Two Essays in Financial Engineering

Linan Yang

Submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2015

ABSTRACT

Two Essays in Financial Engineering

Linan Yang

This dissertation consists of two essays in financial engineering: one on credit valuation adjustment and the other on stock trading with realization utility.

In the first part of this dissertation, we investigate the potential impact of wrong-way risk on calculating credit valuation adjustment (CVA) to a derivatives portfolio. A credit valuation adjustment (CVA) is an adjustment applied to the value of a derivative contract or a portfolio of derivatives to account for counterparty credit risk. Counterparty credit risk measurement integrates two sources of risk: *market risk*, which determines the size of a firm's exposure to a counterparty, and *credit risk*, which reflects the likelihood that the counterparty will default on its obligations. Measuring CVA requires combining models of market and credit risk to estimate the counterparty's risk of default together with the market value of the firm's exposure to the counterparty at default. Wrong-way risk refers to the possibility that a counterparty's likelihood of default increases with the market value of the exposure.

We develop a method for bounding wrong-way risk, holding fixed marginal models for market and credit risk and varying the dependence between them. Given simulated paths of the two models, we solve a linear program to find the worst-case CVA resulting from wrong-way risk. We analyze properties of the solution and prove convergence of the estimated bound as the number of paths increases. The worst case can be overly pessimistic, so we extend the procedure by constraining the deviation of the joint model of market and

credit risk from a reference model. Measuring the deviation through relative entropy leads to a tractable convex optimization problem that can be solved through the iterative proportional fitting procedure. By varying the penalty for deviations, we can sweep out the full range of possible CVA values for different degrees of wrong-way risk. Here, too, we prove convergence of the resulting estimate of the penalized worst-case CVA and the joint distribution that attains it. We consider extensions with additional constraints and illustrate the method with examples. Our method addresses an important source of model risk in counterparty risk measurement.

In the second part, we study investors' trading behavior in a model of realization utility. We assume that investors' trading decisions are driven not only by the utility of consumption and terminal wealth, but also by the utility burst from realizing a gain or a loss. More precisely, we consider a dynamic trading problem in which an investor decides when to purchase and sell a stock to maximize her wealth utility and realization utility with her reference points adapting to the stock's gain and loss asymmetrically.

We study, both theoretically and numerically, the optimal trading strategies and asset pricing implications of two types of agents: *adaptive agents*, who realize prospectively the reference point adaptation in the future, and *naive agents*, who fail to do so. We find that an adaptive agent sells the stock more frequently when the stock is at a gain than a naive agent does, and that the adaptive agent asks for a higher risk premium for the stock than the naive agent does in equilibrium. Moreover, compared to a non-adaptive agent whose reference point does not change with the stock's gain and loss, both the adaptive and naive agents sell the stock less frequently, and the naive agent requires the same risk premium as the non-adaptive agent does.

Contents

List of Figures

List of Tables

Acknowledgements

First and foremost, I must offer my profoundest gratitude to Professor Paul Glasserman and Professor Xuedong He, my academic advisors. It is my honor to work with them and it has been a great experience. Without their patient guidance, continuous support and kind encouragement, I would not have achieved this stage and finished my dissertation.

Special thanks to my committee, Professor Karl Sigma, Professor Tim Leung and Professor Jacob Leshno, for their support, guidance and helpful suggestions. Their effort has helped me to improve the quality of my thesis and I owe them my heartfelt appreciation.

I also want thank Professor David Yao, Ward Whitt, Donald Goldfarb, Cliff Stein, Garud Iyengar and others from Columbia IEOR Department, Professor Assaf Zeevi, Neng Wang, Charles Jones and others from Columbia Business School. Their courses on stochastic processes, optimization, econometrics and asset pricing equipped me with knowledge and tools to facilitate my research. Especially, I want to express my sincere gratitude to professor David Yao, Ward Whitt and Mariana Olvera-Cravioto for encouraging me to pursue doctorate in Operations Research.

It is a wonderful experience when I study at Columbia IEOR Department and share the exciting time with other doctoral students and friends, including Anran Li, Jing Guo, Juan Li, Mu Cun, Jing Dong, Xinyun Chen, Ni Ma, Yanan Pei and others. Finally, I would like to thank my parents for their consistent support and confidence in me.

To my mother and father

Chapter 1

Overview

This dissertation consists of two parts in which we utilize various methods in applied probability, optimization, and stochastic control to solve two problems in the areas of counterparty credit risk and behavioral finance: credit valuation adjustment (CVA) with wrong-way risk and stock trading with realization utility.

In Chapter 2, we focus on credit valuation adjustment and wrong-way risk. Research on counterparty credit risk dated back to nineties, but it takes on heightened importance since the financial crisis in 2008. Counterparty credit risk is taken on by an entity entering an over-the-counter (OTC) contract with a counterparty which has a relevant default probability; if the counterparty were to default, the expected value of all future net payments to this entity turn into a loss for this entity. Counterparty credit risk is associated with all OTC transactions with a defaultable counterparty, and all the agents taking OTC transactions need to take care of counterparty credit risk regardless of the type of transactions.

Credit valuation adjustment (CVA) has become an important tool for managing counterparty credit risk. CVA measures the value of counterparty credit risk, and is contracted as a credit derivative that can be traded and hedged, although most banks take it as a reserve and do not manage it actively. With CVA, an agent doing OTC transactions can focus on their major market risk and transfer counterparty credit risk to another agent, who is specialized in trading CVA, by paying a CVA premium. The OTC transactions can be deemed as counterparty risk free as the second agent would meet the payment obligation if the counterparty were to default.

Although conceptually simple, calculating CVA of a typical portfolio is not an easy task. It is similar as pricing a complex illiquid instrument, and needs to take into account three factors. The first factor is the value of the underlying portfolio, which may consist of multiple transactions across different markets with various counterparties. Portfolio valuation itself can be complicated because multiple market factors and pricing models may be involved. The second factor is the collateral rule, netting agreement, and recovery at default. The first two factors together determine the exposure of the portfolio holder to its counterparties, which may turn into a loss at counterparty's default. The third factor is the credit quality of the counterparty.

Furthermore, co-dependency generally exists between the portfolio exposure and counterparty's credit quality, which brings in further complexity in calculating CVA. The possibility that market risk and counterparty credit risk move together, so that the market exposure increases just as the counterparty's risk of default increases, is referred to as *wrong-way risk*. Wrong-way risk arises, for example, if one bank sells credit default swap protection on another bank with a similar profile. The value of the credit protection increases when the second bank faces financial difficulties; this is likely to be a scenario in which the bank that sold the protection is also at greater risk of default. In practice, the sources and nature of wrong-way risk are often less obvious.

In this dissertation we introduce a method for bounding wrong-way risk in CVA calculation — that is, for finding the largest CVA that is consistent with fixed models for market risk of the portfolio and credit risk of the counterparty, letting the dependence between market and credit risk vary. Our approach builds on a standard simulation framework for CVA calculation: paths of underlying market risk factors are simulated over time; a portfolio is revalued (often using approximations) at fixed dates along each path of the market risk factors; the counterparty's time to default is either simulated from a credit risk model or extracted from a credit curve. Given a set of paths of portfolio exposures and the distribution of the time to default, we find the worst-case CVA by solving a linear programming problem. The linear program finds the assignment of default times to paths that results in the largest possible CVA, given the constraints on the default time distribution and the set of paths simulated from the market model. As a byproduct, the dual variables associated with the constraints on the marginal default time distribution provide sensitivities of the worst-case CVA to the default probabilities.

A strength of this approach is that it yields the largest possible CVA value consistent with given models for market and credit risk. Because it finds the worst-case wrong-way risk, this approach can also be too conservative. We then extend the method by constraining the deviation of the joint model of market and credit risk from a reference model, or equivalently by penalizing deviations from a reference model and finding the resulting tempered CVA. A natural choice for the reference model is to take market and credit risk to be independent of each other.[1] By varying the penalty for deviations, we can sweep out the full range of potential CVA values from the independent case to the worst-case wrong-way risk. The penalized problem can no longer be solved through linear programming, but we formulate it as a tractable convex optimization problem. The special structure of this problem leads to a convenient solution through iterative rescaling of the rows and columns of a matrix. Dual variable are obtained through the iterative rescaling process, providing sensitivity result for the tempered CVA to the default probability.

The result obtained from above optimization method is an estimator of CVA based on simulation outcomes of the market factors and empirical default distribution of the counterparty. We then establish the consistency of this estimator for both worst-case and tempered CVA. In other words, we show that as the sample size of market paths increases, the estimated CVA converges to its true value. At last, we extend the model to include the martingale property of the market risk factors by adding additional equality constraints. We also show that this model can be easily extended to bilateral CVA in which the agent's self-default is also considered in CVA calculation.

To summarize, our model provides a very general framework for CVA calculation to account for the impact of wrong-way risk. It applies to both single transactions and portfolios. In addition, it is computationally efficient, because it reuses simulated exposure paths that need to be generated anyway to estimate CVA even ignoring wrong-way risk.

In Chapter 3, we study investors' trading behavior in a model of realization utility. In

[1] The Basel III standardized approach for CVA assumes independence and then multiplies the resulting CVA by a factor of 1.4.

neoclassical finance, investors are assumed to maximize the utility of their consumption and wealth. Economic models in this regard, however, cannot explain many empirical findings; see for instance Barberis and Thaler [5] and Campbell [20]. For example, Shefrin and Statman [51] find that individual investors are reluctant to sell stocks trading at a loss relative to the price at which they were purchased, a phenomenon called disposition effect, for which models in neoclassical finance fail to provide a satisfactory explanation. Recently, Barberis and Xiong [6] propose a model of realization utility to explain the disposition effect.

Barberis and Xiong [6] assume that investors experience utility directly from realizing a gain or a loss on the sale of the risky assets (e.g., stocks) they hold. As defined by Barberis and Xiong [6], realization utility is a consequence of two cognitive processes: First, instead of viewing their investment history in terms of the investment return, investors often think about it as a series of investment episodes. Second, an investor feels good, i.e., receives positive realization utility, when she sells a stock at a gain because she is creating a positive investment episode; on the other hand, she feels bad, i.e., experiences negative realization utility, when she sells the stock at a loss because she is creating a negative investment episode. Barberis and Xiong [6] employs cumulative prospect theory (Tversky and Kahneman [40, 53]) with a piece-wise linear utility function to measure the realization utility of a gain or a loss. Ingersoll and Jin [38] extends Barberis and Xiong [6] by using an S-shaped utility function.

In this dissertation, we extend the works by Barberis and Xiong [6] and Ingersoll and Jin [38] in two aspects. First, in addition to realization utility, the agent in our model also experiences utility from her terminal wealth. As in neoclassical finance, we use the classical expected utility theory to model the terminal wealth utility. Second, the reference point of the agent, which decides whether the agent is experiencing a gain or a loss at the current stock price, is assumed to be the purchase price at the time of purchase and then adapt to the stock's gain and loss, and the adaptation to the gain is more than to the loss. Instead, Barberis and Xiong [6] assume the reference point to be the purchase price growing at the risk-free rate, and Ingersoll and Jin [38] assume the reference point to be fixed at the purchase price. Experimental evidence in Baucells et al. [9] and Arkes et al. [1, 2] reveals that when selling a stock, most investors choose their reference points to be the purchase price plus a portion of the prior paper gain and loss of the stock. Moreover, the reference point adapts more to a prior gain than to a comparable prior loss. Therefore, our model is more consistent with individuals' behavior observed in the literature.

More precisely, we consider a dynamic trading problem in which an agent decides when to purchase and sell a stock to maximize her realization utility with her reference point adapting to the stock's gain and loss asymmetrically. We formulate the trading problem as an optimal stopping problem and solve it completely. We study, both theoretically and numerically, the optimal trading strategies and asset pricing implications of two types of agents: adaptive agents who realize prospectively the reference point adaptation in the future, and naive agents who fail to do so. We have three main findings: First, when becoming more concerned with the terminal wealth utility, both the naive and adaptive agents sell the stock less frequently and ask for a higher risk premium for the stock in equilibrium. Second, an adaptive agent sells the stock more (less) frequently when the stock is at a gain (at a loss) than a naive agent does, and the adaptive agent asks for a higher risk premium for the stock than the naive agent does in equilibrium. Third, compared to a non-adaptive agent whose reference point does not adapt to the stock's gain and loss, both the adaptive and naive agents sell the stock less frequently, and the naive agent requires the same risk premium as the non-adaptive agent does.

Chapter 2 and Chapter 3 are self-contained and independent of each other.

Chapter 2

Credit Valuation Adjustment and Wrong-Way Risk

2.1 Introduction

When a firm enters into a swap contract, it is exposed to market risk through changes in market prices and rates that affect the contract's cash flows. It is also exposed to the risk that the party on the other side of the contract may default and fail to make payments due on the transaction. Thus, market risk determines the magnitude of one party's exposure to another, and credit risk determines the likelihood that this exposure will become a loss. Derivatives counterparty risk refers to this combination of market and credit risk, and proper measurement of counterparty risk requires integrating market uncertainty and credit uncertainty.

The standard tool for quantifying counterparty risk is the credit valuation adjustment, CVA, which can be thought of as the price of counterparty risk. Suppose firm A has entered into a set of derivative contracts with firm B. From the perspective of firm A, the CVA for this portfolio of derivatives is the difference between the value the portfolio would have if firm B were default-free and the actual value taking into account the credit quality of firm B. More precisely, this is a unilateral CVA; a bilateral CVA adjusts for the credit quality of both firms A and B.

Counterparty risk generally and CVA in particular have taken on heightened importance since the failures of major derivatives dealers Bear Stearns, Lehman Brothers, and AIG Financial Products in 2008. A new CVA-based capital charge for counterparty risk is among the largest changes to capital requirements under Basel III for banks with significant derivatives activity (BCBS [8]). CVA calculations are significant consumers of bank computing resources, typically requiring simulation of all relevant market variables (prices, interest rates, exchanges rates), valuing every derivative at every time step on every path, and integrating these market exposures with a model of credit risk for each counterparty. See Canabarro and Duffie [21] and Gregory [34] for background on industry practice.

Our focus in this chapter is on the effect of dependence between market and credit risk. *Wrong-way risk* refers to the possibility that a counterparty will become more likely to default when the market exposure is larger and the impact of the default is greater; in other words, it refers to positive dependence between market and credit risk. Wrong-way risk arises, for example, if one bank sells put options on the stock of another similar bank. The value of the options increases as the price of the other bank's stock falls; this is likely to be a scenario in which the bank that sold the options is also facing financial difficulty and is less likely to be able to make payment on the options. In practice, the sources and nature of wrong-way risk may be less obvious.

Holding fixed the marginal features of market risk and credit risk, greater positive dependence yields a larger CVA. But capturing dependence between market and credit risk is difficult. There is often ample data available for the separate calibration of market and credit models but little if any data for joint calibration. CVA is calculated under a risk-adjusted probability measure, so historical data is not directly applicable. In addition, for their CVA calculations banks often draw on many valuation models developed for trading and hedging specific types of instruments that cannot be easily integrated with a model of counterparty credit risk. CVA computation is much easier if dependence is ignored. Indeed, the Basel III standardized approach for CVA assumes independence and then multiplies the result by a factor of 1.4; this ad hoc factor is intended to correct for several sources of error, including the lack of dependence information.

Models that explicitly describe dependence between market and credit risk include in CVA calculation include Brigo, Capponi, and Pallavicini [17], Crépey [25], Hull and White [37], and Rosen and Saunders [46]; see Brigo, Morini, and Pallavicini [18] for an extensive overview of modeling approaches. Dependence is usually introduced by correlating default intensities with market risk factors or through a copula. A direct model of dependence is, in principle, the best approach to CVA. However, correlation-based models generally produce weak dependence between market and credit risk, and both techniques are difficult to calibrate.

In this chapter, we develop a method to bound the effect of dependence, holding fixed marginal models of market and credit risk. Our approach uses simulated paths that would be needed anyway for a CVA calculation without dependence. Given paths of market exposures and information (simulated or implied from prices) about the distribution of time to the counterparty's default, we show that finding the worst-case CVA is a linear programming problem. The linear program is easy to solve, and it provides a bound on the potential impact of wrong-way risk. We view this in-sample bound based on a finite set of paths as an estimate of the worst-case CVA for a limiting problem and prove convergence of the estimator. The limiting problem is an optimization over probability measures with given marginals. We also show that the LP formulation has additional useful features. It extends naturally to a bilateral CVA calculation, and it allows additional constraints. Moreover, the dual variables associated with constraints on the marginal default time distribution provide useful information for hedging purposes.

The strength of the LP solution is that it yields the largest possible CVA value — the worst possible wrong-way risk — consistent with marginal information about market and credit risk. This is also a shortcoming, as the worst case can be too pessimistic. We therefore extend the method by constraining or penalizing deviations from a nominal reference model. The reference model could be one in which marginals are independent or linked through some simple model of dependence. A large penalty produces a CVA value close to that obtained under the reference model, and with no penalty we recover the LP solution. Varying the penalty parameter allows us to "interpolate" between the reference model and the worst-case joint distribution.

To penalize deviations from the reference model, we use a relative entropy measure between probability distributions, also known as the Kullback-Leibler divergence. Once we add the penalty, finding the worst-case joint distribution is no longer a linear programming problem, but it is still convex. Moreover, the problem has a special structure that allows convenient solution through iterative rescaling of the rows and columns of a matrix. This iterative rescaling projects a starting matrix onto the convex set of joint distributions with given marginals. Here, too, we prove convergence of the in-sample solution to the solution of a limiting problem as the number of paths increases.

The problem of finding extremal joint distributions with given marginals has a long and rich history. It includes the well-known Fréchet bounds in the scalar case and the multivariate generalization of Brenier [16] and Rüschendorf and Rachev [49]; see the books by Rüschendorf [48] and Villani [54] for detailed treatments and historical remarks. In finance, related ideas have been used to find robust or model-free bounds on option prices; see Cox [23] for a survey. In some versions of the robust pricing problem, one observes prices of simple European options and seeks to bound prices of path-dependent or multi-asset options given the European prices, as in Carr, Ellis, and Gupta [22], Brown, Hobson, and Rogers [19], and Tankov [52], among many others. This has motivated the study of martingale optimal transport problems in Dolinsky and Soner [29], Beiglböck and Juillet [10], Henry-Labordère and Touzi [36]. The literature on price bounds focuses on extremal solutions and does constrain or penalize deviations from a reference model.

Our focus is not on pricing but rather risk measurement. Within the risk measurement literature, questions of joint distributions with given marginals arise in risk aggregation; see, for example, Bernard, Jiang, and Wang [12], Embrechts and Puccetti [31], and Embrechts, Wang, and Wang [32]. A central problem in risk aggregations is finding the worst-case distribution for a sum of random variables, given marginals for the summands.

Our work differs from earlier work in several respects. We focus on CVA, rather than option pricing or risk aggregation. Our marginals may be quite complex and need not be explicitly available — they are implicitly defined through marginal models for market and credit risk. Given the generality of the setting, we do not seek to characterize extremal joint distributions but rather to estimate bounds using samples generated from the marginals. We temper the bounds by constraining deviations from a reference model, drawing on the idea of robustness as developed in economics in Hansen and Sargent [35] and distributional robustness as developed in the optimization literature in Ben-Tal et al. [11] and references there. The methods we develop are easy to implement in practice. The main contribution lies in the formulation and in the convergence analysis. Our general approach to convergence is to use primal and dual optimization problems to get upper and lower bounds.

The rest of the chapter is organized as follows. In Chapter 2.2, we introduce the problem setting, and in Chapter 2.3 we introduce the optimization formulation for the worst case CVA bound and show convergence of the bound estimator. In Chapter 2.4, we extend the problem to a robust formulation with a relative entropy constraint, and we provide numerical examples in Chapter 2.5. In Chapter 2.6, we extend the model further to incorporate expectation constraints.

2.2 Problem Formulation

To help fix ideas, we start with an example. Consider a T-year foreign exchange forward contract between a U.S. bank, which receives U.S. dollar payments, and a foreign bank which receives its local currency. The contract has forward exchange rate K and notional size S. If the foreign currency weakens against the dollar, the foreign bank's credit quality is likely to deteriorate with its currency, just as the U.S. bank's exposure increases, so this transaction exhibits evident wrong-way risk.

Let U_t be the exchange rate, measured as the number of units of the foreign currency paid in exchange for one U.S. dollar at time t. Assume this exchange rate follows an Ornstein-Uhlenbeck process,

$$dU_t = \kappa(\bar{U} - U_t)dt + \sigma dW_t,$$

where $\bar{U}$ is the level toward which the exchange rate mean-reverts, and W_t is a standard Brownian motion.

CVA measures the discounted expected loss of a portfolio at the counterparty's default, so its calculation involves the default time of the counterparty and the discounted exposure value of the portfolio of derivatives with this counterparty at the time of its default. Let τ denote counterparty's default time, and let $V(\tau)$ denote the discounted portfolio exposure at the time of counterparty's default. We assume that $V(\tau)$ accounts for all netting and collateral agreements, as well as recovery. The portfolio exposure could be positive or negative, but only the positive part leads to a loss at default, so the loss at default is denoted as $V^+(\tau)$.

The CVA for a time horizon T is the expected exposure at default,

$$\text{CVA} = \mathbb{E}[V^+(\tau)\mathbf{1}\{\tau \leq T\}], \tag{2.2.1}$$

given a joint law for the default time τ and the exposure V^+. Our focus will be on uncertainty around this joint law, but we first provide some additional details on the problem formulation.

CVA is customarily calculated over a finite set of dates $0 = t_0 < t_1 < \cdots < t_d = T < t_{d+1} = \infty$; for example, the dates can be monthly or quarterly, or the payment dates of the underlying contracts. We limit τ to values in $\{t_1, \ldots, t_d, t_{d+1}\}$ and let q_j, $j = 1, \ldots, d+1$, denote the probability that default occurs at t_k, or, more precisely that it occurs in the interval $(t_{k-1}, t_k]$. The distribution of the counterparty's default time τ may be extracted from credit default swap spreads, or it may be the result of a more extensive credit risk model — for example, a stochastic intensity model.

An underlying simulation of market risk factors generates paths of all relevant market variables and is used to generate exposure paths $(V^+(t_1), \ldots, V^+(t_K))$. In our foreign exchange example,

$$V(t_j) = e^{-\delta t_j}(1 - R) \cdot \mathbb{E}[e^{-\delta(T - t_j)} S(U_T - K)/U_T \,|\, U_{t_j}]$$

where δ is the discount rate and R is the recovery rate. The expectation gives the expected exposure of the contract at time t_j, and this value is discounted to t_0 and adjusted for partial recovery. The market risk model (in this example the exchange rate dynamics) implicitly determines the law of the positive exposure path $(V^+(t_1), \ldots, V^+(t_d))$, and we denote this law by a probability measure p on $\mathbb{R}^d$.

Calculating these exposures is a demanding task because it requires valuing all instruments in a portfolio with a counterparty in each market scenario at each date. In addition, the calculation of each $V(t_j)$ needs to account for netting and collateral agreements with the counterparty and recovery rates if the counterparty were to default. The method we develop takes these calculations as inputs and assumes the availability of independent copies of the exposure paths.

Let X denote this vector of positive exposures at the specified dates, and let Y be a vector of default indicators,

$$X = (V^+(t_1), \ldots, V^+(t_d)) \text{ and } Y = (\mathbf{1}\{\tau = t_1\}, \ldots, \mathbf{1}\{\tau = t_d\}).$$

The problem of calculating CVA would reduce to the problem of calculating the expectation of the inner product

$$< X, Y > = \sum_{j=1}^{d} V^+(t_j)\mathbf{1}\{\tau = t_j\} = V^+(\tau)\mathbf{1}\{\tau \leq T\},$$

if the joint law for X and Y were known.

However the joint law is in general unavailable and difficult to find because of limited data on the dependence between market and credit risk. With the marginals fixed, we need to assign a joint probability between X and Y to calculate CVA. As an upper bound, we seek to evaluate the *worst-case* CVA, defined by

$$\mathrm{CVA}_* := \sup_{\mu \in \Pi(p,q)} \int_{\mathbb{R}^d \times \mathbb{R}^d} < x, y > d\mu(x,y), \tag{2.2.2}$$

where $\Pi(p,q)$ denotes the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals p and q.

The characterization of extremal joint distributions with given marginals has a rich history; see Villani [54] and Rüschendorf [48] for recent treatments with extensive historical remarks. In the scalar case $d = 1$, the largest value of (2.2.2) is attained by the comonotonic construction, which sets $X = F_p^{-1}(U)$ and $Y = F_q^{-1}(U)$, where F_p and F_q are the cumulative distribution functions associated with p and q, and U is uniformly distributed on $[0,1]$. The smallest value of (2.2.2) is attained by setting $Y = F_q^{-1}(1-U)$ instead. In the vector case, a characterization of joint laws maximizing (2.2.2) has been given by Brenier [16] and Rüschendorf and Rachev [49]. It states that under an optimal coupling, Y is a subgradient of a convex function of X, but this provides more of a theoretical description than a practical characterization. Our setting has the added complication that at least p (and possibly also q) is itself unknown and only implicitly specified through a simulation model.

2.3 Worst-Case CVA

2.3.1 Estimation

We develop a simulation procedure to estimate (2.2.2). As we noted earlier, generating exposure paths is the most demanding part of a CVA calculation. Our approach essentially reuses these paths to bound the potential effect of wrong-way risk at little additional computational cost.

Let $X_1, \ldots, X_N$ be N independent copies of X, and let $Y_1, \ldots, Y_N$ be N independent copies of Y. Denote their empirical measures on $\mathbb{R}^d$ by

$$p_N(\cdot) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\{X_i \in \cdot\}, \quad q_N(\cdot) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\{Y_i \in \cdot\}, \tag{2.3.1}$$

For notational simplicity, we will assume that p has no atoms so that, almost surely, there are no repeated values in $X_1, X_2, \ldots$. This allows us to identify the empirical measure p_N on $\mathbb{R}^d$ with the uniform distribution on the set $\{X_1, \ldots, X_N\}$ or on the set of indices $\{1, \ldots, N\}$. The assumption that p has no atoms is without loss of generality because we can expand the dimension of X to include an independent, continuously distributed coordinate X_{d+1} and expand Y by setting $Y_{d+1} \equiv 0$ without changing (2.2.2).

Observe that Y is supported on the finite set $\{y_1, \ldots, y_{d+1}\}$, with $y_1 = (1, 0, \ldots, 0), \ldots, y_d = (0, 0, \ldots, 1)$, and $y_{d+1} = (0, \ldots, 0)$. Each y_j has probability $q(y_j)$. These probabilities may be known or estimated from simulation of N independent copies $Y_1, \ldots, Y_N$ of Y, in which case we denote the empirical frequency of each y_j by $q_N(y_j)$.

We will put a joint mass function P^N_{ij} on the set of pairs $\{(X_i, y_j),\ i = 1, \ldots, N,\ j = 1, \ldots, d+1\}$. We restrict attention to the set $\Pi(p_N, q_N)$ of joint mass functions with marginals p_N and q_N. We estimate (2.2.2) using

$$\widehat{\text{CVA}}_* = \max_{P^N \in \Pi(p_N, q_N)} \sum_{i=1}^{N} \sum_{j=1}^{d+1} P^N_{ij} < X_i, y_j > .$$

Finding the worst-case joint distribution is a linear programming problem:

$$\max_{\{P_{ij}\}} \quad \sum_{i=1}^{N}\sum_{j=1}^{d+1} C_{ij}P_{ij}, \tag{2.3.2}$$

$$\text{subject to} \quad \sum_{j=1}^{d+1} P_{ij} = 1/N,\ i = 1, ..., N, \tag{2.3.3}$$

$$\sum_{i=1}^{N} P_{ij} = q_N(y_j),\ j = 1, ..., d+1 \quad \text{and} \tag{2.3.4}$$

$$P_{ij} \geq 0,\ i = 1, ..., N,\ j = 1, ..., d+1, \tag{2.3.5}$$

with $C_{ij} =< X_i, y_j >$. Constraint (2.3.3) ensures that the paths $X_1, \ldots, X_N$ of market factors get equal weight; constraint (2.3.4) ensures that the default-time distribution in the joint model has the correct marginal distribution. In our running example, we have

$$C_{ij} = (V^i(t_j))^+ = e^{-\delta t_j}(1-R) \cdot \mathbb{E}^+[e^{-\delta(T-t_j)}S(U_T - K)/U_T \,|\, U^i_{t_j}].$$

In particular, this has the structure of a transportation problem, for which efficient algorithms are available, for example a strongly polynomial algorithm; see Kleinschmidt and Schannath [41]. Bilateral CVA, involving the joint distribution of market exposure and the default times of both parties, admits a similar formulation.

To better understand the optimal solution joint probability, we can make the following assumptions: without loss of generality, N is large enough and default time probabilities q_j, for all j, are properly rounded to be multiple of $1/N$. If we let i be row index and j be column index, there is an optimal solution matrix P^* to the above linear program, which, with proper order of rows, is a block diagonal matrix with nonzero values on the diagonal and zeros off the diagonal. This means that default probability is very concentrated for each market scenario, and each exposure path gets assigned only one possible default time.

2.3.2 Sensitivity

To formulate the dual problem, let a_i and b_j be dual variables associated with constraints (2.3.3) and (2.3.4), respectively. The dual problem is then

$$\begin{aligned} \min_{a \in \mathbb{R}^N, b \in \mathbb{R}^{d+1}} \quad & \sum_{i=1}^{N} a_i/N + \sum_{j=1}^{d+1} b_j q_N(y_j) \\ \text{subject to} \quad & a_i + b_j \geq C_{ij}, i = 1, ..., N, \ j = 1, ..., d. \end{aligned}$$

The dual variables are useful because they measure the sensitivity of the estimated worst-case CVA to the marginal constraints. Consider any vector of perturbations $(\Delta q_1, \ldots, \Delta q_{d+1})$ to the mass function q_N with components that sum to zero. Suppose these perturbations are sufficiently small to leave the dual solution unchanged. Then

$$\Delta \widehat{\text{CVA}}_* = \sum_{j=1}^{d+1} b_j \Delta q_j.$$

In particular, we can calculate the sensitivity of the worst-case CVA to a parallel shift in the credit curve by setting $\Delta q_j = \Delta$, $j = 1, \ldots, d$, and $\Delta q_{d+1} = -d\Delta$, for sufficiently small Δ.

2.3.3 Convergence as $N \to \infty$

The solution to the linear program provides an estimate $\widehat{\text{CVA}}_*$ based on N simulated paths. But we are ultimately interested in CVA_* in (2.2.2), the worst-case CVA based on the true marginal laws for market and credit risk, rather than their sample counterparts. We show that our estimate converges to CVA_* almost surely as N increases.

Although in our application Y has finite support, we state the following result more generally. For probability laws p and q on $\mathbb{R}^d$, let p_N and q_N denote the corresponding empirical laws in (2.3.1). Let $\Pi(p,q)$, $\Pi(p_N,q_N)$, and $\Pi(p_N,q)$ denote the sets of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with the indicated arguments as marginals.

Theorem 2.3.1. *Let X and Y be d-dimensional random vectors with distributions p and q respectively such that $\int_{\mathbb{R}^d} \|x\|^2 dp(x) < \infty$, and $\int_{\mathbb{R}^d} \|y\|^2 dq(y) < \infty$. Then*

$$\begin{aligned}
\lim_{N\to\infty} \sup_{\mu\in\Pi(p_N,q_N)} \int_{\mathbb{R}^d\times\mathbb{R}^d} < x, y > \mu(dx,dy) &= \lim_{N\to\infty} \sup_{\mu\in\Pi(p_N,q)} \int_{\mathbb{R}^d\times\mathbb{R}^d} < x, y > \mu(dx,dy) \\
&= \sup_{\mu\in\Pi(p,q)} \int_{\mathbb{R}^d\times\mathbb{R}^d} < x, y > \mu(dx,dy).
\end{aligned}$$

The proof follows from results on optimal transport in Villani [54]; see Appendix A.1.

2.4 Robust Formulation with a Relative Entropy Constraint

The linear program (2.3.2)–(2.3.5) provides a simple way to bound the impact of wrong-way risk and estimate a worst-case CVA, and Theorem 2.3.1 establishes the consistency of this estimate as the number of paths grows. An attractive feature of this approach is that it reuses simulated exposure paths that need to be generated anyway to estimate CVA even ignoring wrong-way risk.

A drawback of the bound CVA_* is that it may be too pessimistic: the worst-case joint distribution may be implausible, even if it is theoretically feasible. To address this concern, we extend our analysis and formulate the problem of bounding wrong-way risk as a question of robustness to model uncertainty. By controlling the degree of uncertainty we can temper the bound on wrong-way risk.

2.4.1 Constrained and Penalized Problems

In this formulation, we start with a reference model for the dependence between the market and credit models and control model uncertainty by constraining deviations from the reference model. To be concrete, we will assume that the reference model takes market and credit risk to be independent, though this is not essential. We use ν to denote the corresponding element of $\Pi(p,q)$ that makes X and Y independent; in other words,

$$\nu(A \times B) = p(A)q(B),$$

for all measurable $A, B \subseteq \mathbb{R}^d$. Under this reference model, we have the independent case CVA given by

$$\text{CVA}_\nu = \int_{\mathbb{R}^d \times \mathbb{R}^d} < x, y > d\nu(x,y) = \int_{\mathbb{R}^d \times \mathbb{R}^d} < x, y > dp(x)dq(y).$$

To constrain deviations from the reference model, we need a notion of "distance" between probability measures. Among the many candidates, relative entropy, also known as the Kullback-Leibler divergence, is particularly convenient. For probability measures P and F on a common measurable space and with $F >> P$, the relative entropy of P to F is defined as

$$D(P|F) = \mathbb{E}_F \left[\frac{dP}{dF} \ln \left(\frac{dP}{dF} \right) \right] = \int \ln \left(\frac{dP}{dF} \right) dP.$$

the subscripts indicating the measure with respect to which the expectation is taken. Relative entropy is frequently used to quantify model uncertainty; see, for example, Hansen and Sargent [35] and Ben-Tal et al. [11]. Relative entropy is not symmetric in its arguments, but this is not necessarily a drawback because we think of the reference model as a favored benchmark. We are interested in the potential impact of deviations from the reference model, but we do not necessarily view nearby alternative models as equally plausible. Relative entropy $D(P|F)$ is convex in P, and this will be important for our application. Also, $D(P|F) = 0$ only if $P = F$.

To find a tempered worst case for wrong-way risk, we maximize CVA with the marginal models p and q held fixed and with a constraint $\eta > 0$ on the relative entropy divergence from the reference joint model ν:

$$\text{CVA}_\eta := \sup_{\mu \in \Pi(p,q)} \int_{\mathbb{R}^d \times \mathbb{R}^d} < x, y > d\mu(x,y), \quad (2.4.1)$$

$$\text{subject to} \int \ln(\frac{d\mu}{d\nu}) d\mu \leq \eta. \quad (2.4.2)$$

At $\eta = 0$, the only feasible solution is the reference model $\mu = \nu$. At $\eta = \infty$, the problem reduces to the worst-case CVA of the previous section. Varying the relative entropy budget η thus controls the degree of confidence in the reference model or the degree of wrong-way risk.

We are actually interested in solving this problem for a range of η values to see how the potential impact of wrong-way risk varies with the degree of model uncertainty. For this purpose, it will be convenient to work with a penalty on relative entropy rather than a constraint. The penalty formulation with parameter $\theta > 0$ is as follows:

$$\sup_{\mu \in \Pi(p,q)} \int_{\mathbb{R}^d \times \mathbb{R}^d} < x, y > d\mu(x,y) - \frac{1}{\theta} \int \ln(\frac{d\mu}{d\nu}) d\mu. \tag{2.4.3}$$

The penalty term subtracted from the linear objective is nonnegative because relative entropy is nonnegative. At $\theta = 0$, the penalty would be infinite unless $\mu = \nu$; at $\theta = \infty$, the penalty drops out and we recover the worst-case linear program of Section 2.3. A related problem appears in Bosc and Galichon [15], but without a reference model ν. The correspondence between the constrained problem (2.4.1)–(2.4.2) and the penalized problem (2.4.3) is established in the following result, proved in the Appendix A.2:

Proposition 2.4.1. *For $\theta > 0$, the optimal solution μ^θ to (2.4.3) is the optimal solution to (2.4.1)–(2.4.2) with*

$$\eta(\theta) = \int \ln(\frac{d\mu^\theta}{d\nu}) d\mu^\theta. \tag{2.4.4}$$

The mapping from θ to $\eta(\theta)$ is increasing, and $\eta(\theta) \in (0, \eta^*]$ for $\theta \in (0, \infty)$, where η^* is (2.4.4) evaluated at the solution to (2.2.2).

In the following, we write CVA_θ instead of $\text{CVA}_{\eta(\theta)}$ for $\theta \in (0, \infty)$. To estimate CVA_θ, we form a sample counterpart, modifying the linear programming formulation (2.3.2)–(2.3.5). We denote the finite sample reference joint probabilities by F_{ij}. In the independent case, these are given by $F_{ij} = q_N(y_j)/N$, $i = 1, \ldots, N$, $j = 1, \ldots, d+1$. Let P^θ denote the optimal solution to the following optimization problem:

$$\max_{\{P_{ij}\}} \quad \sum_{i=1}^{N}\sum_{j=1}^{d+1} C_{ij}P_{ij} - \frac{1}{\theta}\sum_{i=1}^{N}\sum_{j=1}^{d+1} P_{ij}\ln\left(\frac{P_{ij}}{F_{ij}}\right) \quad \text{subject to (2.3.3)-(2.3.5).} \quad (2.4.5)$$

We estimate CVA_θ by

$$\widehat{\text{CVA}}_\theta := \sum_{i=1}^{N}\sum_{j=1}^{d+1} C_{ij}P_{ij}^\theta.$$

In the penalty formulation (2.4.3), if we let $\theta < 0$ and replace sup by inf, we get CVA with right-way risk, in which case the likelihood of default of the counterparty decreases with the market value of exposure.

2.4.2 Iterative Proportional Fitting Procedure

The penalty problem (2.4.5) is a convex optimization problem and can be solved using general optimization methods. However, the choice of relative entropy for the penalty leads to a particularly simple and interesting method through the iterative proportional fitting procedure (IPFP). The method dates to Deming and Stephan [28], yet it continues to generate extensions and applications in many areas.

To apply the method in our setting, we use as initial guess the $N \times (d+1)$ matrix M^θ with entries

$$M_{ij}^\theta = \frac{e^{\theta \cdot C_{ij}} \cdot F_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{d+1} e^{\theta \cdot C_{ij}} \cdot F_{ij}}.$$

As before, F_{ij} is the independent joint distribution with prescribed marginals p_N and q_N, which we take as reference model. Each $C_{ij} =< X_i, y_j >$ is the loss on market risk path i

if the counterparty defaults at time t_j. With $\theta > 0$, the numerator of M_{ij}^θ puts more weight on combinations that produce larger losses. In this sense, M_{ij}^θ is designed to emphasize wrong-way risk.

The denominator of M_{ij}^θ normalizes the entries to sum to 1, but M^θ will not in general have the target marginals. The IPFP algorithm projects a matrix M with positive entries onto the set of joint distribution matrices with marginals p_N and q_N by iteratively renormalizing the rows and columns as follows:

(r) For $i = 1, \ldots, N$ and $j = 1, \ldots, d+1$, set $M_{ij} \leftarrow M_{ij} p_N(i) / \sum_{k=1}^{d+1} M_{ik}$.

(c) For $j = 1, \ldots, d+1$ and $i = 1, \ldots, N$, set $M_{ij} \leftarrow M_{ij} q_N(j) / \sum_{n=1}^{N} M_{nj}$.

This iteration is also known as biproportional scaling, Sinkhorn's algorithm, and the RAS algorithm; see Pukelsheim [45] for an overview of the extensive literature on the theory and application of these methods.

Write $\Phi(M)$ for the result of applying both steps (r) and (c) to M, and write $\Phi^{(n)}$ for the n-fold composition of Φ. For our setting, we need the following result:

Proposition 2.4.2. *The sequence $\Phi^{(n)}(M^\theta)$, $n \geq 1$, converges to the solution P^θ to (2.4.5).*

Proof. It follows from Ireland and Kullback [39] that $\Phi^{(n)}(M^\theta)$ converges to the solution of

$$\min_P \quad \sum_{i=1}^{N} \sum_{j=1}^{d+1} P_{ij} \ln\left(\frac{P_{ij}}{M_{ij}^\theta}\right) \quad \text{subject to (2.3.3)-(2.3.5).}$$

In other words, the IPFP algorithm converges to the feasible matrix (in the sense of (2.3.3)-(2.3.5)) that is closest to the initial matrix in the sense of relative entropy. For our particular choice of M^θ, this minimization problem has the same solution as the maximization problem

$$\max_P \quad \theta \sum_{i=1}^{N} \sum_{j=1}^{d+1} C_{ij} P_{ij} - \sum_{i=1}^{N} \sum_{j=1}^{d+1} P_{ij} \ln(\frac{P_{ij}}{F_{ij}}) - W_\theta^N \quad \text{subject to (2.3.3)-(2.3.5)},$$

with $W_\theta^N = \ln\left(\sum_{i=1}^N \sum_{j=1}^{d+1} e^{\theta \cdot C_{ij}} \cdot F_{ij}\right)$. This follows directly from the definition of M^θ. Because W_θ^N does not depend on P, this maximization problem has the same solution as (2.4.5). □

With $\theta < 0$, the limit of IPFP algorithm solves the penalty problem (2.4.5) with max replaced by min, corresponding to right-way risk in the sense that it minimizes the CVA subject to the marginal constraints and the penalty on deviations from the reference model.

To summarize, we start with the reference model F_{ij}, put more weight on adverse outcomes to get M_{ij}^θ, and then iteratively renormalize the rows and columns of M^θ to match the target marginals. This procedure converges to the penalized worst-case joint distribution defined by (2.4.5) with penalty parameter θ.

2.4.3 Sensitivity Through Dual Variables

Consider the dual of the convex optimization problem in (2.4.5),

$$\min_{a \in \mathbb{R}^N, b \in \mathbb{R}^{d+1}} \sum_{i=1}^{N} a_i/N + \sum_{j=1}^{d+1} b_j q_N(y_j) + \frac{1}{\theta} \sum_{i=1}^{N} \sum_{j=1}^{d+1} F_{ij} e^{\theta(C_{ij} - a_i - b_j)}. \tag{2.4.6}$$

Let (a^*, b^*) denote the optimal dual solution, and consider a vector of small perturbations $(\Delta q_1, \ldots, \Delta q_{d+1})$ to the marginal distribution q_N with components that sum to zero. For perturbations small enough to keep the dual solution unchanged, we can estimate the change in CVA, without resolving problem (2.4.5), using

$$\Delta\widehat{\text{CVA}}_\theta = \sum_{j=1}^{d+1} b_j \Delta q_j.$$

We can calculate the sensitivity to a parallel shift in the credit curve by setting $\Delta q_j = \Delta$, $j = 1, \ldots, d$, and $\Delta q_{d+1} = -d\Delta$, for sufficiently small Δ.

The dual solution can be obtained as a byproduct of the IPFP algorithm. The optimal primal solution takes the form $P^\theta_{ij} = F_{ij} e^{\theta(C_{ij} - a^*_i - b^*_j)}$, where a^* and b^* are optimal dual variables, so we can define scalars u_i and v_j such that

$$P^\theta_{ij} = \frac{M_{ij} w}{u_i v_j},$$

where $w = F_{ij} \sum_{i=1}^{N} \sum_{j=1}^{d+1} e^{\theta \cdot C_{ij}}$ is the normalization term in M_{ij}.

Let $r_i(n)$ be the i-th row sum of $\Phi^{(n)}(M)$ and let $c_j(n)$ be the j-th column sum of $\Phi^{(n)}(M)$ after step (r) in the $(n+1)$-th iteration. By Pukelsheim [45],

$$u_i = \lim_{n\to\infty} \prod_{t=0}^{n} \left(\frac{r_i(t)}{p^N(X_i)} \right) \quad \text{and} \quad v_j = \lim_{n\to\infty} \prod_{t=0}^{n} \left(\frac{c_j(t)}{q^N(y_j)} \right).$$

The optimal dual variables are then given by $a^*_i = \frac{1}{\theta} \ln(u_i) + \frac{1}{2} \ln w$ and $b^*_j = \frac{1}{\theta} \ln(v_j) + \frac{1}{2} \ln w$.

2.4.4 Convergence as $N \to \infty$

We now formulate a convergence result as the number of paths N increases. As before, let $\Pi(p, q)$ denote the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals p and q. Let p_N, q_N

denote the empirical measures in (2.3.1), and let $\Pi(p_N, q_N)$ denote the set of joint laws with these marginals. The independent joint distributions are $\nu \in \Pi(p, q)$ and $\nu_N \in \Pi(p_N, q_N)$; i.e., $d\nu(x, y) = dp(x)dq(y)$ and $d\nu_N(x, y) = dp_N(x)dq_N(y)$.

Fix $\theta > 0$ and define, for a probability measure μ on $\mathbb{R}^d \times \mathbb{R}^d$,

$$G(\mu, \nu) = \int < x, y > \, d\mu - \frac{1}{\theta} D(\mu|\nu),$$

and define $G(\mu, \nu_N)$ accordingly. To show that our simulation estimate of the penalized worst-case CVA converges to the true value, we need to show that

$$\int < x, y > \, d\mu_N^* \to \int < x, y > \, d\mu^*, \quad a.s. \tag{2.4.7}$$

where $\mu_N^* \in \Pi(p_N, q_N)$ maximizes $G(\cdot, \nu_N)$ and $\mu^* \in \Pi(p, q)$ maximizes $G(\cdot, \nu)$.

Theorem 2.4.1. *Suppose the random vectors X and Y satisfy $\mathbb{E}_\nu[e^{\theta < X,Y >}] < \infty$ and that Y has finite support. The following hold as $N \to \infty$.*

1. $\max_{\mu \in \Pi(p_N, q_N)} G(\mu, \nu_N) \longrightarrow \sup_{\mu \in \Pi(p,q)} G(\mu, \nu)$*, a.s.*

2. *The maximizer $\mu_N^* \in \Pi(p_N, q_N)$ of $G(\cdot, \nu_N)$ converges weakly to a maximizer $\mu^* \in \Pi(p, q)$ of $G(\cdot, \nu)$.*

3. *The penalized worst-case CVA converges to the true value, a.s.; i.e., (2.4.7) holds.*

The proof is in Appendix A.3.

2.5 Examples

2.5.1 A Gaussian Example

For purposes of illustration we begin with a simple example in which X and Y are scalars and normally distributed. This example is not intended to fit the CVA application but to illustrate some features of the penalty formulation. It also lends itself to a simple comparison with a Gaussian copula, which is another way of introducing dependence with given marginals.

Suppose then that X and Y have the standard normal distribution on $\mathbb{R}$. Paralleling the definition of the matrix M^θ, consider the bivariate density

$$f_0(x,y) = c' e^{\theta xy} p(x) q(y) = c e^{-\frac{1}{2}x^2 - \frac{1}{2}y^2 + \theta xy}, \tag{2.5.1}$$

where c' and c are normalization constants. This density weights the independent joint density at (x,y) by $\exp(\theta xy)$, so the product xy plays the role that C_{ij} plays in the definition of M^θ.

The reweighting changes the marginals, so now we want to use a continuous version of the IPFP algorithm to project f_0 onto the set of bivariate densities with standard normal marginals. The generalization of the algorithm from matrices to measures has been analyzed in Rüschendorf [47]. The row and column operations become

$$\hat{f}_n(x,y) \leftarrow f_n(x,y) p(x) \Big/ \int f_n(x,y)\, dy$$

and

$$f_{n+1}(x,y) \leftarrow \hat{f}_n(x,y)q(y) \Big/ \int \hat{f}_n(x,y)\,dx\ .$$

An induction argument shows that

$$f_n(x,y) = c_n e^{-\frac{a_n^2}{2}x^2 - \frac{a_n^2}{2}y^2 + \theta xy},$$

for constants c_n and a_n, so each f_n is a bivariate normal density. Then a_n satisfies

$$a_n^2 = \left(1 + \frac{\theta^2}{a_{n-1}^2}\right) \to \frac{1}{2} + \frac{1}{2}\sqrt{1+4\theta^2}, \text{ as } n \to \infty.$$

Some further algebraic simplification then shows that the limit is a bivariate normal density with standard normal marginals and correlation parameter

$$\rho = \frac{2\theta}{1+\sqrt{1+4\theta^2}}, \quad \theta = \frac{\rho}{1-\rho^2}. \tag{2.5.2}$$

This is the bivariate distribution with standard normal marginals that maximizes the expectation of XY with a penalty parameter of θ on the deviation from independence as measured by relative entropy.

Observe that $\rho = 0$ when $\theta = 0$; $\rho \to 1$ as $\theta \to \infty$; and $\rho \to -1$ as $\theta \to -\infty$. Because θ penalizes deviations from independence, it controls the strength of the dependence between X and Y. The relationship between ρ and θ allows us to reinterpret the strength of dependence as measured by θ in terms of the correlation parameter ρ. This is somewhat analogous to the role of a correlation parameter in the Gaussian copula, where it measures the strength of dependence but is not literally the correlation between the marginals except when the marginals are normal.

The fact that the IPFP algorithm projects f_0 to a bivariate normal is a specific feature of the weight $\exp(\theta xy)$ in (2.5.1). For contrast, we consider the weight $\exp(\theta x^2 y)$. The resulting f_0 is no longer integrable for $\theta > 0$, so we work instead with truncated and discretized marginal distributions and apply the IPFP numerically. The result is shown in Figure 2.1. The resulting density has nearly standard normal marginals (up to truncation and discretization), but the joint distribution is clearly not bivariate normal.

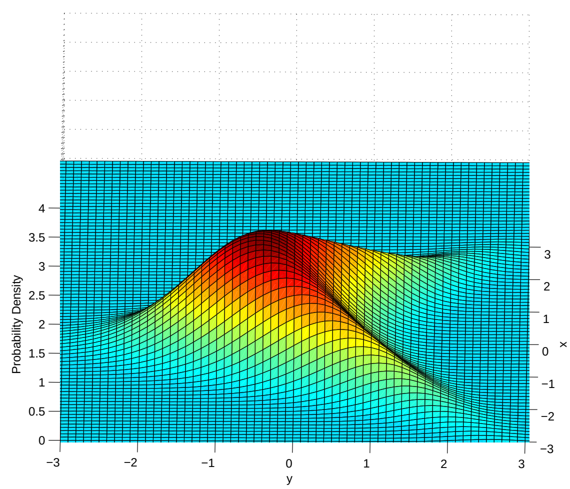

Figure 2.1: Probability mass of joint truncated and discretized normal random variables X and Y, with $\theta = 1$ and initial weight $\exp(\theta x^2 y)$.

The dependence illustrated in the figure is beyond the scope of the Gaussian copula because any joint distribution with Gaussian marginals and a Gaussian copula must be Gaussian. This example thus illustrates the broader point that our approach generates a wider range of dependence than can be achieved with a specific type of copula. For examples of wrong-way risk CVA models based on the Gaussian copula, see Brigo et al. [18], Hull and White [37], and Rosen and Saunders [46].

2.5.2 The Currency Swap Example

In this section, we apply the method on the example in Section 2.2. We take $T = 10$ years, divide time into 20 time steps, and simulate 1000 market scenarios. Expected exposures are adjusted for recoveries and discounted. The sample average positive expected exposure is shown in Figure 2.2. For illustrative purpose, we assume that the counterparty's default time has an exponential distribution with hazard rate $\lambda = 0.04$. We use the following parameters: $(U_{t_0}, \bar{U}, K, \kappa, \sigma, \lambda, \delta, S) = (1000, 1000, 1000, 0.3, 50, 0.04, 0.03, 10^6)$.

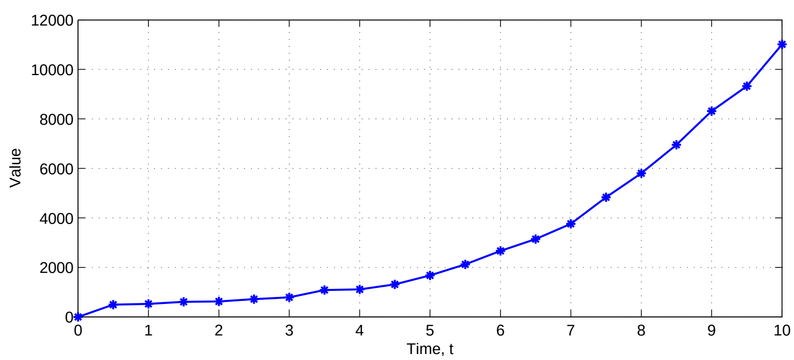

Figure 2.2: Sample Average Positive Exposure

Figure 2.3 shows a CVA stress test for wrong-way risk. It plots CVA against the penalty parameter θ. The numbers are normalized by dividing by the independent market-credit risk CVA, so the independent case $\theta = 0$ is presented as 100%. As θ increases, the positive dependence between market and credit risk increases, approaching the worst-case bound, which is over six times as large as the independent CVA. For $\theta < 0$, we have right-way risk, and the CVA bound approaches zero as θ decreases. The parameter θ could be rescaled using the transformation in (2.5.2) to allow a rough interpretation as a correlation parameter.

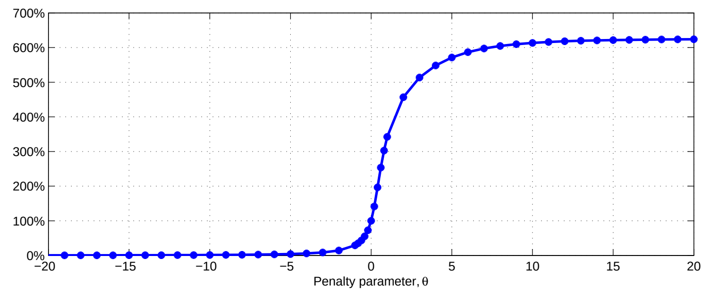

Figure 2.3: CVA Stress Test

The Gaussian copula provides a simple alternative way to vary dependence and measure wrong-way risk; see Rosen and Saunders [46] for details and applications. Figure 2.4 shows how wrong-way risk varies in the Gaussian copula model as the correlation parameter ρ varies from -1 to 1. Comparison with Figure 2.3 shows that constraining dependence to conform to a Gaussian copula significantly underestimates the potential wrong-way risk.

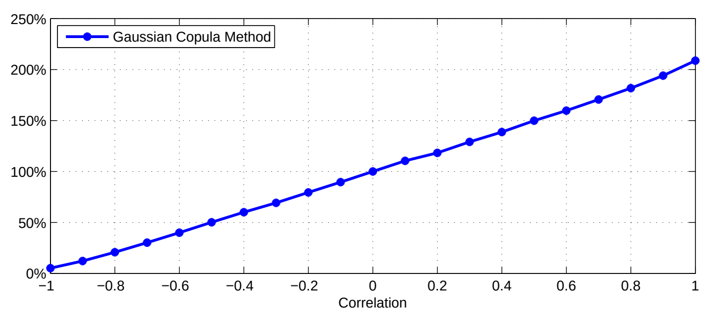

Figure 2.4: CVA Stress Test by Gaussian Copula Method

In Figure 2.5, we show the impact of varying the foreign exchange volatility σ, and the counterparty default hazard rate. Increasing either of these parameters shifts the curve up for $\theta > 0$. In other words, increasing the volatility of the market exposure or the level of the credit exposure in this example increases the potential impact of wrong-way risk, relative to the benchmark of independent market and credit risk.

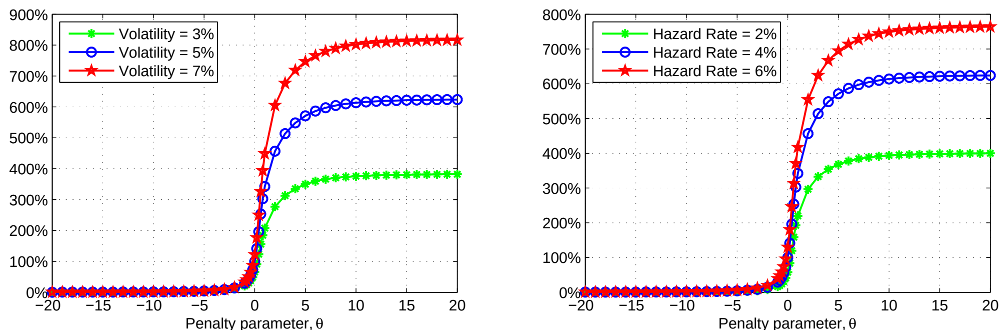

Figure 2.5: CVA with different volatility and hazard rate

2.5.3 Portfolio CVA

We next consider a 10 year fixed-for-floating cross currency swap, in which a U.S. bank receives a fixed rate in dollars and pays a floating rate in foreign currency, with a notional size of $\$5$ million. At the same time, this U.S. bank enters a 3 year and a 6 year foreign exchange forward contract with the same counterparty in the same currency, each with a notional of $\$0.5$ million. We use the Vasicek model for U.S. interest rate,

$$dr_t = \kappa_r(\bar{r} - r_t)dt + \sigma_r d\tilde{W}_t,$$

with parameter values $(r_0, \bar{r}, \sigma_r, \kappa_r) = (0.05, 0.05, 0.0005, 0.8)$.

We consider three different portfolios for the U.S. bank with the same counterparty. The first two portfolios contain multiple transactions of different maturities. The third one contains a single transaction with multiple cash flows.

Portfolio 1 : $5 million 10 year cross currency swap, $0.5 million 3 year and 6 year foreign exchange forwards. The U.S. bank is the U.S. dollar receiver in all transactions.

Portfolio 2 : $5 million 10 year cross currency swap, $0.5 million 3 year and 6 year foreign exchange forwards. The U.S. bank is the U.S. dollar receiver in the cross currency swap and the U.S. dollar payer in the 3 year and 6 year forward contracts.

Portfolio 3 : A simple interest rate swap with notional size $5 million. The U.S. bank receives the floating rate and pays a fixed rate $r_{\text{fix}} = 5\%$.

The sample average positive exposures for these three portfolios are shown in Figure 2.6. In the top two panels, the average positive exposure increases with time because the largest payments are exchanged at maturity. For portfolio 1, the drop in exposure at year 3 and year 6 results from the expiration of the foreign exchange forward contracts. For portfolio 2, since the portfolio is more balanced, the exposure path is smoother. In the bottom panel, for portfolio 3, the average positive exposure decreases to 0 at maturity because the total exposure decreases with time in an interest rate swap.

Figure 2.7 shows CVA values as θ varies. We report CVA as a percentage of the portfolio 2 CVA in the independent case. Portfolio 1 has the greatest sensitivity to wrong-way risk because all its transactions run in the wrong-way direction. Portfolio 2 is better diversified, with both wrong-way and right-way transactions. Because the average positive exposure for portfolio 1 is higher than that of portfolio 2, it has a higher CVA for all θ, and with θ increasing, portfolio 1 attains much higher CVA values near the worst-case bound than does the more diversified portfolio 2. For portfolio 3, because of its lower and less concentrated positive exposure, the CVA bound is much lower compared with the other two portfolios.

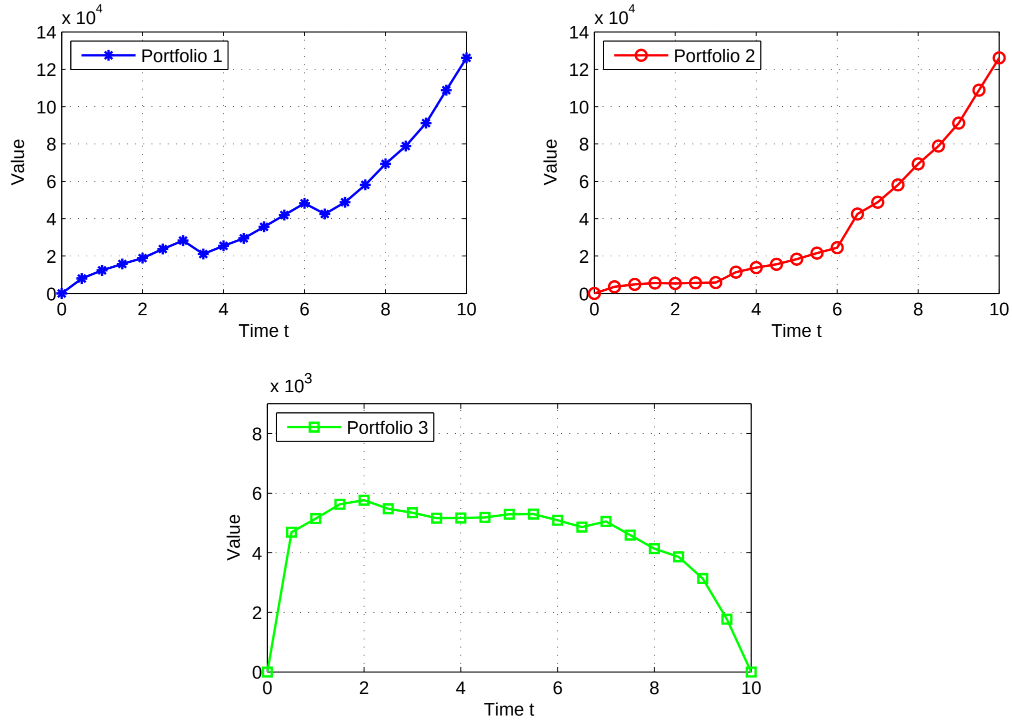

Figure 2.6: Sample Average Positive Exposures for Three Portfolios.

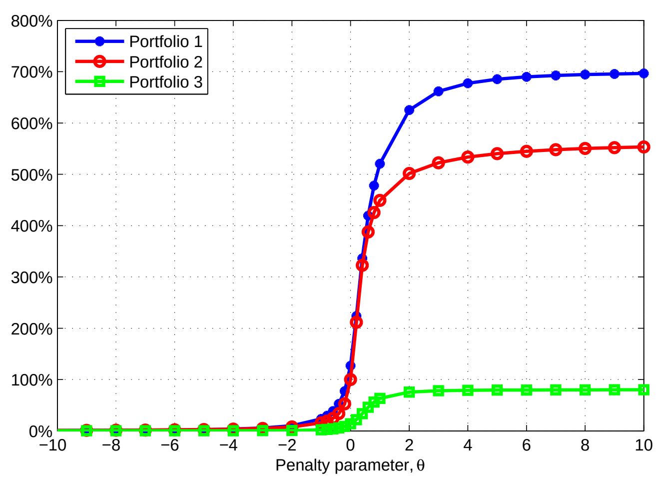

Figure 2.7: CVA Bounds for Three Portfolios.

Figure 2.8 shows the sensitivity of the CVA estimates to a change in the counterparty's default hazard rate. We increase the hazard rate by 1 basis point from $\lambda = 0.04$ to $\lambda' = 0.0401$ and show the estimated change in CVA using dual variables and the actual difference based on resolving the optimization problem at each θ. To put these sensitivities in perspective, the CVA estimate at $\theta = 0$ is \$11,241, and at $\theta = 20$, it is \$62,659. The sensitivities in Figure 2.8 are in dollars. Overall, the dual variables provide good estimates of the change in CVA under a small change in the default probability. Compared with resolving the optimization problem at the perturbed λ, the dual variables slightly underestimate the change in wrong-way scenarios ($\theta > 0$) and overestimate the change in the right-way scenarios ($\theta < 0$).

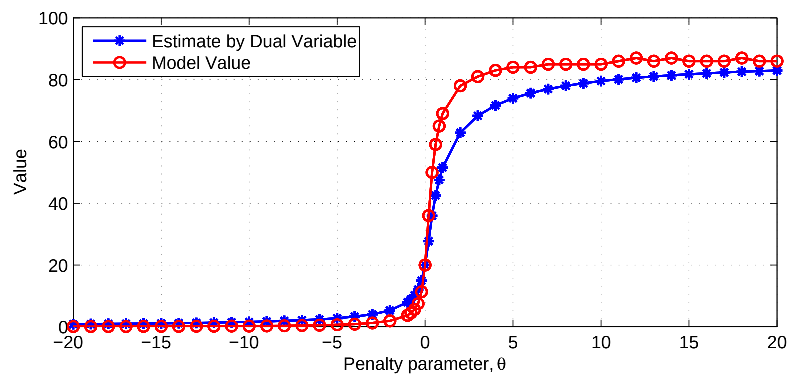

Figure 2.8: Change in CVA for a 1 basis point change in hazard rate at various levels of the penalty parameter θ.

2.6 Adding Expectation Constraints

When additional information is available, we can often improve our CVA bound by incorporating the information through constraints on the optimization problem. Constraints on expectations are linear constraints on joint distributions and thus particularly convenient in our framework.

Recall that we think of the exposure path X as the output of a simulation of a market model. Such a model generates many other market variables, and in specifying the joint distribution between the market and credit models, we may want to add constraints through other variables. Constraints represent relationships between market and credit risk that should be preserved as the joint distribution varies. To incorporate such constraints, we expand the simulation output from X to (X, Z), where the random vector $Z = (Z_1, \ldots, Z_d)$ represents a path of auxiliary variables. The joint law of (X, Z) is determined by the market model. We want to add a constraint of the form $E[Z_\tau \mathbf{1}\{\tau \leq t_d\}] = z_0$, for given z_0, when the expectation is taken with respect to the joint law of the market and credit models. This is a constraint on the expectation of $< Z, Y >$.

As a specific illustration, suppose $\tilde{Z}$ is a martingale generated by the market model and we want to impose the constraint $\mathbb{E}[\tilde{Z}_{\tau \wedge t_d}] = z_0$ on the joint law of $\tilde{Z}$ and τ. This is equivalent to the constraint $\mathbb{E}[(\tilde{Z}_{t_d} - \tilde{Z}_\tau)\mathbf{1}\{\tau \leq t_d\}] = 0$, so we can define $Z_j = \tilde{Z}_d - \tilde{Z}_j$, $j = 1, \ldots, d$, and then impose the constraint $\mathbb{E}[< Z, Y >] = 0$.

To incorporate constraints, we redefine p to denote the joint law of (X, Z) on $\mathbb{R}^d \times \mathbb{R}^d$; we continue to use q for the marginal law of Y. Let $\Pi(p, q)$ be the set of probability measures on $(\mathbb{R}^d \times \mathbb{R}^d) \times \mathbb{R}^d$ with the specified marginals of (X, Z) and Y. We denote by $h_X(x, z) = x$ and $h_Z(x, z) = z$ the projections of (x, z) to x and z respectively. Set

$$\bar{\Pi}(p, q) = \{\mu \in \Pi(p, q) : \int < h_Z(x, z), y > d\mu((x, z), y) = v_0\}. \tag{2.6.1}$$

We will assume that $\bar{\Pi}(p, q)$ is nonempty so that the problem is feasible.

Given independent samples (X_i, Z_i), $i = 1, \ldots, N$, let p_N denote their empirical measure. As before q_N denotes the empirical measure for N independent copies of Y. Even if $\bar{\Pi}(p, q)$ is nonempty, we cannot assume that the equality constraint in (2.6.1) holds for some element of $\Pi(p_N, q_N)$, so for finite N we will need a relaxed formulation. Let $\Pi_\epsilon(p_N, q_N)$ denote the set of joint distributions on $\{((X_i, Z_i), y_j), i = 1, \ldots, N, j = 1, \ldots, d+1\}$ with marginals p_N and $\tilde{q}$, where

$$\max_{1 \leq j \leq d+1} |q_N(y_j) - \tilde{q}(y_j)| < \epsilon,$$

and define

$$\bar{\Pi}_\epsilon(p_N, q_N) = \left\{ \mu \in \Pi_\epsilon(p_N, q_N) : \left| \int < h_Z(x,z), y > d\mu((x,z),y) - v_0 \right| < \epsilon \right\}. \quad (2.6.2)$$

In our convergence analysis, we will let $\epsilon \equiv \epsilon_N$ decrease to zero as N increases.

Let $\nu \in \Pi(p, q)$ denote the independent case $d\nu((x,z),y) = dp(x,z)dq(y)$, and let $\nu_N \in \Pi(p_N, q_N)$ denote the independent case $d\nu_N((x,z),y) = dp_N(x,z)dq_N(y)$. We will assume that v_0 is chosen so that $\nu \in \bar{\Pi}(p, q)$. It then follows that $\nu_N \in \bar{\Pi}_\epsilon(p_N, q_N)$ for all sufficiently large N, for all $\epsilon > 0$.

The worst-case CVA with an auxiliary constraint on Z is

$$c_\infty = \sup_{\mu \in \bar{\Pi}(p,q)} \int_{(\mathbb{R}^d \times \mathbb{R}^d) \times \mathbb{R}^d} < h_X(x,z), y > d\mu((x,z),y) \quad (2.6.3)$$

The corresponding estimator is

$$c_{N,\epsilon} = \max_{\mu \in \bar{\Pi}_\epsilon(p_N, q_N)} \sum_{i=1}^{N} \sum_{j=1}^{d+1} < X_i, y_j > \mu((X_i, Z_i), y_j). \quad (2.6.4)$$

This is a linear programming problem: the objective and the constraints are linear in the variables $\mu((X_i, Z_i), y_j)$. The following result establishes convergence of the estimator.

Theorem 2.6.1. *Suppose the following conditions hold:*

(i) $\int_{\mathbb{R}^d \times \mathbb{R}^d} \|h_X(x,z)\|^2 dp(x,z) < \infty$ *and* $\int_{\mathbb{R}^d \times \mathbb{R}^d} \|h_Z(x,z)\|^2 dp(x,z) < \infty$*;*

(ii) $\bar{\Pi}(p,q)$ *contains the independent joint distribution* ν*.*

Then with $\epsilon_N = 1/N^\alpha$ *for any* $\alpha \in (0, 1/2)$*, the finite sample estimate converges to the constrained worst-case CVA for the limiting problem; i.e.,* $c_{N,\epsilon_N} \to c_\infty$*, a.s.*

We define a penalty formulation with $\theta > 0$ for the limiting problem,

$$\sup_{\mu \in \bar{\Pi}(p,q)} G(\mu, \nu) \quad = \quad \sup_{\mu \in \bar{\Pi}(p,q)} \int_{(\mathbb{R}^d \times \mathbb{R}^d) \times \mathbb{R}^d} < h_X(x,z), y > d\mu((x,z),y) - \frac{1}{\theta} D(\mu|\nu),$$

and with (2.6.2) for the finite problem,

$$\max_{\mu \in \bar{\Pi}_\epsilon(p_N, q_N)} G(\mu, \nu) \quad = \quad \max_{\mu \in \bar{\Pi}_\epsilon(p_N, q_N)} \sum_{i=1}^{N} \sum_{j=1}^{d+1} < h_X(X_i, Z_i), y_j > \mu_N((X_i, Z_i), y_j) - \frac{1}{\theta} D(\mu_N|\nu_N).$$

The corresponding convergence result given by the following theorem.

Theorem 2.6.2. *Suppose the following conditions hold:*

(i) $\int_{\mathbb{R}^d \times \mathbb{R}^d} \|h_X(x,z)\|^2 dp(x,z) < \infty$*,* $\int_{\mathbb{R}^d \times \mathbb{R}^d} \|h_Z(x,z)\|^2 dp(x,z) < \infty$*, and* $\mathbb{E}_\nu[e^{\theta < h_X(X,Z), Y>}] < \infty$*;*

(ii) $\bar{\Pi}(p,q)$ *contains the independent joint distribution* ν*.*

Then with $\epsilon_N = 1/N^\alpha$ *for any* $\alpha \in (0, 1/2)$*, the following hold,*

1. $\max_{\mu \in \bar{\Pi}_{\epsilon_N}(p_N, q_N)} G(\mu, \nu_N) \longrightarrow \sup_{\mu \in \bar{\Pi}(p,q)} G(\mu, \nu)$, *a.s.*

2. *The maximizer* $\bar{\mu}_N^* \in \bar{\Pi}_{\epsilon_N}(p_N, q_N)$ *of* $G(\cdot, \nu_N)$ *converges weakly to a maximizer* $\bar{\mu}^* \in \bar{\Pi}(p, q)$ *of* $G(\cdot, \nu)$.

3. *The penalized worst-case CVA converges to the true value, a.s.; i.e.,*

$$\int < x, y > \, d\bar{\mu}_N^* \to \int < x, y > \, d\bar{\mu}^*, \quad a.s. \tag{2.6.5}$$

2.7 Bilateral CVA

In previous sections, the CVA bound we discussed is for unilateral CVA. In this section we show that this methodology extends easily to bilateral CVA.

We keep the existing notations, and in addition let τ_s denote the default time of the agent itself. Then the bilateral CVA for a time horizon T is the expected positive exposure at the default of the counterparty, which happens before the default of the agent itself,

$$\text{BCVA} = \mathbb{E}[V^+(\tau)\mathbf{1}\{\tau \leq \tau_s \wedge T\}],$$

given joint law for the default times τ and τ_s, and the exposure $V^+(t)$.

There are two ways to approach this problem with different levels of information. If the joint distribution of the counterparty's default time τ and the agent's self-default time τ_s is known, or if it is the result of a more extensive credit model, then we define

$$X = (V^+(t_1), \dots, V^+(t_d)) \text{ and } Y = (\mathbf{1}\{\tau = t_1, \tau_s \geq t_1\}, \dots, \mathbf{1}\{\tau = t_d, \tau_s \geq t_d\}).$$

Note that Y has the support of the finite set $\{y_1, \ldots, y_{d+1}\}$, with $y_1 = (1, 0, \ldots, 0), \ldots, y_d = (0, 0, \ldots, 1)$, and $y_{d+1} = (0, \ldots, 0)$. Each y_j has probability $q(y_j) = P(\tau = t_j, \tau_s \geq t_j)$ for $j = 1, ..., d$, and $q(y_j) = 1 - \sum_{k=1}^{d} q(y_j)$ for $j = d + 1$. These probabilities may be known or estimated from simulation of N independent copies $Y_1, \ldots, Y_N$ of Y, in which case we denote the empirical frequency of each y_j by $q_N(y_j)$. Then the bilateral CVA formulation for the *worst-case* and penalty case follow the same as earlier discussed, except for the changes in Y's definition and marginal.

The joint distribution of default times of both parties is usually not available. In a more general case, we define

$$X = (V^+(t_1), \ldots, V^+(t_d)) \ , \ Y = (\mathbf{1}\{\tau = t_1\}, \ldots, \mathbf{1}\{\tau = t_d\}),$$

and

$$W = (\mathbf{1}\{\tau_s \geq t_1\}, \ldots, \mathbf{1}\{\tau_s \geq t_d\}).$$

The problem of calculating bilateral CVA reduce to the problem of calculating the expectation of the following function,

$$C(X, Y, W) \equiv \sum_{j=1}^{d} X_j Y_j W_j = \sum_{j=1}^{d} V^+(t_j)\mathbf{1}\{\tau = t_j\}\mathbf{1}\{\tau_s \geq t_j\} = V^+(\tau)\mathbf{1}\{\tau \leq \tau_s \wedge T\},$$

if the joint law for X, Y and W were known.

We continue to use p and q to denote the marginal law of X and Y, and let q_s to denote marginal of W. Furthermore, note that W is supported on the finite set $\{w_1, \ldots, w_d\}$, with $w_1 = (1, 0, \ldots, 0), w_2 = (1, 1, 0, \ldots, 0), \ldots w_d = (1, 1, \ldots, 1)$. Each w_j has probability

$q_s(w_j) = P(\tau_s = t_j)$ for $j = 1, ..., d-1$ and $q_s(w_j) = P(\tau_s \geq t_j)$ for $j = d$. Define $\Pi(p, q, q_s)$ to be the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ with marginals p, q and q_s.

With marginals fixed but the joint law unknown, we define the *worst-case* bilateral CVA by

$$\text{BCVA}_* := \sup_{\mu \in \Pi(p,q,q_s)} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} C(x, y, w) d\mu(x, y, w). \tag{2.7.1}$$

We extend the CVA formulation to 3-dimensional, by including the self-default time of the agent, for bilateral CVA. IPFP algorithm still applies to the tempered bilateral CVA.

2.8 Concluding Remarks

We have focused in this chapter on the problem of bounding wrong-way risk in CVA calculation, taking the marginal models for market and credit risk as given and varying the dependence between the two. Put more generally, the problem we have addressed is one of bounding the expected inner product between two random vectors with fixed marginals. A key feature of our setting is that these marginals need not be known explicitly. Instead, they are outputs of the simulation of potentially very complex models, of the type used to model asset prices and default times.

Calculating the worst-case bound for the exact marginal distributions is typically infeasible. But using simulated outcomes, the problem reduces to a tractable linear programming problem. We extend this formulation by penalizing deviations from a reference model, which results in a convex optimization problem. In both cases, we prove convergence of the solutions calculated from simulated outcomes to the corresponding solutions using exact distributions as the sample size grows. The approach is sufficiently general and flexible to be applicable to many other settings in which the nature of dependence between different model components is unknown.

An important practical problem is the choice of the penalty parameter θ, which reflects the user's confidence in the reference model. A large θ leads to more conservative values; a small θ produces values very close to the reference model. Inevitably, the choice of θ involves some judgment. However, this judgment is best anchored in observable data. In our examples, each value of θ implies some level of correlation between the exchange rate and the credit spread for the counterparty. This correlation is a limited measure that cannot determine the full dependence between the market and credit risk models, but it can help pin down an appropriate value for θ.

Chapter 3

Realization Utility with Adaptive Reference Point

3.1 Introduction

In neoclassical finance, investors trade financial assets to maximize their utility of consumption or utility of terminal wealth. Optimal trading in this regard has been studied for decades. Recently, it has been found that in addition to intermediate consumption and terminal wealth, investors are also sensitive to trading gains and losses; see for example Barberis and Huang [4], Barberis and Xiong [6] and Ingersoll and Jin [38].

Realization utility is first explicitly modeled by Barberis and Xiong [6]. They assume that investors think of their investing experience as a series of separate episodes, and they receive utility burst when a gain or a loss is realized. Assuming a piecewise linear realization utility function, the authors find that the investors never voluntarily sell a stock at a loss. Ingersoll and Jin [38] extend the model by assuming an S-shaped realization utility function and find that the investors voluntarily sell a stock both at a gain and at a loss.

In this chapter, we extend the works by Barberis and Xiong [6] and Ingersoll and Jin [38] in two aspects: First, in addition to realization utility, the agent in our model also experiences utility from her terminal wealth. The terminal wealth utility is modeled by the expected utility theory. Second, the reference point that decides whether a payoff is experienced by the agent as a gain or as a loss is assumed to adapt to the stock's prior gain and loss, and the adaptation to the gain is more than to the loss. By contrast, Ingersoll and Jin [38] assume the reference point to be fixed at the purchase price, and Barberis and Xiong [6] assume the reference point to be the purchase price growing at the risk-free rate. Experimental evidence in Baucells et al. [9] and Arkes et al. [1, 2] reveals that when selling a stock, most investors choose their reference points to be the purchase price plus a portion of the prior paper gain and loss of the stock. Moreover, the reference point adapts more to a prior gain than to a prior loss. Therefore, our model is more consistent with individuals' behavior observed in the literature.

We assume that the agent can trade one risk-free asset and multiple stocks with the same constant expected return and volatility, but she can only invest in one of these assets at one time. In other words, the agent can only switch her position between the risk-free asset and a stock. We formulate the agent's trading problem as an optimal stopping problem. We first provide sufficient and necessary conditions under which the optimal value of the agent's trading problem is finite. We then find the optimal purchase time of the stock: the agent either immediately or never re-purchases the stock after selling it. We also prove that it is optimal to hold the stock if it is already at a deep loss. Finally, we prove that the value function of the trading problem is the unique solution to a variational inequality, and we develop an efficient algorithm to solve the value function and the optimal trading strategy.

In the study of trading strategies and asset pricing, we distinguish two types of agents: adaptive agents and naive agents. An adaptive agent knows today that her reference point in the future will adapt to the prior gain and loss of the stock, and thus knows today that the gain and loss she will experience in the future will change accordingly. When making decisions today, the adaptive agent already takes this knowledge into account. For a naive agent, her reference point in the future will also adapt to the stock's gain and loss, but she doesn't realize it prospectively: the naive agent wrongly believes that her reference point will remain constant over time. Therefore, the naive agent plans her trading strategy based on this wrong belief. At each time in the future, however, the naive agent realizes retrospectively that her reference point has changed, but she still fails to realize prospectively that her reference point will continue to adapt to the stock's gain and loss. Because the reference point has changed, when re-examining the trading problem, the naive agent finds that the strategy that was planned before is no longer optimal and thus changes to a new strategy. As time goes by, the naive agent changes her strategy constantly.

Note that the inconsistency in the liquidation strategy between the selves of the naive agent at different times arises from the adaptation of the reference point to the stock's gain and loss. Inconsistent dynamic decisions have been extensively observed and studied in the literature as a result of hyperbolic discounting (O'Donoghue and Rabin [43]), probability weighting (Barberis [3]), and mean-variance criteria (Basak and Chabakauri [7]). As far as we know, our work is the first one to show that time-varying reference points can also lead to dynamic inconsistency. The naive agent studied in this dissertation is similar to those considered in the dynamic inconsistency literature.

We have the following three main findings in the trading strategies of the adaptive and naive agents: First, the adaptive agent sells the stock more (less) frequently when the stock is at a gain (at a loss) than the naive agent does. When comparing to a non-adaptive agent whose reference point does not adapt to the stock's gain and loss, the naive agent sells the stock less frequently both at a gain and at a loss. The adaptive agent sells the stock at a loss less frequently than the non-adaptive agent does as well.

Second, when the reference point adapts more to stock's gain, the naive agent sells the stock at a gain less frequently because she actually experiences less gain for the same amount of increase in stock's price. The naive agent's selling policy when the stock is at a loss, however, is unaffected. Similarly, when the reference point adapts more to the stock's loss, the naive agent sells the stock at a loss less frequently, but the selling policy at a gain does not change. For the adaptive agent, when the reference point adapts more to the stock's gain, she sells the stock at a loss less frequently. When the reference point adapts more to the stock's loss, the adaptive agent sells the stock at a loss more frequently, which is the opposite to the behavior of the naive agent.

Third, when becoming more concerned with the terminal wealth utility, both the adaptive and naive agents sell the stock less frequently both at a gain and at a loss. In the extreme case in which the agents are only concerned with their terminal wealth utility, they will never trade the stock because trading incurs transaction costs without improving wealth (because the stock's expected return and volatility are assumed to be constant). Therefore, trading is generated in our model because of realization utility.

We also study asset pricing implications of our model. For each stock in the market, we define the equilibrium risk premium of the stock as the expected excess return such that the investors in the market are indifferent between the stock and the risk-free asset. We consider two markets: one with investors that are homogeneous adaptive agents, and the other with investors that are homogeneous naive agents. We have four main findings: First, adaptive agents require a higher risk premium than naive agents do.

Second, when the reference point adapts more to the stock's gain, the adaptive agent asks for a higher risk premium for the stock because the same increase in the stock price leads to less realization utility in this case. Similarly, when the reference point adapts more to the stock's loss, the stock is rewarded with a lower risk premium because the same decrease in the stock price yields less realization disutility. On the other hand, the risk premium required by the naive agent is insensitive to the degree to which the reference point adapts to the stock's gain or loss. This is because the naive agent does not realize prospectively that her reference point will change in the future. As a result, when buying the stock, the value of the stock from the naive agent's perspective does not depend on how the reference point actually changes in the future.

Third, the risk premium becomes higher when the naive and adaptive agents become more concerned with the terminal wealth utility. This is because both agents become more risk averse in this case.

Fourth, the risk premium can be positive or negative and can be increasing or decreasing with respect to the stock volatility, depending on the type of agents in the market and on model parameter values. Thus, our model can generate different patterns of return-risk tradeoffs for stocks.

The remainder of this chapter is organized as follows: In Chapter 3.2, we propose our model and formulate the trading problem. In Chapter 3.3, we solve the trading problem. In Chapter 3.4, we compare the trading strategies of the naive and adaptive agents and study the sensitivity of the strategies with respect to model parameters. In Chapter 3.5, we study the asset pricing implications of our model. Chapter 3.6 provides extensions of our model and Chapter 3.7 concludes. Appendix B contains all the proofs and Appendix C provides the algorithm used to solve the trading problem.

3.2 Model

Consider an agent who makes a sequence of purchase and sale decisions to take positions in stocks. Following Barberis and Xiong [6] and Ingersoll and Jin [38], we assume that at any time the agent can invest her money only in one of the stocks or in a risk-free asset. In other words, the agent does not diversify her portfolio and only decides in which asset she invests her money.

The risk-free rate is assumed to be constant r. The prices of the stocks that the agent can trade are assumed to follow geometric Brownain motions with the same appreciation rate μ and volatility σ (though the prices can be driven by different Brownain motions). Because the agent can invest only in one asset at one time, we can simply assume that the agent can only trade one stock with price dynamics $dS_t = \mu S_t dt + \sigma S_t dW_t$, where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion. Note that this setting is the same as in Barberis and Xiong [6] and Ingersoll and Jin [38].

We assume a proportional transaction cost k_p of buying the stock and a proportional transaction cost k_s of selling the stock. Consequently, x dollars to be invested in the stock becomes $x/(1+k_p)$ dollars value of stock after the purchase cost is deducted and y dollars value of the stock position becomes $y(1-k_s)$ dollars after the position is liquidated. We assume no transaction cost of buying and selling the risk-free asset.

The investment horizon of the agent is a Poisson shock time $\tilde{\tau}$ that is independent of the Brownain motion $\{W_t\}_{t\geq 0}$ that drives the stock price. In other words, $\tilde{\tau}$ is exponentially distributed and we assume its rate parameter to be ρ. Consequently, the mean of the investment horizon is $1/\rho$. When the shock time arrives, the agent has to liquidate all the assets she holds.

In the following, we denote $\mathcal{F}_t$ as the information available at time t, i.e., $\mathcal{F}_t$ is the σ-algebra generated by W_s and $\mathbf{1}_{\{\tilde{\tau}>s\}}$, $s \leq t$, and denote $\mathbb{E}_t$ as the expectation conditioning on $\mathcal{F}_t$.

The agent experiences the utility of her wealth at the end of the investment horizon. In addition, the agent also experiences realization utility every time she sells the stock, no matter the sale is voluntary before the shock time $\tilde{\tau}$ or is forced at $\tilde{\tau}$. We use expected utility theory (EUT) to model the terminal wealth utility. Suppose the current time is t and the wealth at $\tilde{\tau} > t$ is X, then the utility of this wealth is defined to be $\mathbb{E}_t[e^{-\delta(\tilde{\tau}-t)}U_W(X)]$,

where

$$U_W(x) = \theta x^{\beta}. \tag{3.2.1}$$

Here, $\delta > 0$ is a discount factor for utility and $\beta \in (0, 1]$ determines the relative risk aversion of the agent regarding random wealth. Later on, we will aggregate the terminal wealth utility and realization utility by adding them together, so parameter $\theta \geq 0$ represents the relative weight of the terminal wealth utility to the latter. If $\theta = 0$, the agent does not experience terminal wealth utility, and if $\theta \to +\infty$, realization utility disappears.

On the other hand, we use cumulative prospect theory (CPT; Tversky and Kahneman [40, 53]) to model realization utility. Suppose the current time is t and the agent sells the stock at $\tau > t$. Then, the realization utility of this sale is $\mathbb{E}_t[e^{-\delta(\tau-t)}U(G_{\tau-}, R_{\tau-})]$. Here, we use the same discount factor δ as for the discounting of the terminal wealth utility. $G_{\tau-}$ stands for the gain and loss experienced by the agent for realization utility when she sells the stock at τ, and $R_{\tau-}$ is the reference point the agent uses at τ to determine that gain and loss. The function U is S-shaped with respect to G and satisfies $U(0, R) = 0$, as suggested in CPT. We will model $G_{\tau-}$ and $R_{\tau-}$ momentarily.

Let X_t be the agent's wealth, i.e., the book value of the agent's position in the stock or in the risk-free asset. Between a purchase time ζ_i of the stock and the following sale time τ_i, X_t grows in the same way as the stock price, i.e.,

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t, \quad t \in [\zeta_i, \tau_i).$$

Between a sale time τ_i of the stock and the following purchase time ζ_{i+1}, X_t grows at the risk-free rate, i.e.,

$$dX_t = rX_t\,dt, \quad t \in [\tau_i, \zeta_{i+1}).$$

Furthermore, $\{X_t\}$ jumps downwards at each purchase and sale times due to the transaction cost, i.e.,

$$X_{\zeta_i} = \frac{1}{1+k_p} \cdot X_{\zeta_i -}, \quad X_{\tau_i} = (1-k_s) \cdot X_{\tau_i -}.$$

Note that we choose the right-continuous version of X, so if there is a purchase or sale at t, X_t and X_{t-} stand for the agent's wealth after and before the transaction cost being deducted, respectively.

Next, let us specify the reference point and the gain and loss of the agent at each sale time. Denote P_t as the purchase price at the latest purchase time prior to t, i.e., $P_t := X_{\zeta_i}$ for $t \in [\zeta_i, \zeta_{i+1})$ and two consecutive purchase times $\zeta_i < \zeta_{i+1}$. Ingersoll and Jin [38] assume that the agent uses the purchase price P_t as the reference point. Barberis and Xiong [6] assume that the reference point is the purchase price growing at the risk-free rate, i.e., $P_t e^{r(t-\zeta_i)}$ for $t \in [\zeta_i, \zeta_{i+1})$.

Experimental evidence in Baucells et al. [9] and Arkes et al. [1, 2], however, reveals that individuals adapt their reference points to prior paper gains and losses of the asset they trade and the adaptation to a gain is more than to a comparable loss. Therefore, we model the reference point as follows:

$$R_t = R(X_t, P_t) := \begin{cases} P_t + \gamma_+(X_t - P_t), & X_t \geq P_t, \\ P_t + \gamma_-(X_t - P_t), & X_t < P_t. \end{cases}$$

Note that $X_t - P_t$ stands for the paper gain and loss of the stock since the last purchase time, so $\gamma_\pm \in [0, 1]$ model the asymmetric adaptation of the reference point to the stock's gain and loss. Consequently, the reference point the agent employs to calculate realization utility when selling the stock at τ_i is R_{τ_i-} and the resulting gain or loss is $G_{\tau_i-} = X_{\tau_i-} - R_{\tau_i-}$.

We assume $U(G, R)$ to be homogeneous of degree $\beta \in (0, 1]$, i.e, $U(G, R) = R^\beta U(G/R, 1)$. Note that β here is the same as the relative risk aversion index of the utility function U_W for terminal wealth. Denote $u(x) := U(x, 1), x \in \mathbb{R}$. We assume that $u(\cdot)$ is strictly increasing and $u(0) = 0$. Furthermore, we assume $u(x)$ is concave for $x \geq 0$ and convex for $x \leq 0$, representing the agent's risk averse attitude regarding gains and risk seeking attitude regarding losses. Finally, the same amount of gain $G > 0$ yields less utility when the reference point R is higher and the same loss $G < 0$ yields more disutility when the reference point R is lower. Thus, we assume that $U(G, R)$ is decreasing in R when $G > 0$ and is increasing in R when $G < 0$. It is straightforward to show that this assumption is equivalent to assuming

$$xu'(x)/u(x) \geq \beta, \quad x \neq 0. \tag{3.2.2}$$

For example, the following function

$$u(x) = [(x + b_G)^{\alpha_G} - b_G^{\alpha_G}]\,\mathbf{1}_{\{x\geq 0\}} - \lambda\,[(-x + b_L)^{\alpha_L} - b_L^{\alpha_L}]\,\mathbf{1}_{\{x<0\}} \tag{3.2.3}$$

with $\lambda > 0$, $\alpha_G, \alpha_L \in [\beta, 1]$, and $b_G, b_L \geq 0$ satisfies (3.2.2). Ingersoll and Jin [38] consider cases of b_G, b_L both equal to 0 and both equal to 1. If $b_G = b_L = 0$ and $\alpha_G = \alpha_L = \beta$, $U(G, R) = G^\beta \mathbf{1}_{\{G\geq 0\}} - \lambda(-G)^\beta \mathbf{1}_{\{G<0\}}$, which is the same as the utility function used in Tversky and Kahneman [53]. On the other hand, the piece-wise exponential utility function, i.e., $u(x) = (1 - e^{-\alpha_G x})\mathbf{1}_{\{x\geq 0\}} - \lambda(1 - e^{\alpha_L x})\mathbf{1}_{\{x<0\}}$ for some $\alpha_G, \alpha_L \geq 0$, does not satisfy (3.2.2).

Because $R(X_t, P_t) = P_t R(X_t/P_t, 1)$, we have

$$\begin{aligned} U(G_t, R_t) = R_t^\beta u(G_t/R_t) &= P_t^\beta R(X_t/P_t, 1)^\beta u\left(\frac{X_t - R_t}{R_t}\right) \\ &= P_t^\beta R(X_t/P_t, 1)^\beta u\left(\frac{X_t}{P_t}\frac{1}{R(X_t/P_t, 1)} - 1\right). \end{aligned}$$

Therefore, if we denote

$$\bar{u}(x) := R(x, 1)^\beta u\left(\frac{x}{R(x, 1)} - 1\right) = \begin{cases} (1 + \gamma_+(x-1))^\beta u\left(\frac{(1-\gamma_+)(x-1)}{1+\gamma_+(x-1)}\right), & x \geq 1, \\ (1 + \gamma_-(x-1))^\beta u\left(\frac{(1-\gamma_-)(x-1)}{1+\gamma_-(x-1)}\right), & x < 1, \end{cases}$$

we have $U(G_t, R_t) = P_t^\beta \bar{u}(X_t/P_t)$. In addition, note that $\bar{u}(x)$ is increasing and continuous in $x \geq 0$.

Define

$$\alpha := \lim_{x\to+\infty} \frac{xu'(x)}{u(x)}, \quad \bar{\alpha} := \lim_{x\to+\infty} \frac{x\bar{u}'(x)}{\bar{u}(x)} = \begin{cases} \alpha & \gamma_+ = 0, \\ \beta & \gamma_+ > 0, \end{cases} \tag{3.2.4}$$

assuming the limit exist. Because $u(x)$ is concave for $x \geq 0$ and $u(0) = 0$, we have $u(x) \geq xu'(x)$, showing that $\alpha \leq 1$. On the other hand, condition (3.2.2) leads to $\alpha \geq \beta$. For $u(\cdot)$ defined as in (3.2.3), $\alpha = \alpha_G$.

Suppose right before time t, the agent is holding the stock. Then, the agent's decision is $\{\mathcal{F}_s\}$-stopping times $t \leq \tau_1 \leq \zeta_2 \leq \tau_2 \leq \zeta_3 \leq \dots$, where ζ_k's and τ_k's are purchase and sale times of the stock, respectively. The agent faces the following optimal liquidation problem[1]

$$\begin{aligned} \sup_{t\leq\tau_1\leq\zeta_2\leq\tau_2\leq\dots} \quad & \mathbb{E}_t\Big[\textstyle\sum_{i=1}^\infty e^{-\delta(\tau_i - t)} U(G_{\tau_i-}, R_{\tau_i-}) \mathbf{1}_{\{\tau_i<\tilde{\tau}\}} \\ & \quad + e^{-\delta(\tilde{\tau}-t)} U_W(X_{\tilde{\tau}-}) \textstyle\sum_{i=1}^\infty \mathbf{1}_{\{\tau_i<\tilde{\tau}\leq\zeta_{i+1}\}} \\ & \quad + e^{-\delta(\tilde{\tau}-t)} \big(U_W((1-k_s)X_{\tilde{\tau}-}) + U(G_{\tilde{\tau}-}, R_{\tilde{\tau}-})\big) \textstyle\sum_{i=1}^\infty \mathbf{1}_{\{\zeta_i<\tilde{\tau}\leq\tau_i\}}\Big] \\ \text{subject to} \quad & dX_s = \textstyle\sum_{i=1}^\infty \Big[(\mu X_s\, ds + \sigma X_s\, dW_s) \mathbf{1}_{\{s\in[\zeta_i,\tau_i)\}} + rX_s\, ds \mathbf{1}_{\{s\in[\tau_i,\zeta_{i+1})\}} \\ & \qquad - k_s X_{s-} \mathbf{1}_{\{s=\tau_i\}} - \frac{k_p}{1+k_p} X_{s-} \mathbf{1}_{\{s=\zeta_{i+1}\}} \Big], \quad s \geq t, \\ & P_s = X_{\zeta_i}, s\in[\zeta_i,\zeta_{i+1}), i\geq 2,\ X_{t-} = x,\ P_{s-} = p, s\in[t,\zeta_2], \\ & R_s = R(X_s, P_s),\ G_s = X_s - R_s, s\geq t. \end{aligned} \tag{3.2.5}$$

The first term in the objective function of (3.2.5) stands for the realization utility when a sale

[1] Here, we set $\zeta_1 := -\infty$.

τ_i occurs before the shock time (i.e., the terminal time) $\tilde{\tau}$. The second term is the utility of wealth at the terminal time if the agent is holding the risk-free asset then. The third term is the realization utility and the utility of wealth at the terminal time if the agent is holding the stock then. Note that in this case, the wealth that after the stock is sold is $(1 - k_s)X_{\tilde{\tau}-}$. The agent's wealth grows in the same way as the stock does when the agent is holding the stock, i.e., when $s \in [\zeta_i, \tau_i)$, and grows at the risk-free rate when the agent is holding the risk-free asset, i.e., when $s \in [\tau_i, \zeta_{i+1})$. In addition, the wealth decreases by factors k_s and $k_p/(1+k_p)$ at each sale and purchase times, respectively, due to the transaction cost. The purchase price is re-adjusted to the agent's wealth at each purchase time of the stock and remains at constant until the next purchase time.

On the other hand, if the agent is holding the risk-free asset right before time t, the agent's decision is $\{\mathcal{F}_s\}$-stopping times $t \leq \zeta_1 \leq \tau_1 \leq \zeta_2 \leq \tau_2 \leq \zeta_3 \leq \dots$, where ζ_k's and τ_k's are purchase and sale times of the stock, respectively. In this case, the agent faces the following optimal purchase problem[2]

$$
\begin{aligned}
\sup_{t\le\zeta_1\le\tau_1\le\zeta_2\le\tau_2\le\ldots} \quad & \mathbb{E}_t\Big[\sum_{i=1}^{\infty} e^{-\delta(\tau_i-t)}U(G_{\tau_i-},R_{\tau_i-})\mathbf{1}_{\{\tau_i<\tilde\tau\}} \\
& \quad +e^{-\delta(\tilde\tau-t)}U_W(X_{\tilde\tau-})\sum_{i=0}^{\infty}\mathbf{1}_{\{\tau_i<\tilde\tau\le\zeta_{i+1}\}} \\
& \quad +e^{-\delta(\tilde\tau-t)}\big(U_W((1-k_s)X_{\tilde\tau-})+U(G_{\tilde\tau-},R_{\tilde\tau-})\big)\sum_{i=1}^{\infty}\mathbf{1}_{\{\zeta_i<\tilde\tau\le\tau_i\}}\Big] \\
\text{subject to} \quad & dX_s=\sum_{i=1}^{\infty}\Big[(\mu X_s\,ds+\sigma X_s\,dW_s)\,\mathbf{1}_{\{s\in[\zeta_i,\tau_i)\}}+rX_s\,ds\mathbf{1}_{\{s\in[\tau_{i-1},\zeta_i)\}} \\
& \quad -k_sX_{s-}\mathbf{1}_{\{s=\tau_i\}}-\frac{k_p}{1+k_p}X_{s-}\mathbf{1}_{\{s=\zeta_i\}}\Big], \quad s\ge t, \\
& P_s=X_{\zeta_i}, s\in[\zeta_i,\zeta_{i+1}), i\ge 1,\ X_{t-}=x, \\
& R_s=R(X_s,P_s),\ G_s=X_s-R_s, \quad s\ge\tau_1.
\end{aligned}
\tag{3.2.6}
$$

Finally, the optimal liquidation and purchase problems can be formulated similarly when there is no shock time, i.e., when $\rho = 0$. Indeed, in this case, we simply replace the objective functions in (3.2.5) and (3.2.6) with $\mathbb{E}_t\Big[\sum_{i=1}^{\infty} e^{-\delta(\tau_i-t)}U(G_{\tau_i-},R_{\tau_i-})\mathbf{1}_{\{\tau_i<+\infty\}}\Big]$. Note that we allow the stopping times to take $+\infty$, which stands for the case in which the agent never purchases or voluntarily sells the stock.

Our model differs from those in Ingersoll and Jin [38] and Barberis and Xiong [6] mainly in two aspects: First, the agent in our model experiences terminal wealth utility in additional to realization utility. Second, the agent's reference point in our model adapts to the stock's prior gain and loss. A detailed comparison is provided in Table 3.1.

[2]Here, we set $\tau_0 = -\infty$.

Table 3.1: Comparison of the model settings in the present dissertation and in Ingersoll and Jin [38] and Barberis and Xiong [6].

Model Settings	The present dissertation	Ingersoll and Jin	Barberis and Xiong
Realization Utility Function	S-shaped	S-shaped	Piece-wise linear
Reference Point	Adapted to gains and losses	Purchase price	Purchase price growing with the risk free rate.
Shock Time	Yes	No	Yes
Terminal Wealth	Yes	No	No
Trading Strategies	Yes	Yes	Yes
Asset Pricing	Yes	No	Yes

3.3 Solution

3.3.1 Well-Posedness Condition

Denote $V(x,p)$ and $\bar{V}(x)$ as the optimal values of (3.2.5) and (3.2.6), respectively. We first show the condition for problems (3.2.5) and (3.2.6) to have finite optimal values. To this end, define $K := (1-k_s)/(1+k_p)$.

Proposition 3.3.1. *1. If $\delta+\rho > \max\{\beta r, \beta\mu - \frac{\beta(1-\beta)}{2}\sigma^2, \bar{\alpha}\mu - \frac{\bar{\alpha}(1-\bar{\alpha})}{2}\sigma^2\}$ and $K<1$, then $V(x,p) < +\infty$ and $\bar{V}(x) < +\infty$ for any $x \geq 0, p > 0$.*

2. *Suppose $\rho > 0$ and $\theta > 0$. If $\delta+\rho < \max\{\beta r, \beta\mu - \frac{\beta(1-\beta)}{2}\sigma^2, \bar{\alpha}\mu - \frac{\bar{\alpha}(1-\bar{\alpha})}{2}\sigma^2\}$, then $V(x,p) = \bar{V}(x) = +\infty$ for any $x > 0$ and $p > 0$.*

3. *Suppose $\rho = 0$. If $\delta+\rho < \max\{\beta r, \beta\mu - \frac{\beta(1-\beta)}{2}\sigma^2, \bar{\alpha}\mu - \frac{\bar{\alpha}(1-\bar{\alpha})}{2}\sigma^2\}$, then $V(x,p) = \bar{V}(x) = +\infty$ for any $x > 0$ and $p > 0$.*

4. *Suppose* $\rho > 0$ *and* $\theta = 0$. *If* $\delta + \rho < \bar{\alpha}\mu - \frac{\bar{\alpha}(1-\bar{\alpha})}{2}\sigma^2$, *then* $V(x,p) = \bar{V}(x) = +\infty$ *for any* $x > 0$ *and* $p > 0$.

Proposition 3.3.1 provides a sufficient condition for the optimal values of problems (3.2.5) and (3.2.6) to be finite, and this condition is nearly necessary in the case in which $\rho > 0$ and $\theta > 0$ and in the case in which $\rho = 0$. To understand this condition, we regard $\delta + \rho$, the sum of the discount rate of the agent's utility and the shock rate, as the effective discount rate of the agent. Suppose $\rho > 0$ and $\theta > 0$. We can consider βr to be the growth rate of the agent's terminal wealth utility if she holds the risk-free asset. For the optimal values of problems (3.2.5) and (3.2.6) to be finite, it is necessary that this growth rate is smaller than the effective discount factor. Similarly, $\beta\mu - \frac{\beta(1-\beta)}{2}\sigma^2$ and $\bar{\alpha}\mu - \frac{\bar{\alpha}(1-\bar{\alpha})}{2}\sigma^2$ can be understood as the growth rates of the agent's terminal wealth utility and realization utility, respectively, if she holds the stock. Thus, it is also necessary that these two growth rates are less than the effective discount rate.

In the case in which $\rho = 0$, for the optimal values of problems (3.2.5) and (3.2.6) to be finite, it is also necessary that the growth rate of the agent's realization utility $\bar{\alpha}\mu - \frac{\bar{\alpha}(1-\bar{\alpha})}{2}\sigma^2$ is smaller than the discount rate δ. On the other hand, βr needs to be smaller than δ as well; otherwise, the agent can hold the risk-free asset to let her wealth grow to infinity and finally invest in the stock to receive positive realization utility for each unit of her wealth. Similarly, $\beta\mu - \frac{\beta(1-\beta)}{2}\sigma^2$ must be smaller than δ too; otherwise, the agent can hold the stock to let her wealth grow to infinity.

Note that the optimal liquidation problem studied by Ingersoll and Jin [38] is a special case of our model with $\delta = 0$ and $\gamma_{\pm} = 0$. Ingersoll and Jin [38, Section C] provide a necessary condition under which the value of their problem is finite, but sufficient conditions are not provided.

In the case in which $\rho > 0$ and $\theta = 0$, there is a gap between the sufficient and necessary conditions, but the gap is small. Indeed, with reasonable parameter values (e.g., $\mu - \frac{1}{2}\sigma^2 \geq r$), we have $\max\{\beta r, \beta\mu - \frac{\beta(1-\beta)}{2}\sigma^2, \bar{\alpha}\mu - \frac{\bar{\alpha}(1-\bar{\alpha})}{2}\sigma^2\} = \bar{\alpha}\mu - \frac{\bar{\alpha}(1-\bar{\alpha})}{2}\sigma^2$ because $\bar{\alpha} \geq \beta$.

Finally, $1 - K$ stands for the transaction cost of buying and selling the stock once. The presence of transaction cost, i.e., assuming $K < 1$, prevents the agent from trading the stock for infinite number of times in a finite time interval.

In view of Proposition 3.3.1, the following assumption is in force throughout the rest of this chapter:

Assumption 3.3.1. *Assume $K < 1$ and $\delta + \rho > \max\{\beta r, \beta\mu - \frac{\beta(1-\beta)}{2}\sigma^2, \bar{\alpha}\mu - \frac{\bar{\alpha}(1-\bar{\alpha})}{2}\sigma^2\}$.*

3.3.2 Optimal Purchase Time

We first show the continuity of the value functions $V(x, p)$ and $\bar{V}(x)$ and the following dynamic programming principle.

Proposition 3.3.2. *$\bar{V}(x)$ and $V(x, p)$ are homogeneous of degree β. Moreover, $V(x, 1)$ is continuous in $x \geq 0$ and $|V(x, 1)| \leq C(1 + x^{\tilde{\alpha}}), \forall x \geq 0$ for some $C > 0$ and $\tilde{\alpha} \in [\bar{\alpha}, 1]$.*

Furthermore, the following dynamic programming principle holds:

$$V(x,p) = \sup_{\tau\geq t} \mathbb{E}_t\Big[e^{-\delta(\tau-t)}(U(G_{\tau-},R_{\tau-})+\bar{V}((1-k_s)X_{\tau-}))\mathbf{1}_{\{\tau<\tilde{\tau}\}}$$
$$+e^{-\delta(\tilde{\tau}-t)}(U(G_{\tilde{\tau}-},R_{\tilde{\tau}-})+U_W((1-k_s)X_{\tilde{\tau}}))\mathbf{1}_{\{\tilde{\tau}\leq\tau\}}\Big],$$
$$\text{subject to}\quad dX_s=\mu X_s\,ds+\sigma X_s\,dW_s,\ s\geq t,\ X_{t-}=x,$$
$$R_s=R(X_s,p), G_s=X_t-p, s\geq t,$$

(3.3.1)

$$\bar{V}(x) = \sup_{\zeta\geq t} \mathbb{E}_t\Big[e^{-\delta(\zeta-t)}V(\tfrac{1}{1+k_p}X_{\zeta-},\tfrac{1}{1+k_p}X_{\zeta-})\mathbf{1}_{\{\zeta<\tilde{\tau}\}}$$
$$+e^{-\delta(\tilde{\tau}-t)}U_W(X_{\tilde{\tau}-})\mathbf{1}_{\{\tilde{\tau}\leq\zeta\}}\Big], \quad (3.3.2)$$
$$\text{subject to}\quad dX_s=rX_s\,ds,\ s\geq t,\ X_{t-}=x.$$

Suppose the agent knows the optimal strategy if she is holding the risk-free asset, i.e., the optimal solution to problem (3.2.6). When the agent is holding the stock, she only needs to decide the first sale time τ. If τ is before the shock time $\tilde{\tau}$, the agent receives realization utility at τ, her wealth becomes $(1-k_s)X_{\tau-}$ in the risk-free asset after the sale of the stock, and the optimal aggregate utility she can receive afterwards is $\bar{V}((1-k_s)X_{\tau-})$. If the shock time occurs first, then the agent is forced to liquidate the stock and receives realization utility and terminal wealth utility. Similarly, if the agent knows the optimal strategy if she is holding the stock, she only needs to decide when to purchase the stock. Therefore, problems (3.3.1) and (3.3.2) are coupled.

In the following, we denote $\Theta := \rho\theta(1-k_s)^{\beta}/(\rho+\delta-\beta r)$.

Proposition 3.3.3. *The optimal value of problem* (3.3.2) *is*

$$\bar{V}(x) = (1 - k_s)^{-\beta} \max(K^{\beta} V(1,1), \Theta) x^{\beta}. \tag{3.3.3}$$

Furthermore,

1. *If* $V(1,1) > \Theta K^{-\beta}$, $\zeta = t$ *is optimal to problem* (3.3.2).
2. *If* $V(1,1) < \Theta K^{-\beta}$, $\zeta = +\infty$ *is optimal to problem* (3.3.2).
3. *If* $V(1,1) = \Theta K^{-\beta}$, *any stopping time* $\zeta \geq t$ *is optimal to problem* (3.3.2).

Proposition 3.3.3 shows that when holding the risk-free asset, it is either optimal to immediately re-purchase the stock or optimal to hold the risk-free asset forever. In other words, the stock is more valuable than the risk-free asset if and only if $V(1,1) > \Theta K^{-\beta}$, i.e., if and only if the optimal utility of holding the stock exceeds some threshold.

3.3.3 Liquidation at A Deep Loss

Note that the objective function in the optimal liquidation problem (3.2.5) is homogeneous in (x, p) of degree β for each fixed decision $t \leq \tau_1 \leq \zeta_2 \leq \dots$ Therefore, we can set $p = 1$ in problem (3.2.5) without loss of generality. In this regard, X_t stands for the wealth-purchase price ratio, and in view of Proposition 3.3.3, we can rewrite the optimal liquidation problem as

$$
\begin{aligned}
v(x) = \quad & \sup_{\tau \geq t} \quad \mathbb{E}_t\Big[e^{-\delta(\tau-t)}\left(\bar{u}(X_{\tau-}) + \max\left(K^\beta v(1), \Theta\right) X_{\tau-}^\beta\right)\mathbf{1}_{\{\tau<\tilde{\tau}\}} \\
& \qquad\qquad + e^{-\delta(\tilde{\tau}-t)}\left(\bar{u}(X_{\tilde{\tau}-}) + \theta(1-k_s)^\beta X_{\tilde{\tau}-}^\beta\right)\mathbf{1}_{\{\tilde{\tau}\leq\tau\}}\Big], \qquad (3.3.4) \\
& \text{subject to} \quad dX_s = \mu X_s\, ds + \sigma X_s\, dW_s,\ s \geq t,\ X_{t-} = x,
\end{aligned}
$$

where $v(x) := V(x, 1)$.

The following proposition shows that it is optimal to hold the stock at a deep loss.

Proposition 3.3.4. *There exist $\epsilon > 0$ such that it is optimal to hold the stock when the wealth-purchase price ratio X_t/P_t is less than ϵ. Moreover,*

$$
\lim_{x \downarrow 0} v(x) = \frac{\rho}{\delta+\rho}\bar{u}(0) = \frac{\rho}{\delta+\rho}(1-\gamma_-)^\beta u(-1).
$$

When the agent's position in the stock is at a deep loss, i.e., when the agent nearly loses all her investment in the stock, the agent's wealth is nearly zero. If the agent liquidates the stock in this case, she will experience negative realization utility immediately, reset the reference point, and experience realization utility and terminal wealth utility in the future. Because the agent's wealth is nearly zero, the utility in the future is also nearly zero, so the total utility of the agent if she liquidates the stock in deep loss is approximately the negative realization utility that will be immediately experienced by the agent at the liquidation time. If the agent does not liquidate the stock, e.g., holds the stock until the shock time, the only possible negative utility that will be experienced by the agent is at the shock time. This utility is less magnificent than the realization utility experienced by the agent if she liquidates the stock immediately for two reasons: First, the agent discounts utility over time (e.g., $\delta > 0$); Second, the possible loss at the shock time is unlikely to be larger than the paper loss today (because the agent already loses nearly all her wealth today). Therefore, liquidating the stock at a deep loss is always strictly suboptimal.

In the model proposed by Barberis and Xiong [6], the agents do not voluntarily sell at any loss level. In the model proposed by Ingersoll and Jin [38], the authors did not discuss specifically whether the agent sells at a deep loss.

3.3.4 Variational Inequality

We first derive the variational inequality satisfied by $v(\cdot)$ heuristically. Comparing two strategies, selling and not selling the stock in $[t, t+dt]$, and noting that $\tilde{\tau} < t+dt$ with probability ρdt, we conclude

$$\begin{aligned} v(x) = \max \Big\{ & \bar{u}(x) + x^{\beta} \max\big\{ K^{\beta} v(1) \, , \Theta \big\} \, , \\ & (1-\rho\, dt)\mathbb{E}_t[e^{-\delta\, dt} v(X_{t+dt})] + \rho\, dt[\bar{u}(x) + \theta(1-k_s)^{\beta} x^{\beta}] \Big\}. \end{aligned}$$

Because

$$\begin{aligned} (1-\rho\, dt)\mathbb{E}_t[e^{-\delta\, dt} v(X_{t+dt})] &= (1-\rho\, dt)(1-\delta\, dt)(v + \mu x v_x\, dt + \frac{1}{2}\sigma^2 x^2 v_{xx}\, dt) \\ &= v - (\rho+\delta) v\, dt + \mu x v_x\, dt + \frac{1}{2}\sigma^2 x^2 v_{xx}\, dt, \end{aligned}$$

We conclude that

$$0 = \min\Big\{v(x) - \big(\bar{u}(x) + x^{\beta}\max\big\{K^{\beta}v(1)\ ,\ \Theta\big\}\big), \\ -\frac{1}{2}\sigma^2 x^2 v_{xx}(x) - \mu x v_x(x) + (\rho+\delta)v(x) - \rho(\bar{u}(x) + \theta(1-k_s)^{\beta}x^{\beta})\Big\}. \tag{3.3.5}$$

In the following theorem, we prove that value function $v(x)$ of problem (3.3.4) is indeed the unique viscosity solution to the variational inequality (3.3.5).[3]

Theorem 3.3.1. *Value function $v(x)$ of problem (3.3.4) is the unique continuous viscosity solution with linear growth to variational inequality (3.3.5).*

3.4 Trading Strategies

3.4.1 Two Types of Agents

Adaptive Agents

In this section, we discuss the trading strategies of two types of agents: adaptive agents and naive agents. An adaptive agent knows today that her reference point in the future will adapt to the stock's gain and loss, and thus knows today that the gain and loss she will experience in the future are benchmarked to this adaptive reference point. Therefore, the adaptive agent's trading strategy follows the optimal solutions to problems (3.2.5) and (3.2.6).

The optimal purchase time of the stock for the adaptive agent has been solved in Proposition 3.3.3: it is either optimal to immediately buy the stock or optimal never to buy it. The optimal sale time of the stock for the adaptive agent is the solution to problem (3.3.4). Recall

[3] For the notion of viscosity solutions, one can refer to Crandall, Ishii and Lions [24].

that the value function $v(x)$ of problem (3.3.4) is continuous and is the unique viscosity solution to the variational inequality (3.3.5). In addition, function $\bar{u}(x) + \max\big(K^{\beta}v(1)\ ,\Theta\big)x^{\beta}$, which stands for the immediate payoff if the agent sells the stock, is also continuous. Denote $\mathcal{H}$ as the holding region, i.e.,

$$\mathcal{H} := \{x > 0 | v(x) > \bar{u}(x) + \max\big(K^{\beta}v(1)\ ,\Theta\big)x^{\beta}\}.$$

Then, the standard optimal stopping theory shows that the optimal sale time of the stock, i.e., the optimal solution to problem (3.3.1), is the first time when the wealth-purchase price ratio X_t/P_t exits the holding region.

Figure 3.1 illustrates the value function $v(x)$ and the holding region; the parameter values used for this figure are provided in Section 3.4.2. In this illustration, the holding region is disconnected: it consists of a neighbourhood of 0 and a neighbourhood of 1. In other words, the agent holds the stock if the stock is at a deep loss (corresponding to the neighbourhood of 0) and if the stock's gain and loss are very small (corresponding to the neighbourhood of 1). Note that the wealth-purchase price ratio X_t/P_t is 1 at the purchase time and evolves continuously in time. Therefore, the optimal sale time, which is the first time when the wealth-purchase price ratio exits the holding region, becomes the first time for this ratio to exit the neighbourhood of 1; the other part of the holding region, i.e., the neighbourhood of 0, does not play a role.

In general, we define

$$x_u := \sup\{x \geq 1 | [1, x) \subseteq \mathcal{H}\}, \quad x_d := \inf\{x \leq 1 | (x, 1] \subseteq \mathcal{H}\}.$$

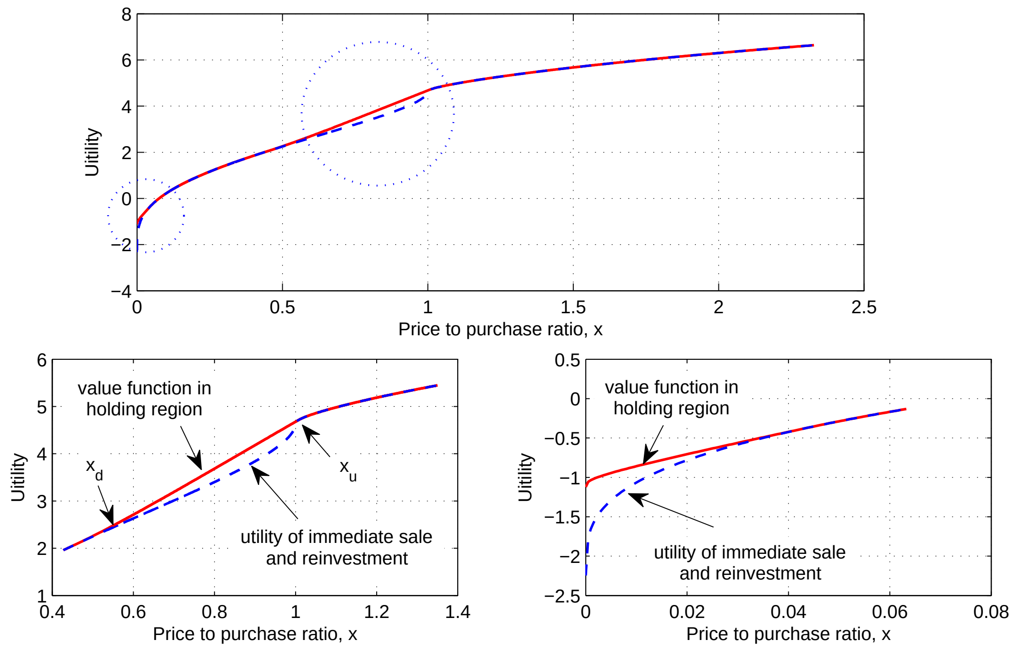

Figure 3.1: Value function, immediate payoff, and holding region. The solid line stands for the optimal value function $v(x)$ of problem (3.3.4) and the dashed line, which is partially covered by the solid line, stands for the immediate payoff function $\bar{u}(x) + \max\left(K^\beta v(1)\,, \Theta\right) x^\beta$. Two graphs in the lower panel are zoom-ins of the circled areas in the upper graph. The holding region is the area in which the dashed line is strictly below the solid line, and it consists of two intervals, one in the neighbourhood of 1 and the other in the neighbourhood of 0. The upper and lower liquidation points x_u and x_d are the end points of the interval in the neighbourhood of 1.

Then, the optimal sale time is the first time when the wealth-purchase price ratio exits the interval (x_d, x_u), and we call x_u and x_d the upper and lower liquidation points, respectively. Figure 3.1 illustrates the upper and lower liquidation points; in that case, the holding region is disconnected, so we have $x_d > 0$. If the holding region is connected, then $x_d = 0$, because it is optimal to hold the stock at a deep loss.

Naive Agents

For a naive agent, her reference point in the future will also adapt to the stock's gain and loss, but she doesn't realize it prospectively: the naive agent wrongly believes that her reference point will remain constant over time. At each purchase time ζ, the naive agent's reference point is the purchase price. The naive agent then believes that her reference point in the future (before selling the asset) will still be this purchase price. Consequently, she believes that her optimal strategy is the optimal solution to problem (3.3.1) with $\gamma_+ = \gamma_- = 0$ (so that the reference point does not adapt to the stock's gain and loss as believed by the agent). Denote the corresponding upper and lower liquidation points with $\gamma_+ = \gamma_- = 0$ as x_u^0 and x_d^0, respectively. Then, the naive agent believes that it is optimal to sell the stock when the wealth-purchase price ratio $\{X_s/P_s\}$ first exits (x_u^0, x_d^0). In particular, the naive agent believes, at the purchase time ζ, that she will sell the stock at time $t > \zeta$ if and only if $X_t/P_t \notin (x_u^0, x_d^0)$.

Now, consider some time t in the future when the naive agent is still holding the stock. Suppose the naive agent re-examines whether she needs to sell the stock. At that time, the naive agent's reference point actually becomes $R_t = P_t + \gamma_+(X_t - P_t)\mathbf{1}_{\{X_t \geq P_t\}} + \gamma_-(X_t - P_t)\mathbf{1}_{\{X_t < P_t\}}$, which was not expected by the agent at the purchase time but is recognized by the agent retrospectively at time t. However, the naive agent still holds the wrong belief that her reference point will remain constant in the future. Consequently, she believes that her optimal strategy is the optimal solution to problem (3.3.1) with $\gamma_+ = \gamma_- = 0$ and with $p = R_t$; i.e., she believes that it is optimal to sell the stock when the wealth-reference price ratio $\{X_s/R_s\}$ first exits (x_u^0, x_d^0). In particular, the naive agent sells the stock at time t if and only if $X_t/R_t \notin (x_u^0, x_d^0)$

Note that the strategy planned by the naive agent at the purchase time ζ is different from her strategy at the future time t when she re-examines the liquidation problem: at time ζ the agent believes that it is optimal for her to sell the stock at time t if and only if $X_t/P_t \notin (x_u^0, x_d^0)$, but at time t, she actually sells the stock if and only if $X_t/R_t \notin (x_u^0, x_d^0)$. Note that $R_t = P_t + \gamma_+(X_t - P_t)\mathbf{1}_{\{X_t \geq P_t\}} + \gamma_-(X_t - P_t)\mathbf{1}_{\{X_t < P_t\}}$ is different from P_t. Therefore, the naive agent is inconsistent in the liquidation strategy between herselves at different times.

We assume the naive agent re-examines the optimal liquidation problem at any time after the purchase time. We further assume that the naive agent is extremely naive in the sense that she knows *retrospectively* that her reference point has adapted to the stock's prior gain and loss but never realizes *prospectively* the adaptation of her reference point to the stock's gain and loss in the future. We then study the strategy that is actually implemented by the naive agent. From the above discussion, we observe that the naive agent implements the following strategy: sell the stock at the first time when X_t/R_t exits (x_u^0, x_d^0). Denote

$$x_u^n := +\infty \mathbf{1}_{\{\gamma_+ \geq 1/x_u^0\}} + x_u^0 \frac{1-\gamma_+}{1-\gamma_+ x_u^0} \mathbf{1}_{\{\gamma_+ < 1/x_u^0\}}, \tag{3.4.1}$$

$$x_d^n := x_d^0 \frac{1-\gamma_-}{1-\gamma_- x_d^0} \mathbf{1}_{\{\gamma_- < 1/x_d^0\}}. \tag{3.4.2}$$

Eventually, the naive agent sells the stock at the first time when X_t/P_t exits (x_u^n, x_d^n).

Dynamic Inconsistency

Note that the inconsistency in the trading strategy between the selves of the naive agent at different times arises from the adaptation of the reference point to the stock's gain and loss. Inconsistent dynamic decisions have been extensively observed and studied in the literature. For instance, in a decision problem formulated by O'Donoghue and Rabin [43], the agent exhibits dynamic inconsistency due to hyperbolic discounting. In a casino gambling model proposed by Barberis [3], the gambler is dynamically inconsistent regarding the gambling strategy because of probability weighting. In Basak and Chabakauri [7], the authors show that a mean-variance maximizer is dynamically inconsistent. As far as we know, our work is the first one to show that time-varying reference points can also lead to dynamic inconsistency.

The naive agent in our model seems to be unrealistic because she knows *retrospectively* that her reference point has changed but never realizes *prospectively* that the reference point will adapt to the stock's gain and loss. However, this agent is the same type of naive agents as defined and studied in the dynamic inconsistency literature; see e.g., O'Donoghue and Rabin [43] and Barberis [3]. On the other hand, another type of agents, named sophisticated agents, are studied in the literature; see for instance O'Donoghue and Rabin [43] (where these agents are called sophisticates) and Barberis [3] (where these agents are called sophisticated agents without pre-commitment). The adaptive agent is related to but different from sophisticated agents. Both these two types of agents anticipate the change of their preferences in the future and thus make their decisions today accordingly. In addition, their strategies are both dynamically consistent. For a sophisticated agent, she realizes her decision in the future does not follow the optimal plan set up today. By anticipating the strategy she will implement in the future, which is not optimal today, the sophisticated agent revises her strategy today accordingly. For the adaptive agent in our model, she realizes prospectively the reference point adaptation in the future, and thus anticipates correctly the gain and loss that she will actually experience in the future. As a result, she makes the decision today based on this anticipation. Note that the adaptive agent does not need to anticipate the decisions of her selves in the future in order to make a decision today.

Finally, another type of agents, named pre-committing agents, are studied in the literature; see for instance Barberis [3] (where these agents are called sophisticated agents with pre-commitment). A pre-committing agent realizes that the optimal strategy set up today is no longer optimal to her future selves, but she is able to commit future selves to today's optimal strategy usually with the help of commitment device. Such agents are unrealistic in our stock liquidation problem, so we do not consider them. Indeed, in our problem, if an agent realizes that her reference point in the future will adapt to the stock's gain and loss, she should realize immediately that the gain and loss that she will experience are benchmarked to the adaptive reference point. Thus, she will not commit herself to follow the strategy that is optimal under the wrong assumption that the reference point does not adapt to the stock's gain and loss.

3.4.2 Comparison of Trading Strategies

In this section we numerically compute the trading strategies of the naive and adaptive agents. Table 3.2 lists the default values we use for the model parameters. Here, we as-

Table 3.2: Values for model parameters.

μ	σ	r	k_p	k_s	δ	ρ	θ
9%	30%	3%	1%	1%	5%	0.1	1
α_G	α_L	b_G	b_L	λ	β	γ_+	γ_-
0.5	0.5	0	0	2.5	0.3	0.6	0.3

sume that the realization utility function is as in (3.2.3). Note that we use the same values as in Ingersoll and Jin [38] for parameters μ, σ, δ, k_p, k_s, α_G, α_L, b_G, b_L, β, and λ. On the other hand, Ingersoll and Jin [38] do not consider the shock time, terminal wealth utility, and reference point adaptation, so parameters ρ, θ, and $\gamma_\pm$ are not present therein. We imply the values for $\gamma_\pm$ from the experimental results in Arkes et al. [1]: in that paper, the authors designed several experiments to test the subjects' reference point adaptation to $\$6$ gain or loss. In the gain situation, the reference point shifted upwards by an amount between \$3.8 to \$5.8 and in the loss situation, the reference point shifted downwards by an amount between \$1.5 to \$3.1. The choice of $\gamma_+ = 0.6$ and $\gamma_- = 0.3$ is consistent with these experimental results. Following Barberis and Xiong [6], we set $\rho = 0.1$, so that the average shock time is 10 years. Finally, there is no clear benchmark for the value of θ. We set $\theta = 1$ for illustration purpose only.

Recall that the strategy implemented by the naive agent is to sell the stock when X_t/P_t exits (x_u^n, x_d^n) and the strategy of the adaptive agent is to sell the stock when X_t/P_t exits (x_u, x_d). Figure 3.2 illustrates the upper and lower liquidation points for the strategies of the naive agent (dashed lines) and of the adaptive agent (solid lines) with respect to μ, σ, δ, k_p, k_s, α_G, α_L, β, and λ, respectively. To compare our results to the literature such as Barberis and Xiong [6] and Ingersoll and Jin [38], we also compute the strategy of a *non-adaptive agent* whose reference point does not adapt to the stock's gain and loss (i.e., whose $\gamma_\pm = 0$), and plot the corresponding liquidation points x_u^0 and x_d^0 in Figure 3.2 (using dotted lines). As discussed in Section 3.4.1, this strategy is the same as the one planned by the naive agent immediately after she purchases the stock.

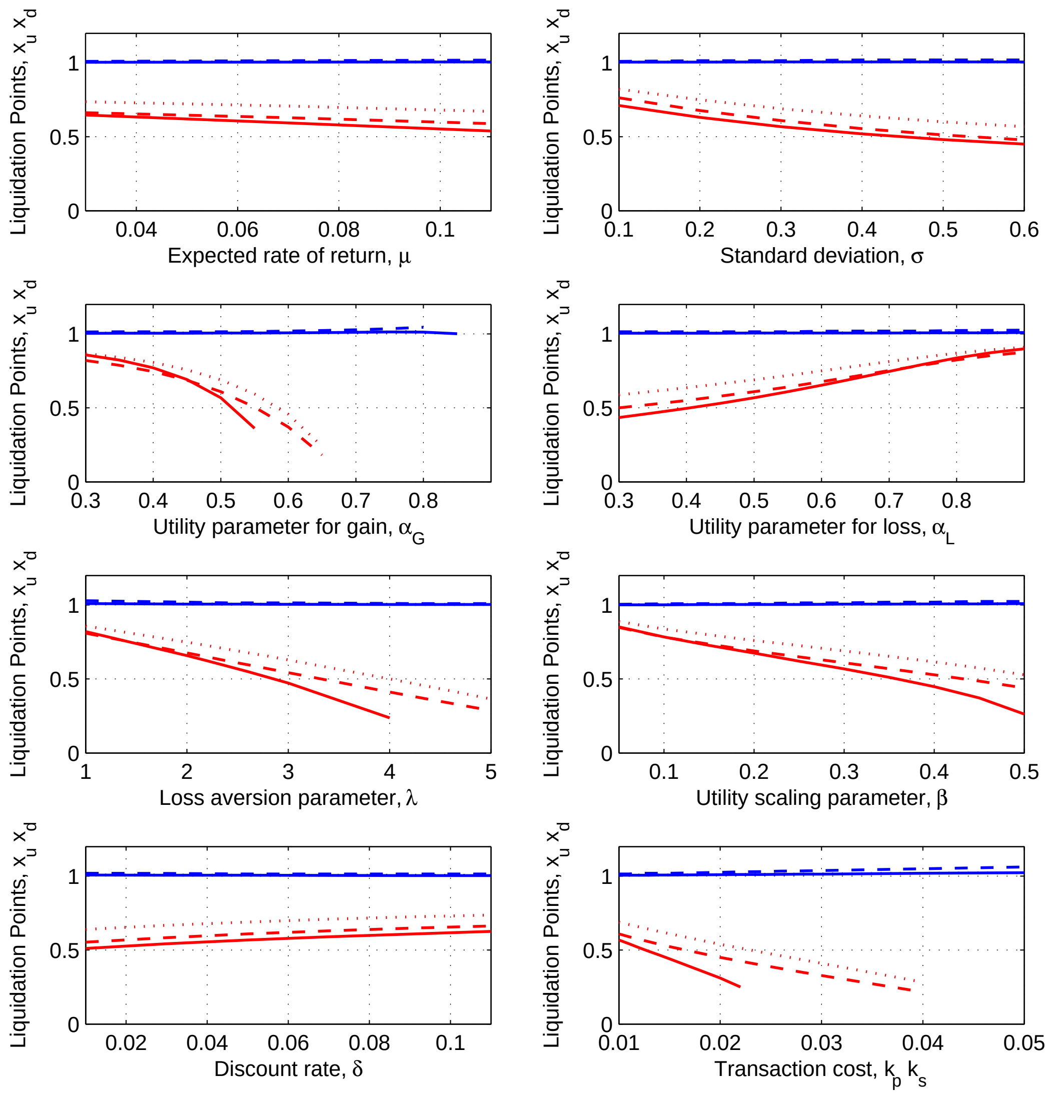

Figure 3.2: Upper and lower liquidation points for the naive agent (dashed lines), adaptive agent (solid lines), and non-adaptive agent (dotted lines) with respect to parameters μ, σ, δ, k_p, k_s, α_G, α_L, β, and λ. The model parameters take values in Table 3.2.

In each panel of Figure 3.2, we study the sensitivity of the liquidation points with respect to one parameter while assuming default values for other parameters. For all the parameter values used in Figure 3.2, Assumption 3.3.1 is satisfied. In some panels, there is a break point of the curve of the lower or upper liquidation point with respect to certain parameter, and this indicates that there is no lower or upper liquidation point when the parameter value is beyond this break point.

We observe from Figure 3.2 that when comparing the strategies of the naive and adaptive agents, except for the cases of a small α_G and a large α_L, both of the lower and upper liquidation points for the adaptive agent are lower than that for the naive agent. In other words, the adaptive agent sells the stock more frequently at a gain and less frequently at a loss than the naive agent does.

In Figure 3.2, the upper liquidation point of the non-adaptive agent (whose $\gamma_{\pm} = 0$) is almost identical to and thus covered by the upper liquidation point of the adaptive agent. We observe from Figure 3.2 that the naive agent sells the stock less frequently than the non-adaptive agent does both at a gain and at a loss. This can also be seen analytically from (3.4.1) and (3.4.2). The adaptive agent sells the stock at a loss less frequently than the non-adaptive agent does as well.

The sensitivity of the naive, adaptive, and non-adaptive agents' strategies with respect to parameters μ, σ, δ, k_p, k_s, α_G, α_L, β, and λ is similar to the findings in Ingersoll and Jin [38, Figure 5]. First, the upper liquidation point is close to one and the lower liquidation point is much smaller than one and is even zero, showing that the agents sell the stock at a gain much more frequently than at a loss. Second, the upper liquidation point is much less affected by parameter changes than the lower liquidation point. Third, the lower liquidation point is decreasing with respect to μ, σ, α_G, λ, β, k_p, and k_s, and is increasing with respect to α_L and δ. In other words, the agents hold the stock at a loss longer if the stock has a higher expected return or a higher volatility, or if the agents are less risk averse with respect to gains, more risk seeking with respect to losses, more loss averse, less risk averse with respect to terminal wealth, or more patient, or if the transaction cost is higher.

In panel 3 of Figure 3.2, we observe that it is optimal for the adaptive and naive agents not to sell the stock at a gain if α_G is close to one. To explain this observation, we consider an agent whose $\alpha_G = 1$ and α_L is much smaller than 1. We further assume $\gamma_\pm = 0$ for simplicity. If the agent sells the stock at a gain, she will reset her reference point. Because $\alpha_G = 1$, the increase in the realization utility due to \$1 increase in the stock price is the same before and after the reference point is reset. The reduction in the realization utility due to \$1 decrease in the stock price, however, is much larger after the reference point is reset. This is because α_L is small and thus the sensitivity of the realization utility with respect to \$1 loss is much higher when the agent is at the break-even point than when she is at a gain. As a result, the agent is unwilling to reset the reference point, i.e., unwilling to sell the stock, when the stock is at a gain. In the models proposed by Barberis and Xiong [6] and Ingersoll and Jin [38], it is assumed that $\alpha_G = \alpha_L = 1$ and that $\rho = 0$, respectively, so it is optimal for the agent in their models to voluntarily sell at certain gain.

Next, we discuss the impact of parameters $\gamma_\pm$, θ, ρ, and r on trading strategies. Note that such impact is not discussed in Barberis and Xiong [6] or Ingersoll and Jin [38]. Similar as in Figure 3.2, we illustrate the naive, adaptive, and non-adaptive agents' strategies by plotting the corresponding upper and lower liquidation points with respect to $\gamma_\pm$, θ, ρ, and r in Figure

3.3 (dash lines for the naive agent's strategy, solid lines for the adaptive agent's strategy, and dotted lines for the non-adaptive agent's strategy). We have the following observations:

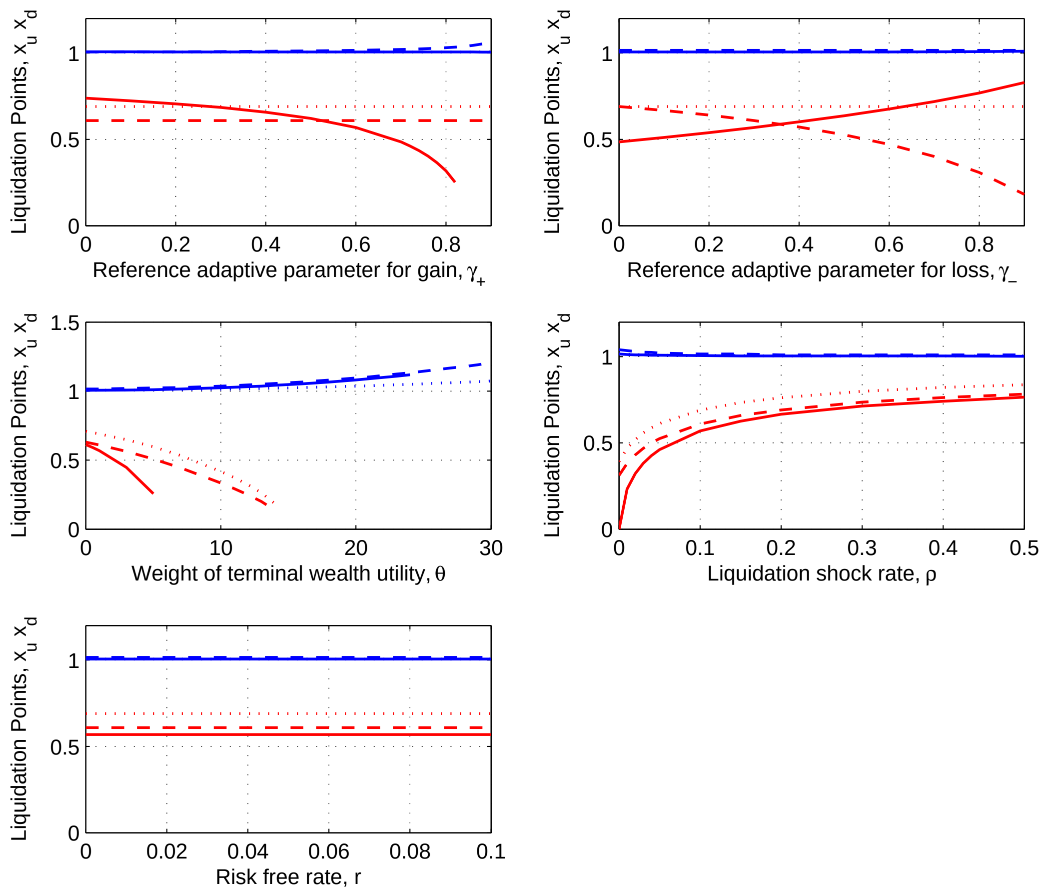

Figure 3.3: Upper and lower liquidation points for the naive agent (dashed lines), adaptive agent (solid lines), and non-adaptive agent (dotted lines) with respect to parameters $\gamma_{\pm}$, θ, ρ, and r. The model parameters take values in Table 3.2.

First, for the naive agent's strategy, the lower liquidation point is independent of γ_+ while the upper liquidation point is increasing with respect to γ_+. This can be seen from (3.4.1) and (3.4.2). Similarly, the lower liquidation point is decreasing with respect to γ_- and the upper liquidation point is independent of γ_-.

Second, for the adaptive agent, the lower liquidation point is decreasing with respect to

γ_+ and increasing with respect to γ_-. If the agent voluntarily sells the stock to realize a loss, she will experience negative realization utility and then reset the reference point. Note the agent's realization utility is less sensitive with respect to \$1 increase in the stock price when she is at a loss than when she is at the break-even point because the diminishing sensitivity of S-shaped utility functions. Thus, the agent is willing to sell the stock so as to reset the reference point. With a larger γ_+, the sensitivity of the agent's realization utility with respect \$1 increase in the stock price, when she is at the break-even point, becomes smaller, so the agent is less motivated to reset the reference point. Consequently, the agent is less willing to sell the stock at a loss. Similarly, with a larger γ_-, the sensitivity of the agent's realization utility with respect \$1 increase in the stock price, when she is at a loss, becomes smaller, so the agent is more willing to sell the stock at a loss and to reset the reference point.

Third, with a large γ_+ and a small γ_-, both the upper and lower liquidation points are smaller for the adaptive agent than for the naive agent. In other words, compared to the naive agent, the adaptive agent sells the stock more frequently at a gain and less frequently at a loss. Note that for most individuals, their reference points adapt more to stock gains than to stock losses, i.e., γ_+ is large and γ_- is small. On the other hand, with a small γ_+ and a large γ_-, which, however, is not true for a typical investor, the adaptive agent sells the stock at a loss more frequently than the naive agent does. When γ_+ becomes even smaller and γ_- becomes even larger, the adaptive agent sells the stock at a loss more frequently than the non-adaptive agent does as well.

Fourth, a larger θ makes all of the adaptive, naive, and non-adaptive agents less willing to sell the stock, i.e., leads to a wider holding region. Note that trading the stock boosts the realization utility but reduces the terminal wealth utility because of transaction costs. When θ is larger, the agents focus more on the terminal wealth utility, so they trade the stock less frequently. When θ exceeds certain threshold, the adaptive agent does not voluntarily sell the stock at all.

Fifth, the lower liquidation boundary of each of the adaptive, naive, and non-adaptive agents is increasing with respect to ρ. Note that if the agent does not sell the stock at any loss, she will hold the stock until voluntarily selling it at a gain or forcefully selling it at the shock time. The sale at the shock time can be possibly at a loss. With a smaller ρ, the shock time becomes longer, and thus the agent experiences less realization disutility of the loss at the shock time because of discounting. Therefore, the agent becomes more willing to hold the stock at a loss today.

Finally, the risk-free rate r has no effect on the upper and lower liquidation points. Indeed, with the parameter values used here, the agent re-purchases the stock immediately after selling it, so the return rate of the risk-free asset does not affect the agent's liquidation policy.

3.5 An Asset Pricing Model

Following Barberis and Xiong [6], we consider a market in which the participants are homogeneous adaptive agents. We define the *equilibrium expected return* of the stock for adaptive agents as the value of μ such that the adaptive agents in the market are indifferent between the stock and the risk-free asset. According to Theorem 3.3.3, this is the case if and only if

$$v(1) = K^{-\beta}\Theta. \tag{3.5.1}$$

We can then solve the equilibrium expected return for adaptive agents from (3.5.1).

Similarly, we can also consider a market with homogeneous naive agents, and define the equilibrium expected return of the stock for naive agents as the value of μ such that the naive agents in the market are indifferent between the stock and the risk-free asset. Recall that at the purchase time, a naive agent believes that her reference point does not change in the future and plans her strategy accordingly. Thus, when deciding whether to purchase the stock, the naive agent also believes that her reference point does not change in the future. Consequently, the equilibrium expected return of the stock for naive agents can be solved from (3.5.1) by setting $\gamma_+ = \gamma_- = 0$.

In the following, we compute the equilibrium risk premium of the stock, i.e., the equilibrium expected return of the stock in excess of the risk-free rate, for naive and adaptive agents. We use the same parameter values as in Table 3.2, except that we reset $\theta = 15$ for illustration purpose. Figure 3.4 shows the risk premium, denoted as μ_{ex}, as functions of the stock volatility for different values of $\gamma_\pm$ and θ. We have the following observations:

First, the risk premium required by an adaptive agent is increasing with respect to γ_+ and decreasing with respect to γ_-. With a larger γ_+, the same increase in the stock price leads to less realization utility of the agent. Consequently, the stock becomes less attractive to the agent and thus is rewarded a higher risk premium in equilibrium. Similarly, with a larger γ_-,

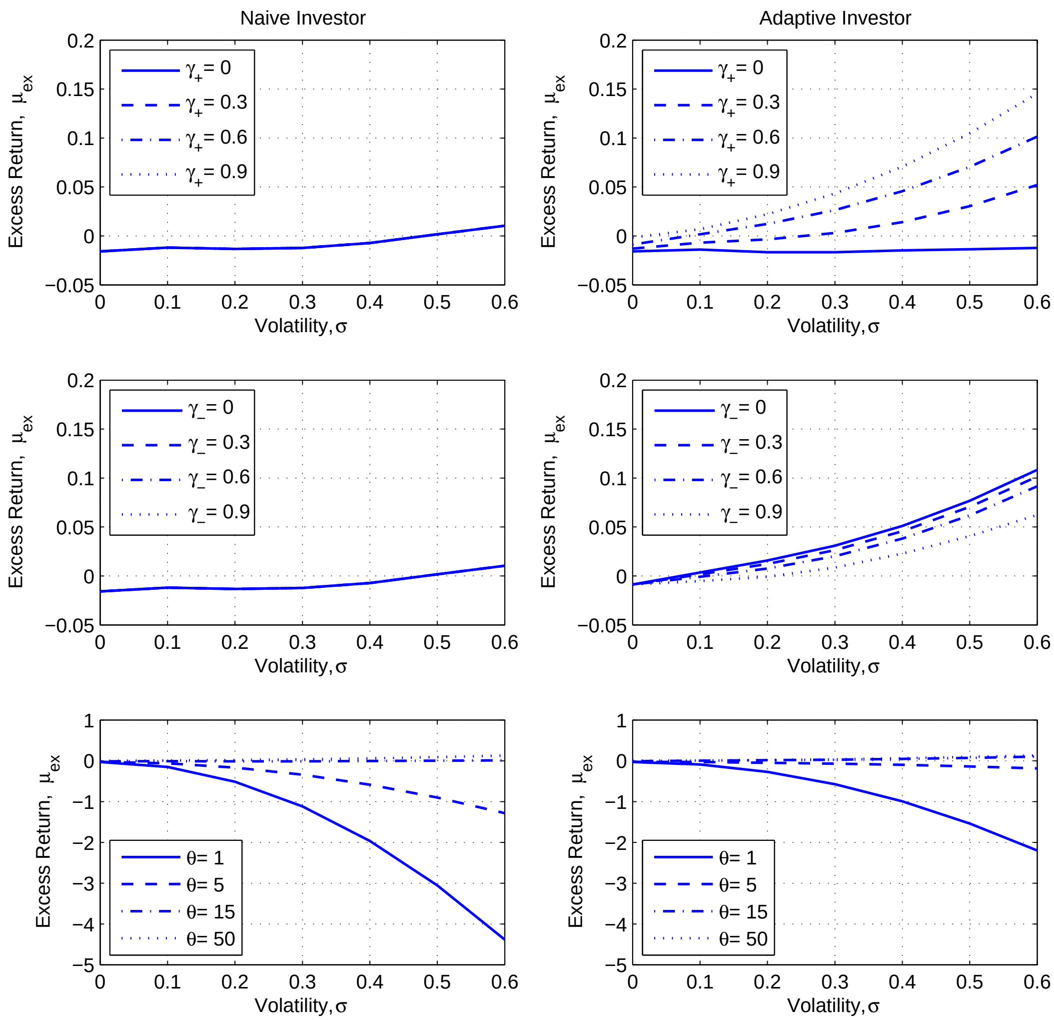

Figure 3.4: Equilibrium risk premium from the perspectives of naive agents (left panel) and adaptive agents (right panel) with respect to parameters $\gamma_{\pm}$ and θ. The default values for model parameters are given as in Table 3.2 except that $\theta = 15$.

the adaptive agent experiences less realization disutility of a loss of the stock, so the stock is more attractive and thus only rewarded with a lower risk premium.

Second, for reasonable values of $\gamma_{\pm}$, i.e., for $\gamma_{+} > \gamma_{-}$, adaptive agents ask for a higher risk premium than naive agents do. When γ_{+} is much smaller than γ_{-}, e.g., $\gamma_{+} = 0$ and $\gamma_{-} = 0.3$, adaptive agents ask for a lower risk premium than naive agents do.

Third, the risk premium is increasing with respect to θ for both naive and adaptive agents. A larger θ implies that the agents are more concerned with their terminal wealth. Because the agents are more risk averse regarding terminal wealth than regarding trading gains and losses, focusing more on the terminal wealth leads to a higher risk premium.

Fourth, the risk premium can be negative for both naive and adaptive agents. Each of the naive and adaptive agents experiences terminal wealth utility and realization utility. Since an agent is risk averse with respect to her terminal wealth, the negative risk premium is a result of the agent's risk seeking behavior related to the realization utility of trading gains and losses. Negative risk premium are also observed in Barberis and Xiong [6].

Fifth, the risk premium can be increasing or decreasing with respect to the stock volatility, depending on the type of agents in the market and on the values of other parameters.[4] In Barberis and Xiong [6], the risk premium is negative and decreasing with respect to the stock volatility. Thus, our model generates more patterns of return-risk tradeoffs for stocks. Actually, with certain parameter values, our model can also generate a risk premium curve that is first increasing and then decreasing with respect to the stock volatility, and the risk premium is positive for high-volatility stocks, see Figure 3.5.

Finally, we illustrate the sensitivity of the risk premium with respect to other parameters in Figures 3.6–3.7.

[4] In Figure 3.4, the risk premium is negative when the stock's volatility tends to zero. This is because in our model the agent's reference point is the purchase price of the stock without growing at the risk-free rate. As a result, if the stock has a positive return, which can be lower than the risk-free return, the agent still experiences positive realization utility. When holding the risk-free asset, however, the agent does not experience any realization utility. Consequently, a zero-volatility stock can have negative risk premium.

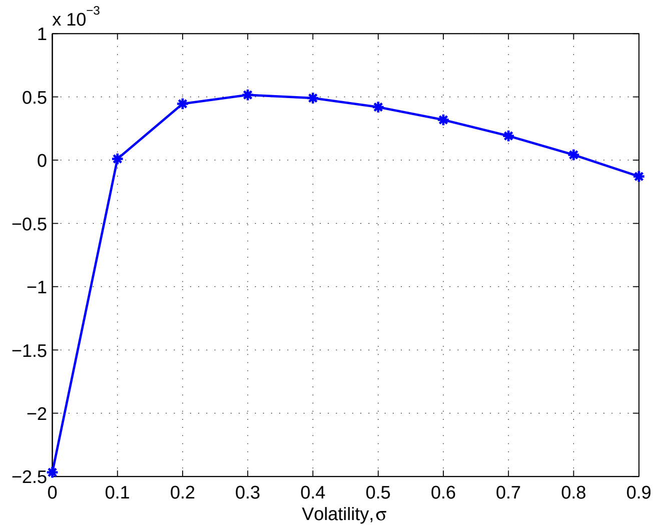

Figure 3.5: Equilibrium risk premium under parameter values $\mu = 9\%$, $\sigma = 30\%$, $r = 3\%$, $k_p = k_s = 1\%$, $\delta = 5\%$, $\rho = 0.05$, $\theta = 52$, $\alpha_G = \alpha_L = 0.35$, $b_G = b_L = 0$, $\lambda = 1$, $\beta = 0.3$, $\gamma_+ = \gamma_- = 0$. Because we choose $\gamma_+ = \gamma_- = 0$, the naive and adaptive agents are the same.

3.6 Extensions

3.6.1 Transaction Cost Effect

In the previous analysis, we use the post-transaction cost purchase price P_t and pre-transaction cost wealth X_t in the calculation of the gain and loss G_t experienced by the agent. In other words, we assume that the agent leaves out the transaction cost of buying and selling the stock in assessing her gain and loss. In the following, we discuss the cases in which the transaction cost is taken into account in the calculation of the agent's gain and loss.

Denote $\tilde{P}_t := (1 + k_p)P_t$ as the pre-transaction cost purchase price and $\tilde{X}_t := (1 - k_s)X_t$ as the post-transaction cost wealth. None, or one, or both of the buying cost and selling cost of the stock can be taken into account when the agent assesses her gain and loss. Thus, we have four combinations, as listed below.

1. Post-transaction cost purchase price P_t and pre-transaction cost wealth X_t.

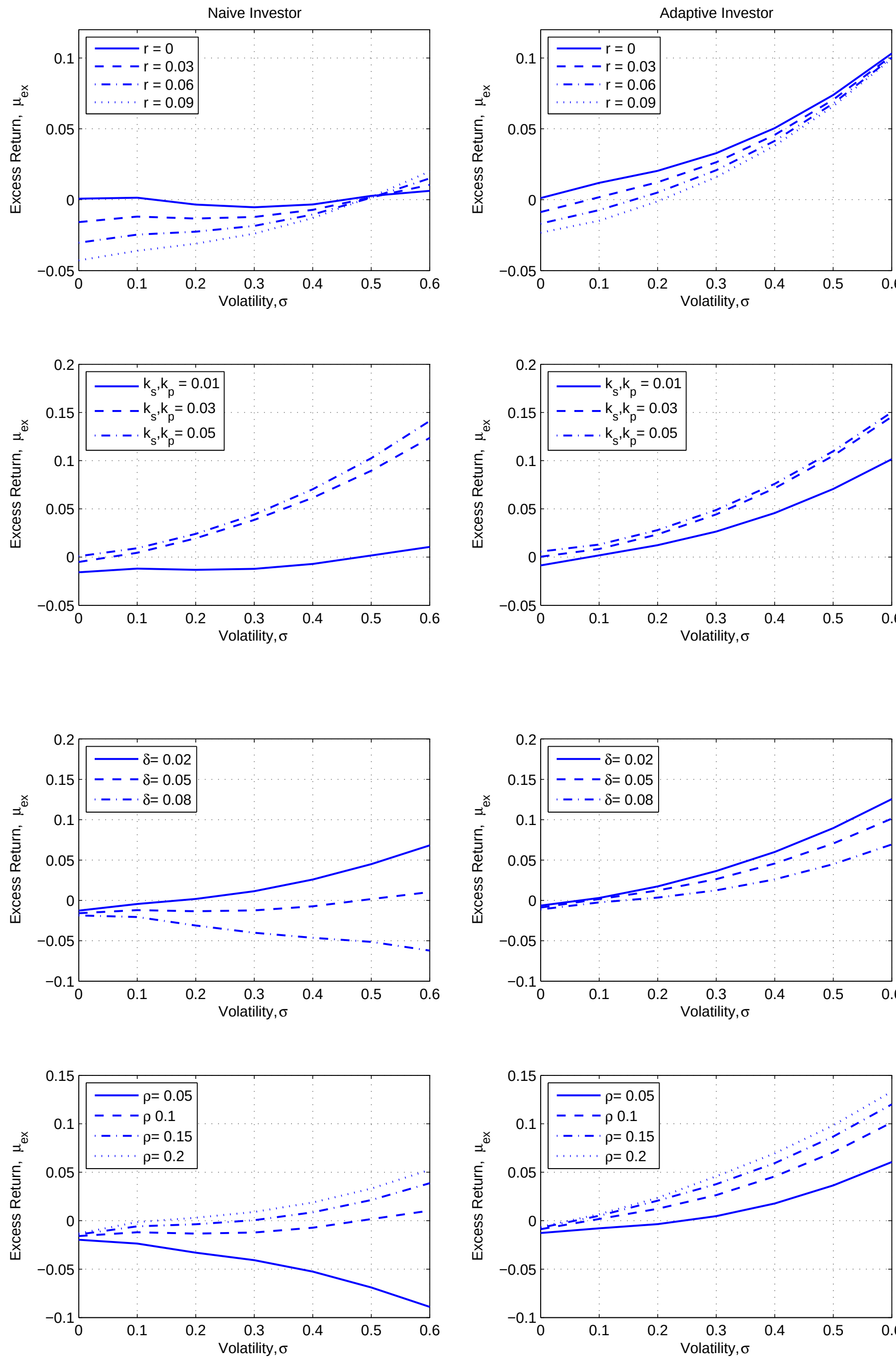

Figure 3.6: Equilibrium risk premium from the perspectives of naive agents (left panel) and adaptive agents (right panel) with respect to parameters r, k_p, k_s, δ, and ρ. The default values for model parameters are given as in Table 3.2 except that $\theta = 15$.

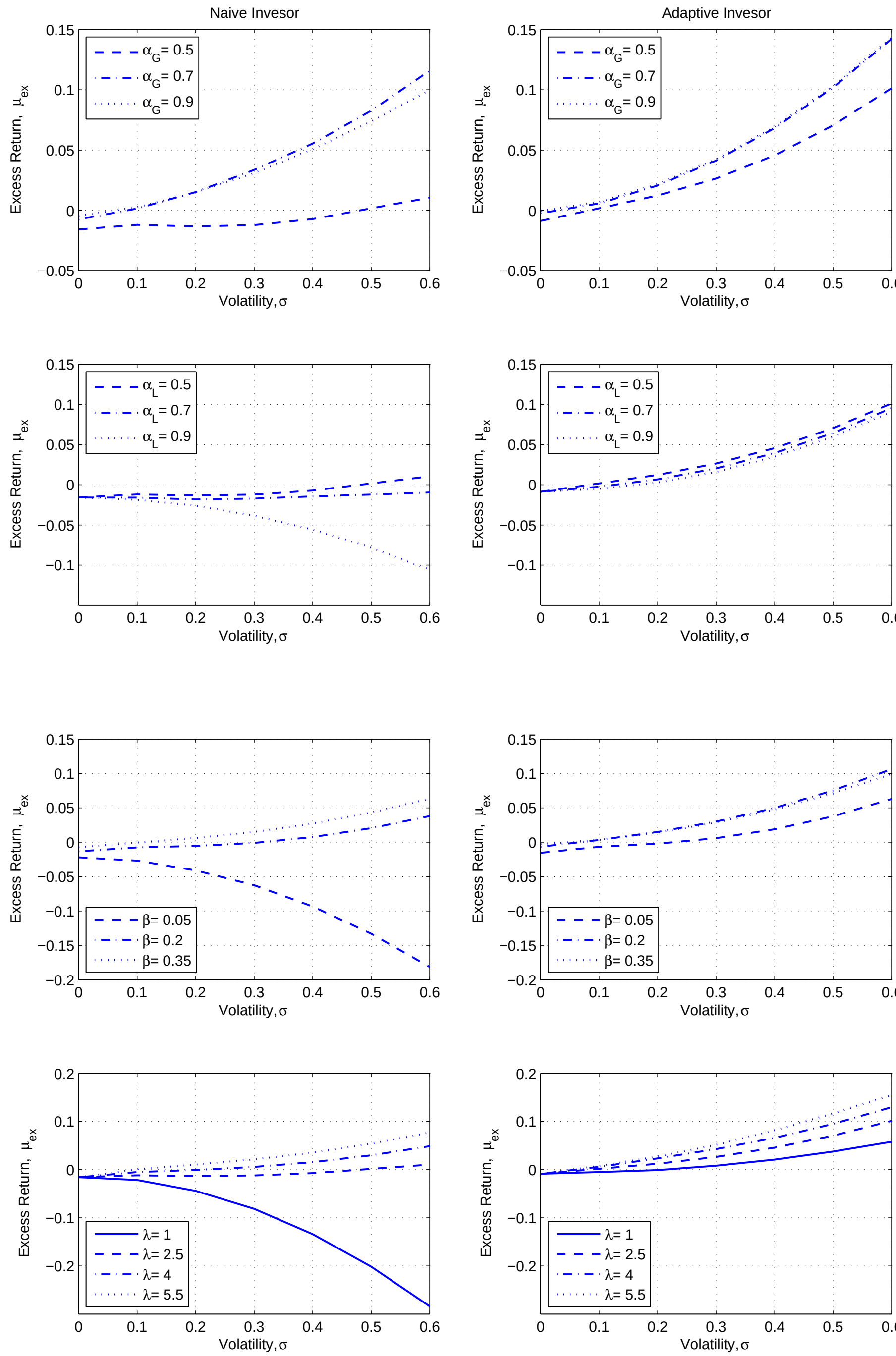

Figure 3.7: Equilibrium risk premium from the perspectives of naive agents (left panel) and adaptive agents (right panel) with respect to parameters α_G, α_L, β, and λ. The default values for model parameters are given as in Table 3.2 except that $\theta = 15$.

In this case, the agent ignores the buying and selling costs of the stock in assessing her gain and loss. Consequently, a sale immediately after a purchase has zero realization utility. This case is explicitly modeled in section 3.2 and analyzed in the previous sections.

2. Pre-transaction cost purchase price $\tilde{P}_t$ and pre-transaction cost wealth X_t.

   In this case, the agent ignores the selling cost but takes the buying cost into account when assessing her gain and loss. In other words, the reference point R_t is given as $R(X_t, \tilde{P}_t)$ in the calculation of the agent's gain and loss. The realization utility can then be written as

$$U(G_t, R_t) = \tilde{P}_t^{\beta}\bar{u}(X_t/\tilde{P}_t) = (1+k_p)^{\beta}P_t^{\beta}\bar{u}((1+k_p)^{-1}X_t/P_t). \tag{3.6.1}$$

   Therefore, the agent's optimal sale time can be solved from (3.3.4) with $\bar{u}(x)$ replaced by $(1+k_p)^{\beta}\bar{u}((1+k_p)^{-1}x)$. We can see that in this case, selling the stock immediately after buying it incurs negative realization utility.

3. Post-transaction cost purchase price P_t and post-transaction cost wealth $\tilde{X}_t$.

   In this case, the agent ignores the buying cost but takes the selling cost into account when assessing her gain and loss. Consequently, the reference point R_t is given as $R(\tilde{X}_t, P_t)$, and the realization utility is

$$U(G_t, R_t) = P_t^{\beta}\bar{u}(\tilde{X}_t/P_t) = P_t^{\beta}\bar{u}((1-k_s)X_t/P_t). \tag{3.6.2}$$

   Therefore, the agent's optimal liquidation time can be solved from (3.3.4) with $\bar{u}(x)$

replaced by $\bar{u}((1-k_s)x)$. Again, in this case, selling the stock immediately after buying it incurs negative realization utility.

4. Pre-transaction cost purchase price $\tilde{P}_t$ and post-transaction cost wealth $\tilde{X}_t$.

   In this case, the agent takes both the buying and selling costs into account when assessing her gain and loss, and thus the reference point R_t is given as $R(\tilde{X}_t, \tilde{P}_t)$. Then, the realization utility is

$$U(G_t, R_t) = \tilde{P}_t^{\beta}\bar{u}(\tilde{X}_t/\tilde{P}_t) = (1+k_p)^{\beta}P_t^{\beta}\bar{u}(KX_t/P_t).$$

   The agent's optimal liquidation time can be solved from (3.3.4) with $\bar{u}(x)$ replaced by $(1+k_p)^{\beta}\bar{u}(Kx)$. Again, in this case, selling the stock immediately after buying it incurs negative realization utility.

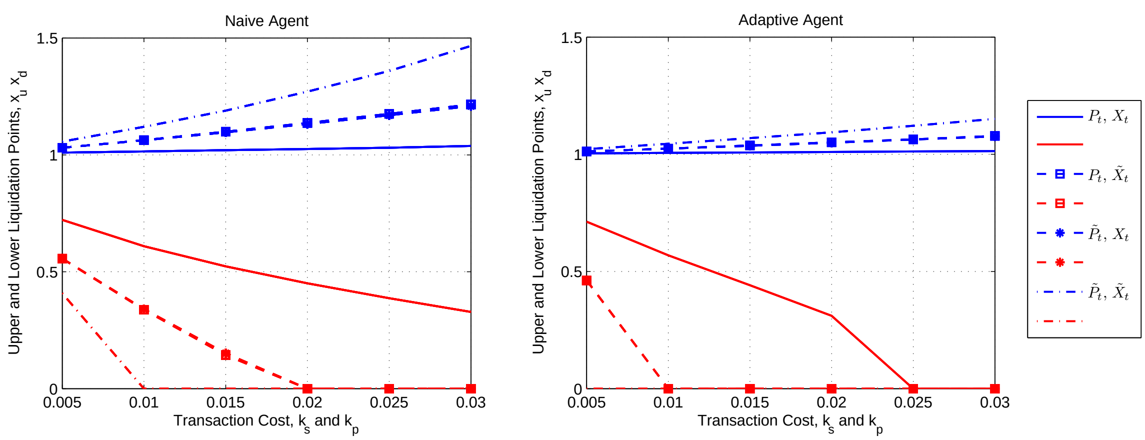

Figure 3.8: Upper and lower liquidation points for the naive agent (left panel) and for the adaptive agent (right panel) when the agents include none, one, or both of the buying and selling costs of the stock in assessing their gains and losses. The parameter values used here are given as in Table 3.2.

Figure 3.8 shows the liquidation points of the naive and adaptive agents. The parameter

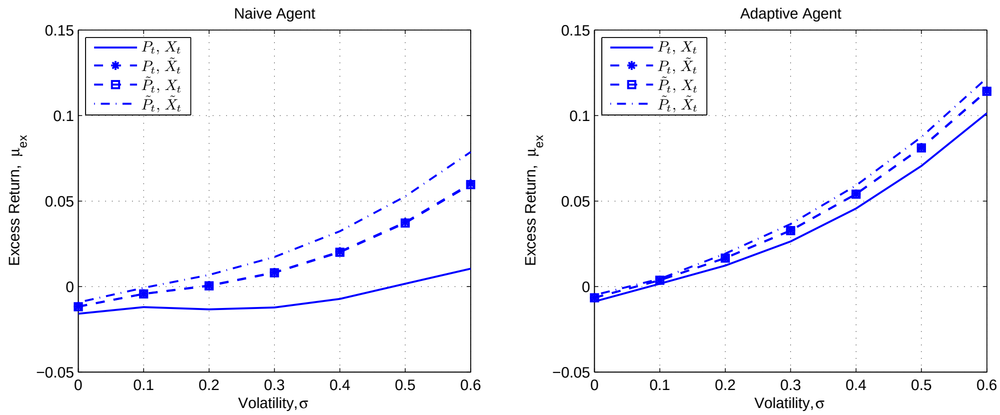

Figure 3.9: Equilibrium risk premium from the perspectives of the naive agent (left panel) and of the adaptive agent (right panel) when the agents include none, one, or both of the buying and selling costs of the stock in assessing their gains and losses. The parameter values used here are given as in Table 3.2 except that $\theta = 15$.

values used here are given as in Table 3.2. The message is clear: the more cost the agent takes into account in assessing her gain and loss, the less frequently she trades the stock both at a gain and at a loss.

Figure 3.9 shows the equilibrium risk premium from the perspectives of the naive agent and of the adaptive agent. The parameter values used are given as in Table 3.2 except that $\theta = 15$. We can observe that the more cost the agents internalizes in determining their gains and losses, the higher risk premium they require for the stock.

3.6.2 Different Choices of Utility Functions

As observed in Ingersoll and Jin [38], with $b_G = b_L = 0$, the realization utility is extremely sensitive to a small increase in the stock price when the agent is break even (i.e., the derivative of $u(x)$ is infinity at $x = 0$). Thus, the agent is willing to realize very small gains frequently; i.e., she sets the upper liquidation point very close to one. Ingersoll and Jin [38] set $b_G = b_L = 1$ in their model and find that the resulting upper liquidation point becomes much higher than in the case $b_G = b_L = 0$. We compute the trading strategies and equilibrium risk premium in the previous sections for the case $b_G = b_L = 1$, and find that compared to the case $b_G = b_L = 0$, for both naive and adaptive agents, the upper liquidation point is evidently higher, the lower liquidation point is lower, and the equilibrium risk premium for the stock is higher.

3.7 Conclusions

In this chapter, we have proposed a trading model in which an agent decides when to sell a stock to maximize her realization utility and terminal wealth utility. Our model extends those by Barberis and Xiong [6] and Ingersoll and Jin [38] in two aspects: First, in addition to realization utility, the agent in our model also experiences utility from her terminal wealth. Second, the reference point in our model adapts to the stock's gain and loss, and the adaptation to the gain is more than to the loss.

We have proved sufficient and necessary conditions under which the optimal value of the agent's trading problem is finite. We have also found the optimal purchase time of the stock: the agent either immediately or never re-purchases the stock after selling it. We have proved that it is optimal to hold the stock if it is already at a deep loss. We have proved that the value function of the trading problem is the unique solution to a variational inequality.

We have considered two types of agents, adaptive agents and naive agents, in the study of trading strategies and asset pricing. An adaptive agent knows today that her reference point in the future will adapt to the prior gain and loss of the stock, but a naive agent fails to do so. We have found that the adaptive agent sells the stock more (less) frequently when the stock is at a gain (at a loss) than the naive agent does. Moreover, when the reference point adapts more to the stock's loss (gain), the naive agent sells the stock at a loss (at a gain) less frequently. The adaptive agent, however, sells the stock at a loss more frequently when the reference point adapts more to the stock's loss. We have also found that when becoming more concerned with the terminal wealth utility, both the adaptive and naive agents sell the stock less frequently both at a gain and at a loss.

We have also studied the risk premium of the stock in equilibrium. We have found that the adaptive agent requires a higher risk premium than the naive agent does. In addition, when the reference point adapts more to the stock's gain (loss), the adaptive agent asks for a higher (lower) risk premium. The risk premium required by the naive agent, however, is insensitive to the degree to which the reference point adapts to the stock's gain or loss. Finally, the risk premium becomes higher when the naive and adaptive agents become more concerned with the terminal wealth utility.

Bibliography

[1] H. R. Arkes, D. Hirshleifer, D. Jiang, and S. Lim. Reference point adaptation: Tests in the domain of security trading. 105(1):67–81, 2008.

[2] H. R. Arkes, D. Hirshleifer, D. Jiang, and S. S. Lim. A cross-cultural study of reference point adaptation: Evidence from China, Korea, and the U.S. 112(2):99–111, 2010.

[3] N. Barberis. A model of casino gambling. 58(1):35–51, Jan. 2012.

[4] N. Barberis, M. Huang, and T. Santos. Prospect theory and asset prices. 116(1):1–53, Feb. 2001.

[5] N. Barberis and R. Thaler. A survey of behavioral finance. In *Handbook of the Economics of Finance*. Elsevier Science B.V., 2003.

[6] N. Barberis and W. Xiong. Realization utility. 104(2):251–271, May 2012.

[7] S. Basak and G. Chabakauri. Dynamic mean-variance asset allocation. 23(8):2970–3016, 2010.

[8] Basel Committee on Banking Supervision. *Results of the comprehensive quantitative impact study*, Switzerland, 2010. Bank for International Settlements, Basel.

[9] M. Baucells, M. Weber, and F. Welfens. Reference-point formation and updating. 57(3):506–519, 2011.

[10] M. Beiglböck and N. Juillet. On a problem of optimal transport under marginal martingale constraints. *ArXiv preprint arXiv:1208.1509*, 2012.

[11] A. Ben-Tal, D. Den Hertog, A. De Waegenaere, B. Melenberg, and G. Rennen. Robust solutions of optimization problems affected by uncertain probabilities. *Management Science*, 59(2):341–357, 2013.

[12] C. Bernard, X. Jiang, and R. Wang. Risk aggregation with dependence uncertainty. *Insurance: Mathematics and Economics*, 54:93–108, 2014.

[13] B. Bhattacharya. An iterative procedure for general probability measures to obtain i-projections onto intersections of convex sets. *Annals of Statistics*, 34(2):878–902, 2006.

[14] P. Billingsley. *Convergence of probability measures*. Wiley, New York, 1968.

[15] D. Bosc and A. Galichon. Extreme dependence for multivariate data. *Quantitative Finance*, 14(7):1187–1199, 2014.

[16] Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Communications on Pure and Applied Mathematics*, 44(4):375–417, 1991.

[17] D. Brigo, A. Capponi, and A. Pallavicini. Arbitrage-free bilateral counterparty risk valuation under collateralization and application to credit default swaps. *Mathematical Finance*, 24(1):125–146, 2014.

[18] D. Brigo, M. Morini, and A. Pallavicini. *Counterparty Credit Risk, Collateral and Funding*. Wiley, Chichester, U.K, 2013.

[19] H. Brown, D. Hobson, and L. C. Rogers. Robust hedging of barrier options. *Mathematical Finance*, 11(3):285–314, 2001.

[20] J. Y. Campbell. Household finance. *The Journal of Finance*, 61(4):1553–1604, 2006.

[21] E. Canabarro and D. Duffie. Measuring and marking counterparty risk. In *Asset/Liability Management for Financial Institutions*. Institutional Investor Books, 2003.

[22] E. K. Carr, P. and V. Gupta. Static hedging of exotic options. *Journal of Finance*, 53(3):1165–1190, 1998.

[23] A. Cox. Arbitrage bounds. In *Encyclopedia of Quantitative Finance*. Wiley, New York, 2010.

[24] M. Crandall, H. Ishii, and P. Lions. User's guide to viscosity solutions of second order partial differential equations. 27:1–67, 6 1992.

[25] S. Crépey. Bilateral counterparty risk under funding constraints — part ii: Cva. *Mathematical Finance*, 25(1):23–50, 2015.

[26] I. Csiszár. I-divergence geometry of probability distributions and minimization problems. *The Annals of Probability*, 3:146–158, 1975.

[27] M. Dai, Y. K. Kwok, and H. You. Intensity-based framework and penalty formulation of optimal stopping problems. 31(12):3860–3880, 2007.

[28] W. Deming and F. Stephan. On a least squares adjustment of a sampled frequency table when the expected marginal totals are known. *Annals of Mathematical Statistics*, 11:427–444, 1940.

[29] Y. Dolinsky and H. M. Soner. Martingale optimal transport and robust hedging in continuous time. *Probability Theory and Related Fields*, 160:391–427, 2014.

[30] P. Dupuis and R. S. Ellis. *A weak convergence approach to the theory of large deviations*. Wiley, New York, 1997.

[31] P. Embrechts and G. Puccetti. Bounds for functions of dependent risks. *Finance and Stochastics*, 10:341–352, 2006.

[32] P. Embrechts, B. Wang, and R. Wang. Aggregation-robustness and model uncertainty of regulatory risk measures. *Working paper, ETH Zurich, Switzerland*, 2014.

[33] P. Forsyth and K. Vetzal. Quadratic convergence for valuing american options using a penalty method. 23(6):2095–2122, 2002.

[34] J. Gregory. *Counterparty Credit Risk and Credit Value Adjustment: A Continuing Challenge for Global Financial Markets*. Wiley, Chichester, U.K, 2006.

[35] L. P. Hansen and T. J. Sargent. *Robustness*. Princeton University Press, Princeton, New Jersey, 2008.

[36] P. Henry-Labordère and N. Touzi. An explicit martingale version of brenier's theorem. SSRN:http://ssrn.com/abstract=2218488, 2013.

[37] J. Hull and A. White. Cva and wrong way risk. *Financial Analysts Journal*, 68(5):58–69, 2012.

[38] J. E. Ingersoll and L. J. Jin. Realization utility with reference-dependent preferences. 26:723–767, 2013.

[39] C. T. Ireland and S. Kullback. Contingency tables with given marginals. *Biometrika*, 55(1):179–188, 1968.

[40] D. Kahneman and A. Tversky. Prospect theory: An analysis of decision under risk. 47(2):263–291, 1979.

[41] P. Kleinschmidt and H. Schannath. A strongly polynomial algorithm for the transportation problem. *Mathematical Programming*, 68:1–13, 1995.

[42] D. Luenberger. *Optimization by Vector Space Methods*. Wiley, New York, 1969.

[43] T. O'Donoghue and M. Rabin. Doing it now or later. 89(1):103–124, Mar. 1999.

[44] H. Pham. *Continuous-time stochastic control and optimization with financial applications*, volume 61. Springer Science & Business Media, Berlin, 2009.

[45] F. Pukelsheim. Biproportional scaling of matrices and the iterative proportional fitting procedure. *Annals of Operations Research*, 215(1):269–283, 2014.

[46] D. Rosen and D. Saunders. Cva the wrong way. *Journal of Risk Management in Financial Institutions*, 5(3):252–272, 2012.

[47] L. Rüschendorf. Convergence of the iterative proportional fitting procedure. *The Annals of Statistics*, pages 1160–1174, 1995.

[48] L. Rüschendorf. *Mathematical Risk Analysis*. Springer-Verlag, Berlin, 2013.

[49] L. Rüschendorf and S. T. Rachev. A characterization of random variables with minimum l^2-distance. *Journal of Multivariate Analysis*, 32(1):48–54, 1990.

[50] L. Rüschendorf and W. Thomsen. Note on the schrödinger equation and i-projections. *Statistics and Probability Letters*, 17(5):369–375, 1993.

[51] H. Shefrin and M. Statman. The disposition to sell winners too early and ride losers too long: Theory and evidence. *The Journal of Finance*, pages 777–790, 1985.

[52] P. Tankov. Improved fréchet bounds and model-free pricing of multi-asset options. *Journal of Applied Probability*, 48:389–403, 2011.

[53] A. Tversky and D. Kahneman. Advances in prospect theory: Cumulative representation of uncertainty. 5(4):297–323, 1992.

[54] C. Villani. *Optimal Transport: Old and New*. Springer, Berlin, 2008.

Appendix A

Additional Proofs for Chapter 2

A.1 Proof of Theorem 2.3.1

The Wasserstein metric of order 2 between probability measures p and q on $\mathbb{R}^d$ is $W_2(p,q)$, where

$$\begin{aligned} W_2^2(p,q) &= \inf_{\pi\in\Pi(p,q)} \int_{\mathbb{R}^d\times\mathbb{R}^d} \|x-y\|^2\, d\pi(x,y) \\ &= \int_{\mathbb{R}^d} \|x\|^2\, dp(x) + \int_{\mathbb{R}^d} \|y\|^2\, dq(y) \\ &\quad -2 \sup_{\pi\in\Pi(p,q)} \int_{\mathbb{R}^d\times\mathbb{R}^d} < x, y > \, d\pi(x,y). \end{aligned} \tag{A.1}$$

The empirical measures p_N and q_N converge weakly to p and q, respectively, a.s., so it follows from Corollary 6.11 of Villani [54] that $W_2^2(p_N, q_N) \to W_2^2(p,q)$, a.s., and $W_2^2(p_N, q) \to W_2^2(p,q)$, a.s. Under the assumed square-integrability conditions, we also have

$$\int_{\mathbb{R}^d} \|x\|^2\, dp_N(x) \to \int_{\mathbb{R}^d} \|x\|^2\, dp(x), \quad \text{a.s.,}$$

and similarly for q_N. The theorem now follows from (A.1). $\square$

A.2 Proof of Proposition 2.4.1

Problem (2.4.3) is equivalent to

$$-\inf_{\mu\in\Pi(p,q)} \frac{1}{\theta} \int \ln\Big(\frac{d\mu}{\exp(\theta < x, y >) d\nu}\Big) d\mu. \tag{A.1}$$

Theorem 3 of Rüschendorf and Thomsen [50] implies the existence of a unique optimal solution to (A.1), which we denote by μ^θ.

First we show that μ^θ is optimal for (2.4.1)–(2.4.2) with $\eta = \eta(\theta)$. Suppose μ^θ is not optimal, then there exists $\mu^{\eta(\theta)}$ such that

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} < x, y > d\mu^{\eta(\theta)}(x, y) > \int_{\mathbb{R}^d\times\mathbb{R}^d} < x, y > d\mu^\theta(x, y),$$

and

$$\int \ln(\frac{d\mu^{\eta(\theta)}}{d\nu}) d\mu^{\eta(\theta)} \le \int \ln(\frac{d\mu^\theta}{d\nu}) d\mu^\theta.$$

But then

$$\int < x, y > d\mu^{\eta(\theta)}(x, y) - \frac{1}{\theta} \int \ln(\frac{d\mu^{\eta(\theta)}}{d\nu}) d\mu^{\eta(\theta)} > \int < x, y > d\mu^\theta(x, y) - \frac{1}{\theta} \int \ln(\frac{d\mu^\theta}{d\nu}) d\mu^\theta,$$

which contradicts the optimality of μ^θ for the penalty problem (2.4.3).

Next we show that the mapping from θ to $\eta(\theta)$ is increasing. For any $\theta_2 > \theta_1 > 0$, let

μ^{θ_1} and μ^{θ_2} denote optimal solution to the penalty problem with θ_1 and θ_2 respectively. If $\mu^{\theta_1} = \mu^{\theta_2}$, then $\eta(\theta_1) = \eta(\theta_2)$. If $\mu^{\theta_1} \neq \mu^{\theta_2}$, then, by unique optimality of μ^{θ_2}, it holds that

$$\int < x, y > d\mu^{\theta_2}(x,y) - \frac{1}{\theta_2} \int \ln(\frac{d\mu^{\theta_2}}{d\nu}) d\mu^{\theta_2} > \int < x, y > d\mu^{\theta_1}(x,y) - \frac{1}{\theta_2} \int \ln(\frac{d\mu^{\theta_1}}{d\nu}) d\mu^{\theta_1}. \tag{A.2}$$

Compare the first term on each side. If $\int_{\mathbb{R}^d \times \mathbb{R}^d} < x, y > d\mu^{\theta_2}(x,y) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} < x, y > d\mu^{\theta_1}(x,y)$, then $\int \ln(\frac{d\mu^{\theta_2}}{d\nu}) d\mu^{\theta_2} < \int \ln(\frac{d\mu^{\theta_1}}{d\nu}) d\mu^{\theta_1}$ by (A.2). Adding $(\frac{1}{\theta_2} - \frac{1}{\theta_1}) \int \ln(\frac{d\mu^{\theta_2}}{d\nu}) d\mu^{\theta_2}$ to the left side and $(\frac{1}{\theta_2} - \frac{1}{\theta_1}) \int \ln(\frac{d\mu^{\theta_1}}{d\nu}) d\mu^{\theta_1}$ to the right side of (A.2), the sign does not change, which means μ^{θ_2} is optimal for the penalty problem with θ_1. However that contradicts the unique optimality of μ^{θ_1}. We conclude that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} < x, y > d\mu^{\theta_2}(x,y) \quad > \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} < x, y > d\mu^{\theta_1}(x,y).$$

Now compare the second term on each side. If $\int \ln(\frac{d\mu^{\theta_2}}{d\nu}) d\mu^{\theta_2} \leq \int \ln(\frac{d\mu^{\theta_1}}{d\nu}) d\mu^{\theta_1}$, then the unique optimality of μ^{θ_1} is again contradicted, so we have

$$\eta(\theta_2) \quad > \quad \eta(\theta_1).$$

Next we show $\eta(\theta) \in (0, \eta^*]$ for $\theta \in (0, \infty)$. Since the relative entropy $\int \ln(\frac{d\mu^{\theta}}{d\nu}) d\mu^{\theta}$ is nonnegative and equals 0 only if $\mu^{\theta} = \nu$, we have $\eta(\theta) > 0$ for $\theta > 0$. Let μ^* denote optimal solution to (2.2.2) and let $\eta^* = \int \ln(\frac{d\mu^*}{d\nu}) d\mu^*$. Since problem (2.2.2) is a relaxation of problem (2.4.1)–(2.4.2), we conclude that for all $\theta > 0$,

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} < x, y > d\mu^*(x, y) \geq \int_{\mathbb{R}^d \times \mathbb{R}^d} < x, y > d\mu^\theta(x, y). \tag{A.3}$$

Suppose there exists $\theta^* > 0$ such that $\eta(\theta^*) = \int \ln(\frac{d\mu^{\theta^*}}{d\nu})d\mu^{\theta^*} > \eta^*$. By adding $-\frac{1}{\theta^*}\eta^*$ to the left and $-\frac{1}{\theta^*}\eta(\theta^*)$ to the right of (A.3), the inequality does not change, which contradicts the optimality of μ^{θ^*}. Thus $\eta(\theta) \leq \eta^*$. $\square$

A.3 Proof of Theorem 2.4.1

We divide the proof into several parts, starting with the convergence of the objective function value asserted in part (i) of the theorem.

A.3.1 Convergence of the Optimal Objective Value

We will first show that for any feasible solution to the limiting problem, we can construct a sequence of approximating solutions that approach the limiting objective function from above. To get the reverse inequality we will use a dual formulation of the limiting objective and show that it is approached from below.

Since Y has finite support, we may assume without loss of generality that $q(y_j) > 0$ for all j. If we had $q(y_j) = 0$ for some j, we could reformulate an equivalent problem by removing the marginal constraint on y_j.

Let $\mu \in \Pi(p, q)$ be any feasible solution to the limiting problem. Write $\mu(dx, y) = p(dx)q(y|x)$, and define the following mass function on the pairs (X_i, y_j), $i = 1, \ldots, N$,

$j = 1, \ldots, d + 1$:

$$\mu_N(X_i, y_j) = \frac{1}{N} q(y_j | X_i). \tag{A.1}$$

If we sum over the y_j for any X_i, we get

$$\sum_{j=1}^{d+1} \mu_N(X_i, y_j) = \frac{1}{N} \sum_{j=1}^{d+1} q(y_j | X_i) = \frac{1}{N}.$$

If we sum over the X_i for any y_j, we get

$$\sum_{i=1}^{N} \mu_N(X_i, y_j) = \frac{1}{N} \sum_{i=1}^{N} q(y_j | X_i) =: \bar{q}_N(y_j).$$

We will not in general have $\bar{q}_N = q_N$, so μ_N is not in general a feasible solution to the finite problem, in the sense that $\mu_N \notin \Pi(p_N, q_N)$. However, by the strong law of large numbers for $\{X_1, X_2, \ldots\}$, for each y_j, $j = 1, \ldots, d + 1$,

$$\bar{q}_N(y_j) = \frac{1}{N} \sum_{i=1}^{N} q(y_j | X_i) \to \int q(y_j | x) \, dp(x) = q(y_j), \text{ a.s.,}$$

because $\mu \in \Pi(p, q)$. Also by the strong law of large numbers, we have $q_N(y_j) \to q(y_j)$, a.s. We will therefore consider a relaxed constraint. Let $\Pi_\epsilon(p_N, q_N)$ denote the set of joint distributions on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals p_N and q', where $|q'(y_j) - q_N(y_j)| \leq \epsilon$, $j = 1, \ldots, d + 1$.

Lemma A.3.1. *As $N \to \infty$,*

$$\varliminf_{N\to\infty} \max_{\mu\in\Pi(p_N,q_N)} G(\mu,\nu_N) \geq \sup_{\mu\in\Pi(p,q)} G(\mu,\nu).$$

Proof: For each N, we are maximizing a concave function over a compact convex set, so the maximum is indeed attained. Write c_N for $\max_{\mu\in\Pi(p_N,q_N)} G(\mu,\nu_N)$ and $c_{N,\varepsilon}$ for $\max_{\mu\in\Pi_\epsilon(p_N,q_N)} G(\mu,\nu_N)$. For any $\mu \in \Pi(p,q)$, define μ_N as in (A.1). Then $\mu_N \in \Pi_\epsilon(p_N,q_N)$ for all sufficiently large N, a.s., and

$$\begin{aligned} c_{N,\varepsilon} &\geq \sum_{i=1}^{N}\sum_{j=1}^{d+1} \mu_N(X_i,y_j) < X_i, y_j > -\frac{1}{\theta} D(\mu_N|\nu_N) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{d+1} \frac{q(y_j|X_i)}{N} < X_i, y_j > -\frac{1}{\theta}\sum_{i=1}^{N}\sum_{j=1}^{d+1} \ln\left(\frac{q(y_j|X_i)/N}{q_N(y_j)/N}\right)\frac{q(y_j|X_i)}{N}. \end{aligned}$$

By the strong law of large numbers, almost surely,

$$\sum_{i=1}^{N}\sum_{j=1}^{d+1} \frac{q(y_j|X_i)}{N} < X_i, y_j > \to \int \sum_{j=1}^{d+1} < x, y_j > q(y_j|x)dp(x) = \int < x, y > d\mu(x,y)$$

and

$$\sum_{i=1}^{N}\sum_{j=1}^{d+1} \ln\left(\frac{q(y_j|X_i)}{q_N(y_j)}\right)\frac{q(y_j|X_i)}{N} \to \int \sum_{j=1}^{d+1} \ln\left(\frac{q(y_j|x)}{q(y_j)}\right) q(y_j|x)dp(x) = \int \ln(\frac{d\mu}{d\nu})d\mu.$$

Since this holds for any $\mu \in \Pi(p,q)$,

$$\varliminf_{N\to\infty} c_{N,\varepsilon} \geq c_\infty \equiv \sup_{\mu\in\Pi(p,q)} G(\mu,\nu). \tag{A.2}$$

Recall $c_N = \max_{\mu \in \Pi(p_N, q_N)} G(\mu, \nu_N)$. We claim that

$$c_{N,\varepsilon} \leq c_N + \varepsilon K_N, \tag{A.3}$$

for

$$K_N = K_1 \cdot \max_{i=1,\dots,N} \max_{j=1,\dots,d+1} | < X_i, y_j > | + \frac{1}{\theta} \cdot K_2,$$

where K_1 and K_2 are constants. We prove (A.3) in Appendix A.3.3.

Under our assumption that $\mathbb{E}_\nu[\exp(\theta < X, Y >)] < \infty$, the sequence K_N satisfies $K_N/N^\alpha \to 0$, for any $\alpha \in (0, 1/2)$. Set $\varepsilon_N = 1/N^\alpha$ so $\varepsilon_N K_N \to 0$. By the law of the iterated logarithm, with probability 1,

$$\max_{1 \leq j \leq d+1} |q_N(y_j) - q(y_j)| < \varepsilon_N/2 \quad \text{and} \quad \max_{1 \leq j \leq d+1} |\bar{q}_N(y_j) - q(y_j)| < \varepsilon_N/2$$

for all sufficiently large N, and then

$$\max_{1 \leq j \leq d+1} |\bar{q}_N(y_j) - q_N(y_j)| < \varepsilon_N$$

as well. In other words, for any $\mu \in \Pi(p, q)$, we have $\mu_N \in \Pi_{\epsilon_N}(p_N, q_N)$ for all sufficiently large N, a.s. We can therefore strengthen (A.2) to

$$\varliminf_{N \to \infty} c_{N,\varepsilon_N} \geq c_\infty.$$

But

$$\underline{\lim}_{N\to\infty} c_{N,\varepsilon_N} \leq \underline{\lim}_{N\to\infty} c_N + K_N\varepsilon_N = \underline{\lim}_{N\to\infty} c_N.$$

So we have shown that

$$\underline{\lim}_{N\to\infty} c_N \geq c_\infty.$$

□

We now establish the reverse inequality.

Lemma A.3.2. *As $N \to \infty$,*

$$\overline{\lim}_{N\to\infty} \max_{\mu\in\Pi(p_N,q_N)} G(\mu,\nu_N) \leq \sup_{\mu\in\Pi(p,q)} G(\mu,\nu), \text{ a.s.}$$

Proof: The supremum of $G(\mu,\nu)$ over $\mu \in \Pi(p,q)$ can be written as

$$-\frac{1}{\theta}\inf_{\mu\in\Pi(p,q)}\int \ln\left(\frac{d\mu(x,y)}{\exp\{\theta < x,y >\}d\nu(x,y)}\right)\,d\mu(x,y) = \frac{1}{\theta}\sup_{\mu\in\Pi(p,q)} -D(\mu|e^{\theta<x,y>}\nu). \tag{A.4}$$

By Theorem 3 of Rüschendorf and Thomsen [50], the optimum in (A.4) is attained at a solution of the form

$$d\mu^*(x,y) = e^{a(x)+b(y)+\theta<x,y>}\,d\nu(x,y), \tag{A.5}$$

for some functions a and b on $\mathbb{R}^d$. Similarly, for finite N, the optimizer of $G(\mu,\nu_N)$ over

$\mu \in \Pi(p_N, q_N)$ has the form

$$d\mu_N^*(x,y) = e^{a_N(x)+b_N(y)+\theta<x,y>}\, d\nu_N(x,y).$$

For the rest of the proof, we will work with the formulation in (A.4), omitting the constant factor of $1/\theta$. We will apply a dual formulation of Bhattacharya [13]. To this end, consider the set Π^* of functions $h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ of the form $h(x,y) = h_1(x) + h_2(y)$ with

$$\int h_1(x)\, dp(x) + \int h_2(y)\, dq(y) \geq 0.$$

The convex cone Π^* is contained within the dual cone of $\Pi(p,q)$, which is the set of functions $h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ that have nonnegative expectations with respect to all $\mu \in \Pi(p,q)$. We consider the dual problem

$$\inf_{h\in\Pi^*} \ln \int e^{h_1(x)+h_2(y)+\theta<x,y>}\, d\nu(x,y).$$

With a and b as in (A.5), set

$$h_1^*(x) = a(x) + c/2, \quad h_2^*(x) = b(x) + c/2,$$

where

$$c = -\int [a(x) + b(y)]\, d\mu^*(x,y).$$

Observe that

$$\int h_1^*(x)\,dp(x) + \int h_2^*(y)\,dq(y) = 0,$$

so this (h_1^*, h_2^*) is dual feasible. Moreover, with this choice of h_1^*, h_2^*, the dual objective function value is

$$\ln \int e^{a(x)+b(x)+c+\theta<x,y>}\,d\nu(x,y) = \ln \int e^c\,d\mu^*(x,y) = c. \tag{A.6}$$

The primal objective in (A.4) evaluated at (A.5) yields

$$-D(\mu^*|e^{\theta<x,y>}\nu) = -\int [a(x)+b(y)]\,d\mu^*(x,y) = c,$$

so the primal and dual objective values agree. It follows from Theorem 2.1 of Bhattacharya [13] that this choice of (h_1^*, h_2^*) is optimal for the dual objective.

Parallel results hold for finite N as well. The maximal value of $G(\cdot, \nu_N)$ is $1/\theta$ times the dual optimum

$$\begin{aligned} c_N \quad &= \quad \inf_{h_1,h_2} \ln \int e^{h_1(x)+h_2(y)+\theta<x,y>}\,d\nu_N, \\ &\text{s.t.} \quad \int h_1(x)\,dp_N(x) + \int h_2(y)\,dq_N(y) \geq 0. \end{aligned} \tag{A.7}$$

For $\varepsilon \geq 0$, define

$$\begin{aligned} c_\infty^\varepsilon \quad &:= \quad \inf_{h_1,h_2} \ln \int e^{h_1(x)+h_2(y)+\theta<x,y>}\,d\nu, \\ &\text{s.t.} \quad \int h_1(x)\,dp(x) + \int h_2(y)\,dq(y) \geq \varepsilon. \end{aligned} \tag{A.8}$$

The infimum is finite because the integral is finite for any constant h_1, h_2. Let $(h_1^\varepsilon, h_2^\varepsilon)$ be feasible for (A.8) and satisfy

$$\ln \int e^{h_1^\varepsilon(x)+h_2^\varepsilon(y)+\theta<x,y>}\, d\nu \leq c_\infty^\varepsilon + \varepsilon.$$

Then, with probability 1,

$$\begin{aligned}\int h_1^\varepsilon(x)\, dp_N(x) + \int h_2^\varepsilon(y)\, dq_N(y) &= \frac{1}{N}\sum_{i=1}^{N}(h_1^\varepsilon(X_i) + h_2^\varepsilon(Y_i)) \\ &\to \int h_1^\varepsilon(x)\, dp(x) + \int h_2^\varepsilon(y)\, dq(y) \geq \varepsilon,\end{aligned}$$

so, for all sufficiently large N,

$$\int h_1^\varepsilon(x)\, dp_N(x) + \int h_2^\varepsilon(y)\, dq_N(y) \geq 0.$$

In other words, $(h_1^\varepsilon, h_2^\varepsilon)$ is feasible for (A.7) for all sufficiently large N, so

$$c_N \leq \ln \int e^{h_1^\varepsilon(x)+h_2^\varepsilon(y)+\theta<x,y>}\, d\nu_N \to \ln \int e^{h_1^\varepsilon(x)+h_2^\varepsilon(y)+\theta<x,y>}\, d\nu \leq c_\infty^\varepsilon + \varepsilon.$$

Hence,

$$\overline{\lim_{N\to\infty}}\, c_N \leq c_\infty^\varepsilon + \varepsilon.$$

By construction,

$$\int h_1^*(x)\, dp(x) + \int h_2^*(y)\, dq(y) = 0,$$

so $(h_1^* + \varepsilon/2, h_2^* + \varepsilon/2)$ is feasible for (A.8) and then

$$c_\infty^\varepsilon \leq \ln \int e^{h_1^*(x)+h_2^*(y)+\varepsilon+\theta<x,y>}\, d\nu$$

and

$$\lim_{\varepsilon \downarrow 0} \ln \int e^{h_1^*(x)+h_2^*(y)+\varepsilon+\theta<x,y>}\, d\nu = c,$$

with c as in (A.6). Thus, since $\varepsilon > 0$ can be taken arbitrarily small,

$$\overline{\lim_{N\to\infty}}\, c_N \leq c.$$

□

Combining Lemmas A.3.1 and A.3.2 proves part (i) of the theorem.

A.3.2 Weak Convergence of Optimal Solutions

Define

$$\Pi^N = \Pi(p,q) \cup \left(\underset{n \geq N}{\cup} \Pi(p_n, q_n) \right)$$

We will show that, almost surely, Π^N is compact (with respect to the topology of weak convergence on $\mathbb{R}^d \times \mathbb{R}^d$) for all sufficiently large N. It will follow that any sequence of optimizers $\{\mu_n^*\}$ is then eventually contained within a compact set, so every subsequence has a convergent subsequence.

Lemma A.3.3. Π^N *is compact for all sufficiently large* N*, a.s.*

Proof: By Prohorov's Theorem (Billingsley [14], p.37) the set Π^N is compact if it is uniformly tight, meaning that for all $\varepsilon > 0$ we can find a compact subset A of $\mathbb{R}^d \times \mathbb{R}^d$ such that $\mu(A) \geq 1 - \varepsilon$, for all $\mu \in \Pi^N$. Let A_1, A_2 be compact subsets of $\mathbb{R}^d$ such that

$$P(X \in A_1) = \int_{A_1} dp(x) \geq 1 - \varepsilon/4, \quad P(Y \in A_2) = \int_{A_2} dq(x) \geq 1 - \varepsilon/4.$$

Then, for any $\mu \in \Pi(p, q)$,

$$\int \mathbf{1}_{\{(x,y) \notin A_1 \times A_2\}} \, d\mu(x, y) \leq P(X \notin A_1) + P(Y \notin A_2) \leq \varepsilon/2.$$

With probability 1, for all sufficiently large N and $\mu \in \Pi(p_N, q_N)$,

$$\int \mathbf{1}_{\{(x,y) \notin A_1 \times A_2\}} \, d\mu(x, y) \leq \frac{1}{N} \sum_{i=1}^{N} \left(\mathbf{1}_{\{X_i \notin A_1\}} + \mathbf{1}_{\{Y_i \notin A_2\}} \right) \leq \varepsilon.$$

Thus, with probability 1, Π^N is uniformly tight for all sufficiently large N, and thus compact.

□

The optimizers μ_N^* are contained in the sets $\Pi(p_N, q_N)$, so for all sufficiently large N, the sequence μ_n^*, $n \geq N$, is contained in a compact set Π^N, and then every subsequence has a further subsequence that converges weakly.

Suppose the subsequence $\mu_{n_k}^*$ converges, say $\mu_{n_k}^* \Rightarrow \tilde{\mu}$. The marginals of $\mu_{n_k}^*$ converge to p and q, so $\tilde{\mu} \in \Pi(p, q)$, making $\tilde{\mu}$ feasible for the limiting problem. We claim that it is optimal. We have, a.s.,

$$\int e^{\theta<x,y>} \, d\mu_{n_k}^* \leq \int \sum_{j=1}^{d+1} e^{\theta<x,y_j>} \, dp_{n_k}(x) \to \int \sum_{j=1}^{d+1} e^{\theta<x,y_j>} \, dp(x),$$

by the strong law of large numbers, because the condition $\mathbb{E}_\nu[e^{\theta<x,y>}] < \infty$ implies that the limit is finite. This is then more than sufficient to ensure that

$$\int < x, y > \, d\mu^*_{n_k}(x, y) \to \int < x, y > \, d\tilde{\mu}(x, y). \tag{A.9}$$

Moreover, relative entropy is lower semi-continuous with respect to weak convergence (Dupuis and Ellis [30], Lemma 1.4.3), so

$$D(\tilde{\mu}|\nu) \leq \varliminf_{k\to\infty} D(\mu^*_{n_k}|\nu_{n_k})$$

and then

$$G(\tilde{\mu}, \nu) \geq \varlimsup_{k\to\infty} G(\mu^*_{n_k}, \nu_{n_k}) = \sup_{\mu\in\Pi(p,q)} G(\mu, \nu),$$

by part (i) of the theorem. Thus, $\tilde{\mu}$ is optimal. Using the equivalence between the optimization of $G(\cdot, \nu)$ and (A.4), we know from Theorem 3 of Rüschendorf and Thomsen [50] that the maximum is uniquely attained by some μ^*, and thus $\tilde{\mu} = \mu^*$.

We have shown that every subsequence of μ^*_n has a further subsequence that converges to μ^*. It follows that $\mu^*_n \Rightarrow \mu^*$. This proves part (ii) of the theorem. The uniform integrability needed for (2.4.7) follows as in (A.9), which proves part (iii).

A.3.3 Proof of Inequality (A.3)

It remains to prove (A.3). First we construct a feasible solution $\hat{\mu}_N$ of $\max_{\mu\in\Pi(p_N,q_N)} G(\mu, \nu_N)$ by modifying the optimal solution $\mu^*_{N,\varepsilon}$ of the relaxed problem $\max_{\mu\in\Pi_\varepsilon(p_N,q_N)} G(\mu, \nu_N)$.

Then we use the difference between $G(\hat{\mu}_N, \nu_N)$ and $G(\mu^*_{N,\varepsilon}, \nu_N)$ to bound the difference between c_N and $c_{N,\varepsilon}$.

Define $\varepsilon^N_j = \sum_{i=1}^N (\mu^*_{N,\varepsilon})_{ij} - q_N(y_j)$, which is the difference between the Y marginal of $\mu^*_{N,\varepsilon}$ and the empirical distribution of Y. Note that $|\varepsilon^N_j| \leq \varepsilon$ for $j = 1, ..., d+1$. We claim that there exists $\{\varepsilon^*_{ij}\}$ for which

$$(\hat{\mu}_N)_{ij} := (\mu^*_{N,\varepsilon})_{ij} - \varepsilon^*_{ij}, \quad i = 1, ..., N \text{ and } j = 1, ..., d+1,$$

satisfies the following conditions:

$$\hat{\mu}_N \in \Pi(p_N, q_N), \tag{A.10}$$

$$\textstyle\sum_{i=1}^N \sum_{j=1}^{d+1} |\varepsilon^*_{ij}| \leq (d+1)\varepsilon, \tag{A.11}$$

$$-C_N \cdot \varepsilon \cdot \tfrac{1}{N} \leq \varepsilon^*_{ij} \leq C_N \cdot \varepsilon \cdot (\mu^*_{N,\varepsilon})_{ij}, \tag{A.12}$$

where $C_N = \max_{j=1,\cdots,d+1}\{1/q_N(y_j)\}$. Since $q(y_j) > 0$ for $j = 1, \cdots, d+1$, we know $q_N(y_j) > 0$ for all j and N large enough, and C_N is well defined.

To see that such $\{\varepsilon^*_{ij}\}$ exist, rearrange $\{\varepsilon^N_j\}$ in descending order $\{\varepsilon^N_{j_k}\}$ for $k = 1, \cdots, d+1$, and let m denote number of nonnegative elements. Note that $\varepsilon^N_{j_k} \geq 0$ for $k = 1, \cdots, m$, and $\varepsilon^N_{j_k} < 0$ for $k = m+1, \cdots, d+1$, and $\sum_{k=1}^m \varepsilon^N_{j_k} = -\sum_{k=m+1}^{d+1} \varepsilon^N_{j_k}$. Let

$$\varepsilon^*_{i,j_k} = \frac{(\mu^*_{N,\varepsilon})_{i,j_k}}{\sum_{i=1}^N (\mu^*_{N,\varepsilon})_{i,j_k}} \cdot \varepsilon^N_{j_k}$$

for $i = 1, \cdots, N$ and $k = 1, \cdots, m$. Let

$$S_i = \sum_{k=1}^{m} \varepsilon^*_{i,j_k}$$

for $i = 1, \cdots, N$. Let

$$\varepsilon^*_{i,j_k} = \frac{\varepsilon^N_{j_k}}{|\sum_{l=m+1}^{d+1} \varepsilon^N_{j_l}|} \cdot S_i$$

for $i = 1, \cdots, N$ and $k = m+1, \cdots, d+1$.

We verify (A.10)-(A.12) for $\{\varepsilon^*_{ij}\}$. Since (A.10) is equivalent to $\sum_{i=1}^N \varepsilon^*_{i,j_k} = \varepsilon^N_{j_k}$ for $k = 1, \cdots, d+1$, we know that by construction it holds for $\{\varepsilon^*_{ij}\}$. Next,

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{d+1} |\varepsilon^*_{ij}| &= \sum_{i=1}^{N}\sum_{k=1}^{d+1} |\varepsilon^*_{i,j_k}| \\
&= \sum_{i=1}^{N}\sum_{k=1}^{m} \varepsilon^*_{i,j_k} - \sum_{i=1}^{N}\sum_{k=m+1}^{d+1} \varepsilon^*_{i,j_k} \\
&= \sum_{i=1}^{N}\sum_{k=1}^{m} \frac{(\mu^*_{N,\varepsilon})_{i,j_k}}{\sum_{i=1}^N (\mu^*_{N,\varepsilon})_{i,j_k}} \cdot \varepsilon^N_{j_k} - \sum_{i=1}^{N}\sum_{k=m+1}^{d+1} \frac{\varepsilon^N_{j_k}}{|\sum_{l=m+1}^{d+1} \varepsilon^N_{j_l}|} \cdot S_i \\
&= \sum_{k=1}^{m} \varepsilon^N_{j_k} - \sum_{k=m+1}^{d+1} \varepsilon^N_{j_k} \\
&\leq (d+1)\varepsilon
\end{aligned}$$

The last equality follows by $\sum_{i=1}^N S_i = \sum_{k=1}^m \varepsilon^N_{j_k} = |\sum_{l=m+1}^{d+1} \varepsilon^N_{j_l}|$. Thus $\{\varepsilon^*_{ij}\}$ satisfy (A.11).

For $k = 1, \cdots, m$,

$$0 \leq \varepsilon^*_{i,j_k} = \frac{(\mu^*_{N,\varepsilon})_{i,j_k}}{q_N(y_{j_k}) + \varepsilon^N_{j_k}} \cdot \varepsilon^N_{j_k} \leq \frac{(\mu^*_{N,\varepsilon})_{i,j_k}}{q_N(y_{j_k})} \cdot \varepsilon^N_{j_k} \leq C_N \cdot \varepsilon \cdot (\mu^*_{N,\varepsilon})_{i,j_k}$$

For $k = m+1, \cdots, d+1$,

$$0 \geq \varepsilon^*_{i,j_k} \geq -S_i \geq -C_N \cdot \varepsilon \cdot \sum_{k=1}^{m} (\mu^*_{N,\varepsilon})_{i,j_k} \geq -C_N \cdot \varepsilon \cdot \frac{1}{N}$$

Thus $\{\varepsilon^*_{ij}\}$ satisfy (A.12).

Because $\hat{\mu}_N$ is feasible but not necessarily optimal, we have

$$G(\hat{\mu}_N, \nu_N) \leq c_N \leq c_{N,\varepsilon}.$$

We will show that

$$c_{N,\varepsilon} - G(\hat{\mu}_N, \nu_N) \leq \varepsilon K_N, \tag{A.13}$$

for

$$K_N = (d+1) \cdot \max_{i=1,\ldots,N} \max_{j=1,\ldots,d+1} | < X_i, y_j > | + \frac{1}{\theta} \cdot K_2,$$

where K_2 is a constant. It then follows that

$$c_{N,\varepsilon} - c_N \leq \varepsilon K_N.$$

To show (A.13), write

$$\begin{aligned} c_{N,\varepsilon} - G(\hat{\mu}_N, \nu_N) &= \Big(\int < x, y > d\mu_{N,\varepsilon} - \int < x, y > d\hat{\mu}_N \Big) \\ &\quad - \frac{1}{\theta} \int \big(\frac{d\mu_{N,\varepsilon}}{d\nu_N} \ln(\frac{d\mu_{N,\varepsilon}}{d\nu_N}) - \frac{d\hat{\mu}_N}{d\nu_N} \ln(\frac{d\hat{\mu}_N}{d\nu_N}) \big) d\nu_N. \end{aligned}$$

The first part has upper bound

$$(d+1)\cdot \max_{i=1,\dots,N}\max_{j=1,\dots,d+1} |<X_i, y_j>|\cdot \varepsilon.$$

Let $x = d\mu_{N,\varepsilon}/d\nu_N$ and $x - \Delta x = d\hat{\mu}_N/d\nu_N$. Drop the factor $-1/\theta$ and rewrite the second part as follows:

$$\begin{aligned}
&\int x\ln x - (x-\Delta x)\ln(x-\Delta x)\,d\nu_N \\
&= \int x\ln x - x\ln(x-\Delta x) + \Delta x\ln(x-\Delta x)\,d\nu_N \\
&= \int -x\ln(1-\frac{\Delta x}{x}) + \Delta x\ln(x-\Delta x)\,d\nu_N \\
&\geq \int -x\cdot(-\frac{\Delta x}{x})\,d\nu_N + \int \Delta x\ln(x-\Delta x)\,d\nu_N \\
&= \int \Delta x\,d\nu_N + \int \mathbf{1}_{\{\Delta x\geq 0\}}\Delta x\ln(x-\Delta x)\,d\nu_N + \int \mathbf{1}_{\{\Delta x<0\}}\Delta x\ln(x-\Delta x)\,d\nu_N \\
&= 0 + \int \mathbf{1}_{\{\Delta x\geq 0\}}\frac{\Delta x}{x-\Delta x}(x-\Delta x)\ln(x-\Delta x)\,d\nu_N + \int \mathbf{1}_{\{\Delta x<0\}}\Delta x\ln(x-\Delta x)\,d\nu_N \\
&\geq \int \mathbf{1}_{\{\Delta x\geq 0\}}\frac{\Delta x}{x-\Delta x}(x-\Delta x-1)\,d\nu_N + \int \mathbf{1}_{\{\Delta x<0\}}\Delta x(x-\Delta x-1)\,d\nu_N \\
&= \int \mathbf{1}_{\{\Delta x\geq 0\}}\Delta x(1-\frac{1}{x-\Delta x})\,d\nu_N + \int \mathbf{1}_{\{\Delta x<0\}}(\Delta x\cdot x-(\Delta x)^2-\Delta x)\,d\nu_N \\
&\geq \int \mathbf{1}_{\{\Delta x\geq 0\}}(-\frac{\Delta x}{x-\Delta x})\,d\nu_N - C_N^2\varepsilon - C_N^3(d+1)\varepsilon^2 \\
&\geq \int \mathbf{1}_{\{\Delta x\geq 0\}}(-\frac{C_N\cdot\varepsilon}{(1-C_N\cdot\varepsilon)})\,d\nu_N - C_N^2\varepsilon - C_N^3(d+1)\varepsilon^2 \\
&\geq -\frac{C_N}{(1-C_N\cdot\varepsilon)}\cdot\varepsilon - C_N^2\varepsilon - C_N^3(d+1)\varepsilon^2 \\
&= -\Big(\frac{C_N}{(1-C_N\cdot\varepsilon)} + C_N^2 + C_N^3(d+1)\varepsilon\Big)\cdot\varepsilon \\
&:= -K_{C_N,\varepsilon}\cdot\varepsilon
\end{aligned}$$

We explain the inequalities in turn. The first inequality follows from $\ln x \leq x-1$ for $x \geq 0$, and the second inequality follows from both $\ln x \le x-1$ for $x \ge 0$ and $x \ln x \ge x-1$ for $x \ge 0$. The third inequality follows by dropping a positive term Δx in the first integral and noting that

$$\begin{aligned}\int \mathbf{1}_{\{\Delta x<0\}} \Delta x \cdot x \, d\nu_N &= \int \mathbf{1}_{\{d\mu_{N,\varepsilon}-d\hat{\mu}_N<0\}} \frac{d\mu_{N,\varepsilon}-d\hat{\mu}_N}{d\nu_N} \cdot \frac{d\mu_{N,\varepsilon}}{d\nu_N} \, d\nu_N \\ &= \sum_{ij} \frac{\mathbf{1}_{\{\varepsilon^*_{ij}<0\}} \varepsilon^*_{ij}}{\frac{1}{N} q_N(y_j)} \cdot (\mu_{N,\varepsilon})_{ij} \\ &\ge \sum_{ij} -C_N^2 \cdot \varepsilon \cdot (\mu_{N,\varepsilon})_{ij} = -C_N^2 \cdot \varepsilon, \quad \text{by (A.12)},\end{aligned}$$

and

$$\begin{aligned}\int \mathbf{1}_{\{\Delta x<0\}} (-\Delta x)^2 \, d\nu_N &\ge -\int \Delta x^2 \, d\nu_N = -\sum_{ij} \frac{(\varepsilon^*_{ij})^2}{\frac{1}{N} q_N(y_j)} \\ &\ge -\sum_{ij} \frac{C_N^2 (\frac{1}{N})^2 \varepsilon^2}{\frac{1}{N} q_N(y_j)}, \quad \text{by (A.12)}, \\ &\ge -C_N^3 (d+1) \varepsilon^2.\end{aligned}$$

The fourth inequality holds because $\Delta x \le x \cdot C_N \cdot \varepsilon$ for $\Delta x \ge 0$, by (A.12).

The coefficient $K_{C_N,\varepsilon}$ is increasing in both C_N and ε. Since $q_N(y_j) \to q(y_j)$ as $N \to \infty$, $C_N \to \max_{j=1,\cdots,d+1}\{1/q(y_j)\}$ as $N \to \infty$, thus we can find a constant $C \ge C_N$ for all N. On the other hand, without loss of generality we can assume that ε is small enough, such that $1 - C \cdot \varepsilon > 1/2$, i.e. $\varepsilon < 1/(2C)$. Choose $K_2 = K_{C,1/(2C)}$. Then

$$\int x \ln x - (x - \Delta x) \ln(x - \Delta x) \, d\nu_N \ge -K_2 \cdot \varepsilon$$

for all N and ε small enough. We thus have

$$\begin{aligned} c_{N,\varepsilon} - G(\hat{\mu}_N, \nu_N) &\leq (d+1) \cdot \max_{i=1,\ldots,N} \max_{j=1,\ldots,d+1} | < X_i, y_j > | \cdot \varepsilon + \frac{1}{\theta} K_2 \cdot \varepsilon \\ &= K_N \cdot \varepsilon, \end{aligned}$$

and ((A.13)) is proved. □

A.4 Proof of Theorem 2.6.1

Let $\mu \in \bar{\Pi}(p,q)$ be any feasible solution to the limiting problem. Write $\mu((dx,dz),y) = p(dx,dz)q(y|x,z)$, and define the mass function μ_N on $((X_i,Z_i),y_j)$, $i = 1,\ldots,N$, $j = 1,\ldots,d+1$, by setting

$$\mu_N((X_i,Z_i),y_j) = \frac{1}{N} q(y_j|(X_i,Z_i)).$$

For each y_j, we get the marginal probability

$$\bar{q}_N(y_j) = \sum_{i=1}^{N} \mu_N((X_i,Z_i),y_j) = \frac{1}{N} \sum_{i=1}^{N} q(y_j|(X_i,Z_i)).$$

The expectation of $< Z_i, y_j >$ with respect to μ_N is given by

$$\bar{v}_0^N = \sum_{i=1}^{N} \sum_{j=1}^{d+1} < Z_i, y_j > \mu_N((X_i,Z_i),y_j).$$

By the strong law of large numbers for the i.i.d. sequence (X_i, Z_i), $i = 1, \ldots, N$, we have $(\bar{q}_N(y_1), \ldots, \bar{q}_N(y_m)) \to (q(y_1), \ldots, q(y_m))$, a.s., and also

$$\begin{aligned}\bar{v}_0^N = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{d+1} < Z_i, y_j > q(y_j|(X_i, Z_i)) \quad &\to \quad \int \sum_{j=1}^{d+1} < h_Z(x,z), y_j > \; q(y_j|(x,z))\, dp(x,z) \\ &= \quad \int < h_Z(x,z), y > \; d\mu((x,z),y) = v_0,\end{aligned}$$

where v_0 is the value in the constraint (2.6.1) because $\mu \in \bar{\Pi}(p,q)$. In fact, by the law of the iterated logarithm, if we set $\epsilon_N = 1/N^{\alpha}$ with $0 < \alpha < 1/2$, then, with probability 1,

$$\max_{1 \le j \le d+1} |\bar{q}_N(y_j) - q(y_j)| < \epsilon_N, \quad \max_{1 \le j \le d+1} |\bar{q}_N(y_j) - q_N(y_j)| < \epsilon_N$$

and, under our square-integrability condition on Z,

$$|\bar{v}_0^N - v_0| < \epsilon_N,$$

for all sufficiently large N. It follows that $\mu_N \in \bar{\Pi}_{\epsilon_N}(p_N, q_N)$, for all sufficiently large N.

A.4.1 Upper Bound

Because μ_N is feasible for all sufficiently large N, it provides a lower bound on the optimal value c_{N,ϵ_N} in (2.6.4),

$$c_{N,\epsilon_N} \ge \sum_{i=1}^{N}\sum_{j=1}^{d+1} \mu_N((X_i, Z_i), y_j) < X_i, y_j >= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{d+1} q(y_j|(X_i, Z_i)) < X_i, y_j > .$$

By the strong law of large numbers

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{d+1} q(y_j|(X_i,Z_i)) < X_i, y_j > \;\rightarrow\; \int_{\mathbb{R}^d\times\mathbb{R}^d}\sum_{j=1}^{d+1} q(y_j|(x,z)) < h_X(x,z), y_j > dp(x,z)$$
$$= \int_{(\mathbb{R}^d\times\mathbb{R}^d)\times\mathbb{R}^d} < h_X(x,z), y > d\mu((x,z),y).$$

So

$$\varliminf_{N\to\infty} c_{N,\epsilon_N} \geq \int_{(\mathbb{R}^d\times\mathbb{R}^d)\times\mathbb{R}^d} < h_X(x,z), y > d\mu((x,z),y)$$

And since this holds for any $\mu \in \bar{\Pi}(p,q)$,

$$\varliminf_{N\to\infty} c_{N,\epsilon_N} \geq c_\infty. \tag{A.1}$$

A.4.2 Lower Bound

To prove a lower bound, we formulate a dual problem for the relaxed finite-N problem (2.6.4) with objective value $d_{N,\epsilon}$, and we formulate a dual for the limiting problem (2.6.3) with objective value d_∞.

The relaxed finite problem in (2.6.4) is a linear program. Its dual can be written as

$$d_{N,\epsilon} \equiv \min_{\Phi,\Psi_1,\Psi_2,\xi_1,\xi_2}\{F_N(\Phi,\Psi_1,\Psi_2,\xi_1,\xi_2) + \epsilon K(\Psi_1,\Psi_2,\xi_1,\xi_2)\} \tag{A.2}$$

with

$$F_N(\Phi,\Psi_1,\Psi_2,\xi_1,\xi_2) = \frac{1}{N}\sum_{i=1}^{N}\Phi_i + \sum_{j=1}^{d+1}(\Psi_{1j}+\Psi_{2j})\cdot q_N(y_j) + (\xi_1+\xi_2)v_0$$

and

$$K(\Psi_1, \Psi_2, \xi_1, \xi_2) = \sum_{j=1}^{d+1} (\Psi_{1j} - \Psi_{2j}) + (\xi_1 - \xi_2),$$

the infimum taken over $\Phi \in \mathbb{R}$, $\Psi_{1j} \geq 0$, $\Psi_{2j} \leq 0$, $\xi_1 \geq 0$, $\xi_2 \leq 0$, satisfying

$$\Phi_i + \Psi_{1j} + \Psi_{2j} + (\xi_1 + \xi_2) \cdot < Z_i, y_j > \geq < X_i, y_j >,$$

for $i = 1, ..., N$, and all $j = 1, ..., d+1$ with $q_N(y_j) > 0$. We have already seen that problem (2.6.4) is feasible for all sufficiently large N, and once it is feasible $c_{N,\epsilon} = d_{N,\epsilon}$ by standard linear programming duality.

We define the dual of the limiting problem (2.6.3) by setting

$$d_\infty = \inf_{\phi, \psi, \xi} F(\phi, \psi, \xi)$$

with

$$F(\phi, \psi, \xi) = \int \phi(x, z)\, dp(x, z) + \sum_{j=1}^{d+1} \psi(y_j) q(y_j) + \xi v_0,$$

the infimum taken over functions $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $\psi : \mathbb{R}^d \to \mathbb{R}$, and a scalar $\xi \in \mathbb{R}$, satisfying, for all (x, z) in the support of p and all y in the support of q,

$$\phi(x, z) + \psi(y) + \xi < z, y > \ \geq \ < x, y >,$$

with $\phi \in L^1(p)$.

For any $\tilde{\epsilon} > 0$, we may pick $\phi_{\tilde{\epsilon}}$, $\psi_{\tilde{\epsilon}}$, and $\xi_{\tilde{\epsilon}}$ feasible for the limiting dual and for which

$$F(\phi_{\tilde{\epsilon}}, \psi_{\tilde{\epsilon}}, \xi_{\tilde{\epsilon}}) \leq d_\infty + \tilde{\epsilon}.$$

We may then define a feasible solution to (A.2) by setting $\Phi_i = \phi_{\tilde{\epsilon}}(X_i, Z_i)$, $\Psi_{1j} = \psi^+_{\tilde{\epsilon}}(y_j)$, $\Psi_{2j} = -\psi^-_{\tilde{\epsilon}}(y_j)$, $\xi_1 = \xi^+_{\tilde{\epsilon}}$, and $\xi_2 = -\xi^-_{\tilde{\epsilon}}$. By the strong law of large numbers, this choice yields

$$F_N(\Phi, \Psi_1, \Psi_2, \xi_1, \xi_2) \to F(\phi_{\tilde{\epsilon}}, \psi_{\tilde{\epsilon}}, \xi_{\tilde{\epsilon}}), \quad \text{a.s.}$$

For any $\bar{\epsilon} > 0$, there is a stochastic $N(\tilde{\epsilon}, \bar{\epsilon})$ such that for all $N > N(\tilde{\epsilon}, \bar{\epsilon})$,

$$F_N(\Phi, \Psi_1, \Psi_2, \xi_1, \xi_2) \leq F(\phi_{\tilde{\epsilon}}, \psi_{\tilde{\epsilon}}, \xi_{\tilde{\epsilon}}) + \bar{\epsilon}, \quad \text{a.s.,}$$

and this $N(\tilde{\epsilon}, \bar{\epsilon})$ does not depend on the ϵ that defines the relaxation (A.2). Thus, we have, for all sufficiently large N,

$$d_{N,\epsilon} \leq d_\infty + \tilde{\epsilon} + \bar{\epsilon} + K(\Psi_1, \Psi_2, \xi_1, \xi_2)\epsilon;$$

and, because $N(\tilde{\epsilon}, \bar{\epsilon})$ does not depend on ϵ,

$$d_{N,\epsilon_N} \leq d_\infty + \tilde{\epsilon} + \bar{\epsilon} + K(\Psi_1, \Psi_2, \xi_1, \xi_2)\epsilon_N,$$

for all $N > N(\tilde{\epsilon}, \bar{\epsilon})$, so

$$\overline{\lim_{N\to\infty}}\, d_{N,\epsilon_N} \leq d_\infty + \tilde{\epsilon} + \bar{\epsilon}.$$

Because $\tilde{\epsilon} > 0$ and $\bar{\epsilon} > 0$ are arbitrary,

$$\overline{\lim_{N\to\infty}}\, d_{N,\epsilon_N} \leq d_\infty.$$

We have already noted that $d_{N,\epsilon_N} = c_{N,\epsilon_N}$ by ordinary linear programming duality. In Appendix A.4.3 we show that that

$$d_\infty = c_\infty. \tag{A.3}$$

Thus,

$$\overline{\lim_{N\to\infty}}\, c_{N,\epsilon_N} = \overline{\lim_{N\to\infty}}\, d_{N,\epsilon_N} \leq d_\infty = c_\infty,$$

which, together with (A.1) proves the result.

A.4.3 A Duality Result

In this section, we prove the equality $c_\infty = d_\infty$ used in Appendix A.4.2. The result follows from Theorem 5.10 of Villani [54], once we show that we can transform the primal problem to an equivalent problem that satisfies the conditions of the theorem. We formulate the equivalent problem using a result of Luenberger [42], for which we adopt his notation.

Let X be the vector space of signed finite measures μ on $(\mathbb{R}^d \times \mathbb{R}^d) \times \mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \|u\| \, \mu(dx, \mathbb{R}^d) < \infty.$$

Let $\Omega \subset X$ be the subset of probability measures with marginals p and q, which is a convex set. For $\mu \in X$, let

$$f(\mu) \;=\; \int_{(\mathbb{R}^d \times \mathbb{R}^d) \times \mathbb{R}^d} - < h_X(x,z), y > \; d\mu((x,z),y).$$

Let $G(\cdot)$ be a mapping from X to $\mathbb{R}$ defined by

$$G(\mu) = \int < h_Z(x,z), y > d\mu((x,z),y) - v_0.$$

The primal problem is

$$c_\infty = - \inf_{\mu \in \Omega, G(\mu)=0} f(\mu).$$

Define

$$L(\xi) = \inf_{\mu \in \Omega} \big\{ \int - < h_X(x,z), y > d\mu((x,z),y) + \xi \cdot G(\mu) \big\}. \tag{A.4}$$

Now apply Theorem 1 of Section 8.6 of Luenberger [42] (with the extension in problem 7 of Section 8.8) to conclude that

$$\inf_{\mu \in \Omega, G(\mu)=0} f(\mu) = \max_{\xi \in \mathbb{R}} \quad L(\xi),$$

and there exists ξ^* such that $L(\xi^*) = -c_\infty$.

Drop the constant term $-\xi^* \cdot v_0$ in $L(\xi^*)$, and denote it by L^*, so

$$\begin{aligned} L^* &= \inf_{\mu \in \Pi(p,q)} \int -< h_X(x,z), y > d\mu((x,z),y) + \xi^* \cdot \int < h_Z(x,z), y > d\mu((x,z),y) \\ &= \inf_{\mu \in \Pi(p,q)} \int \big(-< h_X(x,z), y > +\xi^* \cdot < h_Z(x,z), y > \big) d\mu((x,z),y) \end{aligned}$$

Define the dual problem DL^*,

$$DL^* = \sup_{(\phi,\psi) \in L^1(p) \times L^1(q); \quad -\phi-\psi \leq -c+\xi^* \cdot v} - \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x,z) dp(x,z) - \sum_{j=1}^{d+1} \psi(y_j) q(y_j),$$

where $c((x,z),y) =< h_X(x,z), y >$, and $v((x,z),y) =< h_Z(x,z), y >$.

Let $a(x,z) = \frac{1}{2} < (x, \xi^* z), (x, \xi^* z) >$ and $b(y) = \frac{1}{2} < y, y >$. We have

$$-< h_X(x,z), y > +\xi^* \cdot < h_Z(x,z), y > \quad \geq \quad -a(x,z) - b(y).$$

By condition (i) in Theorem 2.6.1, $a \in L^1(p)$ and $b \in L^1(q)$. It follows from Theorem 5.10 of Villani [54] that strong duality holds, i.e. $L^* = DL^*$.

Since $L^* < +\infty$ and $-< h_X(x,z), y > +\xi^* \cdot < h_Z(x,z), y > \quad \leq \quad a(x,z) + b(y)$, it follows from part (iii) of Theorem 5.10 of Villani [54] that solutions exists for both problems. Let (ϕ^*, ψ^*) denote an optimal solution to DL^*, then (ϕ^*, ψ^*, ξ^*) is a feasible solution to the dual problem

$$d_\infty = \inf_{\phi(x,z)+\psi(y)+\xi v((x,z),y) \geq c((x,z),y)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x,z) dp(x,z) + \sum_{j=1}^{d+1} \psi(y_j) q(y_j) + \xi v_0.$$

Let d^* denote the objective value by substituting (ϕ^*, ψ^*, ξ^*) in the objective function. Note that $d^* = -DL^* + \xi^* v_0 = -L^* + \xi^* v_0 = c_\infty$, so (ϕ^*, ψ^*, ξ^*) is optimal for the dual problem d_∞, and strong duality holds $d_\infty = c_\infty$. $\square$

A.5 Proof of Theorem 2.6.2

We show the convergence result for the penalty problems with the auxiliary constraints in (2.6.1) as $N \to \infty$. We start with the convergence of the objective function value asserted in part (i) of the theorem.

A.5.1 Convergence of the Optimal Objective Value

Let G_∞ denote the optimal value of the penalty limit problem,

$$G_\infty = \sup_{\mu \in \bar{\Pi}(p,q)} G(\mu, \nu) = \sup_{\mu \in \bar{\Pi}(p,q)} \int < x, y > \, d\mu - \frac{1}{\theta} \int \ln(\frac{d\mu}{d\nu}) \, d\mu. \tag{A.1}$$

Let $G_{N,\epsilon}$ be the optimal value of the penalty finite relaxed problem with sample size N,

$$G_{N,\epsilon} = \sup_{\mu \in \bar{\Pi}_\epsilon(p_N, q_N)} G(\mu, \nu_N) = \sup_{\mu \in \bar{\Pi}_\epsilon(p_N, q_N)} \int < x, y > \, d\mu - \frac{1}{\theta} \int \ln(\frac{d\mu}{d\nu_N}) \, d\mu. \tag{A.2}$$

Lemma A.5.1. $\underline{\lim}_{N\to\infty} G_{N,\epsilon_N} \geq G_\infty$*, for* $\epsilon_N = 1/N^\alpha$ *and* $\alpha \in (0, 1/2)$*.*

Proof: Let $\mu \in \bar{\Pi}(p,q)$ be any feasible solution to the limiting problem. Define a mass function on the pairs $((X_i, Z_i), y_j)$, $i = 1, \cdots, N$, $j = 1, \cdots, d+1$:

$$\mu_N((X_i, Z_i), y_j) = \frac{1}{N} q(y_j | (X_i, Z_i)).$$

From the argument in Appendix A.4, we know that $\mu_N \in \bar{\Pi}^N_{\epsilon_N}$, so

$$\begin{aligned} G_{N,\epsilon_N} &\geq \sum_{i=1}^{N}\sum_{j=1}^{d+1} \mu_N((X_i, Z_i), y_j) < X_i, y_j > -\frac{1}{\theta} D(\mu_N|\nu_N) \\ &= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{d+1} q(y_j|(X_i, Z_i)) < X_i, y_j > -\frac{1}{\theta}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{d+1} \ln(\frac{q(y_j|(X_i, Z_i))}{q(y_j)})q(y_j|(X_i, Z_i)) \\ &\to \int < x, y > d\mu((x,z),y) - \frac{1}{\theta}\int \ln(\frac{d\mu}{d\nu})d\mu, \end{aligned}$$

the limit following from the strong law of large numbers. Thus, $\underline{\lim}_{N\to\infty} G_{N,\epsilon_N} \geq G_\infty$. $\square$

We have shown that the limiting objective value is a lower bound for the sequence in part (i) of Theorem 2.6.2. We will use a dual formulation to show the reverse inequality. The argument requires several lemmas.

We reformulate the problem of maximizing $G(\cdot, \nu)$ as

$$-\frac{1}{\theta}\min_{\mu\in\bar{\Pi}(p,q)} \int \left(\frac{d\mu((x,z),y)}{\exp(\theta < x, y >)d\nu((x,z),y)}\right) d\mu((x,z),y) = -\frac{1}{\theta}\min_{\mu\in\bar{\Pi}(p,q)} D(\mu|e^{(\theta<x,y>)}\nu). \tag{A.3}$$

Dropping the constant factor $-1/\theta$ from (A.3), we get the equivalent problem

$$P_\infty = \min_{\mu\in\bar{\Pi}(p,q)} D(\mu|e^{(\theta<x,y>)}\nu) \tag{A.4}$$

Define $\bar{\Pi}^*(p,q)$ to be the set of functions $h : (\mathbb{R}^d \times \mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ of the form

$$h((x,z),y) = h_1(x,z) + h_2(y) + h_3 v((x,z),y) - h_4$$

where

$$v((x,z),y) =< h_Z(x,z), y >=< z, y >,$$

with

$$\int h((x,z),y) d\mu((x,z),y) \geq 0, \quad \text{for all } \mu \in \bar{\Pi}(p,q).$$

Lemma A.5.2. *Let D_∞ be the dual problem to P_∞, defined as*

$$D_\infty = \inf_{h \in \bar{\Pi}^*(p,q)} \ln \int e^{h((x,z),y)+\theta<x,y>} d\nu((x,z),y), \tag{A.5}$$

The following statements hold:

(i) The optimal solution to the primal problem is

$$d\mu^*((x,z),y) = e^{a(x,z)+b(y)+\xi v((x,z),y)+\theta<x,y>} d\nu((x,z),y). \tag{A.6}$$

(ii) The optimal solution to the dual problem is

$$h^*((x,z),y) = h_1^*(x,z) + h_2^*(y) + h_3^* v((x,z),y) - h_4^*,$$

$$h_1^*(x,z) = a(x,z), \quad h_2^*(x) = b(x), \quad h_3^* = \xi,$$

$$h_4^* = \int a(x,z) + b(y) + \xi \cdot v((x,z),y) \, d\mu^*((x,z),y).$$

(iii) Strong duality holds, $P_\infty = -D_\infty$.

Proof: Conclusion (i) follows Theorem 3 of Rüschendorf and Thomsen [50].

To apply the dual formulation in Bhattacharya, we consider the set $\bar{\Pi}^*(p,q)$ of functions $h : (\mathbb{R}^d \times \mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ of the form

$$h((x,z),y) = h_1(x,z) + h_2(y) + h_3 v((x,z),y) - h_4$$

with, for any $\mu \in \bar{\Pi}(p,q)$

$$\begin{aligned} \int h((x,z),y) d\mu((x,z),y) &= \int h_1(x,z)\, dp((x,z)) + \int h_2(y)\, dq(y) + h_3 v_0 - h_4 \\ &\geq 0. \end{aligned}$$

Observe that (the convex cone) $\bar{\Pi}^*(p,q)$ is contained within the dual cone of $\bar{\Pi}(p,q)$. We consider the dual problem

$$\inf_{h \in \bar{\Pi}^*} \ln \int e^{h((x,z),y) + \theta <x,y>}\, d\nu((x,z),y).$$

With μ^*, $a(x,z)$, $b(x)$, ξ as in (A.6), set

$$h_1^*(x,z) = a(x,z), \quad h_2^*(x) = b(x), \quad h_3^* = \xi,$$

$$h_4^* = c \equiv \int a(x,z) + b(y) + \xi \cdot v((x,z),y)\, d\mu^*((x,z),y).$$

Observe that

$$\int h^*((x,z),y)d\mu((x,z),y) = \int h_1^*((x,z))\,dp((x,z)) + \int h_2^*(y)\,dq(y) + h_3^*v_0 - h_4^* = 0,$$

for all $\mu \in \bar{\Pi}(p,q)$, so this (h_1^*, h_2^*, h_3^*, h_4^*) is dual feasible. Moreover, with this choice of h_1^*, h_2^*, h_3^*, h_4^*, the dual objective function value in (A.5) is

$$D_\infty = \ln \int e^{a(x,z)+b(y)+\xi v((x,z),y)-c+\theta<x,y>}\,d\nu((x,z),y) = \ln \int e^{-c}\,d\mu^*(x,y) = -c.$$

The primal objective function value is

$$P_\infty = D(\mu^*|e^{\theta<x,y>}\nu) = \int a(x,z) + b(y) + \xi v((x,z),y)\,d\mu^*((x,z),y) = c.$$

It follows from Theorem 2.1 of Bhattacharya that this choice of $(h_1^*, h_2^*, h_3^*, h_4^*)$ is optimal for the dual problem (A.5), and strong duality holds $P_\infty = -D_\infty$. □

Next we establish a similar result for the discrete problem. Define $\bar{\Pi}^*_{\epsilon_N}(p_N, q_N)$ to be set of functions $h : (\mathbb{R}^d \times \mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ of the form

$$h((x,z),y) = h_1(x,z) + h_2(y) + h_3 v((x,z),y) - h_4$$

with

$$\int h((x,z),y)d\mu_N((x,z),y) \geq 0,$$

for all $\mu_N \in \bar{\Pi}_{\epsilon_N}(p_N, q_N)$.

Lemma A.5.3. *For the primal problem*

$$P_{N,\epsilon_N} = \min_{\mu \in \bar{\Pi}_{\epsilon_N}(p_N,q_N)} \int \left(\frac{d\mu((x,z),y)}{\exp(\theta < x,y >) d\nu_N((x,z),y)} \right) d\mu((x,z),y), \tag{A.7}$$

define the dual

$$D_{N,\epsilon_N} = \inf_{h \in \bar{\Pi}^*_{\epsilon_N}(p_N,q_N)} \ln \int e^{h((x,z),y)+\theta<x,y>} \, d\nu_N((x,z),y). \tag{A.8}$$

The following statements hold:

(i) The optimal solution to the primal problem takes the form

$$d\mu^*_N((x,z),y) = e^{a^N(x,z)+b_1^N(y)+b_2^N(y)+\xi_1^N v((x,z),y)+\xi_2^N v((x,z),y)+\theta<x,y>} \, d\nu_N((x,z),y),$$

where $b_1^N(y) \leq 0$*,* $b_2^N(y) \geq 0$*,* $\xi_1^N \leq 0$*,* $\xi_2^N \geq 0$*.*

(ii) A feasible solution to the dual is $\tilde{h}$*,*

$$\tilde{h}((x,z),y) = \tilde{h}_1(x,z) + \tilde{h}_2(y) + \tilde{h}_3 v((x,z),y) - \tilde{h}_4, \text{ where}$$

$$\tilde{h_1}(x,z) = a(x,z), \quad \tilde{h_2}(x) = b_1(x) + b_2(x), \quad \tilde{h_3} = \xi_1 + \xi_2,$$

$$\begin{aligned}\tilde{h_4} = \quad & \int a(x,z)\, dp_N(x,z) + \int (b_1(y)+b_2(y))\, dq_N(y) + (\xi_1+\xi_2)v_0 \\ & + \sum_{j=1}^{d+1} (b_1(y_j) - b_2(y_j))\epsilon_N + (\xi_1 - \xi_2)\epsilon_N\end{aligned}$$

where $b_1(y) = b(y)^-$, $b_2(y) = b(y)^+$, *and* $\xi_1 = \xi^-$, $\xi_2 = \xi^+$, *for* $a(x,z)$, $b(x)$, ξ *as in (A.6).*

(iii) $D_\infty \geq \overline{\lim}_{N\to\infty} \; D_{N,\epsilon_N}$.

Proof: Conclusion (i) is the discrete form of part (i) in Lemma A.5.2. For (ii), we consider the dual problem

$$\inf_{h\in\bar{\Pi}^*_{\epsilon_N}(p_N,q_N)} \ln \int e^{h((x,z),y)+\theta<x,y>}\, d\nu_N((x,z),y).$$

Let $\tilde{h_1}(x,z) = a(x,z)$, $\tilde{h_2}(x) = b_1(x) + b_2(x)$, $\tilde{h_3} = \xi_1 + \xi_2$ and

$$\begin{aligned}\tilde{h_4} = \tilde{c} \equiv \quad & \int a(x,z)\, dp_N(x,z) + \int b_1(y) + b_2(y)\, dq_N(y) + (\xi_1+\xi_2)v_0 \\ & + \sum_{j=1}^{d+1} (b_1(y_j) - b_2(y_j))\epsilon_N + (\xi_1 - \xi_2)\epsilon_N\end{aligned}$$

where $b_1(y) = b(y)^-$, $b_2(y) = b(y)^+$, and $\xi_1 = \xi^-$, $\xi_2 = \xi^+$, for $a(x,z)$, $b(x)$, ξ as in (A.6).

Notice that

$$\sum_{j=1}^{d+1} (b_1(y_j) - b_2(y_j))\epsilon_N + (\xi_1 - \xi_2)\epsilon_N \leq 0.$$

For any $\mu_N \in \bar{\Pi}_{\epsilon_N}(p_N, q_N)$,

$$
\begin{aligned}
\int \tilde{h}(x,y) d\mu_N((x,z),y) &= \int \tilde{h_1}(x,z)\, dp_N(x,z) \\
&\quad + \int \left(\tilde{h_2}(y) + \tilde{h_3} v((x,z),y)\right) d\mu_N((x,z),y) - \tilde{h_4} \\
&\geq \int \tilde{h_1}(x,z)\, dp_N(x,z) + \int \tilde{h_2}(y)\, dq_N(y) + \tilde{h_3} v_0 - \tilde{h_4} \\
&\quad + \sum_{j=1}^{d+1} (b_1(y_j) - b_2(y_j))\epsilon_N + (\xi_1 - \xi_2)\epsilon_N \\
&= \int a(x,z)\, dp_N(x,z) + \int b_1(y) + b_2(y)\, dq_N(y) + (\xi_1 + \xi_2) v_0 \\
&\quad + \sum_{j=1}^{d+1} (b_1(y_j) - b_2(y_j))\epsilon_N + (\xi_1 - \xi_2)\epsilon_N - \tilde{h_4} \\
&= 0
\end{aligned}
$$

so this ($\tilde{h_1}$, $\tilde{h_2}$, $\tilde{h_3}$, $\tilde{h_4}$) is feasible for the dual problem (A.8).

For (iii), let $\tilde{D}_{N,\epsilon_N}$ denote the objective value in (A.8) with solution $\tilde{h}$. Since $\tilde{h}$ is dual feasible,

$$\tilde{D}_{N,\epsilon_N} \geq D_{N,\epsilon_N}$$

We show that $\tilde{D}_{N,\epsilon_N} \to D_\infty$ as $N \to \infty$. Substituting $\tilde{h}$ in (A.8)

$$\begin{aligned}
\tilde{D}_{N,\epsilon_N} &= \ln \sum_{i=1}^{N} \sum_{j=1}^{d+1} \exp\big(a(X_i, Z_i) + b(y_j) + \xi v((X_i, Z_i), y_j) - \frac{1}{N} \sum_{i=1}^{N} a(X_i, Z_i) \\
&\quad - \int b(y)\, dq_N(y) - \xi v_0 - \sum_{j=1}^{d+1} |b(y_j)|\epsilon_N - |\xi|\epsilon_N + \theta < X_i, y_j > \big) \cdot \frac{1}{N} \cdot q_N(y_j) \\
&= \ln \exp\big(- \frac{1}{N} \sum_{i=1}^{N} a(X_i, Z_i) - \int b(y)\, dq_N(y) - \xi v_0 - \sum_{j=1}^{d+1} |b(y_j)|\epsilon_N - |\xi|\epsilon_N \big) \\
&\quad + \ln \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{d+1} \exp\big(a(X_i, Z_i) + b(y_j) + \xi v(X_i, y_j) + \theta < X_i, y_j > \big) q_N(y_j) \\
&= \big(- \frac{1}{N} \sum_{i=1}^{N} a(X_i, Z_i) - \frac{1}{N} \sum_{j=1}^{N} b(Y_j) - \xi v_0 - \sum_{j=1}^{d+1} |b(y_j)|\epsilon_N - |\xi|\epsilon_N \big) \\
&\quad + \ln \big(\frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \exp(a(X_i, Z_i) + b(Y_j) + \xi v((X_i, Z_i), Y_j) + \theta < X_i, Y_j >) \big) \\
\longrightarrow &- \int a(x,z)\, dp(x,z) - \int b(y)\, dq(y) - \xi v_0 + \ln \int e^{a(x,z) + b(y) + \xi v((x,z),y) + \theta < x,y >}\, d\nu \\
&= D_\infty
\end{aligned}$$

so $D_\infty \geq \overline{\lim}_{N\to\infty} \; D_{N,\epsilon_N}$. □

Lemma A.5.4. $G_\infty \geq \limsup_N G_{N,\epsilon_N}$.

Proof: From strong duality of the continuous problem (A.4) and (A.5) in Lemma A.5.2, we have $G_\infty = -\frac{1}{\theta} P_\infty = \frac{1}{\theta} D_\infty$. By weak duality of the finite relaxed problem (A.7) and (A.8), we have $\frac{1}{\theta} D_{N,\epsilon_N} \geq -\frac{1}{\theta} P_{N,\epsilon_N} = G_{N,\epsilon_N}$. Therefore by Lemma A.5.3, we have

$$G_\infty = \frac{1}{\theta} D_\infty \geq \limsup_N \frac{1}{\theta} D_{N,\epsilon_N} \geq \limsup_N G_{N,\epsilon_N}.$$

□

Combining Lemma A.5.1 and Lemma A.5.4 proves part (i) of Theorem 2.6.2.

A.5.2 Weak Convergence of Optimal Solutions

The argument is similar to that of Section A.3.2. Define

$$\bar{\Pi}^N = \bar{\Pi}(p,q) \cup \Big(\bigcup_{n \geq N} \bar{\Pi}_{\epsilon_n}(p_n, q_n) \Big).$$

By the argument used in Lemma A.3.3, we have

Lemma A.5.5. $\bar{\Pi}^N$ *is compact for all sufficiently large N, a.s.*

The optimizers $\bar{\mu}^*_N$ are contained in the sets $\bar{\Pi}_{\epsilon_N}(p_N, q_N)$, so for all sufficiently large N, the sequence $\bar{\mu}^*_n$, $n \geq N$, is contained in a compact set $\bar{\Pi}^N$, and then every subsequence has a further subsequence that converges weakly.

Suppose the subsequence $\bar{\mu}^*_{n_k}$ converges, say $\bar{\mu}^*_{n_k} \Rightarrow \tilde{\mu}$. The marginals of $\bar{\mu}^*_{n_k}$ converge to p and q, and $\lim_{k\to\infty} \int v((x,z),y)\, d\bar{\mu}^*_{n_k} = v_0$, so $\tilde{\mu} \in \bar{\Pi}(p,q)$, making $\tilde{\mu}$ feasible for the limiting problem. We claim that it is optimal. We have, a.s.,

$$\int e^{\theta < h_x(x,z), y>}\, d\bar{\mu}^*_{n_k} \leq \int \sum_{j=1}^{d+1} e^{\theta < h_x(x,z), y_j>}\, dp_{n_k}(x,z) \to \int \sum_{j=1}^{d+1} e^{\theta < h_x(x,z), y_j>}\, dp(x,z),$$

by the strong law of large numbers, because the condition $\mathbb{E}_\nu[e^{\theta < h_x(x,z), y>}] < \infty$ implies that the limit is finite. This is then more than sufficient to ensure that

$$\int < h_x(x,z), y > \, d\bar{\mu}^*_{n_k}((x,z),y) \to \int < h_x(x,z), y > \, d\tilde{\mu}((x,z),y). \tag{A.9}$$

Moreover, relative entropy is lower semi-continuous with respect to weak convergence (Dupuis and Ellis [30], Lemma 1.4.3), so

$$D(\tilde{\mu}|\nu) \leq \varliminf_{k\to\infty} D(\bar{\mu}^*_{n_k}|\nu_{n_k})$$

and then

$$G(\tilde{\mu},\nu) \geq \varlimsup_{k\to\infty} G(\bar{\mu}^*_{n_k},\nu_{n_k}) = \sup_{\mu\in\bar{\Pi}(p,q)} G(\mu,\nu),$$

by part (i) of the theorem. Thus, $\tilde{\mu}$ is optimal. Using the equivalence between the optimization of $G(\cdot,\nu)$ and (A.4), we know from Theorem 2.1 of Csiszár [26] that the maximum is uniquely attained by some $\bar{\mu}^*$, and thus $\tilde{\mu} = \bar{\mu}^*$.

We have shown that every subsequence of $\bar{\mu}^*_n$ has a further subsequence that converges to $\bar{\mu}^*$. It follows that $\bar{\mu}^*_n \Rightarrow \bar{\mu}^*$. This proves part (ii) of the theorem. The uniform integrability needed for (2.6.5) follows as in (A.9), which proves part (iii). $\square$

Appendix B

Additional Proofs for Chapter 3

B.1 Proof of Proposition 3.3.1

We prove the Proposition for the case of $V(x,p)$; the case of $\bar{V}(x)$ is similar. For each given decision $t \leq \tau_1 \leq \zeta_2 \leq \tau_2 \leq \zeta_3 \leq \dots$, denote

$$\tilde{\mu}_s := \sum_{i=1}^{\infty} \left[\mu \mathbf{1}_{\{s \in [\zeta_i, \tau_i)\}} + r \mathbf{1}_{\{s \in [\tau_i, \zeta_{i+1})\}} \right], \quad \tilde{\sigma}_s := \sum_{i=1}^{\infty} \left[\sigma \mathbf{1}_{\{s \in [\zeta_i, \tau_i)\}} \right], \quad s \geq t,$$

and

$$dZ_s := \tilde{\mu}_s Z_s ds + \tilde{\sigma}_s Z_s dW_s, \; s \geq t, \quad Z_t = 1.$$

Then, for $s \in (\zeta_i, \tau_i]$, we have

$$X_{s-} = xK^{i-1}Z_s, s \in (\zeta_i, \tau_i], \quad X_{s-} = xK^{i-1}(1-k_s)Z_s, s \in (\tau_i, \zeta_{i+1}], \quad i \geq 1,$$
$$P_{s-} = p, s \in [t, \zeta_2], \quad P_{s-} = X_{\zeta_i} = \frac{X_{\zeta_i -}}{1+k_p} = xK^{i-1}Z_{\zeta_i}, s \in (\zeta_i, \zeta_{i+1}], \quad i \geq 2.$$

Part one: We prove that $V(x,p)$ is finite if $\delta+\rho > \max\{\beta r, \beta\mu - \frac{\beta(1-\beta)}{2}\sigma^2, \bar{\alpha}\mu - \frac{\bar{\alpha}(1-\bar{\alpha})}{2}\sigma^2\}$. We prove it only for the case of $\rho > 0$; the case of $\rho = 0$ is similar.

We first show that there exists $\tilde{\alpha} \in [\bar{\alpha}, 1]$ such that $\bar{u}(x) \leq C(1+x^{\tilde{\alpha}}), \forall x > 0$ for some $C > 0$ and $\rho + \delta > \tilde{\alpha}\left(\mu - \frac{1-\tilde{\alpha}}{2}\sigma^2\right)$. This is true when $\bar{\alpha} < 1$ because $\bar{\alpha} = \lim_{x\to+\infty} \frac{x\bar{u}'(x)}{\bar{u}(x)}$ and $\rho + \delta > \bar{\alpha}\left(\mu - \frac{1-\bar{\alpha}}{2}\sigma^2\right)$. When $\bar{\alpha} = 1$, we set $\tilde{\alpha} = 1$, and then we have $\rho + \delta > \tilde{\alpha}\left(\mu - \frac{1-\tilde{\alpha}}{2}\sigma^2\right)$. In addition, in the case that $\gamma_+ = 0$ we have $\bar{u}(x) = u(x) \leq C(1+x), \forall x > 0$ for some $C > 0$ because of the concavity of $u(x)$ in $x > 1$, and in the case that $\gamma_+ > 0$, we also have $\bar{u}(x) \leq C(1+x), \forall x > 0$ for some $C > 0$.

Observe that $U(G_t, R_t) = P_t^\beta \bar{u}(X_t/P_t) \leq CP_t^\beta\left(1 + (X_t/P_t)^{\tilde{\alpha}}\right)$. Therefore, for any stopping times $t \leq \tau_1 \leq \zeta_1 \leq \tau_2 \leq \cdots$, each $i \geq 2$, and each $T \geq t$, we have

$$\begin{aligned}
&\mathbb{E}_t[e^{-\delta(\tau_i - t)}U(G_{\tau_i-}, R_{\tau_i-})\mathbf{1}_{\{\tau_i<\tilde{\tau}\}}|\tilde{\tau} = T] \\
\leq& \mathbb{E}_t[e^{-\delta(\tau_i - t)}CP_{\tau_i-}^\beta\left(1 + (X_{\tau_i-}/P_{\tau_i-})^{\tilde{\alpha}}\right))\mathbf{1}_{\{\tau_i<\tilde{\tau}\}}|\tilde{\tau} = T] \\
=& \mathbb{E}_t[e^{-\delta(\tau_i - t)}C\left(xK^{i-1}Z_{\zeta_i}\right)^\beta\left(1 + (Z_{\tau_i}/Z_{\zeta_i})^{\tilde{\alpha}}\right))\mathbf{1}_{\{\tau_i<T\}}] \\
=& Cx^\beta(K^\beta)^{i-1}\mathbb{E}_t[e^{-\delta(\tau_i - t)}Z_{\zeta_i}^\beta\left(1 + (Z_{\tau_i}/Z_{\zeta_i})^{\tilde{\alpha}}\right))\mathbf{1}_{\{\tau_i<T\}}].
\end{aligned}$$

On the one hand,

$$
\begin{aligned}
&\mathbb{E}_t[e^{-\delta(\tau_i - t)} Z_{\zeta_i}^{\beta} \mathbf{1}_{\{\tau_i < T\}}] \\
=&\mathbb{E}_t\Big[\exp\big(-\delta(\tau_i - t) + \beta \int_t^{\zeta_i} \tilde{\mu}_s ds - \frac{1}{2}\beta(1-\beta)\int_t^{\zeta_i} \tilde{\sigma}_s^2 ds \\
&\quad - \frac{1}{2}\beta^2 \int_t^{\zeta_i} \tilde{\sigma}_s^2 ds + \beta \int_t^{\zeta_i} \tilde{\sigma}_s dW_s\big) \mathbf{1}_{\{\tau_i < T\}}\Big] \\
\leq&\mathbb{E}_t\Big[\exp\big(-\delta(\zeta_i - t) + \max\left(\beta\mu - \frac{1}{2}\beta(1-\beta)\sigma^2, \beta r\right)(\zeta_i - t) \\
&\quad - \frac{1}{2}\beta^2 \int_t^{\zeta_i} \tilde{\sigma}_s^2 ds + \beta \int_t^{\zeta_i} \tilde{\sigma}_s dW_s\big) \mathbf{1}_{\{\tau_i < T\}}\Big] \\
\leq& \exp\left[\max\left(-\delta + \max\left(\beta\mu - \frac{1}{2}\beta(1-\beta)\sigma^2, \beta r\right), 0\right)(T - t)\right] \\
&\times \mathbb{E}_t\Big[\exp\big(-\frac{1}{2}\beta^2 \int_t^{\zeta_i} \tilde{\sigma}_s^2 ds + \beta \int_t^{\zeta_i} \tilde{\sigma}_s dW_s\big) \mathbf{1}_{\{\tau_i < T\}}\Big] \\
\leq& \exp\left[\max\left(-\delta + \max\left(\beta\mu - \frac{1}{2}\beta(1-\beta)\sigma^2, \beta r\right), 0\right)(T - t)\right] \\
&\times \mathbb{E}_t\Big[\exp\big(-\frac{1}{2}\beta^2 \int_t^{\zeta_i \wedge T} \tilde{\sigma}_s^2 ds + \beta \int_t^{\zeta_i \wedge T} \tilde{\sigma}_s dW_s\big)\Big] \\
=& \exp\left[\max\left(-\delta + \max\left(\beta\mu - \frac{1}{2}\beta(1-\beta)\sigma^2, \beta r\right), 0\right)(T - t)\right] \\
:=& \exp[\rho_1(T - t)].
\end{aligned}
$$

On the other hand, similar calculation yields

$$
\begin{aligned}
&\mathbb{E}_t[e^{-\delta(\tau_i-t)}Z_{\zeta_i}^{\beta}(Z_{\tau_i}/Z_{\zeta_i})^{\tilde{\alpha}}\mathbf{1}_{\{\tau_i<T\}}]\\
=&\mathbb{E}_t\Big[\exp\big(-\delta(\tau_i-t)+\beta\int_t^{\zeta_i}\tilde{\mu}_s ds-\frac{1}{2}\beta(1-\beta)\int_t^{\zeta_i}\tilde{\sigma}_s^2 ds-\frac{1}{2}\beta^2\int_t^{\zeta_i}\tilde{\sigma}_s^2 ds+\beta\int_t^{\zeta_i}\tilde{\sigma}_s dW_s\\
&\quad+\tilde{\alpha}\int_{\zeta_i}^{\tau_i}\tilde{\mu}_s ds-\frac{1}{2}\tilde{\alpha}(1-\tilde{\alpha})\int_{\zeta_i}^{\tau_i}\tilde{\sigma}_s^2 ds-\frac{1}{2}\tilde{\alpha}^2\int_{\zeta_i}^{\tau_i}\tilde{\sigma}_s^2 ds+\tilde{\alpha}\int_{\zeta_i}^{\tau_i}\tilde{\sigma}_s dW_s\big)\mathbf{1}_{\{\tau_i<T\}}\Big]\\
\leq&\exp\left[\max\left(-\delta+\max\left(\beta\mu-\frac{1}{2}\beta(1-\beta)\sigma^2,\beta r,\tilde{\alpha}\mu-\frac{1}{2}\tilde{\alpha}(1-\tilde{\alpha})\sigma^2\right),0\right)(T-t)\right]\\
:=&\exp[\rho_2(T-t)].
\end{aligned}
$$

Because $\delta+\rho>\max\{\beta r,\beta\mu-\frac{\beta(1-\beta)}{2}\sigma^2,\tilde{\alpha}\mu-\frac{\tilde{\alpha}(1-\tilde{\alpha})}{2}\sigma^2\}$ and $\rho>0$, we conclude that $\rho>\rho_i, i=1,2$. Consequently,

$$
\begin{aligned}
&\mathbb{E}_t[e^{-\delta(\tau_i-t)}U(G_{\tau_i-},R_{\tau_i-})\mathbf{1}_{\{\tau_i<\tilde{\tau}\}}]\\
=&\int_t^{\infty}\mathbb{E}_t[e^{-\delta(\tau_i-t)}U(G_{\tau_i-},R_{\tau_i-})\mathbf{1}_{\{\tau_i<\tilde{\tau}\}}|\tilde{\tau}=T]\rho e^{-\rho(T-t)}dT\\
\leq&Cx^{\beta}(K^{\beta})^{i-1}\int_t^{T}\left[\exp[\rho_1(T-t)]+\exp[\rho_2(T-t)]\right]\rho e^{-\rho(T-t)}dT\\
=&Cx^{\beta}\left(\frac{\rho}{\rho-\rho_1}+\frac{\rho}{\rho-\rho_2}\right)(K^{\beta})^{i-1}.
\end{aligned}
$$

Similarly, one can show that

$$
\begin{aligned}
&\mathbb{E}_t[e^{-\delta(\tau_1-t)}U(G_{\tau_1-},R_{\tau_1-})\mathbf{1}_{\{\tau_1<\tilde{\tau}\}}] \\
\le\ &\mathbb{E}_t[e^{-\delta(\tau_1-t)}CP^{\beta}_{\tau_1-}(1+(X_{\tau_1-}/P_{\tau_1-})^{\tilde{\alpha}})\mathbf{1}_{\{\tau_1<\tilde{\tau}\}}] \\
=\ &\mathbb{E}_t[e^{-\delta(\tau_1-t)}Cp^{\beta}(1+(x/p)^{\tilde{\alpha}}Z^{\alpha}_{\tau_1})\mathbf{1}_{\{\tau_1<\tilde{\tau}\}}] \\
=\ &Cp^{\beta}\mathbb{E}_t[e^{-\delta(\tau_1-t)}\mathbf{1}_{\{\tau_1<\tilde{\tau}\}}]+Cp^{\beta}(x/p)^{\tilde{\alpha}}\mathbb{E}_t[e^{-\delta(\tau_1-t)}Z^{\tilde{\alpha}}_{\tau_1}\mathbf{1}_{\{\tau_1<\tilde{\tau}\}}] \\
\le\ &Cp^{\beta}(1+(x/p)^{\tilde{\alpha}}\frac{\rho}{\rho-\rho_3}),
\end{aligned}
$$

where $\rho_3 := \max\big(-\delta+\big(\tilde{\alpha}\mu-\frac{1}{2}\tilde{\alpha}(1-\tilde{\alpha})\sigma^2\big),0\big)<\rho$. Consequently,

$$
\begin{aligned}
&\mathbb{E}_t\Big[\sum_{i=1}^{\infty}e^{-\delta(\tau_i-t)}U(G_{\tau_i-},R_{\tau_i-})\cdot\mathbf{1}_{\{\tau_i<\tilde{\tau}\}}\Big] \\
\le &Cx^{\beta}(\frac{\rho}{\rho-\rho_1}+\frac{\rho}{\rho-\rho_2})\cdot\frac{K^{\beta}}{1-K^{\beta}}+Cp^{\beta}(1+(x/p)^{\tilde{\alpha}}\cdot\frac{\rho}{\rho-\rho_3}).
\end{aligned}
$$

Using the same argument, we can show that

$$
\begin{aligned}
&\mathbb{E}_t\Big[e^{-\delta(\tilde{\tau}-t)}\big(U_W(X_{\tilde{\tau}})+U(G_{\tilde{\tau}-},R_{\tilde{\tau}-})\cdot\sum_{i=1}^{\infty}\mathbf{1}_{\{\zeta_i<\tilde{\tau}\le\tau_i\}}\big)\Big] \\
\le &\frac{\theta x^{\beta}\rho}{\rho-\rho_1}+Cx^{\beta}(\frac{\rho}{\rho-\rho_1}+\frac{\rho}{\rho-\rho_2})+Cp^{\beta}(1+(x/p)^{\tilde{\alpha}}\cdot\frac{\rho}{\rho-\rho_3}).
\end{aligned}
$$

Therefore, we conclude that

$$
\begin{aligned}
V(x,p)\le&\left[\frac{\theta\rho}{\rho-\rho_1}+C(\frac{\rho}{\rho-\rho_1}+\frac{\rho}{\rho-\rho_2})\frac{1}{1-K^{\beta}}\right]x^{\beta} \\
&+2Cp^{\beta}(1+(x/p)^{\tilde{\alpha}}\cdot\frac{\rho}{\rho-\rho_3})<+\infty.
\end{aligned}
$$

Part two: We prove $V(x,p) = +\infty$ for any $x > 0$ and $p > 0$ under certain conditions.

We first consider the case in which $\rho > 0$ and $\theta > 0$. If $\delta + \rho < \beta r$, we consider the strategy of holding the risk-free asset until the shock time. The terminal wealth utility for this strategy is

$$\begin{aligned}\mathbb{E}_t[e^{-\delta(\tilde{\tau}-t)}U_W(X_{\tilde{\tau}})] &= \mathbb{E}_t[e^{-\delta(\tilde{\tau}-t)}\theta(1-k_s)^\beta x^\beta e^{\beta r(\tilde{\tau}-t)}] \\ &= \rho\theta(1-k_s)^\beta x^\beta \int_0^\infty e^{(\beta r-(\delta+\rho))t}\,dt \\ &= \infty.\end{aligned}$$

If $\delta + \rho < \beta\mu - \frac{\beta(1-\beta)}{2}\sigma^2$, we consider the strategy of holding the stock until the shock time. The terminal wealth utility for this strategy is

$$\begin{aligned}&\mathbb{E}_t[e^{-\delta(\tilde{\tau}-t)}U_W((1-k_s)X_{\tilde{\tau}_-})] \\ =&\mathbb{E}_t[e^{-\delta(\tilde{\tau}-t)}\theta(1-k_s)^\beta x^\beta e^{\beta(\mu-\frac{1}{2}\sigma^2)(\tilde{\tau}-t)+\beta\sigma(W_{\tilde{\tau}}-W_t)}] \\ =&\theta(1-k_s)^\beta x^\beta \mathbb{E}_t[e^{-\delta(\tilde{\tau}-t)}e^{\beta\mu(\tilde{\tau}-t)-\frac{1}{2}\beta(1-\beta)\sigma^2(\tilde{\tau}-t)}]\cdot\mathbb{E}_t[e^{-\frac{1}{2}\beta^2\sigma^2(\tilde{\tau}-t)+\beta\sigma(W_{\tilde{\tau}}-W_t)}] \\ =&\theta(1-k_s)^\beta x^\beta \mathbb{E}_t[e^{-\delta(\tilde{\tau}-t)}e^{\beta\mu(\tilde{\tau}-t)-\frac{1}{2}\beta(1-\beta)\sigma^2(\tilde{\tau}-t)}] \\ =&\rho\theta(1-k_s)^\beta x^\beta \int_0^\infty \exp(-(\delta+\rho)t + (\beta\mu - \frac{1}{2}\beta(1-\beta)\sigma^2)t)\,dt \\ =&\infty.\end{aligned}$$

If $\delta + \rho < \bar{\alpha}\mu - \frac{\bar{\alpha}(1-\bar{\alpha})}{2}\sigma^2$, we consider the strategy of holding the stock until the shock time and compute the realization utility experienced by the agent at the shock time. Because $\delta + \rho < \bar{\alpha}\mu - \frac{\bar{\alpha}(1-\bar{\alpha})}{2}\sigma^2$, there exists $\hat{\alpha} < \bar{\alpha}$ such that $\delta + \rho < \hat{\alpha}\mu - \frac{\hat{\alpha}(1-\hat{\alpha})}{2}\sigma^2$. Because

$\bar{u}(x) \geq \bar{u}(0) = -\lambda(1-\gamma_-)^\beta$ for $x \geq 0$ and $\bar{\alpha} = \lim_{x\to+\infty} \frac{x\bar{u}'(x)}{\bar{u}(x)}$, there exists $C_1 > 0$ and $C_2 > 0$, such that $\bar{u}(x) \geq C_1(x^{\hat{\alpha}} - C_2), \forall x \geq 0$. Consequently, the realization utility experienced by the agent at the shock time is

$$
\begin{aligned}
&\mathbb{E}_t[e^{-\delta(\tilde{\tau}-t)}U(G_{\tilde{\tau}-}, R_{\tilde{\tau}-})] = \mathbb{E}_t[e^{-\delta(\tilde{\tau}-t)}P^\beta_{\tilde{\tau}-}\bar{u}(X_{\tilde{\tau}-}/P_{\tilde{\tau}-})] \\
=&\mathbb{E}_t[e^{-\delta(\tilde{\tau}-t)}p^\beta\bar{u}((x/p)Z_{\tilde{\tau}-})] \geq \mathbb{E}_t[e^{-\delta(\tilde{\tau}-t)}p^\beta C_1((x/p)^{\hat{\alpha}}Z^{\hat{\alpha}}_{\tilde{\tau}-} - C_2)] \\
=&p^\beta C_1(x/p)^{\hat{\alpha}}\mathbb{E}_t\left[e^{-\delta(\tilde{\tau}-t)}Z^{\hat{\alpha}}_{\tilde{\tau}-}\right] - p^\beta C_1C_2\mathbb{E}_t[e^{-\delta(\tilde{\tau}-t)}] \\
=&p^\beta C_1(x/p)^{\hat{\alpha}}\mathbb{E}_t\left[e^{-\delta(\tilde{\tau}-t)}e^{(\hat{\alpha}\mu-\frac{\hat{\alpha}(1-\hat{\alpha})}{2}\sigma^2)(\tilde{\tau}-t)}\right] - p^\beta C_1C_2\frac{\rho}{\rho+\delta} \\
=&\infty.
\end{aligned}
$$

Next, we consider the case in which $\rho = 0$. If $\delta < \beta\mu - \frac{\beta(1-\beta)}{2}\sigma^2$, consider the following strategy: $\tau_1 = \inf\{s \geq t | Z_s/Z_t \geq \Lambda\}$, $\zeta_i = \tau_{i-1}$, $\tau_i = \inf\{s \geq \zeta_i | Z_s/Z_{\zeta_i} \geq \Lambda\}$, $i \geq 2$, for some $\Lambda > 1$. Denote $g(\Lambda) := \mathbb{E}_t\left[e^{-\delta\tau_1}\mathbf{1}_{\{\tau_1<+\infty\}}\right]$. Then, $g(\Lambda) = \Lambda^\kappa$, where $\kappa := \sigma^{-2}\left[\mu - (1/2)\sigma^2 - \sqrt{(\mu-(1/2)\sigma^2)^2 + 2\delta\sigma^2}\right]$. Furthermore, we have $\mathbb{E}_t[e^{-\delta(\tau_i - t)}] = g(\Lambda)^i$. Now, straightforward computation yields that the realization utility experienced by the agent is

$$
\begin{aligned}
&\mathbb{E}_t\Big[\sum_{i=1}^{\infty} e^{-\delta(\tau_i - t)} U(G_{\tau_i -}, R_{\tau_i -}) \mathbf{1}_{\{\tau_i < +\infty\}}\Big] \\
=&\mathbb{E}_t\Big[e^{-\delta(\tau_1 - t)} p^\beta \bar{u}((x/p) Z_{\tau_1}) \mathbf{1}_{\{\tau_1 < +\infty\}} + \sum_{i=2}^{\infty} e^{-\delta(\tau_i - t)} (x K^{i-1} Z_{\zeta_i})^\beta \bar{u}(Z_{\tau_i}/Z_{\zeta_i}) \mathbf{1}_{\{\tau_i < +\infty\}}\Big] \\
=&\mathbb{E}_t\Big[e^{-\delta(\tau_1 - t)} p^\beta \bar{u}((x/p)\Lambda) \mathbf{1}_{\{\tau_1 < +\infty\}} + \sum_{i=2}^{\infty} e^{-\delta(\tau_i - t)} (x K^{i-1} \Lambda^{i-1})^\beta \bar{u}(\Lambda) \mathbf{1}_{\{\tau_i < +\infty\}}\Big] \\
=&p^\beta \bar{u}((x/p)\Lambda) g(\Lambda) + g(\Lambda) x^\beta \bar{u}(\Lambda) \sum_{i=2}^{\infty} (g(\Lambda) K^\beta \Lambda^\beta)^{i-1} \\
=&p^\beta \bar{u}((x/p)\Lambda) g(\Lambda) + g(\Lambda) x^\beta \bar{u}(\Lambda) \sum_{i=2}^{\infty} (K^\beta \Lambda^{\beta+\kappa})^{i-1}.
\end{aligned}
$$

Because $\delta < \beta\mu - \frac{\beta(1-\beta)}{2}\sigma^2$, we conclude that $\beta + \kappa > 0$. Then, we can choose $\Lambda > 1$ such that $K^\beta \Lambda^{\beta+\kappa} > 1$. Consequently, $\sum_{i=2}^{+\infty} (K^\beta \Lambda^{\beta+\kappa})^{i-1} = +\infty$, i.e., the agent experiences infinite realization utility.

If $\delta < \bar{\alpha}\mu - \frac{\bar{\alpha}(1-\bar{\alpha})}{2}\sigma^2$, consider the following strategy: $\tau_1 = \inf\{s \geq t | Z_s/Z_t \geq \Lambda\}$, $\zeta_2 = +\infty$. Then, the realization utility experienced by the agent is

$$
\mathbb{E}_t\Big[e^{-\delta(\tau_1 - t)} U(G_{\tau_1 -}, R_{\tau_1 -}) \mathbf{1}_{\{\tau_1 < +\infty\}}\Big] = p^\beta \bar{u}((x/p)\Lambda) g(\Lambda).
$$

Recall that we can find $\hat{\alpha} < \bar{\alpha}$ such that $\delta < \hat{\alpha}\mu - \frac{\hat{\alpha}(1-\hat{\alpha})}{2}\sigma^2$ and $\bar{u}(x) \geq C_1(x^{\hat{\alpha}} - C_2), \forall x \geq 0$ for some $C_1 > 0, C_2 > 0$. Then, we can conclude that

$$
\begin{aligned}
\mathbb{E}_t\Big[e^{-\delta(\tau_1 - t)} U(G_{\tau_1 -}, R_{\tau_1 -}) \mathbf{1}_{\{\tau_1 < +\infty\}}\Big] &\geq C_1 p^\beta (x/p)^{\hat{\alpha}} \Lambda^{\hat{\alpha}+\kappa} - C_1 C_2 p^\beta g(\Lambda) \\
&\geq C_1 p^\beta (x/p)^{\hat{\alpha}} \Lambda^{\hat{\alpha}+\kappa} - C_1 C_2 p^\beta.
\end{aligned}
$$

Because $\delta < \hat{\alpha}\mu - \frac{\hat{\alpha}(1-\hat{\alpha})}{2}\sigma^2$, we can show that $\hat{\alpha} + \kappa > 0$. Consequently, the realization utility of the agent goes to infinity as $\Lambda \to +\infty$.

If $\delta < \beta r$, consider the following strategy: $\tau_1 = t, \zeta_2 = n + t, \tau_2 = \inf\{s \geq \zeta_2 | Z_s/Z_{\zeta_2} \geq \Lambda\}$ for some $\Lambda > 1$. The agent's utility for this strategy is

$$
\begin{aligned}
&\mathbb{E}_t\Big[\sum_{i=1}^{\infty} e^{-\delta(\tau_i - t)} U(G_{\tau_i -}, R_{\tau_i -})\mathbf{1}_{\{\tau_i < +\infty\}}\Big] \\
=&\mathbb{E}_t\Big[e^{-\delta(\tau_2 - t)}(xKZ_{\zeta_2})^{\beta}\bar{u}(Z_{\tau_2}/Z_{\zeta_2})\mathbf{1}_{\{\tau_2 < +\infty\}}\Big] \\
=&g(\Lambda)(xK)^{\beta}e^{-(\delta - \beta r)n},
\end{aligned}
$$

which goes to infinity as $n \to +\infty$.

Finally, we consider the case in which $\rho > 0$ and $\theta = 0$. Using the same argument in the case of $\rho > 0$ and $\theta > 0$, we can show that $V(x,p) = +\infty$ if $\delta + \rho < \bar{\alpha}\mu - \frac{\bar{\alpha}(1-\bar{\alpha})}{2}\sigma^2$. $\square$

B.2 Proof of Proposition 3.3.2

Denote $\xi = (\tau_1, \zeta_2, \tau_2, \cdots)$ and $J(x, p, \xi)$ as the sequence of stopping times and the objective function, respectively, in problem (3.2.5). Similarly, denote $\bar{\xi} = (\zeta_1, \tau_1, \zeta_2, \tau_2, \cdots)$ and $\bar{J}(x, \bar{\xi})$ as the sequence of stopping times and the objective function, respectively, in problem (3.2.6).

By careful investigation, one can see that $\bar{J}(x, \bar{\xi}) = x^{\beta}\bar{J}(1, \bar{\xi})$. Therefore, $\bar{V}(x)$ is homogeneous in x of degree β. Next, we prove (3.3.1), i.e., prove $V(x,p) = \sup_{\tau \geq t}\mathbb{E}_t[F(x,p,\tau)]$,

where

$$F(x,p,\tau) := e^{-\delta(\tau-t)}\left[U(G_{\tau-},R_{\tau-})+\bar{V}((1-k_s)X_{\tau-})\right]\mathbf{1}_{\{\tau<\tilde{\tau}\}}$$
$$+\,e^{-\delta(\tilde{\tau}-t)}\left[U(G_{\tilde{\tau}-},R_{\tilde{\tau}-})+U_W((1-k_s)X_{\tilde{\tau}})\right]\mathbf{1}_{\{\tilde{\tau}\le\tau\}}.$$

On the one hand, applying the tower property of expectation operators, we conclude

$$J(x,p,\xi) = \mathbb{E}_t\Big[e^{-\delta(\tau_1-t)}\mathbb{E}_{\tau_1}[H]\mathbf{1}_{\{\tau_1<\tilde{\tau}\}}+e^{-\delta(\tau_1-t)}U(G_{\tau_1-},R_{\tau_1-})\mathbf{1}_{\{\tau_1<\tilde{\tau}\}}$$
$$+\,e^{-\delta(\tilde{\tau}-t)}\big(U_W((1-k_s)X_{\tilde{\tau}-})+U(G_{\tilde{\tau}-},R_{\tilde{\tau}-})\big)\mathbf{1}_{\{\tilde{\tau}\le\tau_1\}}\Big],$$

where

$$H := \sum_{i=2}^{\infty}e^{-\delta(\tau_i-\tau_1)}U(G_{\tau_i-},R_{\tau_i-})\mathbf{1}_{\{\tau_i<\tilde{\tau}\}}+e^{-\delta(\tilde{\tau}-\tau_1)}U_W(X_{\tilde{\tau}-})\sum_{i=1}^{\infty}\mathbf{1}_{\{\tau_i<\tilde{\tau}\le\zeta_{i+1}\}}$$
$$+e^{-\delta(\tilde{\tau}-\tau_1)}\big(U_W((1-k_s)X_{\tilde{\tau}-})+U(G_{\tilde{\tau}-},R_{\tilde{\tau}-})\big)\sum_{i=2}^{\infty}\mathbf{1}_{\{\zeta_i<\tilde{\tau}\le\tau_i\}}.$$

Note that $X_{\tau_1}=(1-k_s)X_{\tau_1-}$ on $\{\tau_1<\tilde{\tau}\}$. In addition, $\{W_{\tau_1+s}\}_{s\ge0}$ is a standard Brownian motion, so there exist $\{\mathcal{F}_s\}$-stopping times $t\le\bar{\zeta}_1\le\bar{\tau}_1\le\bar{\zeta}_2\le\dots$ such that $(\{W_{\tau_1+s}\}_{s\ge0},\zeta_2-\tau_1,\tau_2-\tau_1,\zeta_3-\tau_1,\dots)$ is identically distributed as $(\{W_{t+s}\}_{s\ge0},\bar{\zeta}_1-t,\bar{\tau}_1-t,\bar{\zeta}_2-t,\dots)$. Therefore, denoting $\bar{\xi}=(\bar{\zeta}_1,\bar{\tau}_1,\bar{\zeta}_1,\dots)$, we conclude, for each realization of X_{τ_1-}, that

$$\mathbb{E}_{\tau_1}[H]\mathbf{1}_{\{\tau_1<\tilde{\tau}\}}=\bar{J}((1-k_s)X_{\tau_1-},\bar{\xi})\mathbf{1}_{\{\tau_1<\tilde{\tau}\}}\le\bar{V}((1-k_s)X_{\tau_1-})\mathbf{1}_{\{\tau_1<\tilde{\tau}\}}.$$

Consequently,

$$J(x,p,\xi) \leq \mathbb{E}_t\Big[e^{-\delta(\tau_1-t)}\bar{V}((1-k_s)X_{\tau_1-})\mathbf{1}_{\{\tau_1<\tilde{\tau}\}} + e^{-\delta(\tau_1-t)}U(G_{\tau_1-},R_{\tau_1-})\mathbf{1}_{\{\tau_1<\tilde{\tau}\}}$$
$$+ e^{-\delta(\tilde{\tau}-t)}\big(U_W((1-k_s)X_{\tilde{\tau}-}) + U(G_{\tilde{\tau}-},R_{\tilde{\tau}-})\big)\mathbf{1}_{\{\tilde{\tau}\leq\tau_1\}}\Big]$$

for any $\tau_1 \geq t$, so we conclude $V(x,p) \leq \sup_{\tau\geq t}\mathbb{E}_t[F(x,p,\tau)]$.

On the other hand, for any $\epsilon > 0$, there exist $\tau_1 \geq t$ such that

$$\sup_{\tau\geq t}\mathbb{E}_t[F(x,p,\tau)] - \epsilon \leq \mathbb{E}_t\Big[F(x,p,\tau_1)\Big].$$

In addition, there exist $\bar{\xi} = (\bar{\zeta}_1, \bar{\tau}_1, \bar{\zeta}_1, \dots)$ such that $V(1) - \epsilon \leq \bar{J}(1,\bar{\xi})$. Consequently,

$$F(x,p,\tau_1) = e^{-\delta(\tau_1-t)}\left[U(G_{\tau-},R_{\tau-}) + ((1-k_s)X_{\tau_1-})^\beta\bar{V}(1)\right]\mathbf{1}_{\{\tau_1<\tilde{\tau}\}}$$
$$+ e^{-\delta(\tilde{\tau}-t)}\left[U(G_{\tilde{\tau}-},R_{\tilde{\tau}-}) + U_W((1-k_s)X_{\tilde{\tau}})\right]\mathbf{1}_{\{\tilde{\tau}\leq\tau_1\}}$$
$$\leq e^{-\delta(\tau_1-t)}\left[U(G_{\tau-},R_{\tau-}) + ((1-k_s)X_{\tau_1-})^\beta\bar{J}(1,\bar{\xi})\right]\mathbf{1}_{\{\tau_1<\tilde{\tau}\}}$$
$$+ e^{-\delta(\tilde{\tau}-t)}\left[U(G_{\tilde{\tau}-},R_{\tilde{\tau}-}) + U_W((1-k_s)X_{\tilde{\tau}})\right]\mathbf{1}_{\{\tilde{\tau}\leq\tau_1\}}$$
$$+ \epsilon(1-k_s)^p e^{-\delta(\tau_1-t)}X^\beta_{\tau_1-}\mathbf{1}_{\{\tau_1<\tilde{\tau}\}}.$$

Note that we can find $\{\mathcal{F}_s\}$-stopping times $(\zeta_2,\tau_2,\zeta_3,\dots)$ such that $(\{W_{\tau_1+s}\}_{s\geq0}, \zeta_2 - \tau_1, \tau_2-\tau_1, \zeta_3-\tau_1, \dots)$ is identically distributed as $(\{W_{t+s}\}_{s\geq0}, \bar{\zeta}_1 - t, \bar{\tau}_1 - t, \bar{\zeta}_1 - t, \dots)$. Denote $\xi = (\tau_1, \zeta_2, \tau_2, \cdots)$, then straightforward calculation yields

$$\mathbb{E}_t\Big[F(x,p,\tau_1)\Big] \geq \mathbb{E}_t\Big[J(x,p,\xi)\Big] + \epsilon(1-k_s)^p\mathbb{E}_t\Big[e^{-\delta(\tau_1-t)}X^\beta_{\tau_1-}\mathbf{1}_{\{\tau_1<\tilde{\tau}\}}\Big].$$

Following the proof of Proposition 3.3.1, $C := \sup_{\tau\geq t}\mathbb{E}_t\left[e^{-\delta(\tau-t)}X_{\tau-}^{\beta}\mathbf{1}_{\{\tau<\tilde{\tau}\}}\right] < +\infty$, so we have

$$\sup_{\tau\geq t}\mathbb{E}_t[F(x,p,\tau)] \leq \mathbb{E}_t\Big[J(x,p,\xi)\Big] + (1+C(1-k_s)^p)\epsilon \leq V(x,p) + (1+C(1-k_s)^p)\epsilon.$$

Because ϵ is arbitrary, we conclude that $V(x,p) \geq \sup_{\tau\geq t}\mathbb{E}_t[F(x,p,\tau)]$ and thus (3.3.1) holds.

Similarly, we can verify that $J(ax,ap,\xi) = a^{\beta}J(x,p,\xi)$ for any $a > 0$, so $V(x,p)$ is homogeneous of degree β. Using the same argument as in the proof of (3.3.1), we can prove (3.3.2).

Next, we show that $V(x,1)$ is continuous in $x \geq 0$. Recall $\{Z_s\}_{s\geq t}$ as defined in the proof of Proposition 3.3.1. Then,

$$\begin{aligned}
F(x,1,\tau) =& e^{-\delta(\tau-t)}\left[\bar{u}(X_{\tau-}) + (1-k_s)^{\beta}X_{\tau-}^{\beta}\bar{V}(1)\right]\mathbf{1}_{\{\tau<\tilde{\tau}\}} \\
&+ e^{-\delta(\tilde{\tau}-t)}\left[\bar{u}(X_{\tilde{\tau}-}) + \theta(1-k_s)^{\beta}X_{\tilde{\tau}}^{\beta}\right]\mathbf{1}_{\{\tilde{\tau}\leq\tau\}} \\
=& e^{-\delta(\tau-t)}\left[\bar{u}(xZ_{\tau-}) + (1-k_s)^{\beta}x^{\beta}Z_{\tau-}^{\beta}\bar{V}(1)\right]\mathbf{1}_{\{\tau<\tilde{\tau}\}} \\
&+ e^{-\delta(\tilde{\tau}-t)}\left[\bar{u}(xZ_{\tilde{\tau}-}) + \theta(1-k_s)^{\beta}x^{\beta}Z_{\tilde{\tau}}^{\beta}\right]\mathbf{1}_{\{\tilde{\tau}\leq\tau\}}.
\end{aligned}$$

Because $\bar{u}(x)$ is continuous and increasing in $x \geq 0$, $F(x,1,\tau)$ is continuous and increasing in $x \geq 0$ for each $\tau \geq t$. Consequently, $V(x,1) = \sup_{\tau\geq t}\mathbb{E}_t[F(x,1,\tau)]$ is increasing in $x \geq 0$. Moreover, the monotone convergence theorem shows that $V(x,1)$ is continuous in $x \geq 0$.

Finally, as in the proof of Proposition 3.3.1, there exist $\tilde{\alpha} \in [\bar{\alpha}, 1]$ and $C > 0$ such that

$|\bar{u}(x)| \le C(1+x^{\tilde{\alpha}})$. Consequently.

$$\begin{aligned} |F(x,1,\tau)| \le & e^{-\delta(\tau-t)}\left[C + Cx^{\tilde{\alpha}}Z_{\tau-}^{\tilde{\alpha}} + (1-k_s)^{\beta}x^{\beta}Z_{\tau-}^{\beta}\bar{V}(1)\right]\mathbf{1}_{\{\tau<\tilde{\tau}\}} \\ & + e^{-\delta(\tilde{\tau}-t)}\left[C + Cx^{\tilde{\alpha}}Z_{\tilde{\tau}-}^{\tilde{\alpha}} + \theta(1-k_s)^{\beta}x^{\beta}Z_{\tilde{\tau}}^{\beta}\right]\mathbf{1}_{\{\tilde{\tau}\le\tau\}}. \end{aligned}$$

Following the same proof of Proposition 3.3.1, we can show that $\sup_{\tau\ge t}\mathbb{E}_t[e^{-\delta(\tau-t)}Z_{\tau}^{\tilde{\alpha}}\mathbf{1}_{\{\tau<\tilde{\tau}\}}]$, $\sup_{\tau\ge t}\mathbb{E}_t[e^{-\delta(\tilde{\tau}-t)}Z_{\tilde{\tau}}^{\tilde{\alpha}}\mathbf{1}_{\{\tilde{\tau}\le\tau\}}]$, $\sup_{\tau\ge t}\mathbb{E}_t[e^{-\delta(\tau-t)}Z_{\tau}^{\beta}\mathbf{1}_{\{\tau<\tilde{\tau}\}}]$, and $\sup_{\tau\ge t}\mathbb{E}_t[e^{-\delta(\tilde{\tau}-t)}Z_{\tilde{\tau}}^{\beta}\mathbf{1}_{\{\tilde{\tau}\le\tau\}}]$ are finite. Because $\tilde{\alpha}\ge\bar{\alpha}\ge\beta$, we conclude that $V(x,1)\le C'(1+x^{\tilde{\alpha}})$ for some $C'>0$.

□

B.3 Proof of Proposition 3.3.3

Recall that $V(x,p)$ is homogeneous of degree β. Denote $v(1)=V(1,1)$ and recall Θ. For any purchase time ζ, we have

$$\begin{aligned} & \mathbb{E}_t\Big[e^{-\delta(\zeta-t)}V(\frac{1}{1+k_p}X_{\zeta},\frac{1}{1+k_p}X_{\zeta})\cdot\mathbf{1}_{\{\zeta<\tilde{\tau}\}} + e^{-\delta(\tilde{\tau}-t)}U_W(X_{\zeta})\cdot\mathbf{1}_{\{\tilde{\tau}\le\zeta\}}\Big] \\ = & \mathbb{E}_t\Big[e^{-\delta(\zeta-t)}(\frac{1}{1+k_p})^{\beta}X_{\zeta}^{\beta}v(1)\cdot\mathbf{1}_{\{\zeta<\tilde{\tau}\}} + e^{-\delta(\tilde{\tau}-t)}U_W(X_{\tilde{\tau}})\cdot\mathbf{1}_{\{\tilde{\tau}\le\zeta\}}\Big] \\ = & \mathbb{E}_t\Big[e^{-\delta(\zeta-t)}(\frac{1}{1+k_p})^{\beta}x^{\beta}e^{\beta r(\zeta-t)}v(1)\cdot\mathbf{1}_{\{\zeta<\tilde{\tau}\}} + e^{-\delta(\tilde{\tau}-t)}U_W(X_{\tilde{\tau}})\cdot\mathbf{1}_{\{\tilde{\tau}\le\zeta\}}\Big] \\ = & \int_t^{\infty}\mathbb{E}_t\Big[e^{-\delta(s-t)}(\frac{1}{1+k_p})^{\beta}x^{\beta}e^{\beta r(s-t)}v(1)\cdot\mathbf{1}_{\{s<\tilde{\tau}\}} \\ & +e^{-\delta(\tilde{\tau}-t)}U_W(X_{\tilde{\tau}})\cdot\mathbf{1}_{\{\tilde{\tau}\le s\}}|\zeta=s\Big]dF_{\zeta}(s) \\ = & \int_t^{\infty}\Big\{\mathbb{E}_t\Big[e^{-\delta(s-t)}(\frac{1}{1+k_p})^{\beta}x^{\beta}e^{\beta r(s-t)}v(1)\cdot\mathbf{1}_{\{s<\tilde{\tau}\}}\Big] \\ & +\mathbb{E}_t\Big[e^{-\delta(\tilde{\tau}-t)}U_W(X_{\tilde{\tau}})\cdot\mathbf{1}_{\{\tilde{\tau}\le s\}}\Big]\Big\}dF_{\zeta}(s) \end{aligned}$$

where F_ζ is the distribution function of ζ. The first term in the integral is

$$\mathbb{E}_t\left[e^{-\delta(s-t)}(\frac{1}{1+k_p})^\beta x^\beta e^{\beta r(s-t)}v(1)\cdot \mathbf{1}_{\{s<\tilde{\tau}\}}\right] = e^{(\beta r-\rho-\delta)(s-t)}(\frac{1}{1+k_p})^\beta x^\beta v(1).$$

The second term in the integral is

$$\begin{aligned}
\mathbb{E}_t\left[e^{-\delta(\tilde{\tau}-t)}U_W(X_{\tilde{\tau}})\cdot \mathbf{1}_{\{\tilde{\tau}\le s\}}\right] &= \mathbb{E}_t\left[e^{-\delta(\tilde{\tau}-t)}U_W(xe^{r(\tilde{\tau}-t)})\cdot \mathbf{1}_{\{\tilde{\tau}\le s\}}\right] \\
&= \int_t^s e^{-\delta(u-t)}\theta x^\beta e^{\beta r(u-t)}\rho e^{-\rho(u-t)}\,du \\
&= x^\beta\theta\rho\int_t^s e^{-(\rho+\delta-\beta r)(u-t)}\,du \\
&= \frac{x^\beta\theta\rho}{\rho+\delta-\beta r}\big(1-e^{(\beta r-\delta-\rho)(s-t)}\big).
\end{aligned}$$

As a result,

$$\begin{aligned}
&\mathbb{E}_t\Big[e^{-\delta(\zeta-t)}V(\frac{1}{1+k_p}X_\zeta,\frac{1}{1+k_p}X_\zeta)\cdot \mathbf{1}_{\{\zeta<\tilde{\tau}\}}+e^{-\delta(\tilde{\tau}-t)}U_W(X_\zeta)\cdot \mathbf{1}_{\{\tilde{\tau}\le\zeta\}}\Big] \\
=\ & (1-k_s)^{-\beta}x^\beta\int_t^\infty\Big(e^{(\beta r-\rho-\delta)(s-t)}K^\beta v(1)+\big(1-e^{(\beta r-\delta-\rho)(s-t)}\big)\Theta\Big)dF_\zeta(s) \\
\le\ & (1-k_s)^{-\beta}x^\beta\int_t^\infty \max\Big\{K^\beta v(1)\,,\Theta\Big\}dF_\zeta(s) \\
=\ & (1-k_s)^{-\beta}x^\beta \max\Big\{K^\beta v(1)\,,\Theta\Big\}.
\end{aligned}$$

Furthermore, when $K^\beta v(1) > \Theta$, the inequality becomes equality if and only if $\zeta = t$ with probability 1; i.e., $\zeta = t$ is the unique optimal purchase time in this case. Similarly, when $K^\beta v(1) < \Theta$, $\zeta = +\infty$ is the unique optimal purchase time. Finally, when $K^\beta v(1) = \Theta$, any stopping time $\zeta \ge t$ is optimal. $\square$

B.4 Proof of Proposition 3.3.4

Denote $g(x)$ as the objective value in (3.3.4) with $\tau = t$ (which stands for the immediate liquidation strategy). Then, we have $g(x) = \bar{u}(x) + x^{\beta}\max\{K^{\beta}v(1), \Theta\}$. On the other hand, denote $h(x)$ as the objective value in (3.3.4) with $\tau = +\infty$ (which stands for the strategy of holding the stock until the shock time). Then, we have

$$h(x) = \mathbb{E}_t[e^{-\delta(\tilde{\tau}-t)}(\bar{u}(X_{\tilde{\tau}}) + U_W((1-k_s)X_{\tilde{\tau}_-}))].$$

It is straightforward to see that both g and h are continuous in $x \geq 0$. Furthermore, we have

$$g(0) = \bar{u}(0) < 0, \quad h(0) = \mathbb{E}_t[e^{-\delta(\tilde{\tau}-t)}(\bar{u}(0) + U_W(0))] = \frac{\rho}{\delta+\rho}\bar{u}(0),$$

showing that $h(0) > g(0)$. By the continuity of g and h, there exist $\epsilon > 0$ such that $h(x) > g(x), \forall x \leq \epsilon$. Because $v(x) \geq h(x)$, we conclude $v(x) > g(x), \forall x \leq \epsilon$. Thus, in the liquidation problem (3.3.4), it is optimal to hold the stock when $X_t \leq \epsilon$.

Finally, we compute $v(x)$ when x approaches zero. According to the general theory of optimal stopping, the optimal stopping time to problem (3.3.4) is $\tau^* = \inf\{s \geq t | g(X_s) = v(X_s)\}$. Denote τ_ϵ as the first hitting time of $\{X_s\}$ to ϵ. Because $v(x) > g(x), \forall x \leq \epsilon$, we conclude that $\tau^* \geq \tau_\epsilon$. Then,

$$\lim_{x\downarrow 0}\mathbb{P}(\tau^* < \tilde{\tau} | X_t = x, \tilde{\tau} > t) \leq \lim_{x\downarrow 0}\mathbb{P}(\tau_\epsilon < \tilde{\tau} | X_t = x, \tilde{\tau} > t) = 0.$$

Consequently, straightforward calculation shows that $\lim_{x\downarrow 0} v(x) = h(0) = \frac{\rho}{\delta+\rho}\bar{u}(0)$. □

B.5 Proof of Theorem 3.3.1

For each $a \in \mathbb{R}$ and $x \geq 0$, consider the following optimal stopping problem

$$\begin{aligned} \sup_{\tau \geq t} \quad & \mathbb{E}_t \Big[e^{-\delta(\tau - t)} \left(\bar{u}(X_{\tau-}) + \max\left(K^\beta a, \Theta\right) X_{\tau-}^\beta \right) \cdot \mathbf{1}_{\{\tau < \tilde{\tau}\}} \\ & \quad + e^{-\delta(\tilde{\tau} - t)} \left(\bar{u}(X_{\tilde{\tau}-}) + \theta(1 - k_s)^\beta X_{\tilde{\tau}-}^\beta \right) \cdot \mathbf{1}_{\{\tilde{\tau} \leq \tau\}} \Big], \\ \text{subject to} \quad & dX_s = \mu X_s \, ds + \sigma X_s \, dW_s, \ s \geq t, \ X_{t-} = x. \end{aligned} \tag{B.1}$$

Denote the optimal value as $v(x; a)$. Then, problem (3.3.4) is a special case of problem (B.1) with $a = v(1)$, and the value function of problem (3.3.4) $v(x) = v(x; v(1))$. Using the same argument as in the proof of Proposition 3.3.2, we can show that for each $a \in \mathbb{R}$, $v(x; a)$ is continuous in $x \geq 0$ and there exist $\tilde{\alpha} \in [\bar{\alpha}, 1]$ and $C > 0$ such that $|v(x; a)| \leq C(1 + x^{\tilde{\alpha}}), \forall x \geq 0$. In addition, $\tilde{\alpha}$ can be chosen arbitrarily close to $\bar{\alpha}$. In particular, $v(x; a)$ is of linear growth in $x \geq 0$.

Because the shock time $\tilde{\tau}$ is exponentially distributed and independent of τ and $\{W_s\}$, we can reformulate the objective function of problem (B.1) and rewrite this problem as

$$\begin{aligned} \sup_{\tau \geq t} \quad & \mathbb{E}\Big[\int_0^\tau e^{-\delta' t} \cdot f(X_t) \, dt + e^{-\delta' \tau} \cdot g(X_\tau; a) | X_0 = x \Big], \\ \text{subject to} \quad & dX_t = \mu X_t \, dt + \sigma X_t \, dW_t, \ t \geq 0, \ X_0 = x, \ x \geq 0, \end{aligned}$$

where $\delta' := \delta + \rho$, $f(x) := \rho\left(\bar{u}(x) + \theta(1 - k_s)^\beta x^\beta\right)$, and $g(x; a) := \bar{u}(x) + \max\left(K^\beta a, \Theta\right) \cdot x^\beta$. In the proof of Proposition 3.3.1, we have shown that $\bar{u}(x)$ is of linear growth in $x \geq 0$. Consequently, both $f(x)$ and $g(x; a)$ are of linear growth in $x \geq 0$ for each $a \in \mathbb{R}$. Applying Theorem 5.2.1 in Pham [44], we conclude that for each fixed $a \in \mathbb{R}$, $v(x; a)$ is the unique viscosity solution of linear growth to following variational inequality[1]

$$\min[\delta' v(x) - \mathcal{L}v(x) - f(x), v(x) - g(x;a)] = 0, \tag{B.2}$$

where $\mathcal{L}v(x) := \frac{1}{2}\sigma^2 x^2 v_{xx}(x) + \mu x v_x(x)$. Recalling that the value function of problem (3.3.4) $v(x) = v(x; v(1))$ and noting that equation (3.3.5) is the same as (B.2) with $a = v(1)$, we conclude that v is a viscosity solution of linear growth to equation (3.3.5).

Finally, we show that the solution to (3.3.5) is unique. Consider any $a_1 > a_2$ and the viscosity solutions $v(x; a_1)$ and $v(x; a_2)$ to (B.2) with $a = a_1$ and $a = a_2$, respectively. Because $g(x; a_1) \geq g(x; a_2)$ for any $x \geq 0$, it immediately follows that $v(x; a_1)$ is a viscosity super-solution to (B.2) with $a = a_2$. By the comparison theorem in Pham [44, p. 98], we conclude that $v(x; a_1) \geq v(x; a_2), \forall x \geq 0$.

Next, we define $u_2(x) = v_1(x) - K^\beta(a_1 - a_2)x^\beta$ and show that it is a viscosity sub-solution to (B.2) with $a = a_2$. Take any $x_0 > 0$ and any test function $\phi_2 \in C^2(\mathbb{R})$ such that $0 = (u_2 - \phi_2)(x_0) = \max_{x \geq 0}(u_2 - \phi_2)(x)$. Define $\phi_1(x) := \phi_2(x) + K^\beta x^\beta (a_1 - a_2)$. Then,

$$\max_{x \geq 0}(v_1 - \phi_1)(x) = \max_{x \geq 0}(u_2 - \phi_2)(x) = (u_2 - \phi_2)(x_0) = (v_1 - \phi_1)(x_0) = 0.$$

Because $v(x; a_1)$ is the viscosity solution to (B.2) with $a = a_1$, we conclude by the definition

[1] Theorem 5.2.1 in Pham [44] is presented in the setting in which the domain of the state process $\{X_t\}$ is the whole real line. However, all the proofs can be migrated to the setting in which the domain is the positive real line.

of viscosity solutions that

$$\min\left[\delta'\phi_1(x_0) - \mathcal{L}\phi_1(x_0) - f(x_0), v_1(x_0) - g(x_0; a_1)\right] \leq 0. \tag{B.3}$$

On one hand,

$$\begin{aligned}
&\delta'\phi_2(x_0) - \mathcal{L}\phi_2(x_0) - f(x_0) \\
=&\delta'\phi_1(x_0) - \mathcal{L}\phi_1(x_0) - f(x_0) - (a_1 - a_2)x_0^\beta K^\beta(\delta' - \frac{1}{2}\sigma^2\beta(\beta - 1) - \beta\mu) \\
\leq&(\delta'\phi_1 - \mathcal{L}\phi_1 - f)(x_0),
\end{aligned}$$

where the inequality is the case because of Assumption 3.3.1. On the other hand,

$$\begin{aligned}
u_2(x_0) - g(x_0; a_2) &= v_1(x_0) - K^\beta(a_1 - a_2)x_0^\beta - \left(\bar{u}(x_0) + \max\{K^\beta a_2, \Theta\}x_0^\beta\right) \\
&= v_1(x_0) - \bar{u}(x_0) - \max\{K^\beta a_1, \Theta + K^\beta(a_1 - a_2)\}x_0^\beta \\
&\leq u_2(x_0) - g(x_0; a_1).
\end{aligned}$$

Therefore, we conclude from (B.3) that

$$\min\left[\delta'\phi_2(x_0) - \mathcal{L}\phi_2(x_0) - f(x_0), v_2(x_0) - g(x_0; a_2)\right] \leq 0,$$

showing that u_2 is a viscosity sub-solution to (B.2) with $a = a_2$.

Now, by the comparison theorem in [44, p. 98], we conclude that $u_2(x) \leq v_2(x), \forall x \in$

$\mathbb{R}_+$ and, consequently,

$$0 \le v_1(x) - v_2(x) \le K^\beta x^\beta (a_1 - a_2). \tag{B.4}$$

Denote $\mathbb{X}$ as the space of continuous functions on $\mathbb{R}_+$ with finite $\|\cdot\|$ norm, where $\|f\| := \sup_{x\ge 0} \big| \max(x,1)^{-1} f(x) \big|$. Note that any continuous function on $\mathbb{R}_+$ has finite $\|\cdot\|$ norm if and only if it is of linear growth, so $\mathbb{X}$ is the space of continuous functions of linear growth. In particular, for each $a \in \mathbb{R}$, the viscosity solution to (B.2) is in $\mathbb{X}$.

Define the following mapping on $(\mathbb{X}, \|\cdot\|)$: for each $v \in \mathbb{X}$, define $F(v)$ as the solution to (B.2) with $a = v(1)$. We conclude from (B.4) that

$$\begin{aligned} \max(x,1)^{-1}|F(v_1)(x) - F(v_2)(x)| &\le K^\beta \max(x,1)^{-1} x^\beta |v_1(1) - v_2(1)| \\ &\le K^\beta |v_1(1) - v_2(1)| \le K^\beta \|v_1 - v_2\|, \quad \forall x > 0. \end{aligned}$$

Therefore, $\|F(v_1) - F(v_2)\| \le K^\beta \|v_1 - v_2\|$, showing that F is a contract mapping on $\mathbb{X}$. If we show that $\mathbb{X}$ is a complete normed space, then, by the Banach fixed-point theorem, F has a unique fixed point. Because the solution to equation (3.3.5) is equivalent to the fixed point of F, we conclude that the solution to equation (3.3.5) is unique. Therefore, in the following, we show that $\mathbb{X}$ is complete; i.e., for any Cauchy sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}$, we show that there exists $f \in \mathbb{X}$ such that $\lim_{n\to\infty} \|f_n - f\|_\varepsilon = 0$.

For each $M > 0$, denote $\mathbb{X}^M$ as the normed space of continuous functions on $[0, M]$ with norm $\|\cdot\|$. Then, $\mathbb{X}^M$ is a complete normed space. Define a sequence $\{f_n^M\}_{n\in\mathbb{N}}$ in $\mathbb{X}^M$

with $f_n^M(x) = f_n(x)$ for $x \in [0, M]$. Because $\mathbb{X}^M$ is complete, this sequence has a limit $f^M \in \mathbb{X}^M$.

Note that for $\forall M_1 > 0, M_2 > 0$, $f^{M_1}(x) = f^{M_2}(x)$ for $x \in [0, M_1 \wedge M_2]$. so $f := \lim_{M\to\infty} f^M$ is well-defined and continuous on $[0, \infty)$. Moreover, it is not difficult to see $f_n(x)$ converges to $f(x)$ for each $x \in \mathbb{R}_+$.

We first show that $f \in \mathbb{X}$, i.e., $\|f\| < \infty$. Because $\{f_n\}$ is a Cauchy sequence in $\mathbb{X}$ and $|\|f_n\| - \|f_m\|| \leq \|f_n - f_m\|$, $\{\|f_n\|\}$ is also a cauchy sequence, and thus $\{\|f_n\|\}$ is bounded, i.e., $\|f_n\| \leq C, \forall n \in \mathbb{N}$ for some $C > 0$. Now, for any $M > 0$,

$$\|f^M\| = \lim_{n\to+\infty} \|f_n^M\| \leq \varliminf_{n\to+\infty} \|f_n\| \leq C.$$

Consequently, $\|f\| = \lim_{M\to+\infty} \|f^M\| \leq C < +\infty$.

Next, we show that $\lim_{n\to\infty} \|f_n - f\| = 0$. For any $\epsilon > 0$, there exists N such that $\|f_n - f_m\| < \epsilon$ for any $n > N$ and $m > N$. Consequently, for any $n > N$ and $x \in \mathbb{R}_+$,

$$\max(x, 1)^{-1}|f_n(x) - f(x)| = \lim_{m\to+\infty} \max(x, 1)^{-1}|f_n(x) - f_m(x)| \leq \epsilon.$$

Consequently, $\|f_n - f\| = \sup_{x\geq 0} \max(x, 1)^{-1}|f_n(x) - f(x)| \leq \delta$ for any $n > N$. Therefore, f_n converges to f in $\mathbb{X}$. $\square$

Appendix C

Algorithm to Solve Variational Inequality (3.3.5)

In this section we propose an algorithm to solve the variational inequality (3.3.5). In view of the proof of Theorem 3.3.1, we only need to solve (B.1) for each $a \in \mathbb{R}$.

Following the proof of Proposition 3.3.4, we can also show that $v(0;a) = \frac{\rho}{\delta+\rho}\bar{u}(0)$. On the other hand, recall $\delta + \rho > \bar{\alpha}\mu - \frac{\bar{\alpha}(1-\bar{\alpha})}{2}\sigma^2$, so we can choose ϵ such that $1+\epsilon > \bar{\alpha}$ and $\delta + \rho > (1+\epsilon)\mu + \frac{\epsilon(1+\epsilon)}{2}\sigma^2$. Moreover, because $1+\epsilon > \bar{\alpha}$, we conclude from the proof of Theorem 3.3.1 that $\lim_{x\to+\infty} v(x;a)/(1+x)^{1+\epsilon} = 0$.

C.1 Transformation

Instead of solving (B.1) directly, we apply a transformation first. Define $y = x/(1+x)$ and

$$\bar{v}(y) := v(\frac{y}{1-y};a)(1-y)^{1+\epsilon} = \frac{v(\frac{y}{1-y};a)}{(1+\frac{y}{1-y})^{1+\epsilon}} = \frac{v(x;a)}{(1+x)^{1+\epsilon}}.$$

Then, we have the following boundary conditions for $\bar{v}(y)$:

$$\bar{v}(0) = v(0;a) = \frac{\rho}{\delta+\rho}\bar{u}(0), \quad \bar{v}(1) = \lim_{x\to+\infty}\frac{v(x;a)}{(1+x)^{1+\epsilon}} = 0. \tag{C.1}$$

On the other hand, tedious but straightforward calculation yields the following variational inequality for $\bar{v}$:

$$\max\{A_1(y)y^2\bar{v}_{yy}(y) + A_2(y)y\bar{v}_y(y) + A_3(y)\bar{v}(y) + g(y), h(y) - \bar{v}(y)\} = 0, \tag{C.2}$$

where

$$A_1(y) = \frac{1}{2}\sigma^2(1-y)^2, \tag{C.3}$$

$$A_2(y) = (\sigma^2\cdot\epsilon\cdot y + \mu)(1-y), \tag{C.4}$$

$$A_3(y) = \frac{1}{2}\sigma^2\epsilon(1+\epsilon)y^2 + \mu(1+\epsilon)y - (\rho+\delta), \tag{C.5}$$

$$g(y) = \rho\Big(\bar{u}(\frac{y}{1-y}) + \theta(1-k_s)^\beta(\frac{y}{1-y})^\beta\Big)(1-y)^{1+\epsilon}, \tag{C.6}$$

$$h(y) = \big(\bar{u}(\frac{y}{1-y}) + (\frac{y}{1-y})^\beta\cdot\max\{K^\beta\bar{v}(\frac{1}{2})2^{1+\epsilon}, \Theta\}\big)(1-y)^{1+\epsilon}. \tag{C.7}$$

C.2 Penalty Method

Next, we solve the variational inequality (C.2) with boundary conditions (C.1). We apply the penalty method (Dai et al. [27], Forsyth and Vetzal [33]): fix a penalization parameter P

and solve

$$A_1(y)y^2\bar{v}_{yy}(y) + A_2(y)y\bar{v}_y(y) + A_3(y)\bar{v}(y) + g(y) = -P \cdot \Big(h(y) - \bar{v}(y)\Big)^+. \quad \text{(C.1)}$$

In the following, we use the implicit finite difference method to solve (C.1).

Consider an equally-spaced grid on $[0,1]$: $\Delta y = 1/N_s$ and $y_i = i\Delta y$ for $i = 0, \cdots, N_s$. Plugging

$$v_y(y_i) \approx \frac{v(y_{i+1}) - v(y_{i-1})}{2\Delta y} \quad \text{and} \quad v_{yy}(y_i) \approx \frac{v(y_{i+1}) - 2v(y_i) + v(y_{i-1})}{\Delta y^2}.$$

into the left hand side of (C.1), we obtain

$$\begin{aligned} & A_1(y_i)y_i^2\bar{v}_{yy}(y_i) + A_2(y_i)y_i\bar{v}_{y_i} + A_3(y_i)\bar{v}(y_i) + g(y_i) \\ \approx & (\alpha_i - \beta_i)\bar{v}(y_{i-1}) - (2\alpha_i - \gamma_i)\bar{v}(y_i) + (\alpha_i + \beta_i)\bar{v}(y_{i+1}) + g_i, \end{aligned}$$

where $\alpha_i = A_1(y_i)i^2$, $\beta_i = \frac{1}{2}A_2(y_i)i$, $\gamma_i = A_3(y_i)$ and $g_i = g(y_i)$.

Denote $h_i = h(y_i)$. Denote $\bar{v}_i^k$ as the k-th iteration value of $\bar{v}(y_i)$, $i = 0, 1, \ldots, N_s$. When solving for $\bar{v}_i^{k+1}$, we approximate the right hand side of (C.1) by $(h_i - \bar{v}_i^{k+1})I_{h_i - \bar{v}_i^k > 0}$. Consequently, we have the following equations for $\bar{v}_i^{k+1}$

$$(\alpha_i + \beta_i)\bar{v}_{i+1}^{k+1} - (2\alpha_i - \gamma_i)\bar{v}_i^{k+1} + (\alpha_i - \beta_i)\bar{v}_{i-1}^{k+1} + g_i = -\lambda(h_j - \bar{v}_i^{k+1})I_{h_i - \bar{v}_i^k > 0}. \quad \text{(C.2)}$$

Together with the boundary conditions $\bar{v}_0^{k+1} = \frac{\rho}{\delta+\rho}\bar{u}(0)$ and $\bar{v}_{N_s}^{k+1} = 0$, we can solve $\bar{v}_i^{k+1}$'s efficiently. Finally, we stop once $\bar{v}_i^{k+1}$ and $\bar{v}_i^k$ are sufficiently close.