Computational Methods for Nonlinear Optimization Problems: Theory and Applications

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ABSTRACT

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This dissertation is motivated by the lack of efficient global optimization techniques for polynomial optimization problems. The objective is twofold. First, a new mathematical foundation for obtaining a global or near-global solution will be developed. Second, several case studies will be conducted on a variety of real-world problems. Global optimization, convex relaxation and distributed computation are at the heart of this PhD dissertation. Some of the specific problems to be addressed in this thesis on both the theory and the application of nonlinear optimization are explained below:

Graph theoretic algorithms for low-rank optimization problems: There is a rapidly growing interest in the recovery of an unknown low-rank matrix from limited information and measurements. This problem occurs in many areas of engineering and applied science such as machine learning, control, and computer vision. We develop a graph-theoretic technique in Part I that is able to generate a low-rank solution for a sparse Linear Matrix Inequality (LMI), which is directly applicable to a large set of problems such as low-rank matrix completion with many unknown entries. Our approach finds a solution with a guarantee on its rank, using the recent advances in graph theory.

Resource allocation for energy systems: The flows in an electrical grid are described by nonlinear AC power flow equations. Due to the nonlinear interrelation among physical parameters of the network, the feasibility region represented by power flow equations may be nonconvex and disconnected. Since 1962, the nonlinearity of the network constraints has been studied, and various heuristic and local-search algorithms have been proposed in order to perform optimization over an electrical grid [Baldick, 2006; Pandya and Joshi, 2008]. Part II is concerned with finding convex formulations of the power flow equations using semidefinite programming (SDP). The potential of SDP relaxation for problems in power systems has been manifested in [Lavaei and Low, 2012], with

further studies conducted in [Lavaei, 2011; Sojoudi and Lavaei, 2012]. A variety of graph-theoretic and algebraic methods are developed in Part II in order to facilitate performing fundamental, yet challenging tasks such as optimal power flow (OPF) problem, security-constrained OPF and the classical power flow problem.

Synthesis of distributed control systems: Real-world systems mostly consist of many interconnected subsystems, and designing an optimal controller for them pose several challenges to the field of control theory. The area of *distributed control* is created to address the challenges arising in the control of these systems. The objective is to design a constrained controller whose structure is specified by a set of permissible interactions between the local controllers with the aim of reducing the computation or communication complexity of the overall controller. It has been long known that the design of an optimal distributed (decentralized) controller is a daunting task because it amounts to an NP-hard optimization problem in general [Witsenhausen, 1968; Tsitsiklis and Athans, 1984]. Part III is devoted to study the potential of the SDP relaxation for the optimal distributed control (ODC) problem Our approach rests on formulating each of different variations of the ODC problem as rank-constrained optimization problems from which SDP relaxations can be derived. As the first contribution, we show that the ODC problem admits a sparse SDP relaxation with solutions of rank at most 3. Since a rank-1 SDP matrix can be mapped back into a globally-optimal controller, the low-rank SDP solution may be deployed to retrieve a near-global controller.

Parallel computation for sparse semidefinite programs: While small- to medium-sized semidefinite programs are efficiently solvable by second-order-based interior point methods in polynomial time up to any arbitrary precision [Vandenberghe and Boyd, 1996a], these methods are impractical for solving large-scale SDPs due to computation time and memory issues. In Part IV of this dissertation, a parallel algorithm for solving an arbitrary SDP is introduced based on the alternating direction method of multipliers. The proposed algorithm has a guaranteed convergence under very mild assumptions. Each iteration of this algorithm has a simple closed-form solution, and consists of scalar multiplication and eigenvalue decomposition over matrices whose sizes are not greater than the treewdith of the sparsity graph of the SDP problem. The cheap iterations of the proposed algorithm enable solving real-world large-scale conic optimization problems.

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To the best teachers in the world, my Mom and Dad.

Chapter 1

Introduction

Optimization theory deals with the minimization of an objective function subject to a set of constraints. This area plays a vital role in the design, control, operation, and analysis of real-world systems. The development of efficient optimization techniques and numerical algorithms has been an active area of research for many decades. The goal is to design a robust and scalable method that is able to find a global solution in polynomial time. This question has been fully answered for the class of convex optimization problems that includes all linear and some nonlinear problems [Boyd and Vandenberghe, 2004; Nesterov *et al.*, 1994]. Convex optimization has found a wide range of applications across engineering and economics [Ben-Tal and Nemirovski, 2001]. In the past several years, a great effort has been devoted to casting real-world problems as convex optimization. Nevertheless, several classes of optimization problems, including polynomial optimization and quadratically constrained quadratic program (QCQP) as a special case, are nonlinear, non-convex, and NP-hard in the worst case [Murty and Kabadi, 1987; Klerk, 2008]. In particular, there is no known effective optimization technique for integer and combinatorial optimization as a small subclass of QCQP [Nemhauser and Wolsey, 1988; Papadimitriou and Steiglitz, 1998]. Given a non-convex optimization, there are several techniques to find a solution that is locally optimal. However, seeking a global or near-global solution in polynomial time is a daunting challenge. There is a large body of literature on nonlinear optimization witnessing the complexity of this problem.

This dissertation develops a graph-theoretic basis in order to reduce the computational complexity of sparse rank-constrained and polynomial optimization problems in Part I. The results of

Part I are then illustrated in multiple nonlinear optimization problems for electrical networks and design of distributed controllers in Parts II and III. The proposed algorithms for solving nonlinear optimization problems in this work are based on semidefinite relaxation. Part IV of this dissertation introduces a robust and scalable computational method for solving large scale and sparse semidefinite programs.

1.1 Part I: Rank and Sparsity

Part I is concerned with the problem of finding a low-rank solution of an arbitrary sparse linear matrix inequality (LMI). The structure of the problem is mapped into a graph and multiple graph-theoretic convex programs are proposed in order to obtain a low-rank solution.

1.1.1 Low-rank Solution of Sparse Linear Matrix Inequalities

A large variety of nonlinear optimization problems can be boiled down to the classical problem of searching for a low-rank matrix in a convex set. The procedure through which a QCQP problem can be cast as low-rank matrix optimization problem is called SDP relaxation. The existence of a rank-1 solution for the SDP relaxation guarantees the equivalence of the original QCQP and its relaxed problem. In this case the relaxation is said to be exact. The exactness of the relaxation is substantiated for various applications [Lavaei and Low, 2012; Lasserre, 2001; Kim and Kojima, 2003; Lavaei *et al.*, 2011].

In many applications, the QCQP problem has a sparse underlying structure and as a result, the convex set under study can be characterized by a sparse linear matrix inequality. The SDP relaxation of a sparse QCQP problem often has infinitely many solutions and the conventional numerical algorithms would find a solution with the highest rank. We will verify in Chapter 3 that in many cases the SDP relaxation may have a hidden rank-1 solution that could not be easily found. Hence, a question arises as to whether a low-rank solution of the SDP relaxation of a sparse QCQP can be found efficiently. To address this problem, we capture the sparsity level of the original problem through a graph $\mathcal{G}$ and use graph-theoretic algorithms to offer multiple convex programs that are guaranteed to result in a low-rank solution.

The notions of tree decomposition, minimum semidefinite rank (msr) of a graph, ordered set vertex (OS-vertex) and positive semidefinite zero forcing are utilized in Chapter 2 to find low-rank matrices using convex optimization. We propose a class of convex optimization problem and we show that the rank of every solution of these problems can be upper bounded in terms of the OS and msr of some supergraphs of $\mathcal{G}$. The results of this part can be readily applied to three separate problems of minimum-rank matrix completion, conic relaxation for polynomial optimization, and affine rank minimization. A similar scheme is employed in later chapters for solving real-world problems in power systems and the optimal distributed control problem.

1.2 Part II: Power Networks

A power grid is a large-scale electrical circuit consisting of transmission lines, transformers, and various types of power electronic devices, and is used to transport power from generators (suppliers) to loads (consumers). The real-time operation of a power grid is based on periodically forecasting the time-varying load profiles and accordingly controlling the active devices in the network in order to meet the demand requests while satisfying the network constraints. This is accomplished by adjusting the controllable parameters of the power system (e.g., the mechanical input of each generator and the tap ratio of each transformer) in such a way that the total cost of the energy production is minimized and the stability of the network is ensured.

The real-time operation of a power network depends heavily on several resource allocation problems solved from every few minutes to every year. Optimal power flow (OPF), security constrained OPF (SCOPF), unit commitment (UC), transmission planning, sizing of capacitor banks, and network reconfiguration are some fundamental optimization problems solved for power networks. It is a daunting challenge to solve these problems efficiently due to the nonlinearity/non-convexity created by two different sources: (i) discrete variables such as the ratio of a tap-changing transformer, the on/off status of a line switch, or the commitment parameter of a generator, and (ii) the laws of physics. Due to these non-convexity issues, the existing solvers for energy-related optimization problems either make potentially very conservative approximations to power problems or deploy general-purpose local-search algorithms. Although these OPF-based problems have been studied for 50 years, there does not yet exist a scalable, robust algorithm for this problem, which may incur tens of billions of dollars annually.

The objective of this part of the dissertation is to design a scalable, robust algorithm seeking globally optimal solutions for the above-mentioned power problems. As a preliminary result, we have developed an algorithm and implemented it in a piece of software in [Madani *et al.*, 2014a], which is able to find feasible solutions for several real-world OPF problems with at least 99% global optimality guarantees. For example, the OPF problem for a Polish system with almost 3000 buses (leading to over one million parameters in the SDP relaxation) can be solved on a laptop in a few minutes using our software, resulting in a provably near-global solution.

1.2.1 Convex Relaxation for Optimal Power Flow Problem: Mesh Networks

The optimal power flow involves minimization of an operating cost function in an electrical grid subject to network and physical constraints on loads, powers, voltages and line flows [Momoh, 2001]. The OPF problem is non-convex and NP-hard, due to its possible reduction to the (0,1)-quadratic optimization. Started by the work [Carpentier, 1962] in 1962, many of the existing optimization techniques have been studied for the OPF problem, leading to algorithms based on linear programming, Newton Raphson, quadratic programming, nonlinear programming, Lagrange relaxation, interior point method, artificial intelligence, artificial neural network, fuzzy logic, genetic algorithm, evolutionary programming and particle swarm optimization [Pandya and Joshi, 2008]. Due to the non-convexity of OPF, these algorithms are not robust, lack performance guarantees, and may not find a global optimum. The paper [Lavaei and Low, 2012] proposes a convex relaxation method for solving OPF based on semidefinite programming.

In Chapter 3, we investigate the possibility of finding a global or near-global solution of the OPF problem for mesh networks by solving only a few SDP relaxations. The SDP relaxation for OPF has attracted much attention due to its ability to find a global solution in polynomial time, and it has been applied to various applications in power systems including: voltage regulation in distribution systems [Lam *et al.*, 2012a], state estimation [Weng *et al.*, 2012], calculation of voltage stability margin [Molzahn *et al.*, 2012], economic dispatch in unbalanced distribution networks [Dall'Anese *et al.*, 2013], charging of electric vehicles [Sojoudi and Low, 2011], and power management under time-varying conditions [Ghosh *et al.*, 2011].

The exactness of SDP relaxation has been shown in [Sojoudi and Lavaei, 2012] for acyclic networks with some extra assumption related to the passivity of transmission lines and transformers.

On the other hand, multiple works have reported that SDP relaxation is not always exact even for a basic three-bus cyclic network [Lesieutre *et al.*, 2011]. Valuable test cases are also provided by [Bukhsh *et al.*, 2013] with local solutions that witness inexactness of SDP relaxation for the OPF problem. In the present work, we first consider the three-bus system studied in [Lesieutre *et al.*, 2011] and prove that the problem can be equivalently formulated in order to guarantee the exactness of SDP relaxation. We also prove that the relaxation remains exact for weakly-cyclic networks with cycles of size 3. Furthermore, we substantiate that this type of network has a convex injection region in the lossless case and a non-convex injection region with a convex Pareto front in the lossy case. The importance of this result is that the SDP relaxation works on certain cyclic networks, for example the ones generated from three-bus subgraphs (this type of network is related to three-phase systems).

In the case when the SDP relaxation does not work, an upper bound is provided on the rank of the minimum-rank solution of the SDP relaxation. This bound is related only to the structure of the power network and this number is expected to be very small for real-world power networks. Finally, a heuristic method is proposed to enforce the SDP relaxation to produce a rank-1 solution for general networks (by somehow eliminating the undesirable eigenvalues of the low-rank solution). The efficacy of the proposed technique is elucidated by extensive simulations on IEEE systems as well as a difficult example proposed in [Bukhsh *et al.*, 2013] for which the OPF problem has at least three local solutions.

1.2.2 Promises of Conic Relaxation for Contingency-Constrained Optimal Power Flow Problem

Although OPF is a fundamental problem, a real-world power flow optimization is based on a set of coupled OPFs with a variety of constraints and variables named security-constrained OPF (SCOPF)[Capitanescu *et al.*, 2007; Wood and Wollenberg, 1996]. The SCOPF problem is important in practice, since independent system operators tend to design an operating point that satisfies the demand and network constraints not only under normal operation but also under pre-specified contingencies such as line and generator outages. SCOPF is more challenging than the conventional OPF problem for two reasons. First, the size of the optimization could be prohibitive, depending on the number of contingencies. Second, SCOPF is obtained by coupling a group of non-convex

OPF problems associated with different contingencies and therefore its non-convexity would be much higher than an individual OPF problem. The purpose of Chapter 4 is to propose an efficient computational method that can be applied to not only OPF but also SCOPF.

In Chapter 4, we study the SCOPF problem through an extension of the SDP relaxation method for OPF problem. The existence of a rank-1 SDP solution guarantees the recovery of a global solution of SCOPF. We prove that the relaxation has a matrix solution whose rank is at most the treewidth of the pre-contingency network plus one. The treewidth of real-world networks is perceived to be small due to their (almost) planarity and sparsity [Fomin and Thilikos, 2006]. For example, the treewidth of the graph corresponding to a peak hour setup of a Polish system with over 3000 buses is less than 25.

The major drawback of representing the optimal power flow problem as a semidefinite program is the requirement of defining a square matrix variable, which makes the number of scalar variables of the problem quadratic with respect to the number of network buses. This may yield a very high-dimensional SDP problem for a real-world network. To address this issue, the papers [Lam *et al.*, 2012b; Zhang *et al.*, 2015; Molzahn *et al.*, 2013; Andersen *et al.*, 2014; Jabr, 2012; Molzahn and Hiskens, 2015] have leveraged the sparsity of power networks in order to break down the large-scale semidefinite constraint into small-sized constraints. The simulations performed in those papers, however, suggest that the SDP relaxation would fail to result in a rank one solution for large-scale systems [Molzahn *et al.*, 2013].

In the present work, we reduce the computational complexity of the SDP problem using a tree decomposition method to arrive at a decomposed SDP relaxation with a set of small-sized SDP matrices as opposed to a full-scale SDP matrix. We show that the full-scale SDP relaxation has a solution whose rank is upper bounded by the ranks of the small-sized matrices of the decomposed SDP relaxation. By working on the ranks of these small matrices, we propose a technique to identify the problematic lines of the network for each contingency that may contribute to the inexactness of the SDP relaxation for SCOPF. This diagnosis method may enable us to develop a heuristic method, named penalized SDP relaxation, to find a near-global solution of the problem by penalizing the loss over the problematic lines for each contingency. We perform several simulations on large-scale benchmark systems and verify that the global minima are at most 1% away from the feasible solutions obtained from the proposed penalized relaxation.

1.2.3 Convexification of Power Flow Problem over Arbitrary Networks

Chapter 5 is concerned with a fundamental problem of finding an unknown vector of complex voltages $\mathbf{V} \in \mathbb{C}^n$ for an n-bus power system that satisfies given quadratic constraints associated with known quantities that are measured or specified in the network. This quadratic feasibility problem is central to the analysis and operation of power systems. However, checking the existence of a solution is known to be NP-hard for both transmission and distribution networks due to their reduction to the *subset sum* problem [Lehmann *et al.*, 2014; Verma, 2009].

The classical power flow problem is usually solved *approximately* through linearization or in an *asymptotic* sense using Newton's method, given that the solution belongs to a good regime containing voltage vectors with small angles. The question arises as to whether the PF problem can be cast as the solution of a convex optimization problem over that regime. The objective of Chapter 5 is to show that the answer to the above question is affirmative. More precisely, we propose a class of convex optimization problems with the property that they all solve the PF problem as long as angles are small. Each convex problem proposed in this work is in the form of a semidefinite program.

One important feature of the present approach is that associated with each SDP, we explicitly characterize the set of complex voltages that can be recovered via that convex problem. Since there are infinitely many SDP problems, each capable of recovering a potentially different set of voltages, designing a good SDP problem is cast as a convex problem as well.

1.3 Part III: Distributed Control

The area of distributed control has been extensively studied for the cooperative control of multi-vehicle systems and coordination of autonomous vehicle formations [Murray, 2007; Shamma, 2007; Keviczky *et al.*, 2008]. The design of an optimal distributed controller using an efficient computational method is one of the most fundamental problems in the area of control systems, which remains as an open problem due to its NP-hardness in the worst case. This part of the dissertation aims to proceed with our recent results to develop an efficient computational method for finding a near-global distributed controller for a given large-scale systems.

This part of the thesis is motivated by the computational challenges arising in the control of many complex real-world systems such as communication networks, naval systems, aerospace systems, large-space flexible structures, traffic systems, wireless sensor networks, and various types of multi-agent systems. Each of these dynamical systems can be regarded as an interconnected or multi-channel system composed of several (interconnecting) subsystems. The classical control theory provides a rich mathematical foundation for the design of a centralized controller for such systems. A centralized control framework is concerned with a single control unit responsible for collecting the outputs of all subsystems, processing the acquired information, and generating the inputs of those subsystems. This centralized control approach is not an attractive, if not infeasible, strategy for many real-world systems due to its computation and communication complexity as well as some reliability and security issues.

1.3.1 Convex Relaxation for Optimal Distributed Control Problem

The objective of Chapter 6 is to study the potential of the SDP relaxation for the optimal distributed control (ODC) problem for both finite and infinite-horizon cases. In addition the problem of stochastic ODC (for stochastic systems) is studied in the present work. Our approach rests on formulating each of these problems as a rank-constrained optimization problem from which an SDP relaxation can be derived. With no loss of generality, this part focuses on the design of a static controller. As the first contribution, we show that the ODC problem admits a sparse SDP relaxation with solutions of rank at most 3. Since a rank-1 SDP matrix can be mapped back into a globally-optimal controller, the low-rank SDP solution may be deployed to retrieve a near-global controller.

Since the proposed relaxations are computationally expensive, we also propose computationally cheap SDP relaxations associated with various formulations of the ODC problem. Afterwards, we develop effective heuristic methods to recover a near-optimal controller from the low-rank SDP solution. Note that the computationally-cheap SDP relaxations are exact for the classical (centralized) LQR and H_2 problems. This implies that the relaxations indirectly solve Riccati equations in the extreme case where the controller under design is unstructured. We conduct thousands of simulations on a mass-spring system and 100 random systems to elucidate the efficacy of the proposed relaxations. In particular, the design of numerous near-optimal structured controllers with global optimality guarantees above 99% will be demonstrated.

1.4 Part IV: Parallel Computing

Semidefinite programing is attractive due to its application in approximation of nonlinear and combinatorial optimization problems [Lavaei and Low, 2012; Goemans and Williamson, 1995]. It is known that small- to medium-sized SDP problems can be solved efficiently by interior point methods in polynomial time up to any arbitrary precision [Vandenberghe and Boyd, 1996a]. However, these methods are less practical for large-scale SDPs due to computation time and memory issues.

In Part IV of this work, we show that remarkable features of the Alternating Direction Method of Multipliers (ADMM) make it a suitable choice for designing a distributed and parallel algorithm for solving sparse large-scale semidefinite programs.

1.4.1 ADMM for Sparse Semidefinite Programming

Alternating direction method of multipliers is a first-order optimization algorithm proposed in the mid-1970s by [Gabay and Mercier, 1976] and [Glowinsk and Marroco, 1975]. This method has attracted much attention recently since it can be used for large-scale optimization problems and also be implemented in parallel and distributed computational environments [Wen *et al.*, 2010; Boyd *et al.*, 2011]. Compared to second order methods that are able to achieve a high accuracy via expensive iterations, ADMM relies on low-complex iterations and can achieve a modest accuracy in tens of iterations. Inspired by Nesterov's scheme for accelerating gradient methods [Nesterov, 1983], great effort has been devoted to accelerating ADMM and attaining a high accuracy in a reasonable number of iterations [Goldstein *et al.*, 2014]. Since ADMM's performance is affected by the condition number of the problems data, diagonal rescaling is proposed in [Giselsson and Boyd, 2014] for a class of problems to improve the performance and achieve a linear rate of convergence.

This part aims to develop a fast, parallelizable algorithm for sparse semidefinite programing problems. Based on the alternating direction method of multipliers, we design a numerical algorithm, which has a guaranteed convergence under very mild assumptions. Each iteration of this algorithm has a simple closed-form solution, consisting of scalar multiplications and eigenvalue decompositions performed by individual computing agents. The cheap iterations of the proposed algorithm enable solving real-world large-scale conic optimization problems.

Part I

Rank and Sparsity

Chapter 2

Low-rank Solution of Sparse Linear Matrix Inequalities

This chapter is concerned with the problem of finding a low-rank solution of an arbitrary sparse linear matrix inequality (LMI). To this end, we map the sparsity of the LMI problem into a graph. We develop a theory relating the rank of the minimum-rank solution of the LMI problem to the sparsity of its underlying graph. Furthermore, we propose three graph-theoretic convex programs to obtain a low-rank solution. Two of these convex optimization problems need a tree decomposition of the sparsity graph, which is an NP-hard problem in the worst case. The third one does not rely on any computationally-expensive graph analysis and is always polynomial-time solvable. The results of this work can be readily applied to three separate problems: minimum-rank matrix completion, conic relaxation for polynomial optimization, and affine rank minimization. The results of this chapter are employed in next chapters for two applications of optimal distributed control and nonlinear optimization for electrical networks.

2.1 Introduction

Let $\mathbb{S}_n$ denote the set of $n\times n$ real symmetric matrices and $\mathbb{S}_n^+$ denote the cone of positive semidefinite matrices in $\mathbb{S}_n$. Consider the linear matrix inequality (LMI) problem

$$
\begin{aligned}
&\text{find} && \mathbf{X} \in \mathbb{S}_n && \\
&\text{subject to} && \langle \mathbf{M}_k, \mathbf{X} \rangle \leq a_k, && k = 1, \ldots, p, \quad (2.1\text{a}) \\
& && \mathbf{X} \succeq 0, && \quad (2.1\text{b})
\end{aligned}
$$

where $\succeq$ represents the positive semidefinite sign, the notation $\langle \mathbf{A}, \mathbf{B} \rangle$ denotes the Frobenius inner product of matrices, $\mathbf{M}_1, \ldots, \mathbf{M}_p \in \mathbb{S}_n$ are sparse matrices and $a_1, \ldots, a_p \in \mathbb{R}$ are arbitrary fixed scalars. The objective of this chapter is twofold. First, it is aimed to find a low-rank solution $\mathbf{X}^{\text{opt}}$ of the above LMI (feasibility) problem using a convex program. Second, it is intended to study the relationship between the rank of such a low-rank solution and the sparsity level of the matrices $\mathbf{M}_1, \ldots, \mathbf{M}_k$. To formulate the problem, let $P \subseteq \mathbb{S}_n$ denote the convex polytope characterized by the linear inequalities given in (2.1a). The goal is to design an efficient algorithm to identify a low-rank matrix $\mathbf{X}^{\text{opt}}$ in the set $\mathbb{S}_n^+ \cap P$.

Observe that equality constraints are also covered by the general formulation given in (2.1). The special case where P is an affine subspace of $\mathbb{S}_n$ (i.e., it is only characterized by linear equality constraints) has been extensively studied in the literature [Au-Yeung and Poon, 1979; Barvinok, 2001; Ai *et al.*, 2008]. In particular, the work [Barvinok, 2001] derives an upper bound on the rank of $\mathbf{X}^{\text{opt}}$, which depends on the dimension of P as opposed to the sparsity level of the problem. The paper [Ai *et al.*, 2008] develops a polynomial-time algorithm to find a solution satisfying the bound condition given in [Barvinok, 2001]. However, since the bound obtained in [Barvinok, 2001] is independent of the sparsity of the LMI problem (2.1), it is known not to be tight for several practical examples [Sojoudi and Lavaei, 2014; Lavaei, 2013].

The investigation of the above-mentioned LMI has direct applications in three fundamental problems: (i) minimum-rank positive semidefinite matrix completion, (ii) conic relaxation for polynomial optimization, and (iii) affine rank minimization. In what follows, these problems will be introduced in three separate subsections, followed by an outline of our contribution for each problem.

2.1.1 Low-rank Positive Semidefinite Matrix Completion

The LMI problem (2.1) encapsulates the low-rank positive semidefinite matrix completion problem, which is as follows: given a partially completed matrix with some known entries, the positive semidefinite matrix completion problem aims to design the unknown (free) entries of the matrix in such a way that the completed matrix becomes positive semidefinite. As a classical result, this problem has been fully addressed in [Grone *et al.*, 1984], provided the graph capturing the locations of the known entries of the matrix is chordal. The positive semidefinite matrix completion problem plays a critical role in reducing the complexity of large-scale semidefinite programs [Fukuda *et al.*, 2001; Nakata *et al.*, 2003; Kim *et al.*, 2011; Jabr, 2012; Molzahn *et al.*, 2013; Andersen *et al.*, 2014]. In the case where a minimum-rank completion is sought, the problem is referred to as *minimum-rank positive semidefinite matrix completion.* To formalize this problem, consider a simple graph $\mathcal{G} = (\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G})$ with the vertex set $\mathcal{V}_\mathcal{G}$ and the edge set $\mathcal{E}_\mathcal{G}$. Let $\mathrm{gd}(\mathcal{G})$ denote the Gram dimension of $\mathcal{G}$, defined as the smallest positive integer r such that for every $\widehat{\mathbf{X}} \in \mathbb{S}^+_{|\mathcal{V}_\mathcal{G}|}$, there exists a solution to the following feasibility problem

$$
\begin{aligned}
&\text{find} && \mathbf{X} \in \mathbb{S}_{|\mathcal{V}_\mathcal{G}|} && \\
&\text{subject to} && X_{ij} = \widehat{X}_{ij}, && (i,j) \in \mathcal{E}_\mathcal{G}, \qquad (2.2\text{a})\\
& && X_{kk} = \widehat{X}_{kk}, && k \in \mathcal{V}_\mathcal{G}, \qquad (2.2\text{b})\\
& && \mathbf{X} \succeq 0, && \qquad (2.2\text{c})
\end{aligned}
$$

with rank less than or equal to r. According to the above definition, every arbitrary positive semidefinite matrix $\widehat{\mathbf{X}}$ can be turned into a matrix $\mathbf{X}$ with rank at most $\mathrm{gd}(\mathcal{G})$ by manipulating those off-diagonal entries of $\widehat{\mathbf{X}}$ that correspond to the non-existent edges of $\mathcal{G}$. The paper [Laurent and Varvitsiotis, 2014] introduces the notion of Gram dimension and shows that $\mathrm{gd}(\mathcal{G}) \leq \mathrm{tw}(\mathcal{G}) + 1$ (for real-valued problems), where $\mathrm{tw}(\mathcal{G})$ denotes the treewidth of the graph $\mathcal{G}$.

There is a large body of literature on graph-theoretic parameters regarded as minimum semidefinite rank of a graph over the space of real symmetric and complex Hermitian matrices [Fallat and Hogben, 2007; Barrett *et al.*, 2004; Sinkovic and van der Holst, 2011]. These two parameters, denoted as $\mathrm{msr}_\mathbb{S}(\mathcal{G})$ and $\mathrm{msr}_\mathbb{H}(\mathcal{G})$, are equal to the smallest rank of all positive semidefinite matrices with the same support as the adjacency matrix of $\mathcal{G}$ in the sets of real symmetric and complex Hermitian matrices respectively. It is straightforward from the definition that $\mathrm{msr}_\mathbb{H}(\mathcal{G})$ is a lower bound for $\mathrm{msr}_{\mathbb{S}}(\mathcal{G})$. In [Barioli *et al.*, 2010], an example of a simple graph $\mathcal{G}$ is given, for the first time, with the property $\mathrm{msr}_{\mathbb{H}}(\mathcal{G}) < \mathrm{msr}_{\mathbb{S}}(\mathcal{G})$. The notion of OS-vertex number of $\mathcal{G}$, denoted by $\mathrm{OS}(\mathcal{G})$, is proposed in [Hackney *et al.*, 2009] that serves as a lower bound on $\mathrm{msr}_{\mathbb{H}}(\mathcal{G})$. The paper [Hackney *et al.*, 2009] also shows that $\mathrm{OS}(\mathcal{G}) = \mathrm{msr}_{\mathbb{H}}(\mathcal{G})$ for a chordal graph $\mathcal{G}$. Examples of graphs with $\mathrm{OS}(\mathcal{G}) < \mathrm{msr}_{\mathbb{H}}(\mathcal{G})$ are also introduced in [Mitchell *et al.*, 2010]. The positive semidefinite zero forcing number of a graph $\mathcal{G}$, denoted by $\mathrm{Z}^{+}(\mathcal{G})$, has first been introduced in [Barioli *et al.*, 2010] and is used for computation of msr of certain graphs. It is shown in [Barioli *et al.*, 2010] that

$$\mathrm{Z}^{+}(\mathcal{G}) + \mathrm{OS}(\mathcal{G}) = |\mathcal{G}| \tag{2.3}$$

for every graph $\mathcal{G}$. See [Barioli *et al.*, 2013] and [Fallat and Hogben, 2013] for more comprehensive reviews on the relations between the graph theoretic parameters tw, $\mathrm{msr}_{\mathbb{S}}$, $\mathrm{msr}_{\mathbb{H}}$, OS and Z^{+}.

The matrix completion problem (2.2) can be cast as the LMI problem (2.1) by representing the constraints (2.2a) and (2.2b) as $\langle e_j e_i^*, \mathbf{X}\rangle = \widehat{X}_{ij}$ and $\langle e_k e_k^*, \mathbf{X}\rangle = \widehat{X}_{kk}$ respectively, where $\{e_1, \ldots, e_{|\mathcal{V}_\mathcal{G}|}\}$ is the standard basis for $\mathbb{R}^{|\mathcal{V}_\mathcal{G}|}$. Hence, the minimum-rank positive semidefinite matrix completion problem amounts to finding a minimum-rank matrix in the convex set $\mathbb{S}_n^+ \cap P$. We utilize the notions of tree decomposition, minimum semidefinite rank of a graph, OS-vertex and positive semidefinite zero forcing, to find low-rank matrices in $\mathbb{S}_n^+ \cap P$ using convex optimization. Let $\mathcal{G}$ denote a graph capturing the sparsity of the LMI problem (2.1). Consider the convex problem of minimizing a weighted sum of an arbitrary subset of the free entries of $\mathbf{X}$ subject to the matrix completion constraint of (2.2). We show that the rank of every solution of this problem can be upper bounded in terms of the OS and msr of some supergraphs of $\mathcal{G}$. Our bound depends only on the locations of the free entries minimized in the objective function rather than their coefficients. In particular, given an arbitrary tree decomposition of $\mathcal{G}$ with width t, we show that the minimization of a weighted sum of certain free entries of $\mathbf{X}$ guarantees that **every** solution $\mathbf{X}^{\mathrm{opt}}$ of this problem belongs to $\mathbb{S}_n^+ \cap P$ and satisfies the relation $\mathrm{rank}\{\mathbf{X}^{\mathrm{opt}}\} \leq t+1$, for all possible nonzero coefficients of the objective function. This result holds for both real and complex-valued problems. The problem of finding a tree decomposition of minimum width is NP-complete [Arnborg *et al.*, 1987]. Nevertheless, for a fixed integer t, the problem of checking the existence of a tree decomposition of width t and finding such a decomposition (if any) can be solved in linear time [Matoušek and Thomas, 1991; Bodlaender, 1996]. Whenever a minimal tree decomposition is known, we offer infinitely many optimization problems such that every solution of those problems satisfies the relation $\mathrm{rank}\{\mathbf{X}^{\mathrm{opt}}\} \leq$

$\mathrm{tw}(\mathcal{G}) + 1$.

In the case where a good decomposition of $\mathcal{G}$ with small width is not known, we propose a polynomial-time solvable optimization that is able to find a matrix in $\mathbb{S}_n^+ \cap P$ with rank at most $2(n - \mathrm{msr}_{\mathbb{H}}(\mathcal{G}))$. Note that this solution can be found in polynomial time, whereas our theoretical upper bound on its rank is hard to compute. The upper bound $2(n - \mathrm{msr}_{\mathbb{H}}(\mathcal{G}))$ is a small number for a wide class of sparse graphs [Booth *et al.*, 2008].

2.1.2 Sparse Quadratically-Constrained Quadratic Program

The problem of searching for a low-rank matrix in the convex set $\mathbb{S}_n^+ \cap P$ is important due to its application in obtaining suboptimal solutions of quadratically-constrained quadratic programs (QCQPs). Consider the standard nonconvex QCQP problem

$$\underset{x \in \mathbb{R}^{n-1}}{\text{minimize}} \quad f_0(x) \tag{2.4a}$$

$$\text{subject to} \quad f_k(x) \leq 0, \quad k = 1, \ldots, p, \tag{2.4b}$$

where $f_k(x) = x^T \mathbf{A}_k x + 2b_k^T x + c_k$ for $k = 0, \ldots, p$. Every polynomial optimization can be cast as problem (2.4) and this also includes all combinatorial optimization problems [Shor, 1987; Nesterov *et al.*, 1994]. Thus, the above nonconvex QCQP "covers almost everything" [Nesterov *et al.*, 1994]. To tackle this NP-hard problem, define

$$\mathbf{F}_k \triangleq \begin{bmatrix} c_k & b_k^T \\ b_k & \mathbf{A}_k \end{bmatrix}. \tag{2.5}$$

Each f_k has the linear representation $f_k(x) = \langle \mathbf{F}_k, \mathbf{X} \rangle$ for the following choice of $\mathbf{X}$:

$$\mathbf{X} \triangleq [1 \quad x^T]^T [1 \quad x^T]. \tag{2.6}$$

It is obvious that an arbitrary matrix $\mathbf{X} \in \mathbb{S}_n$ can be factorized as (2.6) if and only if it satisfies the three properties $X_{11} = 1$, $\mathbf{X} \succeq 0$, and $\mathrm{rank}\{\mathbf{X}\} = 1$. Therefore, problem (2.4) can be reformulated as follows:

$$
\begin{aligned}
\underset{\mathbf{X}\in\mathbb{S}_n}{\text{minimize}} \quad & \langle \mathbf{F}_0, \mathbf{X}\rangle && && \text{(2.7a)}\\
\text{subject to} \quad & \langle \mathbf{F}_k, \mathbf{X}\rangle \leq 0 && k = 1, \ldots, p, && \text{(2.7b)}\\
& X_{11} = 1, && && \text{(2.7c)}\\
& \mathbf{X} \succeq 0, && && \text{(2.7d)}\\
& \text{rank}\{\mathbf{X}\} = 1. && && \text{(2.7e)}
\end{aligned}
$$

In the above representation of QCQP, the constraint (2.7e) carries all the nonconvexity. Neglecting this constraint yields a convex problem, known as the semidefinite programming (SDP) relaxation of QCQP [Vandenberghe and Boyd, 1996a; Anstreicher, 2012]. The existence of a rank-1 solution for the SDP relaxation guarantees the equivalence of the original QCQP and its relaxed problem.

The SDP relaxation technique provides a lower bound on the minimum cost of the original problem, which can be used for various purposes such as the branch and bound algorithm [Nesterov *et al.*, 1994]. To understand the quality of the SDP relaxation, this lower bound is known to be at most 14% less than the minimum cost for the MAXCUT problem [Goemans and Williamson, 1995]. In general, the maximum possible gap between the solution of a graph optimization and that of its SDP relaxation is defined as the Grothendieck constant of the graph [Briet *et al.*, 2010; Alon *et al.*, 2006]. This constant is calculated for some special graphs in [Laurent and Varvitsiotis, 2011]. If the QCQP problem and its SDP relaxation result in the same optimal objective value, then the relaxation is said to be exact. The exactness of the relaxation is substantiated for various applications [Lavaei and Low, 2012; Lasserre, 2001; Kim and Kojima, 2003; Lavaei *et al.*, 2011].

By exploring the optimal power flow problem, we will see in chapter 3 that the exactness of the relaxation could be heavily formulation dependent. Indeed, we designed a practical circuit optimization with four equivalent QCQPs, where only one of them had an exact SDP relaxation. In the same context, we have also verified in chapter 3 that the SDP relaxation may have a hidden rank-1 solution that could not be easily found. The reason is that the SDP relaxation of a sparse QCQP problem often has infinitely many solutions and the conventional numerical algorithms would find a solution with the highest rank. Hence, a question arises as to whether a low-rank solution of the SDP relaxation of a sparse QCQP can be found efficiently. To address this problem, let $\widehat{\mathbf{X}}$ denote an arbitrary solution of the SDP relaxation. If the QCQP problem (2.4) is sparse and associated with a sparsity graph $\mathcal{G}$, then every positive semidefinite matrix $\mathbf{X}$ satisfying the matrix completion constraint of (2.2) is another solution of the SDP relaxation of the QCQP problem. Now, the results spelled out in the preceding subsection can be used to find a low-rank SDP solution.

2.1.3 Affine Rank Minimization Problem

Consider the problem

$$\underset{\mathbf{W}\in\mathbb{R}^{m\times r}}{\text{minimize}} \quad \text{rank}\{\mathbf{W}\} \tag{2.8a}$$

$$\text{subject to} \quad \langle \mathbf{N}_k, \mathbf{W}\rangle \leq a_k, \qquad k = 1, \ldots, p, \tag{2.8b}$$

where $\mathbf{N}_1, \ldots, \mathbf{N}_p \in \mathbb{R}^{r\times m}$ are sparse matrices. This is an affine rank minimization problem without any positive semidefinite constraint. A popular convexification method for the above non-convex optimization is to replace its objective with the nuclear norm of $\mathbf{W}$ [Recht *et al.*, 2010b]. This is due to the fact that the nuclear norm $\|\mathbf{W}\|_*$ is the convex envelope for the function rank$\{\mathbf{W}\}$ on the set $\{\mathbf{W} \in \mathbb{R}^{m\times r} \mid \|\mathbf{W}\| \leq 1\}$ [Fazel, 2002]. A special case of Optimization (2.8), known as low-rank matrix completion problem, has been extensively studied in the literature due to its wide applications [Johnson, 1990; Candès and Recht, 2009; Recht *et al.*, 2010b; Keshavan *et al.*, 2010]. In this problem, the constraint (2.8b) determines what entries of $\mathbf{W}$ are known.

A closely related problem is the following: can a matrix $\mathbf{W}$ be recovered by observing only a subset of its entries? Interestingly, $\mathbf{W}$ can be successfully recovered by means of a nuclear norm minimization as long as the matrix is non-structured and the number of observed entries of $\mathbf{W}$ is large enough [Candès and Recht, 2009; Candès and Tao, 2010; Keshavan *et al.*, 2010]. The performance of the nuclear norm minimization method for the problem of rank minimization subject to general linear constraints has also been assessed in [Recht *et al.*, 2011]. Based on empirical studies, the nuclear norm technique is inefficient in the case where the number of free (unconstrained) entries of $\mathbf{W}$ is relatively large. In the present work, we propose a graph-theoretic approach that is able to generate low-rank solutions for a sparse problem of the form (2.8) and for a matrix completion problem with many unknown entries.

Optimization (2.8) can be embedded in a bigger problem of the form (2.1) by associating the matrix $\mathbf{W}$ with a positive semidefinite matrix variable $\mathbf{X}$ defined as

$$\mathbf{X} \triangleq \begin{bmatrix} \mathbf{X}_1 & \mathbf{W} \\ \mathbf{W}^T & \mathbf{X}_2 \end{bmatrix}, \tag{2.9}$$

where $\mathbf{X}_1$ and $\mathbf{X}_2$ are two auxiliary matrices. Note that $\mathbf{W}$ acts as a submatrix of $\mathbf{X}$ corresponding to its first m rows and last r columns. More precisely, consider the nonconvex problem

$$\underset{\mathbf{X}\in\mathbb{S}_{r+m}}{\text{minimize}} \quad \operatorname{rank}\{\mathbf{X}\} \tag{2.10a}$$

$$\text{subject to} \quad \langle \mathbf{M}_k, \mathbf{X}\rangle \leq a_k, \qquad k = 1, \ldots, p, \tag{2.10b}$$

$$\mathbf{X} \succeq 0, \tag{2.10c}$$

where

$$\mathbf{M}_k \triangleq \begin{bmatrix} 0_{m\times m} & \frac{1}{2}\mathbf{N}_k^T \\ \frac{1}{2}\mathbf{N}_k & 0_{r\times r} \end{bmatrix}, \tag{2.11}$$

For every feasible solution $\mathbf{X}$ of the above problem, its associated submatrix $\mathbf{W}$ is feasible for (2.8) and satisfies

$$\operatorname{rank}\{\mathbf{W}\} \leq \operatorname{rank}\{\mathbf{X}\}. \tag{2.12}$$

In particular, it is well known that the rank minimization problem (2.8) with linear constraints is equivalent to the rank minimization (2.10) with LMI constraints [Fazel, 2002; Fazel *et al.*, 2003]. Let $\widehat{\mathbf{X}}$ denote an arbitrary feasible point of optimization (2.10). Depending on the sparsity level of the problem (2.8), some entries of $\widehat{\mathbf{X}}$ are free and do not affect any constraints of (2.10) except for $\mathbf{X} \succeq 0$. Let the locations of those entries be captured by a bipartite graph. More precisely, define $\mathcal{B}$ as a bipartite graph whose first and second parts of vertices are associated with the rows and columns of $\mathbf{W}$, respectively. Suppose that each edge of $\mathcal{B}$ represents a constrained entry of $\mathbf{W}$. In this work, we propose two convex problems with the following properties:

1. The first convex program is constructed from an arbitrary tree decomposition of $\mathcal{B}$. The rank of every solution to this problem is upper bounded by $t+1$, where t is the width of its tree decomposition. Given the decomposition, the low-rank solution can be found in polynomial time.

2. Since finding a tree decomposition of $\mathcal{B}$ with a low treewidth may be hard in general, the second convex program does not rely on any decomposition and is obtained by relaxing the

real-valued problem (2.10) to a complex-valued convex program. The rank of every solution to the second convex problem is bounded by the number $2(r + m - \mathrm{msr}_{\mathbb{H}}\{\mathcal{B}\})$ and such a solution can always be found in polynomial time.

2.1.4 Simple Illustrative Examples

To illustrate some of the main ideas to be discussed in this work, three simple examples will be provided below in the context of low-rank positive semidefinite matrix completion.

Example 1.

Consider a partially-known matrix $\mathbf{X} \in \mathbb{S}_n^+$ with unknown off-diagonal entries and known strictly positive diagonal entries $X_{11}, \ldots, X_{nn}$. The aim is to design the unknown off-diagonal entries of $\mathbf{X}$ to make the resulting matrix as low rank as possible. It can be shown that there are 2^n rank-1 matrices $\mathbf{X} \in \mathbb{S}_n^+$ with the diagonal entries $X_{11}, \ldots, X_{nn}$, each of which can be expressed as xx^T for a vector x with the property that $x_i = \pm\sqrt{X_{ii}}$. A question arises as to whether such matrix completions can be attained via solving a convex optimization. To address this question, consider the problem of finding a matrix $\mathbf{X} \in \mathbb{S}_n^+$ with the given diagonal to minimize an arbitrary weighted sum of the subdiagonal entries of X, i.e., $\sum_{i=1}^{n-1} t_i X_{i+1,i}$ for arbitrary nonzero coefficients $t_1, \ldots, t_{n-1}$. It can be verified that every solution of this optimization problem results in one of the aforementioned 2^n rank-1 matrices X. In other words, there are 2^n ways to fill the matrix X, each of which corresponds to infinitely many easy-to-characterize continuous optimization problems.

Example 2.

Consider a 3×3 symmetric block matrix $\mathbf{X}$ partitioned as

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} & \mathbf{X}_{13} \\ \mathbf{X}_{21} & \mathbf{X}_{22} & \mathbf{X}_{23} \\ \mathbf{X}_{31} & \mathbf{X}_{32} & \mathbf{X}_{33} \end{bmatrix} \tag{2.13}$$

where $\mathbf{X}_{11} \in \mathbb{R}^{\alpha\times\alpha}$, $\mathbf{X}_{22} \in \mathbb{R}^{\beta\times\beta}$ and $\mathbf{X}_{33} \in \mathbb{R}^{\gamma\times\gamma}$, for some positive numbers α, β and γ. Assume that the block $\mathbf{X}_{13}$ is unknown while the remaining blocks of $\mathbf{X}$ are known either partially or completely. Suppose that $\mathbf{X}$ admits a positive definite matrix completion, which implies that

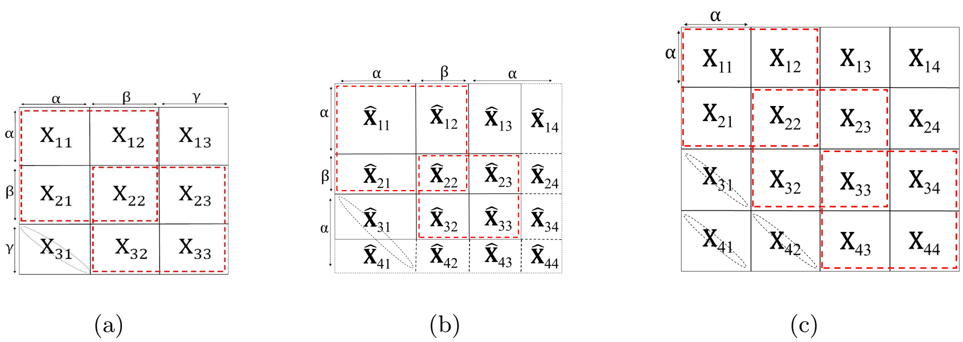


Figure 2.1: (a) The matrix $\mathbf{X}$ studied in Example 2 for $\alpha = \gamma$; (b) the augmented matrix $\widehat{\mathbf{X}}$ obtained from $\mathbf{X}$ in the case where $\alpha > \gamma$; (c) the matrix $\mathbf{X}$ studied in Example 3.

$\text{rank}\{\mathbf{X}\} \geq \max\{\alpha+\beta, \beta+\gamma\}$. The goal is to perform the completion of $\mathbf{X}$ via convex optimization such that $\text{rank}\{\mathbf{X}\} = \max\{\alpha+\beta, \beta+\gamma\}$.

Consider first the scenario where $\alpha = \gamma$. Let $\{(i_1, j_1), \ldots, (i_s, j_s)\}$ denote an arbitrary set of entries of $\mathbf{X}_{13}$ with s elements. Consider the optimization problem of minimizing $\sum_{k=1}^{s} t_k \mathbf{X}_{13}(i_k, j_k)$ subject to the constraint that $\mathbf{X}$ is a positive semidefinite matrix in the form of (2.13), where $t_1, \ldots, t_s$ are nonzero scalars and $\mathbf{X}_{13}(i_k, j_k)$ denotes the (i_s, j_s) entry of $\mathbf{X}_{13}$. Let $\mathbf{X}^{\text{opt}}$ be an arbitrary solution of this problem. In this work, we derive an upper bound on the rank of $\mathbf{X}^{\text{opt}}$, which depends only on the set $\{(i_1, j_1), \ldots, (i_s, j_s)\}$ and is independent of $t_1, \ldots, t_s$. In particular, if $\{(i_1, j_1), \ldots, (i_s, j_s)\}$ corresponds to $s = \alpha$ entries of $\mathbf{X}_{13}$ with no two elements in the same row or column, then it is guaranteed that $\text{rank}\{\mathbf{X}^{\text{opt}}\} = \max\{\alpha+\beta, \beta+\gamma\}$ for all nonzero values of $t_1, t_2, \ldots, t_s$. Figure 2.1(a) shows the blocks of matrix $\mathbf{X}$, where the two 2×2 blocks of $\mathbf{X}$ specified by dashed red lines are known while the block $\mathbf{X}_{31}$ is to be designed. As a special case of the above method, minimizing a weighted sum of the diagonal entries of $\mathbf{X}_{31}$ with nonzero weights leads to a lowest-rank completion.

Consider now the scenario where $\alpha > \gamma$. We add $\alpha - \gamma$ rows and $\alpha - \gamma$ columns to $\mathbf{X}$ and denote the augmented matrix as $\widehat{\mathbf{X}}$. This procedure is demonstrated in Figure 2.1(b), where the added blocks are labeled as $\widehat{\mathbf{X}}_{14}$, $\widehat{\mathbf{X}}_{24}$, $\widehat{\mathbf{X}}_{34}$, $\widehat{\mathbf{X}}_{41}$, $\widehat{\mathbf{X}}_{42}$, $\widehat{\mathbf{X}}_{43}$ and $\widehat{\mathbf{X}}_{44}$. Note that the first $\alpha+\beta+\gamma$ rows and $\alpha+\beta+\gamma$ columns of $\widehat{\mathbf{X}}$ are exactly the same as those of the matrix $\mathbf{X}$. We also set all diagonal entries of $\widehat{\mathbf{X}}_{44}$ to 1. The matrix $\widehat{\mathbf{X}}$ has two partially-known 2×2 blocks of size $\alpha+\beta$ as well as a square non-overlapping block containing $\widehat{\mathbf{X}}_{31}$ and $\widehat{\mathbf{X}}_{41}$. The problem under study now reduces to the matrix completion posed in the previous scenario $\alpha = \gamma$. More precisely, consider the problem of minimizing an arbitrary weighted sum of the diagonal entries of the non-overlapping block $(\widehat{\mathbf{X}}_{31}, \widehat{\mathbf{X}}_{41})$ with nonzero weights over all positive semidefinite partially-known matrices $\widehat{\mathbf{X}}$. Every solution $\widehat{\mathbf{X}}^{\text{opt}}$ of this optimization has rank at most $\alpha + \beta$, and so does its submatrix $\mathbf{X}^{\text{opt}}$.

Example 3.

Consider the 4×4 symmetric block matrix $\mathbf{X}$ shown in Figure 2.1(c) with partially-known blocks $\mathbf{X}_{11}, \mathbf{X}_{21}, \mathbf{X}_{22}, \mathbf{X}_{32}, \mathbf{X}_{33}, \mathbf{X}_{43}, \mathbf{X}_{44}$ and totally-unknown blocks $\mathbf{X}_{31}, \mathbf{X}_{41}, \mathbf{X}_{42}$. The goal is to fill the matrix to a minimum-rank positive semidefinite matrix. For simplicity, assume that the matrix $\mathbf{X}$ admits a positive definite completion and that all 16 blocks $\mathbf{X}_{ij}$ have the same size $\alpha \times \alpha$. It can be verified that the matrix $\mathbf{X}$ admits a positive semidefinite completion with rank 2α. To convert the problem into an optimization, one can minimize the weighted sum of certain entries of $\mathbf{X}_{31}, \mathbf{X}_{41}, \mathbf{X}_{42}$. It turns that if the weighted sum of the diagonal entries of one or all of these three blocks is minimized, the rank would be higher than 2α. However, the minimization of the diagonal entries of the two blocks $\mathbf{X}_{31}$ and $\mathbf{X}_{42}$ always produces a lowest-rank solution.

2.2 Notations and Definitions

Throughout this chapter, the symbols $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{S}_n$ denotes the space of $n \times n$ real symmetric matrices and $\mathbb{H}_n$ denotes the space of $n \times n$ complex Hermitian matrices. Also, $\mathbb{S}_n^+ \subset \mathbb{S}_n$ and $\mathbb{H}_n^+ \subset \mathbb{H}_n$ represent the convex cones of real and complex positive semidefinite matrices, respectively. The set of notations $(\mathbb{F}_n, \mathbb{F}_n^+, \mathbb{F})$ refers to either $(\mathbb{S}_n, \mathbb{S}_n^+, \mathbb{R})$ or $(\mathbb{H}_n, \mathbb{H}_n^+, \mathbb{C})$ depending on the context (i.e., whether the real or complex domain is under study). $\text{Re}\{\cdot\}$, $\text{Im}\{\cdot\}$, $\text{rank}\{\cdot\}$, and $\text{trace}\{\cdot\}$ denote the real part, imaginary part, rank, and trace of a given scalar/matrix. Matrices are shown by capital and bold letters. The symbols $(\cdot)^{\text{T}}$ and $(\cdot)^*$ denote transpose and conjugate transpose, respectively. The notation $\langle \mathbf{A}, \mathbf{B} \rangle$ represents $\text{trace}\{\mathbf{A}^*\mathbf{B}\}$, which is the inner product of $\mathbf{A}$ and $\mathbf{B}$. Also, "$\mathbf{i}$" is reserved to denote the imaginary unit. The notation $\measuredangle x$ denotes the angle of a complex number x. The notation $\mathbf{W} \succeq 0$ means that $\mathbf{W}$ is a Hermitian and positive semidefinite matrix. The (i, j) entry of $\mathbf{W}$ is denoted as W_{ij}, unless otherwise mentioned. Given scalars $x_1, \ldots, x_n$, the notation $\text{diag}\{[x_1, \ldots, x_n]\}$ denotes a $n \times n$ diagonal matrix with the diagonal entries $x_1, \ldots, x_n$. The vertex set and edge set of a

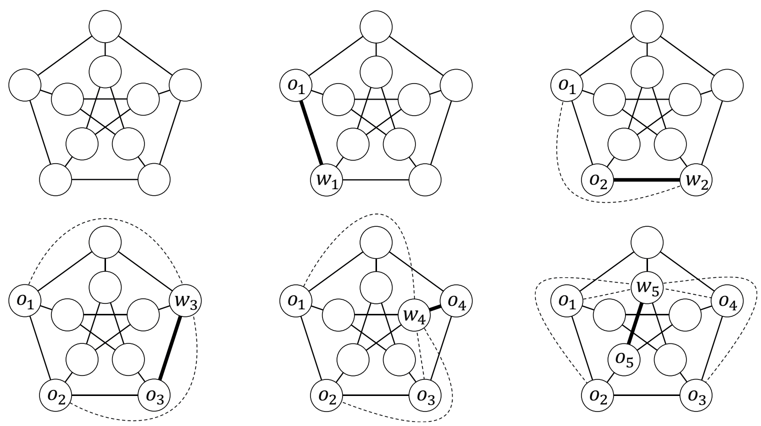


Figure 2.2: A maximal OS-vertex sequence for the Petersen graph

simple undirected graph $\mathcal{G}$ are shown by the notations $\mathcal{V}_\mathcal{G}$ and $\mathcal{E}_\mathcal{G}$, and the graph $\mathcal{G}$ is identified by the pair $(\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G})$. $\mathcal{N}_\mathcal{G}(k)$ denotes the set of all neighbors of the vertex k in the graph $\mathcal{G}$. The symbol $|\mathcal{G}|$ shows the number of vertices of $\mathcal{G}$.

Definition 1. *For two simple graphs $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$, the notation $\mathcal{G}_1 \subseteq \mathcal{G}_2$ means that $\mathcal{V}_1 \subseteq \mathcal{V}_2$ and $\mathcal{E}_1 \subseteq \mathcal{E}_2$. $\mathcal{G}_1$ is called a subgraph of $\mathcal{G}_2$ and $\mathcal{G}_2$ is called a supergraph of $\mathcal{G}_1$. A subgraph $\mathcal{G}_1$ of $\mathcal{G}_2$ is said to be an induced subgraph if for every pair of vertices $v_l, v_m \in \mathcal{V}_1$, the relation $(v_l, v_m) \in \mathcal{E}_1$ holds if and only if $(v_l, v_m) \in \mathcal{E}_2$. In this case, $\mathcal{G}_1$ is said to be induced by the vertex subset $\mathcal{V}_1$.*

Definition 2. *For two simple graphs $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$, the subgraph of $\mathcal{G}_2$ induced by the vertex set $\mathcal{V}_2 \setminus \mathcal{V}_1$ is shown by the notation $\mathcal{G}_2 \setminus \mathcal{G}_1$.*

Definition 3. *For two simple graphs $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2)$ with the same set of vertices, their union is defined as $\mathcal{G}_1 \cup \mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_1 \cup \mathcal{E}_2)$ while the notation $\setminus\!\!\!\!\!\!\;\diagup$ shows their subtraction edge-wise, i.e., $\mathcal{G}_1 \setminus\!\!\!\!\!\!\;\diagup\, \mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_1 \setminus \mathcal{E}_2)$.*

Definition 4. *The representative graph of an $n \times n$ Hermitian matrix $\mathbf{W}$, denoted by $\mathscr{G}(\mathbf{W})$, is a simple graph with n vertices whose edges are specified by the locations of the nonzero off-diagonal entries of $\mathbf{W}$. In other words, two arbitrary vertices i and j are connected if W_{ij} is nonzero.*

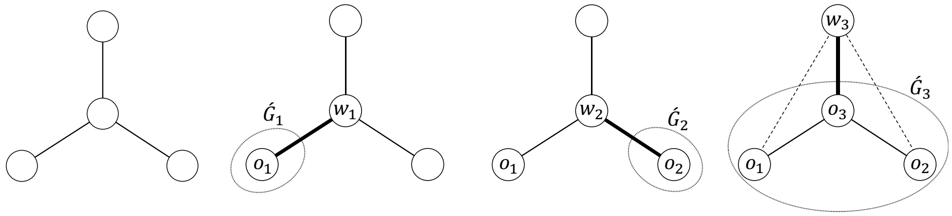


Figure 2.3: A maximal OS-vertex sequence for a tree

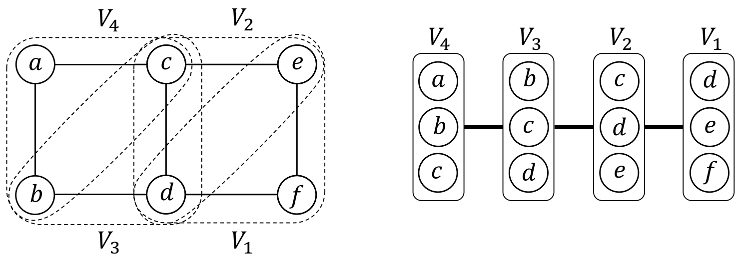


Figure 2.4: A minimal tree decomposition for a ladder

2.3 Connection Between OS and Treewidth

In this section, we study the relationship between the graph parameters of OS and treewidth. For the sake of completeness, we first review these two graph notions.

Definition 5 (OS)**.** *Given a graph $\mathcal{G}$, let $\mathcal{O} = \{o_k\}_{k=1}^{s}$ be a sequence of vertices of $\mathcal{G}$. Define $\mathcal{G}_k$ as the subgraph induced by the vertex set $\{o_1, \ldots, o_k\}$ for $k = 1, \ldots, s$. Let $\mathcal{G}'_k$ be the connected component of $\mathcal{G}_k$ containing o_k. $\mathcal{O}$ is called an OS-vertex sequence of $\mathcal{G}$ if for every $k \in \{1, \ldots, s\}$, the vertex o_k has a neighbor w_k with the following two properties:*

1. *$w_k \neq o_r$ for $1 \leq r \leq k$*

2. *$(w_k, o_r) \notin \mathcal{E}_{\mathcal{G}}$ for every $o_r \in \mathcal{V}_{G'_k} \setminus \{o_k\}$,*

Denote the maximum cardinality among all OS-vertex sequences of $\mathcal{G}$ as $\mathrm{OS}(\mathcal{G})$ *[Hackney* et al.*, 2009] .*

Figure 2.2 shows the construction of a maximal OS-vertex sequence for the Petersen graph. Dashed lines and bold lines highlight nonadjacency and adjacency, respectively, to demonstrate that each w_i satisfies the conditions of Definition 5. Figure 2.3 illustrates the procedure for finding a maximal OS-vertex sequence for a tree. The connected component of each o_k in the subgraph induced by $\{o_1, \ldots, o_k\}$ is also shown in the picture. Notice that although w_2 is connected to o_1, it is a valid choice since o_1 and o_2 do not share the same connected component in G_2.

Definition 6 (Treewidth)**.** *Given a graph $\mathcal{G} = (\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G})$, a tree $\mathcal{T}$ is called a tree decomposition of $\mathcal{G}$ if it satisfies the following properties:*

1. *Every node of $\mathcal{T}$ corresponds to and is identified by a subset of $\mathcal{V}_\mathcal{G}$.*

2. *Every vertex of $\mathcal{G}$ is a member of at least one node of $\mathcal{T}$.*

3. *For every edge (i, j) of $\mathcal{G}$, there exists a node in $\mathcal{T}$ containing vertices i and j simultaneously.*

4. *Given an arbitrary vertex k of $\mathcal{G}$, the subgraph induced by all nodes of $\mathcal{T}$ containing vertex k must be connected (more precisely, a tree).*

Each node of $\mathcal{T}$ is a bag (collection) of vertices of $\mathcal{H}$ and therefore it is referred to as a **bag**. *The width of a tree decomposition is the cardinality of its biggest bag minus one. The treewidth of $\mathcal{G}$ is the minimum width over all possible tree decompositions of $\mathcal{G}$ and is denoted by* $\mathrm{tw}(\mathcal{G})$.

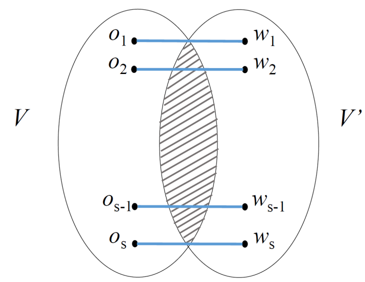
Note that the treewidth of a tree is equal to 1. Figure 2.4 shows a graph $\mathcal{G}$ with 6 vertices named a, b, c, d, e, f, together with its minimal tree decomposition $\mathcal{T}$ with 4 bags V_1, V_2, V_3, V_4. The width of this decomposition is equal to 2.

Definition 7 (Z^+)**.** *Let $\mathcal{G}$ be a simple graph. A subset of vertices $\mathcal{Z} \subseteq \mathcal{V}_\mathcal{G}$ is regarded as a* **positive semidefinite zero forcing set of** *$\mathcal{G}$, if starting from $\mathcal{Z}' := \mathcal{Z}$, it is possible to add all of the vertices of $\mathcal{G}$ to $\mathcal{Z}'$ by repeating the following operation:*

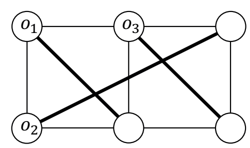
- *Choose a vertex $w \in \mathcal{V}_\mathcal{G} \setminus \mathcal{Z}'$ and let $\mathcal{W}$ be the set of vertices of the connected component of $\mathcal{G} \setminus \mathcal{Z}'$ that contains w. Add w to $\mathcal{Z}'$, if there exist a vertex $u \in \mathcal{Z}'$ such that w is the only neighbor of u in the subgraph of $\mathcal{G}$ induced by $\mathcal{W} \cup \{u\}$.*

The **positive semidefinite zero forcing number of** *$\mathcal{G}$, denoted by $\mathrm{Z}^+(\mathcal{G})$, is the minimum of $|\mathcal{Z}|$ over all positive semidefinite zero forcing sets $\mathcal{Z} \subseteq \mathcal{V}_\mathcal{G}$.*

Definition 8 (Enriched Supergraph)**.** *Given a graph $\mathcal{G}$ accompanied by a tree decomposition $\mathcal{T}$ of width t, $\overline{\mathcal{G}}$ is called an enriched supergraph of $\mathcal{G}$ derived by $\mathcal{T}$ if it is obtained according to the following procedure:*



(a)



(b)

Figure 2.5: (a) This figure illustrates Step 3 of Definition 8 for designing an enriched supergraph. The shaded area includes the common vertices of bags V and V'; (b) OS-vertex sequence $\mathcal{O}$ for the graph $\mathcal{G}$ depicted in Figure 2.4.

1. *Add a sufficient number of (redundant) vertices to the bags of $\mathcal{T}$, if necessary, in such a way that every bag includes exactly $t+1$ vertices. Also, add the same vertices to $\mathcal{G}$ (without incorporating new edges). Denote the new graphs associated with $\mathcal{T}$ and $\mathcal{G}$ as $\overline{\mathcal{T}}$ and $\overline{\mathcal{G}}$, respectively. Set $\mathcal{O}$ as the empty sequence and $\tilde{\mathcal{T}} = \overline{\mathcal{T}}$.*

2. *Identify a leaf of $\tilde{\mathcal{T}}$, named V. Let V' denote the neighbor of V in $\tilde{\mathcal{T}}$.*

3. *Let $V \setminus V' = \{o_1, \ldots, o_s\}$ and $V' \setminus V = \{w_1, \ldots, w_s\}$. Update $\mathcal{O}$, $\overline{\mathcal{G}}$ and $\tilde{\mathcal{T}}$ as*

$$\mathcal{O} := \mathcal{O} \cup \{o_1, \ldots, o_s\}$$
$$\overline{\mathcal{G}} := (\mathcal{V}_{\overline{\mathcal{G}}}, \mathcal{E}_{\overline{\mathcal{G}}} \cup \{(o_1, w_1), \ldots, (o_s, w_s)\})$$
$$\tilde{\mathcal{T}} := \tilde{\mathcal{T}} \backslash V$$

4. *If $\tilde{\mathcal{T}}$ has more than one bag, go to Step 2. Otherwise, terminate.*

The graph $\overline{\mathcal{G}}$ is referred to as an enriched suppergraph of $\mathcal{G}$ derived by $\mathcal{T}$. Moreover, $\mathcal{O}$ serves as an OS-vertex sequence for this supergraph and every bag of $\overline{\mathcal{T}}$ is a positive semidefinite zero forcing set for $\overline{\mathcal{G}}$.

Figure 2.5(a) illustrates Step 3 of the above definition. Figure 2.6 delineates the process of obtaining an enriched supergraph $\overline{\mathcal{G}}$ of the graph $\mathcal{G}$ depicted in Figure 2.4. Bold lines show the added edges at each step of the algorithm. Figure 2.5(b) sketches the resulting OS-vertex sequence

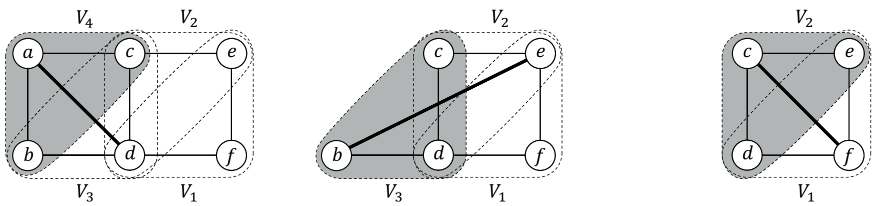


Figure 2.6: An enriched supergraph $\overline{\mathcal{G}}$ of the graph $\mathcal{G}$ given in Figure 2.4

$\mathcal{O}$. Observe that whether or not each non-bold edge exists in the graph, $\mathcal{O}$ still remains an OS-vertex sequence. The next theorem reveals the relationship between OS and treewidth.

Theorem 1. *Given a graph $\mathcal{G}$ accompanied by a tree decomposition $\mathcal{T}$ of width t, consider the enriched supergraph $\overline{\mathcal{G}}$ of $\mathcal{G}$ derived by $\mathcal{T}$ together with the sequence $\mathcal{O}$ constructed in Definition 8. Let $\mathcal{G}_s$ be an arbitrary member of $\left\{\mathcal{G}_s \,\middle|\, (\overline{\mathcal{G}} \setminus \mathcal{G}) \subseteq \mathcal{G}_s \subseteq \overline{\mathcal{G}}\right\}$:*

a) Then, $\mathcal{O}$ is an OS-vertex sequence for $\mathcal{G}_s$ of size $|\mathcal{O}| = |\mathcal{G}_s| - t - 1$.

b) Every bag $\mathcal{Z} \in \mathcal{V}_{\overline{\mathcal{T}}}$ is a positive semidefinite zero forcing set for $\mathcal{G}_s$ of size $|\mathcal{Z}| = t + 1$.

Proof. Consider the procedure described in Definition 8 for the construction of the supergraph $\overline{\mathcal{G}}$. It is easy to verify that $\mathcal{O}$ includes all vertices of $\overline{\mathcal{G}}$ except for those in the only remaining vertex of $\tilde{\mathcal{T}}$ when this process is terminated. Call this bag V_1. Hence,

$$|\mathcal{O}| = |\overline{\mathcal{G}}| - |V_1| = |\overline{\mathcal{G}}| - (t + 1). \tag{2.14}$$

Now, it remains to show that $\mathcal{O}$ is an OS-vertex sequence. To this end, let $\mathcal{G}_s$ be an arbitrary member of $\left\{\mathcal{G}_s \,\middle|\, (\overline{\mathcal{G}} \setminus \mathcal{G}) \subseteq \mathcal{G}_s \subseteq \overline{\mathcal{G}}\right\}$. We use induction to prove that $\mathcal{O}$ is an OS-vertex sequence of $\mathcal{G}_s$.

For $|\mathcal{T}|= 1$, the sequence $\mathcal{O}$ is empty and the statement is trivial. For $|\mathcal{T}|> 1$, consider the first run of the loop in the algorithm. Notice that

$$\{o_1, \ldots, o_s\} \subseteq V \quad \text{and} \quad \{o_1, \ldots, o_s\} \cap V' = \varnothing. \tag{2.15}$$

Let $\overline{\mathcal{T}}_v$ denote the subgraph induced by all bags of $\overline{\mathcal{T}}$ that include an arbitrary vertex $v \in \mathcal{G}$. According to the definition of tree decomposition, we have

$$V \in \overline{\mathcal{T}}_o \quad \text{and} \quad V' \notin \overline{\mathcal{T}}_o \tag{2.16}$$

for every $o \in \{o_1, \ldots, o_s\}$. Since $\overline{\mathcal{T}}_o$ is a connected subgraph of $\overline{\mathcal{T}}$ and V is a leaf, (2.16) implies that $\overline{\mathcal{T}}_o$ has one node and no edges, i.e.,

$$\overline{\mathcal{T}}_o = (\{V\}, \varnothing) \quad \text{for} \quad o \in \{o_1, \ldots, o_s\}. \tag{2.17}$$

On the other hand, since $\{w_1, \ldots, w_s\} \cap V = \varnothing$, we have

$$V \notin \mathcal{V}_{\overline{\mathcal{T}}_w} \quad \text{for} \quad w \in \{w_1, \ldots, w_s\}. \tag{2.18}$$

Given a pair $(i, j) \in \{1, \ldots, s\} \times \{1, \ldots, s\}$, the relations (2.17) and (2.18) yield that the trees $\overline{\mathcal{T}}_{o_i}$ and $\overline{\mathcal{T}}_{w_j}$ do not intersect and therefore $(o_i, w_j) \notin \mathcal{E}_{\mathcal{G}}$. Accordingly, since the edges $(o_1, w_1), \ldots, (o_s, w_s)$ are added to the graph at Step 3 of the algorithm, we have

$$(w_i, o_j) \in \mathcal{E}_{\mathcal{G}_s} \quad \Longleftrightarrow \quad i = j \tag{2.19}$$

This means that the vertex o_i in the sequence $\mathcal{O}$ has a neighbor w_i satisfying the requirements of the OS definition (note that (o_i, w_i) is an edge of $\mathcal{G}_s$).

On the other hand, $\overline{\mathcal{T}} \backslash V$ is a tree decomposition for the subgraph of $\overline{\mathcal{G}}$ induced by the vertex subset $\mathcal{V}_{\overline{\mathcal{G}}} \setminus \{o_1, \ldots, o_s\}$. Hence, according to the induction assumption, the remaining members of the sequence $\mathcal{O}$ satisfy the conditions of Definition 5. This completes the proof.

Proof of Part (b) follows directly from definition.

□

Corollary 1. *For every graph $\mathcal{G}$, there exists a supergraph $\overline{\mathcal{G}}$ with the property that*

$$\max_{\mathcal{G}_s} \left\{ \mathrm{Z}^{+}(\mathcal{G}_s) \,\middle|\, (\overline{\mathcal{G}} \setminus \mathcal{G}) \subseteq \mathcal{G}_s \subseteq \overline{\mathcal{G}} \right\} \leq \mathrm{tw}(\mathcal{G}) + 1 \tag{2.20}$$

Proof. The proof follows directly from Theorem 1 and the equation (2.3). □

2.4 Low-Rank Solutions Via Graph Decomposition

In this section, we develop a graph-theoretic technique to find a low-rank feasible solution of the LMI problem (2.1). To this end, we first introduce a convex optimization problem.

Optimization A: *Let $\mathcal{G}$ and $\mathcal{G}'$ be two graphs such that $\mathcal{V}_{\mathcal{G}} = \{1, \ldots, n\}$, $\mathcal{V}_{\mathcal{G}'} = \{1, \ldots, m\}$, $n \leq m$, and $\mathcal{E}_{\mathcal{G}} \subseteq \mathcal{E}_{\mathcal{G}'}$. Consider arbitrary matrices $\mathbf{X}^{\mathrm{ref}} \in \mathbb{F}_n^{+}$ and $\mathbf{T} \in \mathbb{F}_m$ with the property that*

$\mathscr{G}(\mathbf{T}) = \mathcal{G}'$, where $(\mathbb{F}_n^+, \mathbb{F}_m)$ is either $(\mathbb{S}_n^+, \mathbb{S}_m)$ or $(\mathbb{H}_n^+, \mathbb{H}_m)$. The problem

$$\begin{aligned}
\underset{\overline{\mathbf{X}} \in \mathbb{F}_m}{\text{minimize}} \quad & \langle \mathbf{T}, \overline{\mathbf{X}} \rangle && && \text{(2.21a)}\\
\text{subject to} \quad & \overline{X}_{kk} = X_{kk}^{\text{ref}}, && k \in \mathcal{V}_{\mathcal{G}}, && \text{(2.21b)}\\
& \overline{X}_{kk} = 1 && k \in \mathcal{V}_{\mathcal{G}'} \setminus \mathcal{V}_{\mathcal{G}}, && \text{(2.21c)}\\
& \overline{X}_{ij} = X_{ij}^{\text{ref}} && (i,j) \in \mathcal{E}_{\mathcal{G}}, && \text{(2.21d)}\\
& \overline{\mathbf{X}} \succeq 0, && && \text{(2.21e)}
\end{aligned}$$

is referred to as "Optimization A with the input $(\mathcal{G}, \mathcal{G}', \mathbf{T}, \mathbf{X}^{\text{ref}})$*".*

Optimization A is a convex semidefinite program with a non-empty feasible set containing the point

$$\left[\begin{array}{c|c} \mathbf{X}^{\text{ref}} & 0_{n\times(m-n)} \\ \hline 0_{(m-n)\times n} & \mathbf{I}_{(m-n)} \end{array}\right]. \tag{2.22}$$

Let $\overline{\mathbf{X}}^{\text{opt}} \in \mathbb{F}_m$ denote an arbitrary solution of Optimization A with the input $(\mathcal{G}, \mathcal{G}', \mathbf{T}, \mathbf{X}^{\text{ref}})$ and $\mathbf{X}^{\text{opt}} \in \mathbb{F}_n$ represent its n-th leading principal submatrix. Then, $\mathbf{X}^{\text{opt}}$ is called the *subsolution to Optimization A associated with* $\overline{\mathbf{X}}^{\text{opt}}$. Note that $\mathbf{X}^{\text{opt}}$ and $\mathbf{X}^{\text{ref}}$ share the same diagonal and values for the entries corresponding to the edges of $\mathcal{G}$. Hence, Optimization A is intrinsically a positive semidefinite matrix completion problem with the input $\mathbf{X}^{\text{ref}}$ and the output $\mathbf{X}^{\text{opt}}$.

Definition 9 (msr)**.** *Given a simple graph $\mathcal{G}$, define the real symmetric and complex Hermitian minimum semidefinite rank of $\mathcal{G}$ as*

$$\text{msr}_{\mathbb{S}}(\mathcal{G}) \triangleq \min\left\{\text{rank}(\mathbf{W}) \,|\, \mathscr{G}(\mathbf{W}) = \mathcal{G},\, \mathbf{W} \in \mathbb{S}_n^+\right\} \tag{2.23a}$$

$$\text{msr}_{\mathbb{H}}(\mathcal{G}) \triangleq \min\left\{\text{rank}(\mathbf{W}) \,|\, \mathscr{G}(\mathbf{W}) = \mathcal{G},\, \mathbf{W} \in \mathbb{H}_n^+\right\}. \tag{2.23b}$$

Theorem 2. *Assume that $\mathbf{M}_1, \ldots, \mathbf{M}_p$ are arbitrary matrices in $\mathbb{F}_n$ which is equal to either $\mathbb{S}_n$ or $\mathbb{H}_n$. Suppose that $a_1, \ldots, a_p$ are real numbers such that the feasibility problem*

$$\begin{aligned}
\text{find} \quad & \mathbf{X} \in \mathbb{F}_n && &&\\
\text{subject to} \quad & \langle \mathbf{M}_k, \mathbf{X} \rangle \leq a_k, && k = 1, \ldots, p, && \text{(2.24a)}\\
& \mathbf{X} \succeq 0, && && \text{(2.24b)}
\end{aligned}$$

has a positive-definite feasible solution $\mathbf{X}^{\text{ref}} \in \mathbb{F}_n^+$. Let $\mathcal{G} = \mathscr{G}(\mathbf{M}_1) \cup \cdots \cup \mathscr{G}(\mathbf{M}_p)$.

a) *Consider an arbitrary supergraph $\mathcal{G}'$ of $\mathcal{G}$. Every subsolution $\mathbf{X}^{\rm opt}$ to Optimization A with the input $(\mathcal{G}, \mathcal{G}', \mathbf{T}, \mathbf{X}^{\rm ref})$ is a solution to the LMI problem* (2.24) *and satisfies the relation*

$$\operatorname{rank}\{\mathbf{X}^{\rm opt}\} \leq |\mathcal{G}'| - \min_{\mathcal{G}_s} \left\{ \operatorname{msr}_{\mathbb{F}}(\mathcal{G}_s) \,\middle|\, (\mathcal{G}' \setminus \mathcal{G}) \subseteq \mathcal{G}_s \subseteq \mathcal{G}' \right\} \tag{2.25}$$

b) *Consider an arbitrary tree decomposition $\mathcal{T}$ of $\mathcal{G}$ with width t. Let $\overline{\mathcal{G}}$ be an enriched supergraph of $\mathcal{G}$ derived by $\mathcal{T}$. Every subsolution $\mathbf{X}^{\rm opt}$ to Optimization A with the input $(\mathcal{G}, \overline{\mathcal{G}}, \mathbf{T}, \mathbf{X}^{\rm ref})$ is a solution to* (2.24) *and satisfies the relation*

$$\operatorname{rank}\{\mathbf{X}^{\rm opt}\} \leq t + 1 \tag{2.26}$$

Proof. To prove Part (a), notice that X_{ij} does not play a role in the linear constraint (2.24a) of the LMI problem (2.24) as long as $i \neq j$ and $(i,j) \notin \mathcal{E}_{\mathcal{G}}$. It can be inferred from this property that $\mathbf{X}^{\rm opt}$ is a solution of (2.24). Now, it remains to show the validity of the inequality (2.25). Constraints (2.21b), (2.21c) and (2.21d) imply that for every feasible solution $\overline{\mathbf{X}}$ of Optimization A, the matrix $\overline{\mathbf{X}} - \overline{\mathbf{X}}^{\rm opt}$ belongs to the convex cone

$$C = \{\mathbf{W} \in \mathbb{F}_m | W_{kk} = 0 \quad \text{for} \quad k \in \mathcal{V}_{\mathcal{G}'}, \quad W_{ij} = 0 \quad \text{for} \quad (i,j) \in \mathcal{E}_{\mathcal{G}}\}. \tag{2.27}$$

Therefore, a dual matrix variable $\mathbf{\Lambda}$ could be assigned to these constraints, which belongs to the dual cone

$$C^{\perp} = \left\{ \mathbf{W} \in \mathbb{F}_m \,\middle|\, W_{ij} = 0 \quad \text{for} \quad (i,j) \notin \mathcal{E}_{\mathcal{G}} \text{ and } \mathrm{i} \neq \mathrm{j} \right\}. \tag{2.28}$$

Hence, the Lagrangian of Optimization A can be expressed as

$$\begin{aligned} \mathcal{L}(\overline{\mathbf{X}}, \mathbf{\Lambda}, \mathbf{\Phi}) &= \operatorname{trace}\{\mathbf{T}\overline{\mathbf{X}}\} + \operatorname{trace}\{\mathbf{\Lambda}(\overline{\mathbf{X}} - \overline{\mathbf{X}}^{\rm opt})\} - \operatorname{trace}\{\mathbf{\Phi}\overline{\mathbf{X}}\} \\ &= \operatorname{trace}\{(\mathbf{\Lambda} + \mathbf{T} - \mathbf{\Phi})\overline{\mathbf{X}}\} - \operatorname{trace}\{\mathbf{\Lambda}\overline{\mathbf{X}}^{\rm opt}\} \end{aligned} \tag{2.29}$$

where $\mathbf{\Phi} \succeq 0$ denotes the matrix dual variable corresponding to the constraint $\overline{\mathbf{X}} \succeq 0$. The infimum of the Lagrangian over $\overline{\mathbf{X}}$ is $-\infty$ unless $\mathbf{\Phi} = \mathbf{\Lambda} + \mathbf{T}$. Therefore, the dual problem is as follows:

$$\underset{\mathbf{\Lambda} \in \mathbb{F}_m}{\text{maximize}} \qquad -\langle \mathbf{\Lambda}, \overline{\mathbf{X}}^{\rm opt} \rangle \tag{2.30a}$$

$$\text{subject to} \qquad \Lambda_{ij} = 0 \qquad (i,j) \notin \mathcal{E}_{\mathcal{G}} \text{ and } \mathrm{i} \neq \mathrm{j}, \tag{2.30b}$$

$$\mathbf{\Lambda} + \mathbf{T} \succeq 0. \tag{2.30c}$$

By pushing the diagonal entries of $\mathbf{\Lambda}$ toward infinity, the inequality $\mathbf{\Lambda}+\mathbf{T} \succeq 0$ will become strict. Hence, strong duality holds according to the Slater's condition. If $\mathbf{\Phi} = \mathbf{\Phi}^{\mathrm{opt}}$ denotes an arbitrary dual solution, the complementary slackness condition $\langle \mathbf{\Phi}^{\mathrm{opt}}, \overline{\mathbf{X}}^{\mathrm{opt}} \rangle = 0$ yields that

$$\mathrm{rank}\{\mathbf{\Phi}^{\mathrm{opt}}\} + \mathrm{rank}\{\overline{\mathbf{X}}^{\mathrm{opt}}\} \leq m \tag{2.31}$$

(note that since the primal and dual problems are strictly feasible, $\overline{\mathbf{X}}^{\mathrm{opt}}$ and $\mathbf{\Phi}^{\mathrm{opt}}$ are both finite). On the other hand, according to the equations $\mathbf{\Phi} = \mathbf{\Lambda} + \mathbf{T}$ and $\mathbf{\Lambda} \in C^{\perp}$, we have

$$\mathbf{\Phi}^{\mathrm{opt}}_{ij} \neq 0, \qquad \text{for} \quad (\mathrm{i},\mathrm{j}) \in \mathcal{E}_{\mathcal{G}'} \setminus \mathcal{E}_{\mathcal{G}} \tag{2.32a}$$

$$\mathbf{\Phi}^{\mathrm{opt}}_{ij} = 0, \qquad \text{for} \quad (\mathrm{i},\mathrm{j}) \notin \mathcal{E}_{\mathcal{G}'} \text{ and } \mathrm{i} \neq \mathrm{j}. \tag{2.32b}$$

Therefore,

$$(\mathcal{G}' \setminus \mathcal{G}) \subseteq \mathscr{G}(\mathbf{\Phi}^{\mathrm{opt}}) \subseteq \mathcal{G}' \tag{2.33}$$

The proof of Part (a) is completed by combining (2.31) and (2.33) after noting that $\mathrm{rank}\{\mathbf{X}^{\mathrm{opt}}\} \leq \mathrm{rank}\{\overline{\mathbf{X}}^{\mathrm{opt}}\}$ (recall that $\mathbf{X}^{\mathrm{opt}}$ is a submatrix of $\overline{\mathbf{X}}^{\mathrm{opt}}$).

For Part (b), it follows from Theorem 1 that $\mathrm{OS}(\mathcal{G}_s) \geq |\overline{\mathcal{G}}| - t - 1$ for every $\mathcal{G}_s$ with the property $(\overline{\mathcal{G}} \setminus \mathcal{G}) \subseteq \mathcal{G}_s \subseteq \overline{\mathcal{G}}$. Therefore,

$$\begin{aligned} \mathrm{rank}\{\mathbf{X}^{\mathrm{opt}}\} &\leq |\overline{\mathcal{G}}| - \min\left\{\mathrm{msr}_{\mathbb{F}}(\mathcal{G}_s) \,\middle|\, (\overline{\mathcal{G}} \setminus \mathcal{G}) \subseteq \mathcal{G}_s \subseteq \overline{\mathcal{G}}\right\} \\ &\leq |\overline{\mathcal{G}}| - \min\left\{\mathrm{OS}(\mathcal{G}_s) \,\middle|\, (\overline{\mathcal{G}} \setminus \mathcal{G}) \subseteq \mathcal{G}_s \subseteq \overline{\mathcal{G}}\right\} \\ &\leq |\overline{\mathcal{G}}| - (|\overline{\mathcal{G}}| - t - 1) \\ &\leq t + 1 \end{aligned} \tag{2.34}$$

(note that $\mathrm{OS}(\mathcal{G}) \leq \mathrm{msr}_{\mathbb{F}}(\mathcal{G})$ as proven in [Hackney *et al.*, 2009]). This completes the proof. $\square$

Corollary 2. *The inequality in* (2.25)*, leads to the following upper bound on* $\mathrm{rank}\{\mathbf{X}^{\mathrm{opt}}\}$ *in terms of positive semidefinite zero forcing number:*

$$\mathrm{rank}\{\mathbf{X}^{\mathrm{opt}}\} \leq \max_{\mathcal{G}_s} \left\{\mathrm{Z}^{+}(\mathcal{G}_s) \,\middle|\, (\mathcal{G}' \setminus \mathcal{G}) \subseteq \mathcal{G}_s \subseteq \mathcal{G}'\right\} \tag{2.35}$$

Observe that the objective function of Optimization A is a weighted sum of certain entries of the matrix $\overline{\mathbf{X}}$, where the weights come from the matrix $\mathbf{T}$. Part (a) of Theorem 2 proposes an upper bound on the rank of all subsolutions of this optimization, which is contingent upon the graph of the weight matrix $\mathbf{T}$ without making use of the nonzero values of the weights.

Corollary 3. *If the LMI problem* (2.24) *has a positive-definite feasible solution, then it has a solution* $\mathbf{X}^{\text{opt}}$ *with rank at most* $\text{tw}(\mathcal{G})+1$*.*

Proof. The proof follows immediately from Part (b) of Theorem 2 by considering $\mathcal{T}$ to be a minimal tree decomposition of $\mathcal{G}$. □

Note that Theorem 2 and Corollary 3 both require the existence of a positive-definite feasible solution. This assumption will be reduced to only the feasibility of the LMI problem (2.24) in the next section.

We now revisit Examples 1, 2 and 3 provided earlier and study them using Theorem 2. First, consider Example 1. The graph $\mathcal{G}$ corresponding to a matrix $\mathbf{X}$ with known diagonal entries has the vertex set $\{1,2,\ldots,n\}$ with no edges. An enriched supergraph graph $\overline{\mathcal{G}}$ can be obtained from $\mathcal{G}$ by connecting vertices i and $i+1$ for $i=1,\ldots,n-1$. Consider an arbitrary matrix $\mathbf{T}\in\mathbf{S}^n$ with the representative graph $\overline{\mathcal{G}}$. This matrix is sparse with nonzero subdiagonal and superdiagonal. Using Theorem 2, Optimization A yields a solution such that $\mathbf{X}^{\text{opt}}\leq\text{tw}(\mathcal{G})+1$. Since $\mathcal{G}$ does not have any edges, its treewidth is equal to 0. As a result, every solution of Optimization A has rank 1.

Consider now Example 2 with $\mathbf{X}$ visualized in Figure 2.1. As can be observed, two 2×2 blocks of $\mathbf{X}$ specified by dashed red lines are known and the goal is to design the block $\mathbf{X}_{31}$. The graph $\mathcal{G}$ has $n=\alpha+\beta+\gamma$ vertices with the property that the subgraphs induced by the vertex subsets $\{1,\ldots,\alpha+\beta\}$ and $\{\alpha+1,\ldots,n\}$ are both complete graphs. In the case where $\alpha=\gamma$, an enriched supergraph $\overline{\mathcal{G}}$ can be obtained by connecting vertex i to vertex $\alpha+\beta+i$ for $i=1,2,\ldots,\alpha$. Consider a matrix $\mathbf{T}$ with the representative graph $\overline{\mathcal{G}}$. Optimization A then aims to minimize the weighted sum over the diagonal entries of $\mathbf{X}_{31}$. Consider now the case where $\alpha>\gamma$. A tree decomposition of $\mathcal{G}$ has two bags $\{1,\ldots,\alpha+\beta\}$ and $\{\alpha+1,\ldots,\alpha+\beta+\gamma\}$. Since these bags have disparate sizes, the definition of enriched supergraph requires adding $\alpha-\gamma$ new vertices to the bag with the fewer number of vertices. This can be translated as adding $\alpha-\gamma$ rows and $\alpha-\gamma$ columns to $\mathbf{X}$ in order to arrive at the augmented matrix $\widehat{X}$ depicted in Figure 2.1(b). In this case, Optimization A may minimize a weighted sum of the diagonal entries of the square block including $\widehat{\mathbf{X}}_{31}$ and $\widehat{\mathbf{X}}_{41}$. Regarding Example 3, the matrix $\mathcal{G}$ has the vertex set $\mathcal{V}_{\mathcal{G}}=\{1,\ldots,4\alpha\}$ such that its subgraphs induced by the vertex subsets $\{1,\ldots,2\alpha\}$, $\{\alpha+1,\ldots,3\alpha\}$, and $\{2\alpha+1,\ldots,4\alpha\}$ are all complete graphs. A tree decomposition of $\mathcal{G}$ has three bags $\{1,\dots,2\alpha\}$, $\{\alpha+1,\dots,3\alpha\}$ and $\{2\alpha+1,\dots,4\alpha\}$. Hence, an enriched graph $\overline{\mathcal{G}}$ can be obtained by connecting vertices i and $2\alpha+i$ as well as vertices $i+\alpha$ and $3\alpha+i$ for $i=1,\dots,\alpha$. This implies that Optimization A minimizes a weighted sum of the diagonal entries of the blocks $\mathbf{X}_{31}$ and $\mathbf{X}_{42}$.

2.5 Combined Graph-Theoretic and Algebraic Method

The results derived in the preceding section require the existence of a positive-definite feasible solution for the LMI problem (2.24). The first objective of this part is to relax the above assumption to only the existence of a feasible solution. The second objective is to develop a combined graph-theoretic and algebraic method offering stronger results compared to Theorem 2 and Corollary 3.

Given an arbitrary matrix $\mathbf{M}$ in $\mathbb{F}_n$, we denote its Moore-Penrose pseudoinverse as $\mathbf{M}^+$. If $r=\text{rank}\{\mathbf{M}\}$ and $\mathbf{M}$ admits the eigenvalue decomposition $\mathbf{M}=\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^*$ with $\mathbf{\Lambda}=\text{diag}\{[\lambda_1,\dots,\lambda_r,0,\dots,0]\}$, then $\mathbf{M}^+=\mathbf{Q}\mathbf{\Lambda}^+\mathbf{Q}^*$ where $\mathbf{\Lambda}^+=\text{diag}\{[\lambda_1^{-1},\dots,\lambda_r^{-1},0,\dots,0]\}$. The next lemma is borrowed from [Carlson *et al.*, 1974].

Lemma 1. *Given a 2×2 block matrix*

$$\mathbf{M}=\begin{bmatrix}\mathbf{A} & \mathbf{B}^*\\ \mathbf{B} & \mathbf{C}\end{bmatrix}\in\mathbb{F}_n, \tag{2.36}$$

define its generalized Schur complement as $\mathbf{S}^+\triangleq\mathbf{C}-\mathbf{B}\mathbf{A}^+\mathbf{B}^*$. The relation $\mathbf{M}\succeq 0$ holds if and only if

$$\mathbf{A}\succeq 0,\quad \mathbf{S}^+\succeq 0\quad\text{and}\quad \text{null}\{\mathbf{A}\}\subseteq\text{null}\{\mathbf{B}\}. \tag{2.37}$$

In addition, the equation $\text{rank}\{\mathbf{M}\}=\text{rank}\{\mathbf{A}\}+\text{rank}\{\mathbf{S}^+\}$ *is satisfied if and only if* $\text{null}\{\mathbf{A}\}\subseteq\text{null}\{\mathbf{B}\}$.

Theorem 3. *Consider the block matrix*

$$\mathbf{M}(\mathbf{U})\triangleq\begin{bmatrix}\mathbf{A} & \mathbf{B}_x^* & \mathbf{B}_y^*\\ \mathbf{B}_x & \mathbf{X} & \mathbf{U}^*\\ \mathbf{B}_y & \mathbf{U} & \mathbf{Y}\end{bmatrix} \tag{2.38}$$

where $\mathbf{A}$, $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{B}_x^*$ and $\mathbf{B}_y^*$ are known and the matrix $\mathbf{U}$ is the variable. Define

$$\mathbf{M}_x\triangleq\begin{bmatrix}\mathbf{A} & \mathbf{B}_x^*\\ \mathbf{B}_x & \mathbf{X}\end{bmatrix}\quad\text{and}\quad\mathbf{M}_y\triangleq\begin{bmatrix}\mathbf{A} & \mathbf{B}_y^*\\ \mathbf{B}_y & \mathbf{Y}\end{bmatrix} \tag{2.39}$$

Define also $\mathbf{S}_x^+ \triangleq \mathbf{X} - \mathbf{B}_x\mathbf{A}^+\mathbf{B}_x^*$ *and* $\mathbf{S}_y^+ \triangleq \mathbf{Y} - \mathbf{B}_y\mathbf{A}^+\mathbf{B}_y^*$. *Given a constant matrix* $\mathbf{T}$ *of appropriate dimension, every solution* $\mathbf{U}^{\text{opt}}$ *of the optimization problem*

$$\underset{\mathbf{U}}{\text{minimize}} \qquad \langle \mathbf{T}, \mathbf{U} \rangle \tag{2.40a}$$

$$\text{subject to} \qquad \mathbf{M}(\mathbf{U}) \succeq 0. \tag{2.40b}$$

has the minimum possible rank, i.e.,

$$\text{rank}\{\mathbf{M}\left(\mathbf{U}^{\text{opt}}\right)\} = \max\left\{\text{rank}\{\mathbf{M}_x\}, \text{rank}\{\mathbf{M}_y\}\right\}, \tag{2.41}$$

provided that $\mathbf{S}_y^+\mathbf{T}\mathbf{S}_x^+$ *has the maximum possible rank, i.e.,*

$$\text{rank}\{\mathbf{S}_y^+\mathbf{T}\mathbf{S}_x^+\} = \min\left\{\text{rank}\{\mathbf{S}_x^+\}, \text{rank}\{\mathbf{S}_y^+\}\right\} \tag{2.42}$$

Proof. Let $r_x \triangleq \text{rank}\{\mathbf{S}_x^+\}$ and $r_y \triangleq \text{rank}\{\mathbf{S}_y^+\}$. Consider the following eigenvalue decompositions for $\mathbf{S}_x^+$ and $\mathbf{S}_y^+$:

$$\mathbf{S}_x^+ = \mathbf{Q}_x\mathbf{\Lambda}_x\mathbf{Q}_x^* \quad \text{and} \quad \mathbf{S}_y^+ = \mathbf{Q}_y\mathbf{\Lambda}_y\mathbf{Q}_y^*. \tag{2.43}$$

Let $\mathbf{Q}_x = [\mathbf{Q}_{x1} \quad \mathbf{Q}_{x0}]$ and $\mathbf{Q}_y = [\mathbf{Q}_{y1} \quad \mathbf{Q}_{y0}]$, where $\mathbf{Q}_{x1} \in \mathbb{F}^{n\times r_x}$ and $\mathbf{Q}_{y1} \in \mathbb{F}^{n\times r_y}$. We can also write

$$\mathbf{\Lambda}_x \triangleq \begin{bmatrix} \mathbf{\Lambda}_{x1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \mathbf{\Lambda}_y \triangleq \begin{bmatrix} \mathbf{\Lambda}_{y1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{2.44}$$

where $\mathbf{\Lambda}_{x1}$ and $\mathbf{\Lambda}_{y1}$ are diagonal matrices in $\mathbb{F}_{n_x}$ and $\mathbb{F}_{n_y}$, respectively. Define

$$\mathbf{E}_{ij} \triangleq \mathbf{Q}_{yi}^*(\mathbf{U} - \mathbf{B}_y\mathbf{A}^+\mathbf{B}_x^*)\mathbf{Q}_{xj} \quad \text{for} \quad i,j \in \{1,2\}. \tag{2.45}$$

It can be shown that

$$\begin{aligned} \mathbf{U} - \mathbf{B}_y\mathbf{A}^+\mathbf{B}_x^* =& [\mathbf{Q}_{y1} \quad \mathbf{Q}_{y0}] \begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{10} \\ \mathbf{E}_{01} & \mathbf{E}_{00} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_{x1}^* \\ \mathbf{Q}_{x0}^* \end{bmatrix} \\ =& \mathbf{Q}_{y1}\mathbf{E}_{11}\mathbf{Q}_{x1}^* + \mathbf{Q}_{y1}\mathbf{E}_{10}\mathbf{Q}_{x0}^* + \mathbf{Q}_{y0}\mathbf{E}_{01}\mathbf{Q}_{x1}^* + \mathbf{Q}_{y0}\mathbf{E}_{00}\mathbf{Q}_{x0}^*. \end{aligned} \tag{2.46}$$

Hence,

$$\begin{aligned} \mathbf{S}^+ \triangleq& \begin{bmatrix} \mathbf{X} & \mathbf{U}^* \\ \mathbf{U} & \mathbf{Y} \end{bmatrix} - \begin{bmatrix} \mathbf{B}_x \\ \mathbf{B}_y \end{bmatrix} \mathbf{A}^+ \begin{bmatrix} \mathbf{B}_x^* & \mathbf{B}_y^* \end{bmatrix} \\ =& \begin{bmatrix} \mathbf{Q}_{x1}\mathbf{\Lambda}_{x1}\mathbf{Q}_{x1}^* & \mathbf{U}^* - \mathbf{B}_y\mathbf{A}^+\mathbf{B}_x^* \\ \mathbf{U} - \mathbf{B}_y\mathbf{A}^+\mathbf{B}_x^* & \mathbf{Q}_{y1}\mathbf{\Lambda}_{y1}\mathbf{Q}_{y1}^* \end{bmatrix}. \end{aligned} \tag{2.47}$$

The constraint $\mathbf{M}(\mathbf{U}) \succeq 0$ yields $\mathbf{S}^{+} \succeq 0$ and therefore

$$\begin{aligned}\begin{bmatrix} \mathbf{0} & \mathbf{E}_{ij}^{*} \\ \mathbf{E}_{ij} & \mathbf{0} \end{bmatrix} &= \begin{bmatrix} \mathbf{Q}_{xi}^{*} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{yj}^{*} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_{x1}\mathbf{\Lambda}_{x1}\mathbf{Q}_{x1}^{*} & \mathbf{U}^{*} - \mathbf{B}_{y}\mathbf{A}^{+}\mathbf{B}_{x}^{*} \\ \mathbf{U} - \mathbf{B}_{y}\mathbf{A}^{+}\mathbf{B}_{x}^{*} & \mathbf{Q}_{y1}\mathbf{\Lambda}_{y1}\mathbf{Q}_{y1}^{*} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_{xi} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{yj} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Q}_{xi}^{*} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{yj}^{*} \end{bmatrix} \mathbf{S}^{+} \begin{bmatrix} \mathbf{Q}_{xi} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{yj} \end{bmatrix} \succeq 0 \quad \Longrightarrow \quad \mathbf{E}_{ij} = 0\end{aligned} \tag{2.48}$$

for every $(i, j) \in \{(0, 0), (1, 0), (0, 1)\}$. As a result, the block $\mathbf{U}$ can be written as $\mathbf{U} = \mathbf{B}_{y}\mathbf{A}^{+}\mathbf{B}_{x}^{*} + \mathbf{Q}_{y1}\mathbf{U}_{1}\mathbf{Q}_{x1}^{*}$, where $\mathbf{U}_{1} \triangleq \mathbf{E}_{11} \in \mathbb{F}^{r_{y} \times r_{x}}$. Therefore,

$$\begin{aligned}\mathbf{S}^{+} &\triangleq \begin{bmatrix} \mathbf{X} & \mathbf{U}^{*} \\ \mathbf{U} & \mathbf{Y} \end{bmatrix} - \begin{bmatrix} \mathbf{B}_{x} \\ \mathbf{B}_{y} \end{bmatrix} \mathbf{A}^{+} \begin{bmatrix} \mathbf{B}_{x}^{*} & \mathbf{B}_{y}^{*} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Q}_{x1} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{y1} \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda}_{x1} & \mathbf{U}_{1}^{*} \\ \mathbf{U}_{1} & \mathbf{\Lambda}_{y1} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_{x1}^{*} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{y1}^{*} \end{bmatrix}.\end{aligned} \tag{2.49}$$

Since $\mathbf{M}_{x}, \mathbf{M}_{y} \succeq 0$ according to Lemma 1, one can write

$$\mathrm{null}\{\mathbf{A}\} \subseteq \mathrm{null}\{\mathbf{B}_{x}\} \quad \text{and} \quad \mathrm{null}\{\mathbf{A}\} \subseteq \mathrm{null}\{\mathbf{B}_{y}\} \tag{2.50}$$

and therefore Lemma 1 yields

$$\mathrm{rank}\{\mathbf{M}_{x}\} = \mathrm{rank}\{\mathbf{A}\} + r_{x} \tag{2.51}$$

$$\mathrm{rank}\{\mathbf{M}_{y}\} = \mathrm{rank}\{\mathbf{A}\} + r_{y} \tag{2.52}$$

This implies that

$$\mathbf{M} \succeq 0 \quad \Longrightarrow \quad \begin{bmatrix} \mathbf{\Lambda}_{x1} & \mathbf{U}_{1}^{*} \\ \mathbf{U}_{1} & \mathbf{\Lambda}_{y1} \end{bmatrix} \succeq 0 \tag{2.53}$$

On the other hand,

$$\begin{aligned}\mathrm{trace}\{\mathbf{T}\mathbf{U}^{*}\} &= \mathrm{trace}\{\mathbf{T}(\mathbf{B}_{y}\mathbf{A}^{+}\mathbf{B}_{x}^{*} + \mathbf{Q}_{y1}\mathbf{U}_{1}\mathbf{Q}_{x1}^{*})^{*}\} \\ &= \mathrm{trace}\{\mathbf{B}_{y}^{*}\mathbf{T}\mathbf{B}_{x}\mathbf{A}^{+}\} + \mathrm{trace}\{\mathbf{Q}_{y1}^{*}\mathbf{T}\mathbf{Q}_{x1}\mathbf{U}_{1}^{*}\}\end{aligned} \tag{2.54}$$

Hence, the problem (2.40) is equivalent to

$$\underset{\mathbf{U}_{1}}{\text{minimize}} \qquad \langle \mathbf{T}_{1}, \mathbf{U}_{1} \rangle \tag{2.55a}$$

$$\text{subject to} \qquad \begin{bmatrix} \mathbf{\Lambda}_{x1} & \mathbf{U}_{1}^{*} \\ \mathbf{U}_{1} & \mathbf{\Lambda}_{y1} \end{bmatrix} \succeq 0. \tag{2.55b}$$

where $\mathbf{T}_1 = \mathbf{Q}_{y1}^* \mathbf{T} \mathbf{Q}_{x1}$. Let $\mathbf{U}_1^{\mathrm{opt}}$ be an arbitrary solution of the above problem. It can be easily seen that the dual matrix variable corresponding to the sole constraint of this problem is equal to

$$\begin{bmatrix} \mathbf{\Gamma}_{x1} & \mathbf{T}_1^* \\ \mathbf{T}_1 & \mathbf{\Gamma}_{y1} \end{bmatrix} \succeq 0 \tag{2.56}$$

for some matrices $\mathbf{\Gamma}_{x1}$ and $\mathbf{\Gamma}_{y1}$. It follows from the complementary slackness that

$$\operatorname{trace}\left\{ \begin{bmatrix} \mathbf{\Lambda}_{x1} & \mathbf{U}_1^* \\ \mathbf{U}_1 & \mathbf{\Lambda}_{y1} \end{bmatrix} \begin{bmatrix} \mathbf{\Gamma}_{x1} & \mathbf{T}_1^* \\ \mathbf{T}_1 & \mathbf{\Gamma}_{y1} \end{bmatrix} \right\} = 0, \tag{2.57}$$

implying that

$$\operatorname{rank}\left\{ \begin{bmatrix} \mathbf{\Lambda}_{x1} & \mathbf{U}_1^* \\ \mathbf{U}_1 & \mathbf{\Lambda}_{y1} \end{bmatrix} \right\} + \operatorname{rank}\left\{ \begin{bmatrix} \mathbf{\Gamma}_{x1} & \mathbf{T}_1^* \\ \mathbf{T}_1 & \mathbf{\Gamma}_{y1} \end{bmatrix} \right\} = r_x + r_y. \tag{2.58}$$

Therefore,

$$\operatorname{rank}\left\{ \begin{bmatrix} \mathbf{\Lambda}_{x1} & (\mathbf{U}_1^{\mathrm{opt}})^* \\ \mathbf{U}_1^{\mathrm{opt}} & \mathbf{\Lambda}_{y1} \end{bmatrix} \right\} \leq r_x + r_y - \operatorname{rank}\{\mathbf{T}_1\} = r_x + r_y - \operatorname{rank}\{\mathbf{S}_y^+ \mathbf{T} \mathbf{S}_x^+\} \tag{2.59}$$

Moreover, it can be concluded from (2.50) that

$$\operatorname{null}\{\mathbf{A}\} \subseteq \operatorname{null}\{\mathbf{B}_x\} \cap \operatorname{null}\{\mathbf{B}_y\} = \operatorname{null}\left\{ \begin{bmatrix} \mathbf{B}_x \\ \mathbf{B}_y \end{bmatrix} \right\} \tag{2.60}$$

The proof is now completed by Lemma 1. □

Remark 1. *Note that the condition* (2.42) *required in Theorem 3 is satisfied for a generic matrix* $\mathbf{T}$.

Corollary 4. *Suppose that* $\mathbf{O} \in \mathbb{F}^{r_y \times r_x}$ *is a matrix with 1's on its rectangular diagonal and 0 elsewhere. If the matrix* $\mathbf{M}(\mathbf{U})$ *is completed as*

$$\mathbf{U} = \mathbf{B}_y \mathbf{A}^+ \mathbf{B}_x^* + \mathbf{Q}_{y1} \sqrt{\mathbf{\Lambda}_{y1}}\ \mathbf{O} \sqrt{\mathbf{\Lambda}_{x1}} \mathbf{Q}_{x1}^* \tag{2.61}$$

then, it satisfies the rank property (2.41). *This explicit formula provides an iterative matrix-completion method.*

Definition 10. *For every matrix* $\mathbf{X} \in \mathbb{F}_k$ *and sets* $A, B \subseteq \{1, \ldots, k\}$, *define* $\mathbf{X}(A, B)$ *as a submatrix of* $\mathbf{X}$ *obtained by choosing those rows of* $\mathbf{X}$ *with indices appearing in* A *and those columns of* $\mathbf{X}$ *with indices in* B. *If* $A = B$, *then* $\mathbf{X}(A, B)$ *will be abbreviated as* $\mathbf{X}(A)$.

Assume that $\mathbf{M}_1, \ldots, \mathbf{M}_p$ are arbitrary matrices in $\mathbb{F}_n$, which is equal to either $\mathbb{S}_n$ or $\mathbb{H}_n$. Suppose that $a_1, \ldots, a_p$ are real numbers such that the feasibility problem

$$\begin{aligned} &\text{find} && \mathbf{X} \in \mathbb{F}_n \\ &\text{subject to} && \langle \mathbf{M}_k, \mathbf{X} \rangle \leq a_k, && k = 1, \ldots, p, && \text{(2.62a)} \\ & && \mathbf{X} \succeq 0, && && \text{(2.62b)} \end{aligned}$$

has a feasible solution $\mathbf{X}^{\text{ref}} \in \mathbb{F}_n$. Let $\mathcal{G} = \mathscr{G}(\mathbf{M}_1) \cup \cdots \cup \mathscr{G}(\mathbf{M}_p)$. Consider an arbitrary tree decomposition $\mathcal{T}$ of $\mathcal{G}$ with the set of bags $\mathcal{V}_{\mathcal{T}} = \{V_1, \ldots, V_{|\mathcal{T}|}\}$. Let

$$r \triangleq \max \Big\{ \text{rank}\{\mathbf{X}^{\text{ref}}(V_k)\} \mid 1 \leq k \leq |\mathcal{T}| \Big\} \tag{2.63}$$

and define $\overline{\mathcal{G}}$ as a graph obtained from $\mathcal{G}$ by adding

$$\sum_{k=1}^{|\mathcal{T}|} \Big(r - \text{rank}\{\mathbf{X}^{\text{ref}}(V_k)\} \Big) \tag{2.64}$$

new isolated vertices. Let $\overline{\mathcal{T}} \triangleq (\overline{\mathcal{V}}_{\mathcal{T}}, \mathcal{E}_{\mathcal{T}})$ be a tree decomposition for $\overline{\mathcal{G}}$ with the bags $\overline{V}_1, \ldots, \overline{V}_{|\mathcal{T}|}$, where each bag $\overline{V}_k$ is constructed from V_k by adding $r - \text{rank}\{\mathbf{X}^{\text{ref}}(V_k)\}$ of the new isolated vertices in $\mathcal{V}_{\overline{\mathcal{G}}} \setminus \mathcal{V}_{\mathcal{G}}$ such that $(\overline{V}_i \setminus V_i) \cap (\overline{V}_j \setminus V_j) = \emptyset$ for every $i \neq j$. Let $m \triangleq |\overline{\mathcal{G}}|$ and define the matrix $\overline{\mathbf{X}}^{\text{ref}} \in \mathbb{F}_m$ as

$$\begin{aligned} &\overline{X}^{\text{ref}}_{kk} = X^{\text{ref}}_{kk} && \text{for} && k \in \mathcal{V}_{\mathcal{G}} && \text{(2.65a)} \\ &\overline{X}^{\text{ref}}_{kk} = 1 && \text{for} && k \in \mathcal{V}_{\overline{\mathcal{G}}} \setminus \mathcal{V}_{\mathcal{G}} && \text{(2.65b)} \\ &\overline{X}^{\text{ref}}_{ij} = X^{\text{ref}}_{ij} && \text{for} && (i,j) \in \mathcal{E}_{\mathcal{G}} && \text{(2.65c)} \\ &\overline{X}^{\text{ref}}_{ij} = 0 && \text{for} && (i,j) \notin \mathcal{E}_{\mathcal{G}}. && \text{(2.65d)} \end{aligned}$$

For every pair $i, j \in \{1, \ldots, |\mathcal{T}|\}$, define

$$\begin{aligned} \mathbf{S}^{+}_{ij} \triangleq\ & \overline{\mathbf{X}}^{\text{ref}}(\overline{V}_i \setminus \overline{V}_j) \\ & - \overline{\mathbf{X}}^{\text{ref}}(\overline{V}_i \setminus \overline{V}_j, \overline{V}_i \cap \overline{V}_j) \left(\overline{\mathbf{X}}^{\text{ref}}(\overline{V}_i \cap \overline{V}_j) \right)^{+} \overline{\mathbf{X}}^{\text{ref}}(\overline{V}_i \cap \overline{V}_j, \overline{V}_i \setminus \overline{V}_j) \end{aligned} \tag{2.66}$$

Let the edges of the tree decomposition $\mathcal{T}$ be oriented in such a way that the indegree of every node becomes less than or equal to 1. The resulting directed tree is denoted as $\vec{\mathcal{T}}$. The notation $\mathcal{E}_{\vec{\mathcal{T}}}$ also represents the edge set of this directed tree.

Optimization B: This problem is as follows:

$$\underset{\overline{\mathbf{X}}\in\mathbb{F}_m}{\text{minimize}} \quad \sum_{(i,j)\in\mathcal{E}_{\vec{\mathcal{T}}}} \langle \mathbf{T}_{ij}, \overline{\mathbf{X}}_{ij}\rangle \tag{2.67a}$$

$$\text{subject to} \quad \overline{\mathbf{X}}_k = \overline{\mathbf{X}}_k^{\text{ref}} \qquad k=1,\ldots,|\mathcal{T}|, \tag{2.67b}$$

$$\overline{\mathbf{X}} \succeq 0, \tag{2.67c}$$

where $\mathbf{T}_{ij}$'s are arbitrary constant matrices of appropriate dimensions and

$$\overline{\mathbf{X}}_k \triangleq \overline{\mathbf{X}}(\overline{V}_k), \quad \overline{\mathbf{X}}_k^{\text{ref}} \triangleq \overline{\mathbf{X}}^{\text{ref}}(\overline{V}_k) \quad \text{and} \quad \overline{\mathbf{X}}_{ij} \triangleq \overline{\mathbf{X}}(\overline{V}_i \setminus \overline{V}_j, \overline{V}_j \setminus \overline{V}_i) \tag{2.68}$$

for every $i,j,k \in \{1,\ldots,|\mathcal{T}|\}$.

Let $\overline{\mathbf{X}}^{\text{opt}} \in \mathbb{F}_m$ denote an arbitrary solution of problem (2.67) and $\mathbf{X}^{\text{opt}} \in \mathbb{F}_n$ be equal to $\overline{\mathbf{X}}^{\text{opt}}(\mathcal{V}_{\mathcal{G}})$. Then, $\mathbf{X}^{\text{opt}}$ is called the *subsolution to Optimization B associated with* $\overline{\mathbf{X}}^{\text{opt}}$. Note that $\mathbf{X}^{\text{opt}}$ and $\mathbf{X}^{\text{ref}}$ share the same diagonal and values for the entries corresponding to the edges of $\mathcal{G}$. Hence, Optimization B is a positive semidefinite matrix completion problem with the input $\mathbf{X}^{\text{ref}}$ and the output $\mathbf{X}^{\text{opt}}$.

Theorem 4. *Given an arbitrary solution* $\mathbf{X}^{\text{ref}}$ *of the problem* (2.62)*, every subsolution* $\mathbf{X}^{\text{opt}}$ *of Optimization B has the property*

$$\text{rank}\{\mathbf{X}^{\text{opt}}\} = \max\left\{\text{rank}\{\mathbf{X}^{\text{ref}}(V_k)\} \mid k=1,\ldots,|\mathcal{T}|\right\} \tag{2.69}$$

provided that the following equality holds for every $(i,j)\in\mathcal{E}_{\vec{\mathcal{T}}}$*:*

$$\text{rank}\{\mathbf{S}_{ij}^{+}\mathbf{T}_{ij}\mathbf{S}_{ji}^{+}\} = \min\left\{\text{rank}\{\mathbf{S}_{ij}^{+}\},\, \text{rank}\{\mathbf{S}_{ji}^{+}\}\right\}. \tag{2.70}$$

Proof. The proof follows immediately from Theorem 3 if $|\mathcal{T}|=2$. To prove by induction in the general case, assume that the statement of Theorem 4 holds if $|\mathcal{T}|\leq p$ for an arbitrary natural number p, and the goal is to show its validity for $|\mathcal{T}|=p+1$. With no loss of generality, assume that V_{p+1} is a leaf of $\vec{\mathcal{T}}$ and that (V_p, V_{p+1}) is a directed edge of this tree. Consider a tree decomposition $\mathcal{T}' = (\mathcal{V}_{\mathcal{T}'}, \mathcal{E}_{\mathcal{T}})$ for the sparsity graph of Optimization B with the bags $\overline{V}'_1,\ldots,\overline{V}'_{|\mathcal{T}|}$, where each bag $\overline{V}'_i$ is defined as the union of $\overline{V}_i$ and its parent in the oriented tree $\mathcal{T}$, if any. It results from the chordal theorem that the constraint $\overline{\mathbf{X}}\succeq 0$ in Optimization B can be replaced by the set of constraints $\overline{\mathbf{X}}(\overline{V}'_j) \succeq 0$ for $j = 1, \ldots, p+1$. This implies that Optimization B can be decomposed into $p = |\mathcal{T}| - 1$ independent semidefinite programs:

$$\begin{aligned}
&\underset{\overline{\mathbf{X}}(\overline{V}'_j)}{\text{minimize}} && \langle \mathbf{T}_{ij}, \overline{\mathbf{X}}_{ij} \rangle && \text{(2.71a)}\\
&\text{subject to} && \overline{\mathbf{X}}_i = \overline{\mathbf{X}}_i^{\text{ref}}, && \text{(2.71b)}\\
& && \overline{\mathbf{X}}_j = \overline{\mathbf{X}}_j^{\text{ref}}, && \text{(2.71c)}\\
& && \overline{\mathbf{X}}(\overline{V}'_j) \succeq 0, && \text{(2.71d)}
\end{aligned}$$

for every $(i,j) \in \mathcal{E}_{\vec{\mathcal{T}}}$. Notice that the submatrices $\overline{\mathbf{X}}_1^{\text{ref}}, \ldots, \overline{\mathbf{X}}_{|\mathcal{T}|}^{\text{ref}}$ all have the same rank r. By defining $\overline{V}'_0 \triangleq \overline{V}'_1 \cup \overline{V}'_2 \ldots \cup \overline{V}'_p$, it follows from the induction assumption and the decomposition property of Optimization B that

$$\text{rank}\{\overline{\mathbf{X}}^{\text{opt}}(\overline{V}'_0)\} = \max\left\{\text{rank}\{\overline{\mathbf{X}}_k^{\text{ref}}\} \;\middle|\; k = 1, \ldots, p\right\} = r \tag{2.72}$$

Now, consider the block matrix

$$\mathbf{M}(\mathbf{U}) \triangleq \begin{bmatrix} \overline{\mathbf{X}}^{\text{opt}}(\overline{V}'_0 \cap \overline{V}'_{|\mathcal{T}|}) & \overline{\mathbf{X}}^{\text{opt}}(\overline{V}'_0 \cap \overline{V}'_{|\mathcal{T}|}, \overline{V}'_0 \setminus \overline{V}'_{|\mathcal{T}|}) & \overline{\mathbf{X}}^{\text{opt}}(\overline{V}'_0 \cap \overline{V}'_{|\mathcal{T}|}, \overline{V}'_{|\mathcal{T}|} \setminus \overline{V}'_0) \\ \overline{\mathbf{X}}^{\text{opt}}(\overline{V}'_0 \setminus \overline{V}'_{|\mathcal{T}|}, \overline{V}'_0 \cap \overline{V}'_{|\mathcal{T}|}) & \overline{\mathbf{X}}^{\text{opt}}(\overline{V}'_0 \setminus \overline{V}'_{|\mathcal{T}|}) & \mathbf{U}^* \\ \overline{\mathbf{X}}^{\text{opt}}(\overline{V}'_{|\mathcal{T}|} \setminus \overline{V}'_0, \overline{V}'_0 \cap \overline{V}'_{|\mathcal{T}|}) & \mathbf{U} & \overline{\mathbf{X}}^{\text{opt}}(\overline{V}'_{|\mathcal{T}|} \setminus \overline{V}'_0) \end{bmatrix}$$

According to Theorem 3, it only remains to prove that

$$\text{rank}\{\mathbf{S}^+_{|\mathcal{T}|,0}\, \mathbf{T}_{|\mathcal{T}|,0}\, \mathbf{S}^+_{0,|\mathcal{T}|}\} = \min\left\{\text{rank}\{\mathbf{S}^+_{|\mathcal{T}|,0}\}, \text{rank}\{\mathbf{S}^+_{0,|\mathcal{T}|}\}\right\} \tag{2.73}$$

where

$$\mathbf{T}_{|\mathcal{T}|,0} = \begin{bmatrix} \mathbf{0} & \mathbf{T}_{|\mathcal{T}|-1,|\mathcal{T}|} \end{bmatrix}. \tag{2.74}$$

One can write

$$\begin{aligned}
\text{rank}\{\mathbf{S}^+_{|\mathcal{T}|,0} \begin{bmatrix} \mathbf{0} & \mathbf{T}_{|\mathcal{T}|-1,|\mathcal{T}|} \end{bmatrix} \mathbf{S}^+_{0,|\mathcal{T}|}\} &= \text{rank}\{\mathbf{S}^+_{|\mathcal{T}|-1,|\mathcal{T}|}\, \mathbf{T}_{|\mathcal{T}|-1,|\mathcal{T}|}\, \mathbf{S}^+_{|\mathcal{T}|,|\mathcal{T}|-1}\}\\
&= \min\left\{\text{rank}\{\mathbf{S}^+_{|\mathcal{T}|-1,|\mathcal{T}|}\}, \text{rank}\{\mathbf{S}^+_{|\mathcal{T}|,|\mathcal{T}|-1}\}\right\}\\
&= \text{rank}\{\mathbf{S}^+_{|\mathcal{T}|-1,|\mathcal{T}|}\}\\
&= \text{rank}\{\mathbf{S}^+_{0,|\mathcal{T}|}\}\\
&= \min\left\{\text{rank}\{\mathbf{S}^+_{|\mathcal{T}|,0}\}, \text{rank}\{\mathbf{S}^+_{0,|\mathcal{T}|}\}\right\}. \qquad (2.75)
\end{aligned}$$

This completes the proof. □

Remark 2. *Note that the condition* (2.70) *required in Theorem 4 is satisfied for generic choices of* $\mathbf{T}_{ij}$.

Example 4.

Consider a tree decomposition $\mathcal{T}$ with three bags $V_1 = \{1, 2, 3\}$, $V_2 = \{3, 4, 5\}$ and $V_3 = \{5, 6, 7\}$, and the edge set $\mathcal{E}_{\mathcal{T}} = \{(V_1, V_2),\, (V_2, V_3)\}$. Suppose that the partially known matrix solution is as follows:

$$\mathbf{X}^{\text{ref}} = \begin{bmatrix} 2 & 1 & 1 & u_{11}^* & u_{21}^* & w_{11}^* & w_{21}^* \\ 1 & 1 & 1 & u_{12}^* & u_{22}^* & w_{12}^* & w_{22}^* \\ 1 & 1 & 1 & 1 & 1 & v_{11}^* & v_{21}^* \\ u_{11} & u_{12} & 1 & 1 & 1 & v_{12}^* & v_{22}^* \\ u_{21} & u_{22} & 1 & 1 & 1 & 1 & 1 \\ w_{11} & w_{12} & v_{11} & v_{12} & 1 & 2 & 1 \\ w_{21} & w_{22} & v_{21} & v_{22} & 1 & 1 & 3 \end{bmatrix} \tag{2.76}$$

It can be verified that

$$\text{rank}\{\mathbf{X}^{\text{ref}}(V_1)\} = 2, \quad \text{rank}\{\mathbf{X}^{\text{ref}}(V_2)\} = 1, \quad \text{rank}\{\mathbf{X}^{\text{ref}}(V_3)\} = 3, \tag{2.77}$$

and that there exists only one unique solution for each unknown block

$$\mathbf{X}_{12} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{X}_{23} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \tag{2.78}$$

to meet the constraint $\mathbf{X} \succeq 0$. Hence, the only freedom for the matrix completion problem is on the choice of the remaining block

$$\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}. \tag{2.79}$$

Therefore, optimization problems solved over the blocks $\mathbf{X}_{12}$ and $\mathbf{X}_{23}$ would not result in a rank-3 solution. To resolve the issue, we enrich $\mathbf{X}^{\text{ref}}$ to obtain a matrix $\overline{\mathbf{X}}^{\text{ref}}$ by adding multiple rows and columns to $\mathbf{X}^{\text{ref}}$ in order to make the ranks of all resulting bags equal:

$$\overline{\mathbf{X}}^{\text{ref}} = \begin{bmatrix} 1 & 0 & 0 & 0 & u_{11}^* & u_{21}^* & u_{31}^* & u_{41}^* & w_{11}^* & w_{21}^* \\ 0 & 2 & 1 & 1 & u_{12}^* & u_{22}^* & u_{32}^* & u_{42}^* & w_{12}^* & w_{22}^* \\ 0 & 1 & 1 & 1 & u_{13}^* & u_{23}^* & u_{33}^* & u_{43}^* & w_{13}^* & w_{23}^* \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & v_{11}^* & v_{12}^* \\ u_{11} & u_{12} & u_{13} & 0 & 1 & 0 & 0 & 0 & v_{12}^* & v_{22}^* \\ u_{21} & u_{22} & u_{23} & 0 & 0 & 1 & 0 & 0 & v_{13}^* & v_{23}^* \\ u_{31} & u_{32} & u_{33} & 1 & 0 & 0 & 1 & 1 & v_{14}^* & v_{24}^* \\ u_{41} & u_{42} & u_{43} & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ w_{11} & w_{12} & w_{13} & v_{11} & v_{12} & v_{13} & v_{14} & 1 & 2 & 1 \\ w_{21} & w_{22} & w_{23} & v_{21} & v_{22} & v_{23} & v_{24} & 1 & 1 & 3 \end{bmatrix}. \tag{2.80}$$

Now, we have

$$\overline{V}_1 = \{1,2,3,4\}, \quad \overline{V}_2 = \{4,5,6,7,8\}, \quad \text{and} \quad \overline{V}_3 = \{8,9,10\} \tag{2.81}$$

and

$$\text{rank}\{\overline{\mathbf{X}}_1^{\text{ref}}\} = 3, \quad \text{rank}\{\overline{\mathbf{X}}_2^{\text{ref}}\} = 3, \quad \text{and} \quad \text{rank}\{\overline{\mathbf{X}}_3^{\text{ref}}\} = 3. \tag{2.82}$$

Therefore, the conditions of Theorem 4 hold for generic constant matrices $\mathbf{T}_{12}$ and $\mathbf{T}_{23}$. As a result, every solution $\overline{\mathbf{X}}$ of Optimization B has the property

$$\text{rank}\{\overline{\mathbf{X}}^{\text{opt}}\} = 3. \tag{2.83}$$

As a final step, the deletion of those rows and columns of $\overline{\mathbf{X}}^{\text{opt}}$ with indices 1, 5 and 6 yields a completion of $\mathbf{X}^{\text{ref}}$ with rank 3.

2.6 Low-Rank Solutions via Complex Analysis

Consider the problem of finding a low-rank solution $\mathbf{X}^{\text{opt}}$ for the LMI problem (2.24). Theorem 2 can be used for this purpose, but it needs solving one of the following graph problems: (i) designing a supergraph $\mathcal{G}'$ minimizing the upper bound given in (2.25), or (ii) obtaining a tree decomposition of $\mathcal{G}$ with the minimum width. Although these graph problems are easy to solve for highly sparse and structured graphs, they are NP-hard for arbitrary graphs. A question arises as to whether a low-rank solution can be obtained using a polynomial-time algorithm without requiring an expensive graph analysis. This problem will be addressed in this section.

Definition 11. *Given a complex number z, define*

$$z^{\text{ray}} \triangleq \{\lambda z \,|\, \lambda \in \mathbb{R},\, \lambda \geq 0\}. \tag{2.84}$$

Definition 12. *A finite set $U \subset \mathbb{C}$ is called sign-definite in $\mathbb{C}$ if U and $-U$ can be separated in the complex plane by a line passing through the origin, where $-U \triangleq \{-u \,|\, u \in U\}$ [Sojoudi and Lavaei, 2014]. Moreover, a finite set $U \subset \mathbb{R}$ is called sign-definite in $\mathbb{R}$ if its members are all nonnegative or all nonpositive.*

Optimization C: *Let $\mathcal{G}$ be a simple graph with n vertices and $\mathbb{F}$ be equal to either $\mathbb{R}$ or $\mathbb{C}$. Consider arbitrary matrices $\mathbf{X}^{\text{ref}} \in \mathbb{F}_n^+$ and $\mathbf{T} \in \mathbb{F}_n$ such that $\mathscr{G}(\mathbf{T})$ is a supergraph of $\mathcal{G}$. The problem*

$$\begin{aligned}
&\underset{\mathbf{X}\in\mathbb{F}_n}{\text{minimize}} && \langle \mathbf{T}, \mathbf{X} \rangle && && \text{(2.85a)}\\
&\text{subject to} && X_{kk} = X_{kk}^{\text{ref}} && k \in \mathcal{V}_{\mathcal{G}}, && \text{(2.85b)}\\
& && X_{ij} - X_{ij}^{\text{ref}} \in T_{ij}^{\text{ray}} && (i,j) \in \mathcal{E}_{\mathcal{G}}, && \text{(2.85c)}\\
& && \mathbf{X} \succeq 0, && && \text{(2.85d)}
\end{aligned}$$

*is referred to as "*Optimization C with the input $(\mathcal{G}, \mathbf{X}^{\text{ref}}, \mathbf{T}, \mathbb{F})$*".*

Lemma 2. *Assume that $\mathbf{X}^{\text{ref}}$ is positive definite. Every solution $\mathbf{X}^{\text{opt}}$ of Optimization C with the input $(\mathcal{G}, \mathbf{X}^{\text{ref}}, \mathbf{T}, \mathbb{H})$ satisfies the inequality*

$$\text{rank}\{\mathbf{X}^{\text{opt}}\} \leq n - \text{msr}_{\mathbb{H}}(\mathscr{G}(\mathbf{T})) \tag{2.86}$$

Proof. Constraints (2.85b) and (2.85c) imply that for any feasible matrix $\mathbf{X}$, the matrix $\mathbf{X} - \mathbf{X}^{\text{ref}}$ belongs to the convex cone

$$C = \left\{ \mathbf{W} \in \mathbb{F}_n | W_{kk} = 0 \quad \text{for} \quad k \in \mathcal{V}_{\mathcal{G}}, \quad W_{ij} \in T_{ij}^{\text{ray}} \quad \text{for} \quad (i,j) \in \mathcal{E}_{\mathcal{G}} \right\}. \tag{2.87}$$

Hence, the dual matrix variable $\mathbf{\Lambda}$ is a member of the dual cone

$$C^{\perp} = \left\{ \mathbf{W} \in \mathbb{F}_n | \text{Re}\{W_{ij} T_{ij}^*\} \geq 0 \text{ for } (i,j) \in \mathcal{E}_{\mathcal{G}},\ W_{ij} = 0 \text{ for } (i,j) \notin \mathcal{E}_{\mathcal{G}} \text{ and i} \neq \text{j} \right\}.$$

Therefore, the Lagrangian is equal to

$$\begin{aligned}
\mathcal{L}(\mathbf{X}, \mathbf{\Lambda}, \mathbf{\Phi}) &= \text{trace}\{\mathbf{T}\mathbf{X}\} + \text{trace}\{\mathbf{\Lambda}(\mathbf{X} - \mathbf{X}^{\text{ref}})\} - \text{trace}\{\mathbf{\Phi}\mathbf{X}\} \\
&= \text{trace}\{(\mathbf{\Lambda} + \mathbf{T} - \mathbf{\Phi})\mathbf{X}\} - \text{trace}\{\mathbf{\Lambda}\mathbf{X}^{\text{ref}}\},
\end{aligned} \tag{2.88}$$

where $\mathbf{\Phi} \succeq 0$ denotes the matrix dual variable corresponding to the constraint $\mathbf{X} \succeq 0$. The infimum of the Lagrangian over $\mathbf{X}$ is $-\infty$ unless $\mathbf{\Phi} = \mathbf{\Lambda} + \mathbf{T}$. Therefore, the dual problem is as follows:

$$\begin{aligned}
&\underset{\mathbf{\Lambda}\in\mathbb{H}_n}{\text{maximize}} && -\langle \mathbf{\Lambda}, \mathbf{X}^{\text{ref}} \rangle && && \text{(2.89a)}\\
&\text{subject to} && \text{Re}\{\Lambda_{ij} T_{ij}^*\} \geq 0 && (i,j) \in \mathcal{E}_{\mathcal{G}} && \text{(2.89b)}\\
& && \Lambda_{ij} = 0, && (i,j) \notin \mathcal{E}_{\mathcal{G}} \text{ and i} \neq \text{j} && \text{(2.89c)}\\
& && \mathbf{\Lambda} + \mathbf{T} \succeq 0. && && \text{(2.89d)}
\end{aligned}$$

By pushing the diagonal entries of $\mathbf{\Lambda}$ toward infinity, the inequality $\mathbf{\Lambda} + \mathbf{T} \succeq 0$ will become strict. Hence, strong duality holds according to the Slater's condition. Let $\mathbf{\Phi} = \mathbf{\Phi}^{\text{opt}}$ denote an arbitrary dual solution. The complementary slackness condition $\langle \mathbf{\Phi}^{\text{opt}}, \mathbf{X}^{\text{opt}} \rangle = 0$ yields that

$$\text{rank}\{\mathbf{\Phi}^{\text{opt}}\} + \text{rank}\{\mathbf{X}^{\text{opt}}\} \leq n. \tag{2.90}$$

On the other hand, it can be deduced from the equation $\mathbf{\Phi} = \mathbf{\Lambda}+\mathbf{T}$ together with (2.89b) and (2.89c) that

$$\mathscr{G}(\mathbf{T}) = \mathscr{G}(\mathbf{\Phi}^{\text{opt}}) \tag{2.91}$$

Now, combining (2.90) and (2.91) completes the proof. □

Theorem 5. *Assume that* $\mathbf{M}_1, \ldots, \mathbf{M}_p$ *are arbitrary matrices in* $\mathbb{S}_n$*. Suppose that* $a_1, \ldots, a_p$ *are real numbers such that the LMI problem*

$$\begin{aligned}
&\text{find} && \mathbf{X} \in \mathbb{S}_n && && \\
&\text{subject to} && \langle \mathbf{M}_k, \mathbf{X} \rangle \leq a_k && k = 1, \ldots, p, && \text{(2.92a)}\\
& && \mathbf{X} \succeq 0 && && \text{(2.92b)}
\end{aligned}$$

has a positive-definite feasible solution $\mathbf{X}^{\text{ref}} \in \mathbb{S}_n$*. Let* $\mathbf{T} \in \mathbb{H}_n$ *be an arbitrary matrix such that* $\text{Re}\{\mathbf{T}\} = 0_{n\times n}$ *and* $\mathscr{G}(\mathbf{T})$ *is a supergraph of* $\mathscr{G}(\mathbf{M}_1) \cup \cdots \cup \mathscr{G}(\mathbf{M}_p)$*.*

a) Every solution $\mathbf{X}^{\text{opt}} \in \mathbb{H}_n$ *of Optimization C with the input* $(\mathcal{G}, \mathbf{X}^{\text{ref}}, \mathbf{T}, \mathbb{H})$ *is a solution of the LMI problem* (2.92) *and satisfies the relation*

$$\text{rank}\{\mathbf{X}^{\text{opt}}\} \leq n - \text{msr}_{\mathbb{H}}(\mathscr{G}(\mathbf{T})). \tag{2.93}$$

b) The matrix $\mathrm{Re}\{\mathbf{X}^{\mathrm{opt}}\}$ *is a real-valued solution of the LMI problem* (2.92) *and satisfies the inequality*

$$\mathrm{rank}\{\mathbf{X}^{\mathrm{real}}\} \leq \min\{2(n - \mathrm{msr}_{\mathbb{H}}(\mathscr{G}(\mathbf{T})), n\}. \tag{2.94}$$

Proof. For every feasible solution $\mathbf{X}$ of Optimization C, we have

$$\langle \mathbf{M}_k, \mathbf{X} \rangle = \langle \mathbf{M}_k, \mathbf{X}^{\mathrm{ref}} \rangle \quad \text{for} \quad k = 1, \ldots, p \tag{2.95}$$

Hence, every feasible solution of Optimization C is a solution of the LMI problem (2.92) as well. Now, the proof of Part (a) follows from Lemma 2. For Part (b), it is straightforward to verify that $\mathbf{X}^{\mathrm{real}}$ defined as $\frac{1}{2}\left(\mathbf{X}^{\mathrm{opt}} + (\mathbf{X}^{\mathrm{opt}})^{\mathrm{T}}\right)$ is a feasible solution of (2.92). Moreover,

$$\mathrm{rank}\{\mathbf{X}^{\mathrm{real}}\} \leq \mathrm{rank}\{\mathbf{X}^{\mathrm{opt}}\} + \mathrm{rank}\{(\mathbf{X}^{\mathrm{opt}})^{T}\} = 2\,\mathrm{rank}\{\mathbf{X}^{\mathrm{opt}}\} \tag{2.96}$$

The proof follows from the above inequality and Part (a). □

Corollary 5. *The inequalities* (2.93) *and* (2.94)*, lead to the following upper bounds on* $\mathrm{rank}\{\mathbf{X}^{\mathrm{opt}}\}$ *and* $\mathrm{rank}\{\mathbf{X}^{\mathrm{real}}\}$*, in terms of positive semidefinite zero forcing number:*

$$\mathrm{rank}\{\mathbf{X}^{\mathrm{opt}}\} \leq \mathrm{Z}^{+}(\mathscr{G}(\mathbf{T})), \tag{2.97}$$

$$\mathrm{rank}\{\mathbf{X}^{\mathrm{real}}\} \leq \min\{2\ \mathrm{Z}^{+}(\mathscr{G}(\mathbf{T})), n\}. \tag{2.98}$$

Consider an LMI problem with real-valued coefficients. Theorem 5 states that the complex-valued Optimization C can be exploited to find a real solution of the LMI problem under study with a guaranteed bound on its rank. This bound might be looser than the ones derived in Theorem 2, but is still small for very sparse graphs. Note that although the calculation of the bound given in (2.94) is an NP-hard problem, Optimization C is polynomial-time solvable without requiring any expensive graph preprocessing. In what follows, we improve the bound obtained in Theorem 5 for a *structured LMI problem.*

Lemma 3. *Let* $U = \{u_1, \ldots, u_n\} \subset \mathbb{F}$ *be sign-definite in* $\mathbb{F}$*. Then, the set*

$$\angle U \triangleq \{x \in \mathbb{F} \,|\, \mathrm{Re}\{u_k x\} \leq 0 \text{ for } k = 1, \ldots, n\} \tag{2.99}$$

forms a non-trivial convex cone in $\mathbb{F}$*.*

Proof. In the case $\mathbb{F} = \mathbb{R}$, the set U is either the ray of nonnegative real numbers or non-positive real numbers. Hence, U is a non-trivial convex cone if $\mathbb{F} = \mathbb{R}$. Consider now the case $\mathbb{F} = \mathbb{C}$. The convexity of $\angle U$ results from the fact that this set is described by linear inequalities. $\angle U$ is also a cone because $\lambda x \in \angle U$ for every $x \in \angle U$ and $\lambda \geq 0$. On the other hand, by the definition of a sign-definite set, there exists a line passing through the origin that separates the sets $\{u_1, \ldots, u_n\}$ and $\{-u_1, \ldots, -u_n\}$. Assume that this line makes the angle α with the real axis. Then, one of the two points $\exp\left[\left(\frac{\pi}{2} + \alpha\right)\mathbf{i}\right]$ and $\exp\left[\left(-\frac{\pi}{2} + \alpha\right)\mathbf{i}\right]$ belongs to $\angle U$. As a result, $\angle U$ is non-trivial. $\square$

By leveraging the result of Lemma 3, the bound proposed in Theorem 5 will be improved for a sign-definite LMI problem below.

Theorem 6. *Assume that $\mathbf{M}_1, \ldots, \mathbf{M}_p$ belong to the set $\mathbb{F}_n$ that is equal to either $\mathbb{S}_n$ or $\mathbb{H}_n$. Let $a_1, \ldots, a_m$ be real numbers such that the LMI problem*

$$
\begin{aligned}
&\text{find} && \mathbf{X} \in \mathbb{F}_n && \\
&\text{subject to} && \langle \mathbf{M}_k, \mathbf{X} \rangle \leq a_k && k = 1, \ldots, p, && \text{(2.100a)}\\
& && \mathbf{X} \succeq 0, && && \text{(2.100b)}
\end{aligned}
$$

has a positive-definite feasible solution $\mathbf{X}^{\mathrm{ref}} \in \mathbb{F}_n^+$. Let $\mathcal{G} = \mathscr{G}(\mathbf{M}_1) \cup \cdots \cup \mathscr{G}(\mathbf{M}_p)$ and suppose that the set $\mathscr{M}_{ij}$ composed of the (i,j) entries of $M_1, \ldots, M_p$ is sign-definite for every pair $(i,j) \in \mathcal{E}_{\mathcal{G}}$. Consider a matrix $\mathbf{T} \in \mathbb{F}_n$ such that $\mathscr{G}(\mathbf{T})$ is a supergraph of $\mathcal{G}$ and that $T_{ij} \in \angle \mathcal{M}_{ij}$ for every $(i,j) \in \mathcal{E}_{\mathcal{G}}$. Then, every solution $\mathbf{X}^{\mathrm{opt}} \in \mathbb{F}_n$ of Optimization C with the input $(\mathcal{G}, \mathbf{X}^{\mathrm{ref}}, \mathbf{T}, \mathbb{F})$ is a solution of the LMI problem (2.100) *and satisfies the inequality*

$$\operatorname{rank}\{\mathbf{X}^{\mathrm{opt}}\} \leq n - \operatorname{msr}_{\mathbb{F}}(\mathscr{G}(\mathbf{T})). \tag{2.101}$$

Proof. According to Lemma 3, a matrix $\mathbf{T}$ with the properties mentioned in the theorem always exists. We have $X_{ij}^{\mathrm{opt}} - X_{ij}^{\mathrm{ref}} \in T_{ij}^{\mathrm{ray}} \subseteq \angle \mathcal{M}_{ij}$ for every $(i,j) \in \mathcal{E}_{\mathcal{G}}$. Hence, for $k = 1, \ldots, p$, one can write

$$\operatorname{Re}\{M_k(i,j)(X_{ij}^{\mathrm{opt}} - X_{ij}^{\mathrm{ref}})\} \leq 0 \tag{2.102}$$

or equivalently

$$\langle \mathbf{M}_k, \mathbf{X}^{\mathrm{opt}} \rangle \leq \langle \mathbf{M}_k, \mathbf{X}^{\mathrm{ref}} \rangle \tag{2.103}$$

($M_k(i,j)$ denotes the (i,j) entry of M_k). Consequently, $\mathbf{X}^{\text{opt}}$ is a feasible solution of the LMI problem (2.100) and satisfies the inequality (2.101) in light of Lemma 2. □

Corollary 6. *The inequality* (2.101)*, lead to the following upper bound on* $\text{rank}\{\mathbf{X}^{\text{opt}}\}$*, in terms of positive semidefinite zero forcing number:*

$$\text{rank}\{\mathbf{X}^{\text{opt}}\} \leq \mathrm{Z}^{+}(\mathscr{G}(\mathbf{T})). \tag{2.104}$$

Theorem 6 improves upon the results of Theorem 5 for structured LMI problems in two directions: (i) extension to the complex case, and (ii) reduction of the upper bound by a factor of 2 in the real case.

2.7 Low-Rank Solutions for Affine Problems

In this section, we will generalize the results derived earlier to the affine rank minimization problem.

Definition 13. *For an arbitrary matrix* $\mathbf{W} \in \mathbb{C}^{m\times r}$*, the notation* $\mathscr{B}(\mathbf{W}) = (\mathcal{V}_{\mathcal{B}}, \mathcal{E}_{\mathcal{B}})$ *denotes a bipartite graph defined as:*

1. $\mathcal{V}_{\mathcal{B}}$ *is the union of the first vertex set* $\mathcal{V}_{\mathcal{B}_1} = \{1, \ldots, n\}$ *and the second set vertex set* $\mathcal{V}_{\mathcal{B}_2} = \{1, \ldots, m\}$*, associated with the two parts of the graph.*

2. *For every* $(i,j) \in \mathcal{V}_{\mathcal{B}_1} \times \mathcal{V}_{\mathcal{B}_2}$*, we have* $(i,j) \in \mathcal{E}_{\mathcal{B}}$ *if and only if* $W_{ij} \neq 0$.

Definition 14. *Consider an arbitrary matrix* $\mathbf{X} \in \mathbb{H}_n$ *and two natural numbers* m *and* r *such that* $n \geq m + r$*. The matrix* $\text{sub}_{m,r}\{\mathbf{X}\}$ *is defined as the* $m \times r$ *submatrix of* $\mathbf{X}$ *corresponding to the first* m *rows and the last* r *columns of the* $(m+r)$*-th leading principal submatrix of* $\mathbf{X}$.

Theorem 7. *Consider the feasibility problem*

$$\begin{aligned} &\text{find} && \mathbf{W} \in \mathbb{R}^{m\times r} \\ &\text{subject to} && \langle \mathbf{N}_k, \mathbf{W}\rangle \leq a_k \qquad k = 1, \ldots, p, \end{aligned} \tag{2.105a}$$

where $a_1, \ldots, a_p \in \mathbb{R}$ *and* $\mathbf{N}_1, \ldots, \mathbf{N}_p \in \mathbb{R}^{r\times m}$*. Let* $\mathbf{W}^{\text{ref}} \in \mathbb{R}^{m\times r}$ *denote a feasible solution of this feasibility problem and* $\mathbf{X}^{\text{ref}} \in \mathbb{S}^{+}_{r+m}$ *be a matrix such that* $\text{sub}_{r,m}\{\mathbf{X}^{\text{ref}}\} = \mathbf{W}^{\text{ref}}$*. Define* $\mathcal{G} = \mathscr{B}(\mathbf{N}_1^T) \cup \cdots \cup \mathscr{B}(\mathbf{N}_p^T)$*. The following statements hold:*

a) *Consider an arbitrary supergraph* $\mathcal{G}'$ *of* $\mathcal{G}$ *with* n *vertices, where* $n \geq r + m$*. Let* $\mathbf{X}^{\text{opt}}$ *denote an arbitrary solution of Optimization A with the input* $(\mathcal{G}, \mathcal{G}', \mathbf{T}, \mathbf{X}^{\text{ref}})$*. Then,* $\mathbf{W}^{\text{opt}}$ *defined as* $\text{sub}_{m,r}\{\mathbf{X}^{\text{opt}}\}$ *is a solution of the feasibility problem* (2.105a) *and satisfies the relation*

$$\text{rank}\{\mathbf{W}^{\text{opt}}\} \leq |\mathcal{G}'| - \min\left\{\text{msr}_{\mathbb{S}}(\mathcal{G}_s)\,\middle|\,(\mathcal{G}' \setminus \mathcal{G}) \subseteq \mathcal{G}_s \subseteq \mathcal{G}'\right\} \tag{2.106}$$

b) *Consider an arbitrary tree decomposition* $\mathcal{T}$ *of* $\mathcal{G}$ *with width* t*. If* $\mathcal{G}'$ *in Part (a) is considered as an enriched supergraph of* $\mathcal{G}$ *derived by* $\mathcal{T}$*, then*

$$\text{rank}\{\mathbf{W}^{\text{opt}}\} \leq t + 1 \tag{2.107}$$

c) *Let* $\mathbf{X}^{\text{opt}}$ *denote an arbitrary solution of Optimization C with the input* $(\mathcal{G}, \mathbf{X}^{\text{ref}}, \mathbf{T}, \mathbb{H})$*. Then,* $\mathbf{W}^{\text{opt}}$ *defined as* $\text{sub}_{m,r}\{\text{Re}\{\mathbf{X}^{\text{opt}}\}\}$ *is a solution of the feasibility problem* (2.105a) *and satisfies the relation*

$$\text{rank}\{\mathbf{W}^{\text{real}}\} \leq \min\{2(r + m - \text{msr}_{\mathbb{H}}(\mathscr{G}(\mathbf{T})), r, m\}. \tag{2.108}$$

Proof. The proof follows directly from Theorems 2 and 5, the conversion technique delineated in Subsection 2.1.3, and the inequality

$$\text{rank}\{\text{sub}_{m,r}\{\mathbf{X}\}\} \leq \text{rank}\{\mathbf{X}\} \tag{2.109}$$

for every $\mathbf{X} \in \mathbb{S}_n$. □

Corollary 7. *The inequalities* (2.106) *and* (2.108)*, lead to the following upper bounds on* $\text{rank}\{\mathbf{W}^{\text{opt}}\}$ *and* $\text{rank}\{\mathbf{W}^{\text{real}}\}$*, in terms of positive semidefinite zero forcing number:*

$$\text{rank}\{\mathbf{W}^{\text{opt}}\} \leq \max\left\{\text{Z}^{+}(\mathcal{G}_s)\,\middle|\,(\mathcal{G}' \setminus \mathcal{G}) \subseteq \mathcal{G}_s \subseteq \mathcal{G}'\right\} \tag{2.110a}$$

$$\text{rank}\{\mathbf{W}^{\text{real}}\} \leq \min\{2\text{Z}^{+}(\mathscr{G}(\mathbf{T})), r, m\}. \tag{2.110b}$$

The following corollary is an immediate consequence of Theorem 7.

Corollary 8. *If the feasibility problem* (2.105a) *has a non-empty feasible set, then it has a solution* $\mathbf{W}^{\text{opt}}$ *with rank at most* $\text{tw}\big(\mathscr{B}(\mathbf{N}_1^T) \cup \cdots \cup \mathscr{B}(\mathbf{N}_p^T)\big) + 1$.

As discussed in Subsection 2.1.3, the nuclear norm method is a popular technique for the minimum-rank matrix completion problem. In what follows, we adapt Theorem 7 to improve upon the nuclear norm method by incorporating a weighted sum into this norm and then obtain a guaranteed bound on the rank of every solution of the underlying convex optimization.

Theorem 8. *Suppose that $\mathcal{B}$ is a bipartite graph with $|\mathcal{V}_{\mathcal{B}_1}| = m$ and $|\mathcal{V}_{\mathcal{B}_2}| = r$. Given arbitrary matrices $\mathbf{W}^{\mathrm{ref}}$ and $\mathbf{Q}$ in $\mathbb{R}^{m\times r}$, consider the convex program*

$$\begin{aligned} &\underset{\mathbf{W}\in\mathbb{R}^{m\times r}}{\text{minimize}} && \|\mathbf{W}\|_* + \langle \mathbf{Q}, \mathbf{W}\rangle && && (2.111\text{a})\\ &\text{subject to} && W_{ij} = W_{ij}^{\mathrm{ref}} && (i,j)\in\mathcal{E}_{\mathcal{B}}. && (2.111\text{b}) \end{aligned}$$

Let $\mathcal{B}'$ be defined as the supergraph $\mathcal{B}\cup\mathscr{B}(\mathbf{Q})$. Then, every solution $\mathbf{W}^{\mathrm{opt}}$ of the optimization (2.111) *satisfies the inequality*

$$\operatorname{rank}\{\mathbf{W}^{\mathrm{opt}}\} \leq m + r - \min\left\{\operatorname{msr}_{\mathbb{S}}(\mathcal{B}_s)\,\middle|\,(\mathcal{B}'\setminus\mathcal{B}) \subseteq \mathcal{B}_s \subseteq \mathcal{B}'\right\}. \tag{2.112}$$

Proof. Consider an arbitrary matrix $\mathbf{W}\in\mathbb{R}^{m\times r}$. It has been shown in [Fazel, 2002] that the nuclear norm of $\mathbf{W}$ is equal to the optimal objective value of the optimization

$$\begin{aligned} &\underset{\substack{\mathbf{X}_1\in\mathbb{R}^{m\times m}\\ \mathbf{X}_2\in\mathbb{R}^{r\times r}}}{\text{minimize}} && \frac{1}{2}\operatorname{trace}\{\mathbf{X}_1\} + \frac{1}{2}\operatorname{trace}\{\mathbf{X}_2\} && (2.113\text{a})\\ &\text{subject to} && \begin{bmatrix} \mathbf{X}_1 & \mathbf{W}\\ \mathbf{W}^T & \mathbf{X}_2 \end{bmatrix} \succeq 0. && (2.113\text{b}) \end{aligned}$$

This implies that Optimization (2.111) is equivalent to

$$\begin{aligned} &\underset{\substack{\mathbf{X}_1\in\mathbb{R}^{m\times m}\\ \mathbf{X}_2\in\mathbb{R}^{r\times r}\\ \mathbf{W}\in\mathbb{R}^{m\times r}}}{\text{minimize}} && \frac{1}{2}\operatorname{trace}\{\mathbf{X}_1\} + \frac{1}{2}\operatorname{trace}\{\mathbf{X}_2\} + \langle \mathbf{Q},\mathbf{W}\rangle && && (2.114\text{a})\\ &\text{subject to} && \begin{bmatrix} \mathbf{X}_1 & \mathbf{W}\\ \mathbf{W}^T & \mathbf{X}_2 \end{bmatrix} \succeq 0, && && (2.114\text{b})\\ & && W_{ij} = W_{ij}^{\mathrm{ref}} && (i,j)\in\mathcal{E}_{\mathcal{B}} && (2.114\text{c}) \end{aligned}$$

The proof follows from applying Part (a) of Theorem 7 to the above optimization. □

Corollary 9. *The inequality* (2.112)*, lead to the following upper bound on* $\text{rank}\{\mathbf{W}^{\text{opt}}\}$*, in terms of positive semidefinite zero forcing number:*

$$\text{rank}\{\mathbf{W}^{\text{opt}}\} \leq \max\left\{\mathrm{Z}^{+}(\mathcal{B}_s)\,\middle|\,(\mathcal{B}' \setminus \mathcal{B}) \subseteq \mathcal{B}_s \subseteq \mathcal{B}'\right\}. \tag{2.115}$$

The nuclear norm method reviewed in Subsection 2.1.3 corresponds to the case $\mathbf{Q} = 0$ in Theorem 8. However, this theorem discloses the role of the weight matrix $\mathbf{Q}$. In particular, this matrix can be designed based on the results developed in Section 2.3 to yield a small number for the upper bound given in (2.112), provided $\mathcal{B}$ is a sparse graph.

2.8 Summary

This chapter aims to find low-rank solutions of sparse linear matrix inequality (LMI) problems using convex optimization and graph theory. To this end, the sparsity of a given LMI problem is mapped into a graph and a rigorous theory is developed to connect the rank of the minimum-rank solution of the LMI problem to the sparsity of this graph. Moreover, three graph-theoretic convex programs are proposed to find low-rank solutions of the underlying LMI problem with the property that the rank of *every* solution of these problems has a guaranteed upper bound. Two of these convex optimization problems may need heavy graph computation, whereas the third convex program does not rely on any computationally-expensive graph analysis and is always polynomial-time solvable. The implications of this work are also discussed for three applications: minimum-rank matrix completion, conic relaxation for polynomial optimization, and affine rank minimization. The results are applied to two case studies in the next chapters for electrical power networks and dynamical systems.

Part II

Power Networks

Chapter 3

Convex Relaxation for Optimal Power Flow Problem: Mesh Networks

This chapter is concerned with the optimal power flow (OPF) problem. It has been shown in [Lavaei and Low, 2012] that a convex relaxation based on semidefinite programming (SDP) is able to find a global solution of OPF for IEEE benchmark systems, and moreover this technique is guaranteed to work over acyclic (distribution) networks. The present work studies the potential of the SDP relaxation for OPF over mesh (transmission) networks. First, we consider a simple class of cyclic systems, namely weakly-cyclic networks with cycles of size 3. We show that the success of the SDP relaxation depends on how the line capacities are modeled mathematically. More precisely, the SDP relaxation is proven to succeed if the capacity of each line is modeled in terms of bus voltage difference, as opposed to line active power, apparent power or angle difference. This result elucidates the role of the problem formulation. Our second contribution is to relate the rank of the minimum-rank solution of the SDP relaxation to the network topology. The goal is to understand how the computational complexity of OPF is related to the underlying topology of the power network. To this end, an upper bound is derived on the rank of the SDP solution, which is expected to be small in practice. A penalization method is then applied to the SDP relaxation to enforce the rank of its solution to become 1, leading to a near-optimal solution for OPF with a guaranteed optimality degree. The remarkable performance of this technique is demonstrated on IEEE systems with more than 7000 different cost functions.

3.1 Introduction

The optimal power flow (OPF) problem aims to find an optimal operating point of a power system, which minimizes a certain objective function (e.g., power loss or generation cost) subject to network and physical constraints [Momoh *et al.*, 1999]. Due to the nonlinear interrelation among active power, reactive power and voltage magnitude, OPF is described by nonlinear equations and may have a nonconvex/disconnected feasibility region. Since 1962, the nonlinearity of the OPF problem has been studied, and various heuristic and local-search algorithms have been proposed [Baldick, 2006; Pandya and Joshi, 2008].

The paper [Lavaei and Low, 2012] proposes two methods for solving OPF: (i) to use a convex relaxation based on semidefinite programming (SDP), (ii) to solve the SDP-type Lagrangian dual of OPF. That work shows that the SDP relaxation is exact if and only if the duality gap is zero. More importantly, [Lavaei and Low, 2012] makes the observation that OPF has a zero duality gap for IEEE benchmark systems with 14, 30, 57, 118 and 300 buses, in addition to several randomly generated power networks. This technique is the first method proposed since the introduction of the OPF problem that is able to find a provably global solution for practical OPF problems. The SDP relaxation for OPF has attracted much attention due to its ability to find a global solution in polynomial time, and it has been applied to various applications in power systems including: voltage regulation in distribution systems [Lam *et al.*, 2012a], state estimation [Weng *et al.*, 2012], calculation of voltage stability margin [Molzahn *et al.*, 2012], economic dispatch in unbalanced distribution networks [Dall'Anese *et al.*, 2013], charging of electric vehicles [Sojoudi and Low, 2011], and power management under time-varying conditions [Ghosh *et al.*, 2011].

The paper [Sojoudi and Lavaei, 2012] shows that the SDP relaxation is exact in two cases: (i) for acyclic networks, (ii) for cyclic networks after relaxing the angle constraints (similar result was derived in [Zhang and Tse, 2011] and [Bose *et al.*, 2011] for acyclic networks). This exactness was related to the passivity of transmission lines and transformers. A question arises as to whether the SDP relaxation remains exact for mesh (cyclic) networks without any angle relaxations. To address this problem, the paper [Lesieutre *et al.*, 2011] shows that the relaxation is not always exact for a three-bus cyclic network. More examples can be found in the recent paper [Bukhsh *et al.*, 2013], where the existence of local solutions is studied for the OPF problem. To improve the performance of the above-mentioned convex relaxation, the papers [Gopalakrishnan *et al.*, 2011] and [Phan, 2012] suggest solving a sequence of SDP-type relaxations based on the branch and bound technique. However, it is highly desirable to develop an algorithm needing to solve only a few SDP relaxations in order to guarantee a polynomial-time run for the algorithm. The aim of this chapter is to investigate the possibility of finding a global or near-global solution of the OPF problem for mesh networks by solving only a few SDP relaxations.

In this work, we first consider the three-bus system studied in [Lesieutre *et al.*, 2011] and prove that the exactness of the SDP relaxation depends on the problem formulation. More precisely, we show that there are four (almost) equivalent ways to model the capacity of a power line but only one of these models always gives rise to the exactness of the SDP relaxation. We also prove that the relaxation remains exact for weakly-cyclic networks with cycles of size 3. Furthermore, we substantiate that this type of network has a convex injection region in the lossless case and a non-convex injection region with a convex Pareto front in the lossy case. The importance of this result is that the SDP relaxation works on certain cyclic networks, for example the ones generated from three-bus subgraphs (this type of network is related to three-phase systems).

In the case when the SDP relaxation does not work, an upper bound is provided on the rank of the minimum-rank solution of the SDP relaxation. This bound is related only to the structure of the power network and this number is expected to be very small for real-world power networks. Finally, a heuristic method is proposed to enforce the SDP relaxation to produce a rank-1 solution for general networks (by somehow eliminating the undesirable eigenvalues of the low-rank solution). The efficacy of the proposed technique is elucidated by extensive simulations on IEEE systems as well as a difficult example proposed in [Bukhsh *et al.*, 2013] for which the OPF problem has at least three local solutions. Note that this chapter is concentrated on a basic OPF problem, but the results can be readily extended to a more sophisticated formulation of OPF with security constraints together with variable tap-changing transformers and capacitor banks. This can be carried out using the methodology delineated in [Lavaei, 2011].

Notations: $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{S}_n^+$ and $\mathbb{H}_n^+$ denote the sets of real numbers, positive real numbers, $n \times n$ positive semidefinite symmetric matrices, and $n \times n$ positive semidefinite Hermitian matrices, respectively. $\text{Re}\{\mathbf{W}\}$, $\text{Im}\{\mathbf{W}\}$, $\text{rank}\{\mathbf{W}\}$ and $\text{trace}\{\mathbf{W}\}$ denote the real part, imaginary part, rank and trace of a given scalar/matrix $\mathbf{W}$, respectively. The notation $\mathbf{W} \succeq 0$ means that $\mathbf{W}$ is Hermitian and positive semidefinite. The notation $\measuredangle x$ denotes the angle of a complex number x. The notation "i" is reserved for the imaginary unit. The symbol "*" represents the conjugate transpose operator. Given a matrix $\mathbf{W}$, its (l, m) entry is denoted as W_{lm}. The superscript $(\cdot)^{\text{opt}}$ is used to show the optimal value of an optimization parameter.

Definitions: Given a simple graph $\mathcal{H}$, its vertex and edge sets are denoted by $\mathcal{V}_{\mathcal{H}}$ and $\mathcal{E}_{\mathcal{H}}$, respectively. A "forest" is a simple graph that has no cycles and a "tree" is defined as a connected forest. A graph $\mathcal{H}'$ is said to be a subgraph of $\mathcal{H}$ if $\mathcal{V}_{\mathcal{H}'} \subseteq \mathcal{V}_{\mathcal{H}}$ and $\mathcal{E}_{\mathcal{H}'} \subseteq \mathcal{E}_{\mathcal{H}}$. A subgraph $\mathcal{H}'$ of $\mathcal{H}$ is said to be an induced subgraph if, for every pair of vertices $v_l, v_m \in \mathcal{V}_{\mathcal{H}'}$, $(v_l, v_m) \in \mathcal{E}_{\mathcal{H}'}$ if and only if $(v_l, v_m) \in \mathcal{E}_{\mathcal{H}}$. $\mathcal{H}'$ is said to be induced by the vertex subset $\mathcal{V}_{\mathcal{H}'}$.

3.2 Optimal Power Flow

Consider a power network with the set of buses $\mathcal{N} := \{1, 2, ..., n\}$, the set of generator buses $\mathcal{G} \subseteq \mathcal{N}$, and the set of flow lines $\mathcal{L} \subseteq \mathcal{N} \times \mathcal{N}$, where:

- A known constant-power load with the complex value $P_{D_k} + Q_{D_k}\mathrm{i}$ is connected to each bus $k \in \mathcal{N}$.
- A generator with an unknown complex output $P_{G_k} + Q_{G_k}\mathrm{i}$ is connected to each bus $k \in \mathcal{G}$.
- Each line $(l, m) \in \mathcal{L}$ of the network is modeled as a passive device with an admittance y_{lm} with possible resistance and reactance (the network can be modeled as a general admittance matrix).

We call the network lossless if $\text{Re}\{y_{lm}\} = 0$ for all $(l, m) \in \mathcal{L}$ and call it lossy otherwise. The goal is to design the unknown outputs of all generators in such a way that the load constraints are satisfied. To formulate this problem, named *optimal power flow (OPF)*, define:

- V_k: Unknown complex voltage at bus $k \in \mathcal{N}$.
- P_{lm}: Unknown active power transferred from bus $l \in \mathcal{N}$ to the rest of the network through the line $(l, m) \in \mathcal{L}$.
- S_{lm}: Unknown complex power transferred from bus $l \in \mathcal{N}$ to the rest of the network through the line $(l, m) \in \mathcal{L}$.

- $f_k(P_{G_k})$: Known increasing, convex function representing the generation cost for generator $k \in \mathcal{G}$.

Define $\mathbf{V}$, $\mathbf{P}_G$, $\mathbf{Q}_G$, $\mathbf{P}_D$ and $\mathbf{Q}_D$ as the vectors $\{V_k\}_{k\in\mathcal{N}}$, $\{P_{G_k}\}_{k\in\mathcal{G}}$, $\{Q_{G_k}\}_{k\in\mathcal{G}}$, $\{P_{D_k}\}_{k\in\mathcal{N}}$ and $\{Q_{D_k}\}_{k\in\mathcal{N}}$, respectively. Given the known vectors $\mathbf{P}_D$ and $\mathbf{Q}_D$, OPF minimizes the total generation cost $\sum_{k\in\mathcal{G}} f_k(P_{G_k})$ over the unknown parameters $\mathbf{V}$, $\mathbf{P}_G$ and $\mathbf{Q}_G$ subject to the power balance equations at all buses and some network constraints. To simplify the formulation of OPF, with no loss of generality assume that $\mathcal{G} = \mathcal{N}$. The mathematical formulation of OPF is given as follows

$$
\begin{aligned}
\underset{\substack{\mathbf{V}\in\mathbb{C}^n\\ \mathbf{Q}_G\in\mathbb{R}^n\\ \mathbf{P}_G\in\mathbb{R}^n}}{\text{minimize}} \quad & \sum_{k\in\mathcal{G}} f_k(P_{G_k}) && && (3.1\text{a})\\
\text{subject to} \quad & P_{G_k} - P_{D_k} = \sum_{l\in\mathcal{N}(k)} \mathrm{Re}\left\{V_k(V_k^* - V_l^*)y_{kl}^*\right\}, && k\in\mathcal{N} && (3.1\text{b})\\
& Q_{G_k} - Q_{D_k} = \sum_{l\in\mathcal{N}(k)} \mathrm{Im}\left\{V_k(V_k^* - V_l^*)y_{kl}^*\right\}, && k\in\mathcal{N} && (3.1\text{c})\\
& P_k^{\min} \leq P_k \leq P_k^{\max} && k\in\mathcal{N} && (3.1\text{d})\\
& Q_k^{\min} \leq Q_k \leq Q_k^{\max}, && k\in\mathcal{N} && (3.1\text{e})\\
& V_k^{\min} \leq |V_k| \leq V_k^{\max}, && k\in\mathcal{N} && (3.1\text{f})\\
& \text{A capacity constraint for each line,} && (l,m)\in\mathcal{L} && (3.1\text{g})
\end{aligned}
$$

where:

- (3.1b) and (3.1c) are the power balance equations accounting for the conservation of energy at bus k.
- (3.1d), (3.1e) and (3.1f) restrict the active power, reactive power and voltage magnitude at bus k, for the given limits $P_k^{\min}, P_k^{\max}, Q_k^{\min}, Q_k^{\max}, V_k^{\min}, V_k^{\max}$.
- Each line of the network is subject to a capacity constraint to be introduced later.
- $\mathcal{N}(k)$ denotes the set of all neighboring nodes of bus $k \in \mathcal{N}$.

3.2.1 Convex Relaxation for Optimal Power Flow

Regardless of the unspecified capacity constraint, the above formulation of the OPF problem is non-convex due to the nonlinear terms $|V_k|$'s and $V_k V_l^*$'s. Since this problem is NP-hard in the worst case, the paper [Lavaei and Low, 2012] suggests solving a convex relaxation of OPF. To this end, notice that the constraints of OPF can all be expressed as linear functions of the entries of the quadratic matrix $\mathbf{VV}^*$. This implies that if the matrix $\mathbf{VV}^*$ is replaced by a new matrix variable $\mathbf{W} \in \mathbb{H}_n$, then the constraints of OPF become convex in $\mathbf{W}$. Since $\mathbf{W}$ plays the role of $\mathbf{VV}^*$, two constraints must be added to the reformulated OPF problem in order to preserve the equivalence of the two formulations: (i) $\mathbf{W} \succeq 0$, (ii) $\text{rank}\{\mathbf{W}\} = 1$. Observe that Constraint (ii) is the only non-convex constraint of the reformulated OPF problem. Motivated by this fact, the SDP relaxation of OPF is defined as the OPF problem reformulated in terms of $\mathbf{W}$ under the additional constraint $\mathbf{W} \succeq 0$, which is given as follows:

$$\underset{\substack{\mathbf{W}\in\mathbb{H}_n^+\\ \mathbf{Q}_G\in\mathbb{R}^n\\ \mathbf{P}_G\in\mathbb{R}^n}}{\text{minimize}} \quad \sum_{k\in\mathcal{G}} f_k(P_{G_k}) \tag{3.2a}$$

$$\text{subject to} \quad P_{G_k} - P_{D_k} = \sum_{l\in\mathcal{N}(k)} \text{Re}\left\{(W_{kk} - W_{kl})y_{kl}^*\right\}, \qquad k \in \mathcal{N} \tag{3.2b}$$

$$Q_{G_k} - Q_{D_k} = \sum_{l\in\mathcal{N}(k)} \text{Im}\left\{(W_{kk} - W_{kl})y_{kl}^*\right\}, \qquad k \in \mathcal{N} \tag{3.2c}$$

$$P_k^{\min} \leq P_k \leq P_k^{\max} \qquad k \in \mathcal{N} \tag{3.2d}$$

$$Q_k^{\min} \leq Q_k \leq Q_k^{\max}, \qquad k \in \mathcal{N} \tag{3.2e}$$

$$(V_k^{\min})^2 \leq W_{kk} \leq (V_k^{\max})^2, \qquad k \in \mathcal{N} \tag{3.2f}$$

$$\text{A convexified capacity constraint for each line,} \qquad (l,m) \in \mathcal{L} \tag{3.2g}$$

If the SDP relaxation gives rise to a rank-1 solution $\mathbf{W}^{\text{opt}}$, then it is said that *the relaxation is exact*. The exactness of the SDP relaxation is a desirable property being sought, because it implies the equivalence of the convex SDP relaxation and the non-convex OPF problem.

3.2.2 Four Types of Capacity Constraints

In this part, the line capacity constraint in the formulation of the OPF problem given in (3.1) will be specified. Line flows are restricted in practice to achieve various goals such as avoiding line overheating and guaranteeing the stability of the network. Notice that

i) A thermal limit can be imposed by restricting the line active power flow P_{lm}, the line apparent power flow $|S_{lm}|$, or the line current magnitude $|I_{lm}|$. The maximum allowable limits on these parameters can be determined by analyzing the material characteristics of the line.

ii) A stability limit may be translated into a constraint on the voltage phase difference across the line, i.e., $|\measuredangle V_l - \measuredangle V_m|$.

Hence, each line $(l, m) \in \mathcal{L}$ may be associated with one or multiple capacities constraints, each of which has its own power engineering implication. Four types of capacity constraints are provided in the following inequalities

$$|\theta_{lm}| = |\measuredangle V_l - \measuredangle V_m| \leq \theta_{lm}^{\max} \tag{3.3a}$$

$$|P_{lm}| = |\mathrm{Re}\{V_l(V_l^* - V_m^*)y_{lm}^*\}| \leq P_{lm}^{\max} \tag{3.3b}$$

$$|S_{lm}| = |V_l(V_l^* - V_m^*)y_{lm}^*| \leq S_{lm}^{\max} \tag{3.3c}$$

$$|V_l - V_m| \leq \Delta V_{lm}^{\max} \tag{3.3d}$$

for the given upper bounds $\theta_{lm}^{\max} = \theta_{ml}^{\max}$, $P_{lm}^{\max} = P_{ml}^{\max}$, $S_{lm}^{\max} = S_{ml}^{\max}$ and $\Delta V_{lm}^{\max} = \Delta V_{ml}^{\max}$, where θ_{lm} denotes the angle difference $\measuredangle V_l - \measuredangle V_m$. Note that the constraint (3.3d) is equivalent to the line current limitation constraint in the context of this work, because each line has been modeled as a simple admittance and therefore $V_l - V_m$ is proportional to the line current. Henceforth, we assume that $\theta_{lm}^{\max}$ is less than 90° due to the current practice in power networks. This can be assured by adding the constraint $\mathrm{Re}\{W_{lm}\} > 0$ to the SDP relaxation, if necessary.

The capacity constraints given in (3.3) can all be cast as convex inequalities in $\mathbf{W}$, leading to the reformulated constraints as follows:

$$\mathrm{Im}\{W_{lm}\} \leq \mathrm{Re}\{W_{lm}\} \tan(\theta_{lm}^{\max}) \tag{3.4a}$$

$$\mathrm{Re}\{(W_{ll} - W_{lm})y_{lm}^*\} \leq P_{lm}^{\max} \tag{3.4b}$$

$$|(W_{ll} - W_{lm})y_{lm}^*| \leq S_{lm}^{\max} \tag{3.4c}$$

$$W_{ll} + W_{mm} - W_{lm} - W_{ml} \leq (\Delta V_{lm}^{\max})^2 . \tag{3.4d}$$

To understand how the reformulation from $\mathbf{V}$ to $\mathbf{W}$ is carried out, consider the constraint (3.3a). This constraint is equivalent to $|\measuredangle(V_l V_m^*)| \leq \theta_{lm}^{\max}$ or

$$\left|\frac{\mathrm{Im}\{V_l V_m^*\}}{\mathrm{Re}\{V_l V_m^*\}}\right| \leq \tan(\theta_{lm}^{\max}) \tag{3.5}$$

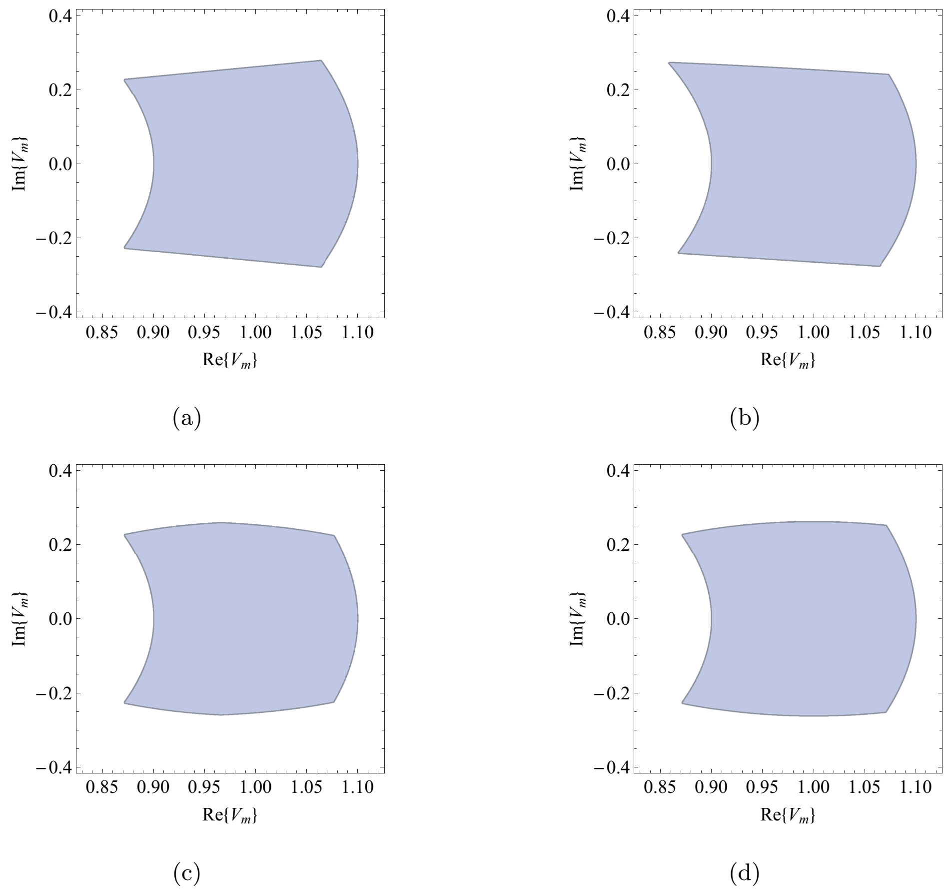


Figure 3.1: Four feasible regions for voltage phasor V_m (in p.u.) associated with the constraints in (3.3) in the case where V_l is fixed at $1\angle 0°$(p.u.) and $0.9 \leq |V_m| \leq 1.1$: (a) region for the line constraint (3.3a); (b) region for the line constraint (3.3b); (c) region for the line constraint (3.3c), and (d) region for the line constraint (3.3d).

Since $\theta_{lm}^{\max}$ is less than $90°$ by assumption, the above inequality can be rewritten as

$$|\mathrm{Im}\{V_l V_m^*\}| \leq \mathrm{Re}\{V_l V_m^*\} \tan(\theta_{lm}^{\max}) \tag{3.6}$$

The convex constraint (3.4a) is obtained from the above inequality by replacing $V_l V_m^*$ with W_{lm} and dropping the absolute value operator from the left side. Note that the absolute value is not important because the two constraints $|\theta_{lm}|, |\theta_{ml}| \leq \theta_{lm}^{\max}$ are equivalent to $\theta_{lm} \leq \theta_{lm}^{\max}$ and $\theta_{ml} \leq \theta_{lm}^{\max}$ all together (recall that $\theta_{lm}^{\max} = \theta_{ml}^{\max}$).

Theorem 9. *Let $\alpha \in [0, \pi/2)$ denote an arbitrary angle. Suppose that all voltage magnitudes are* *fixed at the nominal value of 1 per unit. Then, the capacity constraints in* (3.3) *are all mathematically equivalent and interchangeable through the upper bounds:*

$$\theta_{lm}^{\max}(\alpha) \triangleq \alpha \tag{3.7a}$$

$$P_{lm}^{\max}(\alpha) \triangleq \mathrm{Re}\{(1-e^{\alpha i})y_{lm}^*\} \tag{3.7b}$$

$$S_{lm}^{\max}(\alpha) \triangleq |(1-e^{\alpha i})y_{lm}^*| \tag{3.7c}$$

$$\Delta V_{lm}^{\max}(\alpha) \triangleq \sqrt{2\,(1-\cos(\alpha))}. \tag{3.7d}$$

Proof: In order to prove the equivalence of the constraints (3.3a) and (3.3b) at the nominal voltage magnitudes, notice that

$$P_{lm} = \mathrm{Re}\left\{V_l(V_l^* - V_m^*)y_{lm}^*\right\} = \mathrm{Re}\{(1-e^{\theta_{lm}\mathrm{i}})y_{lm}^*\} = |y_{lm}^*|\left[\cos(\measuredangle y_{lm}^*) - \cos(\theta_{lm} + \measuredangle y_{lm}^*)\right].$$

By inspecting the sinusoidal term inside the expression of P_{lm}, it is straightforward to verify that $|P_{lm}|$ attains its maximum value at $\theta_{lm} = \alpha$. For the constraints (3.3c) and (3.3d), one can write:

$$|S_{lm}|^2 = |V_l(V_l^* - V_m^*)y_{lm}^*|^2 = |y_{lm}^*|^2\left|(1-e^{\mathrm{i}\theta_{lm}})\right|^2 = 2\,|y_{lm}^*|^2\,(1-\cos(\theta_{lm})) \tag{3.8}$$

and

$$|V_l - V_m|^2 = |V_l|^2 + |V_m|^2 - 2|V_l||V_m|\cos(\theta_{lm}) = 2(1-\cos(\theta_{lm})). \tag{3.9}$$

By inspecting the term $\cos(\theta_{lm})$ and using the assumption $\alpha \in [0, \pi/2)$, it follows from the above relations that

$$\theta_{lm} \in [-\alpha, \alpha] \quad \Leftrightarrow \quad |S_{lm}| \le S_{lm}^{\max}(\alpha) \quad \Leftrightarrow \quad |V_l - V_m| \le \Delta V_{lm}^{\max}(\alpha) \tag{3.10}$$

This completes the proof. □

Under relatively tight voltage conditions, the four capacity constraints in (3.3) give rise to very similar feasible regions for (V_l, V_m) if the above upper bounds are employed. Given a certain level of deviation from the nominal voltage magnitude, it is possible to improve the above upper limits of the constraints by incorporating the deviation into these limits via solving a small optimization. In addition, given an upper bound for any of the constraints in (3.3), it is possible to design the upper bounds for the remaining three constraints in such a way that they all imply the constraint with the given upper bound. Since the maximum voltage deviation is usually small and less than 10% in

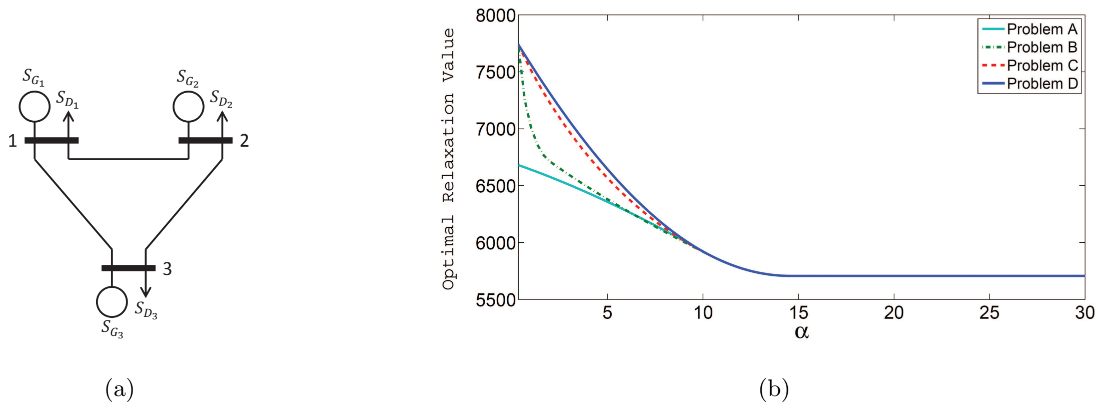


Figure 3.2: (a) Three-bus system studied in Section 3.2.3 ; (b) optimal objective value of the SDP relaxation for Problems A-D.

general, it can be inferred from the above arguments that four common types of capacity constraints with different power engineering implications can be converted to each other with a good accuracy from a mathematical standpoint. To shed light on this fact, Figure 3.1 depicts the feasible region of V_m for each of the constraints in (3.3), where the upper bounds in (3.7) are deployed for the line (l, m) under the following scenario: $\alpha = 15°$, the line admittance $y_{lm} = 1\measuredangle - 80°$ (p.u.), allowing a variable voltage magnitude for V_m with the maximum permissible deviation of 10% from the nominal magnitude, and $V_l = 1\angle 0°$ (p.u.). It can be seen that the feasible regions are very similar and barely distinguishable from each other.

In the following subsection, we will show that this similarity (or equivalence in the extreme case of fixed voltage magnitudes) is no longer preserved after relaxation. In fact, it will be shown that the above capacity constraints behave very differently in the SDP relaxation (i.e., after removing the rank constraint $\text{rank}\{\mathbf{W}\} = 1$).

3.2.3 SDP Relaxation for a Three-bus Network

It has been shown in [Lesieutre *et al.*, 2011] that the SDP relaxation is not exact for a specific three-bus power network with a triangular topology, provided one line has a very limited capacity. The capacity constraint in [Lesieutre *et al.*, 2011] has been formulated with respect to apparent power. It is imperative to study the interesting observation made in [Lesieutre *et al.*, 2011] because if the SDP relaxation cannot handle very simple cyclic networks, its application to mesh networks would be questionable. The result of [Lesieutre *et al.*, 2011] implies that the SDP relaxation is not necessarily exact for cyclic networks if the capacity constraint (3.3c) is employed. The high-level objective of this part is to make the surprising observation that the SDP relaxation becomes exact if the capacity constraint (3.3d) is used instead (this result will be proved later in the chapter). To this end, we explore a scenario for which all four types of capacity constraints provided in (3.3) are equivalent but their convexified counterparts behave very differently. The goal is to show that the SDP relaxation is always exact only for one of these capacity constraints.

Consider the three-bus system depicted in Figure 3.2(a), which has been adopted from [Lesieutre *et al.*, 2011]. The parameters of this cyclic network are provided in Table 3.1, where $z_{lm} = \frac{1}{y_{lm}}$ denotes the impedance of the line (l,m). Assume that lines $(1,2)$ and $(2,3)$ have very high capacities, i.e.,

$$\theta_{12}^{\max} = P_{12}^{\max} = S_{12}^{\max} = \Delta V_{12}^{\max} = \infty, \tag{3.11a}$$

$$\theta_{23}^{\max} = P_{23}^{\max} = S_{23}^{\max} = \Delta V_{23}^{\max} = \infty, \tag{3.11b}$$

while line $(1,3)$ has a very limited capacity. Since there are four ways to limit the flow over this line, we study four problems, each using only one of the capacity constraints given in (3.3) with its corresponding bound from (3.7). To this end, given an angle α belonging to the interval $[0,30°]$, consider the following limits for these four problems:

$$\text{Problem A}: \quad \theta_{13} \leq \theta_{13}^{\max}(\alpha) \tag{3.12a}$$

$$\text{Problem B}: \quad P_{13} \leq P_{13}^{\max}(\alpha) \tag{3.12b}$$

$$\text{Problem C}: \quad S_{13} \leq S_{13}^{\max}(\alpha) \tag{3.12c}$$

$$\text{Problem D}: \quad \Delta V_{13} \leq \Delta V_{13}^{\max}(\alpha) \tag{3.12d}$$

It is straightforward to verify that Problems A-D are equivalent due to the fact that they all lead to the same feasible set for the pair (V_1, V_3). After removing the rank constraint from the OPF problem, these four problems become very distinct. To illustrate this property, we solve four relaxed SDP problems for the network depicted in Figure 3.2(a), corresponding to the equivalent Problems A-D. Figure 3.2(b) plots the optimal objective value of each of the four SDP relaxations as a function of α over the range $\alpha \in [0, 30°]$. Let $f^{\text{opt}}(\alpha)$ denote the solution of the original OPF problem. Each of the curves in Figure 3.2(b) is theoretically a lower bound on the function $f^{\text{opt}}(\alpha)$

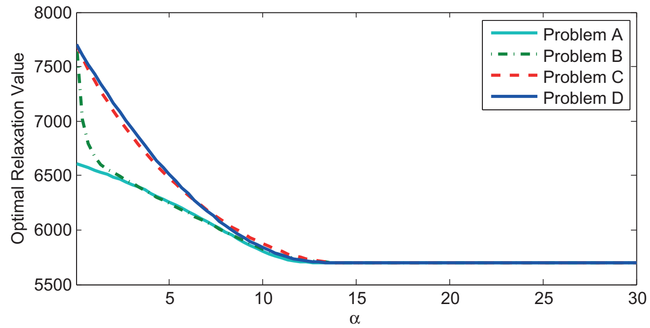


Figure 3.3: Optimal objective value of the SDP relaxation for Problems A-D by allowing 10% off-nominal voltage magnitudes.

in light of removing the non-convex constraint $\text{rank}\{\mathbf{W}\} = 1$. A few observations can be made here:

- The SDP relaxation for Problem D yields a rank-1 solution for all values of α. Hence, the curve drawn in Figure 3.2(b) associated with Problem D represents the function $f^{\text{opt}}(\alpha)$, leading to the true solution of OPF.

- The curves for the SDP relaxations of Problems A-C do not overlap with $f^{\text{opt}}(\alpha)$ if $\alpha \in (0, 7^\circ)$. Moreover, the gap between these curves and the function $f^{\text{opt}}(\alpha)$ is significant for certain values of α.

- Figure 3.3 shows the case when a maximum of 10% off-nominal voltage magnitude is allowed for each bus. In this case, Problem D is the only formulation that always results in a rank-1 solution.

In summary, three types of capacity constraints make the SDP relaxation inexact in general, while the last type of capacity constraint makes the SDP relaxation always exact. The current practice in power systems is to use Problem B (due to its connection to DC OPF), but this example signifies that Problem D is the only one making the SDP relaxation a successful technique. Note that the capacity constraint considered in Problem D is closely related to the thermal loss, and therefore it may be natural to deploy Problem D for solving the OPF problem. Note also that if the OPF is defined in terms of multiple types of capacity constraints, the above reasoning justifies the need for converting the constraints into a single constraint of the form (3.3d).

$$f_1(P_{G_1}) \triangleq 0.11P_{G_1}^2 + 5.0P_{G_1}$$
$$f_2(P_{G_2}) \triangleq 0.085P_{G_2}^2 + 1.2P_{G_2}$$
$$f_3(P_{G_3}) \triangleq 0$$
$$z_{23} = 0.025 + 0.750\text{i}, \qquad S_{D_1} = 110 \text{ MW}$$
$$z_{31} = 0.065 + 0.620\text{i}, \qquad S_{D_2} = 110 \text{ MW}$$
$$z_{12} = 0.042 + 0.900\text{i}, \qquad S_{D_3} = 95 \text{ MW}$$
$$V_k^{\min} = V_k^{\max} = 1 \quad \text{for} \quad k = 1, 2, 3$$
$$(Q_k^{\min}, Q_k^{\max}) = (-\infty, \infty) \quad \text{for} \quad k = 1, 2, 3$$
$$(P_k^{\min}, P_k^{\max}) = (-\infty, \infty) \quad \text{for} \quad k = 1, 2$$
$$P_3^{\min} = P_3^{\max} = 0$$

Table 3.1: Parameters of the three-bus system drawn in Figure 3.2(a) with the base value 100 MVA.

Based on the methodology developed in [Lavaei and Low, 2012] and [Sojoudi and Lavaei, 2012], the above result can be interpreted in terms of the duality gap for OPF: *there are four equivalent non-convex formulations of the OPF problem in the above example with the property that three of them have a nonzero duality gap in general while the last one always has a zero duality gap.* This example reveals the fact that the problem formulation of OPF has a tremendous role in the success of the SDP relaxation, and in particular even equivalent formulations may become distinct after convexification. The observation made in this example will be proved for certain networks below.

Definition 15. *A graph is called weakly cyclic if every edge of the graph belongs to at most one cycle in the graph.*

Theorem 10. *Consider the OPF problem (3.1) with the capacity constraint (3.3d) for a weakly-cyclic network with cycles of size 3. The following statements hold:*

a) The SDP relaxation is exact in the lossless case, provided $Q_k^{\min} = -\infty$ for every $k \in \mathcal{N}$.

b) The SDP relaxation is exact in the lossy case, provided $P_k^{\min} = Q_k^{\min} = -\infty$ and $Q_k^{\max} = +\infty$ for every $k \in \mathcal{N}$.

Proof: The proof is trivial for a 2-bus network. Assume for now that the network is composed of a single cycle of size 3. In order to prove the theorem in this case, consider an arbitrary solution $(\mathbf{P}_G^{\text{init}}, \mathbf{Q}_G^{\text{init}}, \mathbf{W}^{\text{init}})$ of the SDP relaxation. It suffices to show that there exists another solution $(\mathbf{P}_G^{\text{opt}}, \mathbf{Q}_G^{\text{opt}}, \mathbf{W}^{\text{opt}})$ with the same cost as $(\mathbf{P}_G^{\text{init}}, \mathbf{Q}_G^{\text{init}}, \mathbf{W}^{\text{init}})$ such that $\text{rank}\{\mathbf{W}^{\text{opt}}\} = 1$. Alternatively, it is enough to prove that the feasibility problem

$$P_k^{\min} \leq P_{D_k} + \sum_{l \in \mathcal{N}(k)} \text{Re}\left\{(W_{kk} - W_{kl})y_{kl}^*\right\} \leq P_{G_k}^{\text{init}} \tag{3.13a}$$

$$Q_k^{\min} \leq Q_{D_k} + \sum_{l \in \mathcal{N}(k)} \text{Im}\left\{(W_{kk} - W_{kl})y_{kl}^*\right\} \leq Q_{G_k}^{\max} \tag{3.13b}$$

$$(V_k^{\min})^2 \leq W_{kk} \leq (V_k^{\max})^2 \tag{3.13c}$$

$$W_{ll} + W_{mm} - W_{lm} - W_{ml} \leq (\Delta V_{lm}^{\max})^2 \tag{3.13d}$$

$$\mathbf{W} \succeq 0 \tag{3.13e}$$

$\forall\ k \in \mathcal{N},\ (l, m) \in \mathcal{L}$, has a rank-1 solution $\mathbf{W}^{\text{opt}}$. To this end, we convert the above feasibility problem into an optimization by adding the objective function

$$\min_{\mathbf{W} \in \mathbb{H}_n} -\sum_{k \in \mathcal{G}} Q_{G_k} \tag{3.14}$$

to the problem. Let $\underline{\nu}_k, \underline{\lambda}_k, \underline{\mu}_k \in \mathbb{R}^+$, $\overline{\nu}_k, \overline{\lambda}_k, \overline{\mu}_k, \psi_{lm} \in \mathbb{R}^+$, and $\mathbf{A} \in \mathbb{H}_3^+$ denote the Lagrange multipliers corresponding to the lower bounding constraints (3.13a), (3.13b), (3.13c), upper bounding constraints (3.13a), (3.13b), (3.13c), (3.13d), and (3.13e), respectively. It can be shown that

$$\begin{aligned} A_{lm} \quad = \quad & -\text{Im}\{y_{lm}^*\} - \psi_{lm} - \psi_{ml} \\ & -\frac{(\overline{\nu}_l - \underline{\nu}_l)y_{lm}^* + (\overline{\nu}_m - \underline{\nu}_m)y_{lm}}{2} \\ & -\frac{(\overline{\lambda}_l - \underline{\lambda}_l)y_{lm}^* - (\overline{\lambda}_m - \underline{\lambda}_m)y_{lm}}{2\mathbf{i}} \end{aligned} \tag{3.15}$$

for every $(l, m) \in \mathcal{L}$. Define $\nu_k \triangleq \overline{\nu}_k - \underline{\nu}_k$ and $\lambda_k \triangleq \overline{\lambda}_k - \underline{\lambda}_k$, for every $k \in \mathcal{N}$. Then, (3.15) can be rewritten as

$$\begin{aligned} A_{lm} \quad = \quad & -\psi_{lm} - \psi_{ml} \\ & -\text{Re}\{y_{lm}^*\}\left[\frac{\nu_l + \nu_m - (\lambda_l - \lambda_m)\mathbf{i}}{2}\right] \\ & -\text{Im}\{y_{lm}^*\}\left[1 + \frac{\lambda_l + \lambda_m + (\nu_l - \nu_m)\mathbf{i}}{2}\right] \end{aligned} \tag{3.16}$$

for every $(l, m) \in \mathcal{L}$. Moreover, the complementary slackness condition yields that trace$\{\mathbf{W}^{\text{opt}}\mathbf{A}^{\text{opt}}\} = 0$ at optimality. To prove that $\mathbf{W}^{\text{opt}}$ has rank 1, it suffices to show that $\mathbf{A}^{\text{opt}}$ has rank $n-1=2$. To prove the later statement by contradiction, assume that $\mathbf{A}^{\text{opt}}$ has rank 1. Therefore, the determinant of each 2×2 submatrix of $\mathbf{A}^{\text{opt}}$ must be zero. In particular,

$$\det \begin{bmatrix} A_{12}^{\text{opt}} & A_{13}^{\text{opt}} \\ A_{22}^{\text{opt}} & A_{23}^{\text{opt}} \end{bmatrix} = A_{12}^{\text{opt}} A_{23}^{\text{opt}} - A_{13}^{\text{opt}} A_{22}^{\text{opt}} = 0 \tag{3.17}$$

$$\implies \quad \measuredangle A_{12}^{\text{opt}} + \measuredangle A_{23}^{\text{opt}} - \measuredangle A_{13}^{\text{opt}} = \measuredangle A_{22}^{\text{opt}}. \tag{3.18}$$

Since $\mathbf{A}^{\text{opt}}$ is Hermitian, we have

$$\measuredangle A_{22}^{\text{opt}} = 0 \quad \text{and} \quad \measuredangle A_{13}^{\text{opt}} = -\measuredangle A_{31}^{\text{opt}} \tag{3.19}$$

and hence the following relation must hold:

$$\measuredangle A_{12}^{\text{opt}} + \measuredangle A_{23}^{\text{opt}} + \measuredangle A_{31}^{\text{opt}} = 0. \tag{3.20}$$

On the other hand, under the assumptions of the theorem, we have

$$\text{Re}\{y_{lm}^*\} = 0, \quad \lambda_k \geq 0 \tag{3.21}$$

for Part (a) and

$$\lambda_k = 0, \quad \nu_k \geq 0 \tag{3.22}$$

for Part (b). Hence, it can be concluded from (3.16) and each set of equations (3.21) or (3.22) that

$$\text{Re}\{A_{12}^{\text{opt}}\}, \text{Re}\{A_{23}^{\text{opt}}\}, \text{Re}\{A_{31}^{\text{opt}}\} < 0 \tag{3.23a}$$

$$\frac{\text{Im}\{A_{12}^{\text{opt}}\}}{\text{Im}\{y_{12}^*\}} + \frac{\text{Im}\{A_{23}^{\text{opt}}\}}{\text{Im}\{y_{23}^*\}} + \frac{\text{Im}\{A_{31}^{\text{opt}}\}}{\text{Im}\{y_{31}^*\}} = 0. \tag{3.23b}$$

(recall that y_{lm}^* has nonnegative real and imaginary parts due to the positivity assumption of the resistance and reactance of each line). It can be concluded from (3.23b) that the elements of the set $\left\{\text{Im}\{A_{12}^{\text{opt}}\}, \text{Im}\{A_{23}^{\text{opt}}\}, \text{Im}\{A_{31}^{\text{opt}}\}\right\}$ are neither all positive nor all negative. With no loss of generality, it suffice to study the following two cases:

i) If

$$\operatorname{Im}\{A_{12}^{\mathrm{opt}}\}, \operatorname{Im}\{A_{23}^{\mathrm{opt}}\} \geq 0 \quad \text{and} \quad \operatorname{Im}\{A_{31}^{\mathrm{opt}}\} \leq 0, \tag{3.24}$$

then according to (3.23a), we have:

$$\pi/2 < \measuredangle A_{12}^{\mathrm{opt}} \leq \pi \tag{3.25a}$$

$$\pi/2 < \measuredangle A_{23}^{\mathrm{opt}} \leq \pi \tag{3.25b}$$

$$\pi \leq \measuredangle A_{31}^{\mathrm{opt}} < 3\pi/2. \tag{3.25c}$$

ii) If

$$\operatorname{Im}\{A_{12}^{\mathrm{opt}}\}, \operatorname{Im}\{A_{23}^{\mathrm{opt}}\} \leq 0 \quad \text{and} \quad \operatorname{Im}\{A_{31}^{\mathrm{opt}}\} \geq 0, \tag{3.26}$$

then according to (3.23a), we have:

$$\pi \leq \measuredangle A_{12}^{\mathrm{opt}} < 3\pi/2 \tag{3.27a}$$

$$\pi \leq \measuredangle A_{23}^{\mathrm{opt}} < 3\pi/2 \tag{3.27b}$$

$$\pi/2 < \measuredangle A_{31}^{\mathrm{opt}} \leq \pi. \tag{3.27c}$$

Both (3.25) and (3.27) yield that

$$2\pi < \measuredangle A_{12}^{\mathrm{opt}} + \measuredangle A_{23}^{\mathrm{opt}} + \measuredangle A_{31}^{\mathrm{opt}} < 4\pi \tag{3.28}$$

implying that the angle relation (3.20) does not hold. This contradiction completes the proof for both Parts (a) and (b).

For a general network with multiple cycles, let $\mathcal{O}$ denote the set of all 3-bus cyclic subgraphs of the power network. Define $\bar{\mathcal{O}}$ as the set of all bridge edges (i.e., those edges whose removal makes the graph disconnected). By adapting the proof delineated above for a single link and a single cycle, it can be shown that the SDP relaxation has a solution $\mathbf{W}^{\mathrm{opt}}$ with the property

$$\operatorname{rank}\{W^{\mathrm{opt}}(\mathcal{S})\} = 1 \;\text{ for all }\; \mathcal{S} \in \mathcal{O} \cup \bar{\mathcal{O}}, \tag{3.29}$$

where $W^{\mathrm{opt}}(\mathcal{S})$ is a sub-matrix of $\mathbf{W}^{\mathrm{opt}}$ obtained by picking every row and column of $\mathbf{W}^{\mathrm{opt}}$ whose index corresponds to a vertex of the subgraph $\mathcal{S}$. The above relation yields that

$$|W^{\mathrm{opt}}| = \sqrt{W_{ll}^{\mathrm{opt}} W_{mm}^{\mathrm{opt}}}, \qquad (l,m) \in \mathcal{L} \tag{3.30}$$

and that

$$\measuredangle W^{\text{opt}}(\mathcal{S})_{1,2} + \measuredangle W^{\text{opt}}(\mathcal{S})_{2,3} + \measuredangle W^{\text{opt}}(\mathcal{S})_{3,1} = 0 \tag{3.31}$$

for every $\mathcal{S} \in \mathcal{O}$. It follows from the above equation that there exist some angles $\theta_1, \ldots, \theta_n \in [-\pi, \pi]$ such that

$$\theta_l - \theta_m = \measuredangle W^{\text{opt}}_{lm} \;\text{ for all }\; (l, m) \in \mathcal{L} \tag{3.32}$$

Now, it is easy to verity that $\mathbf{V}^{\text{opt}}(\mathbf{V}^{\text{opt}})^*$ is a rank-1 solution of the SDP relaxation, where

$$\mathbf{V}^{\text{opt}} = \left[\sqrt{W^{\text{opt}}_{11}}e^{-\theta_1 \mathrm{i}}, \sqrt{W^{\text{opt}}_{22}}e^{-\theta_2 \mathrm{i}}, \ldots, \sqrt{W^{\text{opt}}_{nn}}e^{-\theta_n \mathrm{i}}\right]^* \tag{3.33}$$

This completes the proof. □

Note that the statement of Theorem 10 cannot be generalized to the capacity constraints (3.3a)-(3.3c). This manifests the importance of the problem formulation and mathematical modeling.

3.3 Injection Region

A power network under operation has a pair of flows (P_{lm}, P_{ml}) over each line $(l, m) \in \mathcal{L}$ and a net injection P_k at each bus $k \in \mathcal{N}$, where P_k is indeed equal to $P_{G_k} - P_{D_k}$. This means that two vectors can be attributed to the network: (i) injection vector $\mathbf{P} = [\; P_1 \; P_2 \; \cdots \; P_n \;]$, (ii) flow vector $\mathbf{F} = [P_{lm} |\; (l, m) \in \mathcal{L}]$. Due to the relation $P_k = \sum_{l \in \mathcal{N}(k)} P_{kl}$, there exists a matrix M such that $\mathbf{P} = M \times \mathbf{F}$.

In order to understand the computational complexity of OPF, it is beneficial to explore the feasible set for the injection vector. To this end, two notions of *flow region* and *injection region* will be defined in line with [Lavaei *et al.*, 2012].

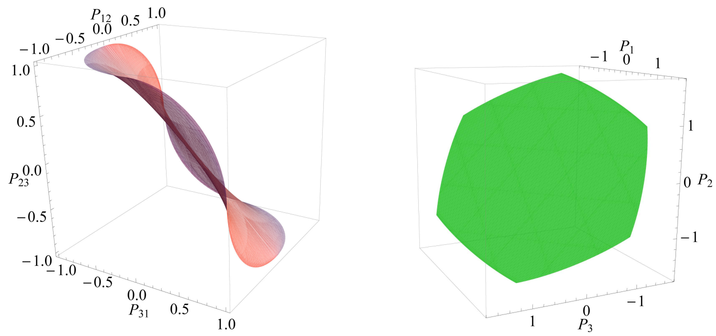
Definition 16. *Define the flow region $\mathcal{F}$ as the set of all flow vectors $\mathbf{F} = [P_{lm} \mid (l, m) \in \mathcal{L}]$ for which there exists a voltage phasors vector $[\; V_1 \; V_2 \; \cdots \; V_n \;]$ such that*

$$P_{lm} = \operatorname{Re}\left\{V_l(V_l^* - V_m^*)y_{lm}^*\right\}, \quad (l, m) \in \mathcal{L} \tag{3.34a}$$

$$|V_l - V_m| \leq \Delta V_{lm}^{\max}, \quad (l, m) \in \mathcal{L} \tag{3.34b}$$

$$V_k^{\min} \leq |V_k| \leq V_k^{\max}, \quad k \in \mathcal{N} \tag{3.34c}$$

Define also the injection region $\mathcal{P}$ as $M \cdot \mathcal{F}$.



(a) (b)

Figure 3.4: (a) The reduced flow region $\mathcal{F}^r$ for a three-bus mesh network; (b) the injection region $\mathcal{P}$ for a three-bus mesh network.

The above definition of the flow and injection regions captures the laws of physics, capacity constraints and voltage constraints. One can make this definition more comprehensive by incorporating reactive-power constraints.

Definition 17. *Define the convexified flow region* $\mathcal{F}_c$ *as the set of all flow vectors* $\mathbf{F} = [P_{lm} \mid (l,m) \in \mathcal{L}]$ *for which there exists a matrix* $\mathbf{W} \in \mathbb{H}_n^+$ *such that*

$$P_{lm} = \mathrm{Re}\left\{(W_{ll} - W_{lm})y_{lm}^*\right\} \tag{3.35a}$$

$$W_{ll} + W_{mm} - W_{lm} - W_{ml} \leq (\Delta V_{lm}^{\max})^2 \tag{3.35b}$$

$$(V_k^{\min})^2 \leq W_{kk} \leq (V_k^{\max})^2 \tag{3.35c}$$

for every $(l,m) \in \mathcal{L}$ *and* $k \in \mathcal{N}$. *Define also the convexified injection region* $\mathcal{P}_c$ *as* $M \cdot \mathcal{F}_c$.

It is straightforward to verify that $\mathcal{P} \subseteq \mathcal{P}_c$ and $\mathcal{F} \subseteq \mathcal{F}_c$.

3.3.1 Lossless Cycles

A lossless network has the property that $P_{lm} + P_{ml} = 0$ for every $(l,m) \in \mathcal{L}$, or alternatively $\mathrm{Re}\{y_{lm}\} = 0$. Since real-world transmission networks are very close to being lossless, we study lossless mesh networks here. The flow region $\mathcal{F}$ has been defined in terms of two flows P_{lm} and P_{ml} for each line $(l,m) \in \mathcal{L}$. Due to the relation $P_{ml} = -P_{lm}$ for lossless networks, one can define a reduced flow region $\mathcal{F}^r$ based on one flow P_{lm} for each line (l,m).

The reduced flow region $\mathcal{F}^r$ has been plotted in Figure 3.4(a) for a cyclic three-bus network under the voltage setting $V_k^{\min} = V_k^{\max}$ for $k = 1,2,3$ and some arbitrary capacity limits. This feasible set is a non-convex 2-dimensional curvy surface in $\mathbb{R}^3$. The corresponding injection region $\mathcal{P}$ can be obtained by applying an appropriate linear transformation to $\mathcal{F}^r$. Surprisingly, this set becomes convex, as depicted in Figure 3.4(b). More precisely, it can be shown that $\mathcal{P} = \mathcal{P}_c$ in this case. The goal of this part is to investigate the convexity of $\mathcal{P}$ for a single cycle. Assume for now that the power network is composed of a single cycle with the links $(1, 2), \ldots, (n-1, n), (n, 1)$.

Theorem 11. *Consider a lossless n-bus cycle with $n \geq 3$. The reduced flow region $\mathcal{F}^r$ is always non-convex if $V_k^{\min} = V_k^{\max}$, $k = 1, 2, ..., n$.*

Proof: The reduced flow region $\mathcal{F}^r$ consists of all vectors of the form

$$(\alpha_1 \sin(\theta_{12}), \alpha_2 \sin(\theta_{23}), \ldots \alpha_n \sin(\theta_{n1})),$$

where $\theta_{12} + \theta_{23} + \cdots + \theta_{n1} = 0$ and $\alpha_k = |V_k||V_{k+1}|\mathrm{Im}\{y^*_{k,k+1}\}$ for $k \in \mathcal{N}$. Therefore, $\mathcal{F}^r$ can be characterized in terms of $n-1$ independent angle differences $\theta_{12}, ..., \theta_{(n-1),n}$. This implies that $\mathcal{F}^r$ is an $(n-1)$-dimensional surface embedded in $\mathbb{R}^n$. On the other hand, this region cannot be embedded in $\mathbb{R}^{n-1}$ due to its non-zero curvature. Thus, $\mathcal{F}^r$ cannot be a convex subset of $\mathbb{R}^n$. □

Since $V_k^{\min} \simeq V_k^{\max}$ in practice, it follows from Theorem 11 that the reduced flow region is expected to be non-convex under a normal operation.

Theorem 12. *Consider a lossless n-bus cycle. The following statements hold:*

a) For $n = 2$ and $n = 3$, the injection region $\mathcal{P}$ is convex and in particular $\mathcal{P} = \mathcal{P}_c$.

b) For $n \geq 5$, the injection region $\mathcal{P}$ is non-convex if

$$\begin{aligned} &V_k^{\min} = V_k^{\max} = V^{\max}, \qquad k \in \mathcal{N} \\ &\Delta V_{lm}^{\max} = \Delta V^{\max}, \qquad (l, m) \in \mathcal{L} \end{aligned} \tag{3.36}$$

for any arbitrary numbers $V^{\max}$ and $\Delta V^{\max}$.

Proof of Part (a): Consider an arbitrary injection vector $\bar{P}$ belonging to the convexified injection region $\mathcal{P}_c$. In order to prove Part (a), it suffices to show that $\bar{P}$ is contained in $\mathcal{P}$. Alternatively, it is enough to prove that the SDP relaxation of OPF with the capacity constraint (3.3d) and the parameters

$$P_k^{\max} = P_k^{\min} = \bar{P}_k \tag{3.37a}$$

$$Q_k^{\max} = +\infty, \qquad Q_k^{\min} = -\infty, \tag{3.37b}$$

has a rank-1 solution $\mathbf{W}$. This follows directly from Part (a) of Theorem 10.

Sketch of Proof for Part (b): Define

$$\theta^{\max} = \cos^{-1}\left(1 - \frac{(\Delta V^{\max})^2}{2}\right) \tag{3.38}$$

As pointed out in the proof of Theorem 11, the reduced flow region $\mathcal{F}^r$ contains all vectors of the form $(\alpha_1 \sin(\theta_{12}), \alpha_2 \sin(\theta_{23}), \ldots \alpha_n \sin(\theta_{n1}))$, where $\theta_{12}+\theta_{23}+\ldots+\theta_{n1} = 0$ and $|\theta_{12}|, ..., |\theta_{n1}| \leq \theta^{\max}$. Four observations can be made here:

i) The mapping from $\mathcal{F}^r$ to $\mathcal{P}$ is linear.

ii) The kernel of the map from $\mathcal{F}^r$ to $\mathcal{P}$ has dimension 1.

iii) Due to (i) and (ii), it can be proved that the restriction of $\mathcal{F}^r$ to the angles $\theta_{12} = \theta^{\max}$ and $\theta_{n1} = -\theta^{\max}$ is a convex set whenever $\mathcal{P}$ is convex.

iv) The restriction of $\mathcal{F}^r$ to the angles $\theta_{12} = \theta^{\max}$ and $\theta_{n1} = -\theta^{\max}$ amounts to the reduced flow region for a single cycle of size $n-2$. In light of Theorem 11, this set is nonconvex if $n-2 \geq 3$.

The proof of Part (b) follows from the above facts. □

Theorem 12 states that the injection region is convex only for small values of n.

3.3.2 Weakly-cyclic Networks

In this part, the objective is to study the convexity of the injection region for a class of mesh networks. Although the class under investigation is simple and far from practical, its study gives rise to a good insight into the complexity of OPF. Notice that the injection region $\mathcal{P}$ is not necessarily convex for lossy networks. For example, the set $\mathcal{P}$ corresponding to a three-bus mesh network with nonzero loss is a curvy 2-dimensional surface in $\mathbb{R}^3$. The objective of this part is to show that the front of this non-convex feasible set is convex in some sense.

Definition 18. *Given a set $\mathcal{A} \subseteq \mathbb{R}^n$, define its Pareto front as the set of all points $(a_1, ..., a_n) \in \mathcal{A}$ for which there does not exist a different point $(b_1, ..., b_n)$ in $\mathcal{A}$ such that $b_i \leq a_i$ for $i = 1, ..., n$.*

Pareto front is an important subset of $\mathcal{A}$ because the solution of an arbitrary optimization over $\mathcal{A}$ with an increasing objective function must lie on the Pareto front of $\mathcal{A}$.

Theorem 13. *The following statements hold for a weakly-cyclic network with cycles of size* 3*:*

a) If the network is lossless, then the injection region $\mathcal{P}$ *is convex and in addition* $\mathcal{P} = \mathcal{P}_c$*.*

b) If the network is lossy, then the injection region $\mathcal{P}$ *and the convexified region* $\mathcal{P}_c$ *share the same Pareto front.*

Proof: The proof of Part (a) of Theorem 12 also works for a general lossless weakly-cyclic network, leading to Part (a) of the present theorem.

In order to prove Part (b), we employ the same strategy as in the proof of Theorem 12. Assume that $\bar{P}$ belongs to the Pareto front of the convexified injection region $\mathcal{P}_c$. Consider the OPF problem (3.1) with the capacity constraint (3.3d) and let

$$P_k^{\max} = \bar{P}_k, \qquad P_k^{\min} = -\infty, \tag{3.39a}$$

$$Q_k^{\max} = +\infty, \qquad Q_k^{\min} = -\infty. \tag{3.39b}$$

The objective function of the OPF problem can be replaced by a certain linear function in such a way that $\bar{P}$ becomes a solution of the SDP relaxation of this problem. On the other hand, it follows form Part (b) of Theorem 10 that there exists a solution $(\mathbf{P}^{\text{opt}}, \mathbf{Q}^{\text{opt}}, \mathbf{W}^{\text{opt}})$ for this problem where $\mathbf{W}^{\text{opt}}$ is a rank-1 matrix. Since $\bar{P}$ belongs to the Pareto front of $\mathcal{P}_c$, we have $\mathbf{P}^{\text{opt}} = \bar{P}$. Hence, $\mathbf{P}^{\text{opt}}$ also belongs to $\mathcal{P}$ and that completes the proof. □

3.4 Penalized SDP Relaxation

So far, it has been shown that the SDP relaxation is exact for certain systems such as weakly-cyclic networks, provided a good mathematical formulation is deployed. Nevertheless, the SDP relaxation may not remain exact for mesh networks with large cycles. The objective of this section is to remedy this shortcoming for general networks. To this end, we first study the rank of the minimum-rank solution of the SDP relaxation and then introduce a penalization technique to enforce the rank of this solution matrix to become one. This will ultimately lead to a near-global solution of OPF with some measure of the optimality degree.

3.4.1 Low-rank Solution for SDP Relaxation

In this part, we first introduce some graph-theoretic parameters and then utilize them to relate the network topology to the existence of a low-rank solution for the SDP relaxation method.

Definition 19. *The representative graph of an $n \times n$ Hermitian matrix $\mathbf{W}$, denoted by $\mathscr{G}(\mathbf{W})$, is a simple graph with n vertices whose edges are specified by the locations of the nonzero off-diagonal entries of $\mathbf{W}$. In other words, two arbitrary vertices i and j are connected if W_{ij} is nonzero.*

Definition 20. *Given a simple graph $\mathcal{H}$, define the complex Hermitian minimum semidefinite rank of $\mathcal{H}$ as*

$$\text{msr}_{\mathbb{H}}(\mathcal{G}) \triangleq \min\left\{\text{rank}(\mathbf{W})\,|\,\mathscr{G}(\mathbf{W}) = \mathcal{G},\, \mathbf{W} \in \mathbb{H}_n^+\right\}. \tag{3.40a}$$

The next theorem studies the rank of a solution of the SDP relaxation of the OPF problem under the load over satisfaction assumption

$$P_k^{\min} = Q_k^{\min} = -\infty \quad \text{for} \quad k \in \mathcal{N}. \tag{3.41}$$

A general version of this theorem with no extra assumption has been developed in chapter 2.

Theorem 14. *Consider the OPF problem given in (3.1) subject to the capacity constraints (3.3a), (3.3b) and (3.3d), under the assumption $P_k^{\min} = Q_k^{\min} = -\infty$ for every $k \in \mathcal{N}$. If this problem is feasible, then its corresponding SDP relaxation has a solution $(\mathbf{W}^{\text{opt}}, \mathbf{P}_G^{\text{opt}}, \mathbf{Q}_G^{\text{opt}})$ such that*

$$\text{rank}\{\mathbf{W}^{\text{opt}}\} \leq |\mathcal{H}| - \text{msr}_{\mathbb{H}}(\mathcal{H}), \tag{3.42}$$

where $\mathcal{H}$ can be any arbitrary simple graph with the property that $\mathcal{V}_{\mathcal{H}} = \mathcal{N}$ and $\mathcal{L} \subseteq \mathcal{E}_{\mathcal{H}}$.

Proof: Since the OPF problem is feasible by assumption, there exists an optimal solution $(\mathbf{W}^0, \mathbf{P}_G^{\text{opt}}, \mathbf{Q}_G^{\text{opt}})$ for this problem. Now, consider the optimization problem:

$$\min_{\mathbf{W} \in \mathbb{H}_n^+} \quad -\sum_{(l,m)\,\in\,\mathcal{E}_{\mathcal{H}}} \text{Re}\{W_{lm}\} \tag{3.43a}$$

$$\text{s.t.} \quad W_{kk} = W_{kk}^0, \qquad k \in \mathcal{N} \tag{3.43b}$$

$$\text{Re}\{W_{lm}\} \geq \text{Re}\{W_{lm}^0\}, \qquad (l,m) \in \mathcal{L} \tag{3.43c}$$

$$\text{Im}\{W_{lm}\} = \text{Im}\{W_{lm}^0\}, \qquad (l,m) \in \mathcal{L} \tag{3.43d}$$

Let $\mathbf{W}^{\mathrm{opt}}$ denote an arbitrary solution of the above optimization. Since the resistance and inductance of each line $(l, m) \in \mathcal{L}$ are considered as nonnegative numbers in this chapter, it is straightforward to verify that $(\mathbf{W}^{\mathrm{opt}}, \mathbf{P}_G^{\mathrm{opt}}, \mathbf{Q}_G^{\mathrm{opt}})$ is an optimal solution of the SDP relaxation under the load over-satisfaction assumption. Now, it remains to prove that $\mathbf{W}^{\mathrm{opt}}$ satisfies the inequality $\mathrm{rank}\{\mathbf{W}^{\mathrm{opt}}\} \leq n - \mathrm{msr}_{\mathbb{H}}(\mathcal{H})$.

To proceed with the proof, we aim to take the Lagrangian of Optimization (3.43). Let $\mathbf{A} \in \mathbb{H}_n^+$ denote the dual variable corresponding to the constraint $\mathbf{W} \succeq 0$. By noting that the positions of the nonzero off-diagonal entries of the matrix $\mathbf{A}$ correspond to the edges of the graph $\mathcal{H}$, it follows from the definition of "msr" that

$$\mathrm{rank}\{\mathbf{A}^{\mathrm{opt}}\} \geq \ \mathrm{msr}_{\mathbb{H}}(\mathcal{H}). \tag{3.44}$$

On the other hand, the complementary slackness condition $\mathrm{trace}\{\mathbf{W}^{\mathrm{opt}}\ \mathbf{A}^{\mathrm{opt}}\} = 0$ yields that

$$\mathrm{rank}\{\mathbf{A}^{\mathrm{opt}}\} + \mathrm{rank}\{\mathbf{W}^{\mathrm{opt}}\} \leq n. \tag{3.45}$$

The proof is completed by combining (3.44) and (3.45). □

Roughly speaking, Theorem 14 aims to relate the computational complexity of the OPF problem to the topology of the power network by quantifying how inexact the SDP relaxation is.

Definition 21. *Define $\eta(\mathcal{H})$ as the minimum number of vertices whose removal from the graph $\mathcal{H}$ eliminates all cycles of the graph.*

To illustrate the definition of η, observe that this number is equal to 0 for a graph representing an acyclic network and is equal to 1 if all cycles of the network share a common node. Two graphs with $\eta = 1$ are depicted in Figure 3.5.

Theorem 15. *Consider the OPF problem given in (3.1) subject to the capacity constraints (3.3a), (3.3b) and (3.3d), under the assumption $P_k^{\min} = Q_k^{\min} = -\infty$ for every $k \in \mathcal{N}$. Let $\mathcal{H}$ be the graph that describes the topology of the power network under study. If the OPF problem is feasible, then its corresponding SDP relaxation has a solution $(\mathbf{W}^{\mathrm{opt}}, \mathbf{P}_G^{\mathrm{opt}}, \mathbf{Q}_G^{\mathrm{opt}})$ such that $\mathrm{rank}\{\mathbf{W}^{\mathrm{opt}}\} \leq \eta(\mathcal{H}) + 1$.*

Proof: Let $\mathcal{J}$ denote an induced subgraph of the power network with no cycles. One can expand $\mathcal{J}$ into a tree $\mathcal{T}$ by adding a minimal set of additional edges to this possibly disconnected subgraph.

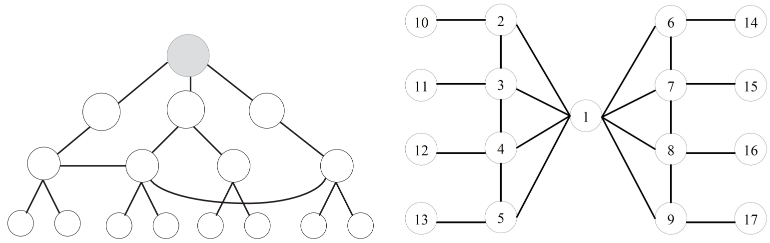


Figure 3.5: Two graphs with $\eta = 1$.

Let $\mathcal{H}' \triangleq (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H} \cup \mathcal{E}_\mathcal{T})$. According to Theorem, 14 there exists a solution $(\mathbf{W}^{\text{opt}}, \mathbf{P}_G^{\text{opt}}, \mathbf{Q}_G^{\text{opt}})$ such that

$$\text{rank}\{\mathbf{W}^{\text{opt}}\} \leq |\mathcal{H}'| - \text{msr}_{\mathbb{H}}(\mathcal{H}'). \tag{3.46}$$

It also follows from [Booth *et al.*, 2008] that

$$\text{msr}_{\mathbb{H}}(\mathcal{H}') \geq |\mathcal{H}'| - \eta(\mathcal{H}') - 1. \tag{3.47}$$

Combining (3.46) and (3.47) yields

$$\text{rank}\{\mathbf{W}^{\text{opt}}\} \leq \eta(\mathcal{H}') + 1. \tag{3.48}$$

For an optimal choice of $\mathcal{J}$ with the maximum number of vertices $|\mathcal{J}| = |\mathcal{H}| - \eta(\mathcal{H})$, we have $\eta(\mathcal{H}') = \eta(\mathcal{H})$. This completes the proof. $\square$

There is a large body of literature on computing η, which signifies that this number is small for a very broad class of graphs, including mostly planar graphs. To illustrate the application of Theorem 15, consider the distribution network depicted in Figure 3.5(a). This network has three cycles, possibly used for exchanging renewable energy between the load buses without going through the feeder (the node shown in gray). Since removing this node eliminates all cycles of the network, it follows from Theorem 15 that the SDP relaxation of OPF has a solution with the property $\text{rank}\{\mathbf{W}^{\text{opt}}\} \leq 2$.

Remark 3. *The power balance equations (3.1b) and (3.1c) are equality constraints. One may relax these equations to inequality constraints so that each bus $k \in \mathcal{N}$ can be oversupplied. This notion is called over-satisfaction and has been considered in a number of papers (see [Lavaei and Low, 2012; Baldick, 2006] and the references therein). The main idea is that whenever a power network operates* *under a normal condition, it is expected that the solution of the OPF problem remains intact or changes insignificantly under the load over-satisfaction assumption. The condition* $P_k^{\min} = Q_k^{\min} = -\infty$ *in Theorem 15 can be supplanted by the load over-satisfaction assumption*

Remark 4. *Given a general graph* $\mathcal{H}$*, finding the parameter* $\eta(\mathcal{H})$ *and its associated maximal induced forest* $\mathcal{J}$ *is known to be an NP-complete problem. Nevertheless, as shown in the proof of Theorem 15, any arbitrary set of nodes whose removal eliminates all cycles of the network leads to a solution* $\mathbf{W}^{opt}$ *together with an upper bound on its rank. In addition, the identification of* $\mathcal{J}$ *is mostly a one-time process and the algorithm proposed in [Razgon, 2006] can be used for that purpose.*

3.4.2 Recovery of Near-optimal Solution for OPF

As discussed in the preceding subsection, the SDP relaxation is expected to have a low-rank solution. This solution may be used to find an approximate rank-1 solution. Another technique is to enforce the SDP relaxation to eliminate the undesirable nonzero eigenvalues of the low-rank solution by incorporating a penalty term into its objective. The recent literature of compressed sensing suggests the penalty term $\varepsilon \times \text{trace}\{\mathbf{W}\}$ for some coefficient $\varepsilon \in \mathbb{R}^+$ [Recht *et al.*, 2010a]. However, this idea fails to work for OPF since all feasible solutions of the SDP relaxation have almost the same trace (because $V_k^{\min}$ and $V_k^{\max}$ are normally close to each other for $k = 1, ..., n$). We propose a different penalty function in this chapter.

Penalized SDP relaxation: This optimization is obtained from the SDP relaxation of the OPF problem by replacing its objective function with

$$\sum_{k\in\mathcal{G}} f_k(P_{G_k}) + \varepsilon \sum_{k\in\mathcal{G}} Q_{G_k} \tag{3.49}$$

for a given positive number ε.

There are two independent reasons behind the introduction of the penalty term $\sum_{k\in\mathcal{G}} Q_{G_k}$:

- Consider a positive semidefinite matrix $\mathbf{X}$ with constant (fixed) diagonal entries $X_{11}, \ldots, X_{nn}$ and variable off-diagonal entries. If we maximize a weighted sum of the off-diagonal entries of X with positive weights, then it turns out that $X_{lm} = \sqrt{X_{ll}X_{mm}}$ for all $l, m \in \{1, \ldots, n\}$, in which case $\mathbf{X}$ becomes rank-1. Motivated by this fact, we employ the idea of elevating the off-diagonal entries of $\mathbf{W}$ to obtain a low-rank solution. For a lossless network, the above penalty term increases the weighted sum of the real parts of the off-diagonal entries of $\mathbf{W}$.

- Denote the set of all feasible vectors $(\mathbf{P}_G, \mathbf{Q}_G)$ satisfying the constraints of OPF as $\mathcal{A}$. The OPF problem minimizes the cost function $\sum_{k\in\mathcal{G}} f_k(P_{G_k})$ over the projection of $\mathcal{A}$ onto the space for $\mathbf{P}_G$, which is referred to as $\mathcal{P}$ in this work. The projection from $\mathcal{A}$ to $\mathcal{P}$ maps multiple (possibly an uncountable number of) points into the same vector $\mathbf{P}_G$. This becomes a critical issue after removing the constraint $\text{rank}\{\mathbf{W}\} = 1$ from OPF. The main reason is that those multiple points with the same projection could correspond to different values of $\mathbf{W}$ with disparate ranks. The penalty term $\varepsilon \sum_{k\in\mathcal{G}} Q_{G_k}$ aims to guide the numerical algorithm by speculating that the right point $(\mathbf{P}_G, \mathbf{Q}_G)$ would cause the lowest reactive loss.

Let $(\mathbf{W}^{\text{opt}}, \mathbf{P}_G^{\text{opt}}, \mathbf{Q}_G^{\text{opt}})$ and $(\mathbf{W}^{\varepsilon}, \mathbf{P}_G^{\varepsilon}, \mathbf{Q}_G^{\varepsilon})$ denote arbitrary solutions of the SDP and penalized SDP relaxations, respectively. Assume that $\mathbf{W}^{\text{opt}}$ does not have rank 1, whereas $\mathbf{W}^{\varepsilon}$ has rank 1. It can be observed that the optimal objective value of OPF is lower and upper bounded by the respective numbers $\sum_{k\in\mathcal{G}} f_k(P_{G_k}^{\text{opt}})$ and $\sum_{k\in\mathcal{G}} f_k(P_{G_k}^{\varepsilon})$. Moreover, $(\mathbf{W}^{\varepsilon}, \mathbf{P}_G^{\varepsilon}, \mathbf{Q}_G^{\varepsilon})$ can be mapped into the feasible solution $(\mathbf{V}^{\varepsilon}, \mathbf{P}_G^{\varepsilon}, \mathbf{Q}_G^{\varepsilon})$ of the OPF problem, where $\mathbf{V}^{\varepsilon}(\mathbf{V}^{\varepsilon})^* = \mathbf{W}^{\varepsilon}$. As a result, whenever the penalized SDP relaxation has a rank-1 solution, a feasible solution of OPF can be readily constructed and its sub-optimality degree can be measured subsequently. Note that a gradient descent algorithm can then be exploited to produce a local (if not global) solution from $(\mathbf{V}^{\varepsilon}, \mathbf{P}_G^{\varepsilon}, \mathbf{Q}_G^{\varepsilon})$. Since the SDP relaxation of OPF possesses a low-rank solution in most cases, it is anticipated that the penalized SDP relaxation generates a global or near-global solution. We conducted extensive simulations on IEEE systems with more than 7000 different cost functions and observed that the penalized SDP relaxation always had a rank-1 solution. In addition, the obtained feasible solution of OPF was not only near optimal but also almost a local solution (satisfying the first order optimality conditions with some small error) in more than 95% of the trials. This observation will be elaborated in the next section. In what follows, we will provide partial theoretical results supporting our penalization technique.

Theorem 16. *Consider a weakly-cyclic network with cycles of size 3. Given an arbitrary strictly positive number ε, every solution of the penalized SDP relaxation with the capacity constraint (3.4d) has rank-1, provided*

IEEE-14					
$\boldsymbol{\epsilon}$		0		0.012	
$\boldsymbol{\lambda_1}$		15.1617		15.1340	
$\boldsymbol{\lambda_2}$		0.0138		0	
Cost		\$316.08		\$316.13	
$\boldsymbol{k}$	$\boldsymbol{c_k}$	$\boldsymbol{P_{G_k}}$	$\boldsymbol{Q_{G_k}}$	$\boldsymbol{P_{G_k}}$	$\boldsymbol{Q_{G_k}}$
1	3	25.36	0	25.38	0.85
2	1	140	25.44	140	22.25
3	4	0	28.77	0	27.11
6	1	100	-6	100	-6
8	4	0	9.16	0	6.42

Table 3.2: Case study for IEEE 14 bus system.

a) $Q_k^{\min} = -\infty$ *for every* $k \in \mathcal{N}$ *in the lossless case;*

b) $P_k^{\min} = Q_k^{\min} = -\infty$ *and* $Q_k^{\max} = \infty$ *for every* $k \in \mathcal{N}$ *in the lossy case.*

Proof: This theorem can be proved in line with the technique developed in the proof of Theorem 10. □

3.5 Simulations

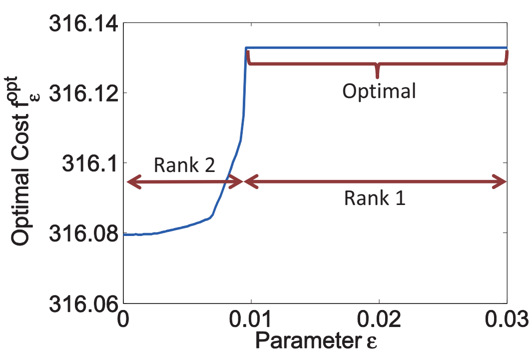
Consider the IEEE 14-bus system with the cost function $\sum_{k \in \mathcal{G}} c_k P_{G_k}$, where the coefficients c_k's are provided in Table 3.2. Let λ_1 and λ_2 denote the two largest eigenvalues of the matrix solution $\mathbf{W}_\varepsilon^{\text{opt}}$ of the penalized SDP relaxation. Solving this relaxation with $\varepsilon = 0$ gives rise to $\lambda_1 = 15.1617$ and $\lambda_2 = 0.0138$, implying that the matrix $\mathbf{W}_\varepsilon^{\text{opt}}$ is nearly rank-1. However, λ_2 being nonzero is an impediment to the recovery of a feasible solution of OPF. To address this issue, we solve the penalized SDP relaxation with $\varepsilon = 0.012$. This leads to a rank-1 matrix $\mathbf{W}_\varepsilon^{\text{opt}}$. The results are summarized in Table 3.2. It can be seen that changing the penalty coefficient ε from 0 to 0.012 has a negligible effect on $\mathbf{P}_G$ but a significant impact on $\mathbf{Q}_G$. As a result, the proposed penalization method corrects the vector of reactive powers and the upshot of this correction is the recovery of a feasible solution for OPF. Notice that the cost for this feasible solution is equal to 316.13, while the optimal cost for the globally optimal solution of OPF is lower bounded by 316.08, i.e., the solution of the SDP relaxation. This means that although it is hard to argue whether the feasible solution

IEEE-30					
$\boldsymbol{\epsilon}$		0		0.55	
$\boldsymbol{\lambda_1}$		30.6789		30.8677	
$\boldsymbol{\lambda_2}$		0.4986		0	
Cost		\$414.34		\$438.40	
$\boldsymbol{k}$	$\boldsymbol{c_k}$	$\boldsymbol{P_{G_k}}$	$\boldsymbol{Q_{G_k}}$	$\boldsymbol{P_{G_k}}$	$\boldsymbol{Q_{G_k}}$
1	1	80	11.11	80	-4.60
2	10	0	39.16	0	-2.10
13	1	40	44.70	40	44.70
33	10	23.98	35.26	27.32	33.36
23	100	0	33.39	0	15.62
27	1	54.55	25.65	45.22	21.33

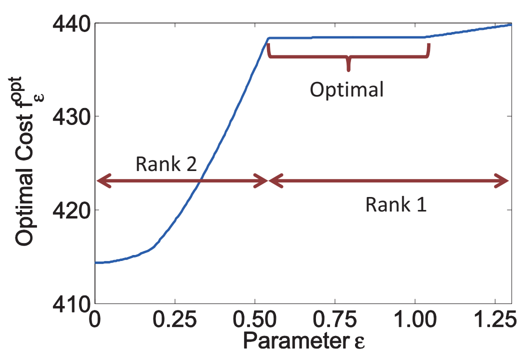
Table 3.3: Case study for IEEE 30 bus system.

IEEE-57					
$\boldsymbol{\epsilon}$		0		1.5	
$\boldsymbol{\lambda_1}$		57.1776		56.8887	
$\boldsymbol{\lambda_2}$		0.0767		0	
Cost		\$259.70		\$272.73	
$\boldsymbol{k}$	$\boldsymbol{c_k}$	$\boldsymbol{P_{G_k}}$	$\boldsymbol{Q_{G_k}}$	$\boldsymbol{P_{G_k}}$	$\boldsymbol{Q_{G_k}}$
1	0.1	575.88	78.60	575.88	111.87
2	0.1	100	50	100	50
3	100	0	60	0	44.29
6	0.1	100	25	100	25
8	10	13.11	117.90	14.41	159.64
9	0.1	100	9	100	9
12	0.1	410	96.91	410	-6.29

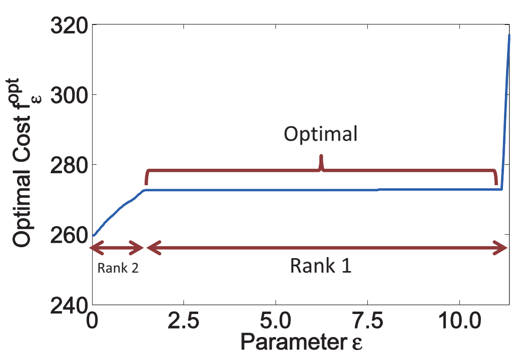
Table 3.4: Case study for IEEE 57 bus system.



(a)



(b)



(c)

Figure 3.6: (a) IEEE-14; (b) IEEE-30; (c) IEEE-57

retrieved from the rank-1 matrix $\mathbf{W}_\varepsilon^{\text{opt}}$ for $\varepsilon = 0.012$ is globally optimal for OPF, its sub-optimality degree is at least %99.98 (this number is obtained by contrasting the cost 316.13 with the lower bound 316.08). It is even more interesting to note that the feasible solution recovered for OPF coincides with the solution found by the interior point method implemented in MATPOWER. This implies that the attained feasible solution is a local near-global (if not global) solution of OPF.

To gain some insight into the selection of the penalty coefficient ε, the cost $f_\varepsilon^{\text{opt}} = \sum_{k \in \mathcal{G}} f_k(P_{G_k}^\varepsilon)$ is plotted in Figure 3.6(a). It can be observed that this function is strictly increasing at the beginning, but there is a breakpoint at which the function becomes almost flat. Interestingly, the matrix $\mathbf{W}_\varepsilon^{\text{opt}}$ has rank 2 before the breakpoint $\varepsilon = 0.012$ and rank 1 after this point. Consequently, there is a range of values for ε (as opposed to a single number) that makes the matrix $\mathbf{W}_\varepsilon^{\text{opt}}$ rank 1 and keeps the cost at the lowest level (due to the almost flat part of the curve $f_\varepsilon^{\text{opt}}$).

The above experiment was repeated on two very extreme cases for IEEE 30 and 57-bus systems with linear cost functions. The results are summarized in Tables 3.3 and 3.4 and Figures 3.6(b)-(c). The observations made for each of these cases conform with the previous ones: (i) there is a turning point at which the cost function $f_\varepsilon^{\text{opt}}$ becomes almost flat and concurrently the matrix $\mathbf{W}_\varepsilon^{\text{opt}}$ becomes rank 1, (ii) the feasible solution of OPF recovered from a rank-1 matrix $\mathbf{W}_\varepsilon^{\text{opt}}$ is not only near-optimal but also a local solution. The phenomenon of the "almost flat part segment" in the curve $f_\varepsilon^{\text{opt}}$ has been observed in numerous cases examined by the authors for which the (unpenalized) SDP relaxation did not have a rank-1 solution.

Some modifications on the IEEE test cases and other well known examples have been proposed in [Bukhsh *et al.*, 2013] and [Gopalakrishnan *et al.*, 2011], which make the SDP relaxation method fail to work. Consider the case "modified 14-bus" from [Gopalakrishnan *et al.*, 2011] and "modified 118-bus" from [Bukhsh *et al.*, 2013] to evaluate the performance of the penalized SDP method:

- For the case "modified 14-bus" from [Gopalakrishnan *et al.*, 2011], the (unpenalized) SDP lower bound on the optimal cost of the solution is 8092.36. A rank-1 solution can be obtained at $\varepsilon = 80$ with the cost 8092.72.

- For the case "modified 118-bus" from [Bukhsh *et al.*, 2013], the diagram of the optimal cost versus the penalty coefficient ε is shown in Figure 3.7. This system has at least 3 local minima with the associated costs 129625.03, 177984.32 and 195695.54. The penalized SDP relaxation gives rise to the best minimum among these local minima for $\varepsilon \simeq 0.2$.

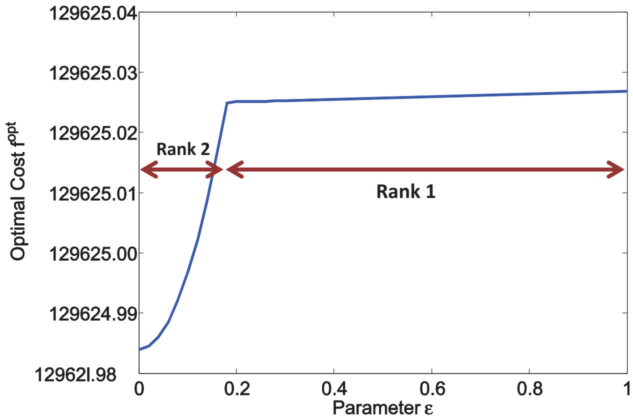


Figure 3.7: The modified 118-bus system

To demonstrate the merit of the penalized SDP relaxation, we generated more than 7000 cost functions for IEEE 14, 30 and 57-bus systems with the network parameters obtained from MATPOWER test data files—including constraints limiting the apparent power for each line—where the cost coefficients c_k's were chosen from the discrete set $\{1, 2, 3, 4\}$. We then conducted the above experiment on all these generated OPF problems and tabulated the findings in the supplement `http://www.columbia.edu/~rm3122/research.html`. The results are encapsulated below:

- There were many cases for which the penalized SDP relaxation with $\varepsilon = 0$ had a rank-1 solution. This means that the unpenalized SDP relaxation was able to find a global solution of OPF in many cases.

- There were cases for which the numerical solution of the SDP relaxation was not rank 1, but the penalized SDP relaxation produced a rank-1 solution for a very small number ε. For example, this occurs for the IEEE-30 bus system with $c_k = 1$ for which $\mathbf{W}^{\text{opt}}$ has two non-zero eigenvalues 32.3437 and 0.0112, while $\mathbf{W}_\epsilon^{\text{opt}}$ has only one nonzero eigenvalue equal to 32.3433 for $\epsilon = 10^{-5}$. Under this circumstance, the SDP relaxation has multiple solutions, including a hidden rank-1 solution that can be obtained through the penalized SDP relaxation with a very small ϵ.

- In many cases, there exists an $\varepsilon_1 > 0$ such that the penalized SDP relaxation always yields a rank-1 solution for every $\varepsilon > \varepsilon_1$ and that there exists an interval $(\varepsilon_1, \varepsilon_2)$ in which the resulting cost changes very slightly (as shown in Figures 3.6 and 3.7). Although the cost can increase dramatically for $\varepsilon > \varepsilon_2$, like the case shown in Figure 3.6(c), we observed that the interval $(\varepsilon_1, \varepsilon_2)$ of interest is relatively large and an ε inside that interval can be spotted with 2 or 3 trial and errors.

- Whenever the SDP relaxation failed to work for each of the generated cases (counting over 7000 OPFs), the penalized SDP relaxation always had a rank-1 solution with a carefully chosen ε. In addition, the recovered near-optimal solution of OPF almost satisfied the KKT conditions (subject to some small error) in 100%, 96.6% and 95.8% of cases for IEEE 14, 30 and 57-bus systems, respectively. This means that these sub-optimal points would be almost globally optimal.

3.6 Summary

It has been shown recently that the semidefinite programming (SDP) can be used to find a global solution of the OPF problem for IEEE benchmark power systems. Although the exactness of the SDP relaxation for acyclic networks has been successfully proved, a recent work has witnessed the failure of this technique for a three-bus cyclic network. Inspired by this observation, this chapter is concerned with understanding the limitations of the SDP relaxation for cyclic power networks. First, it is shown that the injection region of a weakly-cyclic network with cycles of size 3 is convex in the lossless case and has a convex Pareto front in the lossy case. It is then proved that the SDP relaxation works for this type of network. This result implies that the failure of the SDP relaxation for a three-bus network recently reported in the literature can be fixed by utilizing a good modeling of the line capacity. As a more general result, it is then shown that whenever the SDP relaxation does not work, it is expected to have a low-rank solution in practice. Finally, a penalized SDP relaxation is proposed from which a near-global solution of OPF may be recovered. The performance of this method is tested on IEEE systems with over 7000 different cost functions.

Chapter 4

Promises of Conic Relaxation for Contingency-Constrained Optimal Power Flow Problem

This chapter is concerned with the security-constrained optimal power flow (SCOPF) problem, where each contingency corresponds to the outage of an arbitrary number of lines and generators. The problem is studied by means of a convex relaxation, named semidefinite program (SDP). The existence of a rank-1 SDP solution guarantees the recovery of a global solution of SCOPF. We prove that the rank of the SDP solution is upper bounded by the treewidth of the power network plus one, which is perceived to be small in practice. We then propose a decomposition method to reduce the computational complexity of the relaxation. In the case where the relaxation is not exact, we develop a graph-theoretic convex program to identify the problematic lines of the network and incorporate the loss over those lines into the objective as a penalization (regularization) term, leading to a penalized SDP problem. We perform several simulations on large-scale benchmark systems and verify that the global minima are at most 1% away from the feasible solutions obtained from the proposed penalized relaxation.

4.1 Introduction

The classical optimal power flow (OPF) problem aims to find a steady-state operating point of a power system that minimizes a desirable cost function, e.g. power loss or generation cost, and satisfies network and physical constraints on loads, powers, voltages and line flows [Momoh, 2001]. The OPF problem is not only non-convex but also NP-hard, due to its possible reduction to the (0,1)-quadratic optimization. Started by the work [Carpentier, 1962] in 1962, many of the existing optimization techniques have been studied for the OPF problem, leading to algorithms based on linear programming, Newton Raphson, quadratic programming, nonlinear programming, Lagrange relaxation, interior point method, artificial intelligence, artificial neural network, fuzzy logic, genetic algorithm, evolutionary programming and particle swarm optimization [Pandya and Joshi, 2008]. Due to the non-convexity of OPF, these algorithms are not robust, lack performance guarantees, and may not find a global optimum.

Followed by the idea proposed in [Bai *et al.*, 2008] and by exploiting the physical properties of transmission lines, it has been argued in the series of work [Lavaei and Low, 2010; Lavaei and Low, 2012; Sojoudi and Lavaei, 2014; Lavaei *et al.*, 2012; Low, 2014; Zhang and Tse, 2013; Bose *et al.*, 2011] that the classical OPF problem corresponding to a practical power system may be convexified and solved efficiently through a semidefinite programming (SDP) relaxation. In particular, the paper [Sojoudi and Lavaei, 2012] shows that the SDP relaxation is exact in two cases under certain technical assumptions: (i) for acyclic networks, (ii) for cyclic networks after relaxing the angle constraints. However, the SDP relaxation is not always exact for a general mesh network [Lesieutre *et al.*, 2011; Bukhsh *et al.*, 2013; Gopalakrishnan *et al.*, 2011; Phan, 2012]. This issue has been discussed extensively in the literature and several test cases have been contrived that witness the failure of SDP relaxation in obtaining a global optimal solution of the OPF problem [Bukhsh *et al.*, 2013; Louca *et al.*, 2014]. To ameliorate the issue, we have shown in chapter 3 that: (i) the exactness of the SDP relaxation depends on the formulation of the line capacity constraints, and (ii) the penalization of total reactive loss may enable the recovery of a near-global solution (i.e., a solution that is measurably close to a global minimum) for modest-sized systems (as verified in over 7000 simulations).

The major drawback of representing the optimal power flow problem as a semidefinite program is the requirement of defining a square matrix variable, which makes the number of scalar variables of the problem quadratic with respect to the number of network buses. This may yield a very high-dimensional SDP problem for a real-world network. To address this issue, the papers [Lam *et al.*, 2012b; Zhang *et al.*, 2015; Molzahn *et al.*, 2013] have leveraged the sparsity of power networks in order to break down the large-scale semidefinite constraint into small-sized constraints. Similarly, the papers [Andersen *et al.*, 2014], [Jabr, 2012] and [Molzahn and Hiskens, 2015] have exploited the general technique proposed in [Fukuda *et al.*, 2001; Nakata *et al.*, 2003] to reduce the complexity of the SDP relaxation of the OPF problem. The simulations performed in those papers suggest that the SDP relaxation would fail to work properly for large-scale systems [Molzahn *et al.*, 2013].

Although OPF is a fundamental problem studied extensively in the literature for power systems, a real-world power flow optimization is based on a set of coupled OPFs with a variety of constraints and variables. The latter problem is named security-constrained OPF (SCOPF) [Capitanescu *et al.*, 2007; Wood and Wollenberg, 1996]. The SCOPF problem is important in practice, since independent system operators tend to design an operating point that satisfies the demand and network constraints not only under normal operation but also under pre-specified contingencies such as line outages. Depending on the network characteristics, one may adopt preventive or corrective approaches for the SCOPF problem. In the preventive formulation of SCOPF, the under-design state of each generator (e.g., production level) is considered identical for the pre- and post-contingency scenarios. This reflects the fact that mechanical facilities may not be able to respond to the changes in the network fast. In the corrective approach, limited changes in certain control parameters are permitted after the network experiences a fault. SCOPF is more challenging than the conventional OPF problem for two reasons. First, the size of the optimization could be prohibitive, depending on the number of contingencies. Second, SCOPF is obtained by coupling a group of non-convex OPF problems associated with different contingencies and therefore its non-convexity would be much higher than an individual OPF problem. The purpose of this work is to propose an efficient computational method that can be applied to not only OPF but also SCOPF.

In this chapter, we study the SCOPF problem—as a general version of OPF—through a convex relaxation. First, we propose an SDP relaxation for this problem. The existence of a rank-1 SDP solution guarantees the recovery of a global solution of SCOPF. We prove that the relaxation has a matrix solution whose rank is at most the treewidth of the pre-contingency network plus one. The treewidth of real-world networks is perceived to be small due to their (almost) planarity and sparsity [Fomin and Thilikos, 2006]. For example, the treewidth of the graph corresponding to a peak hour setup of a Polish system with over 3000 buses is less than 25. Second, we reduce the computational complexity of the SDP problem using a tree decomposition method to arrive at a decomposed SDP relaxation with a set of small-sized SDP matrices as opposed to a full-scale SDP matrix. We show that the full-scale SDP relaxation has a solution whose rank is upper bounded by the ranks of the small-sized matrices of the decomposed SDP relaxation. By working on the ranks of these small matrices, we propose a technique to identify the problematic lines of the network for each contingency that may contribute to the inexactness of the SDP relaxation for SCOPF. This diagnosis method may enable us to develop a heuristic method, named penalized SDP relaxation, to find a near-global solution of the problem by penalizing the loss over the problematic lines for each contingency. Note that a certain line may be problematic with respect to one contingency and not problematic with respect to another contingency. It is suggested to define the penalty term as a summation of all loss functions for problematic lines under each contingency. Note that a uniform penalty—consisting of losses over all lines for all contingencies—also work for all test systems studied in this chapter, but the resulting SDP solution would have a lower global optimality guarantee compared to the case where the loss over only problematic lines is penalized. We test our method on several benchmark systems with as high as 3000 buses and find a solution with a global optimality guarantee of at least 99% for each case.

Notations: $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}_n$ denote the sets of real numbers, complex numbers, and $n \times n$ Hermitian matrices, respectively. The m by n rectangular identity matrix whose (i, j) entry is equal to the Kronecker delta δ_{ij} is denoted by $\mathbf{I}_{m\times n}$. The notations $\mathrm{Re}\{\mathbf{W}\}$, $\mathrm{Im}\{\mathbf{W}\}$, and $\mathrm{rank}\{\mathbf{W}\}$ denote the real part, imaginary part, and rank of a scalar/matrix $\mathbf{W}$, respectively. The notation $\mathbf{W} \succeq 0$ means that $\mathbf{W}$ is Hermitian and positive semidefinite. The notation $\measuredangle x$ denotes the angle of a complex number x. The notation "i" is reserved for the imaginary unit. The superscripts $(\cdot)^*$ and $(\cdot)^{\mathrm{T}}$ represent the conjugate transpose and transpose operators, respectively. Given a matrix $\mathbf{W}$, its (l, m) entry is denoted as W_{lm}. The subscript $(\cdot)_{\mathrm{opt}}$ is used to show the optimal value of an optimization parameter. Given a matrix $\mathbf{M}$, its Moore Penrose pseudoinverse is denoted as $\mathbf{M}^+$. Given a simple graph $\mathcal{H}$, its vertex and edge sets are denoted by $\mathcal{V}_{\mathcal{H}}$ and $\mathcal{E}_{\mathcal{H}}$, respectively. Given two sets $\mathcal{S}_1$ and $\mathcal{S}_2$, the notation $\mathcal{S}_1 \backslash \mathcal{S}_2$ denotes the set of all elements of $\mathcal{S}_1$ that do not exist in $\mathcal{S}_2$. Given a scalar m and a real-valued set $\mathcal{S}$, define $\mathcal{S} + m$ as a set obtained by adding m to every element of $\mathcal{S}$. Given a Hermitian matrix $\mathbf{M}$ and two sets of natural numbers $\mathcal{S}_1$ and $\mathcal{S}_2$, define $\mathbf{M}\{\boldsymbol{\mathcal{S}_1}, \boldsymbol{\mathcal{S}_2}\}$ as a submatrix of $\mathbf{M}$ obtained through two operations: (i) removing all rows of $\mathbf{M}$ whose indices do not belong to $\mathcal{S}_1$, and (ii) removing all columns of $\mathbf{M}$ whose indices do not belong to $\mathcal{S}_2$. For instance, $\mathbf{M}\left\{\{\mathbf{1},\mathbf{2}\},\{\mathbf{2},\mathbf{3}\}\right\}$ is a 2×2 matrix with the entries $M_{12}, M_{13}, M_{22}, M_{23}$.

4.2 Problem Formulation

Consider a power network with the set of buses $\mathcal{N} := \{1, 2, ..., n\}$ and the set of flow lines $\mathcal{L} \subseteq \mathcal{N}\times\mathcal{N}$. With no loss of generality, each line $(l, m) \in \mathcal{L}$ of the network is modeled as a series admittance y_{lm}. Suppose that a known constant-power load with the complex value $S_{D_k} = P_{D_k} + Q_{D_k}\mathrm{i}$ is connected to bus $k \in \mathcal{N}$, where $P_{D_k}, Q_{D_k} \in \mathbb{R}$. Given a nonnegative integer c, consider a set of c contingencies, where each contingency corresponds to an arbitrary number of pre-specified line/generator outages. In this work, we model a line outage by removing the line from the base case and model a generator outage by enforcing its output to be zero. Define $\mathcal{C} := \{0, 1, \ldots, c\}$ as the set of all pre- and post-contingencies, where the base case is treated as contingency 0. Define $\mathcal{L}^{(0)} = \mathcal{L}$, and $\mathcal{L}^{(r)}$ as the set of lines of the network under contingency $r \in \{1, 2, ..., c\}$.

Consider a contingency scenario $r \in \mathcal{C}$. Assume that a generator is connected to each bus $k \in \mathcal{N}$, whose unknown complex output is denoted as $S^{(r)}_{G_k} = P^{(r)}_{G_k} + Q^{(r)}_{G_k}\mathrm{i}$. Let $f^{(r)}_k(\cdot)$ be a convex function representing the generation cost for generator k in the contingency case $r \in \mathcal{C}$. The unknown complex voltage at bus $k \in \mathcal{N}$ is denoted as $V^{(r)}_k$. Furthermore, define $S^{(r)}_{lm} = P^{(r)}_{lm} + Q^{(r)}_{lm}\mathrm{i}$ as the unknown complex power transferred from bus $l \in \mathcal{N}$ to the rest of the network through the line $(l, m) \in \mathcal{L}^{(r)}$. Define:

$$\begin{aligned}
\mathbf{P}^{(r)}_G &\triangleq \left[P^{(r)}_{G_1}, \ldots, P^{(r)}_{G_n}\right]^{\mathrm{T}}, & \mathbf{Q}^{(r)}_G &\triangleq \left[Q^{(r)}_{G_1}, \ldots, Q^{(r)}_{G_n}\right]^{\mathrm{T}},\\
\mathbf{P}_D &\triangleq \left[P_{D_1}, \ldots, P_{D_n}\right]^{\mathrm{T}}, & \mathbf{Q}_D &\triangleq \left[Q_{D_1}, \ldots, Q_{D_n}\right]^{\mathrm{T}},\\
\mathbf{V}^{(r)} &\triangleq \left[V^{(r)}_1, \ldots, V^{(r)}_n\right]^{\mathrm{T}}.
\end{aligned}$$

Given the known vectors $\mathbf{P}_D$ and $\mathbf{Q}_D$, SCOPF minimizes the generation cost over the unknown parameters $\mathbf{V}^{(r)}$, $\mathbf{P}^{(r)}_G$ and $\mathbf{Q}^{(r)}_G$ for $r = 0, 1, ..., c$ subject to the power balance equations at all buses and certain network constraints. SCOPF is formalized below.

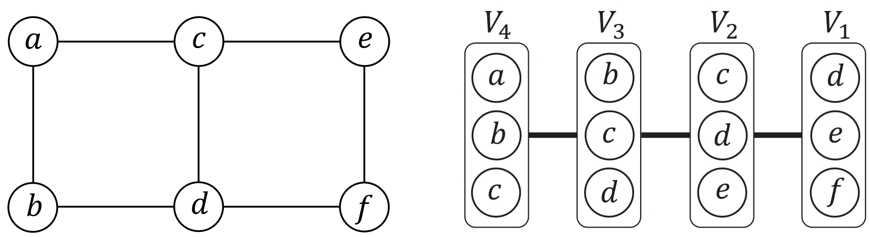


Figure 4.1: A minimal tree decomposition for a ladder

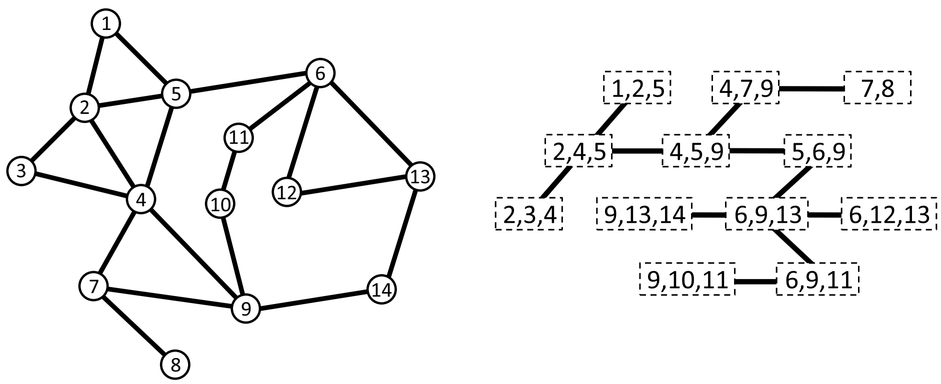


Figure 4.2: The IEEE 14-bus test case and its minimal tree decomposition

SCOPF problem: Minimize

$$\sum_{r \in \mathcal{C}} \sum_{k \in \mathcal{N}} f_k^{(r)} \left(P_{G_k}^{(r)} \right) \tag{4.1}$$

over the variables $\mathbf{P}_G^{(0)}, \mathbf{P}_G^{(1)}, \ldots, \mathbf{P}_G^{(c)} \in \mathbb{R}^n$, $\mathbf{Q}_G^{(0)}, \mathbf{Q}_G^{(1)}, \ldots, \mathbf{Q}_G^{(c)} \in \mathbb{R}^n$ and $\mathbf{V}^{(0)}, \mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(c)} \in \mathbb{C}^n$,

subject to

$$P_{G_k}^{(r)} - P_{D_k} = \sum_{l \in \mathcal{N}_k^{(r)}} \mathrm{Re}\left\{V_k^{(r)} \left(V_k^{(r)} - V_l^{(r)}\right)^* y_{kl}^*\right\} \tag{4.2a}$$

$$Q_{G_k}^{(r)} - Q_{D_k} = \sum_{l \in \mathcal{N}_k^{(r)}} \mathrm{Im}\left\{V_k^{(r)} \left(V_k^{(r)} - V_l^{(r)}\right)^* y_{kl}^*\right\} \tag{4.2b}$$

$$P_{k;\,\min}^{(r)} \leq P_{G_k}^{(r)} \leq P_{k;\,\max}^{(r)} \tag{4.2c}$$

$$Q_{k;\,\min}^{(r)} \leq Q_{G_k}^{(r)} \leq Q_{k;\,\max}^{(r)} \tag{4.2d}$$

$$V_{k;\,\min}^{(r)} \leq |V_k^{(r)}| \leq V_{k;\,\max}^{(r)} \tag{4.2e}$$

$$|\measuredangle V_l^{(r)} - \measuredangle V_m^{(r)}| \leq \theta_{lm;\,\max}^{(r)} \tag{4.2f}$$

$$\left(P_{lm}^{(r)}\right)^2 + \left(Q_{lm}^{(r)}\right)^2 \leq \left(S_{lm;\,\max}^{(r)}\right)^2 \tag{4.2g}$$

$$|P_{G_k}^{(r)} - P_{G_k}^{(0)}| \leq \Delta P_{k;\,\max}^{(r)} \tag{4.2h}$$

$$|Q_{G_k}^{(r)} - Q_{G_k}^{(0)}| \leq \Delta Q_{k;\,\max}^{(r)} \tag{4.2i}$$

$$\left||V_k^{(r)}|^2 - |V_k^{(0)}|^2\right| \leq \left(\Delta V_{k;\,\max}^{(r)}\right)^2 \tag{4.2j}$$

for every $k \in \mathcal{N}$, $r \in \mathcal{C}$ and $(l,m) \in \mathcal{L}^{(r)}$, where

- $\mathcal{N}_k^{(r)}$ denotes the set of all neighboring nodes of bus $k \in \mathcal{N}$ for contingency r.
- (4.2a) and (4.2b) are the power balance equations accounting for the conservation of energy at bus k.
- (4.2c), (4.2d) and (4.2e) restrict the active power, reactive power and voltage magnitude at bus k for contingency r, given the limits $P_{k;\,\min}^{(r)}$, $P_{k;\,\max}^{(r)}$, $Q_{k;\,\min}^{(r)}$, $Q_{k;\,\max}^{(r)}$, $V_{k;\,\min}^{(r)}$ and $V_{k;\,\max}^{(r)}$.
- Each line $(l,m) \in \mathcal{L}^{(r)}$ of the network is subject to two capacity constraints. The constraint (4.2f) restricts the voltage phase difference of buses l and m with the limit $\theta_{lm;\,\max}^{(r)} \in [0, 90°]$. Moreover, (4.2g) restricts the apparent power flow $|S_{lm}^{(r)}|$ with the line capacity $S_{lm;\,\max}^{(r)}$.
- Given the limits $\Delta P_{k;\,\max}^{(r)}$, $\Delta Q_{k;\,\max}^{(r)}$ and $\Delta V_{k;\,\max}^{(r)}$, the constraints (4.2h), (4.2i) and (4.2j) ensure that the potentially controllable parameters $P_{G_k}^{(r)}$, $Q_{G_k}^{(r)}$ and $|V_k^{(r)}|$ vary within permissible ranges after a contingency occurs.

Note that a generator outage can be modeled as

$$P^{(r)}_{k;\min} = P^{(r)}_{k;\max} = Q^{(r)}_{k;\min} = Q^{(r)}_{k;\max} = 0. \tag{4.3}$$

4.2.1 Convex Relaxation for SCOPF

The SCOPF problem includes quadratic constraints such as (4.2a) and (4.2b). Nevertheless, all constraints of (4.2) can be expressed linearly in terms of the entries of the matrix variable $\mathbf{W}$ defined as $\mathbf{W} \triangleq \mathbf{V}\mathbf{V}^*$, where $\mathbf{V}$ denotes a column vector obtained by stacking the voltage vectors $\mathbf{V}^{(0)}, \mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(c)}$. On the other hand, the variable $\mathbf{V}$ can be dropped from the optimization problem by equivalently replacing the consistency constraint $\mathbf{W} = \mathbf{V}\mathbf{V}^*$ with two new constraints: (i) $\mathbf{W} \succeq 0$, and (ii) $\text{rank}\{\mathbf{W}\} = 1$. Observe that Constraint (ii) is the only non-convex constraint of the reformulated SCOPF problem. Motivated by this fact, the SDP relaxation of SCOPF is defined as the optimization problem reformulated in terms of $\mathbf{W}$ under the additional constraint $\mathbf{W} \succeq 0$ without incorporating the rank constraint $\text{rank}\{\mathbf{W}\} = 1$. To formalize this relaxation, let $\mathbf{W}^{(r)} \in \mathbb{H}_n$ denote the submatrix of $\mathbf{W}$ in the intersection of rows $rn+1, ..., rn+n$ with columns $rn+1, ..., rn+n$.

Relaxed SCOPF: Minimize

$$\sum_{r\in\mathcal{C}} \sum_{k\in\mathcal{N}} f^{(r)}_k \left(P^{(r)}_{G_k}\right) \tag{4.4}$$

over the parameters $\mathbf{P}^{(0)}_G, \mathbf{P}^{(1)}_G, \ldots, \mathbf{P}^{(c)}_G \in \mathbb{R}^n$, $\mathbf{Q}^{(0)}_G, \mathbf{Q}^{(1)}_G, \ldots, \mathbf{Q}^{(c)}_G \in \mathbb{R}^n$ and $\mathbf{W} \in \mathbb{H}_{n(c+1)}$, subject to

$$P_{G_k}^{(r)} - P_{D_k} = \sum_{l \in \mathcal{N}_k^{(r)}} \mathrm{Re}\left\{\left(W_{kk}^{(r)} - W_{kl}^{(r)}\right) y_{kl}^*\right\} \tag{4.5a}$$

$$Q_{G_k}^{(r)} - Q_{D_k} = \sum_{l \in \mathcal{N}_k^{(r)}} \mathrm{Im}\left\{\left(W_{kk}^{(r)} - W_{kl}^{(r)}\right) y_{kl}^*\right\} \tag{4.5b}$$

$$P_{k;\min}^{(r)} \leq P_{G_k}^{(r)} \leq P_{k;\max}^{(r)} \tag{4.5c}$$

$$Q_{k;\min}^{(r)} \leq Q_{G_k}^{(r)} \leq Q_{k;\max}^{(r)} \tag{4.5d}$$

$$\left(V_{k;\min}^{(r)}\right)^2 \leq W_{kk}^{(r)} \leq \left(V_{k;\max}^{(r)}\right)^2 \tag{4.5e}$$

$$\mathrm{Im}\left\{W_{lm}^{(r)}\right\} \leq \mathrm{Re}\left\{W_{lm}^{(r)}\right\} \tan\left(\theta_{lm;\max}^{(r)}\right) \tag{4.5f}$$

$$\left(P_{lm}^{(r)}\right)^2 + \left(Q_{lm}^{(r)}\right)^2 \leq \left(S_{lm;\max}^{(r)}\right)^2 \tag{4.5g}$$

$$|P_{G_k}^{(r)} - P_{G_k}^{(0)}| \leq \Delta P_{k;\max}^{(r)} \tag{4.5h}$$

$$|Q_{G_k}^{(r)} - Q_{G_k}^{(0)}| \leq \Delta Q_{k;\max}^{(r)} \tag{4.5i}$$

$$\left|W_{kk}^{(r)} - W_{kk}^{(0)}\right| \leq \left(\Delta V_{k;\max}^{(r)}\right)^2 \tag{4.5j}$$

$$\mathbf{W} \succeq 0 \tag{4.5k}$$

for every $k \in \mathcal{N}$, $r \in \mathcal{C}$ and $(l, m) \in \mathcal{L}^{(r)}$.

The relaxed SCOPF is alternatively referred to as **SDP relaxation** henceforth. Let f_{opt} and $f_{r-\text{opt}}$ denote the optimal objective values of the SCOPF and relaxed SCOPF. As shown in [Lavaei and Low, 2012] and [Lavaei, 2011], the relaxed SCOPF is equivalent to the dual of the dual of the SCOPF problem and therefore it provides a lower bound $f_{r-\text{opt}}$ on the globally minimum solution f_{opt} of the original problem (4.2). Hence, $f_{\text{opt}} - f_{r-\text{opt}}$ represents the duality gap for the non-convex SCOPF problem, which is not necessarily zero or even small [Lesieutre *et al.*, 2011; Bukhsh *et al.*, 2013; Louca *et al.*, 2014]. Zero duality gap is a favorable property because of two reasons: (i) it guarantees the existence of a rank-1 solution $\mathbf{W}_{\text{opt}}$, (ii) it attains the optimal objective value of the SCOPF problem. If $\mathbf{W}_{\text{opt}}$ is obtained numerically, an optimal vector of voltage phasors $\mathbf{V}_{\text{opt}}$ can then be constructed through the decomposition $\mathbf{W}_{\text{opt}} = \mathbf{V}_{\text{opt}}\mathbf{V}_{\text{opt}}^*$. It has been shown in [Lavaei, 2011] that whenever the duality gap of the classical OPF problem is zero for a specific power network, the SCOPF problem also possesses zero duality gap, leading to the presence of a rank-1 solution for the relaxed SCOPF. As investigated in [Gan *et al.*, 2012] and [Sojoudi and Lavaei, 2012], the duality gap for the OPF problem (without any contingency scenarios) is highly correlated with the topology of the network. In addition, the gap heavily depends on the mathematical formulation of the line capacity constraints as shown in chapter 3. Since a power network has a sparse graph in general, the relaxed SCOPF problem may have infinitely many solutions. As an extreme case, the duality gap could be zero and yet there exist a set of rank-1 and higher-rank solutions for the relaxed problem. To alleviate this issue, the paper [Lavaei and Low, 2012] suggests adding a small resistance (10^{-5} per unit) to every ideal transformer with zero resistance.

Using a graph-theoretic approach combined with the SDP relaxation, we aim to study three problems:

- Since the dimension of the matrix variable $\mathbf{W}$ is prohibitive for a large-scale network, how can the computational complexity of the relaxed SCOPF be reduced?
- What is the rank of an optimal solution $\mathbf{W}_{\text{opt}}$ of the relaxed SCOPF and how does it relate to the topology of the power grid?
- If the rank of $\mathbf{W}_{\text{opt}}$ is not 1, how can a near-global solution be recovered for the non-convex SCOPF problem?

4.3 Low-rank SDP Solutions

In this section, the objective is twofold. First, the computational complexity of the relaxed SCOPF will be reduced. Second, the rank of its lowest-rank solution will be studied.

4.3.1 Reduction of Computational Complexity

Definition 22 (Treewidth)**.** *Given a graph $\mathcal{H} = (\mathcal{V}_{\mathcal{H}}, \mathcal{E}_{\mathcal{H}})$, a tree $\mathcal{T}$ is called a tree decomposition of $\mathcal{H}$ if it satisfies the following properties:*

1. *Every node of $\mathcal{T}$ corresponds to and is identified by a subset of $\mathcal{V}_{\mathcal{H}}$.*
2. *Every vertex of $\mathcal{H}$ is a member of at least one node of $\mathcal{T}$.*
3. *$\mathcal{T}_k$ is a connected graph for every $k \in \mathcal{V}_{\mathcal{H}}$, where $\mathcal{T}_k$ denotes the subgraph of $\mathcal{T}$ induced by all nodes of $\mathcal{T}$ containing the vertex k of $\mathcal{H}$.*

4. *If $(i,j) \in \mathcal{E}_{\mathcal{H}}$, then the subgraphs $\mathcal{T}_i$ and $\mathcal{T}_j$ have at least one node in common.*

Each node of $\mathcal{T}$ is a bag (collection) of vertices of $\mathcal{H}$ and hence it is referred to as **bag**. *The width of $\mathcal{T}$ is the cardinality of its biggest bag minus one. The treewidth of $\mathcal{H}$ is the minimum width over all possible tree decompositions of $\mathcal{H}$ and is denoted by* $\mathrm{tw}(\mathcal{H})$.

Note that the treewidth of a tree is equal to 1. Figure 6.1 shows a graph $\mathcal{H}$ with 6 vertices named a, b, c, d, e, f, together with its minimal tree decomposition $\mathcal{T}$. Every node of $\mathcal{T}$ is a set containing three members of $\mathcal{V}_{\mathcal{H}}$. The width of this decomposition is therefore equal to 2. The graph of the IEEE 14-bus system and its minimal tree decomposition are depicted in Figure 4.2. As shown in Table 4.1, the treewidth of IEEE systems and various setups of Polish systems with as high as 3000 buses is at most 24. This empirical evidence signifies that real-world power grids may have a small treewidth, which is leveraged in this work to solve the SCOPF problem.

Definition 23 (Sparsity graph)**.** *The sparsity graph of the relaxed SCOPF problem is defined as a graph with $n(c+1)$ vertices such that (i,j) is an edge of the graph whenever $i \neq j$, $i,j \in \{1,2,...,n(c+1)\}$ and W_{ij} appears in either of the constraints* (4.5a) *and* (4.5b) *with a nonzero coefficient.*

Consider a tree decomposition of the power network in the pre-contingency case and denote its bags (nodes) as $\mathcal{J}_1^{(0)}, \mathcal{J}_2^{(0)}, ..., \mathcal{J}_p^{(0)} \subseteq \mathcal{N}$.

Theorem 17. *The following statements hold:*

i) The sparsity graph of the relaxed SCOPF problem has a tree decomposition with $p(c+1)$ bags given by the set:

$$\left\{ \mathcal{J}_m^{(0)} + nr \;\middle|\; m = 1,...,p \quad \textit{and} \quad r = 0,...,c \right\} \tag{4.6}$$

ii) The optimal objective value of the relaxed SCOPF problem does not change if its constraint $\mathbf{W} \succeq 0$ is replaced by

$$\mathbf{W}\left\{ \mathcal{J}_m^{(0)} + nr, \mathcal{J}_m^{(0)} + nr \right\} \succeq 0 \tag{4.7}$$

or equivalently

$$\mathbf{W}^{(r)}\left\{ \mathcal{J}_m^{(0)}, \mathcal{J}_m^{(0)} \right\} \succeq 0 \tag{4.8}$$

for every $r \in \mathcal{C}$ and $m \in \{1,\ldots,p\}$.

Proof. The proof of Part (i) follows from the fact that the sparsity graph of the relaxed SCOPF problem is composed of $c+1$ disconnected components, each corresponding to one of the contingencies $0, 1, 2..., c$. This is due to the fact that the constraints of the SCOPF problem can all be described in terms of only those entries of $\mathbf{W}$ that appear in one of the submatrices $\mathbf{W}^{(0)}, ..., \mathbf{W}^{(c)}$. Part (ii) is a direct consequence of Part (i) and the chordal theorem in [Grone *et al.*, 1984]. □

Define **decomposed relaxed SCOPF** as a convex optimization obtained from the relaxed SCOPF by replacing $\mathbf{W} \succeq 0$ with the constraints $\mathbf{W}^{(r)}\left\{\mathcal{J}_m^{(0)}, \mathcal{J}_m^{(0)}\right\} \succeq 0$ for every $r \in \mathcal{C}$ and $m \in \{1, \dots, p\}$. Theorem 17 reduces the computational cost of the SDP relaxation dramatically for a large-scale system with a relatively small treewidth. Note that many entries of the matrix variable $\mathbf{W}$ may not appear in the objective or constraints of the decomposed relaxed SCOPF, and those redundant entries can be eliminated. For example, the relaxed OPF for a Polish system has about 9,000,000 scalar variables, while the decomposed relaxed OPF has only about 100,000 parameters. As will be illustrated later, this enables us to solve a large-scale problem efficiently.

4.3.2 Existence of Low-rank Solutions

Let $\mathbf{W}_{\text{ref}} \in \mathbb{H}_{n(c+1)}$ denote an arbitrary solution of the relaxed SCOPF or decomposed relaxed SCOPF. Note that if $\mathbf{W}_{\text{ref}}$ corresponds to the decomposed problem, its redundant entries may not have been found by the numerical algorithm and are regarded as "missing". The following question arises: is it possible to fine-tune the entries of $\mathbf{W}_{\text{ref}}$ or design its missing entries to arrive at a different, but lower rank, solution of the (decomposed) relaxed problem? It is known that there exists a polynomial-time algorithm to fill a partially-known real-valued matrix in such a way that the rank of the resulting matrix becomes equal to the highest rank among all bags [Laurent, 2001; Laurent and Varvitsiotis, 2014]. We extend this result to the complex domain by proposing an iterative algorithm that transforms $\mathbf{W}_{\text{ref}}$ into a solution $\mathbf{W}_{\text{opt}}$ whose rank is upper bounded by the treewidth of the network plus one. To introduce our algorithm, consider a tree decomposition $\mathcal{T}_{\mathcal{C}}$ of the sparsity graph of the relaxed SCOPF problem with the bags specified in (4.6). For simplicity, we name the bags as $\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_{p(c+1)}$.

Matrix completion algorithm:

1. Set $\mathcal{T}' := \mathcal{T}_{\mathcal{C}}$ and $\mathbf{W} := \mathbf{W}_{\text{ref}}$.

2. If $\mathcal{T}'$ has a single node, then consider $\mathbf{W}_{\text{opt}}$ as $\mathbf{W}$ and terminate; otherwise continue to the next step.

3. Choose a pair of bags $\mathcal{J}_x, \mathcal{J}_y$ of $\mathcal{T}'$ such that $\mathcal{J}_x$ is a leaf of $\mathcal{T}'$ and $\mathcal{J}_y$ is its unique neighbor.

4. Define

$$\mathbf{A} \triangleq \mathbf{W}\{\mathcal{J}_x \cap \mathcal{J}_y, \mathcal{J}_x \cap \mathcal{J}_y\} \tag{4.9a}$$

$$\mathbf{B}_x \triangleq \mathbf{W}\{\mathcal{J}_x \setminus \mathcal{J}_y, \mathcal{J}_x \cap \mathcal{J}_y\} \tag{4.9b}$$

$$\mathbf{B}_y \triangleq \mathbf{W}\{\mathcal{J}_y \setminus \mathcal{J}_x, \mathcal{J}_x \cap \mathcal{J}_y\} \tag{4.9c}$$

$$\mathbf{X} \triangleq \mathbf{W}\{\mathcal{J}_x \setminus \mathcal{J}_y, \mathcal{J}_x \setminus \mathcal{J}_y\} \in \mathbb{C}^{d_x \times d_x} \tag{4.9d}$$

$$\mathbf{Y} \triangleq \mathbf{W}\{\mathcal{J}_y \setminus \mathcal{J}_x, \mathcal{J}_y \setminus \mathcal{J}_x\} \in \mathbb{C}^{d_y \times d_y} \tag{4.9e}$$

$$\mathbf{S}_x \triangleq \mathbf{X} - \mathbf{B}_x \mathbf{A}^{+} \mathbf{B}_x^* = \mathbf{Q}_x \mathbf{\Lambda}_x \mathbf{Q}_x^* \tag{4.9f}$$

$$\mathbf{S}_y \triangleq \mathbf{Y} - \mathbf{B}_y \mathbf{A}^{+} \mathbf{B}_y^* = \mathbf{Q}_y \mathbf{\Lambda}_y \mathbf{Q}_y^* \tag{4.9g}$$

where $\mathbf{Q}_x \mathbf{\Lambda}_x \mathbf{Q}_x^*$ and $\mathbf{Q}_y \mathbf{\Lambda}_y \mathbf{Q}_y^*$ denote the eigenvalue decompositions of $\mathbf{S}_x$ and $\mathbf{S}_y$ with the diagonals of $\mathbf{\Lambda}_x$ and $\mathbf{\Lambda}_y$ arranged in descending order. Then, update a part of $\mathbf{W}$ as follows:

$$\mathbf{W}\{\mathcal{J}_y \setminus \mathcal{J}_x, \mathcal{J}_x \setminus \mathcal{J}_y\} := \mathbf{B}_y \mathbf{A}^{+} \mathbf{B}_x^* + \mathbf{Q}_y \sqrt{\mathbf{\Lambda}_y}\ \mathbf{I}_{d_y \times d_x} \sqrt{\mathbf{\Lambda}_x}\ \mathbf{Q}_x^*$$

and update $\mathbf{W}\{\mathcal{J}_x \setminus \mathcal{J}_y, \mathcal{J}_y \setminus \mathcal{J}_x\}$ accordingly to preserve the Hermitian property of $\mathbf{W}$.

5. Update $\mathcal{T}'$ by merging $\mathcal{J}_x$ into $\mathcal{J}_y$, i.e., replace $\mathcal{J}_y$ with $\mathcal{J}_x \cup \mathcal{J}_y$ and then remove $\mathcal{J}_x$ from $\mathcal{T}'$.

6. Go back to step 2.

Theorem 18. *Consider an arbitrary solution $\mathbf{W}_{\text{ref}}$ of the (decomposed) relaxed SCOPF problem. The output of the matrix completion algorithm, denoted as $\mathbf{W}_{\text{opt}}$, is a solution of the relaxed SCOPF problem whose rank is smaller than or equal to:*

$$\max \left\{ \text{rank} \left\{ \mathbf{W}_{\text{ref}}^{(r)}\{\mathcal{J}_m^{(0)}, \mathcal{J}_m^{(0)}\} \right\} \;\middle|\; 1 \leq m \leq p,\ r \in \mathcal{C} \right\}.$$

Proof. See Theorem 4 of Chapter 2 for the proof. □

Note that Theorem 18 is valid for not only relaxed SCOPF but also decomposed relaxed SCOPF. The following three results are the by-products of the above theorem.

Corollary 10. *If the relaxed SCOPF problem is feasible, then it has a solution* $\mathbf{W}_{\text{opt}}$ *whose rank is upper bounded by the treewidth of the power network in the pre-contingency case plus 1.*

Corollary 11. *For every tree network, if the relaxed SCOPF problem is feasible, then it has a solution* $\mathbf{W}_{\text{opt}}$ *whose rank is not greater than* 2.

Corollary 12. *The non-convex SCOPF problem has the same globally optimal value as that of the (decomposed) relaxed SCOPF under the additional constraints*

$$\operatorname{rank}\{\mathbf{W}^{(r)}\{\mathcal{J}_m^{(0)}, \mathcal{J}_m^{(0)}\}\} = 1 \tag{4.10}$$

for every $r \in \mathcal{C}$ *and* $m \in \{1, \ldots, p\}$.

4.4 Recovery of a Near-global Solution

We explored the properties of the decomposed relaxed SCOPF in the preceding section. In this part, we aim to address two problems: (i) how to find a tree decomposition of the power network in order to be able to formulate the decomposed problem, (ii) how to recover a near-global solution of the SCOPF problem through an SDP relaxation.

4.4.1 Tree Decomposition Algorithm

Although the problem of finding the treewidth of an arbitrary graph is known to be NP-hard, there are many efficient algorithms in the literature that provide lower and upper bounds on treewidth [Bodlaender and Koster, 2010; Bodlaender and Koster, 2011]. In what follows, we describe an effective algorithm for finding a tree decomposition that is used in all of the simulations offered in the next section. This algorithm combines the greedy degree and greedy fill-in algorithms presented in [Bodlaender and Koster, 2010] in order to obtain a tree decomposition for a graph with a low maximum clique order.

Consider a graph $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ together with an arbitrary vertex u of this graph. $\delta_\mathcal{H}(u)$ denotes the degree of $u \in \mathcal{V}_\mathcal{H}$. The fill-in of u is defined as the number of edges whose addition to the subgraph formed by the neighbors of u makes the resulting subgraph a clique (complete subgraph). This number is denoted by $\phi_\mathcal{H}(u)$. The vertex u is called simplicial if $\phi_\mathcal{H}(u) = 0$ (i.e., if the neighbors of u are all connected to one another).

Greedy decomposition algorithm:

1. Consider α as an arbitrary constant and define $\mathcal{H}' = \mathcal{H}$. Initialize $\mathcal{T}$ as a graph with no nodes.

2. If $\mathcal{H}'$ has a single vertex, then consider $\mathcal{T}$ as a graph with the single node $\mathcal{V}_{\mathcal{H}}$ and terminate; otherwise continue.

3. Choose a vertex u in $\mathcal{H}'$ according to the following rules:

   - If $\mathcal{H}'$ has a simplicial node, then set u as that vertex.
   - Otherwise, set u as a (not necessarily unique) vertex of $\mathcal{H}'$ that minimizes the function $\phi_{\mathcal{H}'}(u) + \alpha \times \delta_{\mathcal{H}'}(u)$.

4. Define $\mathcal{U}$ as the set of all neighboring vertices of u in $\mathcal{H}'$. Add the bag $\mathcal{U} \cup \{u\}$ to $\mathcal{T}$, and then update the graph $\mathcal{H}'$ by first connecting all vertices in $\mathcal{U}$ to each other and then removing u. Jump to Step 2.

Based on [Bodlaender and Koster, 2010], it is straightforward to show that a set of edges can be added to the nodes of $\mathcal{T}$ to make it a tree decomposition for $\mathcal{H}$. Since the decomposed relaxed SCOPF only needs the bags of $\mathcal{T}$, it is unnecessary to find the edges of the tree decomposition.

4.4.2 Penalization Method

Consider the (decomposed) relaxed SCOPF. Since the mapping from $\mathbf{W}$ to the generating active power levels $P_{G_k}^{(r)}$'s is not bijective, there often exists a space of optimal matrix solutions with disparate ranks. Under such circumstance, commonly-used numerical algorithms would normally find the highest-rank SDP solution, although there may exist a hidden rank-1 solution. To address this issue in the context of OPF, we proposed a method in chapter 3 to penalize the total reactive power generation $\sum_{k\in\mathcal{N}} Q_{G_k}$ in the objective function of the SDP relaxation. This penalty term aims to guide the numerical algorithm by speculating that the right operating point would yield a small reactive loss.

Consider the case where the (decomposed) relaxed SCOPF has no rank-1 solution. Suppose that it is possible to design a convex function $g(\mathbf{W}^{(0)}, ..., \mathbf{W}^{(c)})$ such that the SDP relaxation admits a rank-1 solution whenever the objective of the relaxed SCOPF is replaced by this function. Then, penalizing the objective of the relaxed SCOPF with $\varepsilon \times g(\cdot)$ may lead to an approximate rank-1 SDP solution and subsequently a near-global SCOPF solution, for an appropriate choice of the penalty coefficient ε.

Therefore, the main challenge is to seek a penalization function $g(\cdot)$. The recent literature of compressed sensing suggests a penalty term consisting of a weighted sum of the diagonal entries of $\mathbf{W}$ [Recht *et al.*, 2010a]. However, this idea fails to work for SCOPF since all feasible solutions of the SDP relaxation have similar diagonal values due to a tight voltage control in practice. We propose a different penalty function in this chapter. Consider a positive semidefinite matrix $\mathbf{X}$ with constant (fixed) diagonal entries $X_{11}, \ldots, X_{nn}$ and variable off-diagonal entries. If we maximize a weighted sum of the off-diagonal entries of X with positive weights, then the (l, m) entry of the optimal solution would be $X_{lm} = \sqrt{X_{ll}X_{mm}}$ for all $l, m \in \{1, \ldots, n\}$, in which case $\mathbf{X}$ becomes rank-1. Motivated by this fact, we employ the idea of elevating the off-diagonal entries of $\mathbf{W}$ to obtain a low-rank solution. For a lossless network, any decrease in the total reactive power generation increases the weighted sum of the real parts of the off-diagonal entries of $\mathbf{W}$. Likewise, the penalization of the apparent power loss over the series impedance of the lines of the network (without incorporating the shunt capacitors) plays a similar role for a lossy network. More precisely, the penalization of the loss

$$\begin{aligned} L_{lm}^{(r)} &\triangleq \left| S_{lm}^{(r)} + S_{ml}^{(r)} \right| \\ &= \left| V_l^{(r)} \left(V_l^{(r)} - V_m^{(r)} \right)^* + V_m^{(r)} \left(V_m^{(r)} - V_l^{(r)} \right)^* \right| |y_{lm}^*| \\ &= \left| W_{ll}^{(r)} + W_{mm}^{(r)} - W_{lm}^{(r)} - W_{ml}^{(r)} \right| |y_{lm}^*| \end{aligned} \tag{4.11}$$

associated with the line (l, m) for contingency r enforces the increase of the off-diagonal entries $W_{lm}^{(r)}$ and $W_{ml}^{(r)}$ (relative to those of $W_{ll}^{(r)}$ and $W_{mm}^{(r)}$). Therefore, this penalty term aims for a low-rank solution.

Penalized SDP relaxation: This optimization is obtained from the (decomposed) relaxed SCOPF problem by replacing its objective function with

$$\sum_{\substack{k \in \mathcal{N} \\ r \in \mathcal{C}}} f_k^{(r)} \left(P_{G_k}^{(r)} \right) + \varepsilon_b \sum_{\substack{k \in \mathcal{N} \\ r \in \mathcal{C}}} Q_{G_k}^{(r)} + \varepsilon_l \sum_{(r,l,m) \in \mathcal{L}^{\text{prob}}} L_{lm}^{(r)} \tag{4.12}$$

for given nonnegative numbers ε_b, ε_l and a set of triples $\mathcal{L}^{\text{prob}} \subseteq \big\{(r,l,m)\,|\, r \in \mathcal{C},\ (l,m) \in \mathcal{L}^{(r)}\big\}$, where $L_{lm}^{(r)}$ represents the apparent power loss over the series impedance of the line (l,m) for contingency r.

Let $\mathbf{W}_{\text{opt}}$ and $\mathbf{W}_{\varepsilon}$ denote arbitrary solutions of the SDP and penalized SDP relaxations, respectively. Assume that $\mathbf{W}_{\text{opt}}$ does not have rank 1, whereas $\mathbf{W}_{\varepsilon}$ has rank 1. By decomposing $\mathbf{W}_{\varepsilon}$ as $\mathbf{V}_{\varepsilon}\mathbf{V}_{\varepsilon}^*$, a feasible solution $\mathbf{V}_{\varepsilon}$ of the SCOPF can be obtained. In addition, the optimal value f_{opt} of the SCOPF problem is lower bounded by the optimal value $f_{\text{r-opt}}$ of the SDP relaxation and upper bounded by f_{ε}, where f_{ε} is defined as the total generation cost associated with the operating point $\mathbf{V}_{\varepsilon}$. Define *global optimality guarantee* as

$$100 - \frac{f_{\varepsilon} - f_{\text{r-opt}}}{f_{\varepsilon}} \times 100. \tag{4.13}$$

This number shows the closeness of the feasible solution $\mathbf{V}_{\varepsilon}$ to the unknown globally optimal solution in terms of their costs (in percentage). For example, if the global optimality guarantee is 99%, then the cost associated with the global solution of SCOPF is at most 1% better than the cost corresponding to the obtained feasible point. In summary, if the penalized SDP relaxation has a rank-1 solution, then a feasible solution of SCOPF together with a global optimality guarantee can be computed.

The success of the penalized SDP relaxation is in part related to the choice of $\mathcal{L}^{\text{prob}}$. Sometimes, a good choice is to consider this set as the collection of all lines of the system in pre- and post-contingency cases. In what follows, we propose an effective heuristic method for designing $\mathcal{L}^{\text{prob}}$. Consider a bag $\mathcal{J}_i$ of the tree decomposition $\mathcal{T}_{\mathcal{C}}$ together with a matrix $\mathbf{W}$. $\mathcal{J}_i$ is called a **problematic bag** associated with $\mathbf{W}$ if $\mathbf{W}\{\mathcal{J}_i, \mathcal{J}_i\}$ does not have rank 1. Any line of the pre- or post-contingency network corresponding to an off-diagonal entry of $\mathbf{W}\{\mathcal{J}_i, \mathcal{J}_i\}$ is called a *problematic line* associated with $\mathbf{W}$. It follows from Theorem 18 and Corollary 12 that the SDP relaxation is exact if there is no problematic bags/lines associated with the solution of the decomposed relaxed SCOPF.

Problematic line selection algorithm: Consider the penalized SDP relaxation with $\epsilon_l = 0$. Using a bisection method, find a value for ϵ_b such that the number of problematic lines associated with the penalized SDP solution is small (minimum). A candidate for $\mathcal{L}^{\text{prob}}$ is the set of resulting problematic lines.

Assume that the penalized SDP relaxation results in a rank-1 solution or a low-rank matrix solution with a dominant nonzero eigenvalue. The next algorithm can be used to find an approximate feasible solution of SCOPF.

Recovery algorithm: Given a low-rank solution $\mathbf{W}_{\text{opt}}$ of the penalized SDP relaxation, we obtain an approximate solution for the SCOPF by recovering $\mathbf{V}^{(r)}$ according to the following procedure for every $r \in \mathcal{C}$:

1. Set the voltage magnitude $V_k^{(r)}$ equal to the square root of the (k, k) entry of $W_{\text{opt}}^{(r)}$ for $k = 1, ..., n$.

2. Find the phases of the entries of $\mathbf{V}^{(r)}$ through a convex program by minimizing

$$\sum_{(l,m)\in\mathcal{L}^{(r)}} \left| \measuredangle(\mathbf{W}_{\text{opt}})_{lm}^{(r)} - \measuredangle\mathbf{V}_l^{(r)} + \measuredangle\mathbf{V}_m^{(r)} \right| \tag{4.14}$$

   over the variable $\measuredangle\mathbf{V}^{(r)} \in [-\pi, \pi]^n$ and subject to $\measuredangle V_1^{(r)} = 0$.

Note that the above recovery algorithm retrieves a globally optimal solution of the SCOPF problem in the case where $\text{rank}\{\mathbf{W}_{\text{opt}}\} = 1$. Under that circumstance, we have $\measuredangle(\mathbf{W}_{\text{opt}})_{lm}^{(r)} - \measuredangle\mathbf{V}_l^{(r)} + \measuredangle\mathbf{V}_m^{(r)} = 0$. If the rank of $\mathbf{W}_{\text{opt}}$ is not 1 but this matrix has a dominant nonzero eigenvalue, the above recovery method aims to find a vector $\mathbf{V}$ for which the corresponding line angle differences are as closely as possible to those suggested by the matrix $\mathbf{W}_{\text{opt}}$.

4.5 Simulations Results

In what follows, we offer several simulations for OPF and SCOPF problems. We have written a custom OPF Solver to perform these simulations [Madani *et al.*, 2014a], which is based on CVX and SDPT3.

For all of the cases that will be studied in this section, the penalized SDP has a rank-1 solution with $\varepsilon_b = 0$, a roughly chosen ε_l, and

$$\mathcal{L}^{\text{prob}} = \left\{ (r, l, m) \mid r \in \mathcal{C},\ (l, m) \in \mathcal{L}^{(r)} \right\}. \tag{4.15}$$

This means that the proposed method works at the first try with roughly chosen parameters, leading to a near-optimal solution. Figures 4.3 and 4.4 demonstrate for multiple systems that the

Test cases	α	**TW**	**Prob. bags**	ε_b	ε_l	**Lower bound**	**Upper bound**	**Opt.**	**Time (sec)**
Chow's 9 bus	0	2	2	10	0	5296.68	5296.68	100%	≤ 5
IEEE 14 bus	0	2	0	0	0	8081.53	8081.53	100%	≤ 5
IEEE 24 bus	0	4	0	0	0	63352.20	63352.20	100%	≤ 5
IEEE 30 bus	0	3	1	0.1	0	576.89	576.89	100%	≤ 5
NE 39 bus	0	3	1	10	0	41862.08	41864.40	99.994%	≤ 5
IEEE 57 bus	0	5	0	0	0	41737.78	41737.78	100%	≤ 5
IEEE 118 bus	0	4	61	10	0	129654.61	129660.81	99.995%	≤ 5
IEEE 300 bus	0	6	7	0.1	100	719711.63	719725.10	99.998%	13.9
Polish 2383wp	0	≤ 23	651	3500	3000	1861510.42	1874322.65	99.316%	529
Polish 2736sp	0	≤ 23	1	1500	0	1307882.29	1308270.20	99.970%	701
Polish 2737sop	0	≤ 23	3	1000	0	777626.26	777664.02	99.995%	675
Polish 2746wop	0	≤ 23	1	1000	0	1208273.91	1208453.93	99.985%	801
Polish 2746wp	0	≤ 24	1	1000	0	1631772.83	1632384.87	99.962%	699
Polish 3012wp	1	≤ 24	605	0	10000	2587740.98	2608918.45	99.188%	814
Polish 3120sp	-1.5	≤ 24	20	0	10000	2140765.92	2160800.42	99.073%	910

Table 4.1: Performance of the penalization method for several benchmark systems.

near-optimal cost changes slowly by the increase of ε_l, which points to the high degree of freedom in choosing ε_l. However, a careful choice of problematic lines and the regularization parameters ε_b and ε_l using a bisection approach would lead to a better near-global solution. This will be elaborated in the rest of this section.

3-bus system: Consider the 3-bus system presented in [Lesieutre *et al.*, 2011]. The SDP relaxation may not result in a rank-1 solution for this system if a certain line is under stress (i.e. the capacity constraint of the line is binding at optimality). To address this issue, we use the penalized SDP relaxation with the objective function (4.12), where the parameter ε_b is set to zero and the line under stress is chosen for penalization. The resulting optimal cost is reported in Figure 4.3(a) for different values of ε_l. It can be seen that there exists a relatively large interval for ε_l that makes the penalized SDP relaxation posses a rank-1 solution with a fixed cost. This cost overlaps with the globally optimal cost of the OPF problem. Hence, our method is able to bridge the duality gap reported in [Lesieutre *et al.*, 2011].

Test cases	**Local minima**	**Prob. bags**	ε_b	ε_l	**Lower bound**	**Upper bound**	**Optimality guarantee**
WB2	2	0	0	0	877.78	877.78	100 %
WB3	2	0	0	0	417.25	417.25	100%
WB5	2	3	0	500	946.53	946.58	99.995%
WB5 Mod	3	0	0	0	1482.22	1482.22	100%
LMBM3	5	0	0	0	5694.54	5694.54	100%
LMBM3_50	2	2	0	500	5789.91	5823.86	99.807%
case22loop	2	0	0	0	4538.80	4538.80	100%
case30loop	2	0	0	0	2863.06	2863.06	100%
case30loop Mod	3	0	0	0	2861.88	2861.88	100%
case39 Mod4	3	4	1	0	557.08	557.15	99.999%
case118 Mod1	3	36	10	0	129624.98	129625.19	99.999%
case118 Mod2	2	42	1	0	85987.27	85987.59	100%
case300 Mod2	2	107	0.5	50	474625.99	474643.46	99.996%

Table 4.2: Performance of the penalization method on examples presented in [Bukhsh, 2012].

Test cases	**Prob. bags**	ε_b	ε_l	**Lower bound**	**Upper bound**	**Solution of interior point method**
case14C	12	0.1	10000	6897.02	7289.62	7238.93
case34tree	1	0.1	100	14.49	24.38	24.38

Table 4.3: Performance of the penalization method on examples presented in [Louca *et al.*, 2014].

IEEE and Polish systems: As our second example, we evaluate the penalization method for the OPF problem performed over benchmark systems. The results are reported in Table 4.1. For each system, the following numbers are reported:

- *Treewidth:* exact treewidth or an upper bound on the treewidth (shown as "$\leq$") of the pre-contingency network
- *Problematic bags:* number of problematic bags for the SDP relaxation with $\epsilon_b = \epsilon_l = 0$
- *Lower bound:* lower bound on the globally minimum cost of OPF, corresponding to the cost of the SDP relaxation
- *Upper Bound:* upper bound on the globally optimal cost of OPF, corresponding to the solution recovered from the penalized SDP problem
- *Optimality guarantee:* global optimality guarantee (in percentage)
- *Computation time:* the total computation time (in seconds) including those consumed towards tree decomposition, solving the SDP relaxation, and recovering a solution (the simulations were run on a desktop computer with an Intel Core i7 quad-core 3.4 GHz CPU and 16 GB RAM).

For IEEE systems, we were able to verify that the obtained tree decompositions were all minimal. Note that the permissible feasibility violation for the recovered solution was set to 10^{-6} for all cases reported in Table 4.1, except for Polish 3012wp and Polish 3120sp for which the violation level was set to 1.5×10^{-5}. For Polish 3012wp and Polish 3120sp, we penalized the apparent power loss over all lines of each system. For IEEE 300 bus system, we penalized apparent power loss over two lines: (i) line 38 between buses 9053 and 9533, (ii) line 402 between buses 7023 and 23. These lines are problematic for the penalized SDP problem in the case $\epsilon_b = 0.1$ and $\epsilon_l = 0$.

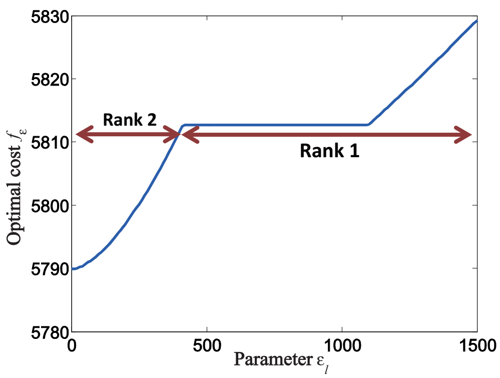
Similarly, the apparent power loss penalization for Polish 2383wp system was performed over problematic lines for the case $\epsilon_b = 3500$ and $\epsilon_l = 0$, leading to the following 9 lines: line 100 between buses 35 and 34, line 101 between buses 34 and 51, line 102 between buses 183 and 34, line 103 between buses 183 and 35, line 104 between buses 617 and 35, line 130 between buses 51 and 50, line 134 between buses 727 and 51, line 819 between buses 546 and 545, and line 821 between buses 727 and 545. It can be observed that the penalized SDP relaxation was able to find feasible solutions with global optimality guarantee above 99% for many benchmark examples.

New England and IEEE 300 bus systems: In order to evaluate the sensitivity of the penalization method to the choice of problematic lines and penalty parameters, we performed an experiment on New England 39 bus and IEEE 300 bus systems for the case where $\varepsilon_b = 0$ and all lines of the network were considered in the penalty factor as "problematic". Figures 4.3(b) and (c) show the performance of the penalized SDP relaxation for a wide range of values for ε_l. It can be seen that it is possible to find a rank-1 solution with a high global optimality guarantee over a large interval for ε_l. However, as shown in Table 4.1, a better optimality guarantee can be obtained by carefully tuning the penalty parameters via solving multiple SDP problems.

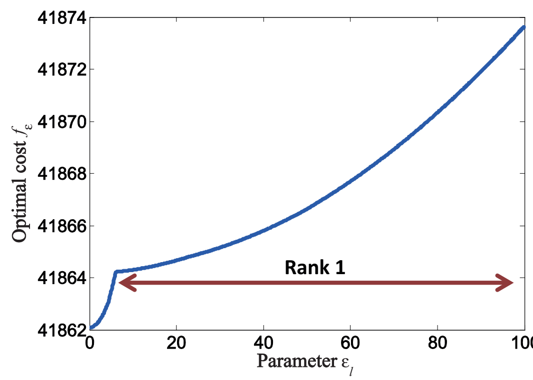
Modified systems 1: We also tested the SDP relaxation method over some of the examples presented in [Bukhsh, 2012]. The results are tabulated in Table 4.2, together with the minimum number of local solutions for each system. For case WB5, line 5 (between buses 4 and 5) was chosen for apparent power loss penalization. Interestingly, the penalization of a different line in the objective function would result in the recovery of a local minimum as opposed to a global solution. For the modified IEEE 300 bus system, the apparent power loss penalization was performed over 116 problematic lines of the network. For LMBM3_50, all lines of the networks were considered as problematic.

Modified systems 2: Two OPF test cases have been recently proposed in [Louca *et al.*, 2014], for which the semidefinite relaxation is inexact. The penalization algorithm proposed here can find a rank-1 solution for each of these cases, as reported in Table 4.3. The test system "case34tree" is of particular interest due to its tree topology. We obtained a rank-1 solution by penalizing the only problematic line of this network.

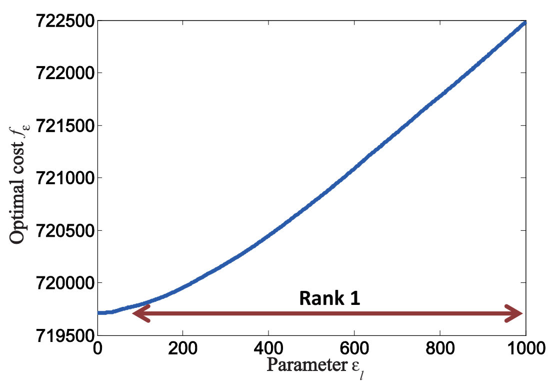
New England system with contingency constraints: Consider the New England test system under 10 contingency scenarios, each representing the outage of one line as described in Table 4.4. Suppose that the objective function of the SCOPF problem only includes the power generation cost for the base case. The corrective active power production for each generator in case of contingency is set to be within 2 MW away from the base case production level. We solve the penalized SDP relaxation by setting ε_b to zero and minimizing the apparent power loss over all lines of the network.



(a)

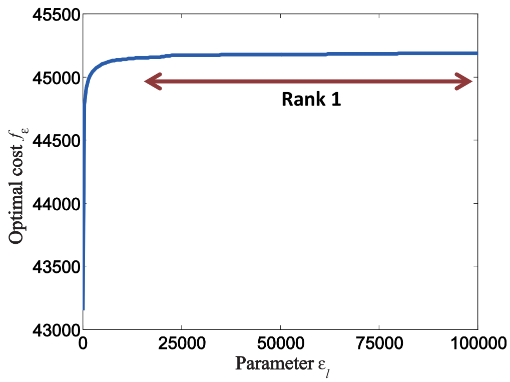


(b)

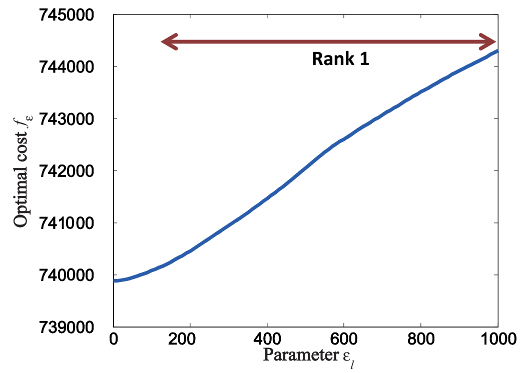


(c)

Figure 4.3: (a) The 3-bus system presented in ; (b) New England system; (c) IEEE 300 bus system.



(a)



(b)

Figure 4.4: (a) Contingency analysis of New England system; (b) contingency analysis of IEEE 300 bus system.

The result is depicted in Figure 4.4(a) for different values of the coefficient ε_l. It can be seen that a near-global solution for the SCOPF problem is associated with the cost 45141.70. This SCOPF cost is 7% higher than the optimal cost of the OPF problem with no contingency.

Contingency number	**Line number**	**Initial node**	**Terminal node**
1	1	1	2
2	2	1	39
3	3	2	3
4	4	2	25
5	6	3	4
6	7	3	18
7	15	7	8
8	20	10	32
9	40	25	26
10	45	28	29

Table 4.4: List of contingencies for New England test system.

IEEE 300 bus system with contingency constraints: Consider the 300 bus system with one contingency scenario associated with the simultaneous outage of three highly congested lines of the base OPF. These lines are listed in Table 4.5. The corrective active power production for each generator in case of contingency is set to be within 1 MW away from the base case production level. We intend to minimize the pre-contingency power generation cost while being secure in the post-contingency scenario. As before, we solve the penalized SDP relaxation by setting ε_b to zero and minimizing the apparent power loss over all lines of the network. The result is depicted in Figure 4.4(b). It can be seen that a near-global solution for the SCOPF problem is associated with the cost 740493.80, which is 3% different from that of the OPF problem.

Contingency number	**Line number**	**Initial node**	**Terminal node**
1	266	19	231
	388	234	236
	400	7130	130

Table 4.5: List of lines outages of the contingency scenario considered for IEEE 300 bus system.

4.6 Summary

This chapter studies the security-constrained optimal power flow (SCOPF) problem by means of a semidefinite programming (SDP) relaxation. The existence of a rank-1 solution guarantees that this convex program will find a globally optimal solution of the SCOPF problem. First, we prove that the SDP relaxation has a solution whose rank is at most equal to the treewidth of the power network plus one, which is expected to be very small for real-world systems. Second, we propose a decomposition method to reduce the computational complexity of the SDP relaxation. In the case where the SDP relaxation fails to work, we develop a graph-theoretic method to identify the problematic lines of the network that make SCOPF difficult to solve. By penalizing the loss over those lines in the SDP relaxation, we develop a rank-enforcing SDP relaxation. We test our relaxation method on several benchmark examples and demonstrate its ability in finding feasible solutions with the property that the global minima are at most 1% away from the obtained solutions.

Chapter 5

Convexification of Power Flow Problem over Arbitrary Networks

Consider an arbitrary power network with PV and PQ buses, where active powers and voltage magnitudes are known at PV buses, and active and reactive powers are known at PQ buses. The classical power flow (PF) problem aims to find the unknown complex voltages at all buses. This problem is usually solved *approximately* through linearization or in an *asymptotic* sense using Newton's method, given that the solution belongs to a good regime containing voltage vectors with small angles. The question arises as to whether the PF problem can be cast as the solution of a convex optimization problem over that regime. The objective of this chapter is to show that the answer to the above question is affirmative. More precisely, we propose a class of convex optimization problems with the property that they all solve the PF problem as long as angles are small. Each convex problem proposed in this work is in the form of a semidefinite program (SDP). Associated with each SDP, we explicitly characterize the set of complex voltages that can be recovered via that convex problem. Since there are infinitely many SDP problems, each capable of recovering a potentially different set of voltages, designing a good SDP problem is cast as a convex problem.

5.1 Introduction

The flows in an electrical grid are described by nonlinear AC power flow equations. This chapter is concerned with the problem of finding an unknown vector of complex voltages $\mathbf{V} \in \mathbb{C}^n$ for an n-bus power system to satisfy m quadratic constraints associated with m known quantities measured or specified in the network. This general power flow problem can be formulated as

$$\begin{array}{lll} \text{find} & \mathbf{V} \in \mathbb{C}^n & \\ \text{subject to} & \langle \mathbf{V}\mathbf{V}^*, \mathbf{M}_i \rangle = X_i, & i = 1, \ldots, m, \end{array} \tag{5.1}$$

where $\langle \cdot, \cdot \rangle$ represents the Frobenius inner product of matrices, $\mathbf{M}_1, \ldots, \mathbf{M}_m$'s are certain $n \times n$ Hermitian matrices obtained from the admittance matrix of the power network, and $X_1, \ldots, X_m$ represent specified physical quantities such as the net active power, reactive power or squared voltage magnitude at a bus or the flow over a line of the network. Checking the existence of a solution to the quadratic feasibility problem (5.1) is NP-hard for both transmission and distribution networks due to their reduction to the *subset sum* problem [Lehmann *et al.*, 2014; Verma, 2009].

Since problem (5.1) is central to the analysis and operation of power systems, its high computational complexity motivates obtaining a tractable formulation of the power flow equations. Addressing this problem facilitates performing several key tasks related to economic dispatch and state estimation for power networks.

5.1.1 Semidefinite Programming Relaxation

To tackle the non-convexity of the feasible set of the power flow problem, the semidefinite programming (SDP) relaxation technique could be deployed. By defining $\mathbf{W} \triangleq \mathbf{V}\mathbf{V}^*$, the quadratic equations in (5.1) can be linearly formulated in terms of $\mathbf{W}$. Motivated by this exact linearization, consider the matrix feasibility problem

$$\begin{array}{lll} \text{find} & \mathbf{W} \in \mathbb{H}_n^+ & \\ \text{subject to} & \langle \mathbf{W}, \mathbf{M}_i \rangle = X_i, & i = 1, \ldots, m, \end{array} \tag{5.2}$$

where $\mathbb{H}_n^+$ is the set of $n \times n$ Hermitian positive semidefinite matrices. Solving the non-convex power flow problem (5.1) is tantamount to finding a rank-1 solution $\mathbf{W}$ for the above linear matrix inequality (because $\mathbf{W}$ could then be decomposed as $\mathbf{V}\mathbf{V}^*$). The problem (5.2) is regarded as a convex *relaxation* of (5.1) since it includes no restriction on the rank of $\mathbf{W}$. Although (5.1) normally has a finite number of solutions whenever $m \geq 2n-1$, its SDP relaxation (5.2) is expected to have infinitely many solutions because the matrix variable $\mathbf{W}$ includes $O(n^2)$ scalar variables as opposed to $O(n)$. However, under some additional assumptions, it is known in some applications, such as phase retrieval, that the relaxed problem would have a unique solution if $m \geq 3n$, and that the solution has automatically rank-1 [Candes *et al.*, 2013]. In the case where the relaxed problem does not have a unique solution, the literature of compressed sensing substantiates that minimizing trace$\{\mathbf{W}\}$ over the feasible set of (5.2) may yield a rank-1 matrix $\mathbf{W}$ under strong technical assumptions [Recht *et al.*, 2010b; Candes *et al.*, 2013; Madani *et al.*, 2014b]. The main purpose of this chapter is to study what objective function should be minimized (instead of trace$\{\mathbf{W}\}$) to attain a rank-1 solution for the power problem (5.2).

The potential of SDP relaxation for finding a global solution of the optimal power flow problem has been manifested in [Lavaei and Low, 2012], with further studies conducted in [Lavaei, 2011; Sojoudi and Lavaei, 2012]. In addition, recent advances in leveraging the sparsity of network graph have made SDP problems computationally more tractable [Lam *et al.*, 2012b; Zhang *et al.*, 2015; Molzahn *et al.*, 2013; Andersen *et al.*, 2014; Jabr, 2012; Madani *et al.*, 2015]. In the case where the SDP relaxation fails to work, we have developed a graph-theoretic penalized SDP problem in chapters 3 and 4, which attempts to identify the problematic lines of the network (sources of non-convexity) through a graph analysis and then penalize the loss over those lines in order to find a near-global solution for the optimal power flow problem. The proposed approach was successful in finding solutions with global optimality guarantees of at least 99% for IEEE and Polish test systems.

5.1.2 Classical Power Flow Problem

Let $\mathcal{N}$ denote the set of buses of the power network under study, and let P_k and Q_k represent the net active and reactive power injections at every bus $k \in \mathcal{N}$. The complex voltage phasor at bus k is denoted by V_k, whose magnitude and phase are shown as $|V_k|$ and $\measuredangle V_k$, respectively. One special case of (5.1) is the classical power flow (PF) problem, for which the number of quadratic constraints (namely m) is equal to $2n-1$. To formulate the problem, three basic types of buses are considered, depending on the parameters that are known at each bus:

- PQ bus: P_k and Q_k are specified.
- PV bus: P_k and $|V_k|$ are specified.
- The slack bus: $|V_k|$ is specified.

Each PQ bus represents a load bus or possibly a generator bus, whereas each PV bus represents a generator bus. In order to resolve the global phase ambiguity, $\measuredangle V_k$ is set equal to zero at any particular bus (for example, the slack bus). Given the specified parameters at every bus of the network, the classical PF problem aims to solve the network equations in order to find an operating point that fits the input values. This problem has been studied extensively for years, without much success in designing an advanced computational method that is able to outperform Newton's method in polar coordinates [Tinney and Hart, 1967; Stott and Alsaç, 1974; Van Amerongen, 1989].

5.1.3 Contributions

This chapter is aligned with the recent body of work on investigating the remarkable promises of SDP relaxations for power optimization problems. The major strength of Newton's method and similar traditional techniques is their local convergence property, meaning that the recovery of a feasible solution is possible as long as the starting point is sufficiently close to a solution. The main objective of this chapter is to develop a similar property for the SDP relaxation. We propose a family of convex optimization problems of the form

$$\underset{\mathbf{W}\in\mathbb{H}_n}{\text{minimize}} \quad \langle \mathbf{W}, \mathbf{M} \rangle \tag{5.3a}$$

$$\text{subject to} \quad \langle \mathbf{W}, \mathbf{M}_i \rangle = X_i, \qquad i = 1, \ldots, m, \tag{5.3b}$$

$$\mathbf{W} \succeq 0, \tag{5.3c}$$

where the matrix $\mathbf{M} \in \mathbb{H}_n^+$ is to be designed. Unlike the compressed sensing literature that assumes $M = I$, we aim for systematically designing $\mathbf{M}$ such that the above problem yields a unique rank-1 solution $\mathbf{W}$ from which a feasible solution for the power flow problem (5.1) can be recovered. Notice that the existence of such a rank-1 solution depends in part on its input specifications $(X_1, X_2, ..., X_m)$.

Definition 24. *It is said that the SDP problem* (5.3) *solves the nonlinear power flow problem* (5.1) *for the input* $(X_1, X_2, ..., X_m)$ *if* (5.3) *has a unique rank-1 solution. Given* $\mathbf{M} \in \mathbb{H}_n$*, define* $\mathcal{M}_X(\mathbf{M}) \subseteq \mathbb{R}^m$ *as the set of all specification vectors* $(X_1, ..., X_m)$ *for which the SDP problem* (5.3) *solves the nonlinear power flow problem* (5.1).

Definition 25. *Given* $\mathbf{M} \in \mathbb{H}_n$*, a voltage vector* $\mathbf{V}$ *is said be recoverable if* $\mathbf{W} = \mathbf{V}\mathbf{V}^*$ *is a solution of the SDP problem* (5.3) *for some* $(X_1, ..., X_m)$ *in* $\mathcal{M}_X(\mathbf{M})$*. Define* $\mathcal{M}_V(\mathbf{M})$ *as the set of all recoverable vectors* $\mathbf{V}$.

Note that the SDP problem (5.3) can be used to find a solution of the nonlinear power flow problem (5.1) if and only if the provided specification set $(X_1, X_2, ..., X_m)$ belongs to $\mathcal{M}_X(\mathbf{M})$. In addition, the set $\mathcal{M}_V(\mathbf{M})$ is indeed the collection of all possible operating points $\mathbf{V}$ that can be found through (5.3) associated with different values of $(X_1, X_2, ..., X_m)$. It is desirable to find out whether there exists a matrix $\mathbf{M}$ for which the recoverable region $\mathcal{M}_V(\mathbf{M})$ covers a large subset of $\mathbb{C}^n$, that is relative to the practical solutions of power flow problem. Addressing this problem is central to this chapter.

Our first contribution is related to the classical power flow problem. We aim to prove that if the matrix $\mathbf{M}$ satisfies the three properties

- $\mathbf{M} \succeq 0$,
- 0 is a simple eigenvalue of $\mathbf{M}$,
- The all-ones vector $\mathbf{1}_n$ belongs to the null space of $\mathbf{M}$,

then the region $\mathcal{M}_V(\mathbf{M})$ contains the nominal point $(1, 1, ..., 1)$ and a ball around it. In other words, as long as the specifications $X_1, \ldots, X_{2n-1}$ correspond to a vector of voltages with small angles, the exact recovery of the solution is guaranteed through the proposed SDP problem, without requiring any assumption on the network topology whatsoever. This implies that although the DC model can be used to find an approximate solution around the nominal point, the SDP relaxation would find an exact solution.

It is desirable to find a matrix $\mathbf{M}$ for which the recoverable set $\mathcal{M}_V(\mathbf{M})$ is as large as possible with respect to some meaningful measure. This design problem is cumbersome due to the non-convexity of $\mathcal{M}_V(\mathbf{M})$. However, we show that the problem of designing a matrix $\mathbf{M}$ for which $\mathcal{M}_V(\mathbf{M})$ contains a pre-specified set of voltage vectors $\mathbf{V}$'s can be cast as a convex program.

Although we develop our results in the context of the classical PF problem, they all hold in the case where $m > 2n-1$, corresponding to redundant specifications. In fact, the proposed approach is flexible in terms of the choice for the types and number of equations, which makes is adoptable for the specification problem. We will demonstrate in simulations that whenever extra equations are available (i.e., $m > 2n-1$), the search for a feasible vector of voltages for problem (5.1) is more likely to be successful via the convex program (5.3).

5.1.4 Notations

The symbols $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{S}_n$ denotes the space of $n \times n$ real symmetric matrices and $\mathbb{H}_n$ denotes the space of $n \times n$ complex Hermitian matrices. $\mathrm{Re}\{\cdot\}$, $\mathrm{Im}\{\cdot\}$, $\mathrm{rank}\{\cdot\}$, $\mathrm{trace}\{\cdot\}$, and $\det\{\cdot\}$ denote the real part, imaginary part, rank, trace, and determinant of a given scalar/matrix. $\mathrm{diag}\{\cdot\}$ denotes the vector of diagonal entries of a matrix. $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. Matrices are shown by capital and bold letters. The symbols $(\cdot)^{\mathrm{T}}$ and $(\cdot)^*$ denote transpose and conjugate transpose, respectively. Also, "$\mathbf{i}$" is reserved to denote the imaginary unit. The notation $\langle \mathbf{A}, \mathbf{B} \rangle$ represents $\mathrm{trace}\{\mathbf{A}^*\mathbf{B}\}$, which is the inner product of $\mathbf{A}$ and $\mathbf{B}$. The notations $\measuredangle x$ and $|x|$ denote the angle and magnitude of a complex number x. The notation $\mathbf{W} \succeq 0$ means that $\mathbf{W}$ is a Hermitian and positive semidefinite matrix. Also $\mathbf{W} \succ 0$ means that it is Hermitian and positive definite. The (i,j) entry of $\mathbf{W}$ is denoted as W_{ij}. $\mathbf{0}_n$ and $\mathbf{1}_n$ denote the $n \times 1$ vectors of zeros and ones, respectively. $\mathbf{0}_{m\times n}$ denotes the $m \times n$ zero matrix and $\mathbf{I}_{n\times n}$ is the $n \times n$ identity matrix. The notation $|\mathcal{X}|$ denotes the cardinality of a set $\mathcal{X}$. For an $m \times n$ matrix $\mathbf{W}$, the notation $\mathbf{W}[\mathcal{X}, \mathcal{Y}]$ denotes the submatrix of $\mathbf{W}$ whose rows and columns are chosen form $\mathcal{X}$ and $\mathcal{Y}$, respectively, for given index sets $\mathcal{X} \subseteq \{1, \ldots, m\}$ and $\mathcal{Y} \subseteq \{1, \ldots, n\}$. Similarly, $\mathbf{W}[\mathcal{X}]$ denotes the submatrix of $\mathbf{W}$ induced by those rows of $\mathbf{W}$ indexed by $\mathcal{X}$. The interior of a set $\mathcal{D} \in \mathbb{C}^n$ is denoted as $\mathrm{int}\{\mathcal{D}\}$.

5.2 Preliminaries

Define $\mathbf{P} = [P_1 \; P_2 \; \cdots \; P_n]^T$ and $\mathbf{Q} = [Q_1 \; Q_2 \; \cdots \; Q_n]^T$ as the vectors containing net injected active and reactive powers, respectively. Define also $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{V}$ as the sets of buses for which active powers, reactive powers and voltage magnitudes are known, respectively. Let $\mathcal{O}$ denote the set of all buses except the slack bus. The admittance matrix of the network is denoted as $\mathbf{Y} = \mathbf{G}+\mathbf{B}\mathbf{i}$, where $\mathbf{G}$ and $\mathbf{B}$ are the conductance and susceptance matrices, respectively. Although the results to be developed in this chapter hold for a general matrix $\mathbf{Y}$, we make a few assumptions to streamline the presentation:

- The network is a connected graph.
- Every line of the network consists of a series impedance with nonnegative resistance and inductance.
- The shunt elements are ignored for simplicity in studying the observability of the network, which ensures that

$$\mathbf{Y} \times \mathbf{1}_n = \mathbf{0}_n. \tag{5.4}$$

The following lemma reveals an interesting property of the matrix $\mathbf{B}$, which will later be used in the chapter.

Lemma 4. *For every $\mathcal{N}' \subseteq \mathcal{N}$, we have*

$$\mathbf{B}[\mathcal{N}', \mathcal{N}'] \preceq 0 \tag{5.5}$$

and $\mathbf{B}[\mathcal{N}', \mathcal{N}']$ is singular if and only if $\mathcal{N}' = \mathcal{N}$.

Proof. $\mathbf{B}$ is symmetric and according to (5.4), we have $\mathbf{B}\mathbf{1}_n = \mathbf{0}_n$ and since every off-diagonal entry of $\mathbf{B}$ is nonnegative, we have

$$-B_{kk} = \sum_{k \neq l} B_{kl} \quad \Rightarrow \quad -B_{kk} \geq \sum_{k \neq l} |B_{kl}| \tag{5.6}$$

for every $k \in \{1, \ldots, n\}$. Therefore $-\mathbf{B}$ is diagonally dominant and positive semidefinite and (5.5) holds for every principal submatrix of $\mathbf{B}$.

Since the network is connected by assumption and every entry of $\mathbf{B}$ corresponding to an existing line of the network is positive, it follows from the Weighted Matrix-Tree Theorem [Duval *et al.*, 2009] that if $|\mathcal{N}'| = n-1$, then $\det\{\mathbf{B}[\mathcal{N}', \mathcal{N}']\} \neq 0$ and consequently $\mathbf{B}[\mathcal{N}', \mathcal{N}'] \prec 0$. In addition, if $|\mathcal{N}'| < n-1$, there exists a set $\mathcal{N}'' \subset \mathcal{N}$ such that $\mathcal{N}' \subset \mathcal{N}''$ and $|\mathcal{N}''| = n-1$. According to the Cauchy Interlacing Theorem, every eigenvalue of $\mathbf{B}[\mathcal{N}', \mathcal{N}']$ is less than or equal to the largest eigenvalue of $\mathbf{B}[\mathcal{N}'', \mathcal{N}'']$, which implies that $\mathbf{B}[\mathcal{N}', \mathcal{N}']$ is non-singular. $\square$

Recall that the power balance equations can be expressed as

$$\mathbf{P} + \mathbf{iQ} = \mathrm{diag}\{\mathbf{V}\mathbf{V}^*\mathbf{Y}^*\}. \tag{5.7}$$

Hence, the SDP program (5.3) associated with the classical PF problem can be written as

$$\begin{aligned}
&\underset{\mathbf{W}\in\mathbb{H}_n}{\text{minimize}} && \langle \mathbf{W}, \mathbf{M}\rangle && && (5.8\text{a})\\
&\text{subject to} && \langle \mathbf{W}, \mathbf{E}_k\rangle = |V_k|^2, && k \in \mathcal{V} && (5.8\text{b})\\
& && \langle \mathbf{W}, \mathbf{Y}_{Q;k}\rangle = Q_k, && k \in \mathcal{Q} && (5.8\text{c})\\
& && \langle \mathbf{W}, \mathbf{Y}_{P;k}\rangle = P_k, && k \in \mathcal{P} && (5.8\text{d})\\
& && \mathbf{W} \succeq 0 && && (5.8\text{e})
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{E}_k &\triangleq e_k e_k^*\\
\mathbf{Y}_{Q;k} &\triangleq \frac{1}{2\mathbf{i}}(\mathbf{Y}^* e_k e_k^* - e_k e_k^* \mathbf{Y})\\
\mathbf{Y}_{P;k} &\triangleq \frac{1}{2}(\mathbf{Y}^* e_k e_k^* + e_k e_k^* \mathbf{Y})
\end{aligned}$$

and $e_1, \ldots, e_n$ denote the standard basis vectors in $\mathbb{R}^n$. The problem (5.8) is in the canonical form (5.3) after noticing that

- $\mathbf{M}_1, \mathbf{M}_2, ..., \mathbf{M}_m$ correspond to the m matrices $\mathbf{E}_1, \mathbf{E}_2,\ ..., \mathbf{E}_{|\mathcal{V}|},\ \mathbf{Y}_{Q;1}, \mathbf{Y}_{Q;2} ..., \mathbf{Y}_{Q;|\mathcal{Q}|}$ and $\mathbf{Y}_{P;1}, \mathbf{Y}_{P;2}, ...,\ \mathbf{Y}_{P;|\mathcal{P}|}$.
- The specifications $X_1, X_2, ..., X_n$ correspond to $|V_k|^2$'s, Q_k's, and P_k's.

Since the voltage angle at the slack bus is set to zero, the operating point of the power system can be characterized in terms of the real-valued vector

$$\overline{\mathbf{V}} \triangleq \left[\mathrm{Re}\{\mathbf{V}[\mathcal{N}]^{\mathrm{T}}\} \quad \mathrm{Im}\{\mathbf{V}[\mathcal{O}]^{\mathrm{T}}\}\right]^{\mathrm{T}} \in \mathbb{R}^{2n-1}.$$

Definition 26. *Define the function $s(\overline{\mathbf{V}}) : \mathbb{R}^{2n-1} \to \mathbb{R}^m$ as the mapping from the state of the power network $\overline{\mathbf{V}}$ to the vector of specifications $\mathbf{X}$. The i-th component of $s(\overline{\mathbf{V}})$ is given by*

$$s_i(\overline{\mathbf{V}}) = \langle \mathbf{V}\mathbf{V}^*, \mathbf{M}_i\rangle, \qquad i = 1, \ldots, m.$$

Define also the sensitivity matrix $\mathbf{J}_s(\overline{\mathbf{V}}) \in \mathbb{R}^{m\times(2n-1)}$ *as the Jacobian of* $s(\overline{\mathbf{V}})$ *at point* $\overline{\mathbf{V}}$*, which is equal to*

$$\mathbf{J}_s(\overline{\mathbf{V}}) = 2\begin{bmatrix}\overline{\mathbf{M}}_1\,\overline{\mathbf{V}} & \overline{\mathbf{M}}_2\,\overline{\mathbf{V}} & \dots & \overline{\mathbf{M}}_m\,\overline{\mathbf{V}}\end{bmatrix}$$

where

$$\overline{\mathbf{M}}_i \triangleq \begin{bmatrix}\mathrm{Re}\{\mathbf{M}_i[\mathcal{N},\mathcal{N}]\} & -\mathrm{Im}\{\mathbf{M}_i[\mathcal{N},\mathcal{O}]\} \\ \mathrm{Im}\{\mathbf{M}_i[\mathcal{O},\mathcal{N}]\} & \mathrm{Re}\{\mathbf{M}_i[\mathcal{O},\mathcal{O}]\}\end{bmatrix}$$

for every $i = 1, \dots, m$*.*

5.3 Main Results

With no loss of generality, we focus on the classical PF problem in this section. Therefore, we assume that $\mathcal{P}$ is the union of PQ and PV buses, $\mathcal{Q}$ is the set of PQ buses, and $\mathcal{V}$ consists of all PV buses as well as the slack bus. Recall that m is equal to $2n-1$ for the classical PF problem.

5.3.1 Invertibility

The point $\overline{\mathbf{V}} = \overline{\mathbf{1}}_n$ (associated with $\mathbf{V} = \mathbf{1}_n$) is often regarded as a convenient state for linearization of the power network. According to the inverse function theorem, the invertibility of $\mathbf{J}_s(\overline{\mathbf{1}}_n)$ guarantees that the inverse of the function $s(\overline{\mathbf{V}})$ exists in a neighborhood of the point $\overline{\mathbf{1}}_n$. Similarly, it follows from Kantorovich Theorem that, under the same invertibility assumption, the power flow problem (5.1) can be solved using Newton's method by starting from the initial point $\mathbf{1}_n$, provided that there exists a solution sufficiently close to this point. The invertibility of $\mathbf{J}_s(\overline{\mathbf{1}}_n)$ is beneficial not only for Newton's method but also for the SDP problem. This condition will be explored below.

Lemma 5. *The matrix* $\mathbf{J}_s(\overline{\mathbf{1}}_n)$ *is non-singular.*

Proof. It is straightforward to verify that

$$\mathbf{J}_s(\overline{\mathbf{1}}_n) = \begin{bmatrix} 2\mathbf{I}_{n\times n}[\mathcal{N},\mathcal{V}] & \mathbf{B}[\mathcal{N},\mathcal{Q}] & \mathbf{G}[\mathcal{N},\mathcal{P}] \\ \mathbf{0}_{(n-1)\times m_V} & -\mathbf{G}[\mathcal{P},\mathcal{Q}] & \mathbf{B}[\mathcal{P},\mathcal{P}]\end{bmatrix}.$$

By Gaussian elimination, $\mathbf{J}_s(\overline{\mathbf{1}}_n)$ reduces to the matrix

$$\mathbf{S} \triangleq \begin{bmatrix}\mathbf{B}[\mathcal{Q},\mathcal{Q}] & \mathbf{G}[\mathcal{Q},\mathcal{P}] \\ -\mathbf{G}[\mathcal{P},\mathcal{Q}] & \mathbf{B}[\mathcal{P},\mathcal{P}]\end{bmatrix}.$$

Hence, it suffices to prove that $\mathbf{S}$ is not singular. To this end, one can write

$$\det\{\mathbf{S}\} = \det\{\mathbf{S}_1\}\det\{\mathbf{S}_2\},$$

where $\mathbf{S}_1 \triangleq \mathbf{B}[\mathcal{P},\mathcal{P}]$ and $\mathbf{S}_2$ is the Schur complement of $\mathbf{S}_1$ in $\mathbf{S}$, i.e.,

$$\mathbf{S}_2 \triangleq \mathbf{B}[\mathcal{Q},\mathcal{Q}] + \mathbf{G}[\mathcal{Q},\mathcal{P}]\mathbf{B}[\mathcal{P},\mathcal{P}]^{-1}\mathbf{G}[\mathcal{P},\mathcal{Q}].$$

On the other hand, $\mathbf{S}_1$ and $\mathbf{S}_2$ are both symmetric, and in addition Lemma 4 yields that $\mathbf{S}_1 \prec 0$ and $\mathbf{B}[\mathcal{Q},\mathcal{Q}] \prec 0$. This implies that $\mathbf{S}_2 \prec 0$ according to the above equation, which leads to the relation $\det\{\mathbf{S}\} \neq 0$. □

Remark 5. *It is straightforward to verify that Lemma 5 is true for arbitrary networks with shunt elements as long as the matrix* $\mathbf{Y}$ *is generic. In other words, if* $\mathbf{J}_s(\overline{\mathbf{1}}_n)$ *is singular for an arbitrary power network, an infinitesimal perturbation of the susceptance values of the existing lines makes the resulting* $\mathbf{J}_s(\overline{\mathbf{1}}_n)$ *non-singular.*

Definition 27. *Define* $\mathcal{J}$ *as the set of all voltage vectors* $\mathbf{V}$ *for which* $\mathbf{J}_s(\overline{\mathbf{V}}_n)$ *is non-singular.*

5.3.2 Region of Recoverable Voltages

Given a matrix $\mathbf{M}$, we intend to characterize $\mathcal{M}_V(\mathbf{M})$, i.e., the set of all voltage vectors that can be recovered using the convex problem (5.3). In what follows, we first explain the main results of this work and then prove them in Subsection 5.3.3.

Definition 28. *For every vector* $\mathbf{V} \in \mathcal{J}$*, define*

$$\Lambda(\mathbf{V},\mathbf{M}) \triangleq -2\mathbf{J}_s(\overline{\mathbf{V}})^{-1}\overline{\mathbf{M}}\,\overline{\mathbf{V}} \tag{5.9}$$

where

$$\overline{\mathbf{M}} = \begin{bmatrix} \mathrm{Re}\{\mathbf{M}[\mathcal{N},\mathcal{N}]\} & -\mathrm{Im}\{\mathbf{M}[\mathcal{N},\mathcal{O}]\} \\ \mathrm{Im}\{\mathbf{M}[\mathcal{O},\mathcal{N}]\} & \mathrm{Re}\{\mathbf{M}[\mathcal{O},\mathcal{O}]\} \end{bmatrix}.$$

Define also

$$f(\mathbf{V},\mathbf{M}) \triangleq \mathbf{M} + \sum_{i=1}^{2n-1} \Lambda_i(\mathbf{V},\mathbf{M})\,\mathbf{M}_i,$$

where $\Lambda_i(\mathbf{V},\mathbf{M})$ *denotes the* i^{th} *entry of* $\Lambda(\mathbf{V},\mathbf{M})$*.*

Definition 29. *For every $\varepsilon \geq 0$, define $\mathcal{D}_\varepsilon$ as the set of all positive semidefinite Hermitian matrices whose sum of two smallest eigenvalues is greater than ε. Also, denote $\mathcal{D}_0$ as $\mathcal{D}$.*

Since the sum of the two smallest eigenvalues of a Hermitian matrix variable is a concave function of that matrix, the set $\mathcal{D}$ is convex. The first result of this work will be provided below.

Theorem 19. *The interior of the set $\mathcal{M}_V(\mathbf{M})$ can be characterized as*

$$\text{int}\{\mathcal{M}_V(\mathbf{M})\} \cap \mathcal{J} = \{\mathbf{V} \in \mathcal{J} \mid f(\mathbf{V}, \mathbf{M}) \in \mathcal{D}\}. \tag{5.10}$$

The above theorem offers a nonlinear matrix inequality to characterize the interior of the set of recoverable voltage vectors, except for a subset of measure zero of this interior at which the Jacobian of $s(\overline{\mathbf{V}})$ loses rank. A question arises as to whether this interior is non-empty. This problem will be addressed next.

Assumption 1. *The matrix $\mathbf{M}$ satisfies the following properties:*

- $\mathbf{M} \succeq 0$
- 0 *is a simple eigenvalue of* $\mathbf{M}$
- *The vector $\mathbf{1}_n$ belongs to the null space of $\mathbf{M}$.*

Theorem 20. *Consider an arbitrary matrix M satisfying the Assumption 1. The region $\mathcal{M}_V(\mathbf{M})$ has a non-empty interior containing the point $\mathbf{1}_n$.*

Due to (5.4), if $\mathbf{M}$ is chosen as $\mathbf{Y}^*\mathbf{Y}$, it will satisfy the Assumption 1. In that case, the objective of the convex problem (5.3) corresponds to $|I_1|^2 + |I_2|^2 + \cdots + |I_n|^2$, where I_k denotes the current at bus k. In that case, Theorem 20 states that as long as the voltage angles are small enough, a solution of the PF problem can be recovered exactly by means of an SDP relaxation whose objective function reflects the minimization of nodal currents.

There are infinitely many $\mathbf{M}$'s satisfying the Assumption 1, each resulting in a potentially different recoverable set $\mathcal{M}_V(\mathbf{M})$. To find the best M, one can search for a set $\mathcal{M}_V(\mathbf{M})$ with the maximum volume or containing the largest ball. However, these problems are indeed hard to solve due to the non-convexity of $\mathcal{M}_V(\mathbf{M})$. Another approach for seeking a good $\mathbf{M}$ is to first choose a set of voltage vectors scattered in $\mathbb{C}^n$ and then find a matrix $\mathbf{M}$ (if any) whose recoverable set $\mathcal{M}_V(\mathbf{M})$ contains all these points. It turns out that this design problem is in fact convex.

Theorem 21. *Given r arbitrary points $\widehat{\mathbf{V}}_1, \widehat{\mathbf{V}}_2, \ldots, \widehat{\mathbf{V}}_r \in \mathcal{J}$, consider the problem*

$$\begin{aligned}
&\text{find} && \mathbf{M} \in \mathbb{H}_n && && \text{(5.11a)}\\
&\text{subject to} && f(\widehat{\mathbf{V}}_l, \mathbf{M}) \in \mathcal{D}_\varepsilon, && l = 1, 2, \ldots, r && \text{(5.11b)}\\
& && \mathbf{M} \in \mathcal{D}_\varepsilon && && \text{(5.11c)}\\
& && \mathbf{M} \times \mathbf{1}_n = \mathbf{0}_n && && \text{(5.11d)}
\end{aligned}$$

where $\varepsilon > 0$ is an arbitrary constant. The following statements hold:

i) The feasibility problem (5.11) is convex.

ii) There exists a matrix $\mathbf{M}$ satisfying the Assumption 1 such that the associated recoverable set $\mathcal{M}_V(\mathbf{M})$ contains $\widehat{\mathbf{V}}_1, \widehat{\mathbf{V}}_2, \ldots, \widehat{\mathbf{V}}_r$ and a ball around each of these points if and only if the convex problem (5.11) has a solution $\mathbf{M}$.

5.3.3 Proofs

In this part, we will prove Theorems 19, 20 and 21. To this end, it is useful to derive the dual of (5.3). This problem can be stated as

$$\begin{aligned}
&\underset{\mathbf{L} \in \mathbb{R}^{2n-1}}{\text{minimize}} && \mathbf{X}^{\mathrm{T}} \mathbf{L} && \text{(5.12a)}\\
&\text{subject to} && \mathbf{M} + \sum_{i=1}^{2n-1} L_i \mathbf{M}_i \succeq 0 && \text{(5.12b)}
\end{aligned}$$

where the dual variable $\mathbf{L}$ is the vector of all Lagrange multipliers $L_1, ..., L_{2n-1}$, and $\mathbf{X} = [X_1 \ X_2 \ \cdots \ X_m]^T$. For every $L \in \mathbb{R}^{2n-1}$, define

$$\mathbf{A}(\mathbf{L}) = \mathbf{M} + \sum_{i=1}^{2n-1} L_i \mathbf{M}_i. \tag{5.13}$$

We need to develop three lemmas before proving Theorem 19.

Lemma 6. *Consider two arbitrary vectors $\mathbf{V} \in \mathcal{J}$ and $\mathbf{L} \in \mathbb{R}^{2n-1}$. The relation*

$$\mathbf{A}(\mathbf{L})\mathbf{V} = 0 \tag{5.14}$$

holds if and only if $\mathbf{L} = \Lambda(\mathbf{V}, \mathbf{M})$.

Proof. Equation (5.14) can be rearranged as

$$\left[\overline{\mathbf{M}}_1\,\overline{\mathbf{V}} \quad \overline{\mathbf{M}}_2\,\overline{\mathbf{V}} \quad \dots \quad \overline{\mathbf{M}}_{2n-1}\,\overline{\mathbf{V}}\right]\mathbf{L} = -\overline{\mathbf{M}}\,\overline{\mathbf{V}}. \tag{5.15}$$

Hence, due to the invertiblity of $\mathbf{J}_s(\overline{\mathbf{V}})$ and Definition 28, $\Lambda(\mathbf{V},\mathbf{M})$ is the unique solution of (5.15). $\square$

The following Lemma studies the recoverability of a voltage vector $\mathbf{V}$ via the convex problem (5.3).

Lemma 7. *Assume that* $\mathbf{V}$ *is a feasible solution of the power flow problem* (5.1) *such that* $\mathbf{J}_s(\overline{\mathbf{V}})$ *is invertible. Then, strong duality holds between the primal problem* (5.3) *and the dual problem* (5.12)*. In addition, the following two statements are equivalent:*

i) $\mathbf{VV}^*$ *is an optimal solution for the primal problem* (5.3)*.*

ii) $\Lambda(\mathbf{V},\mathbf{M})$ *is a feasible point for the dual problem* (5.12)*.*

Proof. By assumption, the matrix $\mathbf{VV}^*$ is a feasible point for the problem (5.3). In order to show the strong duality, it suffices to build a strictly feasible point $\tilde{\mathbf{L}}$ for the dual problem. To this end, we set the Lagrange multipliers corresponding to active power, reactive power and voltage magnitude specifications equal to 0, -1, and a constant c, respectively. Then, one can write

$$\mathbf{A}(\tilde{\mathbf{L}})[\mathcal{Q},\mathcal{Q}] = -\mathbf{B}[\mathcal{Q},\mathcal{Q}] \tag{5.16}$$

$$\mathbf{A}(\tilde{\mathbf{L}})[\mathcal{V},\mathcal{V}] = c\;\mathbf{I}_{|\mathcal{V}|\times|\mathcal{V}|} \tag{5.17}$$

(recall that $\mathbf{A}(\tilde{\mathbf{L}})[\mathcal{Q},\mathcal{Q}]$ denotes a submatrix of $\mathbf{A}(\tilde{\mathbf{L}})$ induced by the index set $\mathcal{Q}$). Since $\mathcal{V}\cup\mathcal{Q}=\mathcal{N}$ and $\mathcal{V}$ has at least one member (the slack bus), $\mathbf{B}[\mathcal{Q},\mathcal{Q}]$ is negative definite according to Lemma 4. Now, the strict positive definiteness of $\mathbf{A}(\tilde{\mathbf{L}})$ can be ensured by choosing a sufficiently large c.

$(i) \Rightarrow (ii)$: Due to strong duality, if $\mathbf{VV}^*$ is primal optimal, there exists a vector $\mathbf{L}'\in\mathbb{R}^{2n-1}$ that is dual feasible:

$$\mathbf{A}(\mathbf{L}')\succeq 0 \tag{5.18}$$

and satisfies the complementary slackness:

$$\langle\mathbf{VV}^*,\mathbf{A}(\mathbf{L}')\rangle = 0. \tag{5.19}$$

Hence,

$$\langle \mathbf{V}\mathbf{V}^*, \mathbf{A}(\mathbf{L}')\rangle = \text{trace}\left\{\mathbf{V}\mathbf{V}^*\mathbf{A}(\mathbf{L}')\right\} = \mathbf{V}^*\mathbf{A}(\mathbf{L}')\mathbf{V} = 0.$$

It follows from this equation and (5.18) that $\mathbf{A}(\mathbf{L}')\mathbf{V} = 0$. The, according to Lemma 6, we have $\mathbf{L}' = \Lambda(\mathbf{V},\mathbf{M})$, which implies that $\Lambda(\mathbf{V},\mathbf{M})$ is dual feasible.

$(ii) \Rightarrow (i)$: It is shown in Lemma 6 that $\mathbf{A}(\Lambda(\mathbf{V},\mathbf{M}))\mathbf{V} = 0$. Therefore,

$$\langle \mathbf{V}\mathbf{V}^*, \mathbf{A}(\Lambda(\mathbf{V},\mathbf{M}))\rangle = 0.$$

As a result, $\Lambda(\mathbf{V},\mathbf{M})$ acts as a dual certificate for the optimality of $\mathbf{V}\mathbf{V}^*$. □

Lemma 8. *Suppose that zero and $\delta > 0$ are the two smallest eigenvalues of an $m \times m$ matrix $\mathbf{A}_1 \in \mathcal{D}$. Then, for every matrix $\mathbf{A}_2 \in \mathbb{H}_m$ with 0 as its eigenvalue, the relation $\mathbf{A}_2 \in \mathcal{D}$ holds if $\|\mathbf{A}_1 - \mathbf{A}_2\|_F < \sqrt{\delta}$.*

Proof. Let $\mathbf{Q}\Delta\mathbf{Q}^*$ denote the eigenvalue decomposition of $\mathbf{A}_1$ such that $\Delta_{mm} = 0$. One can decompose Δ and $\mathbf{A}_2$ as

$$\Delta = \begin{bmatrix} \Delta_1 & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{Q}^*\mathbf{A}_2\mathbf{Q} = \begin{bmatrix} \Delta_2 & \mathbf{u} \\ \mathbf{u}^* & u \end{bmatrix}, \tag{5.20}$$

for some matrices $\Delta_1, \Delta_2 \in \mathbb{H}_{m-1}$, $\mathbf{u} \in \mathbb{C}^{m-1}$ and $u \in \mathbb{R}$. Then, for every vector $\mathbf{x} \in \mathbb{C}^{m-1}$ of length 1, one can write

$$\mathbf{x}^*\Delta_2\mathbf{x} = \mathbf{x}^*\Delta_1\mathbf{x} - \mathbf{x}^*(\Delta_1 - \Delta_2)\mathbf{x} \tag{5.21}$$

$$\geq \delta - \|\Delta_1 - \Delta_2\|_F^2 \tag{5.22}$$

$$\geq \delta - \|\mathbf{A}_1 - \mathbf{A}_2\|_F^2 > 0 \tag{5.23}$$

It can be concluded from the above equation that $\Delta_2 \succ 0$. On the other hand, according to Schur complement, we have

$$0 = \det\{\mathbf{A}_2\} = \det\{\Delta_2\} \times \det\{u - \mathbf{u}^*\Delta_2^{-1}\mathbf{u}\}, \tag{5.24}$$

which implies that $u - \mathbf{u}^*\Delta_2^{-1}\mathbf{u} = 0$. Using Schur complement once more, it can be concluded that $\mathbf{A}_2 \succeq 0$. Moreover,

$$\text{rank}\{\mathbf{A}_2\} = \text{rank}\{\Delta_2\} + \text{rank}\{u - \mathbf{u}^*\Delta_2^{-1}\mathbf{u}\} = n - 1,$$

implying that $\mathbf{A}_2 \in \mathcal{D}$. □

Proof of Theorem 19: We first need to show that $\{\mathbf{V} \in \mathcal{J} \mid f(\mathbf{V}, \mathbf{M}) \in \mathcal{D}\}$ is an open set. Consider a vector $\mathbf{V}$ such that $f(\mathbf{V}, \mathbf{M}) \in \mathcal{D}$ and let δ denote the second smallest eigenvalue of $f(\mathbf{V}, \mathbf{M})$. Due to the continuity of $\det\{\mathbf{J}_s(\cdot)\}$ and $f(\cdot, \mathbf{M})$, there exists a neighborhood $\mathcal{B} \in \mathbb{C}^n$ around $\mathbf{V}$ such that for every $\mathbf{V}'$ within this neighborhood, $f(\mathbf{V}', \mathbf{M})$ is well defined (i.e., $\mathbf{V}' \in \mathcal{J}$) and

$$\|f(\mathbf{V}', \mathbf{M}) - f(\mathbf{V}, \mathbf{M})\|_F < \sqrt{\delta}. \tag{5.25}$$

Hence, according to Lemma 8, we have $f(\mathbf{V}', \mathbf{M}) \in \mathcal{D}$ for every $\mathbf{V}' \in \mathcal{B}$. This proves that $\{\mathbf{V} \in \mathcal{J} \mid f(\mathbf{V}, \mathbf{M}) \in \mathcal{D}\}$ is an open set.

Now, consider a vector $\mathbf{V} \in \mathcal{J}$ such that $f(\mathbf{V}, \mathbf{M}) \in \mathcal{D}$. The objective is to show that $\mathbf{V} \in \text{int}\{\mathcal{M}_V(\mathbf{M})\}$. Notice that since $f(\mathbf{V}, \mathbf{M})$ is assumed to be in the set $\mathcal{D}$, the vector $\Lambda(\mathbf{V}, \mathbf{M})$ is a feasible point for the dual problem (5.12). Therefore, it follows from Lemma 7 that the matrix $\mathbf{V}\mathbf{V}^*$ is an optimal solution for the primal problem (5.3). In addition, every solution $\mathbf{W}$ must satisfy

$$\langle f(\mathbf{V}, \mathbf{M}), \mathbf{W} \rangle = 0. \tag{5.26}$$

According to Lemma 6, $\mathbf{V}$ is an eigenvector of $f(\mathbf{V}, \mathbf{M})$ corresponding to the eigenvalue 0. Therefore, since $f(\mathbf{V}, \mathbf{M}) \succeq 0$ and $\text{rank}\{f(\mathbf{V}, \mathbf{M})\} = n - 1$, every positive semidefinite matrix $\mathbf{W}$ satisfying (5.26) is equal to $c\mathbf{V}\mathbf{V}^*$ for a nonnegative constant c. This concludes that $\mathbf{V}\mathbf{V}^*$ is the unique solution to (5.3), and therefore $\mathbf{V}$ belongs to $\mathcal{M}_V(\mathbf{M})$. Since $\{\mathbf{V} \in \mathcal{J} \mid f(\mathbf{V}, \mathbf{M}) \in \mathcal{D}\}$ is shown to be an open set, the above result can be translated as

$$\{\mathbf{V} \in \mathcal{J} \mid f(\mathbf{V}, \mathbf{M}) \in \mathcal{D}\} \subseteq \text{int}\{\mathcal{M}_V(\mathbf{M})\} \cap \mathcal{J}. \tag{5.27}$$

In order to complete the proof, it is required to show that $\text{int}\{\mathcal{M}_V(\mathbf{M})\} \cap \mathcal{J}$ is a subset of $\{\mathbf{V} \in \mathcal{J} \mid f(\mathbf{V}, \mathbf{M}) \in \mathcal{D}\}$. To this end, consider a vector $\mathbf{V} \in \text{int}\{\mathcal{M}_V(\mathbf{M})\} \cap \mathcal{J}$. This means that $\mathbf{V}\mathbf{V}^*$ is a solution to (5.3), and therefore $f(\mathbf{V}, \mathbf{M}) \succeq 0$, due to Lemma 7. To prove the aforementioned inclusion by contradiction, suppose that $f(\mathbf{V}, \mathbf{M}) \notin \mathcal{D}$, implying that 0 is an eigenvalue of $f(\mathbf{V}, \mathbf{M})$ with multiplicity at least 2. Let $\mathbf{U}$ denote a second eigenvector corresponding to the eigenvalue 0 such that $\mathbf{V}^*\mathbf{U} = 0$. Since $\mathbf{J}_s(\overline{\mathbf{V}}) \neq 0$, in light of the inverse function theorem, there exists a constant $\varepsilon_0 > 0$ with the property that for every $\varepsilon \in [0, \varepsilon_0]$, there is a vector $\mathbf{T}_\varepsilon \in \mathbb{C}^n$ satisfying the relation

$$s(\overline{\mathbf{T}}_\varepsilon) = s(\overline{\mathbf{V}}) + \varepsilon s(\overline{\mathbf{U}}) \tag{5.28}$$

where the function $s(\cdot)$ is defined in Definition 26. This means that the rank-2 matrix

$$\mathbf{W} = \mathbf{V}\mathbf{V}^* + \varepsilon\mathbf{U}\mathbf{U}^* \tag{5.29}$$

is a solution to the problem (5.3) associated with the dual certificate $\Lambda(\mathbf{V}, \mathbf{M})$, and therefore $\mathbf{T}_\varepsilon \notin \mathcal{M}_V(\mathbf{M})$. This contradicts the previous assumption that $\mathbf{V} \in \mathrm{int}\{\mathcal{M}_V(\mathbf{M})\}$. Therefore, we have $f(\mathbf{V}, \mathbf{M}) \in \mathcal{D}$, which completes the proof. ∎

Proof of Theorem 20: It can be inferred from Lemma 5 that $\overline{\mathbf{1}}_n \in \mathcal{J}$. On the other hand, since $\mathbf{M} \times \mathbf{1}_n = 0$, we have

$$\Lambda(\mathbf{1}_n, \mathbf{M}) = \mathbf{0}_{2n-1}, \tag{5.30}$$

which concludes that

$$f(\mathbf{1}_n, \mathbf{M}) = \mathbf{M} \in \mathcal{D}. \tag{5.31}$$

Therefore, it follows from Theorem 19 that $\mathbf{1}_n \in \mathrm{int}\{\mathcal{M}_V(\mathbf{M})\}$. ∎

Proof of Theorem 21: Part (i) is proved by noting that the sum of the two smallest eigenvalues of a matrix is a concave function and that $f(\widehat{\mathbf{V}}_l, \mathbf{M})$ is a linear function with respect to $\mathbf{M}$.

Part (ii) follows immediately from Theorems 19 and 20. ∎

5.4 Illustrative Examples

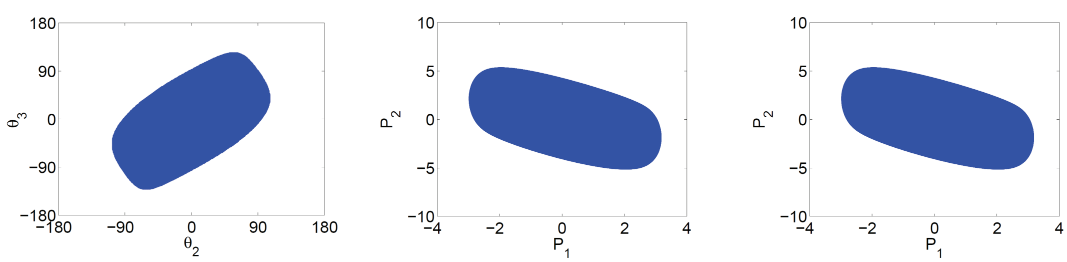


Figure 5.1: These plots show the outcome of the convex problem (5.3) for Example 1. (a): the angle region that can be recovered via problem (5.3), (b): the power region that can be recovered via problem (5.3), (c): the entire feasible power region for the 3-bus system.

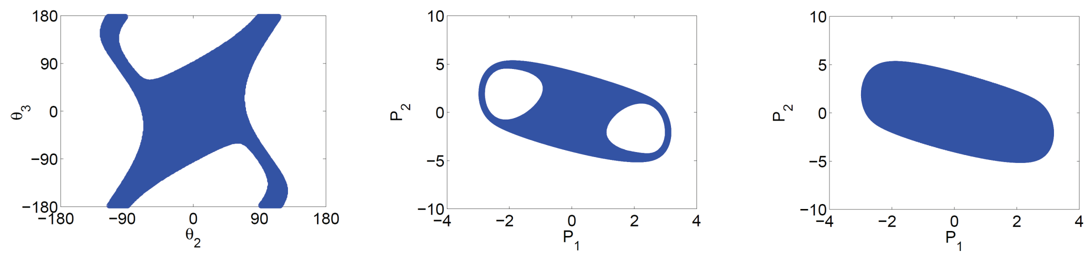


Figure 5.2: These plots show the outcome of the convex problem (5.3) for Example 2. (a): the angle region that can be recovered via problem (5.3), (b): the power region for which problem (5.3), (c): the entire power region for the 3-bus system.

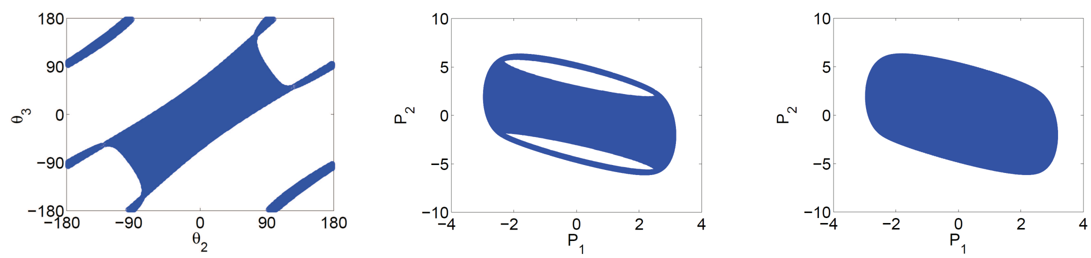


Figure 5.3: These plots show the outcome of the convex problem (5.3) for Example 3. (a): the angle region that can be recovered via problem (5.3), (b): the power region that can be recovered via problem (5.3), (c): the entire feasible power region for the 3-bus system.

Example 1: Consider a 3-bus power network with the line admittances

$$\begin{aligned} y_{12} &= -2\mathbf{i}, \\ y_{13} &= -1\mathbf{i}, \\ y_{23} &= -3\mathbf{i}. \end{aligned} \tag{5.32}$$

Assume that the voltage magnitudes are all equal to 1, and that the active powers P_1, P_2, P_3 are all given. The PF problem aims to find the complex voltage vector $\mathbf{V}$. To this end, let $\mathbf{V}$ be parameterized as

$$\mathbf{V} = \left[\begin{array}{ccc} 1 & e^{-\theta_2 \mathbf{i}} & e^{-\theta_3 \mathbf{i}} \end{array}\right]^*, \tag{5.33}$$

where $\theta_2, \theta_3 \in [-180°, 180°]$. Consider the convex problem (5.3) with M equal to $\mathbf{Y}^*\mathbf{Y}$. This optimization solves the PF problem exactly if and only if (θ_2, θ_3) belongs to the region depicted in Figure 5.1(a). Alternatively, the above convex optimization finds a solution of the PF problem if and only if (P_1, P_2) is contained in the region provided in Figure 5.1(b). On the other hand, the set of all feasible vectors (P_1, P_2) for the power network specified by (5.32) is drawn in Figure 5.1(c). By comparing Figures 5.1(b) and 5.1(c), it can be concluded that the convex optimization (5.3) finds a solution of PF for every feasible vector (P_1, P_2, P_3) (note that $P_3 = -P_1 - P_2$). However, the reason why the angle region in Figure 5.1(a) is not the entire box $[-180, 180] \times [-180, 180]$ is that some instances of the PF problem have multiple solutions (multiple values for the pair (θ_2, θ_3)) and the proposed convex optimization finds just one of those solutions.

Example 2: This example is similar to Example 1 with the only difference that y_{12} is changed to $2\mathbf{i}$. The outcomes are plotted in Figure 5.2. It can be seen in Fig 5.2(a) that the set of values for (θ_2, θ_3) that can be successfully recovered by the convex optimization (5.3) is connected but non-convex. Although this non-convexity is observed here for a non-realistic (negative) line impedance in a 3-bus network, the same phenomenon occurs for larger networks with legitimate positive inductances.

Example 3: This example is similar to Example 1 with the only difference that y_{23} is changed to $4\mathbf{i}$. The outcomes are plotted in Figure 5.3. It can be seen in Fig 5.3(a) that the set of values for (θ_2, θ_3) that can be successfully recovered by the convex optimization (5.3) is disconnected.

5.5 Simulation Results

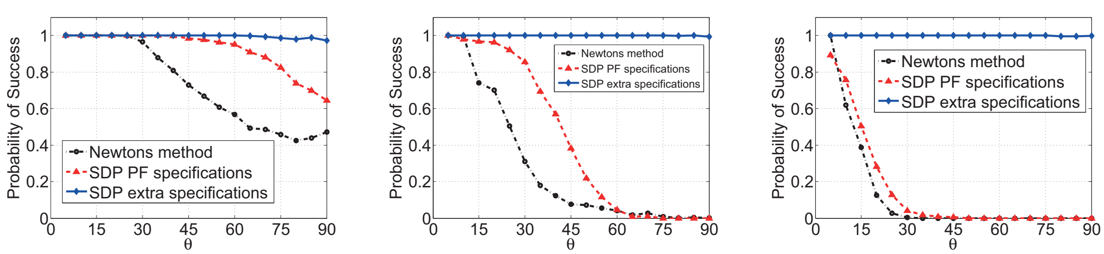


Figure 5.4: These plots show the probability of success for Newton's method, SDP relaxation, and SDP relaxation with extra specifications for (a): IEEE 9-bus system, (b): New England 39-bus system, and (c): IEEE 57-bus system.

In order to evaluate the performance of the SDP relaxation for the PF problem, we perform numerical simulations on the IEEE 9-bus, New England 39-bus, and IEEE 57-bus systems [Zimmerman *et al.*, 2009]. Three recovery methods are considered for each test case:

1. **Newton's method:** We evaluate the probability of convergence for Newton's method in polar coordinates for the classical PF problem with $2n-1$ specifications, where the starting point is $V_k = 1\angle 0^\circ$ for every $k \in \mathcal{N}$.

2. **SDP relaxation:** The probability of obtaining a rank-1 solution for the SDP relaxation (5.3) with $\mathbf{M} = \mathbf{Y}^*\mathbf{Y}$ is evaluated, where the same set of specifications as in Newton's method is used.

3. **SDP relaxation with extra specifications:** The probability of obtaining a rank-1 solution for the SDP relaxation (5.3) with $\mathbf{M} = \mathbf{Y}^*\mathbf{Y}$ is evaluated, under extra specifications compared to the classical PF problem. It is assumed that active powers are measured at PV and PQ buses, reactive powers are measured at PQ buses, and voltages magnitudes are measured at all buses (as opposed to only PV and slack buses).

For different values of θ, we generated 500 specification sets $(X_1, ..., X_m)$ by randomly choosing voltage vectors whose magnitudes and phases are uniformly drawn from the intervals $[0.9, 1.1]$ and $[-\theta, \theta]$, respectively. We then exploited each of the three above methods to find a feasible voltage vector associated with each specification set. The results are depicted in Figure 5.4.

5.6 Summary

In this chapter, the classical power flow (PF) problem is studied by means of a semidefinite programming (SDP) relaxation. The proposed method is based on lifting the nonlinear equations to a higher dimension, where the equations can be cast linearly in terms of a positive semidefinite and rank-one matrix variable. This leads to a family of convex optimization problems, each in the form of a semidefinite program with a linear objective function that captures the rank-one constraint as a proxy. The proposed convex optimization problems are guaranteed to solve the PF problem if the voltage angles are small. The region of complex voltages that can be recovered through each problem is characterized by a nonlinear matrix inequality. Moreover, the problem of finding a convenient objective function for SDP that can recover a given set of voltage vectors and a neighborhood around each vector can itself be cast as a convex problem. The simulation results show the superiority of the proposed method over the traditional Newton's method.

Part III

Distributed Control

Chapter 6

Convex Relaxation for Optimal Distributed Control Problem

This chapter is concerned with the optimal distributed control (ODC) problem for discrete-time deterministic and stochastic systems. The objective is to design a fixed-order distributed controller with a pre-specified structure that is globally optimal with respect to a quadratic cost functional. It is shown that this NP-hard problem has a quadratic formulation, which can be relaxed to a semidefinite program (SDP). If the SDP relaxation has a rank-1 solution, a globally optimal distributed controller can be recovered from this solution. By utilizing the notion of treewidth, it is proved that the nonlinearity of the ODC problem appears in such a sparse way that an SDP relaxation of this problem has a matrix solution with rank at most 3. Since the proposed SDP relaxation is computationally expensive for a large-scale system, a computationally-cheap SDP relaxation is also developed with the property that its objective function indirectly penalizes the rank of the SDP solution. Various techniques are proposed to approximate a low-rank SDP solution with a rank-1 matrix, leading to recovering a near-global controller together with a bound on its optimality degree. The above results are developed for both finite-horizon and infinite horizon ODC problems. While the finite-horizon ODC is investigated using a time-domain formulation, the infinite-horizon ODC problem for both deterministic and stochastic systems is studied via a Lyapunov formulation. The SDP relaxations developed in this work are exact for the design of a centralized controller, hence serving as an alternative for solving Riccati equations. The efficacy of the proposed SDP relaxations is elucidated in numerical examples.

6.1 Introduction

The area of decentralized control is created to address the challenges arising in the control of real-world systems with many interconnected subsystems. The objective is to design a structurally constrained controller—a set of partially interacting local controllers—with the aim of reducing the computation or communication complexity of the overall controller. The local controllers of a decentralized controller may not be allowed to exchange information. The term *distributed control* is often used in lieu of decentralized control in the case where there is some information exchange between the local controllers (possibly distributed over a geographical area). It has been long known that the design of a globally optimal decentralized (distributed) controller is a daunting task because it amounts to an NP-hard optimization problem in general [Witsenhausen, 1968; Tsitsiklis and Athans, 1984]. Great effort has been devoted to investigating this highly complex problem for special types of systems, including spatially distributed systems [D'Andrea and Dullerud, 2003; Bamieh *et al.*, 2002; Langbort *et al.*, 2004; Motee and Jadbabaie, 2008; Dullerud and D'Andrea, 2004], dynamically decoupled systems [Keviczky *et al.*, 2006; Borrelli and Keviczky, 2008], weakly coupled systems [Siljak, 1996], and strongly connected systems [Lavaei, 2012]. Another special case that has received considerable attention is the design of an optimal static distributed controller [Fardad *et al.*, 2009; Lin *et al.*, 2011]. Early approaches for the optimal decentralized control problem were based on parameterization techniques [Geromel *et al.*, 1994; Date and Chow, 1993], which were then evolved into matrix optimization methods [Scorletti and Duc, 2001; Zhai *et al.*, 2001]. In fact, with a structural assumption on the exchange of information between subsystems, the performance offered by linear static controllers may be far less than the optimal performance achievable using a nonlinear time-varying controller [Witsenhausen, 1968].

Due to the recent advances in the area of convex optimization, the focus of the existing research efforts has shifted from deriving a closed-form solution for the above control synthesis problem to finding a convex formulation of the problem that can be efficiently solved numerically [de Castro and Paganini, 2002; Bamieh and Voulgaris, 2005; Qi *et al.*, 2004; Dvijotham *et al.*, 2013; Matni and Doyle, 2013]. This has been carried out in the seminal work [Rotkowitz and Lall, 2006] by deriving a sufficient condition named quadratic invariance, which has been specialized in [Shah and Parrilo, 2013] by deploying the concept of partially order sets. These conditions have been further investigated in several other papers [Lessard and Lall, 2012; Lamperski and Doyle, 2013; Rotkowitz and Martins, 2012]. A different approach is taken in the recent papers [Tanaka and Langbort, 2011] and [Rantzer, 2012], where it has been shown that the distributed control problem can be cast as a convex optimization for positive systems.

There is no surprise that the decentralized control problem is computationally hard to solve. This is a consequence of the fact that several classes of optimization problems, including polynomial optimization and quadratically-constrained quadratic program as a special case, are NP-hard in the worst case. Due to the complexity of such problems, various convex relaxation methods based on linear matrix inequality (LMI), semidefinite programming (SDP), and second-order cone programming (SOCP) have gained popularity [Vandenberghe and Boyd, 1996b; Boyd and Vandenberghe, 2004]. These techniques enlarge the possibly non-convex feasible set into a convex set characterizable via convex functions, and then provide the exact or a lower bound on the optimal objective value. The effectiveness of these techniques has been reported in several papers. For instance, [Goemans and Williamson, 1995] shows how SDP relaxation can be used to find better approximations for maximum cut (MAX CUT) and maximum 2-satisfiability (MAX 2SAT) problems. Another approach is proposed in [Goemans and Williamson, 2004] to solve the max-3-cut problem via a complex SDP. The approaches in [Goemans and Williamson, 1995] and [Goemans and Williamson, 2004] have been generalized in several papers, including [Nesterov, 1998; He *et al.*, 2010].

Semidefinite programming relaxation usually converts an optimization with a vector variable to a convex optimization with a matrix variable, via a lifting technique. The exactness of the relaxation can then be interpreted as the existence of a low-rank (e.g., rank-1) solution for SDP relaxation. Several papers have studied the existence of a low-rank solution to matrix optimizations with linear or nonlinear (e.g., LMI) constraints. For instance, the papers [Pataki, 1998; Madani *et al.*, 2014b] provide upper bounds on the lowest rank among all solutions of a feasible LMI problem. A rank-1 matrix decomposition technique is developed in [Sturm and Zhang, 2003] to find a rank-1 solution whenever the number of constraints is small. It has been shown in [Lavaei and Low, 2012] and [Sojoudi and Lavaei, 2012] that SDP relaxation is able to solve a large class of non-convex energy-related optimization problems performed over power networks. They related the success of the relaxation to the hidden structure of those optimizations induced by the physics of a power grid. Inspired by this positive result, they developed the notion of "nonlinear optimization over graph" in [Sojoudi and Lavaei, 2014; Sojoudi and Lavaei, 2013a; Sojoudi and Lavaei, 2013b]. The technique maps the structure of an abstract nonlinear optimization into a graph from which the exactness of SDP relaxation may be concluded. By adopting the graph technique developed in [Sojoudi and Lavaei, 2014], the objective of the present work is to study the potential of SDP relaxation for the optimal distributed control problem.

In this chapter, we cast the optimal distributed control (ODC) problem as a non-convex optimization problem with only quadratic scalar and matrix constraints, from which an SDP relaxation can be obtained. The goal is to show that this relaxation has a low-rank solution whose rank depends on the topology of the controller to be designed. In particular, we prove that the design of a static distributed controller with a pre-specified structure amounts to a sparse SDP relaxation with a solution of rank at most 3. This positive result is a consequence of the fact that the sparsity graph associated with the underlying optimization problem has a small treewidth. The notion of "treewidth" used in this chapter could potentially help to understand how much approximation is needed to make the ODC problem tractable. This is due to a recent result stating that a rank-constrained optimization problem has an almost equivalent convex formulation whose size depends on the treewidth of a certain graph [Bienstock and Munoz, 2015]. In this work, we also discuss how to round the rank-3 SDP matrix to a rank-1 matrix in order to design a near-global controller.

The results of this work hold true for both a time-domain formulation corresponding to a finite-horizon control problem and a Lyapunov-domain formulation associated with an infinite-horizon deterministic/stochastic control problem. We first investigate the ODC problem for the deterministic systems and then the ODC problem for stochastic systems. Our approach rests on formulating each of these problems as a rank-constrained optimization from which an SDP relaxation can be derived. With no loss of generality, this chapter focuses on the design of a static controller. Since the proposed relaxations with guaranteed low-rank solutions are computationally expensive, we also design computationally-cheap SDP relaxations for numerical purposes. Afterwards, we develop some heuristic methods to recover a near-optimal controller from a low-rank SDP solution. Note that the computationally-cheap SDP relaxations associated with the infinite-horizon ODC are exact in both deterministic and stochastic cases for the classical (centralized) LQR and H_2 problems. Although the focus of the chapter is static controllers, its results can be naturally generalized to the dynamic case as well.

We conduct case studies on a mass-spring system and 100 random systems to elucidate the efficacy of the proposed relaxations. In particular, the design of many near-optimal structured controllers with global optimality degrees above 99% will be demonstrated. An additional study is conducted on electrical power systems in our paper [Kalbat *et al.*, 2014].

This work is organized as follows. The problem is introduced in Section 6.2, and then the SDP relaxation of a quadratically-constrained quadratic program (QCQP) is studied via a graph-theoretic approach. Three different SDP relaxations of the finite-horizon deterministic ODC problem are presented for the static controller design in Section 6.3. The infinite-horizon deterministic ODC problem is studied in Section 6.4. The results are generalized to an infinite-horizon stochastic ODC problem in Section 6.5, followed by a brief discussion on dynamic controllers in Section 6.6. Various experiments and simulations are provided in Section 7.5. Concluding remarks are drawn in Section 7.6.

6.1.1 Notations

$\mathbb{R}$, $\mathbb{S}_n$ and $\mathbb{S}_n^+$ denote the sets of real numbers, $n \times n$ symmetric matrices and $n \times n$ positive semidefinite matrices, respectively. The m by n rectangular identity matrix whose (i,j) entry is equal to the Kronecker delta δ_{ij} is denoted by $I_{m\times n}$ or alternatively I_n when $m = n$. $\text{rank}\{W\}$ and $\text{trace}\{W\}$ denote the rank and trace of a matrix W. The notation $W \succeq 0$ means that W is symmetric and positive semidefinite. Given a matrix W, its (l,m) entry is denoted as W_{lm}. Given a block matrix $\mathbf{W}$, its (l,m) block is shown as $\mathbf{W}_{lm}$. Given a matrix M, its Moore Penrose pseudoinverse is denoted as M^+. The superscript $(\cdot)^{\text{opt}}$ is used to show a *globally optimal value* of an optimization parameter. The symbols $(\cdot)^T$ and $\|\cdot\|$ denote the transpose and 2-norm operators, respectively. The symbols $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_F$ denote the Frobenius inner product and norm of matrices, respectively. The notation $|.|$ shows the size of a vector, the cardinality of a set or the number of vertices a graph, depending on the context. The expected value of a random variable x is shown as $\mathcal{E}\{x\}$. The submatirx of M formed by rows form the set $\mathcal{S}_1$ and columns from the set $\mathcal{S}_2$ is denoted by $M\{\mathcal{S}_1, \mathcal{S}_2\}$. The notation $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ implies that $\mathcal{G}$ is a graph with the vertex set $\mathcal{V}$ and the edge set $\mathcal{E}$.

6.2 Preliminaries

In this chapter, the Optimal Distributed Control (ODC) problem is studied based on the following steps:

- First, the problem is cast as a non-convex optimization problem with only quadratic scalar and/or matrix constraints.
- Second, the resulting non-convex problem is formulated as a rank-constrained optimization.
- Third, a convex relaxation of the problem is derived by dropping the non-convex rank constraint.
- Last, the rank of the minimum-rank solution of the SDP relaxation is analyzed.

Since there is no unique SDP relaxation for the ODC problem, a major part of this work is devoted to designing a sparse quadratic formulation of the ODC problem with a guaranteed low-rank SDP solution. To achieve this goal, a graph is associated to each SDP, which is then sparsified to contrive a problem with a low-rank solution. Note that this chapter significantly improves the recent result in [Lavaei, 2013].

6.2.1 Problem Formulation

The following variations of the Optimal Distributed Control (ODC) problem are studied in this work.

6.2.1.1 Finite-horizon Deterministic ODC Problem

Consider the discrete-time system

$$x[\tau+1] = Ax[\tau] + Bu[\tau], \qquad \tau = 0, 1, \ldots, p-1 \tag{6.1a}$$

$$y[\tau] = Cx[\tau], \qquad \tau = 0, 1, \ldots, p \tag{6.1b}$$

with the known matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$, and $x[0] = x_0 \in \mathbb{R}^n$, where p is the terminal time. The goal is to design a distributed static controller $u[\tau] = Ky[\tau]$ minimizing a quadratic cost function under the constraint that the controller gain K must belong to a given linear subspace $\mathcal{K} \subseteq \mathbb{R}^{m\times r}$. The set $\mathcal{K}$ captures the sparsity structure of the unknown constrained controller and, more specifically, it contains all $m \times r$ real-valued matrices with forced zeros in certain entries. The cost function

$$\sum_{\tau=0}^{p} \left(x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau]\right) + \alpha \|K\|_F^2 \tag{6.2}$$

is considered in this work, where α is a nonnegative scalar, and Q and R are positive-semidefinite matrices. This problem will be studied in Section 6.3.

Remark 6. *The third term in the objective function of the ODC problem is a soft penalty term aimed at avoiding a high-gain controller. Instead of this soft penalty, we could impose a hard constraint $\|K\|_F \leq \beta$, for a given number β. The method to be developed later can be adopted for the modified case.*

6.2.1.2 Infinite-horizon Deterministic ODC Problem

The infinite-horizon ODC problem corresponds to the case $p = +\infty$ subject to the additional constraint that the controller must be stabilizing. This problem will be studied through a Lyapunov domain formulation in Section 6.4.

6.2.1.3 Infinite-horizon Stochastic ODC Problem

Consider the discrete-time stochastic system

$$x[\tau+1] = Ax[\tau] + Bu[\tau] + Ed[\tau], \qquad \tau = 0, 1, \ldots \tag{6.3a}$$

$$y[\tau] = Cx[\tau] + Fv[\tau], \qquad \tau = 0, 1, \ldots \tag{6.3b}$$

with the known matrices A, B, C, E, and F, where $d[\tau]$ and $v[\tau]$ denote the input disturbance and measurement noise, which are assumed to be zero-mean white-noise random processes. The ODC problem for the above system will be investigated in Section 6.5.

The extension of the above results to the design of dynamic controllers will be briefly discussed in Section 6.6.

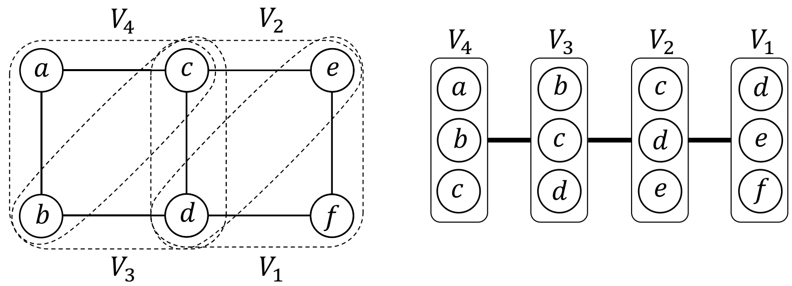


Figure 6.1: A minimal tree decomposition for a ladder graph.

6.2.2 Graph Theory Preliminaries

Definition 30. *For two simple graphs $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2)$ with the same set of vertices, their union is defined as $\mathcal{G}_1 \cup \mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_1 \cup \mathcal{E}_2)$.*

Definition 31. *The representative graph of an $n \times n$ symmetric matrix W, denoted by $\mathcal{G}(W)$, is a simple graph with n vertices whose edges are specified by the locations of the nonzero off-diagonal entries of W. In other words, two disparate vertices i and j are connected if W_{ij} is nonzero.*

Consider a graph $\mathcal{G}$ identified by a set of "vertices" and a set of edges. This graph may have cycles in which case it cannot be a tree. Using the notion to be explained below, we can map $\mathcal{G}$ into a tree $\mathcal{T}$ identified by a set of "nodes" and a set of edges where each node of $\mathcal{T}$ contains a group of vertices of $\mathcal{G}$.

Definition 32 (Treewidth)**.** *Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a tree $\mathcal{T}$ is called* a tree decomposition of *$\mathcal{G}$ if it satisfies the following properties:*

1. *Every node of $\mathcal{T}$ corresponds to and is identified by a subset of $\mathcal{V}$.*

2. *Every vertex of $\mathcal{G}$ is a member of at least one node of $\mathcal{T}$.*

3. *For every edge (i, j) of $\mathcal{G}$, there should be a node in $\mathcal{T}$ containing vertices i and j simultaneously.*

4. *Given an arbitrary vertex k of $\mathcal{G}$, the subgraph induced by all nodes of $\mathcal{T}$ containing vertex k must be connected (more precisely, a tree).*

Each node of $\mathcal{T}$ is a bag (collection) of vertices of $\mathcal{G}$ and hence it is referred to as bag. The width of $\mathcal{T}$ is the cardinality of its biggest bag minus one. The treewidth of $\mathcal{G}$ is the minimum width over all possible tree decompositions of $\mathcal{G}$ and is denoted by $\mathrm{tw}(\mathcal{G})$.

Every graph has a trivial tree decomposition with one single bag consisting of all its vertices. Figure 6.1 shows a graph $\mathcal{G}$ with 6 vertices named a, b, c, d, e, f, together with its minimal tree decomposition $\mathcal{T}$. Every node of $\mathcal{T}$ is a set containing three members of $\mathcal{V}$. The width of this decomposition is therefore equal to 2. Observe that the edges of the tree decomposition are chosen in such a way that every subgraph induced by all bags containing each member of $\mathcal{V}$ is a tree (as required by Property 4 stated before).

Note that if $\mathcal{G}$ is a tree itself, it has a minimal tree decomposition $\mathcal{T}$ such that: each bag corresponds to two connected vertices of $\mathcal{G}$ and every two adjacent bags in $\mathcal{T}$ share a vertex in common. Therefore, the treewidth of a tree is equal to 1. The reader is referred to [Bodlaender, 1994] for a comprehensive literature review on treewidth.

6.2.3 SDP Relaxation

The objective of this subsection is to study SDP relaxation of a quadratically-constrained quadratic program (QCQP) using a graph-theoretic approach. Consider the standard nonconvex QCQP problem

$$\begin{aligned} &\underset{x\in\mathbb{R}^n}{\text{minimize}} && f_0(x) && && (6.4a)\\ &\text{subject to} && f_k(x) \leq 0, && k = 1, \ldots, q, && (6.4b) \end{aligned}$$

where $f_k(x) = x^T A_k x + 2b_k^T x + c_k$ for $k = 0, \ldots, q$. Define

$$F_k \triangleq \begin{bmatrix} c_k & b_k^T \\ b_k & A_k \end{bmatrix}. \tag{6.5}$$

Each f_k has the linear representation $f_k(x) = \langle F_k, W \rangle$ for the following choice of W:

$$W \triangleq [x_0 \quad x^T]^T [x_0 \quad x^T] \tag{6.6}$$

where x_0 is considered as 1. On the other hand, an arbitrary matrix $W \in \mathbb{S}_{n+1}$ can be factorized as (6.6) if and only if it satisfies three properties: $W_{11} = 1$, $W \succeq 0$, and $\text{rank}\{W\} = 1$. In this representation of QCQP, the rank constraint carries all the nonconvexity. Neglecting this constraint yields the convex problem

$$\underset{W \in \mathbb{S}_{n+1}}{\text{minimize}} \quad \langle F_0, W \rangle \tag{6.7a}$$

$$\text{subject to} \quad \langle F_k, W \rangle \leq 0 \qquad k = 1, \ldots, q, \tag{6.7b}$$

$$W_{11} = 1, \tag{6.7c}$$

$$W \succeq 0, \tag{6.7d}$$

known as a semidefinite programming (SDP) relaxation of the QCQP (6.4). The existence of a rank-1 solution for an SDP relaxation guarantees the equivalence of the original QCQP and its relaxed problem.

6.2.4 Connection Between Rank and Sparsity

To explore the rank of the minimum-rank solution of SDP relaxation, define $\mathcal{G} = \mathcal{G}(F_0) \cup \cdots \cup \mathcal{G}(F_q)$ as the **sparsity graph** associated with the problem (6.7). The graph $\mathcal{G}$ describes the zero-nonzero pattern of the matrices $F_0, \ldots, F_q$, or alternatively captures the sparsity level of the QCQP problem (6.4). Let $\mathcal{T} = (\mathcal{V}_\mathcal{T}, \mathcal{E}_\mathcal{T})$ be a tree decomposition of $\mathcal{G}$. Denote its width as t and its bags as $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_{|\mathcal{T}|}$. It is known that given such a decomposition, every solution $W^{\text{ref}} \in \mathbb{S}_{n+1}$ of the SDP problem (6.7) can be transformed into a solution W^{opt} whose rank is upper bounded by $t + 1$ [Madani *et al.*, 2014b]. To perform this transformation, a suitable polynomial-time recursive algorithm will be proposed below.

Rank reduction algorithm:

1. Set $\mathcal{T}' := \mathcal{T}$ and $W := W^{\text{ref}}$.
2. If $\mathcal{T}'$ has a single node, then consider W^{opt} as W and terminate; otherwise continue to the next step.
3. Choose a pair of bags $\mathcal{B}_i, \mathcal{B}_j$ of $\mathcal{T}'$ such that $\mathcal{B}_i$ is a leaf of $\mathcal{T}'$ and $\mathcal{B}_j$ is its unique neighbor.

4. Using the notation $W\{\cdot,\cdot\}$ introduced in Section 6.1.1, define

$$O \triangleq W\{\mathcal{B}_i \cap \mathcal{B}_j, \mathcal{B}_i \cap \mathcal{B}_j\} \tag{6.8a}$$

$$V_i \triangleq W\{\mathcal{B}_i \setminus \mathcal{B}_j, \mathcal{B}_i \cap \mathcal{B}_j\} \tag{6.8b}$$

$$V_j \triangleq W\{\mathcal{B}_j \setminus \mathcal{B}_i, \mathcal{B}_i \cap \mathcal{B}_j\} \tag{6.8c}$$

$$H_i \triangleq W\{\mathcal{B}_i \setminus \mathcal{B}_j, \mathcal{B}_i \setminus \mathcal{B}_j\} \in \mathbb{R}^{n_i \times n_i} \tag{6.8d}$$

$$H_j \triangleq W\{\mathcal{B}_j \setminus \mathcal{B}_i, \mathcal{B}_j \setminus \mathcal{B}_i\} \in \mathbb{R}^{n_j \times n_j} \tag{6.8e}$$

$$S_i \triangleq H_i - V_i O^+ V_i^T = Q_i \Lambda_i Q_i^T \tag{6.8f}$$

$$S_j \triangleq H_j - V_j O^+ V_j^T = Q_j \Lambda_j Q_j^T \tag{6.8g}$$

   where $Q_i\Lambda_i Q_i^T$ and $Q_j\Lambda_j Q_j^T$ denote the eigenvalue decompositions of S_i and S_j with the diagonals of Λ_i and Λ_j arranged in descending order. Then, update a part of W as follows:

$$W\{\mathcal{B}_j \setminus \mathcal{B}_i, \mathcal{B}_i \setminus \mathcal{B}_j\} := V_j O^+ V_i^T + Q_j \sqrt{\Lambda_j}\ \ I_{n_j \times n_i} \sqrt{\Lambda_i}\, Q_i^T$$

   and update $W\{\mathcal{B}_i \setminus \mathcal{B}_j, \mathcal{B}_j \setminus \mathcal{B}_i\}$ accordingly to preserve the Hermitian property of W.

5. Update $\mathcal{T}'$ by merging $\mathcal{B}_i$ into $\mathcal{B}_j$, i.e., replace $\mathcal{B}_j$ with $\mathcal{B}_i \cup \mathcal{B}_j$ and then remove $\mathcal{B}_i$ from $\mathcal{T}'$.

6. Go back to step 2.

Theorem 22. *The output of the rank reduction algorithm, denoted as* W^{opt}*, is a solution of the SDP problem* (6.7) *whose rank is smaller than or equal to* $t+1$*.*

Proof. See Theorem 4 of Chapter 2 for the proof. □

6.3 Finite-horizon Deterministic ODC Problem

The primary objective of the ODC problem is to design a structurally constrained gain K. Assume that the matrix K has l free entries to be designed. Denote these parameters as $h_1, h_2, \ldots, h_l$. To formulate the ODC problem, the space of permissible controllers can be characterized as

$$\mathcal{K} \triangleq \left\{ \sum_{i=1}^{l} h_i N_i \;\middle|\; h \in \mathbb{R}^l \right\}, \tag{6.9}$$

for some (fixed) 0-1 matrices $N_1, \ldots, N_l \in \mathbb{R}^{m\times r}$. Now, the ODC problem can be stated as follows.

Finite-Horizon ODC Problem: Minimize

$$\sum_{\tau=0}^{p} \left(x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau]\right) + \alpha \|K\|_F^2 \tag{6.10a}$$

subject to

$$x[0] = x_0 \tag{6.10b}$$

$$x[\tau+1] = Ax[\tau] + Bu[\tau] \qquad \tau = 0, 1, \ldots, p-1 \tag{6.10c}$$

$$y[\tau] = Cx[\tau] \qquad \tau = 0, 1, \ldots, p \tag{6.10d}$$

$$u[\tau] = Ky[\tau] \qquad \tau = 0, 1, \ldots, p \tag{6.10e}$$

$$K = h_1 N_1 + \ldots + h_l N_l \tag{6.10f}$$

over the variables $\{x[\tau] \in \mathbb{R}^n\}_{\tau=0}^p$, $\{y[\tau] \in \mathbb{R}^r\}_{\tau=0}^p$, $\{u[\tau] \in \mathbb{R}^m\}_{\tau=0}^p$, $K \in \mathbb{R}^{m\times r}$ and $h \in \mathbb{R}^l$.

6.3.1 Sparsification of ODC Problem

The finite-horizon ODC is naturally a QCQP problem. Consider an arbitrary SDP relaxation of the ODC problem and let $\mathcal{G}$ be the sparsity graph of this relaxation. Due to existence of nonzero off-diagonal elements in Q and R, certain edges (and probably cycles) may exist in the subgraphs of $\mathcal{G}$ associated with the state and input vectors $x[\tau]$ and $u[\tau]$. Under this circumstance, the treewidth of $\mathcal{G}$ could be as high as n. To understand the effect of a non-diagonal controller K, consider the case $m = r = 2$ and assume that the controller K under design has three free elements as follows:

$$K = \begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix} \tag{6.11}$$

(i.e., $h_1 = K_{11}$, $h_2 = K_{12}$ and $h_3 = K_{22}$). Figure 6.2 shows a part of the graph $\mathcal{G}$. It can be observed that this subgraph is acyclic for $K_{12} = 0$ but has a cycle as soon as K_{12} becomes a free parameter. As a result, the treewidth of $\mathcal{G}$ is contingent upon the zero pattern of K. In order to guarantee existence of a low rank solution, we diagonalize Q, R and K through a reformulation of the ODC problem. Note that this transformation is redundant if Q, R and K are all diagonal.

Let $Q_d \in \mathbb{R}^{n\times n}$ and $R_d \in \mathbb{R}^{m\times m}$ be the respective eigenvector matrices of Q and R, i.e.,

$$Q = Q_d^T \Lambda_Q Q_d, , \qquad R = R_d^T \Lambda_R R_d \tag{6.12}$$

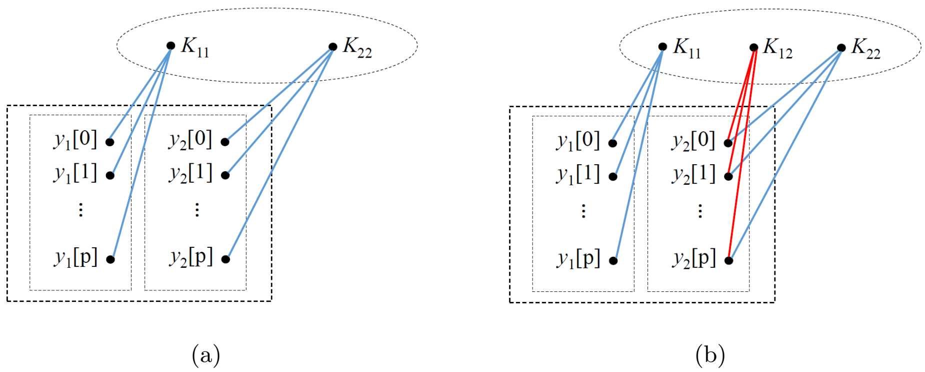


Figure 6.2: Effect of a nonzero off-diagonal entry of the controller K on the sparsity graph of the finite-horizon ODC: (a) a subgraph of $\mathcal{G}$ for the case where K_{11} and K_{22} are the only free parameters of the controller K, (b) a subgraph of $\mathcal{G}$ for the case where K_{12} is also a free parameter of the controller.

where $\Lambda_Q \in \mathbb{R}^{n\times n}$ and $\Lambda_R \in \mathbb{R}^{m\times m}$ are diagonal matrices. Notice that there exist two constant binary matrices $\Phi_1 \in \mathbb{R}^{m\times l}$ and $\Phi_2 \in \mathbb{R}^{l\times r}$ such that

$$\mathcal{K} = \left\{ \Phi_1 \text{diag}\{h\}\Phi_2 \mid h \in \mathbb{R}^l \right\}, \tag{6.13}$$

where diag$\{h\}$ denotes a diagonal matrix whose diagonal entries are inherited from the vector h [Lavaei and Aghdam, 2008]. Now, a sparse formulation of the ODC problem can be obtained in terms of the matrices

$$\bar{A} \triangleq Q_d A Q_d^T, \qquad \bar{B} \triangleq Q_d B R_d^T,$$
$$\bar{C} \triangleq \Phi_2 C Q_d^T, \qquad \bar{x}_0 \triangleq Q_d x_0,$$

and the new set of variables $\bar{x}[\tau] \triangleq Q_d x[\tau]$, $\bar{y}[\tau] \triangleq \Phi_2 y[\tau]$ and $\bar{u}[\tau] \triangleq R_d u[\tau]$ for every $\tau = 0, 1, \ldots, p$.

Reformulated Finite-Horizon ODC Problem: Minimize

$$\sum_{\tau=0}^{p} \left(\bar{x}[\tau]^T \Lambda_Q \bar{x}[\tau] + \bar{u}[\tau]^T \Lambda_R \bar{u}[\tau] \right) + \alpha \|h\|_2^2 \tag{6.14a}$$

subject to

$$\bar{x}[0] = \bar{x}_0 \times z^2 \tag{6.14b}$$

$$\bar{x}[\tau+1] = \bar{A}\bar{x}[\tau] + \bar{B}\bar{u}[\tau] \qquad \tau = 0,1,\ldots,p-1 \tag{6.14c}$$

$$\bar{y}[\tau] = \bar{C}\bar{x}[\tau] \qquad \tau = 0,1,\ldots,p \tag{6.14d}$$

$$\bar{u}[\tau] = R_d\Phi_1\mathrm{diag}\{h\}\bar{y}[\tau] \qquad \tau = 0,1,\ldots,p \tag{6.14e}$$

$$z^2 = 1 \tag{6.14f}$$

over the variables $\{\bar{x}[\tau] \in \mathbb{R}^n\}_{\tau=0}^{p}$, $\{\bar{y}[\tau] \in \mathbb{R}^l\}_{\tau=0}^{p}$, $\{\bar{u}[\tau] \in \mathbb{R}^m\}_{\tau=0}^{p}$, $h \in \mathbb{R}^l$ and $z \in \mathbb{R}$.

To cast the reformulated finite-horizon ODC as a quadratic optimization, define

$$w \triangleq \begin{bmatrix} z \ h^T \ \bar{x}^T \ \bar{u}^T \ \bar{y}^T \end{bmatrix}^T \in \mathbb{R}^{n_w} \tag{6.15}$$

where $\bar{x} \triangleq \begin{bmatrix} \bar{x}[0]^T & \cdots & \bar{x}[p]^T \end{bmatrix}^T$, $\bar{u} \triangleq \begin{bmatrix} \bar{u}[0]^T & \cdots & \bar{u}[p]^T \end{bmatrix}^T$, $\bar{y} \triangleq \begin{bmatrix} \bar{y}[0]^T & \cdots & \bar{y}[p]^T \end{bmatrix}^T$ and $n_w \triangleq 1 + l + (p+1)(n+l+m)$. The scalar auxiliary variable z plays the role of number 1 (it suffices to impose the additional quadratic constraint (6.14f) as opposed to $z = 1$ without affecting the solution).

6.3.2 SDP Relaxations of ODC Problem

In this subsection, two SDP relaxations are proposed for the reformulated finite-horizon ODC problem given in (6.14). For the first relaxation, there is a guarantee on the rank of the solution. In contrast, the second relaxation offers a tighter lower bound on the optimal cost of the ODC problem, but its solution might be high rank and therefore its rounding to a rank-1 solution could be more challenging.

6.3.2.1 Sparse SDP Relaxation

Let $e_1, \ldots, e_{n_w}$ denote the standard basis for $\mathbb{R}^{n_w}$. The ODC problem consists of $n_l \triangleq (p+1)(n+l)$ linear constraints given in (6.14b), (6.14c) and (6.14d), which can be formulated as

$$D^T w = 0 \tag{6.16}$$

for some matrix $D \in \mathbb{R}^{n_w \times n_l}$. Moreover, the objective function (6.14a) and the constraints in (6.14e) and (6.14f) are all quadratic and can be expressed in terms of some matrices $M \in \mathbb{S}_{n_w}$, $\{M_i[\tau] \in \mathbb{S}_{n_w}\}_{i=1,\ldots,m;\,\tau=0,1,\ldots,p}$ and $E \triangleq e_1 e_1^T$. This leads to the following formulation of (6.14).

Sparse Formulation of ODC Problem: Minimize

$$\langle M, ww^T \rangle \tag{6.17a}$$

subject to

$$D^T w = 0 \tag{6.17b}$$

$$\langle M_i[\tau], ww^T \rangle = 0 \qquad i = 1, \ldots, m, \quad \tau = 0, 1, \ldots, p \tag{6.17c}$$

$$\langle E, ww^T \rangle = 1 \tag{6.17d}$$

with the variable $w \in \mathbb{R}^{n_w}$.

For every $j = 1, \ldots, n_l$, define

$$D_j = D_{:,j} e_j^T + e_j D_{:,j}^T \tag{6.18}$$

where $D_{:,j}$ denotes the j-th column of D. An SDP relaxation of (6.17) will be obtained below.

Sparse Relaxation of Finite-Horizon ODC: Minimize

$$\langle M, W \rangle \tag{6.19a}$$

subject to

$$\langle D_j, W \rangle = 0 \qquad j = 1, \ldots, n_l \tag{6.19b}$$

$$\langle M_i[\tau], W \rangle = 0 \qquad i = 1, \ldots, m, \quad \tau = 0, 1, \ldots, p \tag{6.19c}$$

$$\langle E, W \rangle = 1 \tag{6.19d}$$

$$W \succeq 0 \tag{6.19e}$$

with the variable $W \in \mathbb{S}_{n_w}$.

The problem (6.19) is a convex relaxation of the QCQP problem (6.17). The sparsity graph of this problem is equal to

$$\mathcal{G} = \mathcal{G}(D_1) \cup \ldots \cup \mathcal{G}(D_{n_l}) \cup \mathcal{G}(M_1[0]) \cup \ldots \cup \mathcal{G}(M_m[0]) \cup \ldots \cup \mathcal{G}(M_1[p]) \cup \ldots \cup \mathcal{G}(M_m[p])$$

where the vertices of $\mathcal{G}$ correspond to the entries of w. In particular, the vertex set of $\mathcal{G}$ can be partitioned into five vertex subsets, where subset 1 consists of a single vertex associated with the

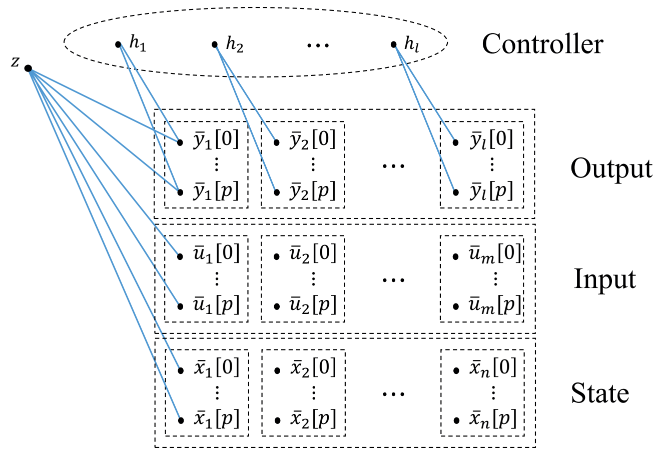


Figure 6.3: Sparsity graph of the problem (6.19) (some edges of vertex z are not shown to improve the legibility of the graph).

variable z and subsets 2-5 correspond to the vectors $\bar{x}$, $\bar{u}$, $\bar{y}$ and h, respectively. The underlying sparsity graph $\mathcal{G}$ for the sparse formulation of the ODC problem is drawn in Figure 6.3, where each vertex of the graph is labeled by its corresponding variable. To maintain the readability of the graph, some edges of vertex z are not shown in the picture. Indeed, z is connected to all vertices corresponding to the elements of $\bar{x}$, $\bar{u}$ and $\bar{y}$ due to the linear terms in (6.17b).

Theorem 23. *The sparsity graph of the sparse relaxation of the finite-horizon ODC problem has treewidth 2.*

Proof. It follows from the graph drawn in Figure 6.3 that removing vertex z from the sparsity graph $\mathcal{G}$ makes the remaining subgraph acyclic. This implies that the treewidth of $\mathcal{G}$ is at most 2. On the other hand, the treewidth cannot be 1 in light of the cycles of the graph. □

Consider the variable W of the SDP relaxation (6.19). The exactness of this relaxation is tantamount to the existence of an optimal rank-1 solution W^{opt} for (6.19). In this case, an optimal vector w^{opt} for the ODC problem can be recovered by decomposing W^{opt} as $(w^{\text{opt}})(w^{\text{opt}})^T$ (note that w has been defined in (6.15)). The following observation can be made.

Corollary 13. *The sparse relaxation of the finite-horizon ODC problem has a matrix solution with rank at most 3.*

Proof. This corollary is an immediate consequence of Theorems 22 and 23. □

Remark 7. *Since the treewidth of the sparse relaxation of the finite-horizon ODC problem* (6.19) *is equal to 2, it is possible to significantly reduce its computational complexity. More precisely, the complicating constraint $W \succeq 0$ can be replaced by positive semidefinite constraints on certain 3×3 submatrices of W, as given below:*

$$W\{\mathcal{B}_i, \mathcal{B}_i\} \succeq 0, \quad k = 1, \dots, |\mathcal{T}| \tag{6.20}$$

where $\mathcal{T}$ is an optimal tree decomposition of the sparsity graph $\mathcal{G}$, and $\mathcal{B}_1, \dots, \mathcal{B}_{|\mathcal{T}|}$ denote its bags. After this simplification of the hard constraint $W \succeq 0$, a quadratic number of entries of W turn out to be redundant (not appearing in any constraint) and can be removed from the optimization [Fukuda et al.*, 2001; Madani* et al.*, 2014b].*

6.3.2.2 Dense SDP Relaxation

Define $D^{\perp} \in \mathbb{R}^{n_w \times (n_w - n_l)}$ as an arbitrary full row rank matrix satisfying the relation $D^T D^{\perp} = 0$. It follows from (6.17b) that every feasible vector w satisfies the equation $w = D^{\perp}\tilde{w}$, for a vector $\tilde{w} \in \mathbb{R}^{(n_w - n_l)}$. Define

$$\tilde{M} = (D^{\perp})^T M D^{\perp} \tag{6.21a}$$

$$\tilde{M}_i[\tau] = (D^{\perp})^T M_i[\tau] D^{\perp} \tag{6.21b}$$

$$\tilde{E} = (D^{\perp})^T e_1 e_1^T D^{\perp}. \tag{6.21c}$$

The problem (6.17) can be cast in terms of $\tilde{w}$ as shown below.

Dense Formulation of ODC Problem: Minimize

$$\langle \tilde{M}, \tilde{w}\tilde{w}^T \rangle \tag{6.22a}$$

subject to

$$\langle \tilde{M}_i[\tau], \tilde{w}\tilde{w}^T \rangle = 0 \qquad i = 1, \dots, m; \quad \tau = 0, 1, \dots, p \tag{6.22b}$$

$$\langle \tilde{E}, \tilde{w}\tilde{w}^T \rangle = 1 \tag{6.22c}$$

over $\tilde{w} \in \mathbb{R}^{(n_w - n_l)}$.

The SDP relaxation of the above formulation is provided next.

Dense Relaxation of Finite-Horizon ODC: Minimize

$$\langle \tilde{M}, \tilde{W} \rangle \tag{6.23a}$$

subject to

$$\langle \tilde{M}_i[\tau], \tilde{W} \rangle = 0 \qquad i = 1, \ldots, m; \quad \tau = 0, 1, \ldots, p \tag{6.23b}$$

$$\langle \tilde{E}, \tilde{W} \rangle = 1 \tag{6.23c}$$

$$\tilde{W} \succeq 0 \tag{6.23d}$$

over $\tilde{W} \in \mathbb{S}_{(n_w - n_l)}$.

Remark 8. *Let $\mathcal{F}_s$ and $\mathcal{F}_d$ denote the feasible sets for the sparse and dense SDP relaxation problems in* (6.19) *and* (6.23)*, respectively. It can be easily seen that*

$$\{D^{\perp}\tilde{W}(D^{\perp})^T \mid \tilde{W} \in \mathcal{F}_d\} \subseteq \mathcal{F}_s \tag{6.24}$$

Therefore, the lower bound provided by the dense SDP relaxation problem (6.23) *is equal to or tighter than that of the sparse SDP relaxation* (6.19)*. However, the rank of the SDP solution of the dense relaxation may be high, which complicates its rounding to a rank-1 matrix. Hence, the sparse relaxation may be useful for recovering a near-global controller, while the dense relaxation may be used to bound the global optimality degree of the recovered controller.*

6.3.3 Rounding of SDP Solution to Rank-1 Matrix

Let W^{opt} either denote a low-rank solution for the sparse relaxation (6.19) or be equal to $D^{\perp}\tilde{W}^{\text{opt}}(D^{\perp})^T$ for a low-rank solution $\tilde{W}^{\text{opt}}$ (if any) of the dense relaxation (6.23). If the rank of W^{opt} is 1, then W^{opt} can be mapped back into a globally optimal controller for the ODC problem through an eigenvalue decomposition $W^{\text{opt}} = w^{\text{opt}}(w^{\text{opt}})^T$. Assume that W^{opt} does not have rank 1. There are multiple heuristic methods to recover a near-global controller, some of which are delineated below.

Direct Recovery Method: If W^{opt} had rank 1, then the $(2,1), (3,1), \ldots, (|h|+1, 1)$ entries of W^{opt} would have corresponded to the vector h^{opt} containing the free entries of K^{opt}. Inspired by this observation, if W^{opt} has rank greater than 1, then a near-global controller may still be recovered from the first column of W^{opt}. We refer to this approach as *Direct Recovery Method*.

Penalized SDP Relaxation: Recall that an SDP relaxation can be obtained by eliminating a rank constraint. In the case where this removal changes the solution, one strategy is to compensate for the rank constraint by incorporating an additive penalty function, denoted as $\Psi(W)$, into the objective of SDP relaxation. A common penalty function $\Psi(\cdot)$ is $\varepsilon \times \text{trace}\{W\}$, where ε is a design parameter. This problem is referred to as *Penalized SDP Relaxation* throughout this chapter.

Indirect Recovery Method: Define x as the aggregate state vector obtained by stacking $x[0], ..., x[p]$. The objective function of every proposed SDP relaxation depends strongly on x and only weakly on k if α is small. In particular, if $\alpha = 0$, then the SDP objective function is not in terms of K. This implies that the relaxation may have two feasible matrix solutions both leading to the same optimal cost such that their first columns overlap on the part corresponding to x and not the part corresponding to h. Hence, unlike the direct method that recovers h from the first column of W^{opt}, it may be advantageous to first recover x and then solve a second convex optimization to generate a structured controller that is able to generate state values as closely to the recovered aggregate state vector as possible. More precisely, given an SDP solution W^{opt}, define $\hat{x} \in \mathbb{R}^{n(p+1)}$ as a vector containing the entries $(|h|+2, 1), (|h|+3, 1), \ldots, (1+|h|+n(p+1), 1)$ of W^{opt}. Define the indirect recovery method as the convex optimization problem

$$\text{minimize} \quad \sum_{\tau=0}^{p} \|\hat{x}[\tau+1] - (A+BKC)\hat{x}[\tau]\|^2 \tag{6.25a}$$

$$\text{subject to} \quad K = h_1 M_1 + \ldots + h_l M_l \tag{6.25b}$$

with the variables $K \in \mathbb{R}^{m \times r}$ and $h \in \mathbb{R}^l$. Let $\hat{K}$ denote a solution of the above problem. In the case where W^{opt} has rank 1 or the state part of the matrix W^{opt} corresponds to the true solution of the ODC problem, $\hat{x}$ is the same as x^{opt} and $\hat{K}$ is an optimal controller. Otherwise, $\hat{K}$ is a feasible controller that aims to make the closed-loop system follow the near-optimal state trajectory vector $\hat{x}$. As tested in [Kalbat *et al.*, 2014], the above controller recovery method exhibits a remarkable performance on power systems.

6.3.4 Computationally-Cheap SDP Relaxation

Two SDP relaxations have been proposed earlier. Although these problems are convex, it may be difficult to solve them efficiently for a large-scale system. This is due to the fact that the size of each SDP matrix depends on the number of scalar variables at all times from 0 to p. There is an efficient approach to derive a computationally-cheap SDP relaxation. This will be explained below for the case where Q and R are non-singular and $r, m \leq n$.

Notation 1. *For every matrix $M \in \mathbb{R}^{n_1 \times n_2}$, define the sparsity pattern of M as follows*

$$\mathcal{S}(M) \triangleq \{S \in \mathbb{R}^{n_1 \times n_2} \mid \forall (i,j)\ M_{ij} = 0 \Rightarrow S_{ij} = 0\} \tag{6.26}$$

With no loss of generality, we assume that C has full row rank. There exists an invertible matrix $\Phi \in \mathbb{R}^{n \times n}$ such that $C\Phi = \begin{bmatrix} I_r & 0 \end{bmatrix}$. Define also

$$\mathcal{K}^2 \triangleq \{\Phi_1 S \Phi_1^T \mid S \in \mathcal{S}(\Phi_2 \Phi_2^T)\}. \tag{6.27}$$

Indeed, $\mathcal{K}^2$ captures the sparsity pattern of the matrix KK^T. For example, if $\mathcal{K}$ consists of block-diagonal (rectangular) matrix, $\mathcal{K}^2$ will also include block-diagonal (square) matrices. Let $\mu \in \mathbb{R}$ be a positive number such that $Q \succ \mu \times \Phi^{-T}\Phi^{-1}$, where Φ^{-T} denotes the transpose of the inverse of Φ. Define

$$\widehat{Q} := Q - \mu \times \Phi^{-T}\Phi^{-1}. \tag{6.28}$$

Using the slack matrix variables

$$X \triangleq [x[0]\ x[1]\ \ldots\ x[p]], \tag{6.29a}$$

$$U \triangleq [u[0]\ u[1]\ \ldots\ u[p]], \tag{6.29b}$$

an efficient relaxation of the ODC problem can be obtained.

Computationally-Cheap Relaxation of Finite-Horizon ODC: Minimize

$$\text{trace}\{X^T \widehat{Q} X + \mu\ \mathbf{W}_{22} + U^T R U\} + \alpha\ \text{trace}\{\mathbf{W}_{33}\} \tag{6.30a}$$

subject to

$$x[\tau+1] = Ax[\tau] + Bu[\tau], \quad \tau = 0, 1, \ldots, p-1, \tag{6.30b}$$

$$x[0] = x_0, \tag{6.30c}$$

$$\mathbf{W} = \left[\begin{array}{c:c:c} I_n & \Phi^{-1}X & [K \quad 0]^T \\ \hdashline X^T\Phi^{-T} & \mathbf{W}_{22} & U^T \\ \hdashline [K \quad 0] & U & \mathbf{W}_{33} \end{array}\right], \tag{6.30d}$$

$$K \in \mathcal{K}, \tag{6.30e}$$

$$\mathbf{W}_{33} \in \mathcal{K}^2, \tag{6.30f}$$

$$\mathbf{W} \succeq 0, \tag{6.30g}$$

over $K \in \mathbb{R}^{m\times r}$, $X \in \mathbb{R}^{n\times(p+1)}$, $U \in \mathbb{R}^{m\times(p+1)}$ and $\mathbf{W} \in \mathbb{S}_{n+m+p+1}$ (note that $\mathbf{W}_{22}$ and $\mathbf{W}_{33}$ are two blocks of the variable $\mathbf{W}$).

Note that the above relaxation can be naturally cast as an SDP problem by replacing each quadratic term in its objective with a new variable and then using the Schur complement. We refer to the SDP formulation of this problem as **computationally-cheap SDP relaxation**.

Theorem 24. *The problem* (6.30) *is a convex relaxation of the ODC problem. Furthermore, the relaxation is exact if and only if it possesses a solution* $(K^{opt}, X^{opt}, U^{opt}, \mathbf{W}^{opt})$ *such that* $\text{rank}\{\mathbf{W}^{opt}\} = n$.

Proof. It is evident that the problem (6.30) is a convex program. To prove the remaining parts of the theorem, it suffices to show that the ODC problem is equivalent to (6.30) subject to the additional constraint $\text{rank}\{\mathbf{W}\} = n$. To this end, consider a feasible solution $(K, X, U, \mathbf{W})$ such that $\text{rank}\{\mathbf{W}\} = n$. Since I_n has rank n, taking the Schur complement of the blocks $(1,1)$, $(1,2)$, $(2,1)$ and $(2,2)$ of $\mathbf{W}$ yields that

$$0 = \mathbf{W}_{22} - X^T\Phi^{-T}(I_n)^{-1}\Phi^{-1}X \tag{6.31}$$

Likewise,

$$0 = \mathbf{W}_{33} - KK^T \tag{6.32}$$

On the other hand,

$$\sum_{\tau=0}^{p} \left(x[\tau]^T Qx[\tau] + u[\tau]^T Ru[\tau]\right) = \text{trace}\{X^T QX + U^T RU\} \tag{6.33}$$

It follows from (6.31), (6.32) and (6.33) that the ODC problem and its computationally cheap relaxation lead to the same objective at the respective points (K, X, U) and $(K, X, U, \mathbf{W})$. In addition, it can be concluded from the Schur complement of the blocks $(1, 1)$, $(1, 2)$, $(3, 1)$ and $(3, 2)$ of $\mathbf{W}$ that

$$U = [K \quad 0]\Phi^{-1}X = KCX \tag{6.34}$$

or equivalently

$$u[\tau] = KCx[\tau] \qquad \text{for} \quad \tau = 0, 1, \ldots, p \tag{6.35}$$

This implies that (K, X, U) is a feasible solution of the ODC problem. Hence, the optimal objective value of the ODC problem is a lower bound on that of the computationally-cheap relaxation under the additional constraint $\text{rank}\{\mathbf{W}\} = n$.

Now, consider a feasible solution (K, X, U) of the ODC problem. Define $\mathbf{W}_{22} = X^T\Phi^{-T}\Phi^{-1}X$ and $K_2 = KK^T$. Observe that $\mathbf{W}$ can be written as the rank-n matrix $W_rW_r^T$, where

$$W_r = \begin{bmatrix} I_n & \Phi^{-1}X & [K \quad 0]^T \end{bmatrix}^T \tag{6.36}$$

Thus, $(K, X, U, \mathbf{W})$ is a feasible solution of the computationally-cheap SDP relaxation. This implies that the optimal objective value of the ODC problem is an upper bound on that of the computationally-cheap SDP relaxation under the additional constraint $\text{rank}\{\mathbf{W}\} = n$. The proof is completed by combining this property with its opposite statement proved earlier. $\square$

The sparse and dense SDP relaxations were both obtained by defining a matrix W as the product of two *vectors*. However, the computationally-cheap relaxation of the finite-horizon ODC Problem is obtained by defining $\mathbf{W}$ as the product of two *matrices*. This significantly reduces the computational complexity. To shed light on this fact, notice that the numbers of rows for the matrix variables of sparse and dense SDP relaxations are on the order of np, whereas the number of rows for the computationally-cheap SDP solution is on the order of $n + p$.

Remark 9. *The computationally-cheap relaxation of the finite-horizon ODC Problem automatically acts as a* <u>*penalized*</u> *SDP relaxation. To explain this remarkable feature of the proposed relaxation, notice that the terms* $\text{trace}\{\mathbf{W}_{22}\}$ *and* $\text{trace}\{\mathbf{W}_{33}\}$ *in the objective function of the relaxation inherently penalize the trace of* $\mathbf{W}$*. This automatic penalization helps significantly with the reduction* *of the rank of* $\mathbf{W}$ *at optimality. As a result, it is expected that the quality of the relaxation will be better for higher values of* α *and* μ*.*

Remark 10. *Consider the extreme case where* $r = n$*,* $C = I_n$*,* $\alpha = 0$*,* $p = \infty$*, and the unknown controller* K *is unstructured. This amounts to the famous LQR problem and the optimal controller can be found using the Riccati equation. It is straightforward to verify that the computationally-cheap relaxation of the ODC problem is always exact in this case even though it is infinite-dimensional. The proof is based on the following facts:*

- *When* K *is unstructured, the constraint* (6.30e) *and* (6.30f) *can be omitted. Therefore, there is no structural constraint on* $\mathbf{W}_{33}$ *and* $\mathbf{W}_{31}$ *(i.e., the* $(3,1)$ *block entry).*
- *Then, the constraint* (6.30d) *reduces to* $\mathbf{W}_{22} = X^T\Phi^{-T}\Phi^{-1}X$ *due to the term* $\text{trace}\{\,\mathbf{W}_{22}\}$ *in the objective function. Consequently, the objective function can be rearranged as* $\sum_{\tau=0}^{\infty}\big(x[\tau]^TQx[\tau] + u[\tau]^TRu$
- *The only remaining constraints are the state evolution equation and* $x[0] = x_0$*. It is known that the remaining feed-forward problem has a solution* (X^{opt}, U^{opt}) *such that* $U^{opt} = K^{opt}X^{opt}$ *for some matrix* K^{opt}*.*

6.3.5 Stability Enforcement

The finite-horizon ODC problem studied before had no stability conditions. It is verified in some simulations in [Kalbat *et al.*, 2014] that the closed-loop stability may be automatically guaranteed for physical systems, provided p is large enough. In this subsection, we aim to directly enforce stability by imposing additional constraints on the proposed SDP relaxations.

Theorem 25. *There exists a controller* $u[\tau] = Ky[\tau]$ *with the structure* $K \in \mathcal{K}$ *to stabilize the system* (6.1) *if and only if there exist a (Lyapunov) matrix* $P \in \mathbb{S}_n$*, a matrix* $K \in \mathbb{R}^{m\times r}$*, and* *auxiliary variables* $L \in \mathbb{R}^{m\times n}$ *and* $\mathbf{G} \in \mathbb{S}_{2n+m}$ *such that*

$$\begin{bmatrix} P - I_n & AP + B\mathbf{G}_{32} \\ PA^T + \mathbf{G}_{23}B^T & P \end{bmatrix} \succeq 0, \tag{6.37a}$$

$$K \in \mathcal{K}, \tag{6.37b}$$

$$\mathbf{G} \succeq 0, \tag{6.37c}$$

$$\mathbf{G}_{33} \in \mathcal{K}^2, \tag{6.37d}$$

$$\text{rank}\{\mathbf{G}\} = n, \tag{6.37e}$$

where

$$\mathbf{G} \triangleq \left[\begin{array}{c:c:c} I_n & \Phi^{-1}P & [K \quad 0]^T \\ \hdashline P\Phi^{-T} & \mathbf{G}_{22} & \mathbf{G}_{23} \\ \hdashline [K \quad 0] & \mathbf{G}_{32} & \mathbf{G}_{33} \end{array}\right] \tag{6.38}$$

Proof. It is well-known that the system (6.1) is stable under a controller $u[\tau] = Ky[\tau]$ if and only if there exists a positive-definite matrix $P \in \mathbb{S}_n$ to satisfy the Lyapunov inequality:

$$(A + BKC)^T P(A + BKC) - P + I_n \preceq 0 \tag{6.39}$$

or equivalently

$$\begin{bmatrix} P - I_n & AP + BKCP \\ PA^T + PK^TC^TB^T & P \end{bmatrix} \succeq 0 \tag{6.40}$$

Due to the analogy between $\mathbf{W}$ and $\mathbf{G}$, the argument made in the proof of Theorem 24 can be adopted to complete the proof of this theorem (note that $\mathbf{G}_{32}$ plays the role of KCP). □

Theorem 25 translates the stability of the closed-loop system into a rank-n condition. Consider one of the aforementioned SDP relaxations of the ODC problem. To enforce stability, it results from Theorem 25 that two actions can be taken: (i) addition of the convex constraints (6.37a)-(6.37d) to SDP relaxations, (ii) compensation for the rank-n condition through an appropriate convex penalization of $\mathbf{G}$ in the objective function of SDP relaxations. Note that the penalization is vital because otherwise $\mathbf{G}_{22}$ and $\mathbf{G}_{33}$ would grow unboundedly to satisfy the condition $\mathbf{G} \succeq 0$.

6.4 Infinite-horizon Deterministic ODC Problem

In this section, we study the infinite-horizon ODC problem, corresponding to $p = +\infty$ and subject to a stability condition.

6.4.1 Lyapunov Formulation

The finite-horizon ODC was investigated through a time-domain formulation. However, to deal with the infinite dimension of the infinite-horizon ODC and its hard stability constraint, a Lyapunov approach will be taken here. Consider the following optimization problem.

Lyapunov Formulation of ODC: Minimize

$$x_0^T P x_0 + \alpha \|K\|_F^2 \tag{6.41a}$$

subject to

$$\begin{bmatrix} G & G & (AG+BL)^T & L^T \\ G & Q^{-1} & 0 & 0 \\ AG+BL & 0 & G & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0, \tag{6.41b}$$

$$\begin{bmatrix} P & I_n \\ I_n & G \end{bmatrix} \succeq 0, \tag{6.41c}$$

$$K \in \mathcal{K}, \tag{6.41d}$$

$$L = KCG, \tag{6.41e}$$

over $K \in \mathbb{R}^{m\times r}$, $L \in \mathbb{R}^{m\times n}$, $P \in \mathbb{S}_n$ and $G \in \mathbb{S}_n$.

It will be shown in the next theorem that the above formulation is tantamount to the infinite-horizon ODC problem.

Theorem 26. *The infinite-horizon deterministic ODC problem is equivalent to finding a controller $K \in \mathcal{K}$, a symmetric Lyapunov matrix $P \in \mathbb{S}_n$, an auxiliary symmetric matrix $G \in \mathbb{S}_n$ and an auxiliary matrix $L \in \mathbb{R}^{m\times n}$ to solve the optimization problem* (6.41).

Proof. Given an arbitrary control gain K, we have:

$$\sum_{\tau=0}^{\infty}\left(x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau]\right) = x[0]^T P x[0] \tag{6.42}$$

where

$$P = (A+BKC)^T P (A+BKC) + Q + (KC)^T R (KC) \tag{6.43a}$$

$$P \succeq 0 \tag{6.43b}$$

On the other hand, it is well-known that replacing the equality sign "=" in (6.43a) with the inequality sign "$\succeq$" does not affect the solution of the optimization problem [Boyd and Vandenberghe, 2004]. After pre- and post-multiplying the Lyapunov inequality obtained from (6.43a) with P^{-1} and using the Schur complement formula, the constraints (6.43a) and (6.43b) can be combined as

$$\begin{bmatrix} P^{-1} & P^{-1} & S^T & P^{-1}(KC)^T \\ P^{-1} & Q^{-1} & 0 & 0 \\ S & 0 & P^{-1} & 0 \\ (KC)P^{-1} & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0 \tag{6.44}$$

where $S = (A+BKC)P^{-1}$. By replacing P^{-1} with a new variable G in the above matrix and defining L as KCG, the constraints (6.41b) and (6.41e) will be obtained. On the other hand, (6.41c) implies that $G \succ 0$ and $P \succeq G^{-1}$. Therefore, the minimization of $x_0^T P x_0$ subject to the constraint (6.41c) ensures that $P = G^{-1}$ is satisfied for at least one optimal solution of the optimization problem. □

Theorem 27. *Consider the special case where $r = n$, $C = I_n$, $\alpha = 0$ and $\mathcal{K}$ contains the set of all unstructured controllers. Then, the infinite-horizon deterministic ODC problem has the same solution as the convex optimization problem obtained from the nonlinear optimization* (6.41) *by removing its non-convex constraint* (6.41e).

Proof. It is easy to verify that a solution $(K^{\text{opt}}, P^{\text{opt}}, G^{\text{opt}}, L^{\text{opt}})$ of the convex problem stated in the theorem can be mapped to the solution $(L^{\text{opt}}(G^{\text{opt}})^{-1}, P^{\text{opt}}, G^{\text{opt}}, L^{\text{opt}})$ of the non-convex problem (6.41) and vice versa (recall that $C = I_n$ by assumption). This completes the proof. □

6.4.2 SDP Relaxation

Theorem 27 states that a classical optimal control problem can be precisely solved via a convex relaxation of the nonlinear optimization (6.41) by eliminating its constraint (6.41e). However, this simple convex relaxation does not work satisfactorily for a general control structure $K = \Phi_1 \text{diag}\{h\}\Phi_2$. To design a better relaxation, define

$$w = \begin{bmatrix} 1 & h^T & \text{vec}\{\Phi_2 CG\}^T \end{bmatrix}^T \tag{6.45}$$

where $\text{vec}\{\Phi_2 CG\}$ is an $nl \times 1$ column vector obtained by stacking the columns of $\Phi_2 CG$. It is possible to write every entry of the bilinear matrix term KCG as a linear function of the entries of the parametric matrix ww^T. Hence, by introducing a new matrix variable $\mathbf{W}$ playing the role of ww^T, the nonlinear constraint (6.41e) can be rewritten as a linear constraint in term of $\mathbf{W}$.

Notation 2. *Define the sampling operator* $\text{samp} : \mathbb{R}^{l\times nl} \to \mathbb{R}^{l\times n}$ *as follows:*

$$\text{samp}\{X\} = \begin{bmatrix} X_{i,(n-1)j+i} \end{bmatrix}_{i=1,\dots,l;\ j=1,\dots,n}. \tag{6.46}$$

Now, one can relax the non-convex mapping constraint $\mathbf{W} = ww^T$ to $\mathbf{W} \succeq 0$ and another constraint stating that the first column of $\mathbf{W}$ is equal to w. This yields the following convex relaxation of problem (6.41).

SDP Relaxation of Infinite-Horizon Deterministic ODC: Minimize

$$x_0^T P x_0 + \alpha\ \text{trace}\{\mathbf{W}_{33}\} \tag{6.47a}$$

subject to

$$\begin{bmatrix} G & G & (AG+BL)^T & L^T \\ G & Q^{-1} & 0 & 0 \\ AG+BL & 0 & G & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0, \tag{6.47b}$$

$$\begin{bmatrix} P & I_n \\ I_n & G \end{bmatrix} \succeq 0, \tag{6.47c}$$

$$L = \Phi_1 \times \text{samp}\{\mathbf{W}_{32}\}, \tag{6.47d}$$

$$\mathbf{W} = \left[\begin{array}{c:c:c} 1 & \text{vec}\{\Phi_2 CG\}^T & h^T \\ \hdashline \text{vec}\{\Phi_2 CG\} & \mathbf{W}_{22} & \mathbf{W}_{23} \\ \hdashline h & \mathbf{W}_{32} & \mathbf{W}_{33} \end{array}\right], \tag{6.47e}$$

$$\mathbf{W} \succeq 0, \tag{6.47f}$$

over $h \in \mathbb{R}^l$, $L \in \mathbb{R}^{m\times n}$, $P \in \mathbb{S}_n$, $G \in \mathbb{S}_n$ and $\mathbf{W} \in \mathbb{S}_{1+l(n+1)}$.

If the relaxed problem (6.47) has the same solution as the infinite-horizon ODC in (6.41), the relaxation is exact.

Theorem 28. *The following statements hold regarding the relaxation of the infinite-horizon deterministic ODC in* (6.47)*:*

i) The relaxation is exact if it has a solution $(h^{opt}, P^{opt}, G^{opt}, L^{opt}, \mathbf{W}^{opt})$ *such that* $\text{rank}\{\mathbf{W}^{opt}\} = 1$.

ii) The relaxation always has a solution $(h^{opt}, P^{opt}, G^{opt}, L^{opt}, \mathbf{W}^{opt})$ *such that* $\text{rank}\{\mathbf{W}^{opt}\} \leq 3$.

Proof. Consider a sparsity graph $\mathcal{G}$ of (6.47), constructed as follows. The graph $\mathcal{G}$ has $1 + l(n+1)$ vertices corresponding to the rows of $\mathbf{W}$. Two arbitrary disparate vertices $i, j \in \{1, 2, \ldots, 1 + l(n+1)\}$ are adjacent in $\mathcal{G}$ if $\mathbf{W}_{ij}$ appears in at least one of the constraints of the problem (6.47) excluding the global constraint $\mathbf{W} \succeq 0$. For example, vertex 1 is connected to all remaining vertices of $\mathcal{G}$. The graph $\mathcal{G}$ with its vertex 1 removed is depicted in Figure 6.4. This graph is acyclic and therefore the treewidth of $\mathcal{G}$ is at most 2. Hence, it follows from Theorem 1 that (6.47) has a matrix solution with rank at most 3. $\square$

Theorem 28 states that the SDP relaxation of the infinite-horizon ODC problem has a low-rank solution. However, it does not imply that every solution of the relaxation is low-rank. Theorem 1 provides a procedure for converting a high-rank solution of the SDP relaxation into a low-rank one.

6.4.3 Computationally-Cheap Relaxation

The aforementioned SDP relaxation has a high dimension for a large-scale system, which makes it less interesting for computational purposes. Moreover, the quality of its optimal objective value can be improved using some indirect penalty technique. The objective of this subsection is to offer a computationally-cheap SDP relaxation for the ODC problem, whose solution outperforms that of the previous SDP relaxation. For this purpose, consider again a scalar μ such that $Q \succ \mu \times \Phi^{-T}\Phi^{-1}$ and define $\hat{Q}$ according to (6.28).

Computationally-Cheap Relaxation of Infinite-horizon Deterministic ODC: Minimize

$$x_0^T P x_0 + \alpha \ \text{trace}\{\mathbf{W}_{33}\} \tag{6.48a}$$

subject to

$$\begin{bmatrix} G-\mu\mathbf{W}_{22} & G & (AG+BL)^T & L^T \\ G & \widehat{Q}^{-1} & 0 & 0 \\ AG+BL & 0 & G & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0, \tag{6.48b}$$

$$\begin{bmatrix} P & I_n \\ I_n & G \end{bmatrix} \succeq 0, \tag{6.48c}$$

$$\mathbf{W} = \left[\begin{array}{c|c|c} I_n & \Phi^{-1}G & [K \quad 0]^T \\ \hline G\Phi^{-T} & \mathbf{W}_{22} & L^T \\ \hline [K \quad 0] & L & \mathbf{W}_{33} \end{array}\right], \tag{6.48d}$$

$$K \in \mathcal{K}, \tag{6.48e}$$

$$\mathbf{W}_{33} \in \mathcal{K}^2, \tag{6.48f}$$

$$\mathbf{W} \succeq 0, \tag{6.48g}$$

over $K \in \mathbb{R}^{m\times r}$, $L \in \mathbb{R}^{m\times n}$, $P \in \mathbb{S}_n$, $G \in \mathbb{S}_n$ and $\mathbf{W} \in \mathbb{S}_{2n+m}$.

The following remarks can be made regarding (6.48):

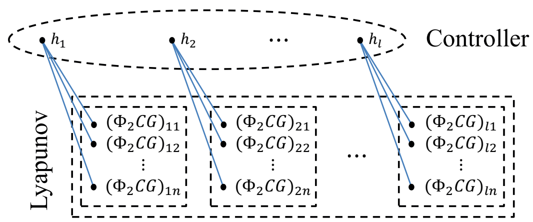


Figure 6.4: The sparsity graph for the infinite-horizon deterministic ODC in the case where $\mathcal{K}$ consists of diagonal matrices (the central vertex corresponding to the constant 1 is removed for simplicity).

- The constraint (6.48b) corresponds to the Lyapunov inequality associated with (6.43a), where $\mathbf{W}_{22}$ in its first block aims to play the role of $P^{-1}\Phi^{-T}\Phi^{-1}P^{-1}$.

- The constraint (6.48c) ensures that the relation $P = G^{-1}$ occurs at optimality (at least for one of the solution of the problem).

- The constraint (6.48d) is a surrogate for the only complicating constraint of the ODC problem, i.e., $L = KCG$.

- Since no non-convex rank constraint is imposed on the problem to maintain the convexity of the relaxation, the rank constraint is compensated in various ways. More precisely, the entries of $\mathbf{W}$ are constrained in the objective function (6.48a) through the term α trace$\{\mathbf{W}_{33}\}$, in the first block of the constraint (6.48b) through the term $G - \mu\mathbf{W}_{22}$, and also via the constraint (6.48e) and (6.48f). These terms aim to automatically penalize the rank of $\mathbf{W}$ indirectly.

- The proposed relaxation takes advantage of the sparsity of not only K, but also KK^T (through the constraint (6.48f)).

Theorem 29. *The problem* (6.48) *is a convex relaxation of the infinite-horizon ODC problem. Furthermore, the relaxation is exact if and only if it possesses a solution* $(K^{opt}, L^{opt}, P^{opt}, G^{opt}, \mathbf{W}^{opt})$ *such that rank*$\{\mathbf{W}^{opt}\} = n$.

Proof. The objective function and constraints of the problem (6.48) are all linear functions of the tuple $(K, L, P, G, \mathbf{W})$. Hence, this relaxation is indeed convex. To study the relationship between this optimization problem and the infinite-horizon ODC, consider a feasible point (K, L, P, G) of the ODC formulation (6.41). It can be deduced from the relation $L = KCG$ that $(K, L, P, G, \mathbf{W})$

is a feasible solution of the problem (6.48) if the free blocks of $\mathbf{W}$ are considered as

$$\mathbf{W}_{22} = G\Phi^{-T}\Phi^{-1}G, \qquad \mathbf{W}_{33} = KK^T \tag{6.49}$$

(note that (6.41b) and (6.48b) are equivalent for this choice of $\mathbf{W}$). This implies that the problem (6.48) is a convex relaxation of the infinite-horizon ODC problem.

Consider now a solution $(K^{\text{opt}}, L^{\text{opt}}, P^{\text{opt}}, G^{\text{opt}}, \mathbf{W}^{\text{opt}})$ of the computationally-cheap SDP relaxation such that $\text{rank}\{\mathbf{W}^{\text{opt}}\} = n$. Since the rank of the first block of $\mathbf{W}^{\text{opt}}$ (i.e., I_n) is already n, a Schur complement argument on the blocks $(1,1)$, $(1,3)$, $(2,1)$ and $(2,3)$ of $\mathbf{W}^{\text{opt}}$ yields that

$$0 = L^{\text{opt}} - [K^{\text{opt}} \quad 0](I_n)^{-1}\Phi^{-1}G^{\text{opt}} \tag{6.50}$$

or equivalently $L^{\text{opt}} = K^{\text{opt}}CG^{\text{opt}}$, which is tantamount to the constraint (6.41e). This implies that $(K^{\text{opt}}, L^{\text{opt}}, P^{\text{opt}}, G^{\text{opt}})$ is a solution of the infinite-horizon ODC problem (6.41) and hence the relaxation is exact. So far, we have shown that the existence of a rank-n solution $\mathbf{W}^{\text{opt}}$ guarantees the exactness of the relaxation. The converse of this statement can also be proved similarly. □

The variable $\mathbf{W}$ in the first SDP relaxation (6.47) had $1 + l(n+1)$ rows. In contrast, this number reduces to $2n + m$ for the matrix $\mathbf{W}$ in the computationally-cheap relaxation (6.48). This significantly reduces the computation time of the relaxation.

Corollary 14. *Consider the special case where $r = n$, $C = I_n$, $\alpha = 0$ and $\mathcal{K}$ contains the set of all unstructured controllers. Then, the computationally-cheap relaxation problem* (6.48) *is exact for the infinite-horizon ODC problem.*

Proof. The proof follows from that of Theorem 27. □

6.4.4 Controller Recovery

In this subsection, two controller recovery methods will be described. With no loss of generality, our focus will be on the computationally-cheap relaxation problem (6.48).

Direct Recovery Method for Infinite-Horizon ODC: A near-optimal controller K for the infinite-horizon ODC problem is chosen to be equal to the optimal matrix K^{opt} obtained from the computationally-cheap relaxation problem (6.48).

Indirect Recovery Method for Infinite-Horizon ODC: Let $(K^{\text{opt}}, L^{\text{opt}}, P^{\text{opt}}, G^{\text{opt}}, \mathbf{W}^{\text{opt}})$ denote a solution of the computationally-cheap relaxation problem (6.48). Given a pre-specified nonnegative number ε, a near-optimal controller $\hat{K}$ for the infinite-horizon ODC problem is recovered by minimizing

$$\varepsilon \times \gamma + \alpha \|K\|_F^2 \tag{6.51a}$$

subject to

$$\begin{bmatrix} (G^{\text{opt}})^{-1} - Q + \gamma I_n & (A+BKC)^T & (KC)^T \\ (A+BKC) & G^{\text{opt}} & 0 \\ (KC) & 0 & R^{-1} \end{bmatrix} \succ 0 \tag{6.51b}$$

$$K = h_1 N_1 + \ldots + h_l N_l. \tag{6.51c}$$

over $K \in \mathbb{R}^{m\times r}$, $h \in \mathbb{R}^l$ and $\gamma \in \mathbb{R}$. Note that this is a convex program. The direct recovery method assumes that the controller K^{opt} obtained from the computationally-cheap relaxation problem (6.48) is near-optimal, whereas the indirect method assumes that the controller K^{opt} might be unacceptably imprecise while the inverse of the Lyapunov matrix is near-optimal. The indirect method is built on SDP relaxation by fixing G at its optimal value and then perturbing Q as $Q - \gamma I_n$ to facilitate the recovery of a stabilizing controller. The underlying idea is that the SDP relaxation depends strongly on G and weakly on P (note that G appears 9 times in the formulation, while P appears only twice to indirectly account for the inverse of G). In other words, there might be two feasible solutions with similar costs for the SDP relaxation whose G parts are identical while their P parts are very different. Hence, the indirect method focuses on G.

6.5 Infinite-Horizon Stochastic ODC Problem

This section is mainly concerned with the stochastic optimal distributed control (SODC) problem, which aims to design a stabilizing static controller $u[\tau] = Ky[\tau]$ to minimize the cost function

$$\lim_{\tau\to+\infty} \mathcal{E}\left(x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau]\right) + \alpha \|K\|_F^2 \tag{6.52}$$

subject to the system dynamics (6.3) and the controller requirement $K \in \mathcal{K}$, for a nonnegative scalar α and positive-definite matrices Q and R. Define two covariance matrices as

$$\Sigma_d = \mathcal{E}\{Ed[0]d[0]^T E^T\} \quad \Sigma_v = \mathcal{E}\{Fv[0]v[0]^T F^T\} \tag{6.53}$$

Consider the following optimization problem.

Lyapunov Formulation of SODC: Minimize

$$\langle P, \Sigma_d \rangle + \langle M + K^T RK, \Sigma_v \rangle + \alpha \|K\|_F^2 \tag{6.54a}$$

subject to

$$\begin{bmatrix} G & G & (AG+BL)^T & L^T \\ G & Q^{-1} & 0 & 0 \\ AG+BL & 0 & G & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0, \tag{6.54b}$$

$$\begin{bmatrix} P & I_n \\ I_n & G \end{bmatrix} \succeq 0, \tag{6.54c}$$

$$\begin{bmatrix} M & (BK)^T \\ BK & G \end{bmatrix} \succeq 0, \tag{6.54d}$$

$$K \in \mathcal{K} \tag{6.54e}$$

$$L = KCG \tag{6.54f}$$

over the controller $K \in \mathbb{R}^{m\times r}$, Lyapunov matrix $P \in \mathbb{S}_n$ and auxiliary matrices $G \in \mathbb{S}_n$, $L \in \mathbb{R}^{m\times n}$ and $M \in \mathbb{S}_r$.

Theorem 30. *The infinite-horizon SODC problem adopts the non-convex formulation* (6.54)*.*

Proof. It is straightforward to verify that

$$x[\tau] = (A+BKC)^\tau x[0] + \sum_{t=0}^{\tau-1} (A+BKC)^{\tau-t-1}(Ed[t] + BKFv[t]) \tag{6.55}$$

for $\tau = 1, 2, \ldots, \infty$. On the other hand, since the controller under design must be stabilizing, $(A+BKC)^\tau$ approaches zero as τ goes to $+\infty$. In light of the above equation, it can be verified that

$$\begin{aligned} \mathcal{E}\left\{ \lim_{\tau\to+\infty} \left(x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau] \right) \right\} &= \mathcal{E}\left\{ \lim_{\tau\to+\infty} x[\tau]^T \left(Q + C^T K^T RKC \right) x[\tau] \right\} \\ &\quad + \mathcal{E}\left\{ \lim_{\tau\to+\infty} v[\tau]^T F^T K^T RKFv[\tau] \right\} \\ &= \langle P, \Sigma_d \rangle + \langle (BK)^T P (BK) + K^T RK, \Sigma_v \rangle \end{aligned} \tag{6.56}$$

where

$$P = \sum_{t=0}^{\infty} \left((A+BKC)^t\right)^T (Q + C^T K^T RKC)(A+BKC)^t$$

Similar to the proof of Theorem 26, the above infinite series can be replaced by the expanded Lyapunov inequality (6.44): After replacing P^{-1} and KCP^{-1} in (6.44) with new variables G and L, it can be concluded that:

- The condition (6.44) is identical to the set of constraints (6.54b) and (6.54f).
- The cost function (6.56) can be expressed as

$$\langle P, \Sigma_d \rangle + \langle (BK)^T G^{-1}(BK) + K^T RK, \Sigma_v \rangle + \alpha \|K\|_F^2$$

- Since P appears only once in the constraints of the optimization problem (6.54) (i.e., the condition (6.54c)) and the objective function of this optimization includes the term $\langle P, \Sigma_d \rangle$, an optimal value of P is equal to G^{-1} (Notice that $\Sigma_d \succeq 0$).
- Similarly, the optimal value of M is equal to $(BK)^T G^{-1}(BK)$.

The proof follows from the above observations. □

The traditional H_2 optimal control problem (i.e., in the centralized case) can be solved using Riccati equations. It will be shown in the next proposition that dropping the nonconvex constraint (6.54f) results in a convex optimization that correctly solves the centralized H_2 optimal control problem.

Proposition 1. *Consider the special case where $r = n$, $C = I_n$, $\alpha = 0$, $\Sigma_v = 0$, and $\mathcal{K}$ contains the set of all unstructured controllers. Then, the SODC problem has the same solution as the convex optimization problem obtained from the nonlinear optimization* (6.54a)-(6.54) *by removing its non-convex constraint* (6.54f).

Proof. It is similar to the proof of Theorem 27. □

Consider the vector w defined in (6.45). Similar to the infinite-horizon ODC case, the bilinear matrix term KCG can be represented as a linear function of the entries of the parametric matrix $\mathbf{W}$ defined as ww^T. Now, a convex relaxation can be attained by relaxing the constraint $\mathbf{W} = ww^T$ to $\mathbf{W} \succeq 0$ and adding another constraint stating that the first column of $\mathbf{W}$ is equal to w.

Relaxation of Infinite-Horizon SODC: Minimize

$$\langle P, \Sigma_d \rangle + \langle M + K^T R K, \Sigma_v \rangle + \alpha \ \mathrm{trace}\{\mathbf{W}_{33}\} \tag{6.57a}$$

subject to

$$\begin{bmatrix} G & G & (AG+BL)^T & L^T \\ G & Q^{-1} & 0 & 0 \\ AG+BL & 0 & G & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0, \tag{6.57b}$$

$$\begin{bmatrix} P & I_n \\ I_n & G \end{bmatrix} \succeq 0, \tag{6.57c}$$

$$K = \Phi_1 \mathrm{diag}\{h\} \Phi_2, \tag{6.57d}$$

$$\begin{bmatrix} M & (BK)^T \\ BK & G \end{bmatrix} \succeq 0, \tag{6.57e}$$

$$L = \Phi_1 \mathrm{samp}\{\mathbf{W}_{32}\}, \tag{6.57f}$$

$$\mathbf{W} = \left[\begin{array}{c:c:c} 1 & \mathrm{vec}\{\Phi_2 C G\}^T & h^T \\ \hdashline \mathrm{vec}\{\Phi_2 C G\} & \mathbf{W}_{22} & \mathbf{W}_{23} \\ \hdashline h & \mathbf{W}_{32} & \mathbf{W}_{33} \end{array}\right], \tag{6.57g}$$

$$\mathbf{W} \succeq 0, \tag{6.57h}$$

over the controller $K \in \mathbb{R}^{m \times r}$, Lyapunov matrix $P \in \mathbb{S}_n$ and auxiliary matrices $G \in \mathbb{S}_n$, $L \in \mathbb{R}^{m \times n}$, $M \in \mathbb{S}_r$, $h \in \mathbb{R}^l$ and $\mathbf{W} \in \mathbb{S}_{1+l(n+1)}$.

Theorem 31. *The following statements hold regarding the convex relaxation of the infinite-horizon SODC problem:*

i) The relaxation is exact if it has a solution $(h^{opt}, K^{opt}, P^{opt}, G^{opt}, L^{opt}, M^{opt}, \mathbf{W}^{opt})$ *such that* $\mathrm{rank}\{W^{opt}\} = 1$.

ii) The relaxation always has a solution $(h^{opt}, K^{opt}, P^{opt}, G^{opt}, L^{opt}, M^{opt}, \mathbf{W}^{opt})$ *such that* $\mathrm{rank}\{W^{opt}\} \leq 3$.

Proof. The proof is omitted (see Theorems 28 and 30). $\square$

As before, it can be deduced from Theorem 31 that the infinite-horizon SODC problem has a convex relaxation with the property that its exactness amounts to the existence of a rank-1 matrix solution $\mathbf{W}^{\text{opt}}$. Moreover, it is always guaranteed that this relaxation has a solution such that $\text{rank}\{\mathbf{W}^{\text{opt}}\} \leq 3$.

A computationally-cheap SDP relaxation for the SODC problem will be derived below. Let μ_1 and μ_2 be two nonnegative numbers such that

$$Q \succ \mu_1 \times \Phi^{-T}\Phi^{-1}, \quad \Sigma_v \succeq \mu_2 \times I_r \tag{6.58}$$

Define $\widehat{Q} := Q - \mu_1 \times \Phi^{-T}\Phi^{-1}$ and $\widehat{\Sigma}_v := \Sigma_v - \mu_2 \times I_r$.

Computationally-Cheap Relaxation of Infinite-Horizon SODC: Minimize

$$\langle P, \Sigma_d \rangle + \langle M, \Sigma_v \rangle + \langle K^T R K, \widehat{\Sigma}_v \rangle + \langle \mu_2 R + \alpha I_m, \mathbf{W}_{33} \rangle \tag{6.59a}$$

subject to

$$\begin{bmatrix} G - \mu_1 \mathbf{W}_{22} & G & (AG+BL)^T & L^T \\ G & \widehat{Q}^{-1} & 0 & 0 \\ AG+BL & 0 & G & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0, \tag{6.59b}$$

$$\begin{bmatrix} P & I_n \\ I_n & G \end{bmatrix} \succeq 0, \tag{6.59c}$$

$$\begin{bmatrix} M & (BK)^T \\ BK & G \end{bmatrix} \succeq 0, \tag{6.59d}$$

$$\mathbf{W} = \left[\begin{array}{c:c:c} I_n & \Phi^{-1}G & [K \quad 0]^T \\ \hdashline G\Phi^{-T} & \mathbf{W}_{22} & L^T \\ \hdashline [K \quad 0] & L & \mathbf{W}_{33} \end{array}\right], \tag{6.59e}$$

$$K \in \mathcal{K}, \tag{6.59f}$$

$$\mathbf{W}_{33} \in \mathcal{K}^2, \tag{6.59g}$$

$$\mathbf{W} \succeq 0, \tag{6.59h}$$

over $K \in \mathbb{R}^{m \times r}$, $P \in \mathbb{S}_n$, $G \in \mathbb{S}_n$, $L \in \mathbb{R}^{m \times n}$, $M \in \mathbb{S}_r$ and $\mathbf{W} \in \mathbb{S}_{2n+m}$.

It should be noted that the constraint (6.59d) ensures that the relation

$$M = (BK)^T G^{-1}(BK) \tag{6.60}$$

occurs at optimality.

Theorem 32. *The problem* (6.59) *is a convex relaxation of the SODC problem. Furthermore, the relaxation is exact if and only if it possesses a solution* $(K^{opt}, L^{opt}, P^{opt}, G^{opt},\ M^{opt}, \mathbf{W}^{opt})$ *such that rank*$\{\mathbf{W}^{opt}\} = n$.

Proof. Since the proof is similar to that of the infinite-horizon case presented earlier, it is omitted here. □

For the retrieval of a near-optimal controller, the direct recovery method delineated for the infinite-horizon ODC problem can be readily deployed. However, the indirect recovery method requires some modifications, which will be explained below. Let $(K^{\text{opt}}, L^{\text{opt}}, P^{\text{opt}}, G^{\text{opt}}, M^{\text{opt}}, \mathbf{W}^{\text{opt}})$ denote a solution of the computationally-cheap relaxation of SODC. A near-optimal controller K for SODC may be recovered by minimizing

$$\langle K^T(B^T(G^{\text{opt}})^{-1}B + R)K, \Sigma_v\rangle + \alpha\|K\|_F^2 + \varepsilon \times \gamma \tag{6.61a}$$

subject to

$$\begin{bmatrix} (G^{\text{opt}})^{-1} - Q + \gamma I_n & (A + BKC)^T & (KC)^T \\ (A + BKC) & G^{\text{opt}} & 0 \\ (KC) & 0 & R^{-1} \end{bmatrix} \succ 0 \tag{6.61b}$$

$$K \in h_1 N_1 + \ldots + h_l N_l. \tag{6.61c}$$

over $K \in \mathbb{R}^{m\times r}$, $h \in \mathbb{R}^l$ and $\gamma \in \mathbb{R}$, where ε is a pre-specified nonnegative number. This is a convex program.

6.6 Extension to Dynamic Controllers

Consider the problem of finding an optimal fixed-order dynamic controller with a pre-specified structure. To formulate the problem, denote the unknown controller as

$$\begin{cases} z_c[\tau + 1] = A_c z_c[\tau] + B_c y[\tau] \\ u[\tau] = C_c z_c[\tau] + D_c y[\tau] \end{cases} \tag{6.62}$$

where $z_c[\tau] \in \mathbb{R}^{n_c}$ represents the state of the controller, and n_c denotes its known degree. The unknown quadruple (A_c, B_c, C_c, D_c) must belong to a given polytope $\mathcal{K}$. More precisely, A_c, B_c, C_c, and D_c are often required to be block matrices with certain forced zero blocks. It is shown in [Lavaei and Aghdam, 2007] how the design of a fixed-order distributed controller for an interconnected system adopts the above formulation. The augmentation of the system (6.1) with the above unknown controller leads to the closed-loop system $\tilde{x}[\tau+1] = \tilde{A}\tilde{x}[\tau]$, where $\tilde{x}[\tau] = \begin{bmatrix} x[\tau+1]^T & z_c[\tau+1]^T \end{bmatrix}^T$ and

$$\tilde{A} = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix} \tag{6.63}$$

Note that this closed-loop system reduces to $x[\tau+1] = (A + BKC)x[\tau]$ in the static case. Since $\tilde{A}$ is a linear structured matrix with respect to (A_c, B_c, C_c, D_c), the state evolution equation $\tilde{x}[\tau+1] = \tilde{A}\tilde{x}[\tau]$ is bilinear, similar to its static counterpart $x[\tau+1] = (A + BKC)x[\tau]$. Hence, the parameterized matrix $\tilde{A}$ plays the role of $A+BKC$, which makes it possible to naturally generalize all results of this work to the dynamic case in both finite- and infinite-horizon cases. Note that the existence of a Lyapunov matrix guarantees the stability of $\tilde{A}$ or the internal stability of the system.

6.7 Numerical Examples

In what follows, we offer multiple experiments on random systems and mass-spring systems. More simulations are provided in [Kalbat *et al.*, 2014].

6.7.1 Random Systems

Consider the system (6.1) with $n = 5$ and $m = r = 3$. The goal is to design a decentralized static controller $u[\tau] = Ky[\tau]$ (i.e., a diagonal matrix K) minimizing the cost function

$$\left(\sum_{\tau=0}^{20} x[\tau]^T x[\tau] + u[\tau]^T u[\tau] \right) + 10^{-3}\|K\|_F \tag{6.64}$$

This function accounts for the state regulation, input energy, and controller gain. The SDP relaxation problems (6.19), (6.23) and (6.30) have a 235×235, 168×168 and 29×29 matrix variables, respectively. According to Corollary 13, it is guaranteed that the sparse SDP relaxation problem (6.19) has a solution W^{opt} with rank at most 3 (i.e., at least 233 eigenvalues of this solution must be zero), independent of the values of the matrices A, B, C, and $x[0]$. Note that this result does

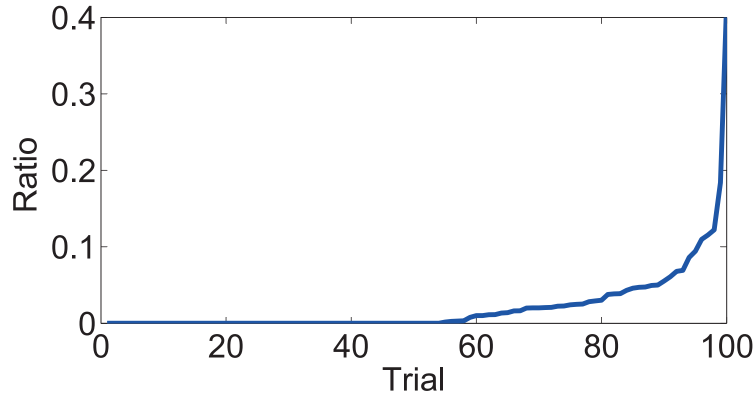


Figure 6.5: The ratio $\frac{\lambda_2}{\lambda_1}$ obtained from the dense SDP relaxation of the finite-horizon ODC Problem (6.23) for 100 random systems.

not imply that all solutions of problem (6.19) have rank at most 3, but Theorem 22 can be used to find such a low-rank solution.

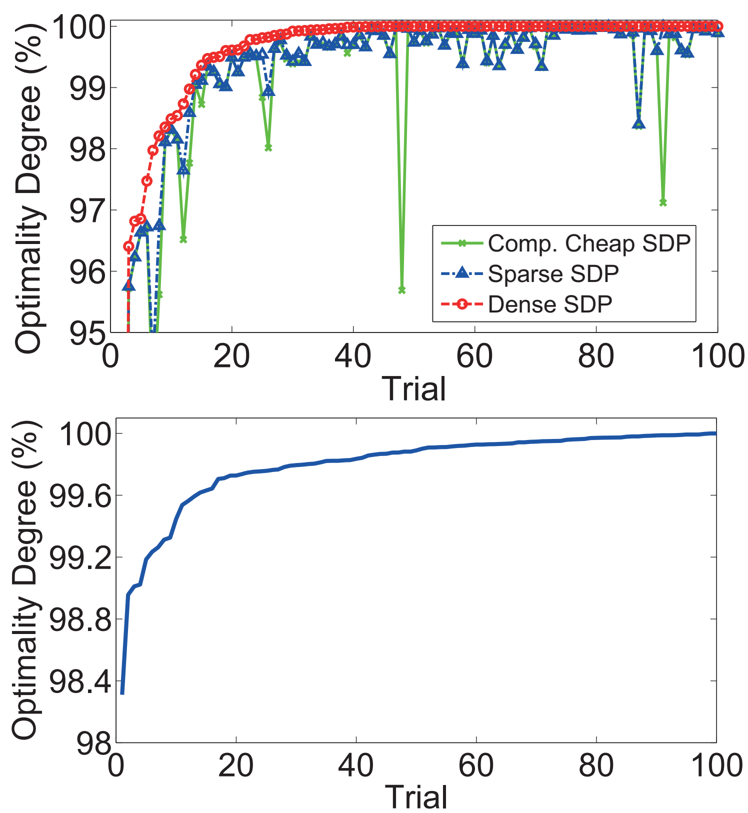


Figure 6.6: Optimal degrees of different relaxations for 100 random systems.

Since real-world systems are normally highly structured in many ways, we consider some structure for the system under study by assuming that B can be expressed as $[b \quad b \quad b]$ for some vector $b \in \mathbb{R}^5$. Assume that A, b, and $x[0]$ are normal random variables with the standard deviations 0.2, 1, and 1, respectively, while C is equal to $[I_3 \quad 0_{3\times 2}]$. We generated 100 random systems according to the above probability distributions for the parameters of the system and checked the rank of a low-rank solution of the sparse, dense, and computationally-cheap SDP relaxation problems for every trial. Let λ_1 and λ_2 denote the largest and the second largest eigenvalues of W^{opt} associated with the dense relaxation. We arranged the obtained 100 ratios $\frac{\lambda_2}{\lambda_1}$ in ascending order and subsequently labeled their corresponding trials as $1, 2, \ldots, 100$. Figure 6.5 plots the ratio $\frac{\lambda_2}{\lambda_1}$ for the ordered trials. It can be observed that this ratio is equal to 0 for 53 trials, implying that the dense SDP relaxation has found the solution of the ODC problem for 53 samples of the system. In addition, $\frac{\lambda_2}{\lambda_1}$ is less than 0.1 in 95 trials. Also, three near-global solutions of the ODC problem were found using different relaxations in all 100 cases. Figure 6.6 (a) depicts the (global) optimality degrees of these solutions after re-arranging the trials based on their associated optimality degrees for the dense SDP relaxation problem. Optimality degree is defined as

$$\text{Optimality degree (\%)} = 100 - \frac{\text{upper bound - lower bound}}{\text{upper bound}} \times 100$$

where "upper bound" and 'lower bound" denote the cost of the near-global controller recovered using the direct method and the optimal SDP cost, respectively. The optimality degree is an upper bound on the closeness of the cost of the near-optimal controller to the minimum cost, which is expressed in percentage. Notice that the employed optimality measure evaluates the global performance within the specified set of controllers. For example, the optimality degree of 100% means that a globally optimal controller is found among all <u>linear static</u> structured controllers.

As an alternative, we solved a penalized SDP relaxation with the penalty term $\Psi(W) = 0.5\ \text{trace}\{W\}$ added to the objective of the SDP relaxation. Interestingly, the matrix $\tilde{W}^{\text{opt}}$ became rank 1 for all of the 100 trials. Figure 6.6 (b) depicts the optimality degrees associated with the penalized dense SDP relaxation problem of the 100 random systems. It can be seen that the optimality degree is greater than 99.8% for 69 trials and is never less than 98.2%.

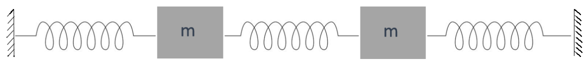


Figure 6.7: Mass-spring system with two masses

6.7.2 Mass-Spring Systems

In this subsection, the aim is to evaluate the performance of the developed controller design techniques in Lyapunov domain on the *Mass-Spring* system, as a classical physical system. Consider a mass-spring system consisting of N masses. This system is exemplified in Figure 6.7 for $N = 2$. The system can be modeled in the continuous-time domain as

$$\dot{x}_c(t) = A_c x_c(t) + B_c u_c(t) \tag{6.65}$$

where the state vector $x_c(t)$ can be partitioned as $[o_1(t)^T \ o_2(t)^T]$ with $o_1(t) \in \mathbb{R}^n$ equal to the vector of positions and $o_2(t) \in \mathbb{R}^n$ equal to the vector of velocities of the N masses. We assume that $N = 10$ and adopt the values of A_c and B_c from [Lin *et al.*, 2013]. The goal is to design a static sampled-data controller with a pre-specified structure (i.e., the controller is composed of a sampler, a static discrete-time structured controller and a zero-order holder). Consider two different control structures shown in Figure 6.8. The free parameters of each controller are colored in red in this figure. Notice that Structure (a) corresponds to a fully decentralized controller, where each local controller has access to the position and velocity of its associated mass. In contrast, Structure (b) allows limited communications between neighboring local controllers. Two ODC problems will be solved for these structures below.

Infinite-Horizon Deterministic ODC: In this experiment, we first discretize the system with the sampling time of 0.1 second and denote the obtained system as

$$x[\tau + 1] = Ax[\tau] + Bu[\tau], \qquad \tau = 0, 1, \ldots \tag{6.66}$$

It is aimed to design a constrained controller $u[\tau] = Kx[\tau]$ to minimize the cost $\sum_{\tau=0}^{\infty}(x[\tau]^T x[\tau] + u[\tau]^T u[\tau])$. Consider 100 randomly-generated initial states $x[0]$ with entries drawn from a normal distribution. We solved the computationally-cheap SDP relaxation of the infinite-horizon ODC problem combined with the direct recovery method to design a controller of Structure (a) minimizing the above cost function. The optimality degrees of the controllers designed for these 100 random trials are depicted in Figure 6.9. As can be seen, the optimality degree is better than 95%

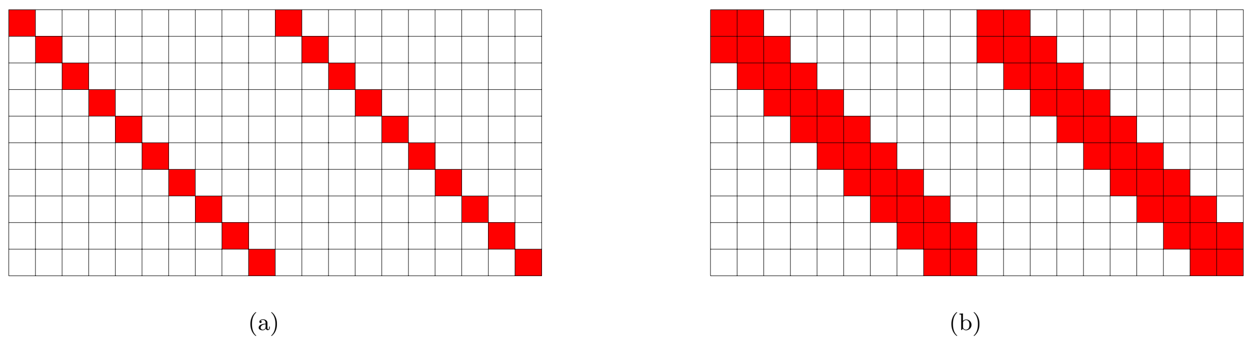


Figure 6.8: (a) Decentralized (b) Distributed. Two different structures (decentralized and distributed) for the controller K: the free parameters are colored in red (uncolored entries are set to zero).

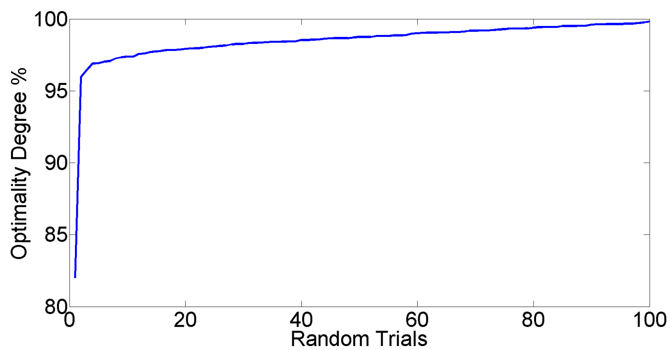
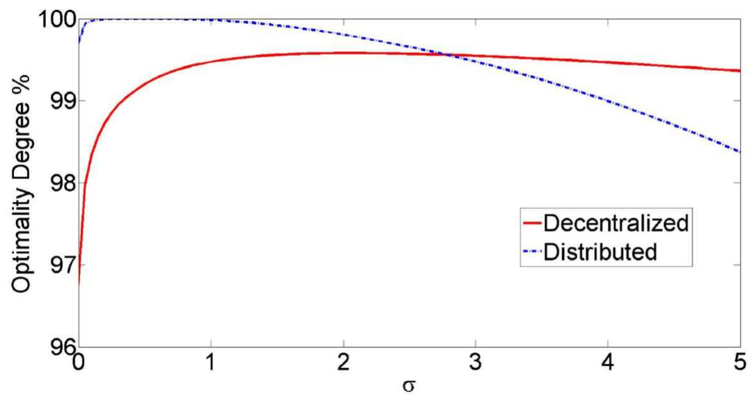


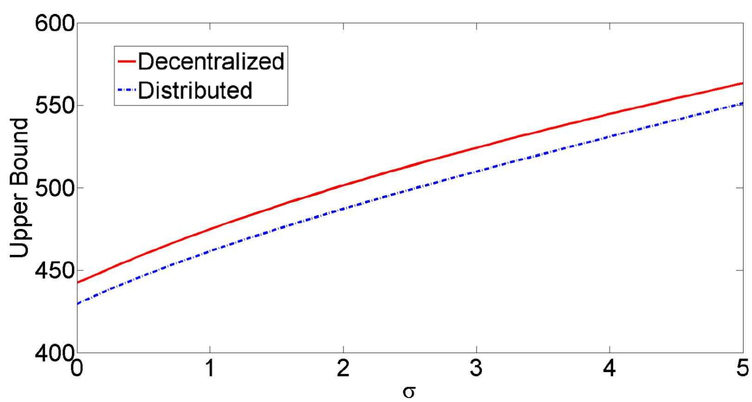
Figure 6.9: Optimality degree (%) of the decentralized controller $\hat{K}$ for a mass-spring system under 100 random initial states.

for more than 98 trials. It should be mentioned that all of these controllers stabilize the system.

Infinite-Horizon Stochastic ODC: Assume that the system is subject to both input disturbance and measurement noise. Consider the case $\Sigma_d = I_n$ and $\Sigma_v = \sigma I_r$, where σ varies from 0 to 5. Using the computationally-cheap relaxation problem (6.59) in conjunction with the indirect recovery method, a near-optimal controller is designed for each of the aforementioned control structures under various noise levels. The results are reported in Figure 6.10. The designed structured controllers are all stable with optimality degrees higher than 95% in the worst case and close to 99% in many cases.



(a)



(b)

Figure 6.10: (a) Optimality degree (b) Cost of near-optimal controller. The optimality degree and optimal cost of the near-optimal controller designed for the mass-spring system for two different control structures. The noise covariance matrix Σ_v is assumed to be equal to σI_r, where σ varies over a wide range.

6.8 Summary

This chapter studies the optimal distributed control (ODC) problem for discrete-time deterministic and stochastic systems. The objective is to design a fixed-order distributed controller with a pre-determined structure to minimize a quadratic cost functional. Both time domain and Lyapunov domain formulations of the ODC problem are cast as rank-constrained optimization problems with only one non-convex constraint requiring the rank of a variable matrix to be 1. We propose semidefinite programming (SDP) relaxations of these problems. The notion of tree decomposition is exploited to prove the existence of a low-rank solution for the SDP relaxation problems with rank at most 3. This result can be a basis for a better understanding of the complexity of the ODC problem because it states that almost all eigenvalues of the SDP solution are zero. Moreover, multiple recovery methods are proposed to round the rank-3 solution to rank 1, from which a near-global controller may be retrieved. Computationally-cheap relaxations are also developed for finite-horizon, infinite-horizon, and stochastic ODC problems. These relaxations are guaranteed to exactly solve the LQR and H_2 problems for the classical centralized control problem. The results are tested on multiple examples. In [Kalbat *et al.*, 2014], we have conducted a case study on electrical power systems to further evaluate the performance of the methods proposed in this chapter.

Part IV

Parallel Computing

Chapter 7

ADMM for Sparse Semidefinite Programming

This chapter designs a distributed algorithm for solving sparse semidefinite programming (SDP) problems, based on the alternating direction method of multipliers (ADMM). It is known that exploiting the sparsity of a large-scale SDP problem leads to a decomposed formulation with a lower computational cost. The algorithm proposed in this work solves the decomposed formulation of the SDP problem using an ADMM scheme whose iterations consist of two subproblems. Both subproblems are highly parallelizable and enjoy closed-form solutions, which make the iterations computationally very cheap. The developed numerical algorithm is also applied to the SDP relaxation of the optimal power flow (OPF) problem, and tested on the IEEE benchmark systems.

7.1 Introduction

While small- to medium-sized semidefinite programs (SDP) are efficiently solvable by second-order-based interior point methods in polynomial time up to any arbitrary precision [Vandenberghe and Boyd, 1996a], these methods are impractical for solving large-scale SDPs due to computation time and memory issues. A promising approach for solving large-scale SDP problems is the alternating direction method of multipliers (ADMM), which is a first-order optimization algorithm proposed in the mid-1970s by [Gabay and Mercier, 1976] and [Glowinsk and Marroco, 1975]. While second-order methods are capable of achieving high accuracy via expensive iterations, a modest accuracy can be achieved through tens of ADMM's low-complex iterations. In order to reach high accuracy in reasonable number of iterations, great effort has been devoted to accelerating ADMM through Nesterov's scheme [Goldstein *et al.*, 2014; Nesterov, 1983]. Because of the sensitivity of the gradient methods to the condition number of the problem's data, diagonal rescaling is proposed in [Giselsson and Boyd, 2014] for a class of problems to improve the performance of ADMM. The $\mathcal{O}(\frac{1}{n})$ worst-case convergence rate of ADMM is proven in [He and Yuan, 2012; Monteiro and Svaiter, 2013] under certain assumptions.

In light of the scalability of ADMM, the main objective of this work is to design an ADMM-based parallel algorithm for solving sparse large-scale SDPs, with a guaranteed convergence under very mild assumptions. We start by defining a representative graph for the large-scale SDP problem, from which a decomposed SDP formulation is obtained using a tree/chordal/clique decomposition technique. This decomposition replaces the large-scale SDP matrix variable with certain submatrices of this matrix. In order to solve the decomposed SDP problem iteratively, a distributed ADMM-based algorithm is derived, whose iterations comprise entry-wise matrix multiplication/division and eigendecomposition on certain submatrices of the SDP matrix. By finding the optimal solution for the distributed SDP, one could recover the solution to the original large-scale SDP formulation using an explicit formula.

This work is related to and improves upon some recent papers in this area. [Wen *et al.*, 2010] applies ADMM to the dual SDP formulation, leading to a centralized algorithm that is not parallelizable and is computationally expensive for large-scale SDPs. [Fukuda *et al.*, 2001] decomposes a sparse SDP into smaller-sized SDPs through a tree decomposition, which are then solved by interior point methods. However, this approach is limited by the large number of consistency constraints. Using a first-order splitting method, [Sun *et al.*, 2014] solves the decomposed SDP problem created by [Fukuda *et al.*, 2001], but the algorithm needs to solve an optimization subproblem at every iteration. In contrast with the above papers, the algorithm proposed in this work is composed of low-complex and parallelizable iterations, which run fast if the treewidth of the representative graph of the SDP problem is small. Since this treewidth is low for real-world power networks, our algorithm is well suited for the SDP relaxation of power optimization problems, and indeed this is the main motivation behind this work. This will be explained below.

This chapter is organized as follows. Some preliminaries and definitions are provided in Section 7.2. An arbitrary sparse SDP is converted into a decomposed SDP in Section 7.3, for which a numerical algorithm is developed in Section 7.4. The application of this algorithm for OPF is investigated and numerical examples are given in Section 7.5, followed by concluding remarks in Section 7.6.

Notations: $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}_n$ denote the sets of real numbers, complex numbers, and $n \times n$ Hermitian matrices, respectively. The notation $\mathbf{X}_1 \circ \mathbf{X}_2$ refers to the Hadamard (entrywise) multiplication of matrices $\mathbf{X}_1$ and $\mathbf{X}_2$. The symbols $\langle \cdot, \cdot \rangle$ and $\| \cdot \|_F$ denote the Frobinous inner product and norm of matrices, respectively. The notation $\|\mathbf{v}\|_2$ denotes the ℓ_2-norm of a vector $\mathbf{v}$. The $m \times n$ rectangular identity matrix, whose (i, j) entry is equal to the Kronecker delta δ_{ij}, is denoted by $\mathbf{I}_{m \times n}$. The notations $\mathrm{Re}\{\mathbf{W}\}$, $\mathrm{Im}\{\mathbf{W}\}$, $\mathrm{rank}\{\mathbf{W}\}$, and $\mathrm{diag}\{\mathbf{W}\}$ denote the real part, imaginary part, rank, and diagonal of a Hermitian matrix $\mathbf{W}$, respectively. Given a vector $\mathbf{v}$, the notation $\mathrm{diag}\{\mathbf{v}\}$ denotes a diagonal square matrix whose entries are given by $\mathbf{v}$. The notation $\mathbf{W} \succeq 0$ means that $\mathbf{W}$ is Hermitian and positive semidefinite. The notation "$\mathbf{i}$" is reserved for the imaginary unit. The superscripts $(\cdot)^*$ and $(\cdot)^{\mathrm{T}}$ represent the conjugate transpose and transpose operators, respectively. Given a matrix $\mathbf{W}$, its (l, m) entry is denoted as W_{lm}. The subscript $(\cdot)_{\mathrm{opt}}$ is used to show the optimal value of an optimization variable. Given a matrix $\mathbf{W}$, its Moore-Penrose pseudoinverse is denoted as $\mathrm{pinv}\{\mathbf{W}\}$. Given a simple graph $\mathcal{H}$, its vertex and edge sets are denoted by $\mathcal{V}_{\mathcal{H}}$ and $\mathcal{E}_{\mathcal{H}}$, respectively. Given two sets $\mathcal{S}_1$ and $\mathcal{S}_2$, the notation $\mathcal{S}_1 \backslash \mathcal{S}_2$ denotes the set of all elements of $\mathcal{S}_1$ that do not exist in $\mathcal{S}_2$. Given a Hermitian matrix $\mathbf{W}$ and two sets of positive integer numbers $\mathcal{S}_1$ and $\mathcal{S}_2$, define $\mathbf{W}\{\boldsymbol{\mathcal{S}_1}, \boldsymbol{\mathcal{S}_2}\}$ as a submatrix of $\mathbf{W}$ obtained through two operations: (i) removing all rows of $\mathbf{W}$ whose indices do not belong to $\mathcal{S}_1$, and (ii) removing all columns of $\mathbf{W}$ whose indices do not belong to $\mathcal{S}_2$. For instance, $\mathbf{W}\left\{\{\mathbf{1}, \mathbf{2}\}, \{\mathbf{2}, \mathbf{3}\}\right\}$ is a 2×2 matrix with the entries $W_{12}, W_{13}, W_{22}, W_{23}$.

7.2 Preliminaries

Consider the semidefinite program

$$
\begin{aligned}
&\underset{\mathbf{X} \in \mathbb{H}_n}{\text{minimize}} && \langle \mathbf{X}, \mathbf{M}_0 \rangle && && \text{(7.1a)}\\
&\text{subject to} && l_s \leq \langle \mathbf{X}, \mathbf{M}_s \rangle \leq u_s, && s = 1, \ldots, p, && \text{(7.1b)}\\
& && \mathbf{X} \succeq 0. && && \text{(7.1c)}
\end{aligned}
$$

where $\mathbf{M}_0, \mathbf{M}_1, \ldots, \mathbf{M}_p \in \mathbb{H}_n$, and

$$(l_s, u_s) \in (\{-\infty\} \cup \mathbb{R}) \times (\mathbb{R} \cup \{+\infty\})$$

for every $s = 1, \ldots, p$. Notice that the constraint (7.1b) reduces to an equality constraint if $l_s = u_s$.

Problem (7.1) is computationally expensive for a large n due to the presence of the positive semidefinite constraint (7.1c). However, if $\mathbf{M}_0, \mathbf{M}_1, \ldots, \mathbf{M}_p$ are sparse, this expensive constraint can be decomposed and expressed in terms of some principal submatrices of $\mathbf{X}$ with smaller dimensions. This will be explained next.

7.2.1 Representative Graph and Tree Decomposition

In order to leverage any possible sparsity of problem (7.1), a simple graph shall be defined to capture the zero-nonzero patterns of $\mathbf{M}_0, \mathbf{M}_1, \ldots, \mathbf{M}_p$.

Definition 33. *Define $\mathcal{G} = (\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G})$ as the representative graph of the SDP problem* (7.1)*, which is a simple graph with n vertices whose edges are specified by the nonzero off-diagonal entries of $\mathbf{M}_0, \mathbf{M}_1, \ldots, \mathbf{M}_p$. In other words, two arbitrary vertices i and j are connected if the (i, j) entry of at least one of the matrices $\mathbf{M}_0, \mathbf{M}_1, \ldots, \mathbf{M}_p$ is nonzero.*

Using a tree decomposition algorithm (also known as chordal or clique decomposition), we can obtain a *decomposed* formulation for problem (7.1), in which the positive semidefinite requirement is imposed on certain principal submatrices of $\mathbf{X}$ as opposed to $\mathbf{X}$ itself.

Definition 34 (Tree decomposition)**.** *A tree graph $\mathcal{T}$ is called a tree decomposition of $\mathcal{G}$ if it satisfies the following properties:*

1. *Every node of $\mathcal{T}$ corresponds to and is identified by a subset of $\mathcal{V}_\mathcal{G}$.*
2. *Every vertex of $\mathcal{G}$ is a member of at least one node of $\mathcal{T}$.*
3. *$\mathcal{T}_k$ is a connected graph for every $k \in \mathcal{V}_\mathcal{G}$, where $\mathcal{T}_k$ denotes the subgraph of $\mathcal{T}$ induced by all nodes of $\mathcal{T}$ containing the vertex k of $\mathcal{G}$.*
4. *The subgraphs $\mathcal{T}_i$ and $\mathcal{T}_j$ have a node in common for every $(i, j) \in \mathcal{E}_\mathcal{G}$.*

Each node of $\mathcal{T}$ is a bag (collection) of vertices of $\mathcal{G}$ and hence it is referred to as a **bag**.

Let $\mathcal{T} = (\mathcal{V}_\mathcal{T}, \mathcal{E}_\mathcal{T})$ be an arbitrary tree decomposition of $\mathcal{G}$, with the set of bags $\mathcal{V}_\mathcal{T} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q\}$. As discussed in the next section, it is possible to cast problem (7.1) in terms of those entries of $\mathbf{X}$ that appear in at least one of the submatrices

$$\mathbf{X}\{\mathcal{C}_1, \mathcal{C}_1\}, \mathbf{X}\{\mathcal{C}_2, \mathcal{C}_2\}, \ldots, \mathbf{X}\{\mathcal{C}_q, \mathcal{C}_q\},$$

These entries of X are referred to as *important entries*. Once the optimal values of the important entries of X are found using an arbitrary algorithm, the remaining entries can be obtained from an explicit (recursive) formula to be stated later.

Among the factors that may contribute to the computational complexity of the decomposed problem are: the size of the largest bag, the number of bags, and the total number of important entries. Finding a tree decomposition that leads to the minimum number of important entries (minimum fill-in problem) or possesses the minimum size for its largest bag (treewidth problem) is known to be NP-hard. Nevertheless, there are many efficient algorithms in the literature that find near-optimal tree decompositions (specially for power networks due to their near planarity) [Bodlaender and Koster, 2010; Bodlaender and Koster, 2011].

7.2.2 Sparsity Pattern of Matrices

Let $\mathbb{F}_n$ denote the set of symmetric $n \times n$ matrices with entries belonging to the set $\{0, 1\}$. The distributed optimization scheme to be proposed in this work uses a group of sparse slack matrices. We identify the locations of nonzero entries of such matrix variables using descriptive matrices in $\mathbb{F}_n$.

Definition 35. *Given an arbitrary matrix $\mathbf{X} \in \mathbb{H}_n$, define its sparsity pattern as a matrix $\mathbf{N} \in \mathbb{F}_n$ such that $N_{ij} = 1$ if and only if $X_{ij} \neq 0$ for every $i, j \in \{1, ..., n\}$. Let $|\mathbf{N}|$ denote the number of nonzero entries of $\mathbf{N}$. Also, define $\mathcal{S}(\mathbf{N})$ as*

$$\mathcal{S}(\mathbf{N}) \triangleq \{\mathbf{X} \in \mathbb{H}_n \mid \mathbf{X} \circ \mathbf{N} = \mathbf{X}\}.$$

Due to the Hermitian property of $\mathbf{X}$, if d denotes the number of nonzero diagonal entries of $\mathbf{N}$, then every $\mathbf{X} \in \mathcal{S}(\mathbf{N})$ can be specified by $(|\mathbf{N}| + d)/2$ real-valued scalars corresponding to $\mathrm{Re}\{\mathbf{X}\}$ and $(|\mathbf{N}| - d)/2$ real scalars corresponding to $\mathrm{Im}\{\mathbf{X}\}$. Therefore, $\mathcal{S}(\mathbf{N})$ is $|\mathbf{N}|$-dimensional over $\mathbb{R}$.

Definition 36. *Suppose that* $\mathcal{T} = (\mathcal{V}_\mathcal{T}, \mathcal{E}_\mathcal{T})$ *is a tree decomposition of the representative graph* $\mathcal{G}$ *with the bags* $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q$*.*

- *For* $r = 1, \ldots, q$*, define* $\mathbf{C}_r \in \mathbb{F}_n$ *as a sparsity pattern whose* (i, j) *entry is equal to* 1 *if* $\{i, j\} \subseteq \mathcal{C}_r$ *and is 0 otherwise for every* $i, j \in \{1, ..., n\}$*.*
- *Define* $\mathbf{C} \in \mathbb{F}_n$ *as an aggregate sparsity pattern whose* (i, j) *entry is equal to* 1 *if and only if* $\{i, j\} \subseteq \mathcal{C}_r$ *for at least one index* $r \in \{1, \ldots, p\}$*.*
- *For* $s = 0, 1, \ldots, p$*, define* $\mathbf{N}_s \in \mathbb{F}_n$ *as the sparsity pattern of* $\mathbf{M}_s$*.*

The sparsity pattern $\mathbf{C}$, which can also be interpreted as the adjacency matrix of a chordal extension of $\mathcal{G}$ induced by $\mathcal{T}$, captures the locations of the important entries of $\mathbf{X}$. The matrix $\mathbf{C}$ will later be used to describe the domain of definition for the variable of decomposed SDP problem.

7.2.3 Indicator Functions

To streamline the formulation, we will replace any positivity or positive semidefiniteness constraints in the decomposed SDP problem by the indicator functions introduced below.

Definition 37. *For every* $l \in \{-\infty\} \cup \mathbb{R}$ *and* $u \in \mathbb{R} \cup \{+\infty\}$*, define the convex indicator function* $\mathcal{I}_{l,u} : \mathbb{R} \to \{0, +\infty\}$ *as*

$$\mathcal{I}_{l,u}(x) \triangleq \begin{cases} 0 & \text{if } \ l \leq x \leq u \\ +\infty & \text{otherwise} \end{cases}$$

Definition 38. *For every* $r \in \{1, 2, \ldots, q\}$*, define the convex indicator function* $\mathcal{J}_r : \mathbb{H}_n \to \{0, +\infty\}$ *as*

$$\mathcal{J}_r(\mathbf{X}) \triangleq \begin{cases} 0 & \text{if } \ \mathbf{X}\{\mathcal{C}_r, \mathcal{C}_r\} \succeq 0 \\ +\infty & \text{otherwise} \end{cases}$$

7.3 Decomposed SDP

Consider the problem

$$\underset{\mathbf{X}\in\mathcal{S}(\mathbf{C})}{\text{minimize}} \quad \langle \mathbf{X}, \mathbf{M}_0 \rangle \tag{7.2a}$$

$$\text{subject to} \quad l_s \leq \langle \mathbf{X}, \mathbf{M}_s \rangle \leq u_s, \qquad s = 1, \ldots, p, \tag{7.2b}$$

$$\mathbf{X}\{\mathcal{C}_r, \mathcal{C}_r\} \succeq 0, \qquad r = 1, \ldots, q \tag{7.2c}$$

which is referred to as *decomposed SDP* throughout this chapter. Due to the chordal theorem [Grone *et al.*, 1984], problems (7.1) and (7.2) lead to the same optimal objective value. Furthermore, if $\mathbf{X}_{\text{ref}} \in \mathcal{S}(\mathbf{C})$ denotes an arbitrary solution of the decomposed SDP problem (7.2), then there exists a solution $\mathbf{X}_{\text{opt}}$ to the SDP problem (7.1) such that $\mathbf{X}_{\text{opt}} \circ \mathbf{C} = \mathbf{X}_{\text{ref}}$.

To understand how $\mathbf{X}_{\text{opt}}$ can be constructed from $\mathbf{X}_{\text{ref}}$, observe that those entries of $\mathbf{X}$ corresponding to the zeros of $\mathbf{C}$ are 0 due to the relation $\mathbf{X}_{\text{ref}} \in \mathcal{S}(\mathbf{C})$. These entries of the matrix variable $\mathbf{X}$ that are needed for SDP but have not been found by decomposed SDP are referred to as *missing entries*. Several completion approaches can be adopted in order to recover these missing entries. An algorithm is proposed in [Fukuda *et al.*, 2001; Nakata *et al.*, 2003] that obtains a completion for $\mathbf{X}_{\text{ref}}$ within the set

$$\{\mathbf{X} \in \mathbb{H}_n \,|\, \mathbf{X} \circ \mathbf{C} = \mathbf{X}_{\text{ref}},\ \mathbf{X} \succeq 0\}$$

whose determinant is maximum. However such a solution may not be favorable for applications that require a low-rank solution such as an SDP relaxation. It is also known that there exists a polynomial-time algorithm to fill a partially-known real-valued matrix in such a way that the rank of the resulting matrix becomes equal to the highest rank among all bags [Laurent, 2001; Laurent and Varvitsiotis, 2014]. In chapter 2, we extended this result to the complex domain by proposing a recursive algorithm that transforms $\mathbf{X}_{\text{ref}} \in \mathcal{S}(\mathbf{C})$ into a solution $\mathbf{X}_{\text{opt}}$ for the original SDP problem (7.1) whose rank is upper bounded by the maximum rank among the matrices $\mathbf{X}_{\text{ref}}\{\mathcal{C}_1, \mathcal{C}_1\}, \mathbf{X}_{\text{ref}}\{\mathcal{C}_2, \mathcal{C}_2\}, \ldots, \mathbf{X}_{\text{ref}}\{\mathcal{C}_q, \mathcal{C}_q\}$. This algorithm is stated below for completeness.

Matrix completion algorithm:

1. Set $\mathcal{T}' := \mathcal{T}$ and $\mathbf{X} := \mathbf{X}_{\text{ref}}$.

2. If $\mathcal{T}'$ has a single node, then consider $\mathbf{X}_{\text{opt}}$ as $\mathbf{X}$ and terminate; otherwise continue to the next step.

3. Choose a pair of bags $\mathcal{C}_x, \mathcal{C}_y$ of $\mathcal{T}'$ such that $\mathcal{C}_x$ is a leaf of $\mathcal{T}'$ and $\mathcal{C}_y$ is its unique neighbor.

4. Define

$$\mathbf{K} \triangleq \text{pinv}\{\mathbf{X}\{\mathcal{C}_x \cap \mathcal{C}_y, \mathcal{C}_x \cap \mathcal{C}_y\}\} \tag{7.3a}$$

$$\mathbf{G}_x \triangleq \mathbf{X}\{\mathcal{C}_x \setminus \mathcal{C}_y, \mathcal{C}_x \cap \mathcal{C}_y\} \tag{7.3b}$$

$$\mathbf{G}_y \triangleq \mathbf{X}\{\mathcal{C}_y \setminus \mathcal{C}_x, \mathcal{C}_x \cap \mathcal{C}_y\} \tag{7.3c}$$

$$\mathbf{E}_x \triangleq \mathbf{X}\{\mathcal{C}_x \setminus \mathcal{C}_y, \mathcal{C}_x \setminus \mathcal{C}_y\} \in \mathbb{C}^{d_x \times d_x} \tag{7.3d}$$

$$\mathbf{E}_y \triangleq \mathbf{X}\{\mathcal{C}_y \setminus \mathcal{C}_x, \mathcal{C}_y \setminus \mathcal{C}_x\} \in \mathbb{C}^{d_y \times d_y} \tag{7.3e}$$

$$\mathbf{S}_x \triangleq \mathbf{E}_x - \mathbf{G}_x\mathbf{K}\mathbf{G}_x^* = \mathbf{Q}_x\mathbf{D}_x\mathbf{Q}_x^* \tag{7.3f}$$

$$\mathbf{S}_y \triangleq \mathbf{E}_y - \mathbf{G}_y\mathbf{K}\mathbf{G}_y^* = \mathbf{Q}_y\mathbf{D}_y\mathbf{Q}_y^* \tag{7.3g}$$

where $\mathbf{Q}_x\mathbf{D}_x\mathbf{Q}_x^*$ and $\mathbf{Q}_y\mathbf{D}_y\mathbf{Q}_y^*$ denote the eigenvalue decompositions of $\mathbf{S}_x$ and $\mathbf{S}_y$ with the diagonals of $\mathbf{D}_x$ and $\mathbf{D}_y$ arranged in descending order. Then, update a part of $\mathbf{X}$ as follows:

$$\mathbf{X}\{\mathcal{C}_y \setminus \mathcal{C}_x, \mathcal{C}_x \setminus \mathcal{C}_y\} := \mathbf{G}_y\mathbf{K}\mathbf{G}_x^* + \mathbf{Q}_y\sqrt{\mathbf{D}_y}\;\mathbf{I}_{d_y \times d_x}\sqrt{\mathbf{D}_x}\;\mathbf{Q}_x^*$$

and update $\mathbf{X}\{\mathcal{C}_x \setminus \mathcal{C}_y, \mathcal{C}_y \setminus \mathcal{C}_x\}$ accordingly to preserve the Hermitian property of $\mathbf{X}$.

5. Update $\mathcal{T}'$ by merging $\mathcal{C}_x$ into $\mathcal{C}_y$, i.e., replace $\mathcal{C}_y$ with $\mathcal{C}_x \cup \mathcal{C}_y$ and then remove $\mathcal{C}_x$ from $\mathcal{T}'$.

6. Go back to step 2.

Theorem 33. *Consider an arbitrary solution $\mathbf{X}_{\text{ref}}$ of the decomposed SDP problem* (7.2)*. The output of the matrix completion algorithm, denoted as $\mathbf{X}_{\text{opt}}$, is a solution of the original SDP problem* (7.1)*. Moreover, the rank of $\mathbf{X}_{\text{opt}}$ is smaller than or equal to:*

$$\max\left\{\text{rank}\left\{\mathbf{X}_{\text{ref}}\{\mathcal{C}_r, \mathcal{C}_r\}\right\} \;\middle|\; r = 1, \ldots, q\right\}.$$

Proof. See Theorem 4 of Chapter 2 for the proof. □

7.4 Alternating Direction Method of Multipliers

Consider the optimization problem

$$\underset{\substack{\mathbf{x}\in\mathbb{R}^{n_x}\\ \mathbf{y}\in\mathbb{R}^{n_y}}}{\text{minimize}} \quad f(\mathbf{x}) + g(\mathbf{y}) \tag{7.4a}$$

$$\text{subject to} \quad \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} = \mathbf{c}. \tag{7.4b}$$

where $\mathbf{c} \in \mathbb{R}^{n_c}$, $\mathbf{A} \in \mathbb{R}^{n_c \times n_x}$ and $\mathbf{B} \in \mathbb{R}^{n_c \times n_y}$ are given matrices. Also $f : \mathbb{R}^{n_x} \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^{n_y} \to \mathbb{R} \cup \{+\infty\}$ are given convex functions. Notice that the variables $\mathbf{x}$ and $\mathbf{y}$ are coupled through the linear constraint (7.4b) while the objective function is separable.

The augmented Lagrangian function for problem (7.4) is equal to

$$\mathcal{L}_\mu(\mathbf{x}, \mathbf{y}, \lambda) = f(\mathbf{x}) + g(\mathbf{y}) + +\lambda^{\mathrm{T}}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} - \mathbf{c}) + (\mu/2)\|\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} - \mathbf{c}\|_2^2, \tag{7.5a}$$

where $\lambda \in \mathbb{R}^{n_c}$ is the Lagrange multiplier associated with the constraint (7.4b), and $\mu \in \mathbb{R}$ is a fixed parameter. ADMM is one approach for solving problem (7.4), which performs the following procedure at each iteration [Boyd *et al.*, 2011]:

$$\mathbf{x}^{k+1} = \underset{\mathbf{x}\in\mathbb{R}^{n_x}}{\arg\min} \quad \mathcal{L}_\mu(\mathbf{x}, \mathbf{y}^k, \lambda^k), \tag{7.6a}$$

$$\mathbf{y}^{k+1} = \underset{\mathbf{y}\in\mathbb{R}^{n_y}}{\arg\min} \quad \mathcal{L}_\mu(\mathbf{x}^{k+1}, \mathbf{y}, \lambda^k), \tag{7.6b}$$

$$\lambda^{k+1} = \lambda^k + \mu(\mathbf{A}\mathbf{x}^{k+1} + \mathbf{B}\mathbf{y}^{k+1} - \mathbf{c}). \tag{7.6c}$$

where $k = 0, 1, 2, ...,$ for an arbitrary initialization $(\mathbf{x}^0, \mathbf{y}^0, \lambda^0)$. In these equations, "argmin" means an arbitrary minimizer of a convex function and does not need any uniqueness assumption. Notice that each of the updates (7.6a) and (7.6b) is an optimization sub-problem with respect to either $\mathbf{x}$ and $\mathbf{y}$, by freezing the other variable at its latest value. We employ the energy sequence $\{\varepsilon^k\}_{k=1}^{\infty}$ proposed in [Goldstein *et al.*, 2014] as measure for convergence:

$$\varepsilon^{k+1} = (1/\mu)\|\lambda^{k+1} - \lambda^k\|_2^2 + \mu\|\mathbf{B}(y^{k+1} - y^k)\|_2^2 \tag{7.7}$$

ADMM is particularly interesting for the cases where (7.6a) and (7.6b) can be performed efficiently through an explicit formula. Under such circumstances, it would be possible to execute a large number of iterations in a short amount of time. In this section, we first cast the decomposed SDP problem (7.2) in the form of (7.4) and then regroup the variables into two blocks $\mathcal{P}_1$ and $\mathcal{P}_2$ playing the roles of $\mathbf{x}$ and $\mathbf{y}$ in the ADMM algorithm.

7.4.1 Projection Into Positive Semidefinite Cone

The algorithm to be proposed in this work requires the projection of q matrices belonging to $\mathbb{H}_{|\mathcal{C}_1|}, \mathbb{H}_{|\mathcal{C}_2|}, \ldots, \mathbb{H}_{|\mathcal{C}_q|}$ onto the positive semidefinite cone. This is probably the most computationally expensive part of each iteration.

Definition 39. *For a given Hermitian matrix $\widehat{\mathbf{Z}}$, define the unique solution to the optimization problem*

$$\underset{\mathbf{Z}\in\mathbb{H}_m}{\text{minimize}} \quad \|\mathbf{Z}-\widehat{\mathbf{Z}}\|_F^2 \tag{7.8a}$$

$$\text{subject to} \quad \mathbf{Z} \succeq 0 \tag{7.8b}$$

as the projection of $\widehat{\mathbf{Z}}$ onto the cone of positive semidefinite matrices, and denote it as $\widehat{\mathbf{Z}}^+$.

The next Lemma reveals the interesting fact that problem (7.8) can be solved through an eigenvalue decomposition of $\widehat{\mathbf{Z}}$.

Lemma 9. *Let*

$$\widehat{\mathbf{Z}} = \mathbf{Q} \times \text{diag}\{(\nu_1 \ldots, \nu_m)\} \times \mathbf{Q}^*$$

denote the eigenvalue decomposition of $\widehat{\mathbf{Z}}$. The solution of the projection problem (7.8) *is given by*

$$\widehat{\mathbf{Z}}^+ = \mathbf{Q} \times \text{diag}\{(\max\{\nu_1, 0\}, \ldots, \max\{\nu_m, 0\})\} \times \mathbf{Q}^*$$

Proof. See [Higham, 1988] for the proof. □

7.4.2 ADMM for Decomposed SDP

We apply ADMM to the following reformulation of the decomposed SDP problem (7.2):

$$\underset{\substack{\mathbf{X}\in\mathcal{S}(\mathbf{C}) \\ \{\mathbf{X}_{N;s}\in\mathcal{S}(\mathbf{N}_s)\}_{s=0}^p \\ \{\mathbf{X}_{C;r}\in\mathcal{S}(\mathbf{C}_r)\}_{r=1}^q \\ \{z_s\in\mathbb{R}\}_{s=0}^p}}{\text{minimize}} \quad z_0 + \sum_{s=1}^{p} \mathcal{I}_{l_s,u_s}(z_s) + \sum_{r=1}^{q} \mathcal{J}_r(\mathbf{X}_{C;r})$$

$$\text{subject to} \quad \mathbf{X} \circ \mathbf{C}_r = \mathbf{X}_{C;r}, \qquad r = 1, 2, \ldots, q, \tag{7.9a}$$

$$\mathbf{X} \circ \mathbf{N}_s = \mathbf{X}_{N;s}, \qquad s = 0, 1, \ldots, p, \tag{7.9b}$$

$$z_s = \langle \mathbf{M}_s, \mathbf{X}_{N;s} \rangle, \qquad s = 0, 1, \ldots, p. \tag{7.9c}$$

If $\mathbf{X}$ is a feasible solution of (7.9) with a finite objective value, then

$$\mathcal{J}_r(\mathbf{X}) = \mathcal{J}_r(\mathbf{X} \circ \mathbf{C}_r) \overset{(7.9a)}{=} \mathcal{J}_r(\mathbf{X}_{C;r}) = 0$$

which concludes that $\mathbf{X}\{\mathcal{C}_r, \mathcal{C}_r\} \succeq 0$. Also,

$$\begin{aligned}\mathcal{I}_{l_s,u_s}(\langle \mathbf{X}, \mathbf{M}_s \rangle) &= \mathcal{I}_{l_s,u_s}(\langle \mathbf{X} \circ \mathbf{N}_s, \mathbf{M}_s \rangle) \\ &\overset{(7.9b)}{=} \mathcal{I}_{l_s,u_s}(\langle \mathbf{X}_{N;s}, \mathbf{M}_s \rangle) \\ &\overset{(7.9c)}{=} \mathcal{I}_{l_s,u_s}(z_s) = 0\end{aligned}$$

which yields that $l_s \leq \langle \mathbf{X}, \mathbf{M}_s \rangle \leq u_s$. Therefore, $\mathbf{X}$ is a feasible point for problem (7.2) as well, with the same objective value. Define

1. $\mathbf{\Lambda}_{C;r} \in \mathcal{S}(\mathbf{C}_r)$ as the Lagrange multiplier associated with the constraint (7.9a) for $r = 1, 2, \ldots, q$,
2. $\mathbf{\Lambda}_{N;s} \in \mathcal{S}(\mathbf{N}_s)$ as the Lagrange multiplier associated with the constraint (7.9b) for $s = 0, 1, \ldots, p$,
3. $\lambda_{z;s} \in \mathbb{R}$ as the Lagrange multiplier associated with the constraint (7.9c) for $s = 0, 1, \ldots, p$.

We regroup the primal and dual variables as

$$\begin{aligned}&(\text{Block 1}) \quad \mathcal{P}_1 = (\mathbf{X}, \{z_s\}_{s=0}^p) \\ &(\text{Block 2}) \quad \mathcal{P}_2 = (\{\mathbf{X}_{C;r}\}_{r=1}^q, \{\mathbf{X}_{N;s}\}_{s=0}^p) \\ &(\text{Dual}) \qquad \mathcal{D} = (\{\mathbf{\Lambda}_{C;r}\}_{r=1}^q, \{\mathbf{\Lambda}_{N;s}\}_{s=0}^p, \{\lambda_s\}_{s=0}^p)\,.\end{aligned}$$

Note that "block 1", "block 2" and "$\mathcal{D}$" play the roles of $\mathbf{x}$, $\mathbf{y}$ and λ in the standard formulation of ADMM, respectively. The augmented Lagrangian can be calculated as

$$\begin{aligned}(2/\mu)\mathcal{L}_\mu(\mathcal{P}_1, \mathcal{P}_2, \mathcal{D}) = \mathcal{L}_D(\mathcal{D})/\mu^2 + \|z_0 - \langle \mathbf{M}_0, \mathbf{X}_{N;0} \rangle + (1 + \lambda_{z;0})/\mu\|_F^2 \quad &(7.11a) \\ + \sum_{s=1}^{p} \|z_s - \langle \mathbf{M}_s, \mathbf{X}_{N;s} \rangle + \lambda_{z;s}/\mu\|_F^2 + \mathcal{I}_{l_s,u_s}(z_s) \quad &(7.11b) \\ + \sum_{r=1}^{q} \|\mathbf{X} \circ \mathbf{C}_r - \mathbf{X}_{C;r} + (1/\mu)\mathbf{\Lambda}_{C;r}\|_F^2 + \mathcal{J}_r(\mathbf{X}_{C;k}) \quad &(7.11c) \\ + \sum_{s=1}^{p} \|\mathbf{X} \circ \mathbf{N}_s - \mathbf{X}_{N;s} + (1/\mu)\mathbf{\Lambda}_{N;s}\|_F^2 \quad &(7.11d)\end{aligned}$$

where

$$\mathcal{L}_D(\mathcal{D}) = -(1+\lambda_{z;0})^2 - \sum_{s=1}^{p} \lambda_{z;s}^2 - \sum_{r=1}^{q} \|\mathbf{\Lambda}_{C;r}\|_F^2 - \sum_{s=1}^{p} \|\mathbf{\Lambda}_{N;s}\|_F^2 \tag{7.12}$$

Using the blocks $\mathcal{P}_1$ and $\mathcal{P}_2$, the ADMM iterations for problem (7.9) can be expressed as follows:

1. The subproblem (7.6a) in terms of $\mathcal{P}_1$ consists of two parallel steps:

   (a) *Minimization in terms of* $\mathbf{X}$: This step consists of $|\mathbf{C}|$ scalar quadratic and unconstrained programs. It possesses an explicit formula that involves $|\mathbf{C}|$ parallel multiplication operations.

   (b) *Minimization in terms of* $\{z_s\}_{s=0}^p$: This step consists of $p+1$ scalar quadratic programs each with a box constraint. It possesses an explicit formula that involves $p+1$ parallel multiplication operations.

2. The subproblem (7.6b) in terms of $\mathcal{P}_2$ also consists of two parallel steps:

   (a) *Minimization in terms of* $\{\mathbf{X}_{C;r}\}_{r=1}^q$: This step consists of q projection problems of the form (7.8). According to Lemma 9, this reduces to q parallel eigenvalue decomposition operations on matrices of sizes $|\mathcal{C}_r| \times |\mathcal{C}_r|$ for $r = 1, \ldots, q$.

   (b) *Minimization in terms of* $\{\mathbf{X}_{N;s}\}_{s=0}^p$: This step consists of p unconstrained quadratic programs of sizes $|\mathbf{N}_s|$ for $s = 0, 1, \ldots, p$. The quadratic programs are parallel and each of them possesses an explicit formula that involves $2|\mathbf{N}_s|$ multiplications.

3. Computation of the dual variables at each iteration, in equation (7.6c), consists of three parallel steps:

   (a) *Updating* $\{\mathbf{\Lambda}_{C;r}\}_{r=1}^q$: Computational costs for this step involves no multiplications and is negligible.

   (b) *Updating* $\{\mathbf{\Lambda}_{N;s}\}_{s=0}^p$: Computational costs for this step involves no multiplications and is negligible.

   (c) *Updating* $\{\lambda_{z;s}\}_{s=0}^p$: This step is composed of $p+1$ parallel inner product computations, each involving $|\mathbf{N}_s|$ multiplications for $s = 0, 1, \ldots, p$.

The fact that every step of the above algorithm has an explicit easy-to-compute formula makes the algorithm very appealing for large-scale SDPs.

Notation 3. *For every* $\mathbf{D}, \mathbf{E} \in \mathbb{H}_n$*, the notation* $\mathbf{D} \oslash_{\mathbf{C}} \mathbf{E}$ *refers to the entrywise division of those entries of* $\mathbf{D}$ *and* $\mathbf{E}$ *that correspond to the ones of* $\mathbf{C}$ *i.e.,*

$$(\mathbf{D} \oslash_{\mathbf{C}} \mathbf{E})_{ij} \triangleq \begin{cases} D_{ij}/E_{ij} & \text{if } \; C_{ij} = 1 \\ 0 & \text{if } \; C_{ij} = 0. \end{cases}$$

Theorem 34. *Assume that Slater's conditions hold for the decomposable SDP problem* (7.2) *and consider the iterative algorithm given in* (7.19)*. The limit of* $\mathbf{X}^k$ *at* $k = +\infty$ *is an optimal solution for* (7.2)*.*

Proof. The convergence of both primal and dual variables is guaranteed for a standard ADMM problem if the matrix $\mathbf{B}$ in (7.4b) has full column rank [He and Yuan, 2014]. After realizing that (7.19) is obtained from a two-block ADMM procedure, the theorem can be concluded form the fact that the equivalent of $\mathbf{B}$ for the algorithm (7.19) is a mapping from the variables $\{\mathbf{X}_{C;r}\}_{r=1}^{q}$ and $\{\mathbf{X}_{N;s}\}_{s=0}^{p}$ to

$$\{\mathbf{X}_{C;r}\}_{r=1}^{q}, \{\mathbf{X}_{N;s}\}_{s=0}^{p} \quad \text{and} \quad \{\langle \mathbf{M}_s, \mathbf{X}_{N;s} \rangle\}_{s=0}^{p}$$

which is not singular, i.e., it has full column rank. The details are omitted for brevity. □

In what follows, we elaborate on every step of the ADMM iterations:

Block 1: The first step of the algorithm that corresponds to (7.6a) consists of the operation

$$\mathcal{P}_1^{k+1} := \arg\min \quad \mathcal{L}_\mu(\mathcal{P}_1, \mathcal{P}_2^k, \mathcal{D}^k).$$

Notice that the minimization of $\mathcal{L}_\mu(\mathcal{P}_1, \mathcal{P}_2^k, \mathcal{D}^k)$ with respect to $\mathcal{P}_1$ is decomposable in terms of the real scalars

$$\mathrm{Re}\{X_{ij}\} \quad \text{for} \quad i = 1, \ldots, n; \quad j = i, \ldots, n \tag{7.14a}$$

$$\mathrm{Im}\{X_{ij}\} \quad \text{for} \quad i = 1, \ldots, n; \quad j = i+1, \ldots, n \tag{7.14b}$$

$$z_s \quad \text{for} \quad s = 1, \ldots, p \tag{7.14c}$$

which leads to the explicit formulas (7.19a), (7.19b) and (7.19c).

Block 2: The second step of the algorithm that corresponds to (7.6b) consists of the operation

$$\mathcal{P}_2^{k+1} = \arg\min \quad \mathcal{L}_\mu(\mathcal{P}_1^{k+1}, \mathcal{P}_2, \mathcal{D}^k)$$

Notice that the minimization of $\mathcal{L}_\mu(\mathcal{P}_1, \mathcal{P}_2^k, \mathcal{D}^k)$ with respect to $\mathcal{P}_1$ is decomposable in terms of the matrix variables

$$\mathbf{X}_{C;r} \quad \text{for} \quad r = 1, 2, \ldots, q \tag{7.16a}$$

$$\mathbf{X}_{N;s} \quad \text{for} \quad s = 0, 1, \ldots, p. \tag{7.16b}$$

Hence, the update of $\mathbf{X}_{C;r}$ reduces to the problem (7.8) for $\widehat{\mathbf{Z}} = \mathbf{X}_{C;r}\{\mathcal{C}_r, \mathcal{C}_r\}$. As shown in Lemma 9, this can be performed via the eigenvalue decomposition of a $|\mathcal{C}_r| \times |\mathcal{C}_r|$ matrix. In addition, the updated value of $\mathbf{X}_{N;s}$ is a minimizer of the function

$$\mathcal{L}_{N;s}(\mathbf{Z}) = \|z_s - \langle \mathbf{M}_s, \mathbf{Z}\rangle + \lambda_{z;s}/\mu\|_F^2 + \|\mathbf{X} \circ \mathbf{N}_s - \mathbf{Z} + (1/\mu)\mathbf{\Lambda}_{N;s}\|_F^2 \tag{7.17}$$

By taking the derivatives of this function, it is possible to find an explicit formula for $\mathbf{Z}_{\text{opt}}$. Define $\mathcal{L}'_{N;s}(\mathbf{Z}) \in \mathcal{S}(\mathbf{N}_s)$ as the gradient of $\mathcal{L}_{N;s}(\mathbf{Z})$ with the following structure:

$$\mathcal{L}'_{N;s}(\mathbf{Z}) \triangleq \left[\frac{\partial \mathcal{L}_{N;s}}{\partial \text{Re}\{Z_{ij}\}} + \mathbf{i}\frac{\partial \mathcal{L}_{N;s}}{\partial \text{Im}\{Z_{ij}\}}\right]_{i,j=1,\ldots,n}$$

Then, we have

$$\mathcal{L}'_{N;s}(\mathbf{Z})/2 = \mathbf{Z} - \mathbf{X} \circ \mathbf{N}_s - (1/\mu)\mathbf{\Lambda}_{N,s} + (-z_s + \langle \mathbf{M}_s, \mathbf{Z}\rangle - \lambda_{z;s}/\mu)\mathbf{M}_s.$$

Therefore,

$$\mathbf{Z}_{\text{opt}} = \mathbf{X} \circ \mathbf{N}_s + (1/\mu)\mathbf{\Lambda}_{N,s} + y_s\mathbf{M}_s, \tag{7.18}$$

where $y_s \triangleq z_s - \langle \mathbf{M}_s, \mathbf{Z}^{\text{opt}}\rangle + \lambda_{z;s}/\mu$. Hence, it only remains to derive the scalar y_s, which can be done by inner multiplying $\mathbf{M}_s$ to the both sides of the equation (7.18). This leads to the equations (7.19e) and (7.19f).

ADMM for Decomposed SDP:

Block 1:

$$\mathbf{X}^{k+1} := \left[\sum_{r=1}^{q} \mathbf{C}_r \circ (\mathbf{X}_{C;r}^k - \mathbf{\Lambda}_{C;r}^k/\mu) + \sum_{s=1}^{p} \mathbf{N}_s \circ (\mathbf{X}_{N;s}^k - \mathbf{\Lambda}_{N;s}^k/\mu)\right] \oslash_{\mathbf{C}} \left[\sum_{r=1}^{q} \mathbf{C}_r + \sum_{s=1}^{p} \mathbf{N}_s\right] \tag{7.19a}$$

$$z_0^{k+1} := \langle \mathbf{M}_0, \mathbf{X}_{N;0}^k\rangle - (\lambda_{z;0}^k + 1)/\mu \tag{7.19b}$$

$$z_s^{k+1} := \max\{\min\{\langle \mathbf{M}_s, \mathbf{X}_{N;s}^k\rangle - \lambda_{z;s}^k/\mu, u_s\}, l_s\} \qquad \text{for} \quad s = 1, 2, \ldots, p \tag{7.19c}$$

Test cases	p	q	**Maximum size of bags**	**Running time of 1000 iterations (sec)**
Chow's 9 bus	27	7	3	6.18
IEEE 14 bus	42	12	3	9.96
IEEE 30 bus	90	18	4	14.66
IEEE 57 bus	171	26	6	21.25
IEEE 118 bus	354	66	5	53.13
IEEE 300 bus	900	111	7	98.95

Table 7.1: Running time of the proposed algorithm for solving the SDP relaxation of OPF problem on IEEE test cases.

Block 2:

$$\mathbf{X}_{C;r}^{k+1} := (\mathbf{X}^{k+1} \circ \mathbf{C}_r + \mathbf{\Lambda}_{C;r}^{k}/\mu)^{+} \qquad \text{for} \quad r = 1, 2, \ldots, q \tag{7.19d}$$

$$y_s^{k+1} := \frac{z_s^{k+1} + \lambda_{z;s}^{k}/\mu - \langle \mathbf{M}_s, \mathbf{N}_s \circ \mathbf{X}^{k+1} + \mathbf{\Lambda}_{N;s}^{k}/\mu \rangle}{1 + \|\mathbf{M}_s\|_F^2} \qquad \text{for} \quad s = 0, 1, \ldots, p \tag{7.19e}$$

$$\mathbf{X}_{N;s}^{k+1} := \mathbf{N}_s \circ \mathbf{X}^{k+1} + \mathbf{\Lambda}_{N,s}^{k}/\mu + y_s^{k+1}\mathbf{M}_s \qquad \text{for} \quad s = 0, 1, \ldots, p \tag{7.19f}$$

Dual:

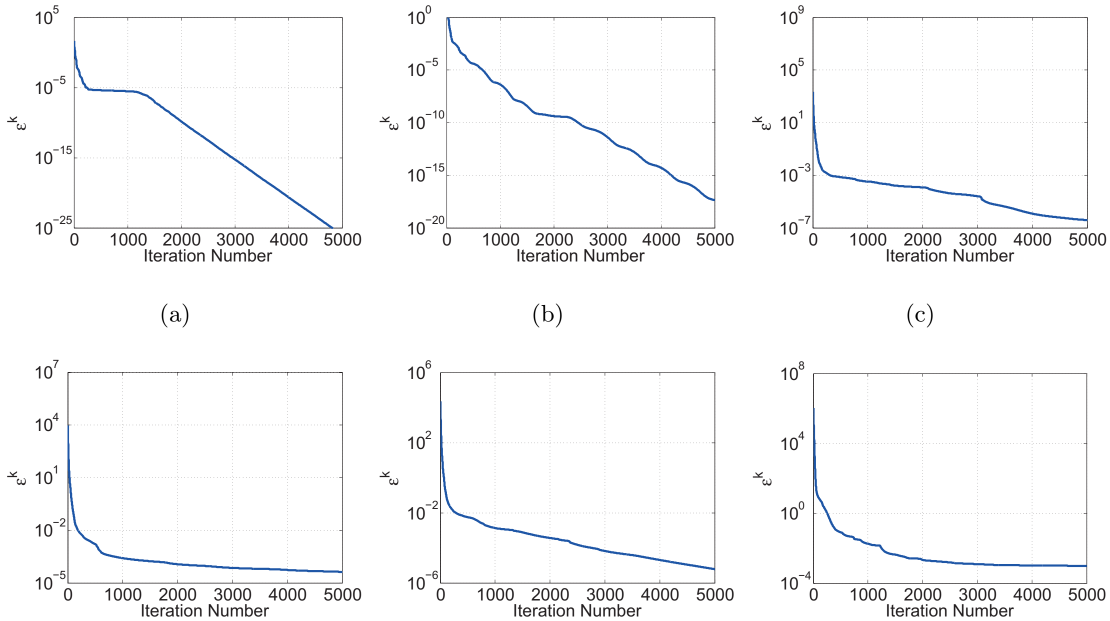
$$\mathbf{\Lambda}_{C;r}^{k+1} := \mathbf{\Lambda}_{C;r}^{k} + \mu(\mathbf{X}^{k+1} \circ \mathbf{C}_r - \mathbf{X}_{C;r}^{k+1}) \qquad \text{for} \quad r = 1, 2, \ldots, q \tag{7.19g}$$

$$\mathbf{\Lambda}_{N;s}^{k+1} := \mathbf{\Lambda}_{N;s}^{k} + \mu(\mathbf{X}^{k+1} \circ \mathbf{N}_s - \mathbf{X}_{N;s}^{k+1}) \qquad \text{for} \quad s = 0, 1, \ldots, p \tag{7.19h}$$

$$\lambda_{z;s}^{k+1} := \lambda_{z;s}^{k} + \mu(z_s^{k+1} - \langle \mathbf{M}_s, \mathbf{X}_{N;s}^{k+1} \rangle) \qquad \text{for} \quad s = 0, 1, \ldots, p \tag{7.19i}$$

7.5 Simulation Results

In this section, we evaluate the performance of the proposed algorithm for solving the SDP relaxation of OPF over IEEE test cases. All simulations are run in MATLAB using a laptop with an Intel Core i7 quad-core 2.5 GHz CPU and 12 GB RAM. As shown in Figure 7.1, the energy function ε^k (as defined in (7.7)) is monotonically decreasing for all simulated cases. In addition, the utmost accuracy of 10^{-25} is ultimately achievable for all these systems. The time per 1000 iteration is between 6.18 and 100 seconds in a MATLAB implementation, which can be reduced significantly in C++ and parallel computing. We have verified that these numbers diminish by



(a) (b) (c)

(d) (e) (f)

Figure 7.1: These plots show the convergence behavior of the energy function ε^k for IEEE test cases. (a): Chow's 9 bus, (b): IEEE 14 bus, (c): IEEE 30 bus, (d): IEEE 57 bus, (e): IEEE 118 bus, (f): IEEE 300 bus.

at least a factor of 3 if certain small-sized bags are combined to obtain a modest number of bags. This shows a trade-off between the chosen granularity for the algorithm and its computation time for a serial implementation (as opposed to a parallel implementation). To elaborate on the algorithm, note that every iteration amounts to a basic matrix operation or an eigendecomposition over matrices of size at most 7×7 for the IEEE 300-bus system. Efficient preconditioning methods could dramatically reduce the number of iterations (as OPF is often very ill-conditioned due to high inductance-to-resistance ratios), and this is left for future work.

7.6 Summary

Motivated by the application of sparse semidefinite programming (SDP) to power networks, the objective of this work is to design a fast and parallelizable algorithm for solving sparse SDPs. To this end, the underling sparsity structure of a given SDP problem is captured using a tree decomposition technique, leading to a decomposed SDP problem. A highly distributed/parallelizable numerical algorithm is developed for solving the decomposed SDP, based on the alternating direction method of multipliers (ADMM). Each iteration of the designed algorithm has a closed-form solution, which involves multiplications and eigenvalue decompositions over certain submatrices induced by the tree decomposition of the sparsity graph. The proposed algorithm is applied to the classical optimal power flow problem, and also evaluated on IEEE benchmark systems.

Chapter 8

Conclusions

This dissertation aims to study real-world nonlinear optimization problems through semidefinite programming (SDP) relaxations combined with graph-theoretic algorithms. First, a method is proposed to study how the underlying structure of an optimization problem reduces the computational complexity of the problem. For this purpose, the structure of the optimization problem is mapped into a graph and it is shown that its SDP relaxation has a solution whose rank can be characterized in terms of the sparsity level of the problem. Two engineering applications of these results in power systems and distributed control are discussed in details. Moreover, a numerical method is developed to answer the need for solving large scale semidefinite programs resulting from the proposed methods.

8.1 Part I: Rank and Sparsity

Part I of this dissertation intends to develop a mathematical foundation for studying nonconvex quadratic optimization problems through a graph theoretic scheme. We cast the problems of interest as finding low-rank solutions of sparse linear matrix inequalities (LMI) and use graph theoretic notions, such as tree decomposition, minimum semidefinite rank, OS-vertex and positive semidefinite zero forcing, for designing convex programs with upper bounds on the rank of every solution. A convex program is also proposed for real-valued problems, which does not rely on any computationally-expensive graph analysis and is always polynomial-time solvable. The implications of this work are also discussed for three applications: minimum-rank matrix completion, conic relaxation for polynomial optimization, and affine rank minimization.

8.2 Part II: Power Networks

The operation planning of large scale power networks is considered in Part II of this dissertation. The problem of optimal power flow (OPF) is studied in Chapter 3 and a theoretical guarantee is developed for the exactness of SDP relaxation of this problem for mesh networks with certain properties. It has been reported in the literature that the SDP relaxation technique can fail even for very basic 3 bus networks. Motivated by this fact, we have shown in Chapter 3 that the performance of SDP relaxation for the OPF problem is highly formulation dependent, and only one of them results in an exact relaxation among equivalent capacity constraints for the original problem. For cases where SDP relaxation fails, an upper bound on the rank of the solution is offered in this work, and then a penalization heuristic is proposed from which a near-global solution of OPF may be recovered. The performance of this method is tested on IEEE systems with over 7000 different cost functions

In chapter 4, the problem of security-constrained optimal power flow (SCOPF) is studied for large scale systems. First, we prove that the SDP relaxation has a solution whose rank is at most equal to the treewidth of the power network plus one, which is expected to be very small for real-world systems. We offer a network decomposition scheme in order to i) reduce the computational cost of solving SDP for large scale systems and ii) to identify lines in the network that result in inexactness of SDP relaxation. We show that for the cases that SDP relaxation fails, a near globally optimal solution may be obtained by penalizing the loss over certain lines of the network. We test our relaxation method on several benchmark examples and demonstrate its ability in finding feasible solutions of SCOPF that are at least 99% globally optimal.

The classical power flow (PF) problem is studied in Chapter 5. We design a family of convex optimization problems, each in the form of a semidefinite program with a linear objective function that captures the rank-one constraint as a proxy. The proposed convex optimization problems are guaranteed to solve the PF problem if the voltage angles are small. The region of complex voltages that can be recovered through each problem is characterized by a nonlinear matrix inequality. The problem of finding a convenient objective function for SDP that can recover a given set of voltage vectors and a neighborhood around each vector can itself be cast as a convex problem. The simulation results show the superiority of the proposed method over the traditional Newton's method.

8.3 Part III: Distributed Control

The optimal distributed control (ODC) problem for discrete-time systems is studied in Part III of this dissertation. The objective is to design a fixed-order distributed controller with a predetermined structure to minimize a quadratic cost functional. Multiple variations of this problem including finite- and infinite-horizon ODC for both deterministic and stochastic systems are studied in Chapter 6. The problem is studied by means of SDP relaxation and the notion of treewidth is exploited to study the rank of the minimum-rank solution of the relaxed problem. A time domain formulation is considered for finite-horizon ODC problem while the infinite-horizon ODC is studied through a Lyapunov domain formulation. Although the problem of designing linear static controller is only considered in this work, we show that the results are readily applicable to the problem of designing dynamic controllers. Multiple heuristic methods are proposed to improve the performance of SDP solution, and the developed results are tested on several random and mass-spring systems.

8.4 Part IV: Parallel Computing

Throughout this dissertation, multiple real-world optimization problems are studied by means of semidefinite programming. Due to the high dimension of those problems, a fast and highly parallelizable numerical algorithm is proposed in Part IV for solving sparse SDP problems. The proposed algorithm is based on two block alternating direction method of multipliers (ADMM) and can be applied on a decomposed formulation of the SDP problem induced from a given tree decomposition. Each iteration involves scalar multiplication and eigenvalue decomposition and can be performed in parallel through multiple agents. As demonstrated in the simulations section, simple iterations of the proposed method enable us to proceed with thousands of iterations in a few seconds, which makes the proposed algorithm a suitable candidate for solving large-scale SDP problems.

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