# CLASSIFICATION OF SPOKEN DISCOURSE IN TEACHING THE CONSTRUCTION OF MATHEMATICAL PROOF

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# ABSTRACT

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The purpose of this study is to analyze the patterns of classroom discourse when high school students move from performing prescribed algorithms in order to solve problems for which the process and solution are well-defined to spoken proof, in which ideas are discussed and arguments are formulated and formalized.

The study uses a modified version of discourse analysis developed by Arno Bellack and refined for usage in a mathematics classroom by James T. Fey. The analysis framework is supplemented by codes borrowed from Maria Blanton, Despina Stylianou, and M. Manuela David (2009), which is in turn a modified version of a coding system developed by Kruger (1993) and Goos, Galbraith and Renshaw (2002).

Twelve mathematics lessons involving two mathematics teachers were recorded, transcribed and coded. Eight of the lessons were classified as "proof-related" and four were designated "non-proof-related." A lesson designated "proof-related" contained more than half activity that was actively concerned with the construction of proof; whereas a lesson in which no proofs were formulated was designated "non-proof." Using the codes described above and a variety of

qualitative and quantitative measures, the transcripts were examined for constructivist behavior on the part of the teachers and modes of participation on the students' part.

The findings suggest a relationship between a teacher's beliefs in constructivist principles and the way in which that teacher instructs proof vs. non-proof. More specifically, a teacher who views her/himself as informed by constructivist pedagogical principles may not evince a sharp distinction between her/his teaching of proof vs. non-proof; but a teacher who does not attempt to incorporate constructivist principles on a daily basis may exhibit more constructivist tendencies when teaching proof.

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# Chapter I. Introduction

## Section 1: Need for the study

> "There is no royal road to geometry." *Euclid*

> "Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures…Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is…."
>
> From "Common Core State Standards Initiative, Standards for Mathematical Practice." <http://www.corestandards.org/Math/Practice/>, retrieved on July 24, 2014.

In contrast to the traditional transmission model of mathematical education that emphasizes transfer of procedures and algorithms from teacher to student, current trends in mathematics education call for students to be able to *construct* understanding by means of sense-making activities. This perspective, known usually as *constructivism*, has been embraced by the National Council of Teachers of Mathematics since 1980, but the release of the Common Core Standards marks the most recent initiative in agreement with this pedagogical orientation.

Central to the idea of sense-making and construction of understanding in the classroom is the concept of proof, considered by some to be "the soul of mathematics": "If problem solving is the 'heart of mathematics,' then proof is its soul" (Schoenfeld 2009). There is an indubitable relationship between problem solving and proof. Problem solving has been defined as "work[ing] problems of significant difficulty and complexity" (Schoenfeld 1992). Selden and Selden (2013) see proof as a subset of problem solving in that it has a "problem-centered part" and a "formal-rhetorical part" (Ibid.). As Lucast

(2003) states: “[P]roving encompasses problem solving…but proving requires more than just arriving at a solution” (49).

Much research has been conducted on the importance of teaching mathematical proof to high school students (Zaslavsky, Nickerson, et al. 2012; Hanna and Barbeau 2008; Schoenfeld 1994). The motivation for doing so is at least twofold: to develop analytical and problem-solving skills by way of logical deduction; and to introduce students to the methods and procedures by which mathematics is done.[1] The literature on proof covers many aspects: why even teachers who believe mathematical proof is important frequently don’t teach proof to their students (Frasier 2014), why and how high school students should be taught to prove in mathematics classes (Hirsch and Lappan 1989), what constitutes proof in a high school classroom (Quinn 2012; deGroot 2001; Hersh 1993, Fawcett 1995/1938), what specific obstacles are encountered by students of proof (Herbst and Brach 2006), common misconceptions held by students and teachers on the nature of proof (Healy and Hoyles 2000, Knuth 2002), and others. What have not been investigated, a deficit this study seeks to correct, are the semantic mechanics of classroom discourse when students study proof; moreover, what discursive stance a teacher may take in order to encourage students to construct their own proofs.

Fortunately, in support of that investigation, there has been a relatively recent proliferation of articles focused on mathematical discourse and, particularly, on discourse in mathematics classrooms. Ryve (2011) cites the fact that there were a total of 24 such articles between 1968 and 1999; whereas in 2000-2009 there were 84 articles that could

[1] Hemmi and Lofwall (2009), upon interviewing ten mathematicians on the functions of proof, found no fewer than six purposes: “conviction, explanation, communication, intellectual challenge, aesthetic and transfer” (204).

be characterized as studies in mathematical discourse. (His study included articles from five journals in total, two of which were the Journal for Research in Mathematics Education and Educational Studies in Mathematics, seen by many as the two most influential journals in the field.) Although the primary focus of his study was to suggest ways in which researchers can and should be more precise in their situation of epistemological principles, it serves to highlight a striking increase in interest in discourse study in mathematics education. Ryve's study also categorizes the types of discourse analysis that have been done, providing options for how one might proceed on such an avenue of inquiry. A more detailed provision of recent discourse analyses in mathematics classrooms will be provided in this study's literature review.

This study attempts to combine these two "trends" in research in mathematics education by coding, tabulating and analyzing spoken exchanges between students and teacher in order to examine the semantic mechanics of classroom-based procedures when teaching high school students how to construct proof. In particular, this study compares such spoken exchanges in a classroom concerned with the construction of proof with those in a classroom in which proof is not being taught.

Since teaching proof is a process whose difficulty has been well-documented (D'Ambrosio, et al. 2010; Epp 2003; Sowder and Harel 1998; Balacheff 1991; Sfard 2000), it is hoped that this study will be of use to those teaching geometric proof at the secondary school level, it is hoped that these findings will be of help to those attempting to adopt constructivist teaching methods in their classrooms, and it is hoped that this study will serve as evidence for the continued inclusion of proof in the teaching of high school geometry.

## Section 2: Purpose of the study

The purpose of this study is to analyze the patterns of classroom discourse when high school students move from performing prescribed algorithms in order to solve problems for which the process and solution are well-defined to spoken proof, in which ideas are discussed and arguments are formulated and formalized. The study will address the following research questions:

1. What semantic features can be shown by coding and analyzing student and teacher utterances when learning to construct proof?
2. What semantic features can be shown by coding and analyzing student and teacher utterances when proof is not being taught?
3. Are there discernible, quantitative differences in the quality or proportions of teacher/student discourse in the teaching of proof?

## Section 3: Overview of the study

The study analyzed audiotapes and transcripts of classroom activity in geometry classes in two high schools in the tri-state area. Both are ninth grade classes in private schools with small classes (12-18 students) and one teacher. Although one school is attended by girls and the other is coeducational, the students are demographically similar.

Both teachers were in their second or third year of teaching. Both teachers had apprenticed as student teachers with the researcher in geometry classes. In the researcher's professional opinion, both teachers were sufficiently proficient at their craft so that lack of experience was not a factor in the study's findings. In fact, the proximity of their training at Teachers College ensured that both had been exposed to the principles of constructivism and were accustomed to incorporating those into their classroom practice.

Data were collected in three stages. The teacher at one school taught proof in the spring and the other in the fall, so both timing and location necessitated two stages of data collection. Two-three weeks of data from each classroom yielded recordings of twenty-four mathematics lessons of which nineteen were chosen for transcription and twelve were chosen for analysis. Some lessons were designated proof-oriented, whereas others were classified as non-proof-oriented, in order to differentiate proof-based instruction from non-proof-based instruction. A lesson designated "proof-related" contained more than half activity that was actively concerned with the construction of proof; whereas a lesson in which no proofs were formulated was designated "non-proof." (It is noted here and elsewhere that only lessons in which at least half of the activity was directly devoted to constructing proof of geometric relationships were classified as "proof-based"; generally speaking, a proof may follow utilize the "statements and reasons" two-column structure although this is not required. What is required to merit the designation "proof" is the establishment of facts to be accepted as "given," a statement or relationship to be proven and the development of the proof using principles of deductive reasoning.) However, it was found that there were not two examples from Teacher B's classroom that could be considered adequately devoid of proof activity, so two of that teacher's algebra 2 lessons were recorded at a later date.

The study utilized linguistic analysis to code audiotaped classroom exchanges between students and teachers. It made additions to the discourse analysis framework developed first by Bellack, et al. (1966) and modified by Fey (1966) to apply to the mathematics classroom.  As expected, Bellack and Fey's coding system did not provide subcategories that were sufficiently specific to classify the nature of teachers' and students' utterances in a mathematics classroom in which the primary occupation is the development of proof, as opposed to a classroom in which the inculcation of mathematical procedures is the primary focus.  These were found elsewhere.  Therefore, in addition to the work of Bellack and Fey, this study leaned heavily on Offenholley (2007), Weinberg (2010), Generazzo (2011) and Blanton, Stylianou and David (2009) for both framework and methodology, including the development of the coding scheme.  The precise codes and their adaptation for this study will be discussed in the methodology section of this paper and also in appendices.

# Chapter II: Literature Review

## Section 1: Constructivist pedagogy

### 1.1: History of support for constructivist pedagogy

In 1980, the National Council of Teachers of Mathematics (NCTM) recommended that "problem solving be the focus of school mathematics" (NCTM 1980). In 1989, NCTM released the 1989 NCTM Standards which stated ""the study of mathematics should emphasize reasoning so that students can believe that mathematics makes sense" (NCTM 1989, p. 29). Following that, NCTM's 2000 Principles and Standards identified five process standards: Problem Solving, Reasoning and Proof, Communication, Connections and Representation (NCTM 2000).

With the publication of Professional Standards for Teaching Mathematics in 1991, NCTM staked its claim that a constructivist approach is an effective, arguably best, way to teach American students all types of mathematics on all levels. Moreover, it is generally held in the mathematics education community that students' engagement in "sense-making" activities will enable them to integrate mathematics with science, engineering and technology on the secondary and collegiate levels; and further down the road, to perform well in STEM professions.

In "Learning to think mathematically" (Grouws/Schoenfeld 1992), Alan Schoenfeld expands upon NCTM's goals, identifying problem solving as "the theme of the 1980's" (2). He clarifies the goal of teaching students to "solve problems," a vague and broad objective, as to engender students' facility in the language of mathematics, to be able to apply skills in a variety of contexts (this is known as the ability to transfer knowledge; Willingham 2012, Hemmi and Lofwall 2009, Fawcett 1938, and many others), to have the ability to recognize patterns and to be able to formulate conjectures (Reshaping School Mathematics, National Research Council, 1990a, quoted in Grouws/Schoenfeld 1992, 4). There is emphasis on the interpretation of data, flexible and analytic thinking, and providing access to STEM professions to all people (especially "women and minorities", 7-8). The way to ensure that these goals are met, according to NCTM and Schoenfeld, is to regard mathematics as a "sense-making" activity, to engage students in dynamic and open-ended activities, and to resist the pull to represent mathematics as "a body of facts and procedures dealing with quantities, magnitudes, and forms, and relationships among them…[in which] knowing mathematics is seen as having 'mastered' these facts and procedures" (3).

It may be daunting, twenty years later, to realize that these goals have not been met, and that these are precisely the same goals of current mathematics reform movements. They bear more than passing resemblance to the goals of the Common Core Standards Initiative, an education reform slated to take full effect in 2014, a portion of whose objectives are as follows:

> **"1. Make sense of problems and persevere in solving them.**
> Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. …Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.
>
> **"2. Reason abstractly and quantitatively.**
> Mathematically proficient students …bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved….
>
> **"3. Construct viable arguments and critique the reasoning of others.**

> Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. ...They justify their conclusions, communicate them to others, and respond to the arguments of others. ...Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is....Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments."
>
> From "Common Core Standards Initiative, Standards for Mathematical Practice." <http://www.corestandards.org/the-standards/mathematics/introduction/standards-for-mathematical-practice/>, retrieved on December 5, 2011.

Although the goal has not been met, much research has been done in the past twenty or so years to explore how a mathematics teacher may induce her/his students to think independently and mathematically, i.e., to regard problems as dynamic, viable representations of the world and how to approach these problems analytically and intelligently. Schoenfeld (1992) and NCTM led the charge; many researchers picked up the mantle.

1.2: The inquiry-based classroom

There is ample research to suggest that creating an atmosphere of dialogue or polyphony, even argument, is conducive to bringing students to comfort with development of their ideas in a community, sometimes critical, of fellows (Yackel and Cobb 1996; Martin, McCrone et al. 2005; Wood 1999; Cobb, Boufi et al. 1997; Elbers 2003; McClain 2002; Bowers, Cobb, McClain 1999; Peressini, et al. 2004; Pape, Bell and Yetkin 2003; Durand-Guerrier, et al. 2012; Cobb, Stephan et al. 2001; Hershkowitz and Schwarz 1999; Zaslavsky 2005). In these models, the instructor takes the role of facilitator (the "guide on the side") instead of transmitter and sole arbiter of knowledge (the "sage on the stage").

Schoenfeld cites the classroom of Harold Fawcett, an educator who modeled his classroom on the ideas of John Dewey. Fawcett believed that "a course in geometric proof can help students learn to reason clearly about a wide variety of situations" (Schoenfeld "Divorce," 1991, 333); in other words, that such learning would transfer to other situations. In Fawcett's classroom, even basic geometric definitions were decided upon as a group in a sense-making process (Ibid., 334-5). Fawcett relied heavily on the provision of counterexamples (which presumed that students understood the power of the counterexample) in order to point out the inadequacy of early versions of definitions, which modeled the meaning of "truth" in mathematics. All activities required the class to act as "jury" for the claims made by students; this was "an active and reflective jury" (335). Also of note in Fawcett's work is the observation that, if students are to integrate their mathematical analytical training into their everyday lives, a phenomenon known as transfer, they must be taught to do so explicitly (Fawcett 1938).

Leone Burton, whose research centers upon interviews with mathematicians in universities in the United Kingdom, cites the prevalence of collaboration in the field of mathematics: "Nowadays it is important to work with others…" (Burton 127); "(t)he participant who claimed only to do individual work was extremely rare – three males and one female out of seventy. What the participants described to me was a cultural practice now widely prevalent in their communities…" (Ibid.), although many also acknowledged the need to publish on one's own.

The proclivity toward collaboration is regarded as a shift in the mathematics community. "Many of the mathematicians who contributed to this picture themselves pointed to a substantial cultural shift in mathematics from a discipline dominated by individualism to one where team work is highly valued…However, a gap has appeared between a (mistaken) public image of how mathematics is done and the current practices of many, possibly most, of those in the mathematics community" Ibid., 131).

The interviewees provide grounds upon which collaboration is essential, all of which transfer well to a classroom community:

> "* talking is a good way to get a problem done,
> "* it shares the work,
> "* you benefit from the experience of others,
> "* it increases the quantity and quality of ideas,
> "* you have someone off whom to bounce ideas,
> "* it enhances the range of skills,
> "* you get into areas that you might not have thought of going into,
> "* you learn a lot from more senior colleagues,
> "* under the pressure of writing up, you mustn't let others down,
> "* there is someone to take over when you reach a dead end,
> "* you share 'the euphoria' with someone,
> "* you feel less isolated,
> "* you can benefit from a novice/expert combination" (128).

Burton finally connects the practice of mathematicians with what could take place in the classroom, citing the prevalence of a "competitive style" in many classrooms (132) which is a "long way from what these research mathematicians are doing" (Ibid.) She states:

> "An individualistic teaching/learning model locates responsibility within the learner and supports a teaching style which is content-based and fragmented. A collaborative model emphasises the group functioning in exploring and negotiating meaning assuming that such meaning is negotiable and non-homogeneous…The world of knowing described by my participants, a world of uncertainties and explorations, and the feelings of excitement, frustration and satisfaction, associated with these journeys, but, above all, a world of connections, relationships and linkages, is a natural outcome of this model and equally available to learners of any sophistication. It compares very unfavourably with the reception learning on which a transmission model is predicated where mathematics is presented to learners in disconnected fragments; if there is any hope of connections being made, that is more often than not left to the individual to achieve. Not only does this fail

> to support learners in constructing any Big, or even smaller, Picture but it also deprives them of the very pleasure of which these research mathematicians speak - the pleasure of making a connection" (Ibid., 138-9).

She concludes with the conjecture that, since many teachers have not worked as professional mathematicians, their lecture style comes from their own classroom experiences, as there is nothing in their experience to convince them that anything else is possible; but she urges teachers to depart from the individually-oriented transmission model.

Marta Civil breaks mathematics down into three types that may or may not be reconciled: "Everyday Mathematics, Mathematicians' Mathematics, and School Mathematics" (Civil 2002). Although she acknowledges that, in many classrooms, we have seen a departure from the traditional Initiation–Response–Feedback, or IRF (or Initiation-Response-Evaluation, IRE) model (Skidmore and Murakami 2012, 201; Pimm 27; Shepherd 7; Schoenfeld 2002, 134), she notes that, even in reform-based classrooms, it is still likely that the teacher (or textbook) will be seen as the final arbiter of what is "true." In addition, students' perceptions of what is supposed to constitute a mathematics lesson or classroom are seriously challenged by standards-based instruction; students must be given a period during which they can be "won over" to the inquiry-based model.

She notes that "the following are some distinguishing characteristics of a classroom environment in which children do mathematics as mathematicians:

> "* The students and the teacher engage in mathematical discussions.
> "* Communication and negotiation of meanings are prominent features of the mathematical activity.
> "* The students collaborate in small groups on challenging mathematical tasks and are encouraged to develop and share their own strategies.
> "* The students are responsible for decisions concerning validity and justification.

"* The teacher encourages the students to be persistent in the mathematical task." (Ibid., 42-3).

She does point out, however, that there are significant differences between mathematicians and schoolchildren: primarily, children are forced to do mathematics, and they generally do not choose the problems on which they work. Consequently, in addressing the project goal of seeing whether ten year-olds could engage in mathematical argumentation, the researchers encountered problems of limited patience in listening to peers' arguments; lack of interest when the discussion flowed from familiar contextualized situations to abstract ones; and a preoccupation with being judged correct: "(W)e also had to work on developing an atmosphere in which being wrong was seen as something valuable to all of us in our route towards learning" (59). Over time, however, a measure of success was achieved (55-6), indicating that it is possible to engage students of relatively young age in "mathematician's mathematics." This finding is shared by several other researchers: that particularly if we don't insist on "formality" in proof, it is valuable to engage in reasoning and informal proof in elementary and middle school (Campbell and Robles 1997, Carpenter et al. 1998, Cobb et al. 1991, Kamii 1994, all cited in Sowder and Harel 1998; deGroot 2001; Schoenfeld 1992). If students are encouraged to experiment and articulate mathematical arguments from an early age, it is less likely that they will conceive of mathematics as solely a set of procedures to be accepted without question and memorized (Schoenfeld 1992, 26-7).

Herbst cites NCTM's *Principles and Standards* in support of teaching proof at all levels of mathematics instruction: "Reasoning and proof are not special activities reserved for special times or special topics in the curriculum but should be a natural, ongoing part of classroom discussions, no matter what topic is being studied" (NCTM 2000, p . 342; quoted in Herbst 2002).

1.3: Convincing teachers to adopt a reform-based classroom practice

"Research has…documented the possibility of establishing alternative, more dialogic patterns of spoken interaction in the classroom, which provide greater opportunities for students to make their thinking explicit, and thus to reach a deeper level of understanding of the subject matter" (Skidmore and Murakami 2012) although admittedly it may be difficult for teachers to shed the "Initiation–Response–Feedback sequence" cited in the last section. Nathan and Knuth (2003) observe that "(f)or teachers attempting to enact discourse-based practices,… the challenge is particularly daunting given that mathematics teacher education and professional development programs typically have not adequately prepared them to enact successfully the lofty expectations set forth in reform documents (Ross 1998)" (203-4). This section provides some suggestions to teachers who wish to incorporate the constructivist model.

Wood, Cobb and Yackel (1991) found that teachers can be brought a realization of the efficacy of constructivist teaching methods by observing their effects on student learning or on their own teaching experience in the classroom. Their research indicates that, not only were second-grade students fully capable of working in pairs and explaining their mathematical processes in whole-class discussion, but that the teacher, previously a practitioner of transmission-based learning, "discovered that [her students'] thinking was far more sophisticated than she had previously assumed…. Listening to their ideas provided opportunities to learn about her student's understanding of mathematics that were not available in her previous pedagogical practice. The unanticipated responses of the children, which were novel to the teacher, acted to stimulate her listening carefully to children and realizing that they have their own ways of knowing. This, in turn, facilitated the development of genuine conversations in which children talked about their mathematics….In describing the change that took place, she commented, 'My teaching is pleasantly different. Rather than being the person with all the answers, the children have been given the opportunity to count on themselves and each other'" (601).

Evidence also suggests that teachers who engage in dialogue about their classroom practice are prone to question their methods and to incorporate successful techniques, which may well include constructivist approaches (Penlington 2008; Vygotsky, Bakhtin, Cazden quoted in Penlington, 1307).

Finally, it is important to address teachers' fears about "relinquishing control" in a standards-based classroom. If teachers attempting to maintain a standards-based classroom are aware that their input is still valued and frequently necessary, their concerns about what they may perceive as giving up their authority may be alleviated and they (and their students) can reap the benefits of discussion-based learning.

## 1.4: Techniques to strengthen reform-based classroom practice

Teachers who believe they are engaging in constructivist behavior have benefited from videotape and close analysis of their classroom practice to see ways in which they are leading the students, despite their sincere intention to allow students to conduct full discovery on their own (Nathan and Knuth 2003; Arsac, Balacheff and Mante 1992). Nathan and Knuth conducted a two-year study involving a teacher eager to scrutinize her own classroom practice including review of videotape and quantitative analysis of classroom interaction. After the first year of discussion about and reflection upon her practice with the researchers, "[the teacher] set out to make the discourse more accessible and more productive for her students, and, in accord with the mathematical reforms, to use the power of these content-centered verbal exchanges to enhance students' understandings. [The teacher's] strategy to remove herself from these interactions so that more student-to-student exchanges would occur seems to have been successful" (197).

Yackel, Rasmussen and King (2000) found that it was necessary to reconstruct social norms in a mathematics classroom in order to encourage the type of student activity they deemed desirable. "[T]he students and the instructor acted in accordance with the normative understandings that students were expected to explain their reasoning and that they were expected to try and make sense of other students' thinking. These two social norms, which were constituted early in the semester, contributed significantly to the climate of the classroom as one in which sense-making and meaning-making prevailed…Moreover, when the classroom norm is that of making sense of other student's reasoning, class discussions often form the basis for students to further their own mathematical development" (280-1). In other words, although it frequently contradicts the norms to which many students (and many teachers) are accustomed, it is possible, and perhaps desirable, for teachers to establish their classrooms as places in which students learn to express, critique, and make arguments. In the classroom studied by Yackel, et al., "… students frequently explained their reasoning without prompting, offered alternative explanations, and attempted to make sense of other students' reasoning and explanations, despite the fact that their prior experiences were with traditional approaches to mathematics instruction" (275-6).

Stylianou and Blanton (2011) cite the usage of "transactive prompts" (143) on the part of the instructor, instead of giving students specific information, to help the students come to the elements of proof on their own. "Clarifications,…explanations,…criticisms…and elaborations" are typical instructor tools to "support students in developing the skill for argumentation in mathematics" (Ibid.). Similarly, "revoicing" and "confirming" students' ideas provide support and guidance in students' development into mathematical thinkers (144). They conclude that students may be led toward deductive arguments and proof if the teacher engages in appropriate discursive behavior. "Analysis of classroom discourse strongly indicates that certain teacher actions, including the types of questions that the teacher asked, supported students' development of proof over time" (Ibid., 143). Wood, Cobb and Yackel (1993) state that "it is understood that the manner in which the teacher acts to direct and control the dynamics of the discourse strongly influences the opportunities for students to be active participants…" and suggest that when "teachers…avoided being directive and instead encouraged the children to engage in the demands of the task by providing prompts and suggestions in their dialogue…the children [were able] to solve the tasks on their own. It would appear, then, that in classroom interactions where the teacher acts to facilitate children's thinking about mathematical problems and where genuine attempts to communicate exist, students have opportunities to engage in dialogue in which they can express their mathematical thinking" (59).

Dawkins and van Dormolen recommend the usage of metaphors to make mathematical concepts more accessible to students; this also permits students to put mathematics in a familiar context (Dawkins 2012, van Dormolen 1991). However, as Civil points out, there may be a difficult transition between familiar contexts and abstraction for which teachers must prepare (Civil 2002).

Hirsch and Lappan recommend the usage of manipulatives to embed imagery in order to build a pattern that may be recognized (Hirsch and Lappan 1989). Again, however, this may complicate the transition from specific to general, which is necessary for proof unless all cases are exhaustively tested.

Arzarello, et al. suggest the usage of construction and dynamic software to provide "observational" representations of geometric ideas. They point to the gradual disappearance of construction from Western classrooms despite its prominence in mathematical history, beginning with the ancient Greeks. They cite as an explanation the 19th century proliferation of disdain for "observational 'intuitive' hidden hypotheses" (Arzarello, et al. 2012, 101) in geometry. However, they make the claim that these activities may create a dialectic between geometric objects and proof, or as they express it, "geometrical construction can serve as a key to accessing the meaning of proof" (102) if the teacher is mindful in directing instruction. In sum, they theorize "artifacts have historically been fruitful in generating the idea of proof and consequently can provide strong didactical support for teaching proofs, specifically, if the teacher acts as a semiotic mediator" (108).

It is also instructive to note that several researchers have noted that teachers should, on occasion, provide specific guidance above their roles as mediators. Zaslavsky et al. (2012) and Lobato, Clarke and Ellis (2005) do not see "telling" as contradictory to the constructivist method although "telling" students has been linked to the transmission pedagogical model. Rather, they advocate "telling actions…when the goal is to develop concepts as opposed to procedure" (Lobato, Clarke and Ellis 2005, 104). They maintain that it is a misconception of constructivism that teachers may never tell students anything; rather, they point out that it is not a question of *whether* teachers should ever convey information, but of *when* to do so.

To assist in this determination, the authors state that "it can be helpful to state facts, share ideas, or identify conflicts, and then examine the sense that students make of them. In addition, introducing new information at critical junctures could help reduce the number of problem features that students must attend to, thus allowing for exploration in new areas" (Ibid., 106). Finally, they clarify that "(s)tudents cannot be expected to reinvent entire bodies of mathematics, regardless of how well each concept is problematized by well-chosen tasks (Clarke 1994; Romagnano 1994). Teachers are expected to enculturate students into the mathematics community, sharing conventional norms associated with mathematical discourse, representation, and forms of argument (Becker & Varela 1995; Cobb & Yackel 1996; Driver 1995). If teachers are to facilitate this enculturation, then making the ideas and conventions of the community available to students is essential. From this perspective, some information must be introduced by the teacher. In short, a telling/not-telling dilemma has emerged. Telling is instructionally important, but has been downplayed due to both perceived inconsistencies with constructivism and historical attempts to develop pedagogical implications of constructivism" (Ibid.). As Sfard cautions, "Rules of language games can only be learned by actually playing the game with experienced players. The profound constructivist principles underlying the current reform movement are only too often misinterpreted as a call to teachers to refrain from any kind of intervention" (Sfard 2000, 185).

Nathan and Knuth (2003) conclude that, although the teacher in their study was able to remove herself effectively, clearing the way for students to discover knowledge, "[the teacher] needed to encourage students to treat other's ideas as objects of discussion in and of themselves -- an analytic scaffolding role -- and critically evaluate the veracity of the ideas presented. When the teacher elected to move away from her analytic role, the team observed that there was nothing added to the classroom culture to fill this gap in the discourse when there were major oversights, or when conflicting views among students arose (200).

As constructivist pedagogues Wood, Cobb and Yackel (1991) have observed, "Teachers must develop the sensitivity to know when to intervene to make appropriate suggestions and when to allow children time to resolve conflicts themselves. This requires listening to the students' explanations and developing an understanding of the underlying conceptual operations that underscore children's thinking" (610). Nathan and Knuth (2003) describe the role of the teacher in a constructivist classroom as follows: "Rather than primarily explaining and demonstrating, the teacher is asked to craft instruction in a nontraditional way, at times leading from behind, at times stepping in as a mathematical authority, and at other times carefully guiding the discussion and activities and seeding ideas" (176).

Elbers (2003), in his presentation of Streefland and Gertsen's deliberate enculturation of a community of inquiry, observes the teachers' activities as having three main purposes: to encourage students to come up with new approaches to an existing problem; to make "global and general" suggestions to students when vocabulary or recognition of a new procedure was required; and as observers of what new solutions merited the class's attention.  Although students initially looked to their teachers for affirmation, they developed as independent learners and came to look to their teachers for guidance, not final approval.  As Confrey (1990) points out, "Ultimately, the student must decide on the adequacy of his/her construction."

1.5: Summary

In sum, it is generally accepted in the mathematics education community that wholly traditional, transmissive modes of instruction (teacher as "sage on the stage") are simply not sufficient nor effective in fostering students' independent intellectual growth. This applies even more emphatically to the realm of proof, which requires independent thought and the willingness to follow lines of reasoning that may not prove fruitful.  Some researchers (Wood 1999, Zaslavsky 2005) have written about the explicit necessity of making the classroom a forum for argumentation in which periods of uncertainty are to be celebrated, not avoided.  As Zaslavsky, Wood and others have found, a successful teacher of proof should create an environment in which students are free to explore and, perhaps counterintuitively, make mistakes.  It is natural and desirable for a teacher to encourage periods of uncertainty (Zaslavsky 2005), argumentation (Wood 1999), and (it is to be hoped) whole-class-negotiated resolution (Wood, Cobb, Yackel 1993; Cobb, Boufi et al. 1997).  As Burton points out, professional research mathematicians experience uncertainty; "although knowing when you know is extremely important, you may have to live with uncertainty" (Burton 134).  Although students initially expect the study of secondary mathematics to consist of a system of algorithms by which they arrive inexorably at "the answer" ("Normally, there's a set way of doing it and you have to do it that way. You can't work out your own way so that you can remember it (Carly, year 11)" (Boaler 1997, p. 23)), students can be brought to appreciate the opportunity to investigate and discover the inner workings of a mathematical situation. Mathematics taught with a constructivist approach encourages students to develop, in Schoenfeld's words, "a way to figure out how things work" (Stylianou/Schoenfeld, Foreword, xiv); in which "even in failure, you may come to a better understanding of the phenomenon you're trying to make sense of [sic]" (Ibid.).

It is equally important for the constructivist teacher to develop an awareness of when it is appropriate and necessary to give her/his "professional opinion" or to convey information to the class. As many have noted, students are not likely to reach high levels of mathematical sophistication on their own. A teacher's role as "guide" must include analytical scaffolding and sufficient information to inspire discussion and discovery. Gonzalez and Herbst (2013) recommend teachers attempt to spur student exploration and discussion as follows:

> "Some teaching actions that may prompt students' construction of arguments in a class include selecting meaningful tasks, bringing attention to a student's ideas, establishing connections between students' solutions and mathematical conventions, requesting students to pay attention to the ideas of others, asking questions about students' conceptual understandings, relating conceptual understandings with procedures, and identifying inconsistencies in students' justifications (Ball & Bass, 2000, 2003; Strom, Kemeny, Lehrer, & Forman, 2001; Zack & Graves, 2001). Thus, specific teaching actions may lead students to develop arguments" (273).

As mentioned above, one of the Common Core's explicit directives and a long-standing goal of NCTM is to have students question each other's ideas and to become comfortable with defense of their own ideas and those of others, to develop the ability to decide for themselves what is useful and true and what is not (from "Common Core Standards Initiative, Standards for Mathematical Practice"). Students' guided initiation into the language of mathematics is an integral part of the process of students learning how to express themselves mathematically – although, as the next section shows, this road is typically fraught with difficulty.

## Section 2: Teachers', students' and mathematicians' conceptions of proof

2.1: The connection between the inquiry-based classroom and proof

"It is desirable for students to bridge the gap between conviction and proof." (Miyakawa and Herbst 2007)

It is reasonable to suppose that "sense-making" explorations that take place in the standards-based classroom will engender analytic and problem-solving skills to enable students to engage in mathematical argumentation, which should lead to the creation of proof, identified as the soul of mathematics: "If problem solving is the 'heart of mathematics,' then proof is its soul" (Schoenfeld, Series Editor's Foreword in Stylianou, Blanton and Knuth 2009).

Zaslavsky et al. (2012) view teaching proof as a natural extension of teaching problem solving. As they point out, even in the professional mathematics community it is desirable to provide multiple proofs; proof is not merely a way of establishing validity of "facts" (217). (If it were, "there would be no need to prove things in multiple ways" (Ibid.).) The mathematics community uses proof as a method of generating new techniques, exploring deeper significance and of providing "a network of connections" (Ibid.); the authors suggest secondary school teachers and students do the same.

As Knuth points out, several mathematical pedagogues have bemoaned the relative absence of proof from secondary school mathematics (Knuth 2002; Wu 1996, Ross 1998 and Schoenfeld 1994 are cited). As stated in the previous section, Schoenfeld in particular finds that proof can and should be embedded in the study of mathematics at ALL levels, not just in high school. Hirsch and Lappan also advocate that "informal explorations of grades 7 and 8…be thought of as building up a rich set of specific instances from which powerful mathematical generalizations and techniques can be abstracted" (Hirsch and Lappan 1989, 614) so that high school may be spent doing mathematics of greater abstraction, and therefore sophistication. Wu (1996) claims that the absence of proof in our teaching of mathematics is "a glaring defect in the present-day mathematics education in high school, namely, the fact that outside geometry there are essentially no proofs....[presenting] a totally falsified view of mathematics itself" (228). Hanna's (2000) opinion is equally strong: "[S]tudents cannot be said to have learned mathematics, or even about mathematics, unless they have learned what proof is" (24).

Herbst and Brach (2006) cite numerous studies in which it is found that "(c)hildren of ages even younger than [14-15] can engage in pursuing deductively the consequences of certain assumptions that they themselves make; students can vary those assumptions, refute arguments, and make conceptual connections (Ball & Bass 2000, 2003; Lampert, 1990; Overton, Ward, Black, Noveck, & O'Brien, 1987; Reid 2002; all cited in Herbst and Brach 2006). They can engage in proving and call their work proving, even if some of the additional values that separate mathematical discourse from the norms of mathematical writing of contemporary mathematicians may not be present in how they represent their arguments (Duval, 1992; Hanna, 1995)" (Ibid., 114-5). Wood, Cobb and Yackel (1991) found that students of mathematics in a second-grade classroom, guided by their teacher, were capable of working in pairs and articulating their findings in whole-class discussion.

Herbst and Brach also provide the suggestion that teachers "engage in some negotiation of how the didactical contract applies [to get students to engage in proving]. Something that could be object [sic] of such negotiation is how a teacher should ask a specific question so as to feel entitled to indeed get the students to work to produce the answer (vs. prompt them to recall a previously studied answer). Another object of negotiation could be what specific affordances for the production of an answer a teacher should provide, so as to stand a chance of observing students' engagement in proving in the context of their work answering the question," both of which are constructivist techniques (Herbst and Brach 2006, 79).

Although there has been much research on constructivism, as evidenced by the last section, there has been a dearth on the mechanics of proof in the classroom. For example, an explicit definition of classroom "proof," the difference between "argumentation" and formal proof, and the particular conceptions students hold about mathematical proof did not receive as much attention as other difficulties in mathematical education (Hoyles and Kuchemann 2002, 195). However, recent studies are addressing this deficiency. The remainder of this section has findings from studies specifically related to the mechanics of proof in the classroom, including teachers' and students' roles and conceptions compared with those of the mathematics community.

### 2.2: Teachers' roles in and conceptions of proof

In contrast to many in the mathematics education community who see an enormous need for the teaching of proof at the secondary school level and prior, many teachers do not. Cabassut et al. (2012) cite numerous studies in which teachers do not see the need for proof AT ALL; some saw the appropriate locus of proof solely in Euclidean geometry in secondary school. (A notable exception is found in Mingus and Grassl's study (Mingus and Grassl 1999, cited in Cabassut et al. 2012, 178), where it is found that 69% of pre-service teachers of secondary school "advocate…the introduction of proof before 10$^{th}$ grade geometry classes" (Ibid.).) In their discussion of the majority of these studies, Cabassut et al. posit that teachers are not themselves comfortable with teaching proof in a constructivist setting because they lack facility with mathematical content and so are anxious about free-form exploration that might reveal their weakness.

Knuth (2002) showed through interviews with 16 in-service high school mathematics teachers that a number of them believed, although a proof of a mathematical proposition had been established, that it was possible that a counterexample might be discovered that would falsify the proposition, showing a flawed understanding of the general nature of mathematical proof. Another six, however, were mathematically savvy enough to point out that a change in the axiomatic system within which the proof had been written would enable the possibility that the proof was now rendered invalid. In that same study, over 93% of teachers' ratings given to arguments that were proofs were correct; however, every teacher rated at least one non-proof as a proof. A preponderance of the teachers in the group avowed the importance of proof in secondary school and in mathematics as a whole.

Schoenfeld (1980) suggests that teachers act as guides or "coaches" in problem solving courses, since "it's the *process* of problem solving that counts" but adds that asking students to think instead of "reciting" subject matter is a "critically important" task. It is worth noting that his entire course in problem solving was taught to motivated, mathematically-minded undergraduates at a top university; Schoenfeld may not be aware of the disparity in mathematical background among schoolteachers, nor is he likely to admit that the motivation (not to mention preparation) of younger students generally does not match that of mathematics majors in college, complicating the whole process even more. Schwarz and Kaiser (2009) recognize the demand placed upon teachers by requiring them to teach proof in a student-centered model: "[They] point out that teaching different types of proof places high demands on teachers and future teachers. They add that teachers should have university-level content knowledge of mathematics, including the abilities to identify different proof structures, to execute proofs on different levels, to know specific alternative mathematical proofs, and to recognise and establish connections between different topic areas" (cited in Dreyfus et al. 2012, 202).

Zaslavsky et al. (2012) and others (Hanna and Jahnke 1996) have also identified the utility of a teacher's guidance since it is not likely that students will "rediscover sophisticated mathematical methods or even the accepted modes of argumentation" (Hanna and Jahnke 1996, 887).)" However, they too acknowledge that "(s)uch practices place strong demands on teachers in terms of the required mathematical knowledge and degree of confidence as well as the challenging and time-consuming task of instructional design. Just as it is unrealistic to expect students to see a need for proof without purposeful and focused actions by the teacher, it is unrealistic to expect teachers to be able to attend to this element of teaching without appropriate preparation and support" (Zaslavsky et al. 2012, 226).

### 2.3: Students' (and some teachers') bias toward empirical evidence in lieu of deductive proof

Within the subject of proof, much has been written on students' conceptions of proof, teachers' conceptions of proof, and how these differ. As every mathematician knows but few students believe, insufficient empirical examples do not constitute proof: "A million, a billion, a trillion positive examples aren't enough. Something is only true for sure when you can prove it's true" (Schoenfeld, foreword, xiv) which mathematically speaking means that a proposition must be shown to be true for the general case over any particular instance.

It is well documented that "in mathematics education…most school students do not find the deductive process straightforward and tend to use inductive reasoning to validate conjectures in mathematics rather than to prove them deductively" (Hoyles and Kuchemann 2002 (193); Bell 1976, Van Dormolen 1977, Balacheff 1988 are cited). In other words, most students will attempt to "prove" the truth of a mathematical idea using examples instead of by deduction from laws and existing theorems. In fact, research shows that even students who recognize the validity of a deductive proof state that they would like to see more analysis of applied examples to "strengthen their confidence" (Fischbein and Kedem 1982, quoted in Zaslavsky et al. 2012, 220).

One explanation for this may be found in Vamvakoussi, who notes that the necessity of deduction is an understandably difficult concept for students who are also studying science. Scientific method dictates that phenomena are observed, a hypothesis is formulated, and more data are collected to support the hypothesis (Vamvakoussi, in Vosniadou, et al. 2007). In certain instances, it is even the case that data that do not support the hypothesis may be discarded. This illustrates the most outstanding difference is the treatment of empirical facts in science and mathematics, which Tsamir and Tirosh, among others, have noted, is epistemologically different (Tsamir and Tirosh, in Vosniadou, et al. 2007). In science, the data are the proof, whereas in mathematics, patterns suggest a hypothesis, but they may not be used to prove truth unless it can be shown that the patterns can be applied generally. In mathematics as a rule, empirical data may not be used in this pursuit as they are specific, not general. In mathematics, as well, data that do not support a hypothesis may not be disregarded (although, as the next section indicates, they may serve the useful purpose of illuminating a need to modify the hypothesis).

Cabassut et al. (2012) corroborate this conception of the polemic of mathematics vs. science. "One reason [for students' preference for empirical "proof"] is the difference between [mathematical] statements and statements made in science courses. Conner and Kittleson (2009) point out that students encounter similar problem situations in mathematics and science, but the ways in which results are established differ between these disciplines. In mathematics, a proof is required to establish a result; in science, results depend on a preponderance of evidence (not accepted as valid in mathematics)" (183).

Chazan also cites numerous sources of evidence of students' "preference for empirical arguments over deductive arguments" (Chazan 1993, 359; Balacheff 1988, Martin and Harel 1989, Porteous 1991, and Williams 1979 are cited). For example, Williams found that 68% of a student sample was willing to accept a set of empirical examples in lieu of deductive proof (quoted in Chazan, 361). In addition, Martin and Harel found more than 50% of prospective elementary school teachers were willing to accept empirical examples as proof.

As recently as 2009, Gabriel Stylianides and Andreas Stylianides noted limitations on research on how to assist students in learning proof, although difficulties with proof have been well documented. They reiterate findings on students' visceral preference of empirical evidence, adding the alarming observation that, in one study, 80% of elementary school teachers also accepted empirical evidence in lieu of mathematical proof (Goetting 1995, quoted in Styliandes and Stylianides 2009).

Knuth (2002) found high school mathematics teachers less likely to accept empirical evidence as proof, but the tendency was not absent. As stated in the last section, this group was relatively proficient at recognizing a valid proof, although generally less so at recognizing an invalid one.

Dreyfus et al. (2012) present several possibilities between empirical "proof" and fully formal proof; these include algorithmic, visual, generic and operative types (203). Although these are admittedly "preformal" (Blum and Kirsch 1991, cited in Dreyfus et al. 2012) and teachers as a rule do not deem them acceptable final products, they may be useful heuristic waystations on the journey from empirical evidence to formal proof.

Hirsch and Lappan, as noted in the previous section, explicitly advocate the usage of manipulatives, particularly among younger students, in order to encourage pattern recognition. They hypothesize that this type of "experiential learning" will set the stage for proof in high school. They fall short, however, in providing techniques to facilitate the transition from pattern recognition (which is a collection of empirical data) to proof; they seem to believe that intellectual maturation and the acquisition of symbolic proficiency will lead organically (magically?) to the creation of proof (Hirsch and Lappan 1989).

Hoyles and Kuchemann cite findings that students are perfectly capable of framing a logical argument, as long as one is not too insistent upon formality (Anderson, Chinn, Chang, Waggoner and Yi 1997, cited in Hoyles and Kuchemann 2002). Hoyles and Kuchemann themselves found that students were more likely to frame arguments deductively after having had a year of development in that area, although they were loath to attribute the change solely to classroom effects. Their clear implication is that an extra year of development may be another reason for the students' intellectual development, echoing the assumption made by Hirsch and Lappan (see above).

There may be good reason to establish and maintain a connection between the observation of examples and the writing of "formal" proof. Dreyfus, Nardi and Leikin (2012) find that it is important to validate visually-observed patterns when teaching students to move from empirical evidence to deductive proof. Furthermore, they find that the transition between these phases is not always as distinct as students perceive. This relationship between the (sometimes blurry) phases of proof has been named "cognitive unity" (Garuti, et al., quoted in Miyakawa and Herbst 2007, 112; Boero et al. 1996, quoted in Arzarello et al. 2012, 118). "The notion of 'cognitive unity' has been proposed to gauge the relationship between conjecturing and proving theorems in classroom activity (Garuti et al., 1998). The notion that there exists 'cognitive unity' characterizes the case in which there is continuity between the two processes of conjecturing and proving. This continuity is visible, for example, in the use of the same arguments during conjecturing process and proving process. We use 'cognitive unity' to examine the relationship between convincing and proving in terms of the geometric objects students might be asked to work with and how they might be asked to work with them" (Garuti et al. 1996, 112).

2.4: Students' justification schemes and students' beliefs about proof

Some studies show that students do not see a need for proof, particularly for propositions they regard as self-evident (D'Ambrosio et al. 2010, Arsac 2007). Herbst and Brach (2006) mention Bourdieu's work with *habitus*, "the collection of dispositions that constitute participants' 'feel for a game' or practical rationality in a particular social practice…lived as inclinations or tendencies that a participant feels compelled to follow as he or she partakes of a practice" (Bourdieu 1984, quoted in Herbst and Brach), implying that students participate in the writing of proof because they have to, not because they want to; when diagrams are provided, students are apt to argue "proof is unnecessary (because figures show what is true)" (Herbst and Brach 2006, 98) or that "proof problems usually require proving a property that is obvious" (Ibid., 99). "Often students do not see why a fact has to be proved when it is either obvious to them or seems to them to have been sufficiently justified by actual measurements" (Hanna and Jahnke 1993, 434).

When they undertake proving as an activity, students misstep frequently and somewhat predictably. A.W. Bell (Bell 1976) is one of the first researchers to have attempted to catalog students' justification schemes. Bell's work identifies three stages in which the first is abstraction (patterns are recognized), the third is proof (proofs may be informal, but are "acceptably complete"; or are empirical, but test all possible cases; 23) and the second, transitional, which is "less easy to define" (Ibid.). He finds that approximately 50% of the student subjects (160 grammar school girls aged 11-18) was in stage 2; 10% of the sample was in stage 3. He also identifies three main functions of proof: verification or justification, illumination, and "systemization," whereby the "deductive chains of reasoning [are made] explicit" (Bell 1976, cited in Coe and Ruthven 1994, 42).

Sowder and Harel (1998) also find that high-achieving secondary school students and university students, even mathematics majors, are likely to attempt incomplete empirical justification. They also enumerate additional "justification schemes" employed by students to justify mathematical claims. They note three types of student proof:

"externally based proof schemes, empirical proof schemes, and analytic proof schemes" (670-1). The first rely upon textbook citations or the work of an "authority," are preoccupied with form or format, or are purely symbolic; the second rely solely upon examples without regard to generality; and the third are the type that most people in the mathematics community regard as the "best" type of proof. Of this last type there are two subgroups: the transformational proof scheme, which begins the transition to the general; and the axiomatic proof scheme, which proceeds deductively from known "facts," or statements which are accepted as known (as in Euclidean geometry) (671-4). The authors also recognize the ability of even primary school students to grasp the concept of proof and reiterate the suggestion that students be exposed to the rudiments of proof and analytical thinking as early as primary school.

Several studies have analyzed proof schemes of undergraduates (Selden and Selden 1995, Moore 1994, Housman and Porter 2003, Coe and Ruthven 1994). These findings reiterate that even university students who have chosen to take advanced mathematics courses experience difficulty in proof. Moore identifies three areas of difficulty: "concept understanding, mathematical language and notation, and getting started on a proof" (Moore 1994, 249). Later he identifies seven specific obstacles: "DI. The students did not know the definitions, that is, they were unable to state the definitions. D2. The students had little intuitive understanding of the concepts. D3. The students' concept images were inadequate for doing the proofs. D4. The students were unable, or unwilling, to generate and use their own examples. D5. The students did not know how to use definitions to obtain the overall structure of proofs. D6. The students were unable to understand and use mathematical language and notation.  D7. The students did not know how to begin proofs" (Moore, 251-2).

A significant weakness of many students is that they are, as a rule, unfamiliar with the precepts of basic logical argumentation (Durand-Guerrier et al. 2012, Moore 1994, Epp 1999, 2003).  Moore (1994) identifies the hypothesis-conclusion nature of a mathematical proposition as a problematic area for students.  Epp has also identified the usage of logical quantifiers, in particular their negation, as difficult for students (Epp 1999, 2003).  "The ability to rephrase statements in alternate, equivalent ways, to recognize that other attractive-looking reformulations are not equivalent, and to have a feeling for truth and falsity of universal and existential statements are crucial mathematical problem-solving tools.  Yet numerous studies show that students do not acquire these abilities spontaneously" (Epp 1999, 2).  One example of this is the naïve belief that the negation of "it is always true that…" is "it is never true that…" instead of "there exists at least one case in which (the proposition) is not true."  Epp suggests that students be introduced to the "language of quantification…[e]ven in the earliest grades" (Ibid., 4) in order to head off these sorts of misconceptions.  Hersh's (1993) claim that "what is really done in day-to-day mathematics has little to do with formal logic" (392) notwithstanding, it is vital that students in secondary school progress in their understanding of logical argumentation if they are to learn how to write mathematical proofs.

Knuth and Elliott (1998) cite Balacheff's four stages of proof: "naive empiricism, crucial experiment, generic example, and thought experiment" (714), in which the fourth level contains the movement from practical proof to intellectual proof.  They observe that instructors should recognize and give credit to work produced in earlier stages, even though it does not pass the test of "mathematical rigor."

Coe and Ruthven also reference Balacheff's four types of "proof approach," describing them as follows:

"* naive empiricism in which the truth of a result is asserted after verifying several cases;

"* the crucial experiment in which a proposition is verified on a particular case recognised to be typical but non-trivial;

"* the generic example in which the reasons for the truth of an assertion are made explicit in a prototypical case; and

"* the thought experiment in which the operations and foundational relations of the proof are indicated in some other way than by the result of their use" (Balacheff 1988, quoted in Coe and Ruthven 1994).

These are also perceived as representing "steps in the learning process" (Ibid.). The movement from "pragmatic" to "conceptual" proof is held as a primary goal, although Coe and Ruthven found, as have many others, that proving mathematically did not come naturally to high-performing sixth-form college students (16-18 years of age), who generally gravitated toward empirical evidence in place of general proof. Coe and Ruthven appear to blame the "epistemic schemas of school mathematics" that had routinized mathematics to the extent that students did not engage in theoretical reflection despite the "rhetoric of reform" (52).

2.5: The usage of counterexamples, or disproof

Just as students are frequently willing to accept a few examples as proof of truth, they are usually unlikely to accept one counterexample as compelling evidence of mathematical falsehood and must be brought to understand the power of the mathematical counterexample. As Cabassut et al. (2002) put it: "To an educated mathematician, it seems nearly unimaginable that the phrase 'for all objects x with a certain property the statement A is true' should present any difficulty of understanding to a learner. Many practical experiences and some recent empirical studies show, however, that it does exactly that. Lee and Smith, in a recent study (2008, 2009) of college students, found that some of their participants held the notion that 'true rules could always allow exceptions' or that 'true means mostly true' or that there might be an 'unknown exception to the rule' (Lee and Smith 2009, pp. 2–24; quoted in Cabassut et al. 2002). This is consistent with the experiences of students frequently not understanding that one counterexample suffices for rejection of a theorem. Galbraith (1981) found, for example, that one third of his 13- to 15-year-old students did not understand the role of counterexamples in refuting general statements (see also Harel and Sowder 1998)" (Cabassut et al. 2002, 184). Again, this may be attributed to students' experience with scientific method in which exceptions do not disprove the hypothesis.

Stylianides and Stylianides (2009) cite their usage of "pivotal counterexamples" in order to introduce students to a state of "cognitive conflict" (Zaslavsky 2005, Zaslavsky et al. 2012). Zaslavsky et al. (2012) believe as well that providing an atmosphere of conflict can motivate students' desire to prove, provided the instructor contributes "appropriate didactical engineering" (224). Students learn not to dismiss contradictory evidence as anomalous while becoming accustomed to the power of the mathematical counterexample.

Epp recommends introduction be first made to disproof, rather than proof, with the usage of counterexamples, perhaps because it is simpler to disprove with an existential statement than to show universal truth; and to do so a few years in advance of geometry, when most students study proof explicitly (Epp 1999, 5).

Epp provides an analysis of the lack of student understanding of the counterexample in logical terms: in order to prove falsehood by means of counterexample, "one must be aware (consciously or unconsciously) that the negation of 'if p then q' is 'p and not q' (Epp 2003, 887). She acknowledges that although "[m]any mathematicians take the reasoning described in these examples for granted,…[she] observed that very few [students] have an intuitive understanding of…[logical] reasoning principles" (Ibid., 888). An illustration is the twin statements by Tabach et al. (2009; quoted in Dreyfus et al. 2012, 200) that "… a universal statement necessitates a general proof and a single counter-example is sufficient to refute the statement. By contrast, an existential statement can be proven by a single supportive example, and a general proof is necessary to refute it," statements which are perfectly transparent to mathematicians but which would almost certainly befuddle most secondary school students.

Epp also cites confusion resulting from the difference in usage of words when in the mathematical register (as opposed to in common speech), a phenomenon that will be discussed in greater depth in the next section.

Finally, students have demonstrated a lack of understanding of the validity of deductive proof by responding to a proof they have acknowledged as correct with the belief that there may still exist a counterexample to prove its falsehood (Zaslavsky et al. 2012, 220).

Proof by contradiction is another situation in which students frequently misunderstand what seems obvious to their teachers. Leron (1985) dissects a familiar such proof, that of the infinitude of primes, pointing out that most students (even mathematics undergraduates) experience "frustration and bewilderment" (321) when they see the proof for the first time, although the proof is lauded by mathematicians as simple and elegant. Of course, it is unrealistic to expect students to grasp the idea of a proof by contradiction when they are not conversant in the usage of the counterexample.

## 2.6: Mathematicians' conceptions of proof

> "Within the community of mathematicians, the truth of a mathematical assertion follows through valid deductive reasoning from established results; proof is a deductive '…demonstration that compels agreement by all who understand the concepts involved.' (Hersh 2008 , p. 100)" (Zaslavsky et al. 2012, 216).

> "Research mathematicians often ignore the aspect of application and confine themselves to the purely deductive notion of proof. The explicit and implicit assumptions behind their mathematical work are taken as given. They are dictated in part by the division of labour between mathematics and the other sciences, and in part they result from a consensus, established by force of habit, among the experts in the field" (Hanna and Jahnke 1993, 432).

We have seen some ways in which students do not begin proof with the same assumptions as mathematicians. In order to put students on the right path, Cabassut et al. suggest that Arsac, et al.'s "explicit 'rules'" be made clear to secondary school mathematics students when introducing proof: "a mathematical assertion is either true or false; a counter-example is sufficient for rejection of an assertion; in mathematics people agree on clearly formulated definitions and properties as warrant of the debate; in mathematics one cannot decide that an assertion is true merely because a majority of persons agree with it;

in mathematics numerous examples confirming an assertion are not sufficient to prove it; in mathematics an observation on a drawing is not sufficient to prove a geometrical assertion" (Arsac et al. 1992, quoted in Cabassut et al. 2012, p. 183). Cabassut, et al. also state: "The issues of the potential certainty of mathematical proof and of the conditionality of the theorems have to be made frequent themes in mathematics education, beginning at the secondary level. Teachers should discuss them with students in various situations if they expect the students to get an adequate understanding of mathematical proof. In particular, they should make students aware of the necessary process of assessing the reliability of a theory" (184).

In addition, Cabassut et al. (2012) make claims about the validity of proof that echo the claims made by Civil about the newly-social mathematics community:

> "Working mathematicians also stress the social process of checking the validity of a proof. As Manin put it: "A proof only becomes a proof after the social act of 'accepting it as a proof'. This is true for mathematics as it is for physics, linguistics, and biology" (Manin 1977 , p. 48). By studying the comments of working mathematicians Hanna came to the conclusion that the public process of accepting a proof not only involves a check of deductive validity, but is also determined by factors like 'fit to the existing knowledge', 'significance of the theorem', the 'reputation of the author' and 'existence of a convincing argument' (Hanna 1983 , p. 70; see also Neubrand 1989). Bell (1976) also stressed the essentially public character of proof.
>
> "All in all, formal definitions of proof cover the meaning of the notion only incompletely, whereas mathematicians are convinced that, in practice, they know precisely what a proof is. This situation is difficult to handle in the teaching of mathematics at schools, since there exist no easy explanations of what proof and proving are that teachers could provide to their pupils. Proof is not a 'stand-alone concept', as Balacheff nicely puts it (2009 , p. 118), and is aligned to the concept of a 'theory' (see also Jahnke 2009b , p. 30)" (Cabassut et al. 2012).

They also point out, contrary to what we might wish about the inalienable nature of proof, “no explicit general definition of a proof is shared by the entire mathematical community.”

Nevertheless, as we can verify from statements from the beginning of this section, there is agreement as to the general necessities inherent in proof among mathematicians. Hersh (1993) characterizes mathematical proof as “convincing argument, as judged by qualified judges” (389), acknowledging with the device of the Ideal Mathematician (Davis and Hersh 1981, 39-40, quoted in Hersh 1993) that this definition is incomplete. He also allows that certain “non-trivial mathematics” defy formal proof, at least at the present time, citing Halmos’s complaints about Haken and Appel’s proof of the Four-Color Theorem. He concludes that “all real-life proofs are to some degree informal” (391). While Hersh’s observations are more philosophical than practical, he brings up points that are part of a much larger discussion, one whose scope is beyond that of the present study.

## 2.7: Summary

In sum, much has been written about the teaching of proof in high school, elementary school, and college. Some have addressed such concerns as what constitutes proof in a high school mathematics classroom, as opposed to among professional mathematicians; others have discussed what sort of environment is appropriate for this pursuit. Most of the literature has focused on how proof should be taught and on the difficulties inherent in the teaching of proof. There has not been very much attention paid to the semantic mechanics of what takes place, nor to possible benefits of teaching proof regardless of whether students emerge having mastered the practice.

**Section 3: Discourse analysis: Qualifying and quantifying the language of the classroom**

The prior section of the literature review addressed common approaches and challenges in teaching proof to high school students. This section will examine what is said in the proof classroom along with a discussion of various approaches to and tools for the analysis of classroom discourse.

3.1: Discourse analysis

It has become common for researchers to consider what is said in a classroom in linguistic terms as well as through a pedagogical lens. The benefits of one or both types of discourse analysis (as it is known) are beginning to pervade the realm of mathematics education.

James Paul Gee, a pioneer in the field of discourse analysis, offered An Introduction to Discourse Analysis in 1999 (third edition, 2011) and How to do Discourse Analysis: A Toolkit (2011) to those wishing to engage in discourse analysis for research purposes. Since Gee's approach is grounded in linguistics and is geared toward linguists, anthropologists and those studying communication, it is too syntactically and syntagmatically precise to be of use in most studies of mathematics education. For example, a dissertation prepared for a Ph.D. in linguistics written by Shepherd (2010) uses sophisticated coding incorporating length of pauses and intonation to examine how, despite relatively recent shifts in power from teacher to students in the course of classroom activity, teachers still wield considerable power as measured by students' hesitation and rising intonation. While this study sheds considerable light on the power dynamics in a third-grade classroom, it does not examine the purposes of utterances and their effects.

Although it is cumbersome logistically and linguistically, Gee's work is notable as it (and other such work) provoked an onslaught of discourse analyses in fields other than linguistics. As noted in the Introduction, there has been a proliferation of discourse analyses in the field of education to various ends.

Atkins (2001) presents a discourse analysis of an EFL classroom in which moves are classified according to Sinclair and Coulthard's Initiation-Response-Follow-up model (1975, revised 1992). In this model, the largest unit of discourse is a lesson and the smallest is an act. Acts are combined to form five classes of moves, which include framing, focusing, opening, answering and follow-up categories. Sinclair and Coulthard's model is intended to describe "micro-interactions" (Rogers, et al. 2005) in classrooms and, like Gee's, is appropriate chiefly for linguistically-based studies such as Atkins's.

Subsequent to Sinclair and Coulthard's work, Cazden (1988/2001) developed a model of classroom discourse analysis that includes a distinction between "answering" and "understanding" questions, which is a step in a useful direction for the current study. However, as Cazden's framework was developed as an amalgam of applied linguistics, anthropology and sociology, and not a means of mapping instruction in a mathematics classroom, it too is of limited use to the current study. Its focus contains references to appropriateness and competence, relevance, intention, knowledge and value that are ideally negotiated by the students and teacher acting as a community. While these are valuable observations, the current study is less concerned with analysis of the classroom as a social object and more concerned with the mechanics of mathematics instruction. For this reason, a framework developed specifically for that purpose was adopted.

For an overview of recent discourse analysis work in the field of mathematics education, see Ryve (2011), who notes that there were a total of 24 such articles between 1968 and 1999; whereas in 2000-2009 there were 84 articles that could be characterized as studies in mathematical discourse. (The study includes articles from five journals in total, two of which were the Journal for Research in Mathematics Education and Educational Studies in Mathematics, seen by many as the two most influential journals in the field; he does not mention dissertations written during that period.) Although the primary focus of his study was to suggest ways in which researchers can and should be more precise in their situation of epistemological principles, it serves to highlight a striking increase in interest in discourse study in mathematics education.

For a sophisticated analysis of discourse between Swedish engineering students, see Ryve (2006), in which utterances are transcribed and interactive flowcharts incorporated in order to answer research questions "Are the students communicating with each other? Are the students discussing mathematics, and if that is the case, how should the quality of the mathematical content be examined?" (193). This particular framework is geared toward analyzing activity in groups (not whole-class discussion) and is far too precise a tool to accomplish the goals of the current study; however, it serves as an instructive example of the heights to which discourse analysis can ascend within certain parameters.

Ryve's sample size is even smaller than that of the current study: a total of four 45-minute discussions (four different groups, each consisting of three students) were videotaped, tape recorded and transcribed.

3.2: A. Bellack and J. Fey

The current study makes additions to a discourse analysis framework developed first by Bellack, et al. (1966) and modified by Fey (1966) to apply to the mathematics classroom. Although Bellack and Fey lack the linguistic rigor that is a hallmark of Gee's (and others') work, their work predates Gee's work by over thirty years and are still among the most useful tools for analyzing discourse in certain contexts, of which the current study is one example.

Bellack, et al. proposed a system of coding classroom interactions, dubbed "moves," that classified utterances as having one of four purposes: structuring, soliciting, responding and reacting. These are reminiscent of Sinclair and Coulthard's "framing, focusing, opening, answering and follow-up," but are condensed and tailored to activities in a high school classroom. Bellack's "structuring" encompasses both "framing" and "focusing"; "soliciting" could be the same as "opening"; "responding" is certainly akin to "answering"; and "reacting" is most certainly related to "follow-up." The difference lies in the relative specificity of Fey and Bellack's terms. Bellack and Fey's "solicitation" can be a question or a request for action, but certainly indicates one or the other; whereas Sinclair and Coulthard's "opening" is descriptive of (an instructor's) intention of the direction in which s/he wishes to lead the lesson but not sufficiently descriptive of the purpose of the utterance. Conversely, Sinclair and Coulthard's "answering" is too specific; Bellack and Fey's "response" may be a statement or an action. "Reacting" can be widely interpreted as a teacher's or student's evaluation of another's statement, or as a teacher's prompt for a student to continue pursuing a current thought, or as a student adding onto another student's train of investigation. "Follow-up" seems to indicate a reaction to a thought or activity that has reached a conclusion.

Since it was judged that Bellack and Fey are well-suited to the mechanics of a high school classroom, this study chose to employ Bellack and Fey's categories of classification.

Fey modified Bellack's system for usage in a mathematics classroom. He adds codes for mathematical content and purpose. His coding extends to analysis of not just number of moves, but length of each move, finding, for example, that an average teacher move was 2.6 lines long, whereas an average student move was 1.5 lines long (38). He finds that, on average, teachers were responsible for 60.2% of moves and 72.9% of lines; whereas students were responsible for 39.8% of moves and 27.1% of lines (40). Table 1, shown below, shows the percentages of classroom discourse devoted to each type of move by both students and teachers:

Table 1. Fey (1966), % of moves and lines devoted to each type of pedagogical move and ratio of lines per move

**TABLE VI**

**% of moves and lines devoted to each type of pedagogical move and ratio of lines per move**

| | *% of Moves* | *% of Lines* | *Lines/Move* |
|---|---|---|---|
| Structuring | 5.2 | 12.2 | 5.2 |
| Soliciting | 32.1 | 33.3 | 2.3 |
| Responding | 31.7 | 21.5 | 1.5 |
| Reacting | 31.0 | 33.0 | 2.3 |

Fey, 43.

Table 2 shows distribution of each type of move with respect to source (student or teacher):

Table 2. Fey (1966), distribution of move types according to move

TABLE VII

Distribution of move types according to source

| | % of Moves by Teacher | % of Lines by Teacher | % of Moves by Student | % of Lines by Student |
|---|---|---|---|---|
| Structuring | 80.3 | 86.0 | 19.7 | 14.0 |
| Soliciting | 92.6 | 93.4 | 7.4 | 6.6 |
| Responding | 4.4 | 9.9 | 95.6 | 90.1 |
| Reacting | 82.9 | 88.2 | 17.1 | 11.8 |

Ibid., 44.

In Fey's suggestions for areas of future study, he asks, "Do variations in the mathematical topic under discussion produce variations in the pattern of mathematical activity or logical process?" (75). This study addresses this question. His sixth question for further research, "Can the behavioral concepts identified in the descriptive study of teachers be used to construct models of teaching behavior, the effectiveness of which can then be compared experimentally?" (76), does not appear to have been addressed by anyone and so remains a robust area for further study.

As noted previously, Bellack and Fey's coding system did not provide subcategories that were appropriate for the current study to classify the nature of teacher and student utterances in a mathematics classroom in which the primary occupation is the development of proof, as opposed to a classroom in which the inculcation of mathematical procedures is the primary focus. In particular, the subcategories "analytic, factual, evaluative, justifying" used by Bellack and Fey's subcategories of "developing, recall, illustration, application and procedural" (Fey 57) were rejected as they require too much assumption on the part of the coder. In addition, those subcategories are oriented toward a reaction the instructor was trying to provoke from one particular student. The current study is more oriented toward instructors attempting to provoke a discussion between several students.

However, the primary four "moves" characterized by Bellack, et al. and also Fey, structuring, soliciting, responding and reacting, are useful and have been retained in this study. Descriptive subcategories were found elsewhere; please see section 3.4 below.

3.3: K. Offenholley and S. Weinberg

As noted previously, this study leans heavily on Offenholley (2007) and Weinberg (2010) for both framework and methodology, particularly in the development of the coding scheme. Offenholley examines the discourse dynamics of online instruction to investigate which types of "moves" encourage student participation. Her study reclassifies the "reacting" move as "evaluative" and disregards "structuring" moves entirely. She also renames certain types of "responding" moves as "explaining," since in Bellack's scheme responses were required to be contiguous to a solicitation for explanation, a specification that does not apply as neatly to an online milieu. She renames "reacting" moves as "evaluating" in certain contexts. Her only subcategories under SOL, EXPL and EVAL are MATH or NO MATH; she deemed only these necessary to characterize all exchanges.

Offenholley finds, using only Fey and Bellack's four moves with minor modification, that "the ratio of teacher to student discourse is far lower in online classes than in face-to-face classes" although there is wide variation between instructors. She shows that, if instructors are cognizant of amount and nature of their postings, students will interact more than they do in face-to-face interactions, and more than they will in online interactions in which instructors are not mindful of their contributions.  Overall, her findings show teacher participation ranging from 0% to 44.3%, much less than that found in Fey and Bellack's observations of classrooms (in terms of number of moves, 61.7% and 60.3%, respectively; in terms of word count, these numbers go up to 71% and 75%, respectively) and of that cited in a 1999 TIMSS report of a study of 8$^{th}$ grade mathematics classes (about 89% of speech was by the instructor) (Offenholley 31-32).

Weinberg also modifies Fey's coding scheme in several ways, the most notable of which is replacing Bellack's categories of "analytic, factual, evaluative, justifying" and Fey's categories of "mathematical activity," which include "developing, recall, illustration, application and procedural" (Fey 57) with seven subcategories including "knowledge, comprehension, application, creative thinking, analysis and evaluation."  Weinberg's subcategories were taken from Brahier (2005) and are useful in describing and categorizing teachers' solicitations made to specific students.  This is appropriate as types of questioning are Weinberg's primary focus.

He ranks these seven subcategories as increasing in levels of cognitive demand.  In conjunction with other tools, Weinberg's study uses Fey and Bellack's "moves" and Brahier's subcategories to analyze teachers' solicitations of students and examines to what extent teachers' epistemological beliefs are in line with their discursive activities in the classroom.  However, Brahier's subcodes were not adopted for use in the current study as they, too, seem to require the coder or reader to make assumptions on the part of the speaker that may not have been true.  In addition, as previously noted, Brahier's subcategories are designed to qualify an instructor's provocation of a specific student's response. The current study is more concerned with whether a type of instructor utterance provokes student discussion of any kind.

<u>3.4: S. Generazzo and M. Blanton, et al.</u>

Generazzo (2011) provides a dissertation presented in the field of mathematics education that is of great interest to the current study. It investigates learning as part of a social process as well as a result of individual constructions, an approach known as the emergent perspective. She pays particular attention to the question "How does the classroom environment shape students' abilities to reason and prove in an inquiry-based, undergraduate geometry classroom?" (5). The subquestion most related to the goal of the current study is her first: "What is the nature of participants' interactions as they engage in proof and reasoning?" (6). She uses a framework developed by Blanton, et al. (2009) which is used to categorize and analyze whole-class activity (Generazzo 2011, 26) with differentiation of codes for student and teacher utterances.

The framework developed by Blanton, et al. (2009) presents a more objective view than Brahier's (adopted by Weinberg) of both teachers' and students' utterances. In addition to objectivity, Blanton's framework codes both teacher and student utterances, which Brahier's and Bellack's do not. Blanton, et al. apply their coding scheme to an undergraduate classroom of mathematics majors in which proving is the mathematical objective.

In this coding scheme, teachers have four primary modes of utterance: directive, didactive, facilitative and transactive. Of these, the first two are considered to engender a lower level of cognitive demand from students whereas the second set are intended to evoke students' participation, discussion and (presumably) understanding. Students have seven modes of speaking, including proposal of a new idea, proposal of a new plan, contribution to an existing idea, posing a transactive question, offering a transactive response, supplying general confirmation, and no code. The first five are seen to evidence access to Vygotsky's zone of proximal development; "no code" does not, and "general confirmation" is considered a gray area. The methodology section of this study details the application of these codes to data in the current study.

Blanton, et al.'s study "assumes that every teacher utterance affects how students learn" (306) and presents its framework to facilitate inquiry into teachers' modes of communication. It is premised on the acquisition of mathematical knowledge as a social activity and also presumes many of the tenets of constructivism to be desirable in a classroom on the college level, but by natural extension also in levels that precede college study. It focuses on the modes of interaction used by teachers to encourage student discussion and the ways in which students contribute to class and group discussion. As such, it is of great use to the current study.

Blanton, et al.'s coding system is based on the work of Kruger (1993) and Goos, Galbraith and Renshaw (2002). Kruger is responsible for the "transactive questioning" and "transactive response" codes, which were developed to describe dyadic dilemma resolution between young people and other young people, or with adults. Goos, et al. apply coding to "senior secondary school mathematics classrooms." As noted, they code "transactive statements," "transactive questions" and passive responses. The sample size is conspicuously small: only three transcripts were coded. The first round of coding parsed the classroom activity into stages including reading, understanding, analysis, exploration, planning, implementation and verification. The second round bears more resemblance to the current study in that it coded "conversational turns of all speakers," called Moves. These conversational moves were, in turn, coded twice, to identify their "contribution to the collaborative structure of the interaction." Goos, et al.'s attention to Kruger's observations on the importance of transactive activity is further articulated in coding all utterances in collaborative transactions as "self-disclosure," "feedback request," or "other-monitoring."

The current study adopts Blanton, et al.'s modification to Kruger and Goos, et al. in that codes were needed to describe not just peer-peer interactions but also those between teachers and students although the focus on conversational "moves" is retained as centrally important.

Generazzo also employs Toulmin's framework to describe student's modes of argumentation. Toulmin's framework, which specifically details the structure and validity of proof, is useful when examining and evaluating those constructs. The current study is concerned more with the pedagogical, conversationally-based methods applied by teachers to encourage proof, discussion and argumentation among students and with students' contributions to whole-class discussion; it is less concerned with the validity or content of proof activities.

Results of tabulated teacher utterances and student utterances are presented below. In contrast to Fey and Bellack, who distinguish between moves and lines, the current study considers only moves with no regard for length of utterance (although, in the current study, overall teacher and student word count, and teacher/student word ratio, will be provided).

Table 3. Blanton, et al. (2009), frequency of forms of instructional scaffolding (teachers)

**Table 17.1** Frequency of forms of instructional scaffolding (teachers)

| *Type of teacher utterance* | *Frequency* | *Percentage* |
|---|---|---|
| Transactive prompt | 37 | 40 |
| Facilitative utterance | 43 | 47 |
| Directive utterance | 7 | 8 |
| Didactive utterance | 5 | 5 |
| **Total** | **92** | **100** |

Blanton, et al. 298.

Table 4. Blanton, et al. (2009), frequency of forms of instructional scaffolding (students)

**Table 17.2** Frequency of forms of instructional scaffolding (students)

| *Type of student utterance* | *Evidence of ZPD access* | *Frequency* | *Percentage* |
|---|---|---|---|
| Proposal of a new idea | Yes | 3 | 4 |
| Proposal of a new plan | Yes | 6 | 8 |
| Contribution to an existing idea | Yes | 29 | 40 |
| Transactive response | Yes | 7 | 10 |
| Transactive questions | Yes | 6 | 8 |
| General confirmations | Not necessarily | 9 | 13 |
| No code | No | 12 | 17 |
| **Total** | | **72** | **100** |
| **Total utterances indicating ZPD access** | | **51** | **70** |

Ibid., 302.

Since Blanton, et al. developed their coding scheme of conversational moves for usage in a classroom in which proof is the objective, it was natural for this study to adopt it. Furthermore, as noted above, codes are assigned to both teachers' and students' utterances, which was necessary for this study. Finally, this coding system is most concerned with instructors' ability to provoke discussion and students' ability to engage in discussion with or without teacher moderation. However, it was necessary to modify their scheme slightly for facility of data presentation; these modifications are discussed in the methodology section of this study.

### 3.5. Summary

In sum, the current study uses discourse analysis techniques that have gained popularity in recent years. It has modified Bellack and Fey's coding scheme, a well-recognized scheme for describing activity in a mathematics or non-mathematics classroom. In addition, it extends Fey and Bellack's scheme with subcategories taken from Blanton, et al. that use coding of conversational moves to describe a mathematics classroom in which proof is the main focus of activity and in which the instructor's intention is to provoke discussion among students.

# Chapter III: Methodology

## Section 1: Overview

The study analyzed audiotapes and transcripts of classroom activity in geometry classes in two high schools in the tri-state area.  Both are ninth grade classes in private schools with small classes (12-18 students) and one teacher.  Although one school is attended solely by girls and the other is coeducational, gender notwithstanding, the students are demographically similar. Teacher A's class was composed of eleven young women.  Seven were Caucasian, two were Latina, one was African-American, and one was of mixed (unknown European/Asian) ethnicity.  Five were of high socioeconomic status, four of medium, and two of low.  Although demographic data were not available from teacher B's school, based on the researcher's observation, it is reasonable to suppose that the students in Teacher B's classes enjoyed similar heterogeneity, although a significant difference is that Teacher B's classes were mixed-gender (approximately half boys and half girls).

Both teachers were in their second or third year of teaching.  Both teachers had apprenticed as student teachers with the researcher in geometry classes.  In the researcher's best professional judgment, both teachers were sufficiently proficient at their craft so that lack of experience was not a factor in the study's findings.  In fact, the proximity of their training at Teachers College ensured that both had been exposed to the principles of constructivism and were accustomed to incorporating those into their classroom practice.  In addition, the researcher recommended both for their first teaching jobs and so it is true that the researcher considers both to be excellent teachers.  These teachers were chosen because they were both teaching in private schools with similar populations and were teaching similar courses in geometry with an emphasis on proof. In addition, the researcher's relationship with the teachers facilitated their willingness to participate in the research project.

Data were collected in three stages. The teacher at one school taught proof in the spring and the other in the fall, so both timing and location necessitated two initial stages of data collection. Two-three weeks of data from each classroom yielded recordings of twenty-four mathematics lessons of which nineteen were chosen for transcription and twelve were chosen for analysis. Of these, eight (four from each teacher) were considered "proof-based," that is, to concern primarily the teaching of proof; and the other four (two from each teacher) were supposed to be composed of non-proof activity. However, upon consideration, there were no examples from Teacher B's classroom that could be considered adequately devoid of proof activity, so two of that teacher's algebra 2 lessons were recorded at a later date.[2]

It should be noted here that only lessons in which at least half of the activity was directly devoted to constructing proof of geometric relationships were classified as "proof-based"; generally speaking, a proof may utilize the "statements and reasons" two-column structure although this is not required. What is required to merit the designation "proof" is the establishment of facts to be accepted as "given," a statement or relationship to be proven and the development of the proof using principles of deductive reasoning. The

[2] In the final analysis, the fact that a second set of data for Teacher B were from a completely different class, one that did not include any emphasis on proof, served to strengthen the hypothesis that Teacher B's instruction did not vary regardless of variables such as content, students, nor year of study.

reader is encouraged to peruse the Results section of this study in which the specific material covered in each lesson is described in more detail.

Finally, it should also be noted that the researcher chose not to code classroom "housekeeping" activity, nor was groupwork coded. In general, only proper whole-class instruction was coded.

**Section 2: On the Development of the Survey**

Both teachers were given a brief survey to provide information on their approach on teaching proof vs. non-proof and general epistemological orientation (constructivist vs. objectivist). The survey is as follows:

1) When you are teaching your classes, please assign a number 1-5 (5 is most) to how highly you prioritize the following:

a) students communicate directly with each other at all times (not just during groupwork)
b) students use correct mathematical terminology in discussion
c) students explore and absorb the skill or concept you are trying to convey
d) you provide time for students to interact and discuss the situation you have presented (separate from whole class instruction)
e) students provide their own goals
f) students ask questions to clarify procedures, concepts or your pedagogical intentions

If you can, please rank the above in order of their relative importance to you.

2) When you think about your teaching and lesson planning, are there differences to how you approach teaching proof lessons, as opposed to non-proof lessons (e.g., midpoint formula, straight algebra)?

The survey was developed in accordance with researchers cited in the literature review. In particular, the work of Bowers and Nickerson (2001), Civil (2002), Schoenfeld

(1992), Skidmore and Murakami (2012), Yackel, Rasmussen and King (2000) and others was seen as instrumental in identifying aspects of instruction that could be seen as important by teachers interested in developing a constructivist atmosphere. Specifically, items a) and e) are seen as most important in a student-centered, constructivist model, which emphasizes exploration and discussion among students; whereas b is more indicative of a teacher-centered, objectivist model in which the teacher is seen as the most knowledgeable person in the room and the arbiter of what is "mathematically correct." The remaining goals are important in both modes of instruction and were provided to draw attention away from the choices intended to expose each teacher's underlying ideological orientation.

The second question is intended to expose whether each teacher regards the teaching of proof as fundamentally different to establish whether Teachers A and B intended for their instruction to differ in order to establish whether difference (if found) was deliberate or not.

**Section 3: On the Size of the Data Set**

Some studies analyze more data than the current study, and some analyze less. This study double-codes twelve lessons of varying lengths with four codes and twelve subcodes. Offenholley's study (2007) codes thirteen online lessons. Although double coding is employed, the coding system is less sophisticated than that of the current study (three codes, two subcodes).

As noted in the Literature Review, Ryve's (2006) sample size is smaller than that of the current study: a total of four 45-minute discussions (four different groups, each consisting of three students) were videotaped, tape recorded and transcribed. Goos, Galbraith and Renshaw (2002) code only three transcripts; these were double-coded with a level of sophistication similar to that in the current study.

## Section 4: On the Development of Tables and Graphs

After the transcripts were transcribed, edited (for student names and other identifying information) and coded, the code tallies were entered into Excel spreadsheets. Two types of data were used. The first, word-related, are in the forms of word ratios and word clouds. The second, utterance-related data, take the form of bar graphs of coded material, analysis of common question-response-reaction patterns, and comparison of proportions of different types of teacher and student utterances. Utterance-related data do not take into account the length of specific utterances, only their code values or (in the case of string lengths) the identity of the speaker.

### 4.1: Word-related figures and tables

Teacher-student word ratios were calculated using Microsoft Word and string lengths tallied by hand. These results were tabulated and can be found in the Results section of the study. Teacher-student word ratios, therefore, consist of a tally of words spoken by each group; since different lessons differed in length, word ratios are calculated as proportions of all words spoken.

The reader will also note the inclusion of “word clouds” for each lesson. Word clouds are a type of weighted list that delivers information about a lesson transcript quickly. The largest word in the cloud is the word that was spoken most often; accordingly, the smallest was spoken the least. Word clouds are not intended to convey detailed information, e.g., exactly how many times each word was spoken; nor is the placement of any particular word in the cloud significant. Rather, they are a means of conveying quickly and easily the words that received the most focus in the lesson, thereby giving an imprecise although effective impression of the nature of the lesson. Word clouds are provided both for all speech in the lesson and for that of the teacher only.[3] The word clouds were created using a free online program at <http://www.wordle.net/>. The font is League Gothic.

4.2: Utterance-related figures and tables

The first of these is string length. Any series of utterances in which a teacher's statement is followed by two or more uninterrupted student responses is called a "string." "TSS" signifies two uninterrupted student utterances, "TSSS" three, and "4 S's," "5 S's" and so forth signify the appropriate number of uninterrupted student utterances following a teacher's initial utterance. These string data are presented in the Results section of the study. There is also an appendix with all string data tabulated on one page.

The second of the utterance-related data is bar graphs. The bar graphs, presented for all proof and non-proof lessons for each teacher and for individual lessons, show the proportion of teacher and student utterances for both types of coding.

Finally, a sequence of SOL-RES-REA (generally, question, response and reaction) was identified as common and informative. In a teacher-centered classroom, it is commonplace for a teacher to initiate the sequence, for a student to respond, and for the teacher to evaluate the student's response. Any deviation in this pattern from teacher

[3] This study's word clouds were created at http://www.wordle.net/create in October and December 2014.

(solicitation)-student (response)-teacher (reaction) was interpreted as relevant to identifying instruction as teacher- or student-centered.

**Section 5: Selection and Development of the Coding Scheme**

As discussed in the introduction and literature review section, this study utilizes a framework developed by Bellack, et al. (1966) and Fey (1966) to code pedagogical "moves" in a classroom. The framework codes interactions as structuring, soliciting, responding and/or reacting. No modification was made to these categories, although subcategories (second level of coding) were affixed as described below.

<u>5.1: First level of coding: purpose of utterance</u> (applies to both teacher and students)

In general, Bellack and Fey's codes, dubbed by this study "purpose of utterance," can be described as follows:

*Structuring*, abbreviated STR, defined as a move that sets the context for activity. It generally initiates or redirects the discussion. It also includes a teacher demonstrating, declaring or administering behavioral prompts. Some examples:

T-STR: *"Now let's talk about how to find the midpoint of a line segment."*

T-STR: *"Here's how you find the circumcenter of a circle."*

S-STR: *"Wait! I have a totally different approach."*

*Soliciting*, abbreviated SOL, is a move that solicits a response. It is generally followed by RES, a response. Some examples:

T-SOL: *"What do you think are the coordinates of this segment's midpoint?"*

T-SOL: *"Can you explain why you did that?"*

S-SOL: *"How should I start this problem?"*

*Responding*, abbreviated RES, is always a response to a SOL move. Some examples:

S-RES: *"I think the circumcenter is at (3,2)."*

T-RES: *"Maybe you should use one of the givens first."*

*Reacting*, abbreviated REA, is a response to something that was said earlier or a continuation of an ongoing process. It is never a response to a SOL. Some examples:

S-REA: *"I agree with ________. The circumcenter is at (3,2)."*

T-REA: *"I think your approach is a good starting point."*

5.2: Second level of coding: nature of utterance

As discussed in the literature review, Bellack uses "analytic, factual, evaluative, justifying" as subcategories, but these were regarded by the current study as too descriptive of utterances. Toulmin's scheme, which details the structure and validity of proof, was also rejected for being too content-oriented. Brahier's subcodes were designed to describe a teacher's desired response from one particular student, which is not in line with the goals of the current study.

Instead, the current study adopts codes developed by Blanton, et al. (2009), chosen because they characterize an instructor's intention to get a quick, factual response or to engender discussion among students. They also classify student responses as requiring higher- or lower-order thinking. In conjunction with tallying strings of student utterances, this subcoding system effectively describes what a teacher has done to engender discussion and whether the discussion takes place.

5.2.1: Teacher codes

Using Blanton, et al.'s codes, one can subcategorize teacher "moves" by one of four modes: facilitative, transactive, directive, or didactive, (abbreviated for this study as FA, TA, DR and DD). Of these, the first two are considered to evoke students' participation and understanding (higher-order demands) whereas the second set engender a lower level of cognitive demand from students. The first two (TA/FA) and last two (DR/DD) have been condensed in the Results section in order to facilitate data presentation.

Transactive (TA) prompts are intended to provoke students' own reasoning, elaboration, justification and so forth. An example of a transactive prompt is: *"Can I have some more examples of that?"*

Facilitative (FA) statements reinforce (sometimes, just revoice) a student's idea or structure discussion. Example: *"So I'm hearing, we should start with a given statement."*

Directive (DR) statements provide immediate feedback or by providing information directly. Example: *"You find that by adding the x-coordinates and dividing by two."* Questions are not discussion-provoking. A question that receives a one-word answer is generally directive. An example is: *"What is the midpoint of the line segment?"*

Didactive (DD) statements reinforce the teacher's position as authority figure by presenting non-negotiative information "on the nature of mathematical knowledge

(Blanton, et al. 2009). Example: *"That's the kind of thing you'll need to know as a math major."*

5.2.2: Student codes

Student "moves" are categorized by seven modes of speaking, which are as follows: proposal of a new idea, proposal of a new plan, contribution to an existing idea, transactive question, transactive response, general confirmation, and no code (PI, PP, CO, TQ, TR, GC or NC). In accordance with Blanton's research, the first five (PI, PP, CO, TQ, TR) are seen to evidence access to Vygotsky's zone of proximal development; "no code" does not, and "general confirmation" is considered a gray area. As with teacher uttterances, the first five (PI, PP, CO, TQ, TR; higher-order utterances) have been condensed in the Results section, as have the last two (GC, NC; lower-order utterances) in order to convey data more meaningfully.

The following describes these categories in greater detail. As noted above, the first five evidence higher-order activity, whereas the latter two evidence lower-order activity.

Proposal of a new idea (PI). Example: *"We should try bringing the angle bisectors into the proof." "How about using different triangles?"*

Proposal of a new plan (PP). Example: *"How about using SAS on these other triangles?"*

Contribution to an existing idea (CO). Example: *"Then use the vertical angles!"*

Transactive question (TQ). is generally a request for clarification, explanation, and so forth. Example: *"Why did you use that segment?"* (N.B.: Most student questions are considered transactive questions although a few merited a "no code" designation.)

Transactive response (TR) is generally a response to a request for clarification, explanation, and so forth. Example: *"I used it because it was shared by two triangles."*

General confirmation (GC). Example: *"I completely agree."*

No code (NC). Example: *"The midpoint is (2, -3)."*

In sum, each student or teacher "move" was coded with S or T (for source), a three-letter code from Fey/Bellack (STR, SOL, RES or REA), and a two-letter code from Blanton, et al.

**Section 6: About the Coders and Intercoder Reliability**

Two coders were chosen and they, along with the researcher, coded paper transcripts, using the margins to record the codes. The coders were both college graduates. Neither was a mathematics teacher. This was done to ensure that the bias for or against the teaching of proof would not influence coding decisions.

The coders were given training for some hours using transcripts that were not being used in the study. (The instructions to coders are appended to this study.) After training, the researcher compared results and provided feedback to assist in the relative consistency of coding. When the researcher judged that an appropriate period of training had passed and adequate feedback had been provided, kappa values were calculated to show both that the coders achieved a high enough measure of intercoder reliability and that the researcher's obvious bias did not render her coding incompatible with that of the objective coders.

Although neither coder was a mathematics teacher, both were aware of the study's hypothesis. Therefore, a third, control coder (C; see below) coded sections of the transcripts already coded by all three coders (A, B, and R; see below) and kappa values were again computed to show that all coding had been relatively bias-free.

Kappa values are a measure of intercoder reliability developed by Cohen (1960). It is notable as it takes into account not only the proportion of common responses but also the probability that two coders would agree purely by chance. The formula is $\kappa = \frac{P(A)-P(E)}{1-P(E)}$. P(A) is the proportion of times the coders agree; P(E) is the proportion of times they would have agreed by chance. Kappa values range from 0 to 1, with 1 representing perfect consistency. In general, a kappa value of .65 or greater is regarded as an indication of very good agreement among coders. A kappa value above .75 is regarded as an indication of excellent intercoder agreement.

Table 5. Intercoder reliability kappas:

| | A and B | A and C | B and C | A and R | B and R | C and R |
|---|---|---|---|---|---|---|
| Fey/Bellack moves | .814 | .809 | 1 | .739 | .92 | .92 |
| Blanton student codes | 1 | .654 | .814 | .76 | .810 | .8 |
| Blanton teacher codes | .75 | .824 | .75 | .714 | .75 | 1 |

N.B.: A and B are coders. C is a control coder who was unaware of the study's hypothesis. R is the researcher who also coded transcripts in addition to training all coders.

**Section 7: On the Application of the Coding Schemes to the Data**

As noted above, no modification was made to Fey and Bellack's basic structure of moves. However, it was necessary to adopt subcodes in order to describe classroom activity in sufficient detail. As noted previously, this study saw Blanton, et al.'s coding scheme as the most appropriate since it was developed to describe activity in a classroom whose primary activity was the development of proof and is focused on a teacher's desire to foster discussion and student contributions to discussion.

It was previously noted that Blanton's codes were developed for usage in an undergraduate classroom; because of this, the current study saw the necessity to consider carefully some of Blanton's codes for application to a high school classroom. In particular, questioning, the majority of which was seen by Blanton, et al. as "transactive," was more frequently encountered in the current study as directive. Since the current study also uses Fey/Bellack's system, it was possible to distinguish clearly between directive and transactive questioning while preserving the occurrence of a question. Another difference can be seen in Blanton's tendency to classify structuring as "facilitative," whereas the current study tended to regard structuring in a high school classroom as directive. Again, the simultaneous usage of both schemes enables a more detailed characterization of structuring (as well as soliciting, responding and reacting) moves.

As noted previously, the appendix entitled "Instructions to coders" details the instructions given when training both coders and the control coder.

# Chapter IV: Results of Study

## Section 1: Overview

This study seeks to answer a number of questions.

1. What semantic features characterize student and teacher utterances when learning to construct proof?

2. What semantic features characterize student and teacher utterances when proof is not being taught?

3. Are there discernible, quantitative differences in the quality or proportions of teacher/student discourse in the teaching of proof?

In addition, this study examines whether the ratio of teacher-student utterances documented by Bellack, Fey, and Offenholley has changed. If so, can any change in this ratio be attributed to the nature of the subject matter, i.e., does the teaching of proof engender different types of student and teacher utterances, and does it affect the ratio of those utterances?

The research indicates that, overall, there can be qualitative and quantitative differences in both teacher and student utterances when proof is being taught, although those differences may vary according to a teacher's epistemological approach to her/his craft. This section will examine each lesson taught by two instructors with different epistemological approaches to instruction in an effort to determine under which conditions changes in traditional teacher-student utterance ratios and nature of utterances are most likely to occur. As stated in the Introduction, it is hoped that these findings will be of help to those attempting to adopt constructivist teaching methods in their classrooms; or conversely, might give those teachers who are befuddled by how to teach proof another approach.

This study acknowledges that students taking a large role in sense-making activities in pursuit of what Bowers and Nickerson (2001) call a proposition-discussion model can be an important feature of constructivist pedagogy. Their formulation cites three types of classroom activity:

1. *Initiation-response-evaluation* (IRE)
2. *Elicit-response-elaborate* (ERE)
3. *Proposition-discussion* (PD)

Franke, Kazemi and Battey (2007) cite the difficulty of departure from the common IRE model (represented in the current coding system as T SOL-S RES-T REA). There are multiple ways that teachers can engender discussion among their students, but it is not an easy process as the following passage suggests:

> Although the kind of discourse we have just described seems like such a necessary and productive part of classroom practice, what teachers must do to support such opportunities is complex and not well characterized in the literature. The way teachers support mathematical discourse matters. Even in what may seem like the simplest form of classroom discourse—large-group discussion to foster students' participation in thinking through a problem—the teacher must attend to many issues. The teacher must attend to who is participating, how they are participating, the mathematical ideas being pursued, the students' linguistic and mathematical backgrounds, the students' current understandings, and the attitudes and identities the students bring to the conversation (Lampert, 2001). The teacher must give each student the opportunity to participate in working through the problem while simultaneously encouraging each student to attend to the solution paths of others, in ways that she can orchestrate opportunities for students to build one another's thinking. While attending to this, the teacher must also actively take a role in making certain that the class gets to the implicit and explicit goals. She needs to make judgments about what to avoid, navigate through solution paths that do not always work, respond to incorrect statements, and watch out for those not participating. She must also find a way to make explicit the underlying mathematical similarities and differences in the solutions in a way that makes sense to her students. All these actions and decisions must of course fit within the given period of time of a lesson, a unit, and a school year.
>
> Teachers are expected to pose problems but not provide answers (Lampert, 1990), stop or slow down the discussion to provide access to more students (Rittenhouse, 1998), model the academic discourse for the students (Ball, 1993; Lampert, 1990; Rittenhouse, 1998), comment and elaborate on student ideas (Rittenhouse, 1998), and question student reasoning so as to foster certain habits of mind (Lampert, 1990; Lampert, Rittenhouse, & Crumbaugh, 1996; Rittenhouse, 1998). Thus, as Ball (1993) pointed out, the teacher is responsible for the students' learning of mathematical content and, at the same time, for fostering a discourse environment that both supports students and helps to create, among them, new identities that include a favorable disposition towards mathematics. It is no wonder IRE remains prevalent.

Ibid., 231.

In order to present an analysis of each teacher's movement from IRE to ERE to PD, this study employs word-based comparisons between a teacher's proof vs. non-proof instruction. As noted in the methodology section, these include string lengths, defined as student exchanges uninterrupted by a teacher; and word ratios, exposing the proportion of a teacher's words to all words spoken. Finally, word clouds are provided to give insight into the content of each lesson.

The study also employs utterance-based analysis, including recurrences of T SOL-S RES-T REA (equivalent to the common initiation-response-evaluation, or IRE, model cited above) as opposed to SOL/RES/REA exchanges from other than teacher/student/teacher (including some rare instances in which solicitation, response and reaction are all from students) and bar codes showing proportions of each type of coded response in all lessons and in individual lessons.

**Section 2: Analysis of Teacher A**

Teacher A is the more traditionally-oriented of the two teachers in this study. She answered questions about her priorities and approach to teaching proof vs. non-proof as follows:

> 1) When you are teaching your classes, please assign a number 1-5 (5 is most) to how highly you prioritize the following:
>
> a) students communicate directly with each other at all times (not just during groupwork) - 4
> b) students use correct mathematical terminology in discussion - 5
> c) students explore and absorb the skill or concept you are trying to convey - 5
> d) you provide time for students to interact and discuss the situation you have presented (separate from whole class instruction) - 5
> e) students provide their own goals - 3
> f) students ask questions to clarify procedures, concepts or your pedagogical intentions - 5
>
> If you can, please rank the above in order of their relative importance to you.
> c, f, d, b, a, e

In other words, she ranks “students explore and absorb the skill or concept you are trying to convey,” “students ask questions to clarify procedures, concepts or your pedagogical intentions,” “you provide time for students to interact and discuss the situation you have presented (separate from whole class instruction),” and “students use correct mathematical terminology in discussion” as most important, followed only then by “students communicate directly with each other at all times (not just during group work)” and “students provide their own goals.” (As stated in the Methdology section, the last two items are indicative of subscription to constructivist ideology.) She believes that students should have the opportunity to ask questions and to communicate with each other, but the latter is relegated to specific sections of class time (“group work”). In general, she places less emphasis placed on the students’ construction of their understanding, more emphasis on an objectivist goal.

> 2) When you think about your teaching and lesson planning, are there differences to how you approach teaching proof lessons, as opposed to non-proof lessons (e.g., midpoint formula, straight algebra)?
>
> > Yes. Proofs require a different type of thought process than straight algebra, as there is not a neat and tidy list of rules to follow. This is why it is so difficult for many students; they crave that checklist that they have become so accustomed to in their math classes. I think in learning proofs it becomes even more important for students to discuss their thought processes, with each other and with me. They should also have the opportunity to really struggle and figure things out for themselves, as I feel that is a requirement for eventually mastering proofs.

She cites the importance of productive “struggle” when learning proof but does not see it as an integral part of the acquisition of more algebraic skills. She sees her instruction as affected by subject matter which is borne out in the analysis of her lessons.

<u>2.1: Summary (all lessons):</u>

The following results will show that, for Teacher A, the teaching of proof may be considered a gateway to a more constructivist teaching approach. There is a marked difference in the ratio of teacher-to-student words uttered, her own utterances, and the number of uninterrupted student exchanges when she is teaching proof as opposed to when she is inculcating more algebraic skills.

Teacher/student word ratio:

The most striking distinction between the proof and non-proof lessons is the extreme difference in teacher/student ratio of speech:

Table 6. Teacher A, teacher/student word ratio, all lessons.

| | Teacher A | Students | T/S word ratio | % teacher speech |
|---|---|---|---|---|
| Lesson 1 - proof | 1422 | 746 | 1.91 | 65.6% |
| Lesson 2 - proof | 1220 | 656 | 1.86 | 65.0% |
| Lesson 3 - proof | 3614 | 2324 | 1.56 | 60.9% |
| Lesson 4 - proof | 1517 | 1752 | .87 | 46.4% |
| | | | | |
| Lesson 1 - no proof | 2776 | 545 | 5.09 | 83.6% |
| Lesson 2 - no proof | 5677 | 1354 | 4.19 | 80.7% |

In other words, when Teacher A is teaching proof, she talks less, and students contribute more, than when she is engaged in transmitting largely arithmetic algorithms.

T SOL/S RES/T REA vs. SOL/RES/REA other than T/S/T:

The following table gives information about Teacher A's proportion of the common SOL/RES/REA sequence (an utterance-related measure). It shows that the proportion of teacher-initiated questioning and evaluation is much greater when she is not teaching proof (even given the relatively high proportion found in proof lesson A1):

Table 7. Teacher A, T SOL/S RES/T REA vs. SOL/RES/REA other than T/S/T, all lessons:

| | T SOL/S RES/T REA | SOL/RES/REA other than T/S/T | Ratio of columns 2 and 3 |
|---|---|---|---|
| A1 | 25.0% | 5.0% | 5 |
| A2 | 20.0% | 13.3% | 1.5 |
| A3 | 18.3% | 11.9% | 1.5 |
| A4 | 18.2% | 16.4% | 1.1 |
| Average | 20.4% | 11.75% | |
| Weighted average | 19.7% | 11.6% | 1.7 |
| Anp1 | 37.9% | 5.8% | 6.5 |
| Anp2 | 34.6% | 6.7% | 5.2 |
| Average | 36.3% | 6.3% | |
| Weighted average | 35.6% | 6.4% | 5.6 |

Bar graphs from codes:

Another surprising feature of the lessons in which proof is taught versus those in which it is not surfaces upon examination of the codes of the pedagogical moves. Although the coding of student utterances appears relatively unchanged regardless of subject matter, there are notable differences in the teacher's utterances. In lessons in which proof is being taught, there is a much less pronounced difference between the directive/didactive bars and the transactive/facilitative. Generally, the teacher's questioning is the most affected area; in general, the solicitative portion of the transactive/facilitative bar dwarfs the solicitative part of the directive/didactive bar in lessons in which proof is being taught. (The exception, lesson #4, is the lesson in which both proof and numerical application is being taught; the difference in the teacher's usage of questioning is apparent. In fact, the teacher/student word ratio is so low (under 1) that it is not surprising that, given how much less the teacher speaks, she also asks fewer questions overall.)

By contrast, in the two lessons engaged in more algebraic activities, the instruction is (predictably and demonstrably) more transmissive. The solicitative portions of the directive/didactive are much larger than those in the transactive/facilitative bars (the reverse of the situation above).

In more common language, when the teacher is teaching proof, she tends to ask questions intended to provoke discussion; when she is not teaching proof, she tends to ask questions with factual, non-negotiable answers.  This is clearly shown below in the bar graphs of the coded material.  There is also a marked difference in the students' utterances. When they are involved in proof-based activities, their own utterances are more thoughtful (PP/PI/CO/TQ/TR) and less automatic (General confirmation/no code) than when they are engaged in more algebraic classwork.

Figure 1.  Teacher A, bar graph of teacher and student utterances, proof, all lessons combined:

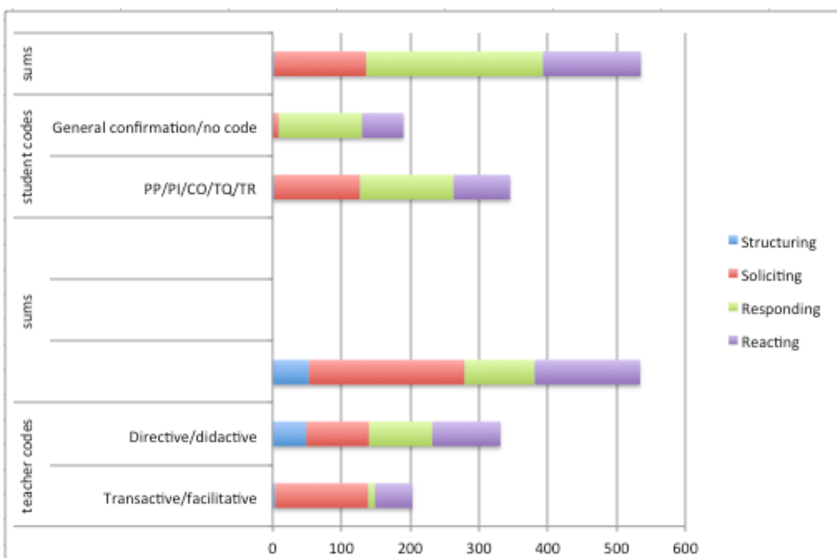
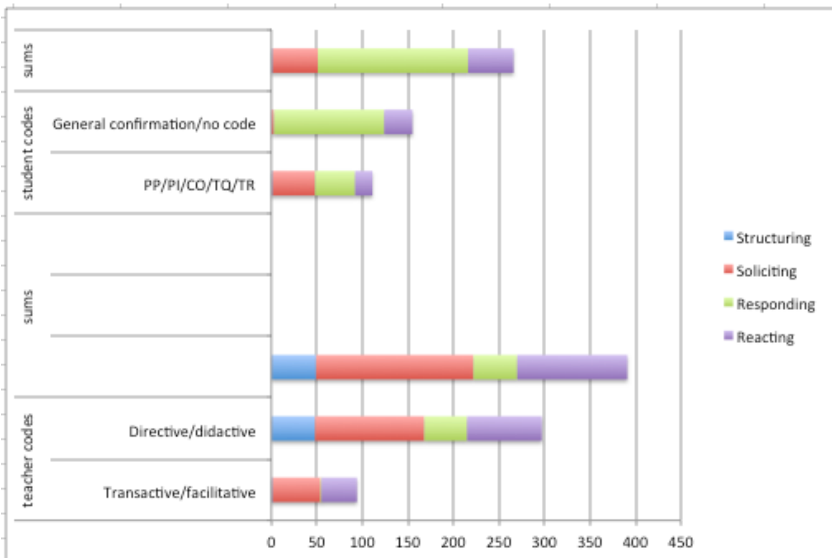


Figure 2. Teacher A, bar graph of teacher and student utterances, non-proof, all lessons combined:




Strings of student discussion:

This study adopts the stance that an important feature of constructivist pedagogy is for students to take a large role in sense-making activities. One way to measure the success of this approach is to examine stretches of discussion between students with little or no teacher mediation. In this pursuit, "strings" of student utterances were tallied and tabulated. As stated in this study's Methodology section, a string is any teacher utterance followed by two or more uninterrupted student utterances. The table below shows the aggregate of such strings from Teacher A's four proof lessons and two non-proof lessons.

Table 8. Teacher A, student string lengths, all lessons combined:

| | |
|---|---|
| Total number of strings, teacher A, proof: | 62 |
| Average string length: | 2.82 |
| Total number of strings, teacher A, non-proof: | 5 |
| Average string length: | 2 |

It is plain from this table that student discussion strings were both more common and of longer length when the students studied proof. In fact, in both non-proof lessons, there were no strings of length longer than two. In sum, Teacher A fosters more uninterrupted student-led inquiry and discussion when she teaches proof.

## 2.2: Individual lessons

Teacher A, lesson 1:

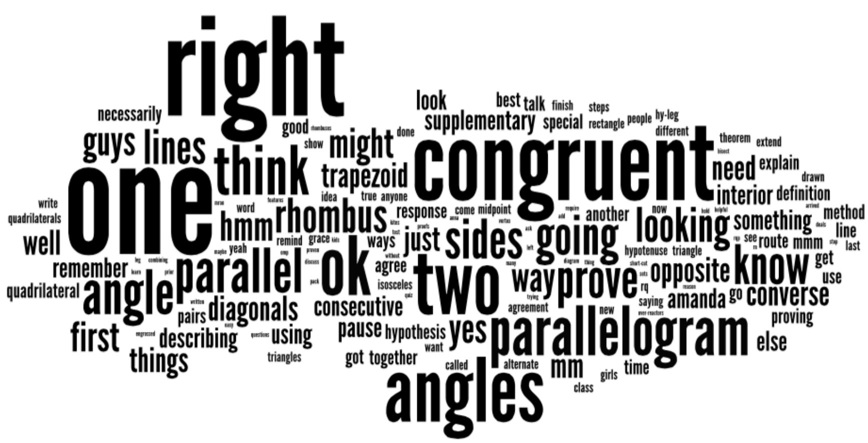
The teacher introduced the lesson by stating that the class would be using triangle congruence in special quadrilaterals. She reviewed the methods by which quadrilaterals could be proven "special" and then referred the class to a sheet with diagrams. The class worked through two proofs involving a parallelogram and a trapezoid.

A word cloud of all words used in the lesson is here. Note the prominence of the word "congruent," which in this lesson and in subsequent lessons frequently signals proof content:

Figure 3. Teacher A, word cloud, teacher and students, lesson A1:


This word cloud features teacher words only. Although the word “congruent” is smaller relative to the other words, there are no major observable differences.

Figure 4. Teacher A, word cloud, teacher words only, lesson A1:




In the lesson, the students were occupied with the study of special quadrilaterals, e.g., parallelograms, rhombuses and so forth. The teacher began by reviewing the types of special quadrilaterals but the bulk of the lesson was taken up with proof that a particular quadrilateral was a special quadrilateral.

An excerpt from the lesson shows how the teacher, using both facilitative and transactive prompts, elicited from the students the methods they needed in order to prove what she wanted them to prove:

> T: Should we remind ourselves of the ways of proving a parallelogram? What are the ways? We said the definition of a parallelogram, right, both pairs of opposite sides parallel. (Agreement.) What else?
> S: One pair of opposite sides both congruent and parallel.
> T: What else?
> S: Wait, say that again.
> S: One set is congruent and parallel.
> T: One pair both congruent and parallel…what else?
> S: Opposite sides congruent.
> T: Both pairs of opposite sides congruent…Both pairs of opposite angles congruent…Diagonals bisect each other…
> S: But you could tell by just the diagonals bisecting each other?
> T: Mm hmm. Well, looking at this diagram, do you have an idea in your head about which one of those might be the best way? (Pause.) Think about it. Think about which, which one might be the best way. I don't see any diagonals there, do you?
> S: No.
> T: So might diagonals be the best route?
> S: I also don't see any parallels. You don't know that they're parallel. You don't know that it's true that they're parallel.
> T: Yes, you do. How do you prove that lines are parallel?
> S: Converse to alternate interior angles.
> T: Converses to all of those things. Yes?
> S: CPCTC. No, you can't…
> T: I don't know! (Pause.)

There are four sections of the lesson during which student work together with little or no assistance from their teacher. Periodically, the teacher would bring the class back together to solidify approaches that students had found to be successful.

Table 9. Teacher A, word counts, lesson A1:

| |
|---|
| Teacher word count: 1422 |
| Student word count: 746 |
| T/S word ratio: 1.91 (65.6%) |

A bar graph for all types of coded utterances is here. Although the teacher's reactions were more directive than transactive, her solicitations were more transactive than directive. The majority of student utterances are PP/PI/CO/TQ/TR, evidencing higher-order activity.

Figure 5. Teacher A, bar graph of teacher and student utterances, lesson A1:

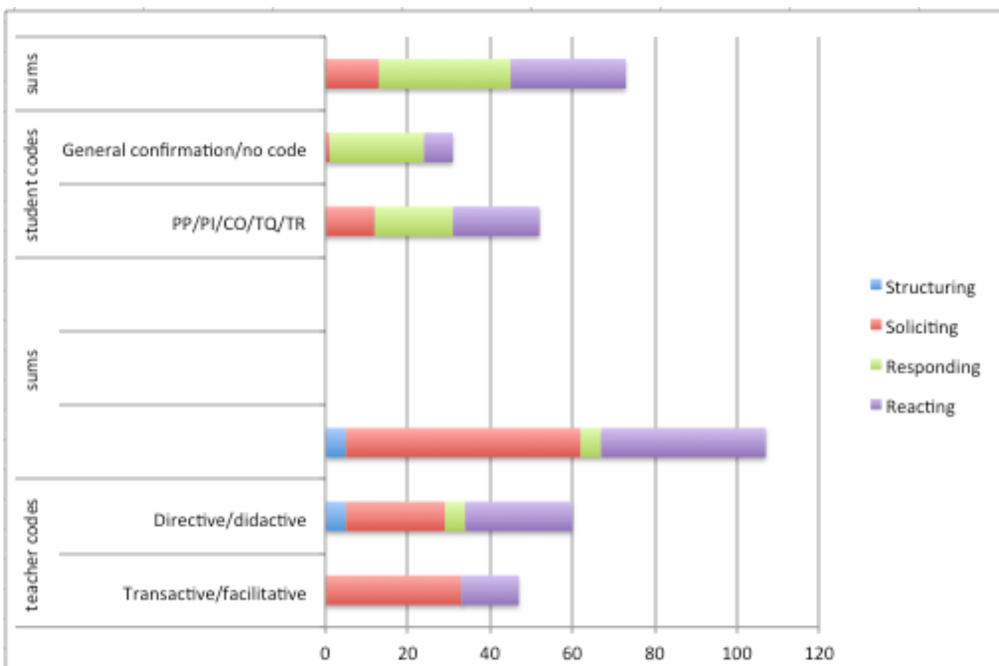


Table 10. Teacher A, lesson A1, string lengths:

| A1 strings: | |
|---|---|
| TSS | 5 |
| 4 S’s | 1 |
| 5 S’s | 1 |

The following table indicates a relatively high proportion of teacher-initiated questioning, student response, and teacher evaluation.

Table 11. Teacher A, T SOL/S RES/T REA vs. SOL/RES/REA other than T/S/T, lesson A1:

| | T SOL/S RES/T REA | SOL/RES/REA other than T/S/T | Ratio of columns 2 and 3 |
|---|---|---|---|
| A1 | 25.0% | 5.0% | 5 |

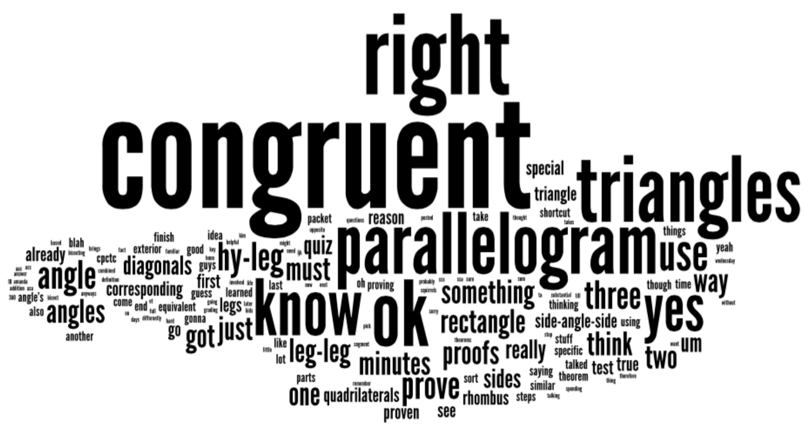
Teacher A, lesson 2:

This lesson consists of review for a test involving congruence and special quadrilaterals. The teacher began with a quick verbal review of methods of proving congruence and then led the class through two proofs. The lesson’s word cloud is here:

Figure 6. Teacher A, word cloud, teacher and students, lesson A2:


Here is the teacher-only word cloud. There are no major differences although the word “ok” is bigger.

Figure 7. Teacher A, word cloud, teacher words only, lesson A2:




As in the last lesson, excerpts show students able to take the lead and reason out their own hypotheses. In the first, a student makes an observation that the teacher didn’t expect:

> S: If you know that it's a quadrilateral, to prove that it's a rectangle, do you need to prove three sides, or four sides? I mean, four angles?
>
> T: Four angles.

S: All four angles? Cause there's this theorem that said that three, says you only needed the three, because by extension the fourth had to be a right angle?

T: I guess that's true, based on the fact that quadrilaterals have to sum up to 360, right, and you've already got three of them, ummmm, but I would err on the side of doing…I think three is fine, I guess.

In this excerpt, the students have done much of the work needed but are stuck. By asking a facilitative question, the teacher spurs inquiry and discussion among students:

T: How are you thinking of doing it?

S: I was going to say…so right now I have ST is congruent to UR, and I was planning on using hy-leg, but I can't figure out how to prove that SE equals AR, because you're not given any information about…

S: What I said, is that, um, SU is congruent to TR by the properties of a parallelogram, and that SU is equal to SE plus EU and that TR is equal to TA plus AR, and that therefore by subtraction SE is congruent to…

T: But you don't know that EU and TA are congruent.

S: Yeah.

T: So hy-leg may not be the best way. Can you think of, um, another property of a parallelogram that might help you?

S: Opposite angles are congruent. I used that.

T: You know angle S is congruent to angle R, right, so what can you use that to prove there?

S: Then you can use AAS.

T: Yes, as well.

Table 12. Teacher A, word counts, lesson A2:

| |
|---|
| Teacher word count: 1220 |
| Student word count: 656 |
| T/S word ratio: 1.86 (65.0%) |

The bar graph shows the relative prevalence of both teacher and student solicitation and response. The teacher's questioning was more transactive than directive and students' questioning was similarly transactive. As in lesson one, the majority of student contributions are higher-order and the word count shows a healthy ratio of 1.86 teacher to student words spoken (65% of all speech is by the teacher).

Figure 8. Teacher A, bar graph of teacher and student utterances, lesson A2:

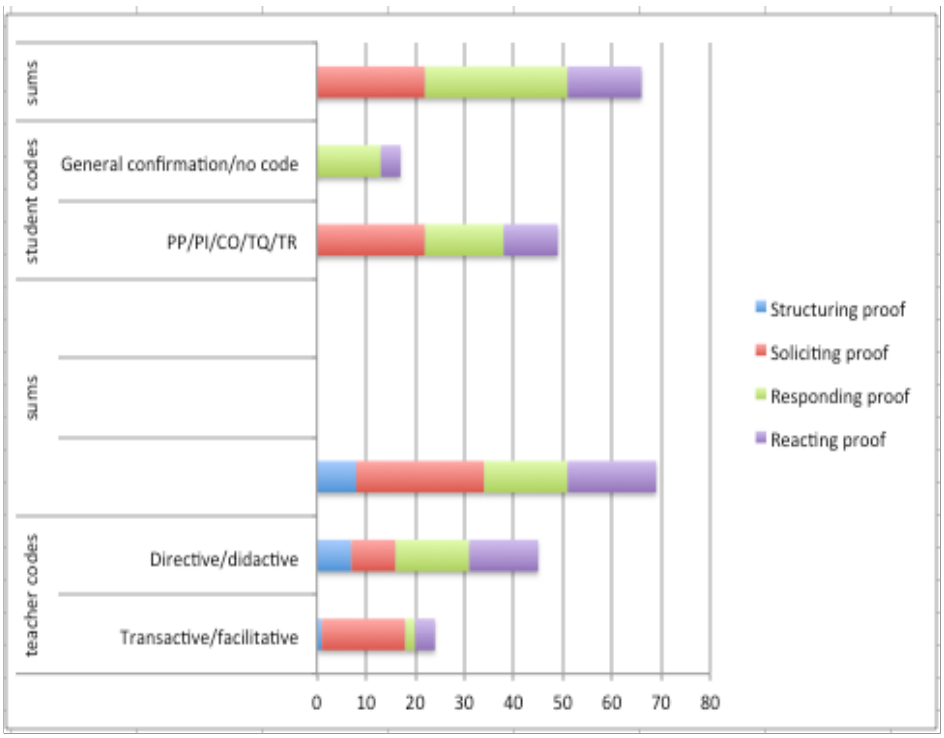



Table 13. Teacher A, student string lengths, lesson A2:

| A2 strings: | |
|---|---|
| TSS | 2 |
| TSSS | 4 |
| 9 S's | 1 |

The following table shows the relative proportion of teacher-initiated questioning, student response and teacher evaluation is relatively low:

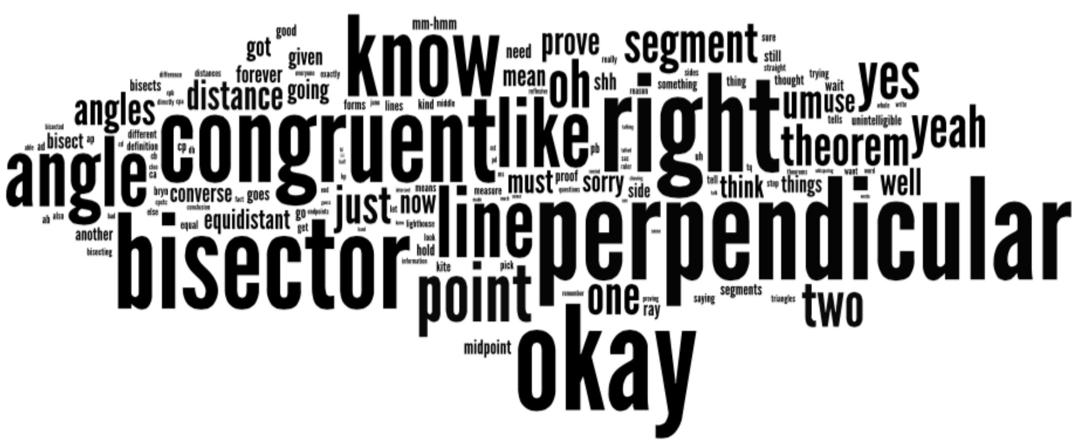
Table 14. Teacher A, T SOL/S RES/T REA vs. SOL/RES/REA other than T/S/T, lesson A2:

| | T SOL/S RES/T REA | SOL/RES/REA other than T/S/T | Ratio of columns 2 and 3 |
|---|---|---|---|
| A2 | 20.0% | 13.3% | 1.5 |

Teacher A, lesson 3:

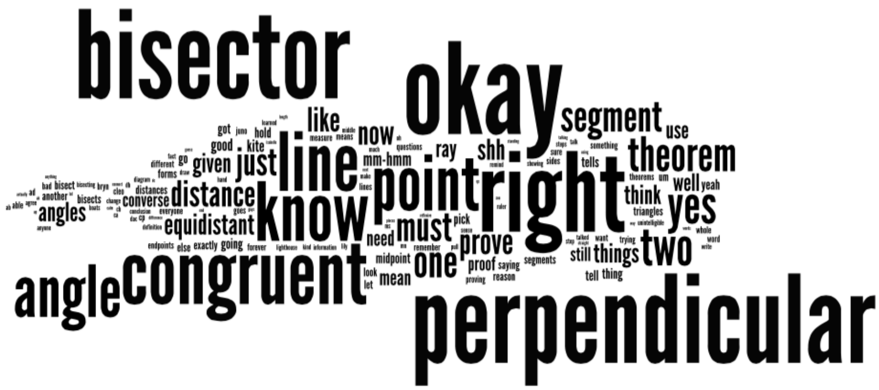
This lesson, which is concerned with perpendicular bisectors, begins with the teacher eliciting a definition of a perpendicular bisector.  The class proceeds to use characteristics of a perpendicular bisector to prove conjectures involving congruent triangles. The lesson's word cloud, below, indicates a focus on perpendicularity but also on congruence:

Figure 9.  Teacher A, word cloud, teacher and students, lesson A3:




Here is the teacher-only word cloud.  The words "know" and "congruent" are a little smaller relative to the other words, but there are no other major differences:

Figure 10. Teacher A, word cloud, teacher words only, lesson A3:




Students engage in discussion to determine the definition of a bisector. In order to promote understanding the teacher engages in transactive questioning and the students respond transactively, both to her and to each other. There is a string of seven uninterrupted student utterances (ten if one disregards the teacher's facilitative "Shh"). The initial statement is a transactive solicitation:

T: No. Can you?
S: No you can't. No.
S: Yeah.
S: No. I said "No."
T: Shh.
S: Yeah you can you --
S: No.
S: -- can like kind of like -- [cross talk]
S: No you can't because it goes on forever that means your middle will always be changing.
S: Unless it's -- unless it has an end point that's specifically --
S: You can't measure forever.
S: Yeah, so I'm rocking. I'm right.
T: Let other people think about it.
S: Sorry. [cross talk]

Another string of six uninterrupted student utterances is found later in the transcript (seven if one again disregards a facilitative teacher "Shh") during which students and teacher are occupied with the construction of a proof. The teacher's initial utterance is a transactive response:

> T: Well we don't know it's a perpendicular bisector. We're trying to prove it's a perpendicular bisector.
> S: Okay.
> S: How do you get past this point?
> S: I put PD is equivalent to PD.
> S: Yeah, that's how far I've gotten to.
> S: Me too.
> S: Can you say if like the angles -- [cross talk]
> T: Shh.
> S: -- then doesn't it have to --
> T: We don't know that it has any right angles right now.

Table 15. Teacher A, word counts, lesson A3:

| |
|---|
| Teacher word count: 3614 |
| Student word count: 2324 |
| T/S word ratio: 1.56 (60.9%) |

The lesson's bar graph indicates a majority of directive/didactive utterances by the teacher, but there is also a large amount of transactive questioning overall. Again, the students exhibit a majority of higher-order behavior, and the teacher/student word ratio of 1.56 continues to show high student involvement:

Figure 11. Teacher A, bar graph of teacher and student utterances, lesson A3:

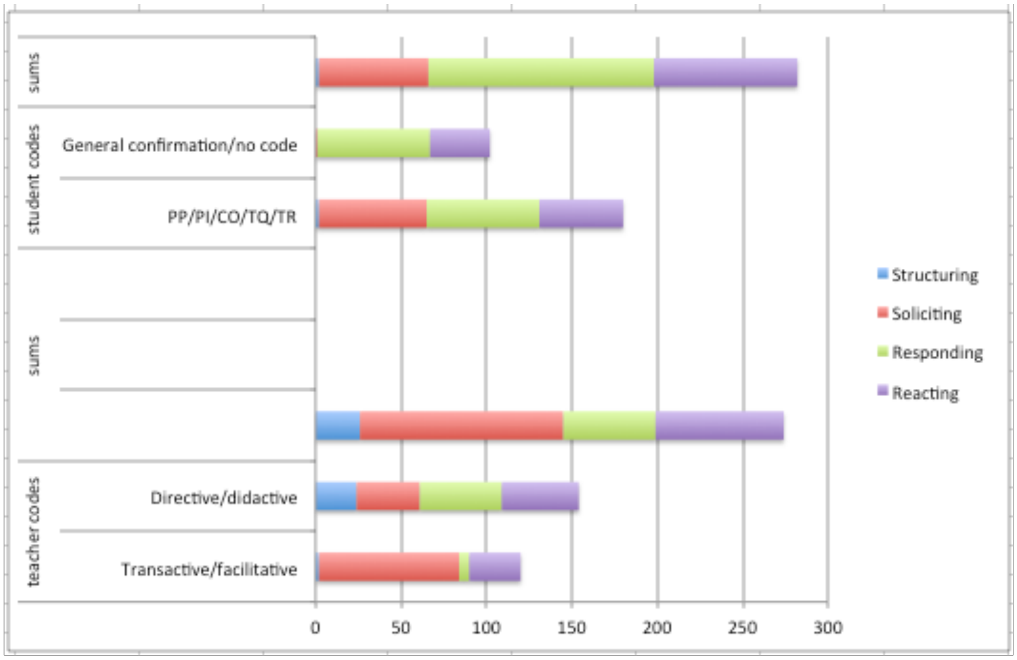



Table 16. Teacher A, student string lengths, lesson A3:

| A3 strings: | |
|---|---|
| TSS | 27 |
| TSSS | 7 |
| 4 S's | 2 |
| 5 S's | 1 |
| 6 S's | 2 |
| 7 S's | 1 |

The following table shows the proportion of teacher-initiated questioning, student response and teacher evaluation is relatively low:

Table 17. Teacher A, T SOL/S RES/T REA vs. SOL/RES/REA other than T/S/T, lesson A3:

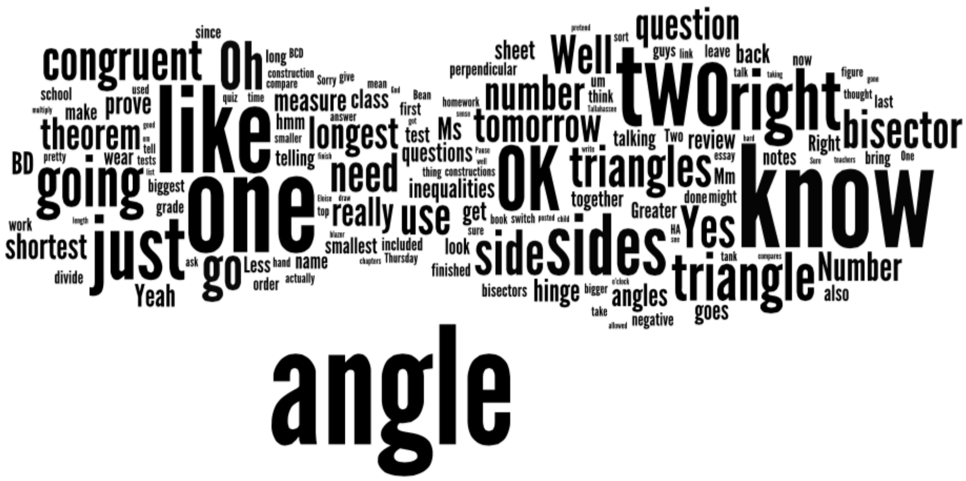
| | T SOL/S RES/T REA | SOL/RES/REA other than T/S/T | Ratio of columns 2 and 3 |
|---|---|---|---|
| A3 | 18.3% | 11.9% | 1.5 |

Teacher A, lesson 4:

This lesson's focus is the hinge theorem as it relates to the relationships between sides and angles of triangles. There is a mixture of applying the hinge theorem in problems with different sized triangles and proof.

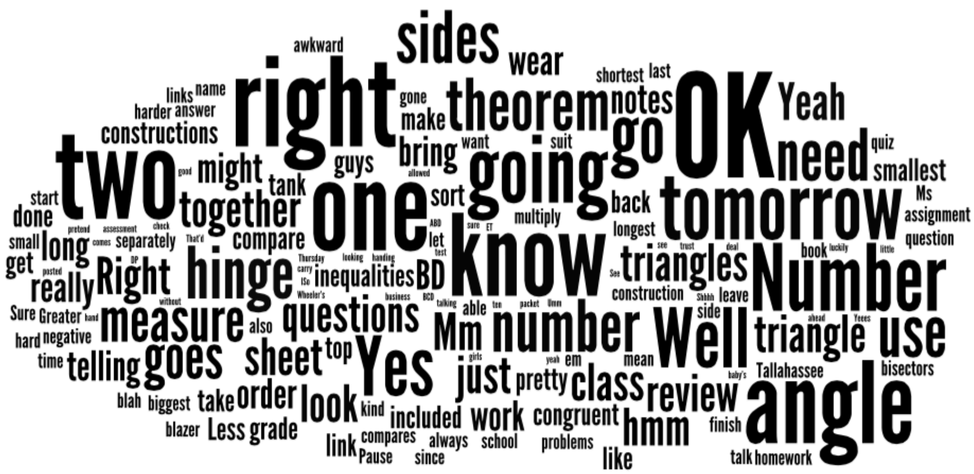
This lesson's word cloud indicates less of a focus on congruence, which is not surprising since hinge theorem is about inequalities in triangles vis a vis lengths of sides relative to measures of angles. The word "congruent" still appears, although it is smaller; and smaller still in the teacher-only word cloud.

Figure 12. Teacher A, word cloud, teacher and students, lesson A4:



Here is the teacher-only word cloud.

Figure 13. Teacher A, word cloud, teacher words only, lesson A4:




A string of five student utterances is introduced by Teacher A's issuance of a transactive structuring move:

T: S., you have a book right there, so you could use the book.
S: OK.
S: Did you get this?
S: So that's a yes?...Because you know that that's a right angle.
S: Uh, I think you go like this…Oh, wait…Noooo!!!!
S: Ms. X?
T: Mm hmm.
S: For 14, could I prove that the smaller triangles are congruent and then use that DB is congruent to…triangle?

The following bar graph and word counts show that the teacher/student word ratio is at an absolute minimum (in this study) of .87, which is to say, the students talked more than their teacher. Despite this, but more likely because of it, the teacher's directive/didactive index is high. In other words, the students may have already assimilated their own methods of working and so elicited much less transactive prompting. The students nevertheless engage in a lot of general confirmation/no code activity, the bulk of it in responses.  Inspection of the transcript shows that this is because they are frequently engaged in affirming each other's work.  Also, although the teacher engages in so much more directive/didactive activity than in other proof lessons, the students nevertheless engage in more PP/PI/CO/TQ/TP activity, the bulk of it in solicitations, than general confirmation/no code (if only marginally so):

Table 18. Teacher A, word counts, lesson A4:

| |
|---|
| Teacher word count: 1517 |
| Student word count: 1752 |
| T/S word ratio: .87 (46.4%) |

Figure 14.  Teacher A, bar graph of teacher and student utterances, lesson A4:

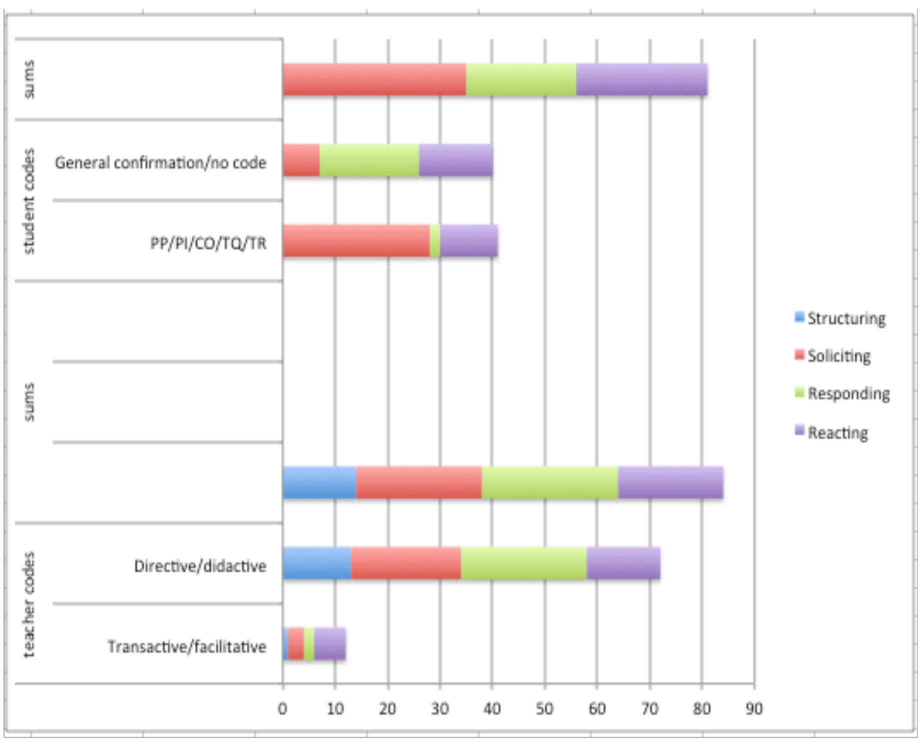


Table 19. Teacher A, student string lengths, lesson A4:

| A4 strings: | |
|---|---|
| TSS | 4 |
| TSSS | 2 |
| 5 S's | 2 |

The following table shows the proportion of teacher-initiated questioning, student response and teacher evaluation is relatively low:

Table 20. Teacher A, T SOL/S RES/T REA vs. SOL/RES/REA other than T/S/T, lesson A4:

| | T SOL/S RES/T REA | SOL/RES/REA other than T/S/T | Ratio of columns 2 and 3 |
|---|---|---|---|
| A4 | 18.2% | 16.4% | 1.1 |

Teacher A, lesson np1:

This lesson is completely absorbed in finding slopes of lines. As such, it spends no time on proof. It is all about calculating slopes on the coordinate plane.

Figure 15. Teacher A, word cloud, teacher and students, lesson Anp1:

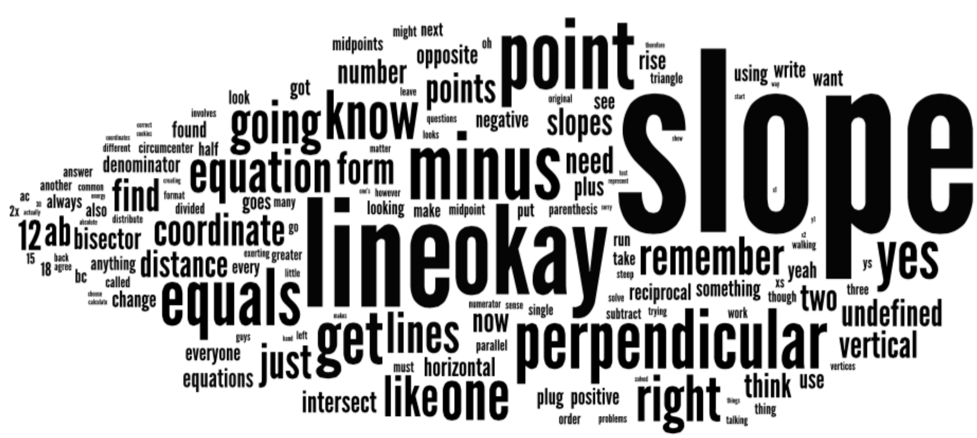


Figure 16. Teacher A, word cloud, teacher words only, lesson Anp1, practically identical to the one above:

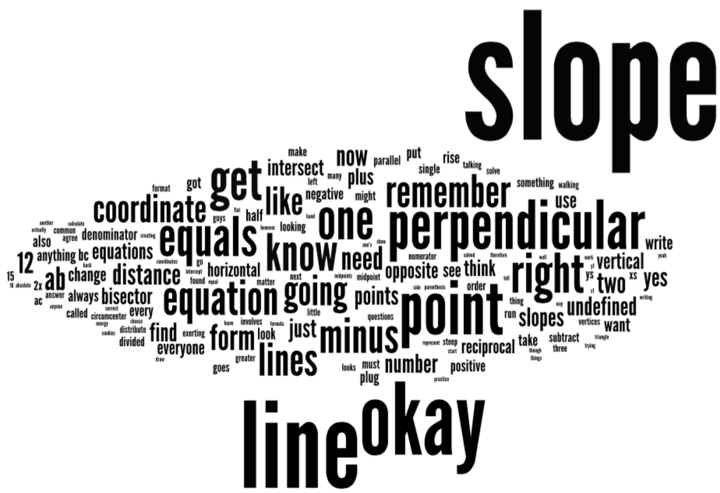
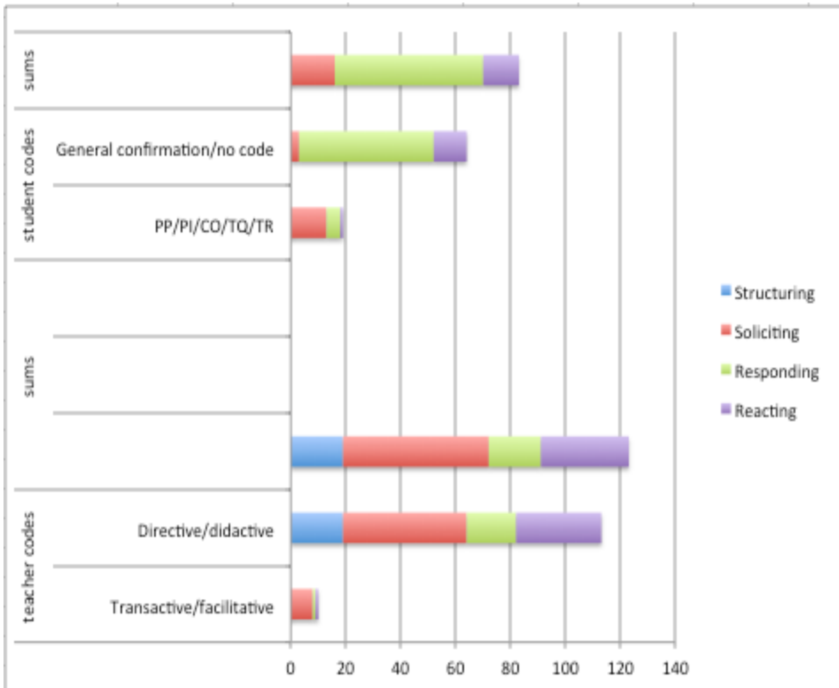



Table 21. Teacher A, word counts, lesson Anp1:

| |
|---|
| Teacher word count: 2776 |
| Student word count: 545 |
| T/S word ratio: 5.09 (83.6%) |

The teacher-to-student word ratio is very high: 5.09. In addition, the students engage in much less PP/PI/CO/TQ/TR activity than GC/NC and the teacher's transactive/facilitative activity is absolutely dwarfed by her directive/didactive utterances:

Figure 17. Teacher A, bar graph of teacher and student utterances, lesson Anp1:




As noted, the word ratio favors teacher speech. The longest string of uninterrupted student speech is 2.

Table 22. Teacher A, student string lengths, lesson Anp1:

| Anp1 strings: | |
|---|---|
| TSS | 2 |

The following illustrates the relatively high proportion of teacher-initiated questioning, student response, and teacher evaluation.

Table 23. Teacher A, T SOL/S RES/T REA vs. SOL/RES/REA other than T/S/T, lesson Anp1:

| | T SOL/S RES/T REA | SOL/RES/REA other than T/S/T | Ratio of columns 2 and 3 |
|---|---|---|---|
| Anp1 | 37.9% | 5.8% | 6.5 |

Teacher A, lesson np2:

This lesson is dedicated to finding lengths and midpoints of line segments. There is some application of these formulae (two lines) to proving quadrilaterals on the coordinate plane are special. However, the bulk of the lesson is on numerical calculation. The teacher-to-student word ratio is 4.19 (80.7%), lower than in the previous lesson but still higher than in all four proof lessons.

Table 24. Teacher A, word counts, lesson Anp2:

| |
|---|
| Teacher word count: 5677 |
| Student word count: 1354 |
| T/S word ratio: 4.19 (80.7%) |

As in the previous non-proof lesson, the teacher's directive/didactive activity far exceeds her transactive/facilitative utterances, although the difference is less pronounced. Surprisingly, however, students' activity is evenly split between general confirmation/no code and PP/PI/CO/TQ/TR, although notably their higher-order activity is equal parts solicitations and responses. Their uncoded/general confirmation speech patterns indicate no solicitations, only responses and reactions, a significant difference from the bar graphs from lesson 4 (which had similar bar lengths but different distributions):

Figure 18. Teacher A, bar graph of teacher and student utterances, lesson Anp2:

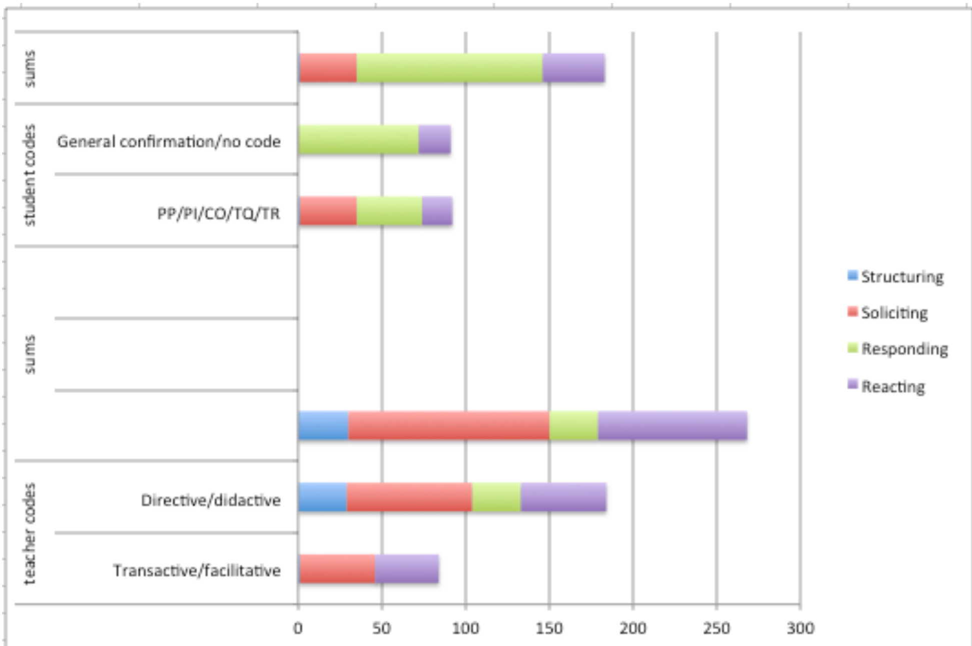



Figure 19. Teacher A, word cloud, teacher and students, lesson Anp2

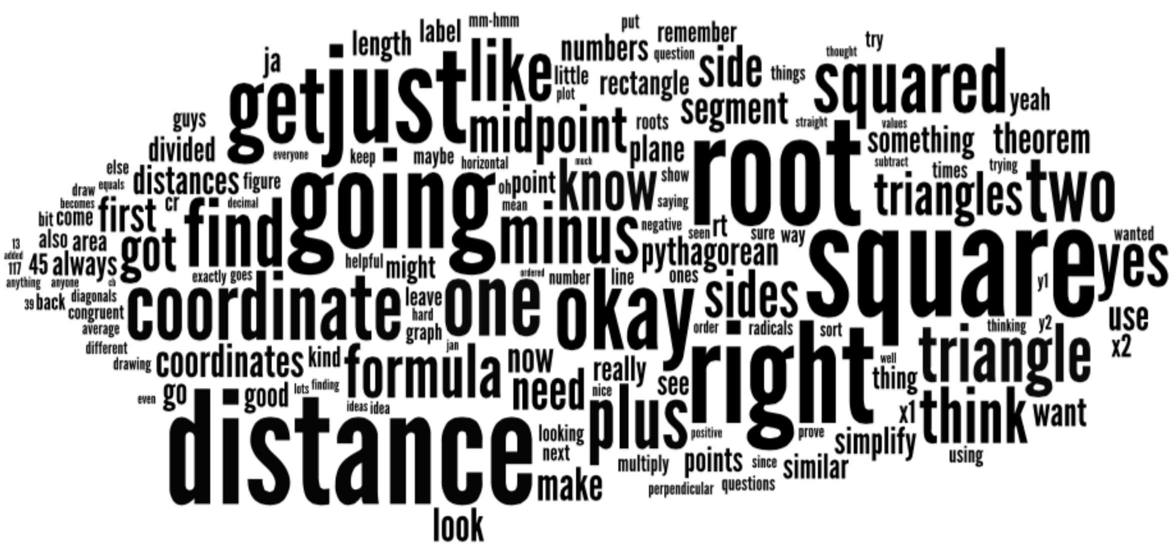



Figure 20. Teacher A, word cloud, teacher words only, lesson Anp2, very similar to one above:

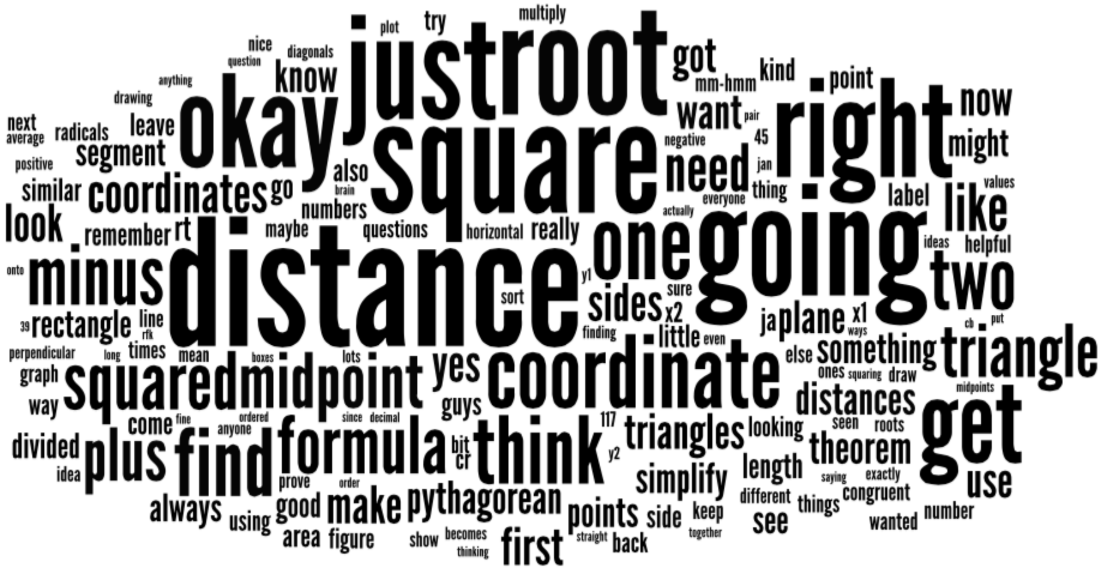



Table 25. Teacher A, student string lengths, lesson Anp2:

| Anp2 strings: | |
|---|---|
| TSS | 3 |

The following illustrates the relatively high proportion of teacher-initiated questioning, student response, and teacher evaluation.

Table 26. Teacher A, T SOL/S RES/T REA vs. SOL/RES/REA other than T/S/T, lesson Anp2

| | T SOL/S RES/T REA | SOL/RES/REA other than T/S/T | Ratio of columns 2 and 3 |
|---|---|---|---|
| Anp2 | 34.6% | 6.7% | 5.2 |

**Section 3: Analysis of Teacher B**

Teacher B considers himself an adherent to constructivist principles. He actively tries to create a constructivist atmosphere at all times regardless of the subject matter of the lesson. Here are his responses to the survey about priorities and differences in his teaching of proof vs. non-proof in which he clearly prioritizes the principles of constructivism above objectivist principles:

> 1) When you are teaching your classes, please assign a number 1-5 (5 is most) to how highly you prioritize the following:
> a) students communicate directly with each other at all times (not just during groupwork)
> b) students use correct mathematical terminology in discussion
> c) students explore and absorb the skill or concept you are trying to convey
> d) you provide time for students to interact and discuss the situation you have presented (separate from whole class instruction)
> e) students provide their own goals
> f) students ask questions to clarify procedures, concepts or your pedagogical intentions
>
> If you can, please rank the above in order of their relative importance to you.
>
> > Number 1: This is so tricky! I feel that what I prioritize is not what I do. Ideally, I would order things like this: e, a, c (but with "you" swapped with "they"), d (but with "you have presented" swapped with "they have presented), f, then b. Still, I will put down what I think I do: c, a, f, e, d, b. This is a great question. It helps me realize how many things I need to work on. I feel like this question will help others, too. I'm sharing this with my colleagues.

He sees students' own goals as ultimately most important, followed by students communicating directly with each other at all times; followed by students' opportunity to explore and absorb skills they convey to each other; then, time is provided for students to interact and discuss situations that they have presented; next, students can ask questions, and finally, the usage of correct mathematical terminology is relegated to last place. (Recall, Teacher A's ranking is almost completely opposite.)

He then acknowledges that, in his actual practice, the acquisition of skills and concepts takes first priority, but still places unmediated communication between students ahead of all other objectives.

Although in reality he acknowledges that the priority of goals shifts, he still does not see the need for scheduling specific time for students to interact as a top priority; nor does he find the correct usage of mathematical notation and terminology to be relatively important.

> 2) When you think about your teaching and lesson planning, are there differences to how you approach teaching proof lessons, as opposed to non-proof lessons (e.g., midpoint formula, straight algebra)?
>
> > Number 2: No. I want my students to try to justify everything they are doing all of the time!

He does not consider the study of proof to be any different than any other study of mathematics, unlike Teacher A. And, in fact, Teacher B's desire to teach all of his lessons the same regardless of subject is borne out in the following analysis of his practice.

<u>3.1: Summary (all lessons):</u>

As just noted, Teacher B's proof/non-proof lessons do not bear the sharp distinction found in Teacher A's. Although, on the whole, Teacher A features lower teacher/student word ratios, more pronounced differences between the general confirmation/no code and PP/PI/CO/TQ/TR (these measure "low-order" and "high-order" student activity, respectively) bars and higher average string length when she is teaching proof, she also (both from the data and her own statement) does not teach in as constructivist a manner when she is not teaching proof.

Teacher/student word ratio:

In contrast with Teacher A, there is much less difference in teacher/student ratio of speech. In fact, on average, the ratio of teacher to student speech is better when he is not teaching proof (72.65% vs. 65.85%).

Table 27. Teacher B: teacher/student word ratio, all classes:

| | Teacher B | Students | T/S word ratio | % teacher speech |
|---|---|---|---|---|
| Lesson 1 - proof | 1497 | 383 | 3.91 | 79.6% |
| Lesson 2 - proof | 1901 | 637 | 2.98 | 74.9% |
| Lesson 3 - proof | 3381 | 983 | 3.44 | 77.5% |
| Lesson 4 - proof | 1340 | 947 | 1.41 | 58.6% |
| | | | | |
| Lesson 1 - no proof | 1232 | 679 | 1.81 | 64.5% |
| Lesson 2 - no proof | 5331 | 2607 | 2.04 | 67.2% |

T SOL/S RES/T REA vs. SOL/RES/REA other than T/S/T:

The following table gives information about Teacher B's proportion of the common SOL/RES/REA sequence, an utterance-based measure. It shows that the proportion of teacher-initiated questioning and evaluation does not differ much when he is not teaching proof, even given the relatively high proportion found in proof lessons B1 and B4.

Table 28. Teacher B, T SOL/S RES/T REA vs. SOL/RES/REA other than T/S/T, all lessons:

| | T SOL/S RES/T REA | SOL/RES/REA other than T/S/T | Ratio of columns 2 and 3 |
|---|---|---|---|
| B1 | 20.3% | 5.1% | 4.0 |
| B2 | 21.3% | 13.0% | 1.6 |
| B3 | 19.9% | 7.2% | 2.8 |
| B4 | 32.5% | 7.2% | 4.5 |
| Average | 23.5% | 8.1% | |
| Weighted average | 23.5% | 9.0% | 2.6 |
| Bnp1 | 15.7% | 6.7% | 2.3 |
| Bnp2 | 23.9% | 10.4% | 2.3 |
| Average | 19.8% | 8.6% | |
| Weighted average | 22.7% | 9.9% | 2.29 |

It should be noted that, as with all other measures, Teacher B's data indicate his instruction to be more teacher-centered than Teacher A when she is teaching proof, but remarkably less so when she is not teaching proof.

Bar graphs from codes:

Although he is generally more DD/DR than FA/TA (as is Teacher A), Teacher B's students are, on average, more engaged in higher-order activities and discussions than lower-order.

Figure 21. Teacher B, bar graph of teacher and student utterances, proof, all lessons combined:

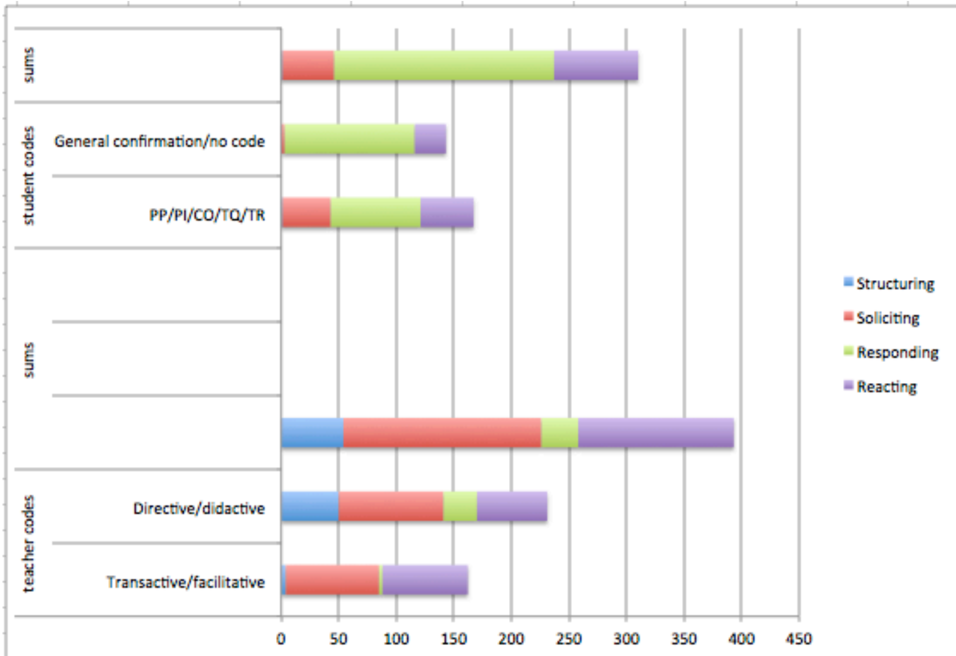
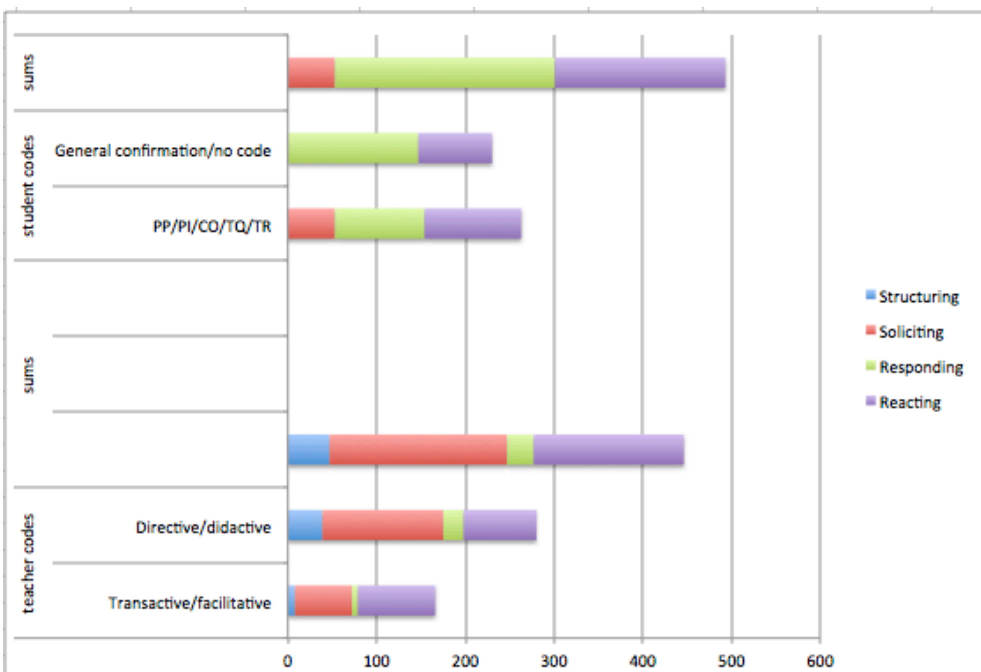



Figure 22. Teacher B, bar graph of teacher and student utterances, non-proof, all lessons combined:




Strings of student discussion:

We will now try to situate what Teacher B is trying to accomplish in the context of Bowers and Nickerson's proposition-discussion model with an eye to the difficulties cited by Franke, Kazemi and Battey (see "Strings of student discussion" at the beginning of this chapter). Teacher B knows that incorporating the principles of constructivism is not as simple as to pose a question and let the students take the reins; rather, he is aware of the importance of revoicing, repacing the discussion, modeling correct terminology, and questioning student reasoning. However, in light of the coding scheme, these teacher functions will increase a teacher's utterances and may not always be coded as facilitative/transactive (since they may be telling students how to modify their discussion, to slow down, and so forth). In sum, although Teacher B may use his utterances wisely in order to foster student understanding and conversation, the measures that showed Teacher A's discursive patterns and strengths may not highlight the strengths of Teacher B's pedagogy.

The "strings of student discussion" table for Teacher B is presented below. The four rows represent the aggregate of strings from his four proof lessons and two non-proof lessons.

Table 29. Teacher B, student string lengths, all lessons combined:

| | |
|---|---|
| Total number of strings, teacher B, proof: | 43 |
| Average string length: | 2.35 |
| Total number of strings, teacher B, non-proof: | 106 |
| Average string length: | 2.73 |

As noted above, a comparison with Teacher A's string lengths does not yield any striking information.  Teacher A's average string length for proof was 2.82 (longer than either type of lesson for Teacher B) but 2 for non-proof (shorter than either type of lesson for Teacher B).

Individual lessons

Teacher B, lesson 1:

This class wrote a proof together.  They also discussed the transitive, symmetric and reflexive properties of congruence and how they are used in proof.

Figure 23.  Teacher B, word cloud, teacher and students, lesson B1:

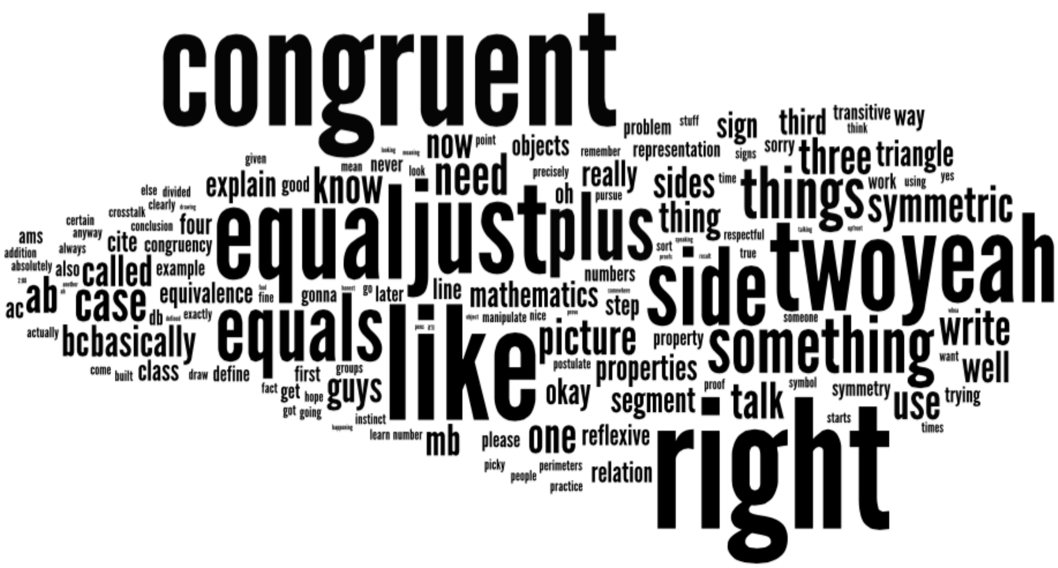


Figure 24. Teacher B, word cloud, teacher words only, lesson B1, not very different:

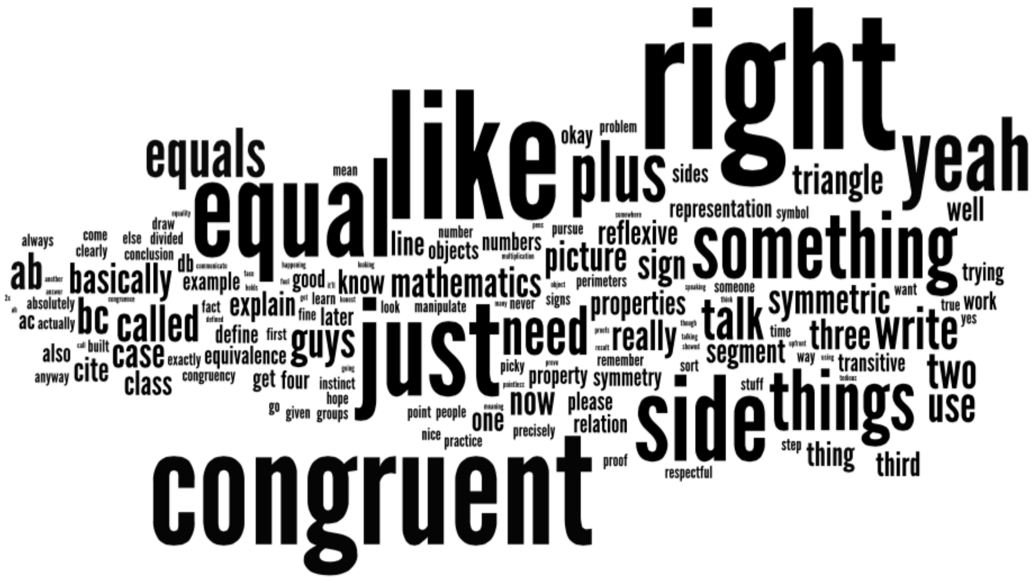



Table 30. Teacher B, word counts, lesson B1:

| |
|---|
| Teacher word count: 1497 |
| Student word count: 383 |
| T/S word ratio: 3.91 (79.6%) |

An example of a directive, although informative, speech is found in this lesson:

> *T:* Yeah. Yeah. So we need congruency to basically be something called an equivalence relation. We need it to basically be equal, right. These things are basically equal. In fact, they kind of are equal, but it's convention not to write objects – even though they are the same object, we don't write it with an equals sign. If it's numbers, we use equals signs; objects: congruent signs.
>
> But they're just saying, like, we can actually sort of manipulate objects, right. Like these two pens are identical. They are congruent. They're still congruent, right. If this was congruent to this, and this was congruent to a third, then the first one is congruent to the third. So they're just like trying to define, where you can say, "Congruence is –."

Teacher B frequently tries to convey truths about mathematics, and the way mathematics are done, to his students. Although he professes not to place the usage of correct mathematical terminology above, for example, students communicating directly with each other, he does take pains to present an accurate view of mathematics as illustrated by the following passage:

> T: It's just advanced mathematics is very, very picky about how you define things because if they're not picky, then later disaster happens as you build on things. Like, if the equal sign wasn't defined this way, and we said, like, two is equal to four divided by two, but four divided by two is not equal to two. That would be a problem in mathematics.

In this lesson, his students spend more time on PP/PI/CO/TQ/TR activity than GC/NC, although his instruction is more DD/DR than TA/FA.

Figure 25. Teacher B, bar graph of teacher and student utterances, lesson B1:

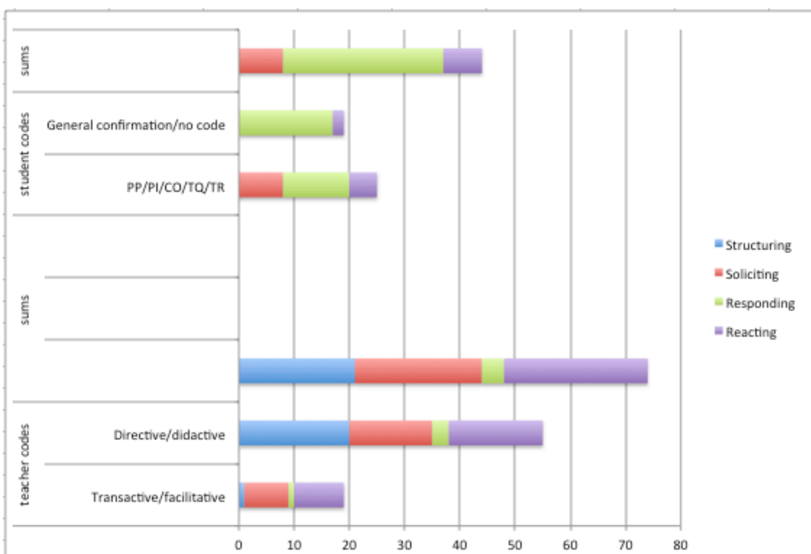


Table 31. Teacher B, student string lengths, lesson B1:

| B1 strings: | |
|---|---|
| TSS | 3 |
| TSSS | 3 |

The following table indicates a relatively high proportion of teacher questioning, student response, and teacher evaluation.

Table 32. Teacher B, T SOL/S RES/T REA vs. SOL/RES/REA other than T/S/T, lesson B1:

| | T SOL/S RES/T REA | SOL/RES/REA other than T/S/T | Ratio of columns 2 and 3 |
|---|---|---|---|
| B1 | 20.3% | 5.1% | 4.0 |

Teacher B, lesson 2:

This class was also engaged in writing geometric proof. They derived the theorem that, if two angles are congruent and supplementary, then they are right angles. They also engaged in discussion about the definitions of complementary and supplementary angles.

There is a five-length string of student discussion that begins with a transactive solicitation by the teacher that could be considered longer (eight) in light of the nature of the teacher's intervention, which is purely transactive. His last utterance, which is directive, ends the discussion:

> *T:* All right, *student*, ______. So what else do I know based on my—shh. *Student*, what else do I know based on my picture?
> *S:* That angle measure PQR is equal to measure—angle measure SQR.
> *S:* No it's not.
> *S:* No it's not.
> *S:* Wait, PQR?

*S:* The entire thing is equal to ______.
*T:* Shh—is equal to the measure of what?

*S:* Same thing as SQR.
*T:* Look, look, look—
*S:* I think I'm missing something.
*T:* PQR—
*S:* Oh, wait—sorry, sorry.
*T:* Here's SQR. It is equal to—SQR is in there—
*S:* Oh!
*[Crosstalk, laughter]*

Table 33. Teacher B, word counts, lesson B2:

| |
|---|
| Teacher word count: 1901 |
| Student word count: 637 |
| T/S word ratio: 2.98 (74.9%) |

Figure 26. Teacher B, word cloud, teacher and students, lesson B2:

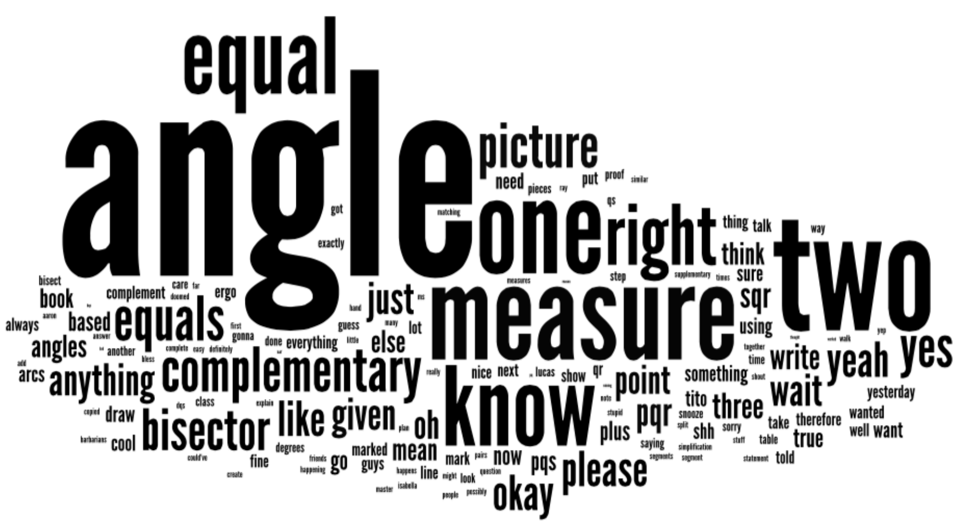
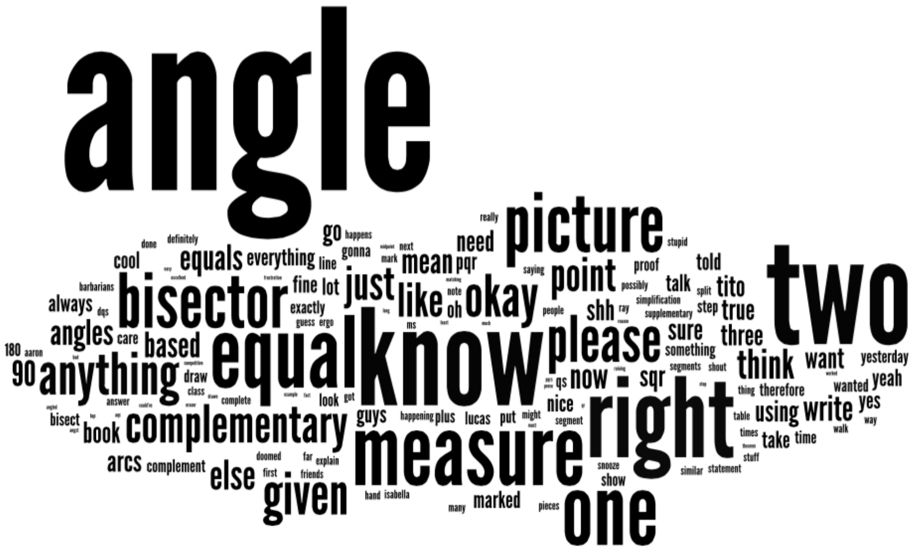


Figure 27. Teacher B, word cloud, teacher words only, lesson B2, very similar:




As in the first lesson, the students engage in mostly high-order interaction although the teacher is more directive/didactive than facilitative/transactive as shown in the following bar graph.

Figure 28. Teacher B, bar graph of teacher and student utterances, lesson B2:

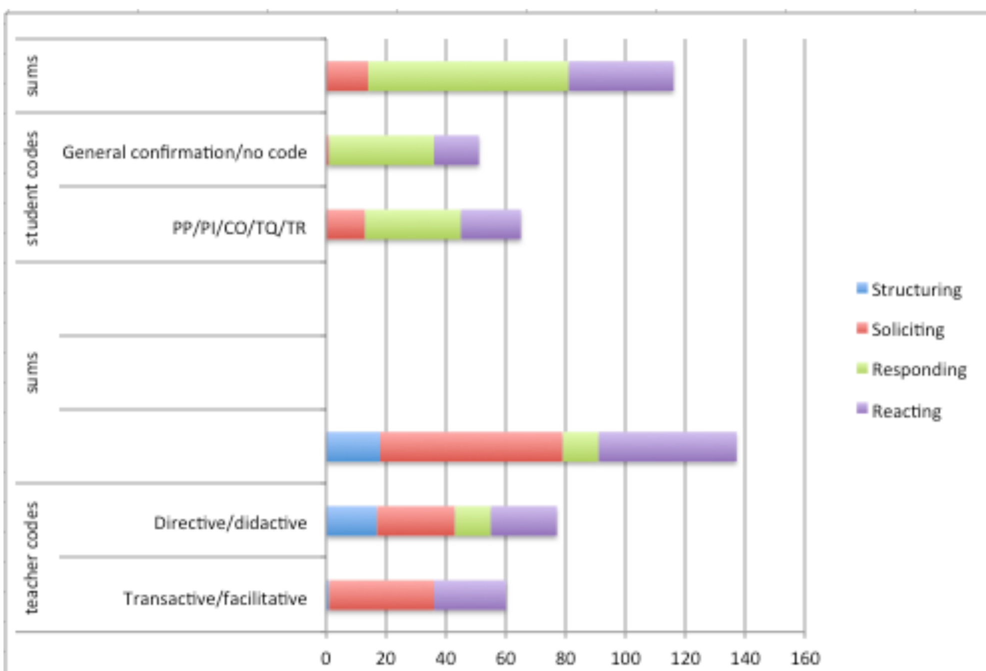


Table 34. Teacher B, student string lengths, lesson B2:

| B2 strings: | |
|---|---|
| TSS | 14 |
| TSSS | 5 |
| 4 S's | 1 |
| 5 S's | 1 |

The following table indicates a relatively low proportion of teacher questioning, student response, and teacher evaluation.

Table 35. Teacher B, T SOL/S RES/T REA vs. SOL/RES/REA other than T/S/T, lesson B2:

| | T SOL/S RES/T REA | SOL/RES/REA other than T/S/T | Ratio of columns 2 and 3 |
|---|---|---|---|
| B2 | 21.3% | 13.0% | 1.6 |

Teacher B, lesson 3:

This class continued to examine the definitions and properties of complementary and supplementary angle pairs. The class also considered the potentially transitive nature of such relationships as adjacency and equality. In the last section, they attempt to prove (in groups) that two angles that complement the same angle are congruent.

There are no long uninterrupted strings of student discussion although there are nine strings of length two.

The following passage contains more examples of Teacher B conveying didactive information about how mathematics is done:

> *T:* So in mathematics, if something does occur one time, do we say it's true, false every time?
> *S:* If it's not true, it's false.

*T:* Right. So adjacency does not apply – there is no transitivity with adjacency. So they don't hold, right. It works very nicely for equality. It equals ___, right.

But just because A and B are – angle B and angle – angle B and angle B are adjacent, angle B and angle C are adjacent, therefore angle A and angle C are adjacent.

*S:* I have a question.

*T:* Hold on. Hold on. I have a question for you. If angle A and angle B are adjacent, and we have angle B an angle C are adjacent, can I conclude that angle A and angle C are adjacent?

*S:* You would need a picture.

*T:* I would need a picture. So can I conclude this? Based on these statements, can I conclude that this is true? Remember in math we said it's true when?

*S:* When it's all *[inaudible].*

*T:* Right. So if sometimes if this is false, what do we say?

*S:* It's false.

*T:* So sometimes false, therefore always false. We don't like coincidences. We like set things that we can always say are true. It's what makes mathematics work. If we just worked with sometimes, then we wouldn't be dealing with that then. So do you want to draw a picture of where it could be true?

*S:* Yeah.

Table 36. Teacher B, word counts, lesson B3:

| |
|---|
| Teacher word count: 3381 |
| Student word count: 983 |
| T/S word ratio: 3.44 (77.5%) |

Figure 29. Teacher B, word cloud, teacher and students, lesson B3:

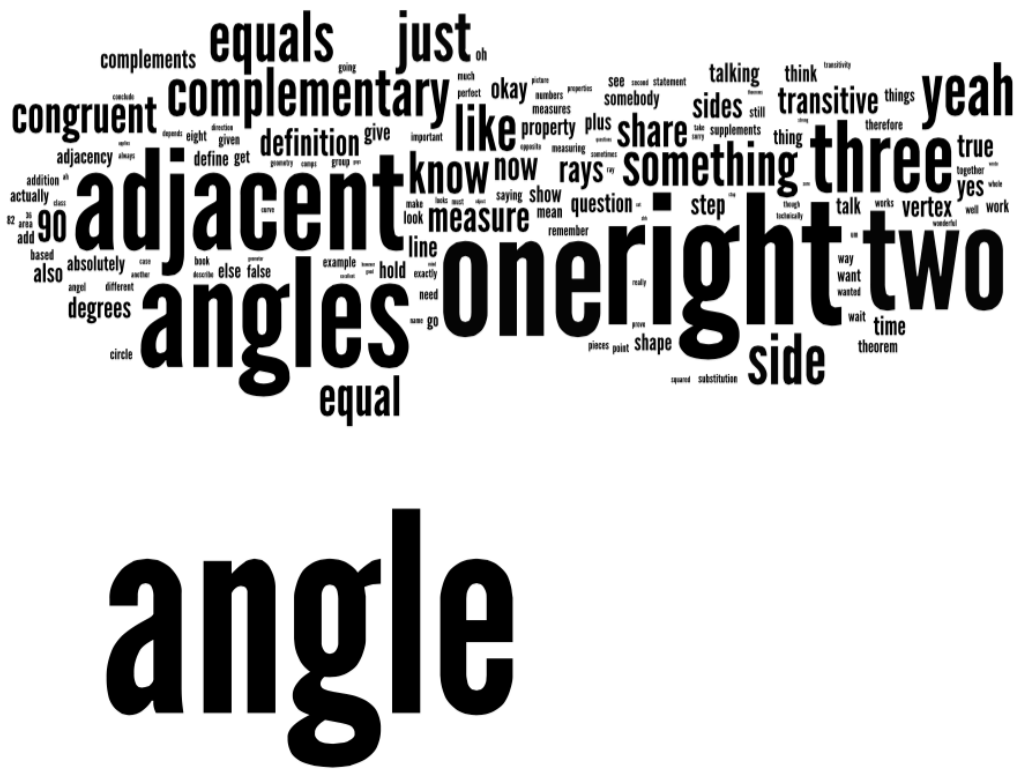
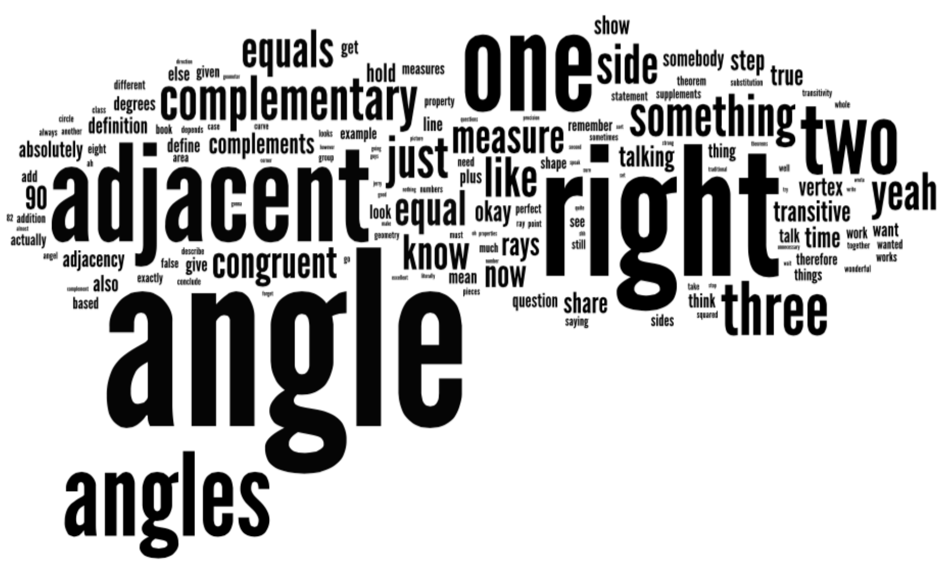



Figure 30. Teacher B, word cloud, teacher words only, lesson B3 (very similar):




As the bar graph shows, this is the one lesson in which Teacher B's students are more GC/NC than high-order. He also is more DD/DR than FA/TA.

Figure 31. Teacher B, bar graph of teacher and student utterances, lesson B3:

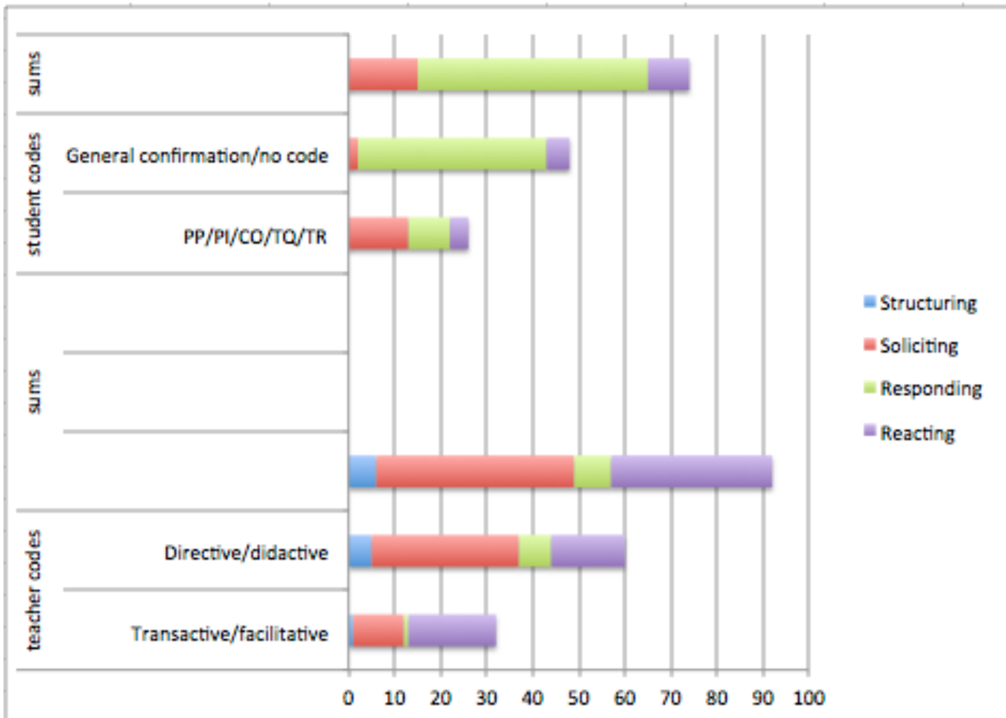



Table 37. Teacher B, student string lengths, lesson B3:

| B3 strings: | |
|---|---|
| TSS | 9 |

The following table indicates a relatively low-to-moderate proportion of teacher questioning, student response, and teacher evaluation.

Table 38. Teacher B, T SOL/S RES/T REA vs. SOL/RES/REA other than T/S/T, lesson B3:

| | T SOL/S RES/T REA | SOL/RES/REA other than T/S/T | Ratio of columns 2 and 3 |
|---|---|---|---|
| B3 | 19.9% | 7.2% | 2.8 |

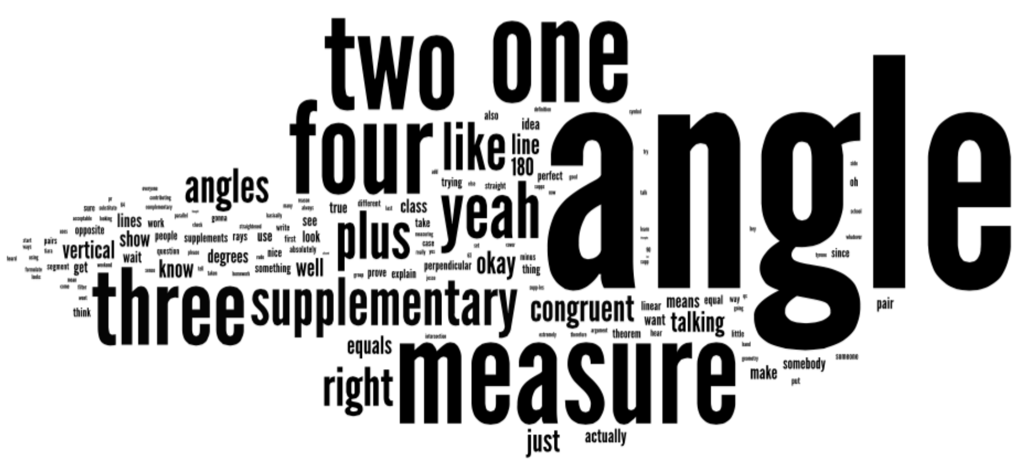
Teacher B, lesson 4:

This lesson begins with the proof of the theorem that vertical angles pairs are congruent using the idea of supplementary angles. There is a second proof done in groups.

Table 39. Teacher B, word counts, lesson B4:

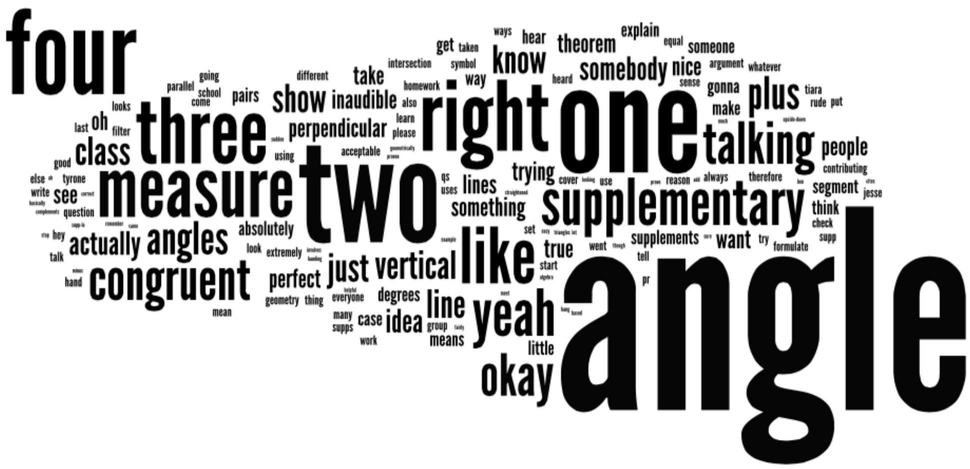
| |
|---|
| Teacher word count: 1340 |
| Student word count: 947 |
| T/S word ratio: 1.41 (58.6%) |

Figure 32. Teacher B, word cloud, teacher and students, lesson B4:




Below, the word "measure" is relatively smaller than in the previous cloud, but aside from that they are similar.

Figure 33. Teacher B, word cloud, teacher words only, lesson B4:




As the bar graph shows, the teacher is more transactive/facilitative than directive/didactive. This is the only lesson in the study in which this is true. In addition, the students are more higher-order than lower-order in their activity and the difference between the bars is the most pronounced of all such differences in the study. The percentage of the ratio of teacher/student utterances, 58.6%, is the lowest in all of Teacher B's lessons (proof or non-proof).

Figure 34.  Teacher B, bar graph of teacher and student utterances, lesson B4:

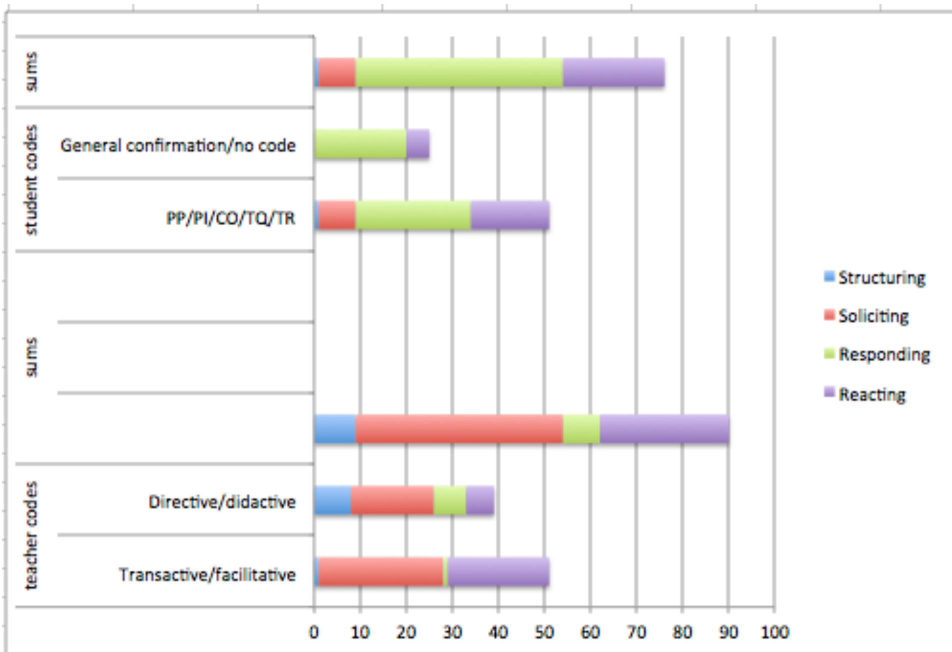



Table 40.  Teacher B, student string lengths, lesson B4:

| B4 strings: | |
|---|---|
| TSS | 6 |
| 4S's | 1 |

The following table indicates a relatively high proportion of teacher questioning, student response, and teacher evaluation.

Table 41. Teacher B, T SOL/S RES/T REA vs. SOL/RES/REA other than T/S/T, lesson B4:

| | T SOL/S RES/T REA | SOL/RES/REA other than T/S/T | Ratio of columns 2 and 3 |
|---|---|---|---|
| B4 | 32.5% | 7.2% | 4.5 |

Teacher B, lesson np1:

This lesson, as is the next, is composed of students and teacher matching equivalent factored and non-factored quadratic expressions. They do this numerically the first day and algebraically the second day (Bnp2).

"Plugging in" numbers to see if two expressions are equivalent does not generally lend itself to discussion, but, as per the excerpt below, Teacher B attempts to incorporate constructivist principles into every lesson.

Table 42. Teacher B, word counts, lesson Bnp1:

| |
|---|
| Teacher word count: 1232 |
| Student word count: 679 |
| T/S word ratio: 1.81 (64.5%) |

Figure 35. Teacher B, word cloud, teacher and students, lesson Bnp1:

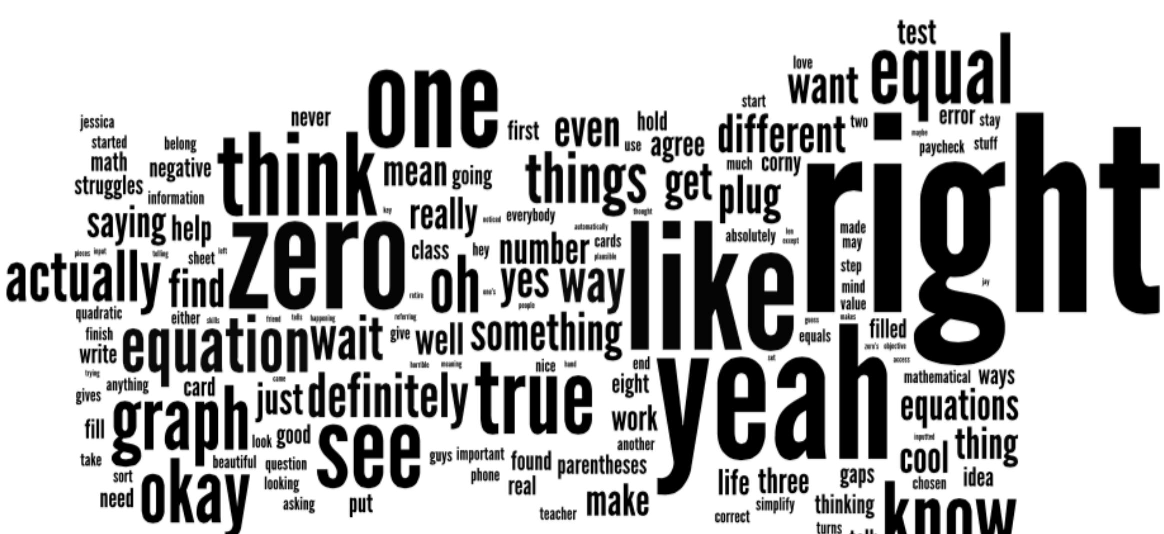
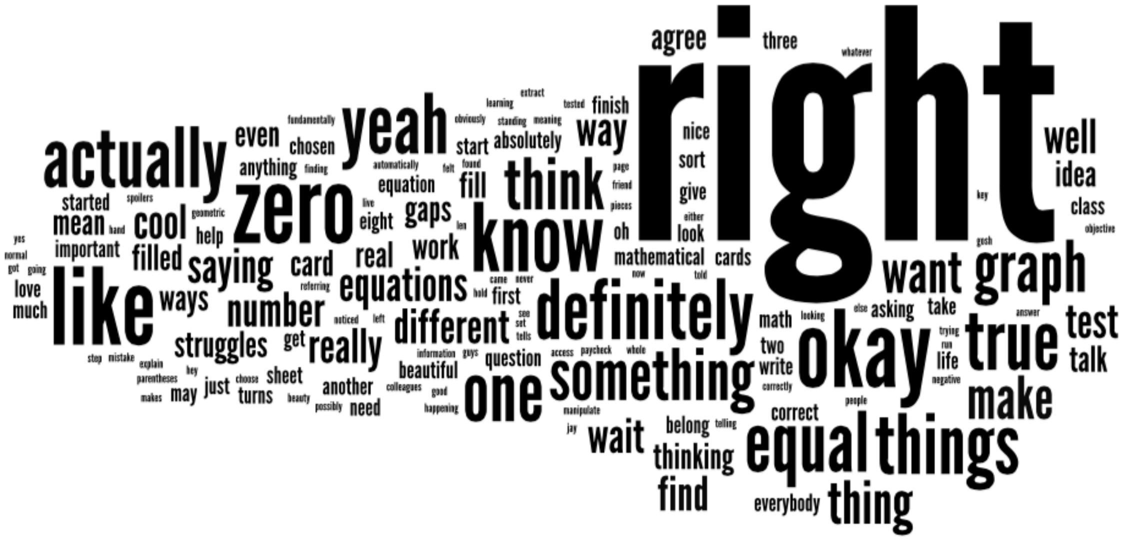


Figure 36. Teacher B, word cloud, teacher words only, lesson Bnp1 ("zero" and "like" are smaller; the word "thinking" appears):




There is a string of twelve uninterrupted student utterances ushered in by Teacher B stating that he made an error (introducing uncertainty as per Zaslavsky 2005, Wood 1999, Wood, Cobb, Yackel 1993; Cobb, Boufi et al. 1997; see literature review for more specifics). His opening statement is a transactive solicitation. The student-only exchange ends with Teacher B's directive solicitation.

*T:* What if I told you I made an error in one of these?
*S:* You'd have to find the error.

*T:* Did I?
*S:* Oh, I see it.
*S:* I see it.
*S:* Wait, hold up.
*S:* Yeah, I see it.
*S:* _____, please. Thank you.
*S:* It's going to be a hundred.
*S:* Oh my God.
*S:* Hold the phone.
*S:* Oh, I see it.
*S:* I'm looking.
*S:* Oh, I understand what you're doing.
*S:* Oh, I know. Okay, I see what you did there.
*T:* Raise your hand when you think you've found the answer.

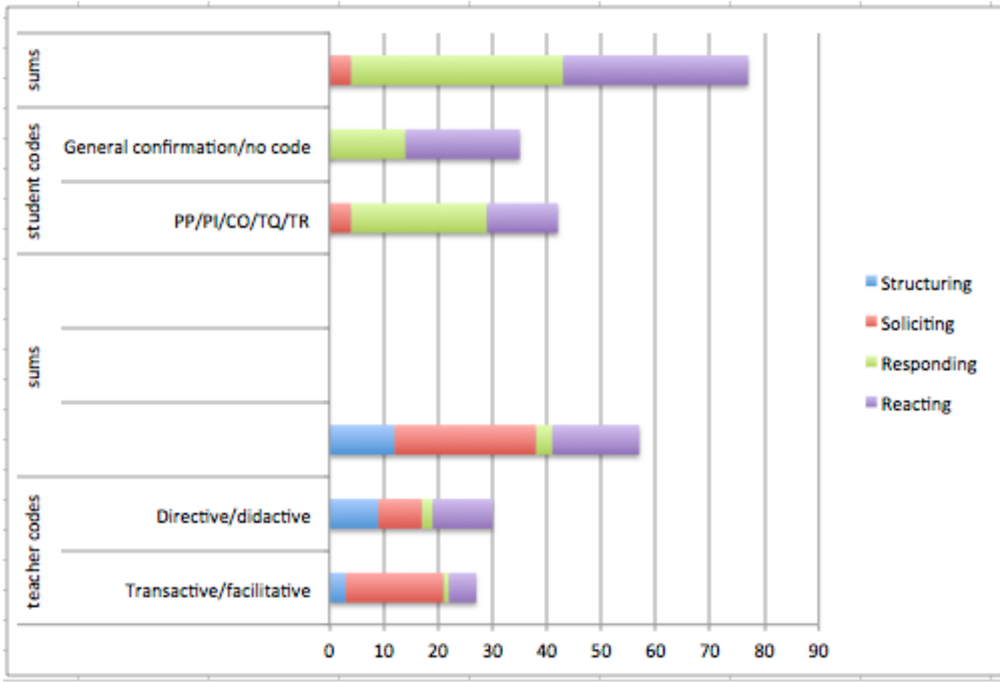
*S:* I think I've found it. I don't know.
*S:* I can't find it.
*S:* I thought I found it but I think it might be the right way to do it.
*T:* All right. S's thinking that actually I didn't make a mistake.
*S:* Oh, never mind.
*S:* Yeah, I think you did it right.
*S:* I think you did it right.
*S:* Never mind.
*T:* What if I'm telling you that I did make …
*S:* But I feel like if you did …
*S:* I don't think he made an error.
*[Crosstalk]*

Table 43. Teacher B, student string lengths, lesson Bnp1:

| Bnp1 strings: | |
|---|---|
| TSS | 15 |
| TSSS | 2 |
| 4 S's | 3 |
| 12 S's | 1 |

The bar graph indicates more higher-order student utterances than lower-order although Teacher B is slightly more directive/didactive than facilitative/transactive.

Figure 37. Teacher B, bar graph of teacher and student utterances, lesson Bnp1:




The following table indicates a low-to-moderate proportion of teacher questioning, student response, and teacher evaluation:

Table 44. Teacher B, T SOL/S RES/T REA vs. SOL/RES/REA other than T/S/T, lesson Bnp1:

| | T SOL/S RES/T REA | SOL/RES/REA other than T/S/T | Ratio of columns 2 and 3 |
|---|---|---|---|
| Bnp1 | 15.7% | 6.7% | 2.3 |

Teacher B, lesson np2:

This lesson is occupied with matching quadratic graphs for which the equations are presented in both factored and non-factored form. This is done algebraically. Teacher B motivates this by saying, “”If I don’t have access to a graphing utility and I don’t feel like drawing axes and plotting points, is there another way? Is there any other way to tell if they’re equal?”

Although the practice of showing that a quadratic can be presented in factored or non-factored form does not necessarily lend itself to the tenets of constructivism, Teacher B actively adheres to those tenets.

Table 45. Teacher B, word counts, lesson Bnp2:

| |
|---|
| Teacher word count: 5331 |
| Student word count: 2607 |
| T/S word ratio: 2.04 (67.2%) |

Figure 38.  Teacher B, word cloud, teacher and students, lesson Bnp2:


Figure 39.  Teacher B, word cloud, teacher words only, lesson Bnp2 (in which the word “right” is noticeably larger, but the word “negative” is noticeably smaller):


There is a string of 10 uninterrupted student utterances that begins with Teacher B's facilitative reaction and ends with another facilitative reaction. In general, Teacher B's "interruptions" are facilitative or transactive:

> *T:* Everyone here should be able to verify – even *you* – everyone here should be able to verify S's correctness or incorrectness.
> *S:* Negative…
> *T:* What, S?
> *S:* He's right.
> *T:* S proclaims!
> *S:* And I have reason to believe why.
>
> *T:* And thus… yeah.
> *S:* But how is the second one –?
> *S:* How is it right?
> *S:* Yeah, the second one, it doesn't –
> *S:* Oh, wait – never mind. I'm sorry. It's multiplication.
> *S:* Pass the baton.
> *S:* Oh, okay.
> *S:* Oh, that's a track thing.
> *S:* No, wait. It's not.
> *S:* You started on top.
> *S:* Put a negative mark –
> *T:* Do whatever you want.
>
> *S:* After the negative mark –
> *T:* Maybe if I use a marker…
> *S:* – and then, the negative one that's in front of the first square in parentheses, you put that in parentheses.
> *T:* Hold on. Are you clear on what S and S are saying?

*S:* No.
*S:* No.
*S:* No.
*T:* Like, S, I know you were visualizing, I'm sure, these things as you were saying them, but not everybody –
*S:* Did you understand this?
*T:* I had no idea. Could you put it on a board perhaps?
*S:* When you say "a board," do you mean *the* board or *a* board?
*T:* *A* board.
*S:* Oh, like, "a board."
*T:* Yeah, so that we can follow what you're saying.

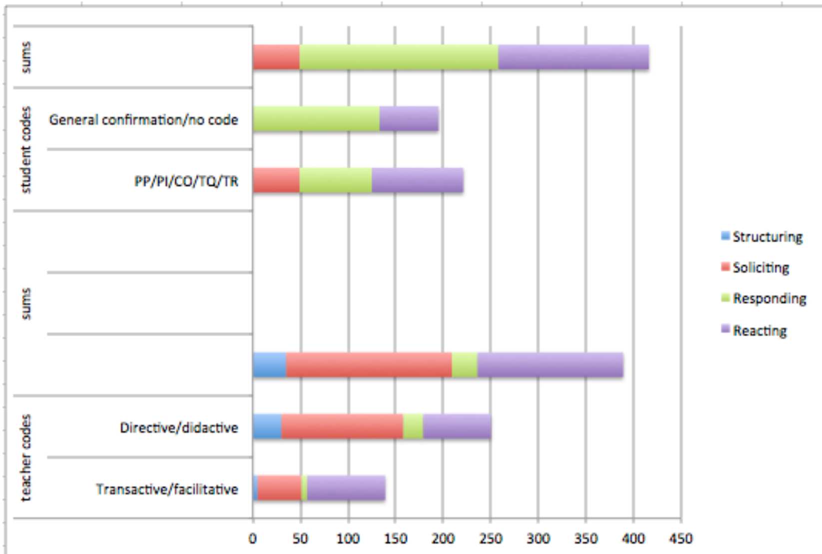
In sum, Teacher B tries to ensure that students are listening to each other's explanations and that they are constructing their own knowledge although he is not teaching proof, but rather a potentially algorithmic, automatic procedure.

Table 46. Teacher B, student string lengths, lesson Bnp2:

| Bnp2 strings: | |
|---|---|
| TSS | 50 |
| TSSS | 24 |
| 4 S's | 6 |
| 5 S's | 3 |
| 8 S's | 1 |
| 10 S's | 1 |

The bar graph shows that, on the whole, the students are engaged in slightly more higher-order activity than lower-order, although Teacher B is much more directive/didactive than facilitative/transactive.

Figure 40.  Teacher B, bar graph of teacher and student utterances, lesson Bnp2:




The following table indicates a low-to-moderate proportion of teacher questioning, student response, and teacher evaluation.

Table 47. Teacher B, T SOL/S RES/T REA vs. SOL/RES/REA other than T/S/T, lesson Bnp2:

| | T SOL/S RES/T REA | SOL/RES/REA other than T/S/T | Ratio of columns 2 and 3 |
|---|---|---|---|
| Bnp2 | 23.9% | 10.4% | 2.3 |

**Section 4: Statistical Comparison of Teachers A and B**

The data appear to show that Teacher A, an educator who does not identify all of her teaching as informed by constructivist ideology, teaches proof differently than the manner in which she teaches non-proof. In particular, pronounced differences between her own directive/didactive and transactive/facilitative utterances and the students' general confirmation/no code and PP/PI/CO/TQ/TR (these measure "low-order" and "high-order" student activity, respectively); teacher-led questioning-student response-teacher evaluation sequences; and her teacher/student word ratios can be shown to differ significantly when she is teaching proof vs. not teaching proof.

By comparison, these same measures show that Teacher B's teaching does not differ significantly depending on subject matter (proof vs. non-proof). Teacher B identifies himself as pedagogically informed by constructivist principles no matter what he is teaching.

Table 48. Teacher A, differences in teacher and student codes:

| A | nonproof | proof | z-value | p-value |
|---|---|---|---|---|
| teacher codes | 0.240409207 | 0.380149813 | -4.50 | p<.0001 |
| student codes | 0.417293233 | 0.644859813 | -6.12 | p<.0001 |

These statistical analyses were performed using one-tailed z-tests. The p-values for Teacher A indicates that the difference in her teaching is not likely to happen by chance.

Table 49. Teacher B, differences in teacher and student codes:

| B | nonproof | proof | z-value | p-value |
|---|---|---|---|---|
| teacher codes | 0.372197309 | 0.41221374 | -1.18 | .1190000000 |
| student codes | 0.53346856 | 0.538709677 | -0.14 | 0.444329995 |

Teacher B, on the contrary, shows a high probability of difference due to chance; therefore, it can be concluded that his teaching is not significantly different according to subject matter.

The z-scores show Teacher A's teacher utterances are 4.5 and 6.12 standard deviations apart respectively.  Teacher B's are 1.18 and .14. The first set are significantly different; the second are not.

It can also be shown that Teachers A and B show great difference in the way they teach content other than proof, but no great statistical difference when teaching proof:

Table 50.  Teachers A and B: Differences from each other in teaching of non-proof vs. proof content, teacher codes:

| | p_A | p_B | z-value | p-value |
|---|---|---|---|---|
| nonproof | 0.240409207 | 0.372197309 | -4.11 | p<.0001 |
| proof | 0.380149813 | 0.41221374 | -0.99 | 0.16108706 |

Examination of the common SOL/RES/REA sequence yields the information that Teacher A is more likely to feature the traditional teacher solicitation-student response-teacher reaction (as opposed to SOL/RES/REA from other than teacher/student/teacher) when she is not teaching proof; for Teacher B, this is not the case.

Table 51. Teachers A and B: Differences in the occurrence of T SOL/S RES/T REA vs. SOL/RES/REA other than T/S/T:

| | p-value | $x^2$ value |
|---|---|---|
| Teacher A | p<.0001 | 36.0000 |
| Teacher B | .487 | .48 |

For these data, a chi-square test with a 2x2 matrix was utilized. As above, the p-values show Teacher A exhibits great variation in the teaching of proof vs. non-proof whereas Teacher B does not.

Table 52. Teachers A and B: Differences in word ratios:

| | Non-proof | Proof | z-value | p-value |
|---|---|---|---|---|
| Teacher A | 0.404870624 | 0.500467727 | -3.87 | p<.0001 |
| Teacher B | 0.525026624 | 0.440967283 | 3.37 | 0.999624159 |

As in the first two analyses, a one-tailed z-test was employed. As before, Teacher A exhibits huge variation between the teaching of proof and not of proof, whereas Teacher B does not show a lot of variation.  It is worth noting, however, that Teacher B actually talks less when not teaching proof.

## Chapter V: Summary, Conclusions, Implications and Suggestions for Future Study

### Section 1: Summary and conclusions

The purpose of this study is to analyze the patterns of classroom discourse when high school students move from performing prescribed algorithms in order to solve problems for which the process and solution are well-defined to spoken proof, in which ideas are discussed and arguments are formulated and formalized.

The study uses a modified version of discourse analysis developed by Arno Bellack and refined for usage in a mathematics classroom by James T. Fey. Bellack and Fey's framework categorizes utterances by students or teacher as having one of four purposes: structuring, soliciting, responding, or reacting. Their analysis framework is supplemented by codes borrowed from Maria Blanton, Despina Stylianou, and M. Manuela David (2009), which is in turn a modified version of a coding system developed by Kruger (1993) and Goos, Galbraith and Renshaw (2002). Blanton, et al.'s codes were adopted to describe the nature of a teacher's or student's utterance. Their scheme characterizes a teacher's utterance as directive/didactive, or transactive/facilitative in nature. The latter teacher codes are seen to seek student contribution to discussion whereas the former are designed to evoke a factual (likely brief) response. The students' utterances are characterized as proposal of a new idea, proposal of a new plan, contribution to an existing idea, transactive questioning, or transactive response; these are seen as evidencing higher-order thinking. Student responses coded as "general confirmation" or "no code" denote factual responses (they may be seen as analogous to the directive/didactive code cited above).

Twelve mathematics lessons involving two mathematics teachers were recorded, transcribed and coded. Eight of the lessons were classified as "proof-related" and four were designated "non-proof-related." A lesson designated "proof-related" contained more than half activity that was actively concerned with the construction of proof; whereas a lesson during which no proofs were formulated was designated "non-proof." Using the codes described above and a variety of qualitative and quantitative measures, the transcripts were examined for constructivist behavior on the part of the teachers and modes of participation on the students' part.

The data in the previous section show that, on the whole, Teacher A (compared with Teacher B) achieves lower teacher/student word ratios, a lower ratio of teacher-led questioning-student response-teacher evaluation sequences, more pronounced differences between the general confirmation/no code and PP/PI/CO/TQ/TR (these measure "low-order" and "high-order" student activity, respectively) bars and higher average student string length when she is teaching proof. However, the data also suggest that she does not teach in as constructivist a manner when she is not teaching proof. By her own admission, she regards the teaching of proof as fundamentally different from the teaching of non-proof. Her belief's effect on her practice is borne out by data and statistical analysis.

As noted, the data show that Teacher B's proof/non-proof classes do not bear the sharp distinction found in Teacher A's. His intention – to provide constructivist instruction at all times – may or may not be realized, but data-based statistical examination of his instruction show that his approach is not determined by subject matter to the same degree as Teacher A's, and that, in the default case of non-proof, his instruction is statistically distinct from Teacher A's, whereas their teaching in the proof case is statistically non-distinct.

These findings suggest a relationship between a teacher's beliefs in constructivist principles and the way in which that teacher instructs proof vs. non-proof. More specifically, a teacher who views her/himself as informed by constructivist pedagogical principles may not evince a sharp distinction between her/his teaching of proof vs. non-proof; whereas a teacher who does not attempt to incorporate constructivist principles on a daily basis may exhibit more constructivist tendencies when teaching proof.

**Section 2: Comparison of the results to other studies**

Unfortunately, it is difficult to compare all of the results from the current study with previous studies, but some areas lend themselves to comparison, specifically word ratios, which may be compared to Bellack, Fey and Offenholley's results; and the proportions of Blanton's codes of teacher and student utterances.

Word ratio:

Teacher A's word ratio table (Table 6, reproduced below) bears comparison to the findings of Bellack, Fey and Offenholley. These percentages cannot compete with Offenholley's range of 0% to 44.3%; but the percentages of teacher speech found during proof lessons compare favorably to Fey and Bellack's teacher word count percentages of 71% and 75%, respectively. The "no proof" class percentages of 83.6% and 80.7% clearly show a greater percentage of teacher speech than those found in Fey and Bellack.

Table 6. Teacher A, teacher/student word ratio, all lessons:

| | Teacher A | Students | T/S word ratio | % teacher speech |
|---|---|---|---|---|
| Lesson 1 - proof | 1422 | 746 | 1.91 | 65.6% |
| Lesson 2 - proof | 1220 | 656 | 1.86 | 65.0% |
| Lesson 3 - proof | 3614 | 2324 | 1.56 | 60.9% |
| Lesson 4 - proof | 1517 | 1752 | .87 | 46.4% |
| | | | | |
| Lesson 1 - no proof | 2776 | 545 | 5.09 | 83.6% |
| Lesson 2 - no proof | 5677 | 1354 | 4.19 | 80.7% |

In contrast with Teacher A, Teacher B shows much less difference in teacher/student ratio of speech. In fact, on average, the ratio of teacher to student speech is better when he is not teaching proof (72.65% vs. 65.85%).

Table 27: Teacher B, teacher/student word ratio, all lessons:

| | Teacher B | Students | T/S word ratio | % teacher speech |
|---|---|---|---|---|
| Lesson 1 - proof | 1497 | 383 | 3.91 | 79.6% |
| Lesson 2 - proof | 1901 | 637 | 2.98 | 74.9% |
| Lesson 3 - proof | 3381 | 983 | 3.44 | 77.5% |
| Lesson 4 - proof | 1340 | 947 | 1.41 | 58.6% |
| | | | | |
| Lesson 1 - no proof | 1232 | 679 | 1.81 | 64.5% |
| Lesson 2 - no proof | 5331 | 2607 | 2.04 | 67.2% |

It was observed that Teacher B's lessons contained some relatively long didactive speeches but their removal did not show much effect.

Table 53. Teacher B, teacher/student word ratio all lessons after DD/DR speeches removed:

| | Teacher B | Students | T/S word ratio | % teacher speech |
|---|---|---|---|---|
| Lesson 1 - proof | 1076 | 383 | 2.81 | 73.7% |
| Lesson 2 - proof | 1714 | 637 | 2.69 | 72.9% |
| Lesson 3 - proof | 3105 | 983 | 3.16 | 76.0% |
| Lesson 4 - proof | 1288 | 947 | 1.36 | 57.6% |
| | | | | |
| Lesson 1 - no proof | 1086 | 679 | 1.60 | 61.5% |
| Lesson 2 - no proof | 5105 | 2607 | 1.96 | 66.2% |

Now we will compare Teacher B's results to those found by Bellack, Fey and Offenholley. As above, Teacher B's percentages cannot compete with Offenholley's range of 0% to 44.3%; but are still comparable to Fey and Bellack's teacher word count percentages of 71% and 75%, respectively. The "no proof" lesson percentages of 61.5% and 66.2% show slightly less teacher speech than what is found in the data of Fey and Bellack.

In sum, neither Teacher A nor Teacher B was able to achieve the level of student participation found by Offenholley, although their ratios compare favorably (although not markedly so) with those found by Bellack and Fey.

Also worth mentioning is the relative unimportance, in this study, of "string lengths," which proved crucial in Offenholley's study. It may bear further investigation to see why online instruction engenders uninterrupted student exchanges, whereas it appears that classroom instruction does not to any great extent.

Proportion of teacher and student utterances:

The percentages found in the current study do not stand up well in comparison with those found by Blanton, et al.

Teacher utterances:

Table 3. Blanton, et al. (2009), frequency of forms of instructional scaffolding (teachers)

**Table 17.1** Frequency of forms of instructional scaffolding (teachers)

| *Type of teacher utterance* | *Frequency* | *Percentage* |
|---|---|---|
| Transactive prompt | 37 | 40 |
| Facilitative utterance | 43 | 47 |
| Directive utterance | 7 | 8 |
| Didactive utterance | 5 | 5 |
| **Total** | **92** | **100** |

Blanton, et al. 298.

The percentage sum of TR/FA is 87% and the sum for DR/DD is 13%.

By comparison, in Teacher A's teaching the breakdowns were as follows.

Table 54. Teacher A, percentage of TR/FA vs. DR/DD:

| | Proof | Non-proof |
|---|---|---|
| Percentage of TR/FA | 38.0% | 24.0% |
| Percentage of DR/DD | 62.0% | 76.0% |

Teacher B did not fare much better.

Table 55. Teacher B, percentage of TR/FA vs. DR/DD:

| | Proof | Non-proof |
|---|---|---|
| Percentage of TR/FA | 41.2% | 37.2% |
| Percentage of DR/DD | 58.8% | 62.8% |

Student utterances:

The proportion of student utterances in the current study also fared poorly in comparison with Blanton, et al.

Table 4. Blanton, et al. (2009), frequency of forms of instructional scaffolding (students):

**Table 17.2** Frequency of forms of instructional scaffolding (students)

| *Type of student utterance* | *Evidence of ZPD access* | *Frequency* | *Percentage* |
|---|---|---|---|
| Proposal of a new idea | Yes | 3 | 4 |
| Proposal of a new plan | Yes | 6 | 8 |
| Contribution to an existing idea | Yes | 29 | 40 |
| Transactive response | Yes | 7 | 10 |
| Transactive questions | Yes | 6 | 8 |
| General confirmations | Not necessarily | 9 | 13 |
| No code | No | 12 | 17 |
| **Total** | | **72** | **100** |
| **Total utterances indicating ZPD access** | | **51** | **70** |

Ibid., 302.

Blanton, et al.'s sums for PI/PP/CO/TQ/TR are 70%; only 30% are GC or NC, as compared to the following:

Table 56. Teacher A, percentage of PI/PP/CO/TQ/TR vs. GC/NC:

| | Proof | Non-proof |
|---|---|---|
| Percentage of PI/PP/CO/TQ/TR | 41.2% | 37.2% |
| Percentage of GC/NC | 58.8% | 62.8% |

Table 57. Teacher B, percentage of PI/PP/CO/TQ/TR vs. GC/NC:

| | Proof | Non-proof |
|---|---|---|
| Percentage of PI/PP/CO/TQ/TR | 41.2% | 37.2% |
| Percentage of GC/NC | 58.8% | 62.8% |

However, in light of the fact that Blanton, et al. were conducting research in a college classroom dedicated to the study of proof (presumably for mathematics majors and other self-directed mathematics students), perhaps the comparison is not as bad as it appears.

**Section 3: Implications for teacher training and practice**

It is as critical as it has ever been to examine the ways in which teachers may be brought to teach in a more student-centered way, at least occasionally. Several states have adopted frameworks for evaluating teacher practice; many of these are informed by constructivist principles. Many states, New York among them, have adopted Charlotte Danielson's framework for teaching which presupposes a student-centered ideology. However, although many states have chosen a student-centered framework by which to judge teachers, it is not generally in accord with the way in which teachers are currently accustomed to teaching.

This study has attempted to show that one of the variables that is not generally considered when trying to introduce student-centeredness into classrooms is the content of lessons or courses. It is hoped that this study has demonstrated that, for example, the inclusion of proof in a geometry curriculum may afford even relatively traditionally-oriented teachers the opportunity to use discussion-based learning techniques in their classrooms. On the other hand, it is possible that it is not appropriate to teach all content in a constructivist manner. Although Teacher B has shown that it is possible to teach proof and content other than proof in the same relatively constructivist way, Teacher A's decision not to do so may indeed show sound professional judgment, particularly in light of standardized assessments that tend to measure mastery of techniques, not understanding of concepts.

In addition to considering content areas in which teachers may be able to introduce inquiry-based techniques, some researchers have other specific suggestions of how to do so on a more regular basis. Not surprisingly, Stylianou and Blanton (2011) suggest teachers use "transactive prompts" instead of directive solicitations in order to encourage student engagement. Various other studies (Hirsch and Lappan 1989, Arzarello et al. 2012, Dawkins 2012, van Dormolen 1991, Zaslavsky 2005, Wood 1999 and others) suggest the usage of manipulatives, dynamic software, metaphors and even the deliberate seeding of uncertainty in order to stimulate student thinking and exploration.

As shown in the literature review of this study, it is not a small task to begin teaching in a constructivist manner. As Civil (2002) points out, most students are not likely to embrace an inquiry-based model without being "won over." In general, it is not the way they are used to learning mathematics and they will likely exhibit suspicion at a situation in which the teacher is no longer expected to arbitrate mathematical "truth." Yackel, Rasmussen and King (2000 cite the necessity of reconstructing social classroom norms in order to enculturate this type of discussion and exploration. In other words, the establishment of a classroom as a place in which students are expected to construct their own understanding by working together under the guidance of a teacher is just that: a process, which can be difficult and time-consuming.

As Nathan and Knuth (2003) observe, teachers should receive training in order to begin teaching in accordance with these reforms. As they state, "the challenge is particularly daunting given that mathematics teacher education and professional development programs typically have not adequately prepared [teachers] to enact successfully the lofty expectations set forth in reform documents (Ross 1998)" (203-4). As Nathan and Knuth further observe, continued support is also helpful; it is not sufficient for administrations to expose teachers to the concepts of constructivism and leave it at that since at least one teacher who believed she was incorporating constructivist principles was not doing so to the extent she believed herself to be. It required a quantitative use of data and a video camera to show her the ways in which she was missing opportunities for constructivist activity. (It should also be noted that the teacher in question was interested in developing the inquiry basis of her instruction.)

Yet another important note is that embracing the principles of constructivism is not equivalent to a teacher abdicating all responsibility, leaving the establishment of all goals and activities to the students. As Zaslavsky et al. (2012), Lobato, Clarke and Ellis (2005), Sfard (2000), Clarke (1994), Romagnano (1994) and others have indicated, the teacher must be experienced enough to know when s/he should stand back, when s/he should intervene, and when take an action somewhere in the middle. As Lobato, Clarke and Ellis eloquently summarize the findings of several researchers, "it can be helpful [for a teacher] to state facts, share ideas, or identify conflicts, and then examine the sense that students make of them. In addition, introducing new information at critical junctures could help reduce the number of problem features that students must attend to, thus allowing for exploration in new areas…[S]tudents cannot be expected to reinvent entire bodies of mathematics, regardless of how well each concept is problematized by well-chosen tasks (Clarke 1994; Romagnano 1994). Teachers are expected to enculturate students into the mathematics community, sharing conventional norms associated with mathematical discourse, representation, and forms of argument (Becker & Varela 1995; Cobb & Yackel 1996; Driver 1995). If teachers are to facilitate this enculturation, then making the ideas and conventions of the community available to students is essential. From this perspective, some information must be introduced by the teacher. In short, a telling/not-telling dilemma has emerged. Telling is instructionally important, but has been downplayed due to both perceived inconsistencies with constructivism and historical attempts to develop pedagogical implications of constructivism" (Ibid.).

Finally, perhaps it would be productive to consider differences between the standards to which public and private school students are accustomed. The private school students observed in the current study did not seem to question the standards-based classroom, nor did they appear to question the validity of their own argumentation. This may be due to their having been placed in the hands of well-trained educators who knew how to introduce student-centered activity, but it is also worth mentioning that the private school students may have been more used to being given relatively free-rein in educational settings because of smaller class sizes and other variables. In addition, private school students are not subject to the same battery of standardized assessments as public school students. In other words, expecting teachers in public school settings to begin teaching their classes in a constructivist manner may present an even bigger set of challenges than one might expect.

However, as Wood, Cobb and Yackel (1991) observed, second graders can be brought to learn in this manner, given an adequately-trained and properly-supported teacher. It is possible that, if school districts and teacher training programs are willing to consider the challenges of teaching teachers how to consider incorporating constructivist principles into their ideology and are also committed to training and supporting teachers properly, and if state assessment bodies align test material more closely to the curricula and standards to which teachers are bound, perhaps inquiry-based instruction can be accomplished more broadly and meaningfully than it has been in the past.

**Section 4: Suggestions for future study**

There are a few ways in which this study was limited in scope and content. Most obvious is the size of the data set, which was described in the methodology section of this study as typical of studies in which discourse analysis is employed; but this does not mean that this study and others wouldn't benefit from the collection, coding and analysis of more data.

In addition, this study does not examine the effects of gender of students (or teacher) on the phenomenon at hand although one set of students was mixed gender and the other was female only.  Despite this confounding variable, this study's results are strong enough to warrant future study; a different study could consider gender as a mitigating factor.

The way in which lessons were considered either inclusive of proof activity (more than half) or not was satisfactory for the scope and depth of the current study, but it does raise a question of the degree to which teachers are likely to modify their instruction based on the title of the course.  For example, a teacher teaching geometric proof vs. the midpoint formula in the context of a geometry syllabus may not vary her daily instruction as much as she would between a course called Geometric Proof vs. one called Advanced Algebra. There are few courses in most high schools with such specific designations (as opposed to colleges, in which an entire course on proof is common); but if such courses were given in high schools, this question could foster future study.

Finally, there remain vast areas of inquiry regarding the relative efficacy of constructivist vs. objectivist teaching methods.  Do students who have learned by developing their own ideas and by taking full part in a community of thinkers perform better on standardized tests such as the NAEP, New York State Regents exams, or the SAT? Will they perform better on the PARCC and SBAC tests being introduced this year and in subsequent years?

It is naïve to assume that students will perform better on standardized tests without objectivist training, the purpose of which is to bring students to mastery of most standardized tests.  Even so, there may still be other, long-lasting benefits to a discussion-based model of instruction.  For example, is there a measurable difference in students' level of comfort when engaged in mathematics while they are in high school, and if so, does this effect persist in later years?  Are students who learn mathematics in this way more likely to become mathematics majors in college? If research can show that a constructivist approach is successful in convincing students that mathematics is not an impermeable morass of formulae but rather a real set of tools with which to apprehend the physical and intellectual world, then, as an approach, it has merit.

**Works Cited**


Arsac, G., Balacheff, N., & Mante, M. (1992). Teacher's role and reproducibility of didactical situations. *Educational Studies In Mathematics*, 23(1), 5-29. Springer.

Arsac, G. (2007). Origin of mathematical proof: History and epistemology. In P. Boero (Ed.), *Theorems in schools: From history epistemology and cognition to the classroom practice* (pp. 27–42). Rotterdam: Sense Publishers.

Arzarello, F., Bussi, M., Leung, A., Mariotti, M., & Stevenson, I. (2012). Experimental approaches to theoretical thinking: artefacts and proofs. Hanna, G., & De Villiers, M. (2012). *Proof and proving in mathematics education* (1st ed.). 97-143. Dordrecht: Springer.

Atkins, Andrew (2001). Sinclair and Coulthard's 'IRF' model in a one-to-one classroom: an analysis. Retrieved December 26, 2014 from < http://r.search.yahoo.com/_ylt=A0LEViiT2Z1UE_QA5v4PxQt.;_ylu=X3oDMTByMG04Z2o2BHNlYwNzcgRwb3MDMQRjb2xvA2JmMQR2dGlkAw--/RV=2/RE=1419659795/RO=10/RU=http%3a%2f%2fwww.birmingham.ac.uk%2fDocuments%2fcollege-artslaw%2fcels%2fessays%2fcsdp%2fAtkins4.pdf/RK=0/RS=dHpk7njmi7Au.HWf5CCkfHA3kFk->.

Balacheff, N. (1991). Benefits and limits of social interaction: The case of teaching mathematical proof. *Mathematical knowledge: Its growth through teaching*, Bishop A., Mellin-Olsen S., Van Dormolen J., eds. (pp.175-192). Dordrecht : Kluwer Academic Publisher.

Becker, J., & Varela, M. (1995). Assisting construction: the role of the teacher in assisting the learner's construction of preexisting cultural knowledge. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp.433-446). Hillsdale, NJ: Erlbaum.

Bellack, A.A.; Kliebard, H.M.; Hyman, R.T. & Smith, F.L. (1966). *The language of the classroom*. New York: Teachers College Press.

Blanton, Maria, Despina Stylianou, and M. Manuela David (2009). Understanding Instructional Scaffolding in Classroom Discourse on Proof. In Stylianou, D., Blanton, M., & Knuth, E. (2009). *Teaching and learning proof across the grades* (1st ed.). New York: Routledge.

Boaler, J. (1997). *Experiencing school mathematics* (1st ed.). Buckingham: Open University Press.

Bowers, J., Cobb, P., & McClain, K. (1999). The evolution of mathematical practices: A case study. *Cognition And Instruction*, 17(1), 25-66.

Bowers, Janet S., and Susan Nickerson (2001). Identifying Cyclic Patterns Of Interaction To Study Individual And Collective Learning. *Mathematical Thinking and Learning* 3.1 (2001): 1-28. Web.

Brahier, D.J. (2005). *Teaching secondary and middle school mathematics.* Boston: Allyn & Bacon.

Burton, L. (1998). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies In Mathematics*, 37(2), 121-143. Springer.

Cabassut, R., Conner, A., Iscimen, F., Furinghetti, F., Jahnke, H., & Morselli, F. (2012). Conceptions of Proof. In Hanna, G., & De Villiers, M. (2012). *Proof and proving in mathematics education* (1st ed.). Dordrecht: Springer.

Campbell, Patricia F., and Josepha Robles (1997). "Project IMPACT: Increasing the Mathematical Power of All Children and Teachers." In *Reflecting on Our Work: NSF Teacher Enhancement in K-6 Mathematics*, edited by Susan N. Friel and George W. Bright, 179-86. Lanham, Md.: University Press of America, 1997.

Carpenter, T.P and Franke, M.L. (1998). Teachers as Learners. *Principled Practice,* 2, 1-3.

Cazden, Courtney B. (1988). *Classroom Discourse*. New Hampshire: Heinemann. ISBN 0-435-08445-3.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies In Mathematics*, 24(4), 359-387. Springer.

Civil, M. (2002). Chapter 4: Everyday Mathematics, Mathematicians' Mathematics, and School Mathematics: Can We Bring Them Together? *Journal For Research In Mathematics Education.* Monograph, 40-62.

Clarke, D.J. (1994). Why Don't We Just Tell Them? Beesey, C., & Rasmussen, D. (1994). *Mathematics without limits* (1st ed.) 11-19. Brunswick, Vic.: Mathematical Association of Victoria.

Cobb, Paul; Terry Wood, Erna Yackel, John Nicholls, Grayson Wheatley, Beatriz Trigatti, and Marcella Perlwitz (1991). Assessment of a Problem-Centered Second Grade Mathematics Project. *Journal for Research in Mathematics Education* 22 (January 1991): 3-29.

Cobb, P., Boufi, A., McClain, K., & Whitenack, J. (1997). Reflective discourse and collective reflection. *Journal For Research In Mathematics Education*, Vol. 28, No. 3 (May 1997), 258-277.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, 31(3-4), 175-190.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2011). Participating in classroom mathematical practices. *The Journal of the Learning Sciences*, Vol. 10, No. 1/2, 117-163. Taylor & Francis, Ltd.

Coe, R., & Ruthven, K. (1994). Proof practices and constructs of advanced mathematics students. *British Educational Research Journal*, 20(1), 41-53.

Cohen, J. (1960). A coefficient for nominal scales. *Educational and Psychological Measurement,* 20:37.

Common Core State Standards Initiative. Standards for Mathematical Practice. Retrieved on July 24, 2014 from <http://www.corestandards.org/Math/Practice/>.

Confrey, J. (1990). Chapter 8: What constructivism implies for teaching. *Journal For Research In Mathematics Education. Monograph*, 107-210.

Conner, A., & Kittleson, J. (2009). Epistemic understandings in mathematics and science: Implications for learning. *In Mathematics Education*, Vol. 1, 106.

D'Ambrosio, B.; Kastberg, S.; Viola dos Santos, J. Learning Student Approaches to Algebraic Proofs. *The Mathematics Teacher*, Vol. 103, No. 7 (March 2010), pp. 489-495. NCTM.

Dawkins, P. (2012). Metaphor as a possible pathway to more formal understanding of the definition of sequence convergence. *The Journal Of Mathematical Behavior*, 31(3), 331-343.

deGroot, Cornelius (2001). From Description to Proof. *Mathematics Teaching in the Middle School*, Vol. 7, No. 4 (December 2001), pp. 244- 248. NCTM.

Dreyfus, T., Nardi, E., & Leikin, R. (2012). Forms of Proof and Proving in the Classroom. In Hanna, G., & De Villiers, M. (2012). *Proof and proving in mathematics education* (1st ed.). Dordrecht: Springer.

Driver, R. (1995). Constructivist approaches to science teaching. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 385-400). Hillsdale, NJ: Erlbaum.

Durand-Guerrier, Viviane; Boero, Paolo; Douek, Nadia; Epp, Susanna; Tanguay, Denis (2012). Argumentation and Proof in the Mathematics Classroom. In Hanna, G., & De Villiers, M. (2012). *Proof and proving in mathematics education* (1st ed.). Dordrecht: Springer.

Elbers, Ed (2003). Classroom Interaction as Reflection: Learning and Teaching Mathematics in a Community of Inquiry. *Educational Studies in Mathematics*, Vol. 54, No. 1 (2003), pp. 77-99. Springer.

Epp, S. (1999). The language of quantification in mathematics instruction. In Stiff, L., & Curcio, F. (1999). *Developing mathematical reasoning in grades K-12* (1st ed.). 188-197. Reston, Va.: National Council of Teachers of Mathematics.

Epp, S. (2003). The Role of Logic in Teaching Proof. *The American Mathematical Monthly*, Vol. 110, No. 10 (December 2003), pp. 886-899. Mathematical Association of America.

Fawcett, Harold P. (1938, 1995). *The Nature of Proof.* Bureau of Publications, Teachers College, Columbia University (1938). Reprinted, *Thirteenth Yearbook of the National Council of Teachers of Mathematics* (1995). Reston, VA: National Council of Teachers of Mathematics.

Fey, J. (1966). *Patterns of verbal communication in mathematics classrooms*. New York: Teachers College Press.

Franke, M. L., Kazemi, E. & Battey, D. (2007). Mathematics teaching and classroom practice. In Lester, F. K. Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 225-256). Charlotte, NC: Information Age Publishing.

Frasier, B. J. (2014). Improving how secondary mathematics teachers are prepared to teach proof. European Journal of Educational Sciences, 2 (2), 1-7.

Gee, James Paul (2011). *How to do Discourse Analysis: A Toolkit*. New York: Routledge.

Gee, James Paul (1999, third edition 2011). *An Introduction to Discourse Analysis: Theory and Method*. New York: Routledge.

Geertzen, J. (2012). *Inter-Rater Agreement with multiple raters and variables*. Retrieved July 26, 2014, from <https://mlnl.net/jg/software/ira/>.

Generazzo, Susan D. (2011). *Proof and reasoning in an inquiry-oriented class: The impact of classroom discourse* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses. (Accession Order No. AAT 3467351)

González, G., & Herbst, P. (2013). An Oral Proof in a Geometry Class: How Linguistic Tools Can Help Map the Content of a Proof. *Cognition And Instruction*, 31(3), 271-313.

Goos, M., Galbraith, P., & Renshaw, P. (2002). Socially mediated metacognition: Creating collaborative zones of proximal development in small group problem solving. *Educational Studies In Mathematics*, 49(2), 193-223. Springer.

Hanna, G. (2000). A critical examination of three factors in the decline of proof. Interchange 31(1), 21-33.

Hanna, G., & Barbeau, E. (2008). Proofs as bearers of mathematical knowledge. *The International Journal on Mathematics Education*, 40(3), 345-353. ZDM.

Hanna, G., & Jahnke, H. (1993). Proof and application. *Educational Studies In Mathematics*, 24(4), 421-438. Springer.

Healy, L. & Hoyles, C. (2000). A Study of Proof Conceptions in Algebra. *Journal for Research in Mathematics Education*, Vol. 31, No. 4 (July 2000), pp. 396-428. NCTM.

Hemmi, K., & Löfwall, C. (2009). Why do we need proof? *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education*. WG 2. January 28th-February 1st 2009, Lyon (France). Institut National de Recherche Pedagogique.

Herbst, P. & Brach, C. (2006). Proving and Doing Proofs in High School Geometry Classes: What Is It That Is Going On for Students? *Cognition and Instruction*, 24 (1), 73-122. Lawrence Erlbaum Associates, Inc.

Herbst, P. (2002). Engaging students in proving: A double bind on the teacher. *Journal For Research In Mathematics Education*, Vol. 33, No. 3 (May 2002), 176-203.

Hersh, Reuben (1993). Proving is Convincing and Explaining. *Educational Studies in Mathematics*, Vol. 24, No. 4, Aspects of Proof (1993), pp. 389- 399. Springer.

Hershkowitz, R., & Schwarz, B. (1999). Reflective processes in a mathematics classroom with a rich learning environment. *Cognition And Instruction*, *17*(1), 65-92.

Hirsch, C. & Lappan, G. (1989). Implementing the "Standards": Transition to High School Mathematics. *Mathematics Teacher*, Vol. 82 (8), November 1989, pp. 614-618. NCTM.

Housman, D., & Porter, M. (2003). Proof schemes and learning strategies of above-average mathematics students. *Educational Studies In Mathematics*, 53(2), 139-158. Springer.

Hoyles, C., & Kuchemann, D. (2002). Students' understandings of logical implication. *Educational Studies In Mathematics*, 51(3), 193-223. Springer.

Kamii, Constance. *Young Children Continue to Reinvent Arithmetic*. New York: Teachers College Press, 1994.

King, J., Orihuela, Y. and Robinson, E. (2010). *Focus in High School Mathematics: Reasoning and Sense Making in Geometry*. NCTM.

Knuth, E. (2002). Secondary School Teachers' Conceptions of Proof. *Journal for Research in Mathematics Education*, Vo. 33, No. 5 (Nov. 2002), pp. 379-405. NCTM.

Knuth, E., & Elliott, R. (1998). Characterizing students' understandings of mathematical proof. *The Mathematics Teacher*, Vol. 91, No. 8 (November 1998), pp. 714-717. NCTM.

Kruger, A. (1993). Peer collaboration: conflict, cooperation, or both? *Social Development*, *2*(3), 165-182.

Lehet, B. (2013). *The value of the clinical interview in interpreting elementary mathematics assessment results.* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses. (Accession Order No. AAT 3590539)

Leron, U. (1985). A direct approach to indirect proofs. *Educational Studies In Mathematics*, 16(3), 321-325. Springer.

Lobato, J., Clarke, D., & Ellis, A. (2005). Initiating and eliciting in teaching: A reformulation of telling. *Journal For Research In Mathematics Education*, Vol. 36, No. 2 (March 2005), 101-136. NCTM.

Lucast, E. K. (2003). Proof as Method: A New Case for Proof in Mathematics Curricula. Unpublished masters thesis. Pittsburgh, PA, USA: Carnegie Mellon University.

Martin, T., McCrone, S., Bower, M., & Dindyal, J. (2005). The interplay of teacher and student actions in the teaching and learning of geometric proof. *Educational Studies In Mathematics*, *60*(1), 95-124. Springer.

McClain, K. (2002). Teacher's and students' understanding: The role of tools and inscriptions in supporting effective communication. *Journal Of The Learning Sciences*, 11(2-3), 217-249.

Miyakawa, T., & Herbst, P. (2007). Geometry teachers' perspectives on convincing and proving when installing a theorem in class. In Lamberg, T., & Wiest, L. R. (Eds.). (2007). *Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* Stateline (Lake Tahoe), NV: University of Nevada, Reno, 110-117.

Moore, R. (1994). Making the transition to formal proof. *Educational Studies In Mathematics*, 27(3), 249-266. Springer.

Nathan, M., & Knuth, E. (2003). A study of whole classroom mathematical discourse and teacher change. *Cognition And Instruction*, 21(2), 175-207.

National Council of Teachers in Mathematics, 1980. An Agenda for Action: Recommendations for School Mathematics of the 1980's. Retrieved on July 25, 2014 from < http://www.nctm.org/standards/content.aspx?id=17278>.

National Council of Teachers in Mathematics, 1989. Curriculum and Evaluation Standards for School Mathematics. Retrieved on July 25, 2014 from < http://www.nctm.org/standards/content.aspx?id=26629>.

National Council of Teachers in Mathematics, 1991. Professional Standards for Teaching Mathematics. Retrieved on July 25, 2014 from < http://www.nctm.org/standards/content.aspx?id=26628>.

National Council of Teachers in Mathematics, 2000. Principles and Standards for School Mathematics Online. Retrieved on July 25, 2014 from < http://www.nctm.org/standards/content.aspx?id=26792>.

Offenholley, K.H. (2007). *A Discourse Analysis of the Online Mathematics Classroom*. Teachers College, Columbia University.

Pape, Stephen; Bell, Clare; Yetkin, Iffet Elif (2003). Developing Mathematical Thinking and Self-Regulated Learning: A Teaching Experiment in a Seventh-Grade Mathematics Classroom. *Educational Studies in Mathematics,* Vol. 53, No. 3, 179-202. Springer.

Penlington, C. (2008). Dialogue as a catalyst for teacher change: A conceptual analysis. *Teaching And Teacher Education*, 24(5), 1304-1316.

Peressini, D., Borko, H., Romagnano, L., Knuth, E., & Willis, C. (2004). A conceptual framework for learning to teach secondary mathematics: A situative perspective. *Educational Studies In Mathematics*, 56(1), 67-96. Springer.

Pimm, David (1987) *Speaking Mathematically: Communication in Mathematics Classrooms*, Routledge & Kegan Paul Ltd., London

Quinn, Frank (2012). Contemporary Proofs for Mathematics Education. *Proof and proving in mathematics education* (1st ed.). 97-143. Dordrecht: Springer.

Rogers, R., Malancharuvil-Berkes, E., Mosley, M., Hui, D., & Joseph, G. (2005). Critical Discourse Analysis in Education: A Review of the Literature. *Review Of Educational Research*, 75(3), 365-416.

Romagnano, L. (1994). *Wrestling with change* (1st ed.). Portsmouth, NH: Heinemann.

Ryve, Andreas (2006). Making Explicit the Analysis of Students' Mathematical Discourses: Revisiting a Newly Developed Methodological Framework. *Educational Studies in Mathematics*, Vol. 62, No. 2 (Jun., 2006), pp. 191-209.

Ryve, Andreas (2011) Discourse Research in Mathematics Education: A Critical Evaluation of 108 Journal Articles. *The Journal for Research in Mathematics Education* (JRME): March 2011, Volume 42, Issue 2.

Schoenfeld, Alan H. (1980). Teaching problem-solving skills. *American Mathematical Monthly*, Vol. 87, No. 10 (Dec., 1980), pp. 794-805. MAA.

Schoenfeld, Alan H. (1991) On Mathematics as Sense-Making: An Informal Attack On the Unfortunate Divorce of Formal and Informal Mathematics. *Informal Reasoning and Education*, James F. Voss, David N. Perkins, and Judith W. Segal, eds. Lawrence Erlbaum Associates, Inc., 1991, pp. 311-343.

Schoenfeld, Alan H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 334-370). New York: MacMillan.

Schoenfeld, Alan H. (1994). Reflections on doing and teaching mathematics*. Mathematical thinking and problem solving*, (Studies in Mathematical Thinking and Learning Series). 53-75. Erlbaum, Hillsdale, NJ.

Schoenfeld, Alan H. (2002). A highly interactive discourse structure. *Advances In Research On Teaching*, 9, 131-170.

Schoenfeld, Alan H. (2009). Series Editor's Foreword in Stylianou, D., Blanton, M., & Knuth, E. (2009). *Teaching and learning proof across the grades* (1st ed.). New York: Routledge.

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies In Mathematics*, 29(2), 123-151. Springer.

Selden, J., & Selden, A. (2013). Proof and Problem Solving at University Level. *The Mathematics Enthusiast*, ISSN 1551-3440, Vol. 10, nos.1&2, pp.303-334. The Author(s) and Dept. of Mathematical Sciences -- The University of Montana.

Sfard, Anna (2000). On Reform Movements and the Limits of Mathematical Discourse. *Mathematical Thinking and Learning*, 2:3, 157-189.

Shepherd, Michael Andrew (2010). *A discourse analysis of teacher-student classroom interactions* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses. (Accession Order No. AAT 3418169)

Skidmore, D., & Murakami, K. (2012). Claiming our own space: Polyphony in teacher-student dialogue. *Linguistics And Education*, 23(2), 200-210.

Solomon, Yvette (2006). Deficit or Difference? The Role of Students' Epistemologies of Mathematics in Their Interactions with Proof. *Educational Studies in Mathematics*, Vol. 61, No. 3 (Mar., 2006), pp. 373-393. Springer.

Sowder, L. & Harel, G. (1998). Types of Students' Justifications. *The Mathematics Teacher*, Vol. 91, No. 8 (November 1998), pp. 670-675. NCTM.

Stylianides, G., & Stylianides, A. (2009). Facilitating the transition from empirical arguments to proof. *Journal For Research In Mathematics Education*, Vol. 40, No. 3 (May, 2009), pp. 314- 352. NCTM.

Stylianou, D., Blanton, M., & Knuth, E. (2009). *Teaching and learning proof across the grades* (1st ed.). New York: Routledge.

Stylianou, D., & Blanton, M. (2011). Developing Students' Capacity for Constructing Proofs through Discourse. *Mathematics Teacher*, 105(2), 140-145.

Toulmin, S. E. (2003). *The uses of argument* (Updated edition). New York, NY: Cambridge University Press.

van Dormolen, Joop. Metaphors Mediating the Teaching and Understanding of Mathematics. *Mathematical Knowledge: Its Growth through Teaching*, edited by Alan J. Bishop, Stieg Mellin-Olsen, and Joop van Dormolen, pp. 89-106. Dordrecht, Netherlands: Kluwer Academic Press, 1991.

Vosniadou, S., Baltas, A., & Vamvakoussi, X. (2007). *Re-framing the conceptual change approach in learning and instruction* (1st ed.). Amsterdam: Elsevier, in association with the European Association for Learning and Instruction.

Weinberg, Stuart A. (2010). *Student Teachers' Epistemological Stance, Expectations of Pupil Success, and Instructional Practice in Mathematics*. Teachers College, Columbia University.

Willingham, D. (2012) Yes, Algebra is Necessary. Retrieved on July 25, 2014 from < http://www.danielwillingham.com/daniel-willingham-science-and-education-blog/yes-algebra-is-necessary>.

Wood, Terry (1999). Creating a Context for Argument in Mathematics Class. *Journal for Research in Mathematics Education,* Vol. 30, No. 2 (Mar., 1999), pp. 171- 191. NCTM.

Wood, T., Cobb, P., & Yackel, E. (1991). Change in teaching mathematics: A case study. *American Educational Research Journal*, 28(3), 587-616.

Wood, T., Cobb, P., & Yackel, E. (1993). Chapter 6: The Nature of Whole-Class Discussion. *Journal For Research In Mathematics Education. Monograph*, Vol. 6, Rethinking Elementary School Mathematics: Insights and Issues (1993), pp. 55-68 and 115-122
55-122.

Wu, H. (1996). The role of Euclidean geometry in high school. *The Journal Of Mathematical Behavior*, 15(3), 221-237.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal For Research In Mathematics Education*, 458-477.

Yackel, E., Rasmussen, C., & King, K. (2000). Social and sociomathematical norms in an advanced undergraduate mathematics course. *The Journal Of Mathematical Behavior*, 19(3), 275-287.

Zaslavsky, O. (2005). Seizing the opportunity to create uncertainty in learning mathematics. *Educational Studies In Mathematics*, 60(3), 297-321. Springer.

Zaslavsky, O.; Nickerson, S.D.; Stylianides, A.J.; Kidron, I.; Winicki-Landman, G. (2012). The Need for Proof and Proving: Mathematical and Pedagogical Perspectives. *Proof and proving in mathematics education* (1st ed.). 97-143. Dordrecht: Springer.

## **Works Consulted**

Derrida, Jacques (1973) *Speech and Phenomena and Other Essays on Husserl's Theory of Signs*, Northwestern University Press, Evanston, IL

Fowler, D.H. (1973) *Introducing Real Analysis*, Transworld, London.

Gasparov, Boris (1978). Ustnaya Rech' Kak Semioticheskij Ob'ekt. *Semantika Nominatsij i Semiotika Ustnoj Rechi: Lingvisticheskaya Semantika i Semiotika*, Issue 442, pp. 63-121.

Gee, J. P. (1996). *Social linguistics and literacies: Ideology in discourses* (2nd ed). London: Falmer Press.

Iser, Wolfgang (1972) The Reading Process: A Phenomenological Approach. *New Literary History*, Vol. 3, No. 2, On Interpretation: I (Winter, 1972), pp. 279-299.

Jakobson, Roman (1960) Linguistics and Poetics. *Style in Language*, New York, Wiley and Sons

Jakobson, Roman (1990) *On Language*, Harvard University Press, Cambridge, Massachusetts.

Lakatos, I. (1963). Proofs and refutations (IV). *British Journal For The Philosophy Of Science*, Vol. 14, No. 56 (February 1964), 296-342.

Nachlieli, Talli and Herbst, Patricio; with Gloriana Gonzalez (2009). Seeing a Colleague Encourage a Student to Make an Assumption While Proving: What Teachers Put in Play When Casting an Episode of Instruction. *Journal for Research in Mathematics Education*, Vol. 40, No. 4, 427-459. NCTM.

Prus-Wisniowska, Ewa Anna (1995). *Cognitive, metacognitive, and social aspects of mathematical proof with respect to calculus* (Doctoral dissertation). Syracuse University, New York, United States. Retrieved December 6, 2011, from Dissertations & Theses: Full Text.(Publication No. AAT 9616296).

Ryve, A. (2006). Making explicit the analysis of students' mathematical discourses--Revisiting a newly developed methodological framework. *Educational Studies In Mathematics*, 62(2), 191-209. Springer.

de Saussure, Ferdinand (1966; orig. copyright 1959) *Course in General Linguistics,* McGraw-Hill Book Company, United States of America

Shklovsky, Victor (1917). Art as Technique. *Russian Formalist Criticism: Four Essays*, Lee T. Lemon and Marion J. Reis, eds. University of Nebraska Press, 1965, pp. 3-24.

Uebersax, John.  Kappa Coefficients: A Critical Appraisal.  Retrieved on August 1, 2014 from < http://www.john-uebersax.com/stat/kappa.htm>.

Wetherell, M., Taylor, S., & Yates, S. J. (Eds.). (2001a). *Discourse theory and practice: A reader.* London: Sage.

Zaiontz, Charles.  Real Statistics Using Excel.  Retrieved on August 1, 2014 from < http://www.real-statistics.com/reliability/cohens-kappa/>.

## Appendix A - Instructions for Coders

This study utilizes a framework developed by Bellack, et al. (1966) and Fey (1966) to code pedagogical "moves" in a classroom. The first part of any code is the source of the "move"; these are T (teacher) or S (student). Then the framework codes interactions as structuring (STR), soliciting (SOL), responding (RES) or reacting (REA) ("Codes"). In addition, this study subcategorizes teacher utterances as transactive, facilitative, directive or didactive; and student utterances as a proposal of a new idea, proposal of a new plan, contribution to an existing idea, transactive response, transactive question, general confirmation, or with no code in accordance with Blanton, et al. (2009). For more discussion of these subcategories, please see "subcategories" below.

**Codes (purpose of utterance):**

STR: defined as a move that sets the context for activity. It generally initiates or redirects the discussion. It also includes a teacher demonstrating, declaring or administering behavioral prompts. Some examples:

T-STR: *"Now let's talk about how to find the midpoint of a line segment."*

T-STR: *"Here's how you find the circumcenter of a circle."*

S-STR: *"Wait! I have a totally different approach."*

SOL: is a move that solicits a response. It is generally followed by RES, a response. Some examples:

T-SOL: *"What do you think are the coordinates of this segment's midpoint?"*

T-SOL: *"Can you explain why you did that?"*

S-SOL: *"How should I start this problem?"*

RES: is always a response to a SOL move. Some examples:

S-RES: *"I think the circumcenter is at (3,2)."*

T-RES: *"Maybe you should use one of the givens first."*

REA: is a response to something that was said earlier or a continuation of an ongoing process. It is never a response to a SOL. Some examples:
S-REA: *"I agree with ________. The circumcenter is at (3,2)."*
T-REA: *"I think your approach is a good starting point."*

**Subcategories (nature of utterance):**

For teachers:

Transactive (TA) statements are intended to provoke students' own reasoning, elaboration, justification and so forth. Example: *"Can I have some more examples of that?"*

Facilitative (FA) statements reinforce (sometimes, just revoice) a student's idea or structure discussion. Example: *"So I'm hearing, we should start with a given statement."*

Directive (DR) statements provide immediate feedback or by providing information directly. Questions are not discussion-provoking. A question with a one-word answer is probably directive. Example: *"You find that by adding the x-coordinates and dividing by two."*

Didactive (DD) statements reinforce the teacher's position as authority figure by presenting non-negotiative information "on the nature of mathematical knowledge (Blanton, et al. 2009). Example: *"That's the kind of thing you'll need to know as a math major."*

For students:

Proposal of a new idea (PI). Example: *"We should try bringing the angle bisectors into the proof." "How about using different triangles?"*

Proposal of a new plan (PP). Example: *"How about using SAS on these other triangles?"*

Contribution to an existing idea (CO). Example: *"Then use the vertical angles!"*

Transactive question (TQ). is generally a request for clarification, explanation, and so forth. Example: *"Why did you use that segment?"*

Transactive response (TR) is generally a response to a request for clarification, explanation, and so forth. Example: *"I used it because it was shared by two triangles."*

General confirmation (GC). Example: *"I completely agree."*

No code (NC). Example: *"Two."*

**Example of double coding:**

The following example encompasses everything (purpose and nature):

*T: So a perpendicular bisector, now -- the bisector -- like the thing that is doing the bisecting can be a line. It can be a ray. Or it can be a segment. That's the thing that's doing the bisecting. But what is actually being bisected can only be what? What did we decide?*

*S: Segment.*

*T: A segment. So a perpendicular bisector is a segment, ray or line or it can be a plane but we aren't really focusing too much on planes. Okay? We're going to deal more in 2-D. The segment, ray or line that is A -- perpendicular, so forms right angles and B -- bisects or forms two congruent segments.*

*S: I just -- what does it mean that is A perpendicular? Oh, oh, sorry, the A. Okay. But that has the definition like --*

*T: Yes but I --*

*S: Okay.*

*T: -- clarified for it.*

*S: Yeah.*

*S: Yes.*

is coded as follows:

T-STR-DR, T-SOL-DR
S-RES-NC
T-REA-DR, T-STR-DR
S-SOL-TQ
T-RES-DR
S-REA-GC
T-RES-DR
S-REA-GC
S-REA-GC

## Appendix B – Raw data spreadsheets

Single lessons by teacher:

Teacher A, proof:

| A1 | | Structuring | Soliciting | Responding | Reacting |
|---|---|---|---|---|---|
| teacher codes | Transactive/facilitative | | 33 | 0 | 14 |
| | Directive/didactive | 5 | 24 | 5 | 26 |
| **sums** | | **5** | **57** | **5** | **40** |
| | | | | | |
| | | | | | |
| student codes | PP/PI/CO/TQ/TR | 0 | 12 | 19 | 21 |
| | General confirmation/no code | | 1 | 23 | 7 |
| **sums** | | **0** | **13** | **32** | **28** |

| A2 | | Structuring | Soliciting | Responding | Reacting |
|---|---|---|---|---|---|
| teacher codes | Transactive/facilitative | 1 | 17 | 2 | 4 |
| | Directive/didactive | 7 | 9 | 15 | 14 |
| **sums** | | **8** | **26** | **17** | **18** |
| | | | | | |
| | | | | | |
| student codes | PP/PI/CO/TQ/TR | 0 | 22 | 16 | 11 |
| | General confirmation/no code | | 0 | 13 | 4 |
| **sums** | | **0** | **22** | **29** | **15** |

| A3 | | Structuring | Soliciting | Responding | Reacting |
|---|---|---|---|---|---|
| teacher codes | Transactive/facilitative | 2 | 82 | 6 | 30 |
| | Directive/didactive | 24 | 37 | 48 | 45 |
| **sums** | | **26** | **119** | **54** | **75** |
| | | | | | |
| | | | | | |
| student codes | PP/PI/CO/TQ/TR | 2 | 63 | 66 | 49 |
| | General confirmation/no code | | 1 | 66 | 35 |
| **sums** | | **2** | **64** | **132** | **84** |

| A4 | | Structuring | Soliciting | Responding | Reacting |
|---|---|---|---|---|---|
| teacher codes | Transactive/facilitative | 1 | 3 | 2 | 6 |
| | Directive/didactive | 13 | 21 | 24 | 14 |
| **sums** | | **14** | **24** | **26** | **20** |
| | | | | | |
| | | | | | |
| student codes | PP/PI/CO/TQ/TR | 0 | 28 | 2 | 11 |
| | General confirmation/no code | | 7 | 19 | 14 |
| **sums** | | **0** | **35** | **21** | **25** |

Teacher A, non-proof:

| Anp1 | | Structuring | Soliciting | Responding | Reacting |
|---|---|---|---|---|---|
| teacher codes | Transactive/facilitative | 0 | 8 | 1 | 1 |
| | Directive/didactive | 19 | 45 | 18 | 31 |
| **sums** | | **19** | **53** | **19** | **32** |
| | | | | | |
| | | | | | |
| student codes | PP/PI/CO/TQ/TR | 0 | 13 | 5 | 1 |
| | General confirmation/no code | | 3 | 49 | 12 |
| **sums** | | **0** | **16** | **54** | **13** |

| Anp2 | | Structuring | Soliciting | Responding | Reacting |
|---|---|---|---|---|---|
| teacher codes | Transactive/facilitative | 1 | 45 | 0 | 38 |
| | Directive/didactive | 29 | 75 | 29 | 51 |
| **sums** | | **30** | **120** | **29** | **89** |
| | | | | | |
| | | | | | |
| student codes | PP/PI/CO/TQ/TR | 1 | 34 | 39 | 18 |
| | General confirmation/no code | | 0 | 72 | 19 |
| **sums** | | **1** | **34** | **111** | **37** |

Teacher B, proof:

| B1 | | Structuring | Soliciting | Responding | Reacting |
|---|---|---|---|---|---|
| teacher codes | Transactive/facilitative | 1 | 8 | 1 | 9 |
| | Directive/didactive | 20 | 15 | 3 | 17 |
| **sums** | | **21** | **23** | **4** | **26** |
| | | | | | |
| | | | | | |
| student codes | PP/PI/CO/TQ/TR | 0 | 8 | 12 | 5 |
| | General confirmation/no code | | | 17 | 2 |
| **sums** | | **0** | **8** | **29** | **7** |

| B2 | | Structuring | Soliciting | Responding | Reacting |
|---|---|---|---|---|---|
| teacher codes | Transactive/facilitative | 1 | 35 | | 24 |
| | Directive/didactive | 17 | 26 | 12 | 22 |
| **sums** | | **18** | **61** | **12** | **46** |
| | | | | | |
| | | | | | |
| student codes | PP/PI/CO/TQ/TR | 0 | 13 | 32 | 20 |
| | General confirmation/no code | | 1 | 35 | 15 |
| **sums** | | **0** | **14** | **67** | **35** |

| B3 | | Structuring | Soliciting | Responding | Reacting |
|---|---|---|---|---|---|
| teacher codes | Transactive/facilitative | 1 | 11 | 1 | 19 |
| | Directive/didactive | 5 | 32 | 7 | 16 |
| **sums** | | **6** | **43** | **8** | **35** |
| | | | | | |
| | | | | | |
| student codes | PP/PI/CO/TQ/TR | 0 | 13 | 9 | 4 |
| | General confirmation/no code | 0 | 2 | 41 | 5 |
| **sums** | | **0** | **15** | **50** | **9** |

| B4 | | Structuring | Soliciting | Responding | Reacting |
|---|---|---|---|---|---|
| teacher codes | Transactive/facilitative | 1 | 27 | 1 | 22 |
| | Directive/didactive | 8 | 18 | 7 | 6 |
| **sums** | | **9** | **45** | **8** | **28** |
| | | | | | |
| | | | | | |
| student codes | PP/PI/CO/TQ/TR | 1 | 8 | 25 | 17 |
| | General confirmation/no code | | | 20 | 5 |
| **sums** | | **1** | **8** | **45** | **22** |

Teacher B, non-proof:

| Bnp1 | | Structuring | Soliciting | Responding | Reacting |
|---|---|---|---|---|---|
| teacher codes | Transactive/facilitative | 3 | 18 | 1 | 5 |
| | Directive/didactive | 9 | 8 | 2 | 11 |
| **sums** | | **12** | **26** | **3** | **16** |
| | | | | | |
| | | | | | |
| student codes | PP/PI/CO/TQ/TR | 0 | 4 | 25 | 13 |
| | General confirmation/no code | 0 | 0 | 14 | 21 |
| **sums** | | **0** | **4** | **39** | **34** |

| Bnp2 | | Structuring | Soliciting | Responding | Reacting |
|---|---|---|---|---|---|
| teacher codes | Transactive/facilitative | 5 | 46 | 6 | 82 |
| | Directive/didactive | 30 | 128 | 21 | 71 |
| **sums** | | **35** | **174** | **27** | **153** |
| | | | | | |
| | | | | | |
| student codes | PP/PI/CO/TQ/TR | 0 | 49 | 76 | 96 |
| | General confirmation/no code | | 0 | 133 | 62 |
| **sums** | | **0** | **49** | **209** | **158** |

Combined lessons by teacher:

| Teacher A proof | | Structuring | Soliciting | Responding | Reacting |
|---|---|---|---|---|---|
| teacher codes | Transactive/facilitative | 4 | 135 | 10 | 54 |
| | Directive/didactive | 49 | 91 | 92 | 99 |
| **sums** | | **53** | **226** | **102** | **153** |
| | | | | | |
| | | | | | |
| student codes | PP/PI/CO/TQ/TR | 2 | 125 | 136 | 82 |
| | General confirmation/no code | | 9 | 121 | 60 |
| **sums** | | **2** | **134** | **257** | **142** |

| Teacher A non-proof | | Structuring | Soliciting | Responding | Reacting |
|---|---|---|---|---|---|
| teacher codes | Transactive/facilitative | 1 | 53 | 1 | 39 |
| | Directive/didactive | 48 | 120 | 47 | 82 |
| **sums** | | **49** | **173** | **48** | **121** |
| | | | | | |
| | | | | | |
| student codes | PP/PI/CO/TQ/TR | 1 | 47 | 44 | 19 |
| | General confirmation/no code | | 3 | 121 | 31 |
| **sums** | | **1** | **50** | **165** | **50** |

| Teacher B proof | | Structuring | Soliciting | Responding | Reacting |
|---|---|---|---|---|---|
| teacher codes | Transactive/facilitative | 4 | 81 | 3 | 74 |
| | Directive/didactive | 50 | 91 | 29 | 61 |
| **sums** | | **54** | **172** | **32** | **135** |
| | | | | | |
| | | | | | |
| student codes | PP/PI/CO/TQ/TR | 1 | 42 | 78 | 46 |
| | General confirmation/no code | | 3 | 113 | 27 |
| **sums** | | **1** | **45** | **191** | **73** |

| Teacher B non-proof | | Structuring | Soliciting | Responding | Reacting |
|---|---|---|---|---|---|
| teacher codes | Transactive/facilitative | 8 | 64 | 7 | 87 |
| | Directive/didactive | 39 | 136 | 23 | 82 |
| **sums** | | **47** | **200** | **30** | **169** |
| | | | | | |
| | | | | | |
| student codes | PP/PI/CO/TQ/TR | 0 | 53 | 101 | 109 |
| | General confirmation/no code | | | 147 | 83 |
| **sums** | | **0** | **53** | **248** | **192** |

## Appendix C – Tables of T/S word ratio

| | Teacher A | Students | T/S word ratio | % teacher speech |
|---|---|---|---|---|
| Lesson 1 - proof | 1422 | 746 | 1.91 | 65.6% |
| Lesson 2 - proof | 1220 | 656 | 1.86 | 65.0% |
| Lesson 3 - proof | 3614 | 2324 | 1.56 | 60.9% |
| Lesson 4 - proof | 1517 | 1752 | .87 | 46.4% |
| | | | | |
| Lesson 1 - no proof | 2776 | 545 | 5.09 | 83.6% |
| Lesson 2 - no proof | 5677 | 1354 | 4.19 | 80.7% |

| | Teacher B | Students | T/S word ratio | % teacher speech |
|---|---|---|---|---|
| Lesson 1 - proof | 1497 | 383 | 3.91 | 79.6% |
| Lesson 2 - proof | 1901 | 637 | 2.98 | 74.9% |
| Lesson 3 - proof | 3381 | 983 | 3.44 | 77.5% |
| Lesson 4 - proof | 1340 | 947 | 1.41 | 58.6% |
| | | | | |
| Lesson 1 - no proof | 1232 | 679 | 1.81 | 64.5% |
| Lesson 2 - no proof | 5331 | 2607 | 2.04 | 67.2% |

Table after DD/DR speeches removed

| | Teacher B | Students | T/S word ratio | % teacher speech |
|---|---|---|---|---|
| Lesson 1 - proof | 1076 | 383 | 2.81 | 73.7% |
| Lesson 2 - proof | 1714 | 637 | 2.69 | 72.9% |
| Lesson 3 - proof | 3105 | 983 | 3.16 | 76.0% |
| Lesson 4 - proof | 1288 | 947 | 1.36 | 57.6% |
| | | | | |
| Lesson 1 - no proof | 1086 | 679 | 1.60 | 61.5% |
| Lesson 2 - no proof | 5105 | 2607 | 1.96 | 66.2% |

## Appendix D – Tables of T SOL/S RES/T REA vs. SOL/RES/REA other than T/S/T

| | T SOL/S RES/T REA | Percentage of total | SOL/RES/REA other than T/S/T | Percentage of total | Total # of moves | Ratio of percent-ages |
|---|---|---|---|---|---|---|
| A1 | 45 | 25.0% | 9 | 5.0% | 180 | 5 |
| A2 | 27 | 20.0% | 18 | 13.3% | 135 | 1.5 |
| A3 | 102 | 18.3% | 66 | 11.9% | 556 | 1.5 |
| A4 | 30 | 18.2% | 27 | 16.4% | 165 | 1.1 |
| **Total moves** | **204** | | **120** | | **1036** | |
| Average | | 20.4% | | 11.75% | | |
| Weighted average | | 19.7% | | 11.6% | | 1.7 |
| Anp1 | 78 | 37.9% | 12 | 5.8% | | 6.5 |
| Anp2 | 156 | 34.6% | 30 | 6.7% | | 5.2 |
| **Total moves** | **234** | | **42** | | **657** | |
| Average | | 36.3% | | 6.3% | | |
| Weighted average | | 35.6% | | 6.4% | | 5.6 |

| | T SOL/S RES/T REA | Percentage of total | SOL/RES/REA other than T/S/T | Percentage of total | Total # of moves | Ratio of percent-ages |
|---|---|---|---|---|---|---|
| B1 | 24 | 20.3% | 6 | 5.1% | 118 | 4 |
| B2 | 54 | 21.3% | 33 | 13.0% | 253 | 1.6 |
| B3 | 33 | 19.9% | 12 | 7.2% | 166 | 2.8 |
| B4 | 54 | 32.5% | 12 | 7.2% | 166 | 4.5 |
| **Total moves** | **165** | | **63** | | **703** | |
| Average | | 23.5% | | 8.1% | | |
| Weighted average | | 23.5% | | 9.0% | | 2.6 |
| Bnp1 | 21 | 15.7% | 9 | 6.7% | 134 | 2.3 |
| Bnp2 | 192 | 23.9% | 84 | 10.4% | 805 | 2.3 |
| **Total moves** | **213** | | **93** | | **939** | |
| Average | | 19.8% | | 8.6% | | |
| Weighted average | | 22.7% | | 9.9% | | 2.29 |

## Appendix E – Tables of string length

| A1 strings: | |
|---|---|
| TSS | 5 |
| 4 S's | 1 |
| 5 S's | 1 |

| A2 strings: | |
|---|---|
| TSS | 2 |
| TSSS | 4 |
| 9 S's | 1 |

| A3 strings: | |
|---|---|
| TSS | 27 |
| TSSS | 7 |
| 4 S's | 2 |
| 5 S's | 1 |
| 6 S's | 2 |
| 7 S's | 1 |

| A4 strings: | |
|---|---|
| TSS | 4 |
| TSSS | 2 |
| 5 S's | 2 |

| B1 strings: | |
|---|---|
| TSS | 3 |
| TSSS | 3 |

| B2 strings: | |
|---|---|
| TSS | 14 |
| TSSS | 5 |
| 4 S's | 1 |
| 5 S's | 1 |

| B3 strings: | |
|---|---|
| TSS | 9 |

| B4 strings: | |
|---|---|
| TSS | 6 |
| 4 S's | 1 |

| | |
|---|---|
| Total number of strings, teacher A, proof: | 62 |
| Average string length: | 2.82 |
| Total number of strings, teacher A, non-proof: | 5 |
| Average string length: | 2 |

| | |
|---|---|
| Total number of strings, teacher B, proof: | 43 |
| Average string length: | 2.35 |
| Total number of strings, teacher B, non-proof: | 106 |
| Average string length: | 2.73 |

| Anp1 strings: | |
|---|---|
| TSS | 2 |

| Anp2 strings: | |
|---|---|
| TSS | 3 |

| Bnp1 strings: | |
|---|---|
| TSS | 15 |
| TSSS | 2 |
| 4 S's | 3 |
| 12 S's | 1 |

| Bnp2 strings: | |
|---|---|
| TSS | 50 |
| TSSS | 24 |
| 4 S's | 6 |
| 5 S's | 3 |
| 8 S's | 1 |
| 10 S's | 1 |