUT-TO-STATE REPRESENTATIONS VIA STATE OBSERVERS

Reference 12 first introduced the ELM method via an approximate IOSR obtained by perturbation theory and requiring special combinations of sinusoidal inputs. References 13, 14 developed more general IOSRs that can be used for any excitation input. The key concept behind their derivation is that of interaction matrix. The interaction matrices were originally formulated by Minh Q. Phan in the context of linear system identification of lightly-damped large flexible space structures. The interaction matrices provide a mechanism to find a compressed but equivalent dynamic representation of such structures. The compression can be exact and extremely efficient. Later development revealed that the interaction matrix in the state-space system identification problem could be interpreted as an observer gain. In the absence of noise the gain corresponds to a deadbeat observer. In the presence of noise it corresponds to the Kalman filter that is the optimal state observer with respect to the system and the (unknown) process and measurement noise statistics embedded in the input-output data. This development led to the Observer/Kalman filter IDentification (OKID) algorithm (Reference 1). Afterwards, interaction matrices found applications in many other contexts, e.g. in model predictive control and iterative learning control (Reference 23).

Following the intuition provided by the linear case, References 16, 17 formulated a full-order bilinear state observer and rigorously established the connection between the interaction matrices in bilinear system identification and the gains of the full-order bilinear observer. The IOSRs developed in References 13, 14 can now be explained via full-order bilinear observers, providing an interpretation of the existing bilinear system identification methods that is more intuitive especially for engineers and researchers familiar with control theory and state estimation. The derivation of the IOSRs via full-order bilinear state observers is presented below.

IOSRs via Full-Order Bilinear State Observers

The fact that the system state history is unknown represents a typical challenge in state-space model identification. Only input and output are measured. Therefore, in the state-space equations the unknown state multiplies unknown matrices, making the identification problem nonlinear. For the bilinear state-space model in Eq. (1a), the nonlinear terms are $Ax(k)$, $Nx(k)u(k)$, and $Cx(k)$. As illustrated in Reference 24, the key intuition of the OKID approach is to use an observer to find an estimate for the state history and exploit such an estimate to turn the identification problem into linear in the unknown system matrices. The concept can be applied to bilinear systems as follows.

Let us introduce the following state observer with bilinear structure (References 16, 17)

$$\begin{aligned}\hat{x}(k+1) &= A\hat{x}(k) + N\hat{x}(k)u(k) + Bu(k) \\ &\quad + M_1\left(y(k) - \hat{y}(k)\right) + M_2\left(y(k) - \hat{y}(k)\right)u(k) \qquad (14a)\\ \hat{y}(k) &= C\hat{x}(k) + Du(k) \qquad (14b)\end{aligned}$$

where $\hat{y}(k) \in \mathbb{R}^{q\times 1}$ is the observer output based on the observer state or system estimated state

$\hat{x}(k) \in \mathbb{R}^{n\times 1}$, and $M_1, M_2 \in \mathbb{R}^{n\times q}$ are the observer gains. Defining

$$\bar{A} = A - M_1 C \qquad \bar{N} = N - M_2 C \tag{15a}$$

$$\bar{B} = \begin{bmatrix} B - M_1 D & M_1 & -M_2 D & M_2 \end{bmatrix} \tag{15b}$$

$$v(k) = \begin{bmatrix} u(k) \\ y(k) \\ u^2(k) \\ u(k)y(k) \end{bmatrix} \tag{15c}$$

the bilinear observer in equation (14) can be written in the following equivalent form

$$\hat{x}(k+1) = \bar{A}\hat{x}(k) + \bar{N}\hat{x}(k)u(k) + \bar{B}v(k) \tag{16a}$$

$$\hat{y}(k) = C\hat{x}(k) + Du(k) \tag{16b}$$

The bilinear observer can be used to obtain an estimate of the state of the bilinear system in Eq. (1) at the future time step $(k+1)$ from the current input and output values. However, the estimate of the state at the time step k is also needed to directly use Eq. (16). To eliminate the need to know the current value of the state, Eq. (16a) can be propagated forward in time. In the following, for brevity of presentation, we will terminate the propagation of Eq. (16a) at time step $k+2$, but it is possible to generalize the termination at any time step $k+p$. Propagating Eq. (16a) one step forward, we get

$$\begin{aligned} \hat{x}(k+2) &= \bar{A}\hat{x}(k+1) + \bar{N}\hat{x}(k+1)u(k+1 + \bar{B}v(k+1) \\ &= \bar{A}^2\hat{x}(k) + \bar{A}\bar{N}\hat{x}(k)u(k) + \bar{A}\bar{B}v(k) + \bar{N}\bar{A}\hat{x}(k)u(k+1) \\ &\quad + \bar{N}^2\hat{x}(k)u(k)u(k+1) + \bar{N}\bar{B}v(k)u(k+1) + \bar{B}v(k+1) \end{aligned} \tag{17}$$

Equation (17) can be written more conveniently as

$$\hat{x}(k+2) = T_2 z_2(k+2) + \mathcal{S}_2(k+2)\hat{x}(k) \tag{18}$$

by defining

$$T_2 = \begin{bmatrix} \bar{A}\bar{B} & \bar{N}\bar{B} & \bar{B} \end{bmatrix} \tag{19a}$$

$$z_2(k+2) = \begin{bmatrix} v(k) \\ v(k)u(k+1) \\ v(k+1) \end{bmatrix} \tag{19b}$$

$$\mathcal{S}_2(k+2) = \bar{A}^2 + \bar{A}\bar{N}u(k) + \bar{N}\bar{A}u(k+1) + \bar{N}^2u(k)u(k+1) \tag{19c}$$

where the subscript 2 remarks that the state estimate depends on input-output data 2 steps back in time.

As previously mentioned, Eq. (17) can be propagated further in time, leading to a higher-order $(p > 2)$ representation of the form

$$\hat{x}(k+p) = T_p z_p(k+p) + \mathcal{S}_p(k+p)\hat{x}(k) \tag{20}$$

By shifting the time index backward by p steps, we obtain

$$\hat{x}(k) = T_p z_p(k) + \mathcal{S}_p(k)\hat{x}(k-p) \tag{21}$$

As will be proven later, suppose the sum of all the terms in Eq. (21) depending on the state estimate $\hat{x}(k-p)$ vanishes. Equation (21) can then be written in the following form, similar to Eq. (3)

$$\hat{x}(k) = T_p z_p(k) \tag{22}$$

In Reference 13, the general pattern to construct the superstate $z_p(k)$ from input-output data is derived. Letting $\mathrm{C}_k^n = \binom{n}{k}$ denote the combinations of k out of n terms, commonly referred to as n-choose-k, the general pattern for the entries of $z_p(k)$ is

- $v(k-p), v(k-p+1), \ldots, v(k-1)$
- $v(k-p)$ multiplied with products of $u(k-p+1)$ to $u(k-1)$ in all possible combinations $\mathrm{C}_1^{(p-1)}$, $\mathrm{C}_2^{(p-1)}$, …, $\mathrm{C}_{(p-1)}^{(p-1)}$ of $\{u(k-p+1), u(k-p+2), \ldots, u(k-1)\}$
- $v(k-p+1)$ multiplied with products of $u(k-p+2)$ to $u(k-1)$ in all possible combinations $\mathrm{C}_1^{(p-2)}$, $\mathrm{C}_2^{(p-2)}$, …, $\mathrm{C}_{(p-2)}^{(p-2)}$ of $\{u(k-p+2), u(k-p+3), \ldots, u(k-1)\}$

  $\vdots$
- $v(k-3)$ multiplied with products of $u(k-2)$ to $u(k-1)$ in all possible combinations C_1^2, C_2^2 of $\{u(k-p+3), u(k-p+4), \ldots, u(k-1)\}$
- $v(k-2)$ multiplied with C_1^1 of $u(k-1)$, which of course is $u(k-1)$

The condition to be satisfied to obtain Eq. (22) is that the sum of all the terms depending on $\hat{x}(k-p)$ in the propagated equation vanishes. This condition is symbolically indicated as

$$\mathcal{S}_p(k)\left(\bar{A}, \bar{N}, \{u(k-p) \ \ \ldots \ \ u(k-1)\}\right) = 0 \tag{23}$$

and for $p = 2$ it takes the explicit form

$$\bar{A}^2 + \bar{A}\bar{N}u(k-2) + \bar{N}\bar{A}u(k-1) + \bar{N}^2 u(k-2)u(k-1) = 0 \tag{24}$$

In Reference 14 it was shown how there exist bilinear systems that satisfy Eq. (23) exactly for $p \geq n$ (referred to as ideal bilinear systems). It was also proven that for arbitrary bilinear systems with observable linear part* Eq. (23) is asymptotically satisfied as $p \gg n$, provided the magnitude of the input u is bounded. The following theorem summarizes the result.

<u>Theorem 0</u>. *Given any bilinear system of the form of Eq. (1), if (A, C) is an observable pair then there exist gains M_1 and M_2 and a value γ such that, for $|u(k)| < \gamma$ for all k, Eq. (23) is asymptotically satisfied as p increases.*

In order to use Eq. (22) as an IOSR, we still need to prove that the state estimation error

$$e(k) = x(k) - \hat{x}(k) \tag{25}$$

*A bilinear system is said to have observable linear part if (A, C) is an observable pair, in accordance with linear system theory.

converges to 0 as more input-output data are collected. In References 16, 17 it was proven that the dynamics of the state estimation error directly depends on $\mathcal{S}_p(k)$. Indeed, starting the observer at time step $k-p$, the estimation error evolves as

$$e(k) = \mathcal{S}_p(k)e(k-p) \tag{26}$$

Therefore, Theorem 0 also ensures that if a sufficiently large* number p of time steps elapses, then $e(k) \to 0$. More generally,

$$\hat{x}(k) \to x(k) \qquad \text{for } p \geq n \tag{27}$$

Therefore $\hat{x}(k) = T_p z_p(k)$ in Eq. (22) can be rewritten as an IOSR, i.e. as $x(k) = T_p z_p(k)$. The IOSR derived via the bilinear observer in Eq. (14) can then be used to implement the ELM or IS methods to identify the matrices A, N, B, C, and D from measured input-output data.

IOSRs via Minimal-Order Bilinear State Observers

Generally a state-space model is defined with fewer outputs than states, i.e. $q < n$. As the choice of the state variables is defined within a coordinate transformation, without loss of generality we can choose

$$\begin{aligned} x_1 &= y_1 \\ x_2 &= y_2 \\ &\vdots \\ x_q &= y_q \end{aligned}$$

$x_{q+1}, \ldots, x_n$ are unmeasured state variables. For convenience, by partitioning the state vector in its measured ($x_m \in \mathbb{R}^q$) and unmeasured ($x_u \in \mathbb{R}^{n-q}$) subsets

$$x(k) = \begin{bmatrix} x_m(k) \\ x_u(k) \end{bmatrix} = \begin{bmatrix} y(k) \\ x_u(k) \end{bmatrix} \tag{29}$$

we can rewrite Eq. (1) in the following block matrix form

$$\begin{bmatrix} x_m(k+1) \\ x_u(k+1) \end{bmatrix} = \begin{bmatrix} A_{mm} & A_{mu} \\ A_{um} & A_{uu} \end{bmatrix} \begin{bmatrix} x_m(k) \\ x_u(k) \end{bmatrix} + \begin{bmatrix} N_{mm} & N_{mu} \\ N_{um} & N_{uu} \end{bmatrix} \begin{bmatrix} x_m(k) \\ x_u(k) \end{bmatrix} u(k) + \begin{bmatrix} B_m \\ B_u \end{bmatrix} u(k) \tag{30a}$$

$$y(k) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_m(k) \\ x_u(k) \end{bmatrix} + Du(k) \tag{30b}$$

where I is the identity matrix of dimension q and 0 indicates a null matrix of appropriate dimension, in this specific case $q \times (n-q)$.

Since q out of n state variables are directly measured, it makes sense to take the measured outputs as the estimates for x_m and use an observer to estimate x_u only. Reference 25 provides a useful formalization of the concept of low-order state observer for linear systems. Following the same approach, Reference 18 studied minimal-order state observers for bilinear systems, in particular those with estimation error dynamics independent on the input†. In this regard, note how the error

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*$p \geq n$ for ideal bilinear systems, $p \gg n$ for arbitrary bilinear systems.

†For the sake of clarity, Reference 18 studied minimal-order bilinear observers in the continuous-time case. Nevertheless the results can be extended to the discrete-time case.

dynamics of the full-order observer in Eq. (14) depends in general on the input, since $\mathcal{S}_p(k)$ in Eq. (26) is a function of $u(k-p)$, $u(k-p+1)$, ..., $u(k-1)$. Because minimal-order observers are at the core of the new IOSRs (and the corresponding new bilinear system identification algorithms) developed in this paper, the existence of minimal-order bilinear observers is addressed in a dedicated section. In this section, we show how to use them to derive IOSRs.

The minimal-order bilinear observer formulated in Reference 18 is*

$$\hat{x}_u(k+1) = \tilde{A}\hat{x}_u(k) + \tilde{N}\hat{x}_u(k)u(k) + \tilde{F}u(k) + \tilde{M}_1 y(k) + \tilde{M}_2 y(k)u(k) + \tilde{H}u^2(k) \tag{31}$$

Note how the matrices in Eq. (31) are denoted with a tilde to remark how they are analogous to the matrices in Eq. (16a) only in that they multiply similar terms. However, these matrices are different even in dimension. For instance, $\tilde{A} \in \mathbb{R}^{(n-q)\times(n-q)}$ whereas $\bar{A} \in \mathbb{R}^{n\times n}$. To emphasize the symmetry with Eq. (16a), Eq. (31) can be rewritten as

$$\hat{x}_u(k+1) = \tilde{A}\hat{x}_u(k) + \tilde{N}\hat{x}_u(k)u(k) + \tilde{B}v(k) \tag{32}$$

where $v(k)$ is defined exactly as in Eq. (15c) and $\tilde{B}$ is given by

$$\tilde{B} = \begin{bmatrix} \tilde{F} & \tilde{M}_1 & \tilde{H} & \tilde{M}_2 \end{bmatrix} \tag{33}$$

Equation (32) can now be treated exactly like Eq. (16a). By propagating it $p-1$ time steps forward and shifting back by p steps, we obtain

$$\hat{x}_u(k) = \tilde{T}_p z_p(k) + \tilde{S}_p(k)\hat{x}_u(k-p) \tag{34}$$

where z_p is constructed as shown below Eq. (22), $\tilde{T}_p$ is formally defined as T_p in Eq. (22), e.g. for $p = 2$ it becomes

$$\tilde{T}_2 = \begin{bmatrix} \tilde{A}\tilde{B} & \tilde{N}\tilde{B} & \tilde{B} \end{bmatrix} \tag{35}$$

The unmeasured state-dependent terms, grouped in $\tilde{\mathcal{S}}_p(k)$, vanish for bilinear systems that satisfy the conditions for the existence of minimal-order state observers (see next section). Equation (22) then becomes

$$\hat{x}_u(k) = \tilde{T}_p z_p(k) \tag{36}$$

To obtain an IOSR as defined in Eq. (3), we use Eq. (36) to express the entire vector state. Recalling Eq. (29), we can write

$$\hat{x}(k) = \begin{bmatrix} I & 0 \\ 0 & \tilde{T}_p \end{bmatrix} \begin{bmatrix} y(k) \\ z_p(k) \end{bmatrix} \tag{37}$$

Additionally (see next section), if a deadbeat minimal-order bilinear observer exists, the state estimation error converges to exactly 0 in exactly $n-q$ time steps. If a minimal-order bilinear observer exists but is not deadbeat, then the state estimation error converges to 0 as p is increased. More generally,

$$\hat{x}(k) \to x(k) \qquad \text{for} \quad p \geq n-q \tag{38}$$

*Actually, Reference 18 did not take into account the possibility of the bilinear system having a non-zero direct influence matrix D. The latter gives rise to the term $Hu^2(k)$ in Eq. (31).

Existence of Minimal-Order Bilinear Observers

The theoretical foundation of the IOSRs developed in this paper and applied to bilinear system identification is found in the work from S. Hara and K. Furuta (Reference 18). They studied the conditions for the existence of minimal-order bilinear observers for bilinear systems, whose error dynamics does not depend on the input u. The work in Reference 18 refers to continuous-time bilinear systems and does not take into account the possible of a non-zero direct influence matrix D. The results can be extended to the discrete-time domain and to the case with $D \neq 0$. As such, they are summarized below in the form of three theorems.

Before plunging into such theorems, it is convenient to recall that, similarly to Eq. (26), the state estimation error dynamics is given by

$$e_u(k) = x_u(k) - \hat{x}_u(k) = \tilde{\mathcal{S}}_p(k) e_u(k-p) \tag{39}$$

Splitting $\tilde{\mathcal{S}}_p(k)$ into two parts, one depending only on the state and the other one also function of the input, we can write

$$e_u(k) = \tilde{A} e_u(k-p) + \tilde{Q}_p(k) e_u(k-p) \tag{40}$$

where, for $p = 2$, $\tilde{Q}_p(k)$ becomes

$$\tilde{Q}_2(k) = \tilde{A}\tilde{N}u(k-2) + \tilde{N}\tilde{A}u(k-1) + \tilde{N}^2 u(k-2)u(k-1) \tag{41}$$

As a consequence of focusing on minimal-order observers whose state estimation error dynamics is independent of the input, $\tilde{Q}_p(k) = 0$ for all k. Hence, Eq. (34) becomes

$$\hat{x}_u(k) = \tilde{T}_p z_p(k) + \tilde{A}\hat{x}_u(k-p) \tag{42}$$

The matrix multiplying the past state estimate can be expressed as

$$\tilde{A} = A_{uu} - L A_{mu} \tag{43}$$

where $L \in \mathbb{R}^{(n-q)\times q}$ is interpreted as the gain of the minimal-order observer.

Theorem 1. *A minimal-order state observer for the bilinear system in Eq. (1) exists if and only if there is a coordinate system where Eq. (1) can be written as*

$$\begin{bmatrix} x_m(k+1) \\ x_u(k+1) \end{bmatrix} = \begin{bmatrix} A_{mm} & A_{mu} \\ A_{um} & A_{uu} \end{bmatrix} \begin{bmatrix} x_m(k) \\ x_u(k) \end{bmatrix} + \begin{bmatrix} N_{mm} & N_{mu} \\ N_{um} & 0 \end{bmatrix} \begin{bmatrix} x_m(k) \\ x_u(k) \end{bmatrix} u(k) + \begin{bmatrix} B_m \\ B_u \end{bmatrix} u(k) \tag{44a}$$

$$y(k) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_m(k) \\ x_u(k) \end{bmatrix} + Du(k) \tag{44b}$$

and there is a matrix L such that $LN_{mu} = 0$ and all the eigenvalues of $A_{uu} - LA_{mu}$ lie within the unit circle.

Note that the conditions in Theorem 1 do not guarantee the existence of a deadbeat observer. They guarantee the existence of a stable (convergent) observer. If L satisfying $LN_{mu} = 0$ is such that all the eigenvalues of $A_{uu} - LA_{mu}$ can be arbitrarily chosen, then a deadbeat observer is guaranteed to exist and is given by the matrix L placing the eigenvalues of $A_{uu} - LA_{mu}$ at the origin. Note that being able to write the bilinear system in the form of Eq. (44) is a necessary but not sufficient condition. The intuition for the necessity is the following. $\tilde{N}$ in Eq. (31) corresponds to N_{uu} and the corresponding term $\tilde{N}\hat{x}_u(k)u(k)$ needs to vanish for the error dynamics to be independent from the input. Hence, N_{uu} must be equal to 0. However, the error dynamics might be unstable, from which the additional condition on the eigenvalues of the error dynamics matrix $\tilde{A} = A_{uu} - LA_{mu}$.

Theorem 2. *If the bilinear system in Eq. (1) is observable and can be transformed into the form*

$$\begin{bmatrix} x_m(k+1) \\ x_u(k+1) \end{bmatrix} = \begin{bmatrix} A_{mm} & A_{mu} \\ A_{um} & A_{uu} \end{bmatrix} \begin{bmatrix} x_m(k) \\ x_u(k) \end{bmatrix} + \begin{bmatrix} N_{mm} & 0 \\ N_{um} & 0 \end{bmatrix} \begin{bmatrix} x_m(k) \\ x_u(k) \end{bmatrix} u(k) + \begin{bmatrix} B_m \\ B_u \end{bmatrix} u(k) \tag{45a}$$

$$y(k) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_m(k) \\ x_u(k) \end{bmatrix} + Du(k) \tag{45b}$$

a minimal-order bilinear observer exists for it and all its eigenvalues can be arbitrarily chosen.

As a corollary of Theorem 2, for an observable bilinear system that can be written in the form of Eq. (45), a deadbeat bilinear observer exists.

Theorem 3. *If a minimal-order linear observer exists for the bilinear system in Eq. (1), there is a coordinate system where Eq. (1) can be written as*

$$\begin{bmatrix} x_m(k+1) \\ x_u(k+1) \end{bmatrix} = \begin{bmatrix} A_{mm} & A_{mu} \\ A_{um} & A_{uu} \end{bmatrix} \begin{bmatrix} x_m(k) \\ x_u(k) \end{bmatrix} + \begin{bmatrix} N_{mm} & N_{mu} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_m(k) \\ x_u(k) \end{bmatrix} u(k) + \begin{bmatrix} B_m \\ B_u \end{bmatrix} u(k) \tag{46a}$$

$$y(k) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_m(k) \\ x_u(k) \end{bmatrix} + Du(k) \tag{46b}$$

Note that Theorem 3 only provides a necessary condition. Also, it concerns the existence of a *linear* observer, i.e. an observer in the form of Eq. (31) with $\tilde{N}$, $\tilde{M}_2$ and $\tilde{H}$ equal to 0. In such a case, T_p and z_p in Eq. (42) take the following, simpler form

$$T_p = \begin{bmatrix} \tilde{A}^{p-1}\tilde{B} & \dots & \tilde{A}\tilde{B} & \tilde{B} \end{bmatrix} \tag{47a}$$

$$z_p(k) = \begin{bmatrix} u(k-p) \\ y(k-p) \\ \vdots \\ u(k-2) \\ y(k-2) \\ u(k-1) \\ y(k-1) \end{bmatrix} \tag{47b}$$

where $\tilde{B} = \begin{bmatrix} \tilde{F} & \tilde{M}_1 \end{bmatrix}$ is also simplified with respect to Eq. (33).

Relationship between IOSRs from Minimal- and Full-Order Observers

The IOSRs derived from full- and minimal-order observers share common features. The super-state $z_p(k)$ in Eqs. (22) and (36) is defined exactly in the same way and the construction of T_p and $\tilde{T}_p$

follows the same pattern (to get the latter from the former, it is sufficient to replace the bar matrices in Eq. (16a) with the tilde matrices in Eq. (31)). By writing the IOSRs obtained in the two cases in the following general form

$$x(k) = T_{FO} z_{FO}(k) \tag{48a}$$
$$x(k) = T_{MO} z_{MO}(k) \tag{48b}$$

where the subscripts FO and MO stand for full-order and minimal-order, respectively, we can highlight the differences:

$$T_{FO} = T_p \qquad z_{FO}(k) = z_p(k) \qquad \text{for} \quad p \geq n \tag{49a}$$
$$T_{MO} = \begin{bmatrix} I & 0 \\ 0 & \tilde{T}_p \end{bmatrix} \qquad z_{MO}(k) = \begin{bmatrix} y(k) \\ z_p(k) \end{bmatrix} \qquad \text{for} \quad p \geq n - q \tag{49b}$$

The advantage of minimal-order observers in bilinear system identification is now apparent. As the IOSRs based on minimal-order observers benefit from the use of the measurements of q state variables, the parameter p can in general be lower than in the case of IOSRs derived from full-order observers. To better understand the significance of the advantage, one needs to go back to the pattern to construct z_p, below Eq. (22). As p increases, the dimension (number of entries) in $z_p(k)$ grows exponentially. This leads to the computational issues discussed in References 14, 15, the latter of which provides a way to avoid the curse of dimensionality to the expense of requiring a specialized input. Instead, the IOSRs developed in the present paper are valid for arbitrary excitation input, similarly to the ones from Reference 14. The computational advantage offered by the IOSRs from minimal-order observers is more significant as more outputs are measured (larger q). In the special case of Theorem 3, the existence of a linear observer makes the computational advantage even more striking. As shown in Eq. (47), the dimension of the superstate z_p grows linearly with p. The curse of dimensionality is completely eliminated and high-order bilinear models can be identified using an arbitrary excitation input.

The second limitation of the IOSRs based on full-order observers is that the theorem ensuring their asymptotic convergence (Theorem 0) requires the magnitude of the excitation input to be bound within a threshold. Moreover, the threshold depends on the bilinear system itself. Therefore it is unknown before identification, potentially leading to an iterative process where the level of excitation needs to be decreased to let the IOSRs at the core of the identification approach converge. Instead, the minimal-order observers studied in Reference18 have error dynamics independent from the input. The level of excitation can then be arbitrarily chosen, since the identification method does not place any constraint on it. The approach to system identification presented in this paper is therefore suitable for arbitrary excitation, both in the temporal profile and in the amplitude.

EXAMPLES

In order to validate the new IOSRs based on minimal-order observers and their application to bilinear system identification, they are shown in action via the ELM method. The examples are based on the system analyzed in Reference 14, together with some variants aiming to illustrate the three cases arising from the results of Reference 18. In each example, measured input-output data are simulated by generating a random input sequence $\{u(k)\}$ of $l = 3000$ samples (from a uniform distribution between 0 and 1) and using it to obtain the output history $\{y(k)\}$ via the state-space model equations of the system to be identified. The sequences $\{u(k)\}$ and $\{y(k)\}$ are the input to the following identification algorithm based on the ELM method outlined above.

Algorithm

1. *Construction of ELM Input.* Form $w(k)$ time history, Eq. (5b), for $k = p, p+1, \ldots, l-1$ with $z(k) = \begin{bmatrix} y^T(k) & z_p^T(k) \end{bmatrix}$ as in Eq. (37). $z_p(k)$ is constructed according to the pattern below Eq. (22). The order p of the IOSR is chosen greater than $n - q$, the difference between the assumed order n of the system and the number q of measured outputs.

2. *Identification of ELM.* Identify the ELM of Eq. (4) by any linear system identification technique. In the examples below, the intersection subspace method presented above is used, modified for a linear system (see also Reference 14).

3. *Reconstruction of State History.* Simulate the identified ELM driven by the input $w(k)$ to reconstruct the state history $x(k)$ for $k = 0, 1, \ldots, l-1$; since the initial state is unknown, discard the reconstructed samples corresponding to the initial transient. In the examples below, the first $k_{ss} = 500$ time steps are discarded.

4. *Estimation of Bilinear System Matrices.* The matrices A, C, and D of the identified ELM directly provide an estimate of the corresponding matrices of the identified bilinear model. The B matrix of the bilinear model is extracted from B_{ELM} in accordance with the partitioning in Eq. (5a). Finally, use the reconstructed state history $x(k)$, $k = k_{ss}, \ldots, l-1$, to form and solve the least-squares problem of Eq. (6) and find the estimate for N.

To validate the identified bilinear model, *another* random input sequence $\{u(k)\}$ is generated (independently from the one used for identification) and used to drive both the original system and the identified model, starting from the same initial conditions (zero for convenience). The difference in their outputs, also known as prediction error, is analyzed to gauge how accurate the identification algorithm was. Among the metrics chosen to rigorously compare different identified model are the mean and the root-mean-square (rms) of their prediction error.

Example 1

With reference to the following bilinear system

$$A = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & A_{uu} \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = 0 \quad (50)$$

consider first the case when $A_{uu} = 0$. A deadbeat full-order observer exists for the system in Eq. (50) (References 16, 17). The IOSR in Eq. (48a) is then exact for $p \geq 2$ and so is the identification via the ELM method based on such an IOSR (Reference 14). The system is also in the form of Eq. (44). According to Theorem 1 and the associated observer design procedure given in Reference 18, a minimal-order bilinear observer exists with $L = 0$. Because $L = 0$, the dynamics of the minimal-order observer is given by the eigenvalues of A_{uu}. Since $A_{uu} = 0$, the minimal-order

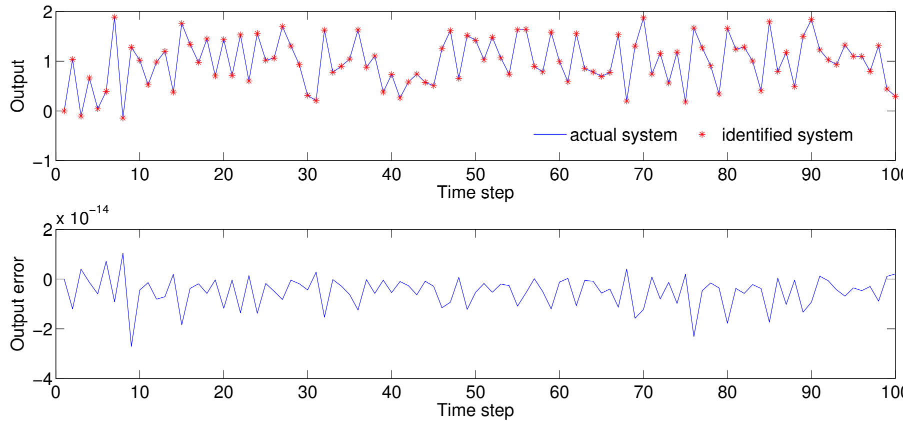
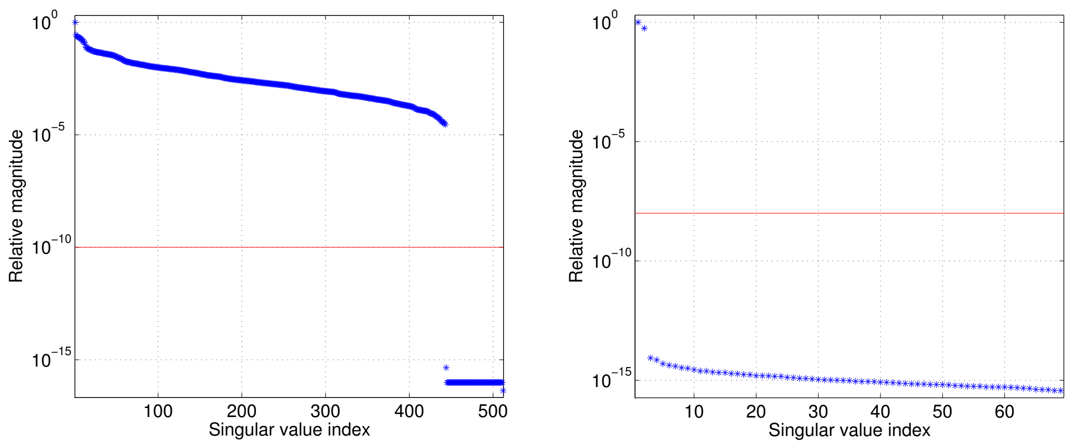


Figure 1: **Example 1 with $A_{uu} = 0$: comparison of output from actual system and identified model.**



(a) **Intersection of superspaces Z_c and Z_a (zero singular values are selected).**

(b) **Minimum basis for the intersection subspace (non-zero singular values are selected).**

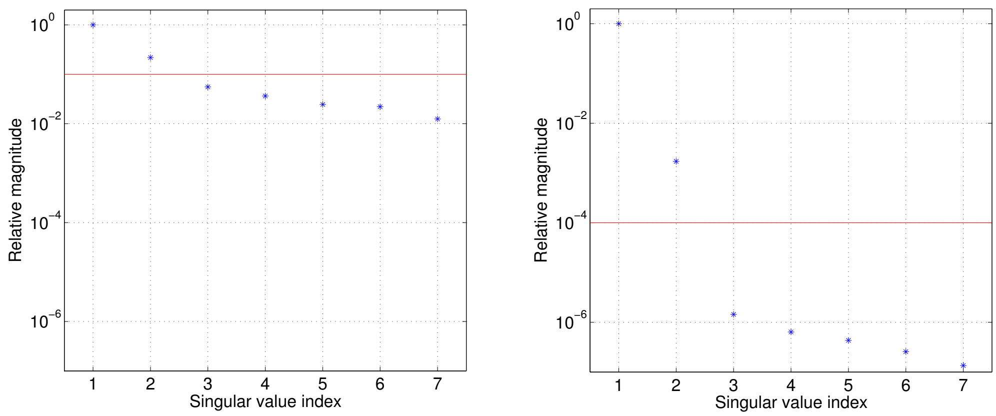
Figure 2: **Example 1 with $A_{uu} = 0$: singular value plots.**

observer is deadbeat (estimation error converging to exactly zero in $n - q = 1$ time steps) and the IOSR in Eq. (48b) is expected to be exact for $p \geq 1$. We can verify this by applying the above identification algorithm. Setting $p = 4$, the error between the true system and the identified model output on the validation dataset can is indeed zero (10^{-14} is Matlab® numerical zero, see Figure 1). Additionally, note how clear the gap between zero and non-zero singular values is in the singular value decompositions necessary for the identification of the ELM (Figure 2): 15 orders of magnitude (small singular values close to Matlab® numerical zero) are a sign of exact identification. It is worth adding that plots for validation and singular value decomposition similar to those in Figures 1 and 2 are obtained if the direct influence matrix D is changed to 0.5. This confirms that the results from Reference 18 can be extended to the case when $D \neq 0$.

It is of interest to analyze what happens when the system in Eq. (50) is modified to $A_{uu} = 0.1$.

Table 1: **Example 1 with $A_{uu} \neq 0$: improvement of identification accuracy as the order p of the IOSR increases.**

p	2	3	4	5	6
mean $_{output\ error}$	3.7×10^{-2}	3.4×10^{-4}	3.4×10^{-4}	-3.3×10^{-4}	-5.8×10^{-5}
rms $_{output\ error}$	1.0×10^{-1}	1.4×10^{-2}	1.4×10^{-2}	9.8×10^{-4}	1.6×10^{-4}



(a) $p = 2$.

(b) $p = 6$.

Figure 3: **Example 1 with $A_{uu} \neq 0$: singular value plots to find the minimum basis for the intersection subspace.**

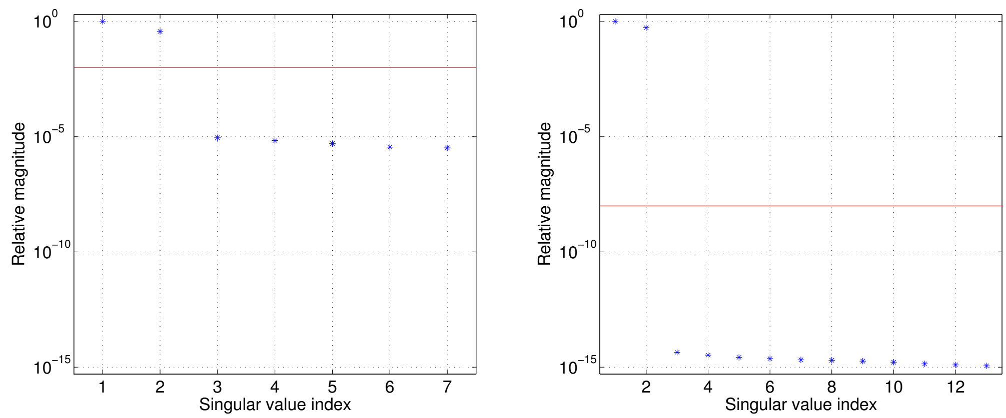
The minimal-order observer still exists: L is again null, but the observer eigenvalue is now different from 0 since so is A_{uu}. Nevertheless, 0.1 lies within the unit circle, hence the observer is stable. The approximation of the corresponding IOSR is due to a term depending on A_{uu}^p and so is the state estimation error. Therefore the approximation $\hat{x}_u(k) \approx x(k)$ is expected to significantly improve as p is increased. This is shown in Table 1 in terms of mean and root mean square of the output prediction error of the identified model over the validation dataset Correspondingly, the gap between zero and non-zero singular values in the SVDs becomes more clear as shown in Figure 3. The example with $A_{uu} = 0.1$ is good to illustrate the case where a minimal-order bilinear observer exists but its eigenvalues cannot be changed to make it deadbeat. This is in contrast with the previous, more fortunate example ($A_{uu} = 0$), where the only existing minimal-order bilinear observer was deadbeat.

Example 2

Consider now the following bilinear system

$$A = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = 0 \qquad (51)$$

which belongs to the class of bilinear systems that can be represented in the form of Eq. (45). According to Theorem 2, a minimal-order bilinear observer exists and its eigenvalues can be arbitrarily



(a) $p = 6$. (b) $p = 15$.

Figure 4: **Example 3: singular value plots to find the minimum basis for the intersection subspace.**

placed. Among all the possible choices, the identification algorithm will automatically use the dead-beat observer as that is the one minimizing (to zero) the approximation $\hat{x}_u(k) \approx x(k)$. Indeed the bias and the rms value of the output error between the true and identified models are -9.2×10^{-15} and 1.2×10^{-13}, respectively.

Example 3

Finally, consider the following bilinear system

$$A = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = 0 \tag{52}$$

which belongs to the class of bilinear systems that can be represented in the form of Eq. (46). According to Theorem 3 and the associated observer design procedure given in Reference 18, a minimal-order *linear* observer exists and its error dynamics is governed by $A_{uu} = 0.1$. Since the observer is linear, the structure of z_p is significantly simpler. In particular, as shown in Eq. (47), the dimension of z_p increases linearly with p as opposed to the exponential growth as in the case of a bilinear observer. The presented algorithm is then implemented with z_p constructed as in Eq. (47), with the advantage of being able to set p as large as desired. As shown in Figure 4, with a value of $p = 15$ the identification can already be considered exact. It is worth remarking that a value of $p = 15$ is not computationally achievable with standard computing power when constructing the IOSR with a strictly bilinear observer. Generally, for systems in the form of Eq. (46) like the one in Eq. (52), a full-order linear observer does not exist. This example emphasizes the advantage of exploiting the results on minimal-order observers over using the full-order bilinear observers proposed in References 14, 17.

CONCLUSIONS

At the core of this paper is a new formulation of existing methods for discrete-time bilinear system identification. Originally formulated by interaction matrices, these methods have now been explained in terms of full-order bilinear state observers. The new formulation has the merit of providing the intuition for the application, for the first time, of minimal-order state observers to bilinear system identification.

Exploiting results already available in the literature on minimal-order bilinear observers, this paper has extended the family of Input-Output-to-State Representations (IOSRs) for bilinear systems, i.e. relationships expressing the bilinear state as a linear combination of input-output data only. Such IOSRs have been used to develop new algorithms for discrete-time bilinear system identification. The new IOSRs feature some benefits over the ones available in the literature. In particular, they do not impose any restriction on the excitation input (nor its profile or its magnitude) and offer computational advantages over the IOSRs based on full-order observers. The curse of dimensionality affecting the latter is completely overcome for a specific class of bilinear systems for which the minimal-order bilinear observer degenerates into linear.

The proposed approach to bilinear discrete-time system identification can then play an important role in the identification of bilinear models to approximate more general nonlinear systems, paving the way for an effective way to handle more general identification and control problems such as satellite attitude control. Numerical examples have been given to validate the new identification algorithms and illustrate their main features and advantages.

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