Anisotropic inverse problems with internal measurements

Chenxi Guo

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ABSTRACT

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This thesis concerns the hybrid inverse problem of reconstructing a tensor-valued conductivity from knowledge of internal functionals. This problem finds applications in the medical imaging modalities Current Density Imaging and Magnetic Resonance Electrical Impedance Tomography.

In the first part of the thesis, we investigate the reconstruction of the anisotropic conductivity σ in a second-order elliptic partial differential equation from knowledge of internal current densities. We show that the unknown coefficient can be uniquely and stably reconstructed via explicit inversion formulas with a loss of one derivative compared to errors in the measurement. This improves the resolution of quantitative reconstructions in Calderón's problem(i.e. reconstruction problems from knowledge of boundary measurements). We then extend the problem to the full anisotropic Maxwell system and show that the complex-valued anisotropic admittivity $\gamma = \sigma + \iota\omega\epsilon$ can be uniquely reconstructed from knowledge of several internal magnetic fields. We also proved a unique continuation property and Runge approximation property for an anisotropic Maxwell system.

In the second part, we performed some numerical experiments to demonstrate the computational feasibility of the reconstruction algorithms and assess their robustness to noisy measurements.

Contents

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Chapter 1

Introduction

Mathematically, many inverse problems find interpretations in terms of linear and nonlinear (systems of) partial differential equations(PDE's). They consist in the reconstruction of the unknown parameter of a given PDE from knowledge of functionals that depend on these parameters. In mathematical terms, an inverse problem is devoted to inverting a functional relation of the form

$$\mathfrak{y} = \mathfrak{M}(\mathfrak{c}), \quad \text{for } \mathfrak{c} \in \mathfrak{X},\ \mathfrak{y} \in \mathfrak{Y}. \tag{1.1}$$

Here, $\mathfrak{c}$ denotes the unknown coefficients and $\mathfrak{X}$ is a subset of a Banach space in which the unknown coefficients are defined. $\mathfrak{M}$ denotes the measurement operator.

Hybrid inverse problems are extensively studied in the bio-engineering community. Such inverse problems aim to combine a high-contrast modality, such as Electrical Impedance Tomography(EIT) or Optical Tomography(OT), with a high-resolution modality, such as Magnetic Resonance Imaging (MRI) or ultrasound. The high-contrast modality EIT aims to locate unhealthy tissues by reconstructing their electrical conductivity γ from knowledge of boundary functionals. This leads to an inverse problem known as the Calderón's problem.

Extensive studies have been made on uniqueness properties and reconstructions methods for this inverse problem [57, 58, 59, 60]. But the problem is mathematically severely ill-posed and the corresponding stability estimates are of logarithmic type, which results in a low resolution for the reconstructions. Moreover, well-known obstructions show that the anisotropic conductivities cannot be uniquely reconstructed from boundary measurements [2, 4].

It is sometimes possible to leverage the physical coupling of the above high contrast modality with a high-resolution modality, which provides high-resolution internal functionals of the unknown conductivity [3, 13, 56]. Thus reconstructions in hybrid inverse problems typically involve two steps. In the first step, an inverse problem involving the high-resolution modality needs to be solved to provide internal functionals. In this thesis, we assume that this first step has been performed. Our interest is in the second step of the procedure, which consists of reconstructing the coefficients that display high contrasts from the mappings obtained during the first step. These mappings involve internal functionals of the coefficients of interest.

The reconstruction methods in hybrid inverse problems depend on the physical model of interest. However, it is natural to ask several common questions:

1. Uniqueness: are the coefficients uniquely characterized by the internal measurements? To answer the question, we must verify the following property,

$$\mathfrak{M}(\mathfrak{c}) = \mathfrak{M}(\tilde{\mathfrak{c}}) \Rightarrow \mathfrak{c} = \tilde{\mathfrak{c}}, \quad \text{for all} \quad \mathfrak{c}, \tilde{\mathfrak{c}} \in \mathfrak{X}.$$

2. Stability: is an inverse problem well-posed or at least mildly ill-posed? The stability estimates are usually written in Lipschitz type,

$$\|\tilde{\mathfrak{c}} - \mathfrak{c}\|_{H^s} \leq C \|\mathfrak{M}(\tilde{\mathfrak{c}}) - \mathfrak{M}(\mathfrak{c})\|_{H^{s+t}},$$

for some constant C and integer s, t.

3. Which component of the anisotropic coefficient can be reconstructed with a better stability and which specific boundary conditions should be prescribed at the boundary of the domain of interest?

The goal of the thesis is to derive explicit reconstructions for anisotropic coefficients in PDE's and obtain Lipschitz-type stability estimates for such reconstructions. For the special isotropic case, a scalar coefficient may be reconstructed with a better stability. The mathematical techniques in this work provides a class of prescribed boundary conditions for which the reconstructions to the hybrid anisotropic inverse problems are shown to be uniquely and stably determined by the internal functionals. The main application of the results in this manuscript is medical imaging, where the reconstructions with internal measurements greatly improve the resolution of images. We also performed numerical simulations to validate the theories and reconstruction algorithms proposed in the thesis.

1.1 Reconstruction of tensor-valued coefficients in second-order elliptic equations

We consider the tensor-valued second-order elliptic equation:

$$\nabla \cdot (\gamma \nabla u) = 0 \quad (X), \qquad u|_{\partial X} = g, \tag{1.2}$$

with a real symmetric tensor γ verifying the ellipticity condition for $\kappa \geq 1$,

$$\kappa^{-1}\|\xi\|^2 \leq \xi \cdot \gamma\xi \leq \kappa\|\xi\|^2, \tag{1.3}$$

such that the above equation admits a unique solution in $H^1(X)$ for $g \in H^{\frac{1}{2}}(\partial X)$. Here X is an open bounded domain in $\mathbb{R}^n$ with smooth boundary ∂X.

1.1.1 Linearized conductivity with power densities

A problem that has received a lot of attention recently concerns the reconstruction of the conductivity tensor γ in the second-order elliptic equation (1.2) from knowledge of internal power density measurements of the form $\nabla u \cdot \gamma \nabla v$, where u and v both solve (1.2) with possibly different boundary conditions. This problem is motivated by a coupling between electrical impedance imaging and ultrasound imaging and also finds applications in thermo-acoustic imaging.

Explicit reconstruction procedures for the above non-linear problem have been established in [17, 8, 47, 46, 44], successively in the 2D, 3D, and nD isotropic case, and then in the 2D and nD anisotropic case. In these articles, the number of functionals may be quite large. The analyses in [44] were recently summarized and pushed further in [45]. If one decomposes γ into the product of a scalar function $\tau = (\det \gamma)^{\frac{1}{n}}$ and a scaled anisotropic structure $\tilde{\gamma}$ such that $\det \tilde{\gamma} = 1$, the latter reference establishes explicit reconstruction formulas for both quantities with Lipschitz stability for τ in $W^{1,\infty}$ norm, and involving the loss of one derivative for $\tilde{\gamma}$.

In the isotropic case, several works study the above problem in the presence of a lesser number of functionals. The case of one functional is addressed in [6], whereas numerical simulations show good results with two functionals in dimension $n = 2$ [24]. Theoretical and numerical analyses of the linearized inverse problem are considered in [37, 38]. The stabilizing nature of a class of internal functionals containing the case of power densities is demonstrated in [38] via micro-local analysis of the linearized inverse problem. The above inverse problem is recast as a system of nonlinear partial differential equations in [7] and its linearization is analyzed by means of theories of elliptic systems of equations. It is shown in the latter reference that $n+1$ functionals, where n is spatial dimension, is sufficient to reconstruct a scalar coefficient γ with elliptic regularity, i.e., with no loss of derivatives, from power density measurements. This was confirmed by two-dimensional simulations in [12]. All known explicit reconstruction procedures require knowledge of a larger number of internal functionals.

In this work, we study the linearized version of this inverse problem in the anisotropic case, i.e. we write an expansion of the form $\gamma^\varepsilon = \gamma_0 + \varepsilon\gamma$ with γ_0 known and $\varepsilon \ll 1$, and study the reconstructibility of γ from *linearized power densities* (LPD). We first proceed by supporting the perturbation γ away from the boundary ∂X and analyze microlocally the symbol of the linearized functionals, and show that, as in [38], a large enough number of functionals allows us to construct a left-parametrix and set up a Fredholm inversion. The main difference between the isotropic and anisotropic settings is that the anisotropic part of the conductivity is reconstructed with a loss of one derivative. Such a loss of a derivative is optimal since our estimates are elliptic in nature. It is reminiscent of results obtained for a similar problem in [15].

Secondly, we show how the explicit inversion approach presented in [44, 45] carries through linearization, thus allowing for reconstruction of fully anisotropic tensors supported up to the boundary of X. In this case, we derive reconstruction formulas that require a smaller number of power densities than in the non-linear case, giving possible room for improvement in the non-linear inversion algorithms. The results are presented in Chapter 2.

1.1.2 Inversion via current densities

In this section, we consider the Current Density Impedance Imaging problem (CDII), also called Magnetic Resonance Electrical Impedance Tomography (MREIT) of reconstructing an anisotropic conductivity γ in the second-order elliptic equation (1.2) from knowledge of internal current densities of the form $H = \gamma\nabla u$, where u solves (1.2). Here X is an open bounded domain with a $\mathcal{C}^{2,\alpha}$ or smoother boundary ∂X. Internal current density functionals H can be obtained by the technique of current density imaging. The idea is to use Magnetic Resonance Imaging (MRI) to determine the magnetic field B induced by an input current I. The current density is then defined by $H = \nabla \times B$. We thus need to measure all components of B to calculate H. See [31, 54] for details.

A perturbation method to reconstruct the unknown conductivity in the linearized case was presented in [32]. In dimension $n = 2$, a numerical reconstruction algorithm based on the construction of equipotential lines was given in [39]. Kwon *et al* [40] proposed a J-substitution algorithm, which is an iterative algorithm. Assuming knowledge of only the magnitude of only one current density $|H| = |\gamma\nabla u|$, the problem was studied in [48, 49, 51] (see the latter reference for a review) in the isotropic case and more recently in [29, 43] in the anisotropic case with anisotropy known. In [34, 42], Nachman *et al.* and Lee independently found a explicit reconstruction formula for visualizing $\log\gamma$ at each point in a domain. The reconstruction with functionals of the form $\gamma^t\nabla u$ is shown in [36] in the isotropic case. For $t = 0$, the functionals are given by solutions of (1.2), then a more general complex-valued tensor in the anisotropic case was presented in [15]. In [55], assuming that the magnetic field B is measurable, Seo *et al.* gave a reconstruction for a complex-valued coefficient in the isotropic case.

In this work, we show that a minimum number of current desities equal to $n+2$, where n is the spatial dimension, is sufficient to guarantee a local reconstruction. γ can be uniquely reconstructed with a loss of one derivative compared to errors in the measurement of H. In the special case where γ is scalar, it can be reconstructed with no loss of derivatives. We provide a precise statement of what components may be reconstructed with a loss of zero or one derivatives. The results are presented in Chapter 3.

1.2 Reconstruction of complex-valued tensors in the Maxwell system

We consider the following system of Maxwell's equations:

$$\begin{cases} \nabla \times E + \iota\omega\mu_0 H = 0 \\ \nabla \times H - \gamma E = 0, \end{cases} \tag{1.4}$$

with the boundary condition

$$\nu \times E|_{\partial X} = f. \tag{1.5}$$

Here $\gamma = \sigma + \iota\omega\varepsilon$ and the smooth anisotropic electric permittivity, conductivity, and the constant isotropic magnetic permeability are respectively described by $\varepsilon(x)$, $\sigma(x)$ and μ_0, where $\varepsilon(x)$, $\sigma(x)$ are tensors and μ_0 is a constant scalar, known, coefficient. Let X be a bounded domain with smooth boundary in $\mathbb{R}^3$ and ν be the exterior unit normal vector on the boundary ∂X. The frequency $\omega > 0$ is fixed. E and H denote the electric and magnetic fields inside the domain X with a harmonic time dependence. We assume that $\varepsilon(x)$ and $\sigma(x)$ satisfy the uniform ellipticity condition for some $\kappa > 0$,

$$\kappa^{-1}\|\xi\|^2 \le \xi \cdot \varepsilon\xi \le \kappa\|\xi\|^2, \quad \kappa^{-1}\|\xi\|^2 \le \xi \cdot \sigma\xi \le \kappa\|\xi\|^2. \tag{1.6}$$

In this section, we consider a hybrid inverse problem where, in addition to boundary data, we have access to the internal magnetic field H in order to reconstruct the complex-valued tensor γ. Internal magnetic fields can be measured using a Magnetic Resonance Imaging (MRI) scanner; see [31] for the experimental details. In [55], assuming that the magnetic field H is measurable, Seo *et al* gave a reconstruction for the conductivity in the isotropic case. This thesis generalizes the reconstruction of an arbitrary (symmetric) complex-valued tensor and gives an explicit reconstruction procedure for $\gamma = \sigma + \iota\omega\varepsilon$. The explicit reconstructions we propose require that all components of the magnetic field H be measured. This is challenging in many practical settings as it requires a rotation of the domain being imaged or of the MRI scanner. The reconstruction of γ from knowledge of only some components of H, ideally only one component for the most practical experimental setup, is open at present. We propose sufficient conditions on the choice of boundary conditions f such that the reconstruction of γ is unique and satisfies elliptic stability estimates. To derive local reconstruction formulas for a more general γ, we need to control the local behavior of solutions by well-chosen boundary conditions. This is done by means of a Runge approximation. We will prove the Runge approximation for an anisotropic Maxwell system using the unique continuation property. The results are presented in Chapter 4.

1.3 Imaging of tensor-valued coefficients with internal data

We first study the special case of reconstructing the anisotropic conductivity γ with current densities $H = \gamma\nabla u$ in the second-order elliptic equation (1.2) in two dimensions. The explicit reconstruction method provided in [10] requires that some matrices constructed from available data satisfy appropriate conditions of linear independence. We will show that in $\mathbb{R}^2$, such assumptions can be globally guaranteed with a set of well-chosen illuminations based on the construction of Complex Geometrical Optics (CGO) solutions, provided that one can prescribe Dirichlet (or other) conditions over the full boundary. Several numerical experiments confirm the theoretical predictions. The numerical simulations in [11] show that the reconstruction procedure works well for different types of tensors containing both smooth and discontinuous coefficients. Using the decomposition $\gamma = \beta\tilde{\gamma}$ with $\beta = (\det\gamma)^{\frac{1}{2}}$, the simulation results also show that both the isotropic and the anisotropic parts of the tensor can be stably reconstructed, with a better robustness to noise for the scalar β. This is consistent with theoretical results in [10], where the stability of the inversion on β is better than on the anisotropy $\tilde{\gamma}$. Our CGO-based theoretical results exhibit a specific class of boundary conditions that ensure stable reconstructions. In practice, a much larger class of boundary conditions than those that can be analyzed mathematically still provide stable reconstructions. Yet, when only a part of the boundary conditions is accessible for current injection, the linear independence of specific matrices needed in the reconstruction deteriorates. The reconstructions then become unstable in some parts of the domain. This phenomenon is demonstrated in several numerical simulations. All simulations are performed in two dimensions of space, although we expect the conclusions to still hold qualitatively in higher dimensions as well. Such results are presented in Chapter 5.

Secondly, we perform numerical simulations of reconstructing (σ, ε) in the Maxwells system (1.2) from the internal magnetic fields H. We find that the reconstructions are more sensitive to noise than the previous case with current densities [27]. This is consistent with theoretical results in [26], where the reconstructions suffer from a loss of two derivatives from errors in the acquisition H. The results are presented in Chapter 6.

Academic publications by the author:

- Linearized internal functionals for anisotropic conductivities, (with Guillaume Bal and François Monard), Inv. Probl. and Imaging 8.1 (2014).
- Inverse anisotropic conductivity from internal current densities, (with Guillaume Bal and François Monard), Inverse Problems 30.2 (2014).
- Reconstruction of complex-valued tensors in the Maxwell system from knowledge of internal magnetic field, (with Guillaume Bal), Inv. Probl. and Imaging 8.4 (2014).
- Imaging of anisotropic conductivities from current densities in two dimensions, (with Guillaume Bal and François Monard), SIAM J. Imaging Sci. 7.4 (2014).
- Imaging of complex-valued tensors in the Maxwell system from internal magnetic fields, (with Guillaume Bal), preprint.

Chapter 2

Linearized inverse conductivity from power densities

In this chapter, we study the reconstruction of the conductivity tensor γ in the elliptic equation,

$$\nabla \cdot (\gamma \nabla u) = 0 \quad (X), \qquad u|_{\partial X} = g, \tag{2.1}$$

from knowledge of internal power density measurements of the form $\nabla u \cdot \gamma \nabla v$, where u and v both solve (2.1) with possibly different boundary conditions and where γ is a symmetric tensor satisfying the uniform ellipticity condition

$$\kappa^{-1}\|\xi\|^2 \le \xi \cdot \gamma \xi \le \kappa \|\xi\|^2, \quad \xi \in \mathbb{R}^n, \quad \text{for some } \kappa \ge 1. \tag{2.2}$$

We focus on the linearized version of this inverse problem in the anisotropic case. We write an expansion of the form $\gamma^\varepsilon = \gamma_0 + \varepsilon\gamma$ with γ_0 known and $\varepsilon \ll 1$, and study the reconstructibility of γ from linearized power densities (LPD).

2.1 Modeling of the problem

Consider the conductivity equation (2.1), where $X \subset \mathbb{R}^n$ is open, bounded and connected with $n \geq 2$, and where γ^ε is a uniformly elliptic conductivity tensor over X.

We set boundary conditions $(g_1, \ldots, g_m)$ and call u_i^ε the unique solution to (2.1) with $u_i^\varepsilon|_{\partial X} = g_i$, $1 \leq i \leq m$ and conductivity γ^ε. We consider the measurement functionals

$$H_{ij}^\varepsilon : \gamma^\varepsilon \mapsto H_{ij}^\varepsilon(\gamma^\varepsilon) = \nabla u_i^\varepsilon \cdot \gamma^\varepsilon \nabla u_j^\varepsilon(x), \quad 1 \leq i, j \leq m, \quad x \in X. \tag{2.3}$$

Considering an expansion of the form $\gamma^\varepsilon = \gamma_0 + \varepsilon\gamma$, where the background conductivity γ_0 is known, uniformly elliptic and ε so small that the total γ^ε remains uniformly elliptic. Expanding the solutions u_i^ε accordingly as

$$u_i^\varepsilon = u_i + \varepsilon v_i + \mathcal{O}(\varepsilon^2), \quad 1 \leq i \leq m,$$

the PDE (2.1) at orders $\mathcal{O}(\varepsilon^0)$ and $\mathcal{O}(\varepsilon^1)$ gives rise to two relations

$$-\nabla \cdot (\gamma_0 \nabla u_i) = 0 \quad (X), \qquad u_i|_{\partial X} = g_i, \tag{2.4}$$

$$-\nabla \cdot (\gamma_0 \nabla v_i) = \nabla \cdot (\gamma \nabla u_i) \quad (X), \qquad v_i|_{\partial X} = 0. \tag{2.5}$$

The measurements then look like

$$H_{ij}^\varepsilon = \nabla u_i \cdot \gamma_0 \nabla u_j + \varepsilon \left(\nabla u_i \cdot \gamma \nabla u_j + \nabla u_i \cdot \gamma_0 \nabla v_j + \nabla u_j \cdot \gamma_0 \nabla v_i \right) + \mathcal{O}(\varepsilon^2). \tag{2.6}$$

Therefore, the component dH_{ij} of the Fréchet derivative of H at γ_0 is

$$dH_{ij}(\gamma) = \nabla u_i \cdot \gamma \nabla u_j + \nabla u_i \cdot \gamma_0 \nabla v_j + \nabla u_j \cdot \gamma_0 \nabla v_i, \quad x \in X, \tag{2.7}$$

where the v_i's are linear functions in γ according to (2.5).

In both subsequent approaches, reconstruction formulas are established under the following two assumptions about the behavior of solutions related to the conductivity of reference γ_0. The first hypothesis deals with having a basis of gradients of solutions of (2.4) over a certain subset $\Omega \subseteq X$.

Hypothesis 2.1.1. *For an open set $\Omega \subseteq X$, there exist $(g_1, \dots, g_n) \in H^{\frac{1}{2}}(\partial X)^n$ such that the corresponding solutions $(u_1, \dots, u_n)$ of (2.4) with boundary condition $u_i|_{\partial X} = g_i$ $(1 \le i \le n)$ satisfy*

$$\inf_{x\in\Omega} \det(\nabla u_1, \dots, \nabla u_n) \ge c_0 > 0.$$

Once Hypothesis 2.1.1 is satisfied, any additional solution u_{n+1} of (2.4) gives rise to a $n \times n$ matrix

$$Z = [Z_1|\dots|Z_n], \qquad \text{where} \qquad Z_i := \nabla \frac{\det(\nabla u_1, \dots, \overbrace{\nabla u_{n+1}}^{i}, \dots, \nabla u_n)}{\det(\nabla u_1, \dots, \nabla u_n)}. \tag{2.8}$$

As seen in [44, 45], such matrices can be computed from the power densities $\{\nabla u_i \cdot \gamma_0 \nabla u_j\}_{i,j=1}^{n+1}$ and help impose orthogonality conditions on the anisotropic part of γ_0. Once enough such conditions are obtained by considering enough additional solutions, then the anisotropy is reconstructed explicitly via a generalization of the usual cross-product defined in three dimensions. In the linearized setting, we find that *one* additional solution such that Z has full rank is enough to reconstruct the linear perturbation γ. We thus formulate our second crucial assumption here:

Hypothesis 2.1.2. *Assume that Hypothesis 2.1.1 holds over some fixed $\Omega \subseteq X$. There exists $g_{n+1} \in H^{\frac{1}{2}}(\partial X)$ such that the solution u_{n+1} of (2.4) with boundary condition $u_{n+1}|_{\partial X} = g_{n+1}$ has a full-rank matrix Z (as defined in (2.8)) over Ω.*

Remark 2.1.3 (Case γ_0 constant)**.** *In the case where γ_0 is constant, then it is straightforward to see that $g_i = x_i|_{\partial X}$ ($1 \le i \le n$) fulfill Hypothesis 2.1.1 over X. Moreover, if $Q = \{q_{ij}\}_{i,j=1}^n$ denotes an invertible constant matrix such that $Q : \gamma_0 = 0$, then the boundary condition $g_{n+1} := \frac{1}{2} q_{ij} x_i x_j|_{\partial X}$ fulfills Hypothesis 2.1.2, since we have $Q = Z$.*

Throughout the chapter, we use for (real-valued) square matrices A and B the contraction notation $A : B = \operatorname{tr} AB^T = \sum_{i,j} A_{ij} B_{ij}$, with B^T the transpose matrix of A.

Remark 2.1.4. *In the treatment of the non-linear case [8, 47, 44, 45], it has been pointed out that Hypothesis 2.1.1 may not be systematically satisfied globally in dimension $n \ge 3$. A more general hypothesis to consider would come from picking a larger family (of cardinality $> n$) of solutions whose gradients have maximal rank throughout X. While this additional technical point would not alter qualitatively the present reconstruction algorithms, it would add complexity in notation which the authors decided to avoid.*

2.1.1 Past work and heuristics for the linearization

In the reconstruction approach developped in [46, 44, 45] for the non-linear problem, it was shown that not every part of the conductivity was reconstructed with the same stability. Namely, consider the decomposition of the tensor γ' into the product of a scalar function $\tau = (\det \gamma')^{\frac{1}{n}}$ and a scaled anisotropic structure $\tilde{\gamma}'$ with $\det \tilde{\gamma}' = 1$. The following results were then established. Starting from n solutions whose gradients form a basis of $\mathbb{R}^n$ over a subset $\Omega \subset X$, it was shown that under knowledge of a $W^{1,\infty}(X)$ anisotropic structure $\tilde{\gamma}'$, the scalar function $\log \det \gamma'$ was uniquely and Lipschitz-stably reconstructible in $W^{1,\infty}(\Omega)$ from $W^{1,\infty}$ power densities. Additionally, if one added a finite number of solutions $u_{n+1}, \dots, u_{n+l}$ such that the family of matrices $Z_{(1)}, \dots, Z_{(l)}$ defined as in (2.8) imposed enough orthogonality constraints on $\tilde{\gamma}'$, then the latter was explicitely reconstructible over Ω from the mutual power densities of $(u_1, \dots, u_{n+l})$. The latter reconstruction was stable in L^∞ for power densities in $W^{1,\infty}$ norm, thus it involved the loss of one derivative.

Passing to the linearized setting now (recall $\gamma^\varepsilon = \gamma_0 + \varepsilon\gamma$), and anticipating that one scalar quantity may be more stably reconstructible than the others, this quantity should be the linearized version of $\log\det\gamma^\varepsilon$. Standard calculations yield

$$\log\det(\gamma_0+\varepsilon\gamma) = \log\det\gamma_0 + \log\det(\mathbb{I}_n + \varepsilon\gamma_0^{-1}\gamma) = \log\det\gamma_0 + \varepsilon\text{tr }(\gamma_0^{-1}\gamma) + \mathcal{O}(\varepsilon^2),$$

and thus the quantity that should be stably reconstructible is $\text{tr }(\gamma_0^{-1}\gamma)$. The linearization of the product decomposition $(\tau, \tilde{\gamma}')$ above is now a spherical-deviatoric one of the form

$$\gamma = \frac{1}{n}\text{tr }(\gamma_0^{-1}\gamma)\gamma_0 + \gamma_d, \quad \gamma_d := \gamma_0(\gamma_0^{-1}\gamma)^{\text{dev}}, \tag{2.9}$$

where $^{\text{dev}}$ is the linear projection onto the hyperplane of traceless matrices $A^{\text{dev}} := A - \frac{\text{tr }A}{n}\mathbb{I}_n$.

2.1.2 Microlocal inversion

The above inverse problem in (2.5)-(2.7) may be seen as a system of partial differential equations for $(\gamma, \{v_j\})$. This is the point of view considered in [7]. However, $\{v_j\}$ may be calculated from (2.5) and the expression plugged back into (2.7). This allows us to recast dH as a linear operator for γ, which is smaller than the original linear system for $(\gamma, \{v_j\})$, but which is no longer differential and rather pseudo-differential. The objective in this section is to show, following earlier work in the isotropic case in [38], that such an operator is elliptic under appropriate conditions.

We first fix $\Omega' \subset\subset X$ and assume that $\text{supp}\gamma \subset \Omega'$, so that the integral $\int_{\mathbb{R}^n} e^{\iota x\cdot\xi} p(x,\xi) : \hat{\gamma}(\xi) d\xi$ is well-defined, where $p(x,\xi)$ is a matrix-valued symbol whose entries are polynomials in ξ and the hat denotes the Fourier Transform $\hat{\gamma}(\xi) = \int_{\mathbb{R}^n} e^{-\iota x\cdot\xi}\gamma(x)\ dx$. We also assume that $\gamma_0 \in \mathcal{C}^\infty(\Omega')$ and can be extended smoothly by $\gamma_0 = \mathbb{I}_n$ outside Ω'. As pointed out in [38], in order to treat this problem microlocally, one must introduce cutoff versions of the dH_{ij} operators, which in turn extend to pseudo-differential operators (ΨDO) on $\mathbb{R}^n$. Namely, if Ω'' is a domain satisfying $\Omega' \subset\subset \Omega'' \subset\subset X$ and χ_1 is a smooth function supported in X which is identically equal to 1 on a neighborhood of $\overline{\Omega''}$, the operator $\gamma \mapsto \chi_1 dH_{ij}(\chi_1\gamma)$ can be made a ΨDO upon considering $L_0 = -\nabla\cdot(\gamma_0\nabla)$ as a second-order operator on $\mathbb{R}^n$ and using standard pseudo-differential parametrices to invert it. We will therefore not distinguish the operators dH_{ij} from their pseudo-differential counterparts. The task of this section is then to determine conditions under which a given collection of such functions becomes an elliptic operator of γ over Ω'.

Using relations (2.5) and (2.7), we aim at writing the operator dH_{ij} in the following form

$$dH_{ij}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\iota\xi\cdot(x-y)} M_{ij}(x,\xi) : \gamma(y)\; d\xi\; dy, \tag{2.10}$$

with symbol $M_{ij}(x,\xi)$ (pseudo-differential terminology is recalled in Sec. 2.2.1). We first compute the main terms in the symbol expansion of dH_{ij} (call this expansion $M_{ij} = M_{ij}|_0 + M_{ij}|_{-1} + \mathcal{O}(|\xi|^{-2})$ with $M_{ij}|_p$ homogeneous of degree p in ξ). From these expressions, we then directly deduce microlocal properties on the corresponding operators.

The first lemma shows that the principal symbols $M_{ij}|_0$ can never fully invert for γ, no matter how many solutions u_i we pick. When Hypothesis 2.1.1 is satisfied, then the characteristic directions of the principal symbols $\{M_{ij}(x,\xi)\}_{1\le i,j\le n}$ reduce to a $n-1$-dimensional subspace of $S_n(\mathbb{R})$. Here and below, we recall that the colon ":" denotes the inner product $A : B = \text{tr}\,(AB^T)$ for $(A,B) \in S_n(\mathbb{R})$ and $\odot$ denotes the symmetric outer product $U \odot V = \frac{1}{2}(U\otimes V + V\otimes U)$ for $U, V \in \mathbb{R}^n$.

Lemma 2.1.5. *(i) For any i, j and $x \in X$, the symbol $M_{ij}|_0$ satisfies*

$$M_{ij}|_0 : (\gamma_0 \xi \odot \eta) = 0, \quad \textit{for all } \eta \in \mathbb{S}^{n-1} \textit{ satisfying } \quad \eta \cdot \xi = 0. \tag{2.11}$$

(ii) Suppose that Hypothesis 2.1.1 holds over some $\Omega \subseteq X$. Then for any $x \in \Omega$, if $P \in S_n(\mathbb{R})$ is such that

$$M_{ij}|_0 : P = 0, \quad 1 \leq i \leq j \leq n, \tag{2.12}$$

then P is of the form $P = \gamma_0 \xi \odot \eta$ for some vector η satisfying $\eta \cdot \xi = 0$.

Since an arbitrary number of zero-th order symbols can never be elliptic with respect to γ, we then consider the next term in the symbol expansion of dH_{ij}. We must also add one solution u_{n+1} to the initial collection, exhibiting appropriate behavior, i.e. satisfying Hypothesis 2.1.2. The collection of functionals we consider below is thus of the form

$$dH := \{dH_{ij} \,|\, 1 \leq i \leq n, \; i \leq j \leq n+1\}, \tag{2.13}$$

and emanates from $n+1$ solutions $(u_1, \ldots, u_{n+1})$ of (2.4) satisfying Hypotheses 2.1.1 and 2.1.2.

In order to formulate the result, we assume to construct a family of unit vector fields

$$\hat{\xi}_0(x,\xi) := \widehat{A_0(x)\xi}\,,\; \hat{\xi}_1(x,\xi)\,,\; \ldots\,,\; \hat{\xi}_{n-1}(x,\xi),$$

homogeneous of degree zero in ξ, smooth in x and everywhere orthonormal. We then define the family of scalar elliptic zeroth-order ΨDO

$$T : \gamma \mapsto T\gamma = \{T_{pq}\gamma\}_{0\le p\le q\le n-1}, \quad T_{pq}\gamma(x) := (2\pi)^{-n}\int_{\mathbb{R}^n} e^{\iota\xi\cdot x} A_0^{-1}\hat{\xi}_p \odot \hat{\xi}_q A_0^{-1} : \hat{\gamma}(\xi)\ d\xi, \tag{2.14}$$

which can be thought of as a microlocal change of basis after which the operator $dH(\gamma)$ becomes both diagonal and elliptic. Indeed, we verify (see section 2.2.4) that for any $k \ge 1$ and γ sufficiently regular, we have

$$\|\gamma\|_{H^k(\Omega')} \le C\|T\gamma\|_{H^k(\Omega')} + C_2\|\gamma\|_{L^2(\Omega')} \le C_3\|\gamma\|_{H^k(\Omega')}. \tag{2.15}$$

The above estimates come from standard result on pseudo-differential operators. The presence of the constant C_2 indicates that T can be inverted microlocally, but may not injective.

Composing the measurements dH_{ij} with appropriate scalar ΨDO of order 0 and 1, we are then able to recover each component of the operator (2.14). The well-chosen "parametrices" are made possible by the fact that the collection of symbols $M_{ij}|_0 + M_{ij}|_{-1}$ becomes elliptic over Ω' when Hypotheses 2.1.1 and 2.1.2 are satisfied. Rather than using the full collection of measurements dH (2.13), we will consider the smaller collection $\{dH_{ij}\}_{1\le i,j\le n}$ augmented with the n measurement operators

$$L_i(\gamma) = \sum_{j=1}^{n} \mu_j\ dH_{ij}(\gamma) + \mu\ dH_{i,n+1}(\gamma), \quad 1 \le i \le n, \tag{2.16}$$

where $(\mu_1, \dots, \mu_n, \mu)(x)$, known from the measurements $\{H_{ij}\}_{i,j=1}^{n+1}$, are the coefficients in the relation of linear dependence

$$\mu_1\nabla u_1 + \cdots + \mu_n\nabla u_n + \mu\nabla u_{n+1} = 0.$$

We also define the operator $L_0^{\frac{1}{2}} \in \Psi^1$ with principal symbol $-\boldsymbol{\iota}\|A_0\xi\|$. Our conclusions may be formulated as follows:

Proposition 2.1.6. *Let the measurements dH defined in* (2.13) *satisfy Hypotheses 2.1.1 and 2.1.2.*

(i) For $(\alpha,\beta)=(0,0)$ and $1\le\alpha\le\beta\le n-1$, there exist $\{Q_{\alpha\beta ij}\}_{1\le i\le j\le n}\in\Psi^0$ such that

$$\sum_{1\le i,j\le n} Q_{\alpha\beta ij}\circ dH_{ij} = T_{\alpha\beta} \qquad \mathrm{mod}\ \Psi^{-1}. \tag{2.17}$$

(ii) For any $1\le\alpha\le n-1$, there exist $\{B_{\alpha i}\}_{1\le i\le n}\in\Psi^0$ such that the following relation holds

$$L_0^{\frac{1}{2}}\circ B_{\alpha i}\circ L_i - R_\alpha\circ R = T_{0\alpha} \qquad \mathrm{mod}\ \Psi^{-1}, \tag{2.18}$$

where the remainder $R_\alpha\circ R$ can be expressed as a zeroth-order linear combination of the components T_{00} and $\{T_{pq}\}_{1\le p\le q\le n-1}$ reconstructed in (i).

The presence of the $L_0^{\frac{1}{2}}$ term in part *(ii)* of Prop. 2.1.6 accounts for the loss of one derivative in the inversion process. From Prop. 2.1.6, we can then obtain stability estimates of the form

$$\|T_{00}\gamma\|_{H^{k+1}(\Omega')} + \sum_{1\le p\le q\le n-1}\|T_{pq}\gamma\|_{H^{k+1}(\Omega')} + \sum_{1\le p\le n-1}\|T_{0p}\gamma\|_{H^{k}(\Omega')} \le C\|dH\|_{H^{k+1}(\Omega')} + C_2\|\gamma\|_{L^2(\Omega)}. \tag{2.19}$$

The above stability estimate holds for $k=0$ using the results of Proposition 2.1.6 and in fact for any $k\ge 0$ updating by standard methods the parametrices in (2.17) and (2.18) to inversions modulo operators in Ψ^{-k} provided that the coefficients $(\gamma_0,\{u_j\})$ are sufficiently smooth. The presence of the constant C_2 indicates that the reconstruction of γ may be performed up to the existence of a finite dimensional kernel as an application of the Fredholm theory as in [38].

Equation (2.19) means that some components of γ are reconstructed with a loss of one derivative while other components are reconstructed with no loss. The latter components are those that can be spanned by the components $T_{00}\gamma$ and $\{T_{\alpha\beta}\gamma\}_{1\le\alpha,\beta\le n-1}$. Some algebra shows that the only such linear combination is $\sum_{i=0}^{n-1} T_{ii}\gamma$, which, using the fact that $\sum_{i=0}^{n-1} \hat{\xi}_i \otimes \hat{\xi}_i = \mathbb{I}_n$, can be computed as

$$\sum_{i=0}^{n-1} T_{ii}\gamma = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\iota x\cdot\xi} A_0^{-1}\mathbb{I}_n A_0^{-1} : \hat{\gamma}(\xi)\ d\xi = \gamma_0^{-1} : \left((2\pi)^{-n} \int_{\mathbb{R}^n} e^{\iota x\cdot\xi}\hat{\gamma}(\xi)\ d\xi\right) = \mathrm{tr}\ (\gamma_0^{-1}\gamma),$$

confirming the heuristics of Sec. 2.1.1. It can be shown that all other components of γ (i.e. any part of γ_d in (2.9)) are, to some extent, spanned by the components $T_{0\alpha}\gamma$, and as such cannot be reconstructed with better stability than the loss of one derivative in light of (2.19). Combining the above results with (2.15), we arrive at the main stability result of the chapter:

$$\|\mathrm{tr}\ (\gamma_0^{-1}\gamma)\|_{H^k(\Omega')} + \|\gamma_d\|_{H^{k-1}(\Omega')} \le C\|dH\|_{H^k(\Omega')} + C_2\|\gamma\|_{L^2(\Omega')}. \tag{2.20}$$

Such an estimate holds for any $k \ge 1$.

The above estimate holds with $C_2 = 0$ when $\gamma \mapsto dH(\gamma)$ is an injective (linear) operator. Injectivity cannot be verified by microlocal arguments since all inversions are performed up to smoothing operators; see [7] in the isotropic setting. In the next section, we obtain an injectivity result, which allows us to set $C_2 = 0$ in the above expression. However, the above stability estimate (2.20) is essentially optimal. An optimal estimate, which follows from the above and the equations for $(\gamma, \{v_j\})$ is the following:

$$\|M_{|0}\gamma\|_{H^k(\Omega')}+\|\gamma\|_{H^{k-1}(\Omega')} \le C\|dH\|_{H^k(\Omega')}+C_2\|\gamma\|_{L^2(\Omega')} \le C'(\|M_{|0}\gamma\|_{H^k(\Omega')}+\|\gamma\|_{H^{k-1}(\Omega')}).$$

The left-hand-side inequality is a direct consequence of (2.20) and the expression of dH. The right-hand side is a direct consequence of the expression of dH. The above estimate is clearly optimal. The operator $M_{|0}$ is of order 0. If it were elliptic, then γ would be reconstructed with no loss of derivative. However, $M_{|0}$ is not elliptic and the loss of ellipticity is precisely accounted for by the results in Lemma 2.1.5. As we discussed above, it turns out that the only spatial coefficient controlled by $M_{|0}\gamma$ is tr $(\gamma_0^{-1}\gamma)$, and hence (2.20).

2.1.3 Explicit inversion:

Now, allowing γ to be supported up to the boundary, we present a variation of the non-linear resolution technique used in [44, 45]. First considering n solutions generated by boundary conditions fulfilling Hypothesis 2.1.1, we establish an expression for γ in terms of the remaining unknowns $(v_1, \dots, v_n)$:

$$\gamma = \gamma_0([\nabla U]H^{-1}dHH^{-1}[\nabla U]^T - [\nabla V]H^{-1}[\nabla U]^T - [\nabla U]H^{-1}[\nabla V]^T)\gamma_0, \tag{2.21}$$

where $[\nabla U]$ and $[\nabla V]$ denote $n \times n$ matrices whose j-th columns are ∇u_j and ∇v_j, respectively, and where $H = \{H_{ij}\}_{i,j=1}^n$ and $dH = \{dH_{ij}\}_{i,j=1}^n$. In particular we find from (2.21) the relation

$$\text{tr } (\gamma_0^{-1}\gamma) = \text{tr } (H^{-1}dH) - 2\text{tr } M, \quad M := ([\nabla V][\nabla U]^{-1})^T. \tag{2.22}$$

Plugging (2.21) back into the second equation in (2.1) for $1 \le i \le n$, one can deduce a gradient equation for the quantity tr $(\gamma_0^{-1}\gamma)$ which in turn allows to reconstruct tr $(\gamma_0^{-1}\gamma)$ in a Lipschitz-stable manner with respect to the LPD $\{dH_{ij}\}_{i,j=1}^n$ (i.e. without loss of derivative).

Now turning to the full reconstruction of γ, we consider an additional solution u_{n+1} generated by a boundary condition fulfilling Hyp. 2.1.2. The following proposition then establishes how to reconstruct $(v_1, \dots, v_n)$ from dH:

Proposition 2.1.7. *Assume that $(g_1, \dots, g_{n+1})$ fulfill Hypotheses 2.1.1 and 2.1.2 over X and consider the linearized power densities $dH = \{dH_{ij}: \ 1 \le i \le j \le n+1, \ i \ne n+1\}$. Then the solutions $(v_1, \dots, v_n)$ satisfy a strongly coupled elliptic system of the form*

$$\nabla \cdot (\gamma_0 \nabla v_i) + W_{ij} \cdot \nabla v_j = f_i(dH, \nabla(dH)) \quad (X), \quad v_i|_{\partial X} = 0, \quad 1 \le i \le n, \tag{2.23}$$

where the vector fields W_{ij} are known and only depend on the behavior of γ_0, Z and $u_1, \dots, u_n$, and where the functionals f_i are linear in the data dH_{ij}.

When the vector fields W_{ij} are bounded, system (2.23) satisfies a Fredholm alternative from which we deduce that if (2.23) with a trivial right-hand side admits no non-trivial solution, then $(v_1, \dots, v_n)$ is uniquely reconstructed from (2.23). We can then reconstruct γ from (2.21).

Remark 2.1.8 (Case γ_0 constant)**.** *In the case where γ_0 is constant, choosing solutions as in Remark 2.1.3, one arrives at a system of the form* (2.23) *where $W_{ij} = 0$ if $i \ne j$, so that the system is decoupled and clearly injective.*

The conclusive theorem for the explicit inversion is thus given by

Theorem 2.1.9. *Assume that $(g_1, \dots, g_{n+1})$ fulfill Hypotheses 2.1.1 and 2.1.2 over X and consider the linearized power densities $dH = \{dH_{ij}: \ 1 \le i \le j \le n+1, \ i \ne n+1\}$. Assume further that the system* (2.23) *with trivial right-hand sides has no non-trivial solution. Then γ is uniquely determined by dH and we have the following stability estimate*

$$\|tr\ (\gamma_0^{-1}\gamma)\|_{H^1(X)} + \|\gamma\|_{L^2(X)} \le C\|dH\|_{H^1(X)}. \tag{2.24}$$

2.2 Microlocal inversion

2.2.1 Preliminaries

Linear algebra. In the following, we consider the $n \times n$ matrices $M_n(\mathbb{R})$ with the inner product structure

$$A : B = \text{tr } (AB^T) = \sum_{i,j=1}^{n} A_{ij} B_{ij}, \tag{2.25}$$

for which $M_n(\mathbb{R})$ admits the orthogonal decomposition $A_n(\mathbb{R}) \oplus S_n(\mathbb{R})$. For two vectors $U = (u_1, \dots, u_n)^T$ and $V = (v_1, \dots, v_n)^T$ in $\mathbb{R}^n$ we denote by $U \otimes V$ the matrix with entries $\{u_i v_j\}_{i,j=1}^n$, and we also define the symmetrized outer product

$$U \odot V := \frac{1}{2}(U \otimes V + V \otimes U). \tag{2.26}$$

With $\cdot$ denoting the standard dotproduct on $\mathbb{R}^n$, we have the following identities

$$2U \odot V : X \odot Y = (U \cdot X)(V \cdot Y) + (U \cdot Y)(V \cdot X), \qquad U, V, X, Y \in \mathbb{R}^n, \tag{2.27}$$

$$U \cdot MU = M : U \otimes U = M : U \odot U, \qquad U \in \mathbb{R}^n, M \in M_n(\mathbb{R}). \tag{2.28}$$

Pseudo-differential calculus. Recall that we denote the set of *symbols* of order m on X by $S^m(X)$, which is the space of functions $p \in C^\infty(X \times \mathbb{R}^n)$ such that for all multi-indices α and β and every compact set $K \subset X$ there is a constant $C_{\alpha,\beta,K}$ such that

$$\sup_{x \in K} |D_x^\beta D_\xi^\alpha p(x, \xi)| \le C_{\alpha,\beta,K}(1 + |\xi|)^{m-|\alpha|}.$$

We denote the operator $p(x,D)$ as

$$p(x,D)\gamma(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\iota x\cdot\xi} p(x,\xi)\hat{\gamma}(\xi) d\xi$$

and the set of *pseudo-differential operators* (ΨDO) of order m on X by $\Psi^m(X)$, where

$$\Psi^m(X) = \{p(x,D) : p \in S^m(X)\}.$$

Suppose $\{m_j\}_0^\infty$ is strictly decreasing and $\lim m_j = -\infty$, and suppose $p_j \in S^{m_k}(X)$ for each j. We denote an *asymptotic expansion* of the symbol $p \in S^{m_0}(X)$ as $p \sim \sum_0^\infty p_j$ if

$$p - \sum_{j<k} p_j \in S^{m_k}(X), \quad \text{for all } k > 0.$$

Given two ΨDO P and Q with respective symbols σ_P and σ_Q and orders d_P and d_Q, we will make repetitive use of the symbol expansion of the product operator $QP \equiv Q \circ P$

$$\sigma_{QP}(x,\xi) \sim \sigma_Q\sigma_P + \frac{1}{\iota}\nabla_\xi\sigma_Q \cdot \nabla_x\sigma_P + \mathcal{O}(|\xi|^{d_Q+d_P-2}), \tag{2.29}$$

where $\mathcal{O}(|\xi|^\alpha)$ denotes a symbol of order at most α. As we will need to compute products of three ΨDO R, P and Q, we write the following formula for later use, obtained by iteration of (2.29)

$$\begin{aligned}\sigma_{RQP} = \sigma_R\sigma_Q\sigma_P + \frac{1}{\iota}(\sigma_R\nabla_\xi\sigma_Q \cdot \nabla_x\sigma_P + \sigma_Q\nabla_\xi\sigma_R \cdot \nabla_x\sigma_P + \sigma_P\nabla_\xi\sigma_R \cdot \nabla_x\sigma_Q) \\ + \mathcal{O}(|\xi|^{d_R+d_Q+d_P-2}).\end{aligned} \tag{2.30}$$

In the next derivations, some operators have matrix-valued principal symbols. However we will only compose them with operators with scalar symbols, so that the above calculus remains valid.

2.2.2 Symbol calculus for the LPD and proof of Lemma 2.1.5

Writing $v_i(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\boldsymbol{\iota} x\cdot\xi} \hat{v}_i(\xi)\, d\xi$ and $\gamma(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\boldsymbol{\iota} x\cdot\xi} \hat{\gamma}(\xi)\, d\xi$ (understood in the componentwise sense), we have

$$\begin{aligned} L_0 v_i := -\nabla\cdot(\gamma_0 \nabla v_i) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\boldsymbol{\iota} x\cdot\xi} \left(\xi\cdot\gamma_0\xi - \boldsymbol{\iota}(\nabla\cdot\gamma_0)\cdot\xi\right) \hat{v}_i(\xi)\, d\xi, \\ P_i\gamma \equiv \nabla\cdot(\gamma\nabla u_i) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\boldsymbol{\iota} x\cdot\xi} \left(\boldsymbol{\iota}\xi \odot \nabla u_i + \nabla^2 u_i\right) : \hat{\gamma}(\xi)\, d\xi. \end{aligned}$$

Thus equation (2.5) reads $L_0 v_i = P_i\gamma$, where the operators $L_0 := -\nabla\cdot(\gamma_0\nabla)$ and P_i have respective symbols

$$\sigma_{L_0} = l_2 + l_1, \qquad l_2 := \xi\cdot\gamma_0\xi \in S^2 \quad \text{and} \quad l_1 := -\boldsymbol{\iota}(\nabla\cdot\gamma_0)\cdot\xi \in S^1, \tag{2.31}$$

$$\sigma_{P_i} = p_{i,1} + p_{i,0}, \qquad p_{i,1} := \boldsymbol{\iota}\xi\odot\nabla u_i \in (S^1)^{n\times n} \quad \text{and} \quad p_{i,0} := \nabla^2 u_i \in (S^0)^{n\times n}. \tag{2.32}$$

For Y a smooth vector field, we will also need in the sequel to express the operator $Y\cdot\nabla$ as ΨDO, the symbol of which is denoted $\sigma_{Y\cdot\nabla} = \sigma_{Y\cdot\nabla}|_1 := \boldsymbol{\iota}\xi\cdot Y$.

We now write dH_{ij} as a ΨDO of γ with symbol M_{ij} as in (2.10). dH_{ij} belongs to $\Psi^0(X)$ and we will compute in this chapter the first two terms in the expansion of M_{ij} (call them $M_{ij}|_0$ and $M_{ij}|_{-1}$), which in turn relies on constructing parametrices of L_0 of increasing order and doing some computations on symbols of products of ΨDO based on formula (2.29). If Q is a parametrix of L_0 modulo Ψ^{-m}, i.e. $K \equiv QL_0 - Id \in \Psi^{-m}$, then straightforward computations based on the relation $L_0 v_i = P_i\gamma$ yield the following relation

$$dH_{ij}(\gamma) = \gamma : \nabla u_i \odot \nabla u_j + (\gamma_0\nabla u_i\cdot\nabla)\circ Q\circ P_j\gamma + (\gamma_0\nabla u_j\cdot\nabla)\circ Q\circ P_i\gamma + K_{ij}\gamma, \tag{2.33}$$

$$\text{where} \quad K_{ij} := (\gamma_0\nabla u_i\cdot\nabla)\circ KL_0^{-1}P_j + (\gamma_0\nabla u_j\cdot\nabla)\circ KL_0^{-1}P_i. \tag{2.34}$$

For any i, $L_0^{-1}P_i$ denotes the operator $\gamma \mapsto v_i$ where v_i solves (2.5), and standard elliptic theory allows to claim that $L_0^{-1}P_i$ smoothes by one derivative so that the error operator K_{ij} defined in (2.34) smoothes by m derivatives. In particular, upon computing a parametrix Q of L_0 modulo Ψ^{-m}, the first three terms in (2.33) are enough to construct the principal part of the symbol M_{ij} modulo Ψ^{-m}.

In light of the last remark, we first compute a parametrix Q of L_0 modulo Ψ^{-1}, that is, since $L_0 \in \Psi^2$, we look for a principal symbol of the form $\sigma_Q = q_{-2} + \mathcal{O}(|\xi|^{-3})$. Clearly, we easily obtain $q_{-2} = l_2^{-1} = (\xi \cdot \gamma_0 \xi)^{-1}$. In this case, the principal symbol of dH_{ij} at order zero is given by, according to (2.33) and (2.29),

$$\begin{aligned} M_{ij}|_0 &= \nabla u_i \odot \nabla u_j + (\sigma_{\gamma_0 \nabla u_i \cdot \nabla}|_1)\ q_{-2}\ p_{j,1} + (\sigma_{\gamma_0 \nabla u_j \cdot \nabla}|_1)\ q_{-2}\ p_{i,1} \\ &= \nabla u_i \odot \nabla u_j - \frac{1}{\xi \cdot \gamma_0\ \xi} \left((\gamma_0 \nabla u_i \cdot \xi)(\xi \odot \nabla u_j) + (\gamma_0 \nabla u_j \cdot \xi)(\xi \odot \nabla u_i) \right). \end{aligned}$$

$M_{ij}|_0$ admits a somewhat more symmetric expression if pre- and post-multiplied by A_0, the unique positive squareroot of γ_0, so that we may write,

$$M_{ij}|_0(x, \xi) = A_0^{-1} \left(V_i \odot V_j - (\hat{\xi}_0 \cdot V_i)\hat{\xi}_0 \odot V_j - (\hat{\xi}_0 \cdot V_j)\hat{\xi}_0 \odot V_i \right) A_0^{-1}, \tag{2.35}$$

where we have defined $\xi_0 := A_0 \xi$ and $\hat{x} := |x|^{-1}x$ for any $x \in \mathbb{R}^n - \{0\}$ as well as $V_i := A_0 \nabla u_i$. This last expression motivates the proof of Lemma 2.1.5.

Proof of Lemma 2.1.5. Proof of *(i)*: Let η such that $\eta \cdot \xi = 0$, and denote $\eta' := A_0^{-1}\eta$

so that $\eta' \cdot \xi_0 = 0$. Then using identity (2.27) and (2.35), we get

$$
\begin{aligned}
2\|\xi_0\|^{-1} M_{ij}|_0(x,\xi) : \gamma_0 \xi \odot \eta &= 2\left[V_i \odot V_j - (\hat{\xi}_0 \cdot V_i)\hat{\xi}_0 \odot V_j - (\hat{\xi}_0 \cdot V_j)\hat{\xi}_0 \odot V_i\right] : \hat{\xi}_0 \odot \eta' \\
&= (V_i \cdot \hat{\xi}_0)(V_j \cdot \eta') + (V_i \cdot \eta')(V_j \cdot \hat{\xi}_0) - (\hat{\xi}_0 \cdot V_i)(\hat{\xi}_0 \cdot \hat{\xi}_0)(V_j \cdot \eta') \\
&\quad - (\hat{\xi}_0 \cdot V_i)(\hat{\xi}_0 \cdot \eta')(V_j \cdot \hat{\xi}_0) - (\hat{\xi}_0 \cdot V_j)(\hat{\xi}_0 \cdot \hat{\xi}_0)(V_i \cdot \eta') \\
&\quad - (\hat{\xi}_0 \cdot V_j)(\hat{\xi}_0 \cdot \eta')(\hat{\xi}_0 \cdot V_i) \\
&= 0,
\end{aligned}
$$

where we have used $\hat{\xi}_0 \cdot \hat{\xi}_0 = 1$ and $\hat{\xi}_0 \cdot \eta' = 0$, thus *(i)* holds.

Proof of *(ii)*: Recall that

$$
M_{ij}|_0 : P = \left[V_i \odot V_j - (\hat{\xi}_0 \cdot V_i)\hat{\xi}_0 \odot V_j - (\hat{\xi}_0 \cdot V_j)\hat{\xi}_0 \odot V_i\right] : A_0^{-1} P A_0^{-1}.
$$

We write $S_n(\mathbb{R})$ as the direct orthogonal sum of three spaces:

$$
S_n(\mathbb{R}) = \left(\mathbb{R}\ \hat{\xi}_0 \otimes \hat{\xi}_0\right) \oplus \left(\{\hat{\xi}_0\}^\perp \odot \{\xi_0\}^\perp\right) \oplus \left(\hat{\xi}_0 \odot \{\hat{\xi}_0\}^\perp\right), \tag{2.36}
$$

with respective dimensions 1, $n(n-1)/2$ and $n-1$. Decomposing $A_0^{-1} P A_0^{-1}$ uniquely into this sum, we write $A_0^{-1} P A_0^{-1} = P_1 + P_2 + P_3$. Direct calculations then show that

$$
M_{ij}|_0 : A_0^{-1} P A_0^{-1} = V_i \odot V_j : (-P_1 + P_2), \quad 1 \le i \le j \le n.
$$

Since $\{V_i\}_{i=1}^n$ is a basis of $\mathbb{R}^n$, $\{V_i \odot V_j\}_{1 \le i \le j \le n}$ is a basis of $S_n(\mathbb{R})$ and thus (2.12) implies that

$$
-P_1 + P_2 = 0, \qquad \text{i.e.} \qquad P_1 = P_2 = 0.
$$

Therefore $P = A_0 P_3 A_0$ with $P_3 = \hat{\xi}_0 \odot \eta'$ for some $\eta' \cdot \hat{\xi}_0 = 0$, so $P = \gamma_0 \xi \odot \eta$ with η proportional to $A_0 \eta'$, i.e. such that $\eta \cdot \xi = 0$, thus the proof is complete.

In other words, **all** symbols of order zero $M_{ij}|_0(x,\xi)$ are orthogonal to the (x,ξ)-dependent $n-1$-dimensional subspace of symmetric matrices $\gamma_0 \xi \odot \{\xi\}^\perp$. One must thus compute the next term in the symbol exampansion of the operators dH_{ij}, i.e. $M_{ij}|_{-1}$. We will then show that enough symbols of the form $M_{ij}|_0 + M_{ij}|_{-1}$ will suffice to span the entire space $S_n(\mathbb{R})$ for every $x \in \Omega'$ and $\xi \in \mathbb{S}^1$, so that the corresponding family of operators is elliptic as a function of γ.

2.2.3 Computation of $M_{ij}|_{-1}$

As the previous section explained, the principal symbols $M_{ij}|_0$ can never span $S_n(\mathbb{R})$. Therefore, we compute the next term $M_{ij}|_{-1}$ in their symbol expansion. We must first construct a parametrix Q of L_0 modulo Ψ^{-2}, i.e. of the form

$$\sigma_Q = q_{-2} + q_{-3} + \mathcal{O}(|\xi|^{-4}), \quad q_i \in S^i. \tag{2.37}$$

Lemma 2.2.1. *The symbols q_{-2} and q_{-3} defined in* (2.37) *have respective expressions*

$$q_{-2} = l_2^{-1} = (\xi \cdot \gamma_0 \xi)^{-1}, \tag{2.38}$$

$$q_{-3} = l_2^{-3} \boldsymbol{\iota}\ \xi_p \xi_q \xi_j \left([\gamma_0]_{pq} \partial_{x_i} [\gamma_0]_{ij} - 2[\gamma_0]_{ij} \partial_{x_i} [\gamma_0]_{pq} \right). \tag{2.39}$$

Proof of Lemma 2.2.1. Using formula (2.29) with $(Q,P) \equiv (Q,L_0)$, and using the expansions of σ_Q and σ_{L_0}, we get

$$\sigma_{QL_0} \sim q_{-2} l_2 + (q_{-2} l_1 + q_{-3} l_2 + \frac{1}{\boldsymbol{\iota}} \nabla_\xi q_{-2} \cdot \nabla_x l_2) + \mathcal{O}(|\xi|^{-2}).$$

In order to match the expansion $1 + 0 + \mathcal{O}(|\xi|^{-2})$, the expansion above must satisfy, for large ξ,

$$q_{-2}l_2 = 1 \quad \text{and} \quad q_{-2}l_1 + q_{-3}l_2 + \frac{1}{\boldsymbol{\iota}}\nabla_\xi q_{-2} \cdot \nabla_x l_2 = 0,$$

that is, $q_{-2} = l_2^{-1} = (\xi \cdot \gamma_0 \xi)^{-1}$ and

$$q_{-3} = l_2^{-1}\left(-q_{-2}l_1 - \frac{1}{\boldsymbol{\iota}}\nabla_\xi q_{-2} \cdot \nabla_x l_2\right) = l_2^{-3}\left(-l_2 l_1 - \boldsymbol{\iota}\nabla_\xi l_2 \cdot \nabla_x l_2\right).$$

Now, we easily have $\nabla_\xi l_2 = 2\gamma_0\xi$ and $\nabla_x l_2 = \partial_{x_i}[\gamma_0]_{pq}\xi_p\xi_q\mathbf{e}_i$, where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is the natural basis of $\mathbb{R}^n$. We thus deduce the expression of q_3

$$q_{-3} = l_2^{-3}\boldsymbol{\iota}\left([\gamma_0]_{pq}\xi_p\xi_q\partial_{x_i}[\gamma_0]_{ij}\xi_j - 2[\gamma_0]_{ij}\xi_j\partial_{x_i}[\gamma_0]_{pq}\xi_p\xi_q\right),$$

from which (2.39) holds. q_{-3} is clearly in S^{-3} from this expression, since l_2^{-3} is of order -6. The proof is complete □

We now give the expression of $M_{ij}|_{-1}$ (or rather, that of $A_0\ M_{ij}|_{-1}\ A_0$).

Proposition 2.2.2 (Expression of $A_0 M_{ij}|_{-1} A_0$)**.** *Given any (i,j) the symbol $A_0\ M_{ij}|_{-1}\ A_0$ admits the following expression*

$$\begin{aligned} A_0\ M_{ij}|_{-1}(x,\xi)\ A_0 = {} & \boldsymbol{\iota}\|\xi_0\|^{-1}\left((\hat{\xi}_0 \cdot V_j)(\mathbb{H}_i - 2\hat{\xi}_0 \odot \mathbb{H}_i\hat{\xi}_0) + \hat{\xi}_0 \odot \mathbb{H}_i V_j\right) \\ & + \boldsymbol{\iota}\|\xi_0\|^{-1}\left((\hat{\xi}_0 \cdot V_i)(\mathbb{H}_j - 2\hat{\xi}_0 \odot \mathbb{H}_j\hat{\xi}_0) + \hat{\xi}_0 \odot \mathbb{H}_j V_i\right) \\ & + \boldsymbol{\iota}\|\xi_0\|^{-1}\left(V_j \cdot G(x,\xi)(\hat{\xi}_0 \odot V_i) + V_i \cdot G(x,\xi)(\hat{\xi}_0 \odot V_j)\right), \end{aligned} \tag{2.40}$$

where we have defined $V_i := A_0\nabla u_i$, $\mathbb{H}_i := A_0\ \nabla^2 u_i\ A_0$, as well as the vector field

$$G(x,\xi) := \|\xi_0\|^2(\boldsymbol{\iota} q_{-3}\xi_0 + A_0\nabla_x q_{-2}) \in (S^0)^n. \tag{2.41}$$

Proof of Prop. (2.40). Assume Q is a parametrix of L_0 modulo Ψ^{-2} and consider formula (2.33). Since the term $\gamma : \nabla u_i \odot \nabla u_j$ is of order zero, the computation of $M_{ij}|_{-1}$ consists in computing the second term in the symbol expansion of $R_i \circ Q \circ P_j$, and the same term with i, j permuted, where we denote $R_i := \gamma_0 \nabla u_i \cdot \nabla$ with symbol $r_{i,1} = \boldsymbol{\iota}\gamma_0 \nabla u_i \cdot \xi$. Plugging $\sigma_{R_i} = r_{i,1}$, $\sigma_Q = q_{-2} + q_{-3}$ and $\sigma_{P_i} = p_{i,1} + p_{i,0}$ into (2.30) and keeping only the terms that are homogeneous of degree -1 in ξ, we arrive at the expression

$$\sigma_{R_iQP_j}|_{-1} = r_{i,1}(q_{-3}p_{j,1} + q_{-2}p_{j,0}) + \frac{1}{\boldsymbol{\iota}}(p_{j,1}\nabla_\xi r_{i,1} \cdot \nabla_x q_{-2} + \nabla_\xi(q_{-2}r_{i,1}) \cdot \nabla_x p_{j,1}). \quad (2.42)$$

Note that the multiplications commute because the symbols of Q and R_i are scalar, while that of P_j is matrix-valued. Since $M_{ij}|_{-1} = \sigma_{R_iQP_j}|_{-1} + \sigma_{R_jQP_i}|_{-1}$, equation (2.40) will be proved when we show that

$$A_0\ \sigma_{R_iQP_j}|_{-1}\ A_0 = \boldsymbol{\iota}\|\xi_0\|^{-1}\left((\hat{\xi}_0 \cdot V_i)(\mathbb{H}_j - 2\hat{\xi}_0 \odot \mathbb{H}_j\hat{\xi}_0) + \hat{\xi}_0 \odot \mathbb{H}_j V_i + V_i \cdot G(x,\xi)(\hat{\xi}_0 \odot V_j)\right). \quad (2.43)$$

Proof of (2.43)**.** Starting from (2.42), plugging the expression $r_{i,1} = \boldsymbol{\iota}(V_i \cdot \xi_0)$, using the identity

$$\nabla_\xi(q_{-2}r_{i,1}) \cdot \nabla_x p_{j,1} = \boldsymbol{\iota}\xi \odot (\nabla^2 u_j \nabla_\xi(q_{-2}r_{i,1})),$$

and pre- and post-multiplying by A_0 yields the relation

$$\begin{aligned} A_0\ \sigma_{R_iQP_j}|_{-1}\ A_0 = {} & \boldsymbol{\iota}(V_i \cdot \xi_0)(q_{-3}\boldsymbol{\iota}\xi_0 \odot V_j + q_{-2}\mathbb{H}_j) \\ & + (V_i \cdot A_0\nabla_x q_{-2})\boldsymbol{\iota}\xi_0 \odot V_j + \xi_0 \odot \mathbb{H}_j A_0^{-1}\nabla_\xi(q_{-2}r_{i,1}). \end{aligned} \quad (2.44)$$

Gathering the first and third terms recombines into $\boldsymbol{\iota}\|\xi_0\|^{-1}V_i \cdot G(\hat{\xi}_0 \odot V_j)$ (the last term of

(2.43)). On to the second and fourth terms, we first compute

$$A_0^{-1}\nabla_\xi(r_{i,1}q_{-2}) = \boldsymbol{\iota}(V_i\cdot\xi_0)(-\|\xi_0\|^{-4})2\xi_0 + \|\xi_0\|^{-2}\boldsymbol{\iota}V_i = \boldsymbol{\iota}\|\xi_0\|^{-2}(V_i - 2(V_i\cdot\hat{\xi}_0)\hat{\xi}_0).$$

Using this calculation, the second and fourth terms in (2.44) recombine into

$$\boldsymbol{\iota}\|\xi_0\|^{-1}\left((\hat{\xi}_0\cdot V_i)(\mathbb{H}_j - 2\hat{\xi}_0\odot\mathbb{H}_j\hat{\xi}_0) + \hat{\xi}_0\odot\mathbb{H}_jV_i\right),$$

thus the argument is complete. □

2.2.4 Proof of Proposition 2.1.6

Preliminaries: By virtue of Hypothesis 2.1.1, ∇u_{n+1} may be decomposed into the basis $\nabla u_1, \dots, \nabla u_n$ by means of scalars $\mu_1, \dots, \mu_n, \mu$ such that

$$\sum_{i=1}^n \frac{\mu_i}{\mu}\nabla u_i + \nabla u_{n+1} = 0. \tag{2.45}$$

As seen in [44, 45], the coefficients $\mu_1, \dots, \mu_{n+1}$ are directly computible from the power densities $\{\nabla u_i\cdot\gamma_0\nabla u_j\}_{1\le i\le j\le n+1}$ and on the other hand, we have the relation

$$\frac{\mu_i}{\mu} = \frac{\det(\nabla u_1, \dots, \overbrace{\nabla u_{n+1}}^{i}, \dots, \nabla u_n)}{\det(\nabla u_1, \dots, \nabla u_m)}, \quad 1\le i\le n,$$

thus $\nabla\frac{\mu_i}{\mu} = Z_i$ as defined in (2.8) for $1\le i\le n$. In the next proofs, we will use the following

Lemma 2.2.3. *Under hypotheses 2.1.1 and 2.1.2, the following matrix-valued function*

$$\mathbb{M} := \mu_i\mathbb{H}_i + \mu\mathbb{H}_{n+1} \tag{2.46}$$

is symmetric and uniformly invertible.

Proof. Symmetry of $\mathbb{M}$ is obvious by definition. Taking gradient of (2.45), we arrive at

$$\sum_{i=1}^{n} Z_i \otimes \nabla u_i + \frac{\mu_i}{\mu}\nabla^2 u_i + \nabla^2 u_{n+1} = 0.$$

Pre- and post-multiplying by A_0, we deduce that

$$\mathbb{M} = \mu_i \mathbb{H}_i + \mu \mathbb{H}_{n+1} = -\mu A_0 Z_i \otimes V_i = -\mu A_0 Z \mathbb{V}^T,$$

where $\mathbb{V} := [V_1| \dots |V_n]$. The proof is complete since Hyp. 2.1.1 ensures that μ never vanishes and $\mathbb{V}$ is uniformly invertible, and Hyp. 2.1.2 ensures that Z is uniformly invertible. □

The T_{pq} operators. Proof of Prop. 2.1.6: As advertised in Sec. 2.1.2, because of the algebraic form of the symbols of the linearized power density operators, it is convenient for inversion purposes to define the microlocal change of basis $T\gamma = \{T_{pq}\gamma\}_{1\le p\le q\le n}$ as in (2.14), i.e.

$$T_{pq}\gamma(x) := (2\pi)^{-n}\int_{\mathbb{R}^n} e^{\iota\xi\cdot x} A_0^{-1}\hat{\xi}_p \odot \hat{\xi}_q A_0^{-1} : \hat{\gamma}(\xi)\ d\xi.$$

To convince ourselves that this collection forms a microlocally invertible operator of γ, let us introduce the zero-th order ΨDOs P_{ijpq} with scalar principal symbol $\sigma_{P_{ijpq}} := (\mathbf{e}_i \cdot A_0\hat{\xi}_p)(\mathbf{e}_j \cdot A_0\hat{\xi}_q)$ for $1 \le i,j,p,q \le n$. Then for any $1 \le i \le j \le n$, the composition of operators $\sum_{p,q=1}^n P_{ijpq} \circ T_{pq}$ has principal symbol

$$\sum_{p,q=1}^{n} (\mathbf{e}_i \cdot A_0\hat{\xi}_p)(\mathbf{e}_j \cdot A_0\hat{\xi}_q) A_0^{-1}\hat{\xi}_p \odot \hat{\xi}_q A_0^{-1} = \sum_{p=1}^{n}(\mathbf{e}_i \cdot A_0\hat{\xi}_p)A_0^{-1}\hat{\xi}_p \odot \sum_{q=1}^{n}(\mathbf{e}_j \cdot A_0\hat{\xi}_q)A_0^{-1}\hat{\xi}_q = \mathbf{e}_i \odot \mathbf{e}_j,$$

where we have used the following property, true for any smooth vector field V:

$$V = \sum_{p=1}^{n} (V \cdot A_0 \hat{\xi}_p) A_0^{-1} \hat{\xi}_p.$$

Thus for any $1 \le i \le j \le n$, the composition $\sum_{p,q=1}^{n} P_{ijpq} \circ T_{pq}$ recovers $\gamma_{ij} = \gamma : \mathbf{e}_i \otimes \mathbf{e}_j$ up to a regularization term. This in particular justifies the estimates (2.15) and the subsequent inversion procedure. We are now ready to prove Proposition 2.1.6.

Proof of Proposition 2.1.6. From the fact that $(V_1, \dots, V_n)$ is a basis at every point and given their dotproducts $H_{ij} = V_i \cdot V_j$, we have the following formula, true for every vector field W:

$$W = H^{pq} (W \cdot V_p) V_q. \tag{2.47}$$

Proof of *(i)*: Reconstruction of the components $T_{00}\gamma$ and $\{T_{\alpha\beta}\gamma\}_{1\le\alpha\le\beta\le n-1}$. We work with $\widetilde{M_{ij}}|_0 := A_0\ M_{ij}|_0\ A_0 = V_i \odot V_j - (\hat{\xi}_0 \cdot V_i)\hat{\xi}_0 \odot V_j - (\hat{\xi}_0 \cdot V_j)\hat{\xi}_0 \odot V_i$. Using (2.47) with $W \equiv \hat{\xi}_\alpha$, straightforward computations yield

$$\begin{aligned} \sum_{i,j,p,q} H^{qj} (\hat{\xi}_\alpha \cdot V_q) H^{pi} (\hat{\xi}_\beta \cdot V_p) \widetilde{M_{ij}}|_0 &= \hat{\xi}_\alpha \odot \hat{\xi}_\beta - (\hat{\xi}_0 \cdot \hat{\xi}_\alpha)\hat{\xi}_0 \odot \hat{\xi}_\beta - (\hat{\xi}_0 \cdot \hat{\xi}_\beta)\hat{\xi}_0 \odot \hat{\xi}_\alpha \\ &= \begin{cases} -\hat{\xi}_0 \odot \hat{\xi}_0 & \text{if} \quad \alpha = \beta = 0, \\ 0 & \text{if} \quad 0 = \alpha \neq \beta, \\ \hat{\xi}_\alpha \odot \hat{\xi}_\beta & \text{if} \quad \alpha \neq 0, \beta \neq 0. \end{cases} \end{aligned}$$

which means that upon defining $Q_{\alpha\beta ij} \in \Psi^0$ with scalar principal symbols

$$\sigma_{Q_{00ij}} := -\sum_{p,q} H^{qj}(\hat{\xi}_0 \cdot V_q) H^{pi}(\hat{\xi}_0 \cdot V_p),$$

$$\sigma_{Q_{\alpha\beta ij}} := \sum_{p,q} H^{qj}(\hat{\xi}_\alpha \cdot V_q) H^{pi}(\hat{\xi}_\beta \cdot V_p), \quad 1 \le \alpha \le \beta \le n-1,$$

relation (2.17) is satisfied in the sense of operators since the previous calculation amounts to computing the principal symbol of the composition of operators in (2.17).

Proof of *(ii)*: Reconstruction of the components $\{T_{0\alpha}\gamma\}_{1\le\alpha\le n-1}$**.** It remains to construct appropriate operators that will map $dH(\gamma)$ to the components $T_{0\alpha}\gamma$ for $1 \le \alpha \le n-1$, which is where the additional measurements $dH_{i,n+1}$ come into play. Let $(\mu_1, \dots, \mu_n, \mu)$ as in (2.45) and construct the ΨDO $\{L_i(\gamma)\}_{i=1}^n$ as in (2.16). It is easy to see that, since the μ_i are only functions of x, the terms of fixed homogeneity in the symbol expansion of L_i satisfy

$$\sigma_{L_i}|_k = \mu_j M_{ij}|_k + \mu M_{i,n+1}|_k, \qquad k = 0, -1, -2, \dots.$$

Then from equation (2.35) and relation (2.45), we deduce that $\sigma_{L_i}|_0 = 0$, so that $L_i \in \Psi^{-1}$. Moreover, using equation (2.40) together with relation (2.45), we deduce that

$$\tilde{\sigma}_{L_i}|_{-1} = A_0 \ \sigma_{L_i}|_{-1} \ A_0 = \boldsymbol{\iota}\|\xi_0\|^{-1} \left((\hat{\xi}_0 \cdot V_i)(\mathbb{M} - 2\hat{\xi}_0 \odot \mathbb{M}\hat{\xi}_0) + \hat{\xi}_0 \odot \mathbb{M} V_i \right)$$

is now the principal symbol of L_i. Using relation (2.47) with $W \equiv \mathbb{M}^{-1}\hat{\xi}_\alpha$, the symmetry of $\mathbb{M}$ and multiplying by $\mathbb{M}$, we have the relation

$$\hat{\xi}_\alpha = H^{pq}(\hat{\xi}_\alpha \cdot \mathbb{M}^{-1} V_p) \mathbb{M} V_q.$$

Using this relation, we deduce the following calculation, for $1 \leq \alpha \leq n-1$

$$H^{pi}(\hat{\xi}_\alpha \cdot \mathbb{M}^{-1} V_p)\ \tilde{\sigma}_{L_i}|_{-1} = \boldsymbol{\iota} \|\xi_0\|^{-1} \left((\hat{\xi}_0 \cdot \mathbb{M}^{-1}\hat{\xi}_\alpha)(\mathbb{M} - 2\hat{\xi}_0 \odot \mathbb{M}\hat{\xi}_0) + \hat{\xi}_0 \odot \hat{\xi}_\alpha \right). \tag{2.48}$$

While the second term gives us the missing components $T_{0\alpha}\gamma$, we claim that the first one is spanned by $\hat{\xi}_0 \odot \hat{\xi}_0$ and $\{\hat{\xi}_\alpha \odot \hat{\xi}_\beta\}_{1 \leq \alpha \leq \beta \leq n-1}$. Indeed we have

$$\begin{aligned}
(\mathbb{M} - 2\hat{\xi}_0 \odot \mathbb{M}\hat{\xi}_0) : (\hat{\xi}_0 \odot \hat{\xi}_\alpha) &= 0, \quad 1 \leq \alpha \leq n-1, \\
(\mathbb{M} - 2\hat{\xi}_0 \odot \mathbb{M}\hat{\xi}_0) : (\hat{\xi}_0 \odot \hat{\xi}_0) &= -\hat{\xi}_0 \cdot \mathbb{M}\hat{\xi}_0, \\
(\mathbb{M} - 2\hat{\xi}_0 \odot \mathbb{M}\hat{\xi}_0) : (\hat{\xi}_\alpha \odot \hat{\xi}_\beta) &= \hat{\xi}_\alpha \cdot \mathbb{M}\hat{\xi}_\beta, \quad 1 \leq \alpha \leq \beta \leq n-1,
\end{aligned}$$

so we deduce that

$$\mathbb{M} - 2\hat{\xi}_0 \odot \mathbb{M}\hat{\xi}_0 = -(\hat{\xi}_0 \cdot \mathbb{M}\hat{\xi}_0)\ \hat{\xi}_0 \odot \hat{\xi}_0 + \sum_{1 \leq \alpha, \beta \leq n-1} (\hat{\xi}_\alpha \cdot \mathbb{M}\hat{\xi}_\beta)\ \hat{\xi}_\alpha \odot \hat{\xi}_\beta. \tag{2.49}$$

In light of these algebraic calculations, we now build the parametrices. Let $L_0^{\frac{1}{2}} \in \Psi^1$, $B_{\alpha i} \in \Psi^0$, $R \in (\Psi^0)^{n \times n}$, $R_\alpha \in \Psi^0$ and $R_{\alpha\beta} \in \Psi^0$ the ΨDOs with respective principal symbols

$$\begin{aligned}
&\sigma_{L_0^{\frac{1}{2}}} = -\boldsymbol{\iota}\|\xi_0\|, \quad \sigma_{B_{\alpha i}} = H^{pi}(\hat{\xi}_\alpha \cdot \mathbb{M}^{-1} V_p), \quad \sigma_R = \mathbb{M} - 2\hat{\xi}_0 \odot \mathbb{M}\hat{\xi}_0, \\
&\sigma_{R_\alpha} = \hat{\xi}_0 \cdot \mathbb{M}^{-1}\hat{\xi}_\alpha, \quad \sigma_{R_{\alpha\beta}} = \hat{\xi}_\alpha \cdot \mathbb{M}\hat{\xi}_\beta.
\end{aligned}$$

Then the relation (2.48) implies (2.18) at the principal symbol level. The operator R can indeed be expressed as the following zero-th order linear combination of the components

T_{00} and $\{T_{\alpha\beta}\}_{1\le\alpha,\beta\le n-1}$:

$$\begin{aligned} R &= -R_{00}T_{00} + \sum_{1\le\alpha,\beta\le n-1} R_{\alpha\beta}T_{\alpha\beta} \\ &= \sum_{i,j=1}^{n} \left(-R_{00}Q_{00ij} + \sum_{1\le\alpha,\beta\le n-1} R_{\alpha\beta}Q_{\alpha\beta ij} \right) \circ dH_{ij} \mod \Psi^{-1}, \end{aligned}$$

so that the left-hand side of (2.18) is expressed as a post-processing of measurement operators dH_{ij} only. The proof is complete. □

2.3 Explicit inversion

2.3.1 Preliminaries and notation

For a matrix A with columns $A_1, \dots, A_n$ and $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ the canonical basis, one has the following representation

$$A = \sum_{j=1}^{n} A_j \otimes \mathbf{e}_j \quad \text{and} \quad A^T = \sum_{j=1}^{n} \mathbf{e}_j \otimes A_j.$$

More generally, for two matrices $A = [A_1|\dots|A_n]$ and $B = [B_1|\dots|B_n]$, we have the relation

$$\sum_{j=1}^{n} A_j \otimes B_j = AB^T.$$

Finally, for A a matrix and $V = [V_1|\dots|V_n]$, the sum $A_{ij}V_j$ is nothing but the i-th column of the matrix VA^T.

2.3.2 Derivation of (2.21) from Hypothesis 2.1.1:

Let us start from n solutions $(u_1, \dots, u_n)$ fulfilling Hypothesis 2.1.1, and let $(v_1, \dots, v_n)$ the corresponding solutions of (2.5). We also denote $[\nabla U] := [\nabla u_1|\dots|\nabla u_n]$ and $[\nabla V]$

similarly. We first mention that for any vector field V, we have the following formulas

$$V = H^{pq}(V \cdot \gamma_0 \nabla u_p)\nabla u_q = H^{pq}(V \cdot \nabla u_p)\gamma_0 \nabla u_q, \tag{2.50}$$

which also amounts to the following matrix relations

$$H^{pq}(\nabla u_p \otimes \nabla u_q)\gamma_0 = H^{pq}\gamma_0(\nabla u_p \otimes \nabla u_q) = \mathbb{I}_n. \tag{2.51}$$

From the relation

$$dH_{ij} = (\gamma \nabla u_i + \gamma_0 \nabla v_i) \cdot \nabla u_j + \gamma_0 \nabla v_j \cdot \nabla u_i, \quad 1 \leq i, j \leq n,$$

we deduce, using (2.50),

$$\gamma \nabla u_i + \gamma_0 \nabla v_i = H^{pq} \left(dH_{ip} - \gamma_0 \nabla v_p \cdot \nabla u_i \right) \gamma_0 \nabla u_q, \quad 1 \leq i \leq n. \tag{2.52}$$

The previous equation allows us express γ in terms of the remaining unknowns $(v_1, \ldots, v_n)$. Indeed, taking the tensor product of (2.52) with $H^{ij}\gamma_0 \nabla u_j$ and summing over i yields

$$\begin{aligned} \gamma + \gamma_0 \nabla v_i \otimes \nabla u_j \gamma_0 H^{ij} &= H^{pq} \left(dH_{ip} - \gamma_0 \nabla v_p \cdot \nabla u_i \right) \left(\gamma_0 \nabla u_q \otimes \nabla u_j \gamma_0 H^{ij} \right) \\ &= dH_{ip}\gamma_0 (H^{pq} \nabla u_q \otimes H^{ij} \nabla u_j)\gamma_0 - \gamma_0 \nabla u_q \otimes \nabla v_p \gamma_0 H^{pq}, \end{aligned}$$

where we have used the identity (2.50) in the last right-hand side. We may rewrite this as

$$\gamma = \gamma_0 \left(dH_{ip}(H^{pq} \nabla u_q \otimes H^{ij} \nabla u_j) - 2H^{ij} \nabla v_i \odot \nabla u_j \right) \gamma_0. \tag{2.53}$$

One may notice that the above expression is indeed a symmetric matrix. In matrix notation, using the preliminaries, we arrive at the expression (2.21).

2.3.3 Algebraic equations obtained by considering additional solutions:

Let us now add another solution u_{n+1} with corresponding solution v_{n+1} at order $\mathcal{O}(\varepsilon)$. By virtue of Hypothesis 2.1.1, as in section 2.2.4, ∇u_{n+1} may be expressed in the basis $(\nabla u_1, \dots, \nabla u_n)$ as

$$\sum_{i=1}^{n} \frac{\mu_i}{\mu} \nabla u_i + \nabla u_{n+1} = 0, \tag{2.54}$$

where the coefficients μ_i can be expressed as ratios of determinants, or equivalently, computable from the power densities at order ε^0, see [44, Appendix A.3]. For $1 \le i \le n$, we define $Z_i := \nabla(\mu^{-1}\mu_i)$, and notice that we have the following two algebraic relations

$$\sum_{i=1}^{n} Z_i \cdot \gamma_0 \nabla u_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} Z_i^{\flat} \wedge du_i = 0. \tag{2.55}$$

The first one is obtained obtained after applying the operator $\nabla \cdot (\gamma_0 \cdot)$ to (2.54) and the second one is obtained after applying an exterior derivative to (2.54) .

Moving on to the study of the corresponding v_{n+1} solution, we write

$$dH_{n+1,j} + \frac{\mu_i}{\mu} dH_{ij} = \left(\nabla v_{n+1} + \frac{\mu_i}{\mu} \nabla v_i \right) \cdot \gamma_0 \nabla u_j, \quad 1 \le j \le n,$$

where we have cancelled sums of the form (2.54). Using the identity (2.50), we deduce that

$$\nabla v_{n+1} + (\mu^{-1}\mu_i) \nabla v_i = H^{pq} \left(dH_{n+1,p} + (\mu^{-1}\mu_i) dH_{ip} \right) \nabla u_q. \tag{2.56}$$

Taking exterior derivative of the previous relation yields

$$Z_i^{\flat} \wedge dv_i = d \left(H^{pq} (dH_{n+1,p} + (\mu^{-1}\mu_i) dH_{ip}) \right) \wedge du_q. \tag{2.57}$$

We now apply $\nabla\cdot(\gamma_0\cdot)$ to (2.56), the left-hand side becomes

$$\begin{aligned}\nabla\cdot(\gamma_0(\nabla v_{n+1}+(\mu^{-1}\mu_i)\nabla v_i)) &= \nabla\cdot(\gamma_0\nabla v_{n+1})+Z_i\cdot\gamma_0\nabla v_i+(\mu^{-1}\mu_i)\nabla\cdot(\gamma_0\nabla v_i)\\ &= -\nabla\cdot(\gamma\nabla u_{n+1})+Z_i\cdot\gamma_0\nabla v_i-(\mu^{-1}\mu_i)\nabla\cdot(\gamma\nabla u_i)\\ &= Z_i\cdot\gamma_0\nabla v_i-\nabla\cdot(\gamma(\nabla u_{n+1}+(\mu^{-1}\mu_i)\nabla u_i))+Z_i\cdot\gamma\nabla u_i\\ &= Z_i\cdot(\gamma_0\nabla v_i+\gamma\nabla u_i),\end{aligned}$$

thus we arrive at the equation

$$Z_i\cdot(\gamma_0\nabla v_i+\gamma\nabla u_i)=\nabla\left(H^{pq}(dH_{n+1,p}+(\mu^{-1}\mu_i)dH_{ip})\right)\cdot\gamma_0\nabla u_q=:Y_q\cdot\gamma_0\nabla u_q,$$

where the vector fields

$$Y_q:=\nabla\left(H^{pq}(dH_{n+1,p}+(\mu^{-1}\mu_i)dH_{ip})\right),\quad 1\le q\le n, \tag{2.58}$$

are known from the data dH. Combining the latter equation with (2.52), we obtain

$$(Z_i\cdot\gamma_0\nabla u_q)H^{pq}\left(dH_{ip}-\gamma_0\nabla v_p\cdot\nabla u_i\right)=Y_q\cdot\gamma_0\nabla u_q,$$

which we recast as

$$(Z_i\cdot\gamma_0\nabla u_q)H^{pq}(\gamma_0\nabla v_p\cdot\nabla u_i)=(Z_i\cdot\gamma_0\nabla u_q)H^{pq}dH_{ip}-Y_q\cdot\gamma_0\nabla u_q.$$

The left-hand side can be considerably simplified by noticing that the second equation of (2.55) implies $[\nabla U]Z^T=Z[\nabla U]^T$. With this fact in mind, the left-hand side looks like

$X_p \cdot \nabla v_p$, where we compute

$$X_p = H^{pq}\gamma_0 \nabla u_i \otimes Z_i \gamma_0 \nabla u_q = \gamma_0 [\nabla U] Z^T \gamma_0 [\nabla U] H^{-1} \mathbf{e}_p = \gamma_0 Z [\nabla U]^T [\nabla U]^{-T} \mathbf{e}_p = \gamma_0 Z_p.$$

Finally, we obtain the more compact equation

$$\sum_{p=1}^{n} \gamma_0 Z_p \cdot \nabla v_p = f, \qquad \text{where} \qquad f := (H^{pq} dH_{ip}\ Z_i - Y_q) \cdot \gamma_0 \nabla u_q, \tag{2.59}$$

with Y_q given in (2.58).

Remark 2.3.1 (On algebraic inversion)**.** *In equations* (2.57) *and* (2.59)*, the only unknown is the matrix* $[\nabla V] := [\nabla v_1, \dots, \nabla v_n]$*. Equations* (2.57) *and* (2.59) *give us the projection of that matrix onto the space* $ZA_n(\mathbb{R})$ *and onto the line* $\mathbb{R}\gamma_0 Z$ *respectively. As in the non-linear case [44, 45], we expect that a rich enough set of such equations provided by a certain number of additional solutions* $(u_{n+1}, \dots, u_{n+l})$ *leads to a pointwise, algebraic reconstruction of* $[\nabla V]$*, however we do not follow that route here.*

2.3.4 Proof of Proposition 2.1.7 and Theorem 2.1.9

We now show that provided that we use *one* additional solution u_{n+1} (on top of the basis $(u_1, \dots, u_n)$) such that the matrix Z is of full rank, then we can reconstruct $(v_1, \dots, v_n)$ via a strongly coupled elliptic system of the form (2.23), after which we can reconstruct γ from $(\nabla v_1, \dots, \nabla v_n)$ by formula (2.21). We now show how to derive this elliptic system.

Proof of Proposition 2.1.7. According to Hypothesis 2.1.2, the matrix $Z = [Z_1|\dots|Z_n]$ has full rank and we recall the important equations

$$\sum_{p=1}^{n} \gamma_0 Z_p \cdot \nabla v_p = f \quad \text{and} \quad \sum_{i=1}^{n} Z_i^\flat \wedge dv_i = \omega, \quad \text{where} \tag{2.60}$$

$$\omega = Y_q^\flat \wedge du_q, \qquad Y_q := \nabla(H^{pq}(dH_{n+1,p} + (\mu^{-1}\mu_i)dH_{ip})), \tag{2.61}$$

and where f is given in (2.59). Assuming that Z has full rank, the family $(Z_1, \ldots, Z_n)$ is a frame with dotproducts defined as $\Xi_{ij} = Z_i \cdot Z_j$, and in this case we define its dual frame $Z_i^\star := \Xi^{ij} Z_j$ for $1 \le i \le n$, such that $Z_i^\star \cdot Z_j = \delta_{ij}$, i.e. with $Z^\star$ the matrix with columns $Z_j^\star$, we have the relation $Z^\star = Z^{-T}$. The second equation of (2.60) may be rewritten as

$$Z_q^\star \cdot \nabla v_p - Z_p^\star \cdot \nabla v_q = \omega(Z_p^\star, Z_q^\star), \quad 1 \le p, q \le n. \tag{2.62}$$

Applying the differential operator $Z_i^\star \cdot \nabla$ to the first equation of (2.60), we obtain

$$\sum_{p=1}^{n} (Z_i^\star \cdot \nabla)(\gamma_0 Z_p \cdot \nabla) v_p = (Z_i^\star \cdot \nabla) f. \tag{2.63}$$

Using (2.62), we may rewrite the left-hand side of (2.63) as

$$\begin{aligned}(Z_i^\star \cdot \nabla)(\gamma_0 Z_p \cdot \nabla) v_p &= [Z_i^\star, \gamma_0 Z_p] \cdot \nabla v_p + (\gamma_0 Z_p \cdot \nabla)(Z_i^\star \cdot \nabla) v_p \\ &= [Z_i^\star, \gamma_0 Z_p] \cdot \nabla v_p + (\gamma_0 Z_p \cdot \nabla)(Z_p^\star \cdot \nabla) v_i + (\gamma_0 Z_p \cdot \nabla)(\omega(Z_p^\star, Z_i^\star)),\end{aligned}$$

where we have introduced the Lie bracket of two vector fields, which may be written in the Euclidean connection

$$[X, Y] := (X \cdot \nabla) Y - (Y \cdot \nabla) X. \tag{2.64}$$

Plugging the last calculation into (2.63)

$$\sum_{p=1}^{n}(\gamma_0 Z_p\cdot\nabla)(Z_p^\star\cdot\nabla)v_i+\sum_{p=1}^{n}[Z_i^\star,\gamma_0 Z_p]\cdot\nabla v_p=(Z_i^\star\cdot\nabla)f-\sum_{p=1}^{n}(\gamma_0 Z_p\cdot\nabla)(\omega(Z_p^\star,Z_i^\star)). \tag{2.65}$$

We now look more closely at the principal part of this equation. The first term may be written as ($\sum$ is implicit, here)

$$\gamma_0 Z_p\otimes Z_p^\star:\nabla^2 v_i+((\gamma_0 Z_p\cdot\nabla)Z_p^\star)\cdot\nabla v_i=\gamma_0:\nabla^2 v_i+((\gamma_0 Z_p\cdot\nabla)Z_p^\star)\cdot\nabla v_i,$$

where we have used that $Z_p\otimes Z_p^\star=\mathbb{I}_n$. We thus obtain a strongly coupled elliptic system of the form (2.23), where

$$W_{ij}:=(\nabla\cdot\gamma_0-((\gamma_0 Z_p\cdot\nabla)Z_p^\star))\ \delta_{ij}-[Z_i^\star,\gamma_0 Z_j],\quad 1\le i,j\le n, \tag{2.66}$$

$$f_i:=-Z_i^\star\cdot\nabla f+(\gamma_0 Z_p\cdot\nabla)(\omega(Z_p^\star,Z_i^\star)),\quad 1\le i\le n. \tag{2.67}$$

This concludes the proof. □

In order to assess the properties of system (2.23), we recast it as an integral equation as follows: Let us call $L_0:=-\nabla\cdot(\gamma_0\nabla)$, and define $L_0^{-1}:H^{-1}(X)\ni f\mapsto u\in H_0^1(X)$, where u is the unique solution to the equation

$$-\nabla\cdot(\gamma_0\nabla u)=f\quad (X),\quad u|_{\partial X}=0. \tag{2.68}$$

By the Lax-Milgram theorem (see e.g. [23]), one can establish that such solutions satisfy an estimate of the form $\|u\|_{H_0^1(X)}\le C\|f\|_{H^{-1}(X)}$, where C only depends on X and the constant of ellipticity of γ_0, thus $L_0^{-1}:H^{-1}(X)\to H_0^1(X)$ is continuous, and by Rellich imbedding (i.e. the fact that the injection $L^2\to H^{-1}$ is compact), $L_0^{-1}:L^2(X)\to H_0^1(X)$

is compact.

Applying the operator L_0^{-1} to (2.23), we arrive at the integral system

$$v_i + \sum_{j=1}^{n} L_0^{-1}(W_{ij} \cdot \nabla v_j) = h_i := L_0^{-1} f_i \quad (X), \quad 1 \le i \le n, \tag{2.69}$$

where it is easy to establish that for $1 \le i, j \le n$, the operator

$$P_{ij} : H_0^1(X) \ni v \to P_{ij} v := L_0^{-1}(W_{ij} \cdot \nabla v) \in H_0^1(X) \tag{2.70}$$

is compact whenever the vector fields W_{ij} are bounded. In vector notation, if we define the vector space $\mathcal{H} = (H_0^1(X))^n$, $\mathbf{v} = (v_1, \ldots, v_n)$, $\mathbf{h} = (h_1, \ldots, h_n)$ and for $\mathbf{v} \in \mathcal{H}$,

$$\mathbf{Pv} := (P_{1j} v_j, P_{2j} v_j, \ldots, P_{nj} v_j) \in \mathcal{H}, \tag{2.71}$$

we have that $\mathbf{P} : \mathcal{H} \to \mathcal{H}$ is a compact linear operator, and the system (2.23) is reduced to the following Fredholm (integral) equation

$$(\mathbf{I} + \mathbf{P})\mathbf{v} = \mathbf{h}. \tag{2.72}$$

Note here that the operator $\mathbf{P}$ defined in (2.71) depends only on γ_0 and the solutions u_i, so that the injectivity properties depend on the γ_0 around which we pose the problem, in particular, whether one can fulfill hypotheses 2.1.1 and 2.1.2.

Injectivity and stability. Equation (2.72) satisfies a Fredholm alternative. In particular, if -1 is not an eigenvalue of $\mathbf{P}$, (2.72) admits a unique solution $\mathbf{v} \in \mathcal{H}$ (injectivity), $(\mathbf{I} + \mathbf{P})^{-1} : \mathcal{H} \to \mathcal{H}$ is well-defined and continuous and $\mathbf{v}$ satisfies the estimate

$$\|\mathbf{v}\|_{\mathcal{H}} \le \|(\mathbf{I} + \mathbf{P})^{-1}\|_{\mathcal{L}(\mathcal{H})} \|\mathbf{h}\|_{\mathcal{H}}, \tag{2.73}$$

from which we deduce stability below. In the statement of Theorem 2.1.9, the fact that "system (2.23) with trivial right-hand sides admits no non-trivial solution" precisely means that -1 is not an eigenvalue of the operator $\mathbf{P}$.

Remark 2.3.2 (Injectivity when γ_0 is constant)**.** *When* γ_0 *is constant, constructing* $(u_1, \dots, u_{n+1})$ *as in Remark 2.1.3 yields* $Z = Q$ *a constant matrix. In particular, the commutators* $[Z_i^\star, \gamma_0 Z_j]$ *vanish in the expression of* W_{ij}*. Thus system* (2.23) *is decoupled and clearly injective. By continuity, we also obtain that* (2.23) *is injective for* γ_0 *(not necessarily scalar) sufficiently close to a constant.*

We now prove Theorem 2.1.9.

Proof of Theorem 2.1.9. Starting from the integral version (2.72) of the elliptic system (2.23) in the case where $-1 \notin \text{sp}(\mathbf{P})$, then the Fredholm alternative implies (2.73). In order to translate inequality (2.73) into a stability statement, we must bound $\mathbf{h}$ in terms of the measurements $\{dH_{ij}\}$. We have for $1 \le i \le n$,

$$\|h_i\|_{H_0^1(X)} \le \|L_0^{-1}\|_{\mathcal{L}(H^{-1}, H_0^1)} \|f_i\|_{H^{-1}(X)},$$

and since f_i, expressed in (2.67) involves the dH_{ij} and their derivatives up to second order, if we assume all other multiplicative coefficients to be uniformly bounded, we obtain an estimate of the form

$$\|h_i\|_{H_0^1(X)} \le C\|dH\|_{H^1(X)}, \quad \text{where} \quad \|dH\|_{H^1(X)} := \sum_{1\le i\le n,\ i\le j\le n+1} \|dH_{ij}\|_{H^1(X)},$$

thus we obtain in the end, an estimate of the form

$$\|\mathbf{v}\|_{H_0^1(X)} \le C\|dH\|_{H^1(X)}. \tag{2.74}$$

Once $\mathbf{v}$ is reconstructed, we can reconstruct γ uniquely from dH and $[\nabla V]$ using formula (2.21), with the stability estimate

$$\|\gamma\|_{L^2(X)} \le C\|dH\|_{H^1(X)}. \tag{2.75}$$

Getting one derivative back on tr $(\gamma_0^{-1}\gamma)$: In order to see that tr $(\gamma_0^{-1}\gamma)$ satisfies a gradient equation that improves the stability of its reconstruction, the quickest way is to linearize [45, Equation (7)] derived in the non-linear case, which reads as follows:

$$\nabla \log\det\gamma^\varepsilon = \nabla \log\det H^\varepsilon + 2\left((\nabla(H^\varepsilon)^{jl})\cdot\gamma^\varepsilon\nabla u_l^\varepsilon\right)\nabla u_j^\varepsilon,$$

where H^ε is the $n\times n$ matrix of power densities $H_{ij}^\varepsilon = \nabla u_i^\varepsilon\cdot\gamma^\varepsilon\nabla u_j^\varepsilon$ and $(H^\varepsilon)^{jl}$ is the (j,l)-th entry of $(H^\varepsilon)^{-1}$. Plugging the expansions $\gamma^\varepsilon = \gamma_0+\varepsilon\gamma$, $u_i^\varepsilon = u_i+\varepsilon v_i$, $H_{ij}^\varepsilon = H_{ij}+\varepsilon dH_{ij}$, and using the fact that

$$(H^\varepsilon)^{jl} = H^{jl} - \varepsilon(H^{-1}dHH^{-1})^{jl} + \mathcal{O}(\varepsilon^2),$$

the linearized equation at $\mathcal{O}(\varepsilon)$ reads

$$\begin{aligned}\frac{1}{2}\nabla\text{tr }(\gamma_0^{-1}\gamma) = {}& \frac{1}{2}\nabla\text{tr }(H^{-1}dH) + (\nabla H^{jl}\cdot\gamma_0\nabla u_l)\nabla v_j + (\nabla H^{jl}\cdot\gamma_0\nabla v_l)\nabla u_j \\ & + (\nabla H^{jl}\cdot\gamma\nabla u_l)\nabla u_j - \left(\nabla(H^{-1}dHH^{-1})^{jl}\cdot\gamma_0\nabla u_l\right)\nabla u_j.\end{aligned}$$

From this equation, and using the stability estimates (2.74) and (2.75), it is straighforward to establish the estimate

$$\|\text{tr }(\gamma_0^{-1}\gamma)\|_{H^1(X)} \le C\|dH\|_{H^1(X)},$$

and thus the proof is complete. □

Chapter 3

Inverse conductivity from current densities

In this chapter, we study the Current Density Impedance Imaging problem (CDII) proposed in Section 1.1.2. We aim at reconstructing an anisotropic conductivity tensor in the second-order elliptic equation,

$$\nabla \cdot (\gamma \nabla u) = \sum_{i,j=1}^{n} \partial_i (\gamma^{ij} \partial_j u) = 0 \quad (X), \qquad u|_{\partial X} = g, \tag{3.1}$$

from knowledge of internal current densities of the form $H = \gamma \nabla u$, where u solves (3.1). The above equation has a symmetric tensor γ satisfying the uniform ellipticity condition (1.3) so that (3.1) admits a unique solution in $H^1(X)$ for $g \in H^{\frac{1}{2}}(\partial X)$. We propose sufficient conditions on the choice of $\{g_j\}_{\le j \le m}$ such that the reconstruction of γ is unique and satisfies elliptic stability estimates.

3.1 Modeling of the problem

For $X \subset \mathbb{R}^n$, we denote by $\Sigma(X)$ the set of conductivity tensors with bounded components satisfying the uniform ellipticity condition. Then for $k \geq 1$ an integer and $0 < \alpha < 1$, we denote

$$\mathcal{C}^{k,\alpha}_{\Sigma}(X) := \{\gamma \in \Sigma(X) | \quad \gamma_{pq} \in \mathcal{C}^{k,\alpha}(X), \quad 1 \leq p \leq q \leq n\}.$$

In what follows, by "solution of (3.1)" we may refer to the solution itself or the boundary condition that generates it, i.e. $g = u|_{\partial X} \in H^{\frac{1}{2}}(\partial X)$. We will consider collections of measurements of the form

$$H_i : \gamma \mapsto H_i(\gamma) = \gamma \nabla u_i, \qquad 1 \leq i \leq m, \tag{3.2}$$

where u_i solves (3.1) with boundary condition g_i. We decompose γ into the product of a scalar factor β with an anisotropic structure $\tilde{\gamma}$

$$\gamma := \beta \tilde{\gamma}, \qquad \beta = (\det \gamma)^{\frac{1}{n}}, \qquad \det \tilde{\gamma} = 1. \tag{3.3}$$

Since γ satisfies the uniform elliptic condition, β is bounded away from zero.

From knowledge of a sufficiently large number of current densities, the reconstruction formulas for β and $\tilde{\gamma}$ can be locally established in terms of the current densities and their derivatives up to first order.

3.1.1 Main hypotheses

We begin with the main hypotheses that allow us to setup a few reconstruction procedures.

The first hypothesis aims at making the scalar factor β in (3.3) locally reconstructible via a gradient equation.

Hypothesis 3.1.1. *There exist two solutions* (u_1, u_2) *of* (3.1) *and* $X_0 \subset X$ *convex satisfying*

$$\inf_{x\in X_0} \mathcal{F}_1(u_1, u_2) \geq c_0 > 0 \quad \textit{where} \quad \mathcal{F}_1(u_1, u_2) := |\nabla u_1|^2 |\nabla u_2|^2 - (\nabla u_1 \cdot \nabla u_2)^2. \tag{3.4}$$

On to the hypotheses for local reconstructibility of $\tilde{\gamma}$, we first need to have, locally, a basis of gradients of solutions of (3.1).

Hypothesis 3.1.2. *There exist* n *solutions* $(u_1, \dots, u_n)$ *of* (3.1) *and* $X_0 \subset X$ *satisfying*

$$\inf_{x\in X_0} \mathcal{F}_2(u_1, \dots, u_n) \geq c_0 > 0, \quad \textit{where} \quad \mathcal{F}_2(u_1, \dots, u_n) := \det(\nabla u_1, \dots, \nabla u_n). \tag{3.5}$$

Let us now pick $u_1, \cdots, u_n$ satisfying Hypothesis 3.1.2 and consider additional solutions $\{u_{n+k}\}_{k=1}^m$. Each additional solution decomposes in the basis $(\nabla u_1, \dots, \nabla u_n)$ as

$$\nabla u_{n+k} = \sum_{i=1}^n \mu_k^i \nabla u_i, \quad 1 \leq k \leq m, \tag{3.6}$$

where, as shown in [9] for instance, the coefficients μ_k^i take the expression

$$\mu_k^i = -\frac{\det(\nabla u_1, \dots, \overbrace{\nabla u_{n+k}}^{i}, \dots, \nabla u_n)}{\det(\nabla u_1, \dots, \nabla u_n)} = -\frac{\det(H_1, \dots, \overbrace{H_{n+k}}^{i}, \dots, H_n)}{\det(H_1, \dots, H_n)},$$

in particular, these coefficients are *accessible from current densities.* The subsequent algorithms will make extensive use of the matrix-valued quantities

$$Z_k = [Z_{k,1}|\cdots|Z_{k,n}], \quad \text{where} \quad Z_{k,i} := \nabla \mu_k^i, \quad 1 \leq k \leq m. \tag{3.7}$$

In particular, the next hypothesis, formulating a sufficient condition for local reconstructibility of the anisotropic part of γ is that, locally, a certain number of matrices Z_k (at least two) satisfies some rank maximality condition.

Hypothesis 3.1.3. *Assume that Hypothesis 3.1.2 holds for some* $(u_1, \dots, u_n)$ *over* $X_0 \subset X$ *and denote by* H *the matrix with columns* $H_1, \dots, H_n$*. Then there exist* $u_{n+1}, \dots, u_{n+m}$ *solutions of* (3.1) *and some* $X' \subseteq X_0$ *such that the* x*-dependent space*

$$\mathcal{W} := span\left\{(Z_k H^T \Omega)^{sym}, \quad \Omega \in A_n(\mathbb{R}), 1 \le k \le m\right\} \subset S_n(\mathbb{R}) \tag{3.8}$$

has codimension one in $S_n(\mathbb{R})$ *throughout* X'.

An alternate approach to reconstruct γ is to set up a coupled system for $u_1, \dots, u_n$ satisfying Hyp. 3.1.2 globally. This system of PDEs can be derived under the following hypothesis (part A). From this system and under an additional hypothesis (part B), we can derive an elliptic system from which to reconstruct $u_1, \dots, u_n$.

Hypothesis 3.1.4. *A. Suppose that Hypothesis 3.1.2 is satisfied over* $X_0 = X$ *for some solutions* $(u_1, \dots, u_n)$*. There exists an additional solution* u_{n+1} *of* (3.1) *whose matrix* Z_1 *defined by* (3.7) *is uniformly invertible over* X*, i.e.*

$$\inf_{x \in X} \det Z_1 \ge c_0 > 0, \tag{3.9}$$

for some positive constant c_0.

B. There exist $n+2$ *solutions* $u_1, \dots, u_{n+2}$ *such that* $(u_1, \dots, u_n, u_{n+2})$ *satisfy (A), and two* $A_n(\mathbb{R})$*-valued functions* $\Omega_1(x)$, $\Omega_2(x)$ *such that the matrix*

$$S = \left(Z_2^\star Z_1^T \Omega_1(x) + H Z_1^T \Omega_2(x)\right)^{sym} \qquad (\textit{with } Z_2^\star := Z_2^{-T}) \tag{3.10}$$

satisfies the ellipticity condition (2.2).

The first important result to note is that the hypotheses stated above remain satisfied under some perturbations of the boundary conditions or the conductivity tensor for smooth enough topologies.

Proposition 3.1.5. *Assume that Hypothesis 3.1.1, 4.1.1, 3.1.3 or 3.1.4 holds over some* $X_0 \subseteq X$ *for a given number* m *of solutions of* (3.1) *with boundary conditions* $g_1, \dots, g_m$*. Then for any* $0 < \alpha < 1$*, there exists a neighborhood of* $(g_1, \dots, g_m, \gamma)$ *open for the* $\mathcal{C}^{2,\alpha}(\partial X)^m \times \mathcal{C}^{1,\alpha}(X)$ *topology where the same hypothesis holds over* X_0*. In the case of 3.1.4.B, it still holds with the same* $A_n(\mathbb{R})$*-valued functions* Ω_1 *and* Ω_2*.*

3.1.2 Reconstruction algorithms and their properties

Reconstruction of β knowing $\tilde{\gamma}$. Under knowledge of $\tilde{\gamma}$ and using two measurements H_1, H_2 coming from two solutions satisfying Hyp. 3.1.1 over some $X_0 \subset X$, we can derive the following gradient equation for $\log \beta$

$$\begin{aligned}\nabla \log \beta = \frac{1}{D|H_1|^2} \left(|H_1|^2 \; d(\tilde{\gamma}^{-1} H_1) - (H_1 \cdot H_2) \; d(\tilde{\gamma}^{-1} H_2) \right) (\tilde{\gamma} H_1, \tilde{\gamma} H_2) \tilde{\gamma}^{-1} H_1 \\ - \frac{1}{|H_1|^2} \; d(\tilde{\gamma}^{-1} H_1)(\tilde{\gamma} H_1, \cdot), \qquad x \in X_0,\end{aligned} \tag{3.11}$$

where $D := |H_1|^2 |H_2|^2 - (H_1 \cdot H_2)^2$ is bounded away from zero over X_0 thanks to Hyp. 3.1.1, and where the exterior calculus notations used here are recalled in Appendix 3.5.

Equation (3.11) allows us to reconstruct β under the knowledge of $\beta(x_0)$ at one fixed point in X_0 by integrating (3.11) over any curve starting from some $x_0 \in X_0$. This leads to a unique and stable reconstruction with no loss of derivatives, as formulated in the following proposition. This generalizes the result in [34] to an anisotropic tensor.

Proposition 3.1.6 (Local uniqueness and stability for β)**.** *Consider two tensors* $\gamma = \beta \tilde{\gamma}$ *and* $\gamma' = \beta' \tilde{\gamma}'$*, where* $\tilde{\gamma}, \tilde{\gamma}' \in W^{1,\infty}(X)$ *are known. Suppose that Hypothesis 3.1.1 holds over the same* $X_0 \subset X$ *for two pairs* (u_1, u_2) *and* (u_1', u_2')*, solutions of* (3.1) *with conductivity* γ

and γ', respectively. Then the following stability estimate holds for any $p \geq 1$

$$\|\log\beta - \log\beta'\|_{W^{p,\infty}(X_0)} \leq \epsilon_0 + C\left(\sum_{i=1,2}\|H_i - H_i'\|_{W^{p,\infty}(X)} + \|\tilde{\gamma} - \tilde{\gamma}'\|_{W^{p,\infty}(X)}\right). \quad (3.12)$$

Where $\epsilon_0 = |\log\beta(x_0) - \log\beta'(x_0)|$ is the error committed at some fixed $x_0 \in X_0$.

Algebraic, local reconstruction of $\tilde{\gamma}$: On to the local reconstruction of the anisotropic structure, we start from $n+m$ solutions $(u_1, \dots, u_{n+m})$ satisfying hypotheses 3.1.2 and 3.1.3 over some $X_0 \subset X$. In particular, the linear space $\mathcal{W} \subset S_n(\mathbb{R})$ defined in (3.8) is of codimension one in $S_n(\mathbb{R})$. We will see that the tensor $\tilde{\gamma}$ must be orthogonal to $\mathcal{W}$ for the inner product $\langle A, B\rangle := A_{ij}B_{ij} = \text{tr }(AB^T)$. Together with the conditions that $\det\tilde{\gamma} = 1$ and $\tilde{\gamma}$ is positive, the space $\mathcal{W}$, known from the measurements $H_1, \dots, H_{n+m}$ completely determines $\tilde{\gamma}$ over X_0. In light of these observations, a constructive reconstruction algorithm based on a generalization of the cross-product is proposed in section 3.3.2. This approach was recently used in [45] in the context of inverse conductivity from power densities. This algorithm leads to a unique and stable reconstruction in the sense of the following proposition.

Proposition 3.1.7 (Local uniqueness and stability for $\tilde{\gamma}$)**.** *Consider two uniformly elliptic tensors γ and γ'. Suppose that Hypotheses 3.1.2 and 3.1.3 hold over the same $X_0 \subset X$ for two $n+m$-tuples $\{u_i\}_{i=1}^{n+m}$ and $\{u_i'\}_{i=1}^{n+m}$, solutions of* (3.1) *with conductivity γ and γ', respectively. Then the following stability estimate holds for any integer $p \geq 0$*

$$\|\tilde{\gamma} - \tilde{\gamma}'\|_{W^{p,\infty}(X_0)} \leq C\sum_{i=1}^{n+m}\|H_i - H_i'\|_{W^{p+1,\infty}(X)}. \quad (3.13)$$

Joint reconstruction of $(\tilde{\gamma}, \beta)$, stability improvement for $\nabla \times \gamma^{-1}$. Judging by the stability estimates (3.13) and (3.12), reconstructing β after having reconstructed $\tilde{\gamma}$ is less stable (with respect to current densities) than when knowing $\tilde{\gamma}$. This is because in the former case, errors on $W^{p,\infty}$-norm in $\tilde{\gamma}$ are controlled by errors in $W^{p+1,\infty}$ norm in current densities. In particular, on the $W^{p,\infty}$ scale, stability on β is no better than that of $\tilde{\gamma}$, and joint reconstruction of $(\tilde{\gamma}, \beta)$ using the preceding two algorithms displays the following stability, with $\gamma = \beta\tilde{\gamma}$

$$\|\gamma - \gamma'\|_{W^{p,\infty}(X_0)} \le C \sum_{i=1}^{n+m} \|H_i - H_i'\|_{W^{p+1,\infty}(X)}. \tag{3.14}$$

However, once γ is reconstructed, some linear combinations of first-order partials of γ^{-1} can be reconstructed with better stability. These are the exterior derivatives of the columns of γ^{-1}, a collection of $n^2(n-1)/2$ scalar functions which we denote $\nabla\times\gamma^{-1}$ and is reconstructed via the formula

$$\partial_q \gamma^{pl} - \partial_p \gamma^{ql} = H^{il}(\gamma^{qj}\partial_p H_{ji} - \gamma^{pj}\partial_q H_{ji}), \quad 1 \le l \le n, \quad 1 \le p < q \le n, \tag{3.15}$$

derived in Sec. 3.3.3 and assuming that we are working with a basis of solutions satisfying Hypothesis 3.1.2. The stability statement (3.14) is thus somewhat improved into a statement of the form

$$\|\gamma - \gamma'\|_{W^{p,\infty}(X_0)} + \|\nabla \times (\gamma^{-1} - \gamma'^{-1})\|_{W^{p,\infty}(X_0)} \le C \sum_{i=1}^{n+m} \|H_i - H_i'\|_{W^{p+1,\infty}(X)}, \tag{3.16}$$

where we have defined

$$\|\nabla \times (\gamma^{-1} - \gamma'^{-1})\|_{W^{p,\infty}(X_0)} := \sum_{l=1}^{n} \sum_{1\le i<j\le n} \|\partial_j \gamma^{il} - \partial_i \gamma^{jl}\|_{W^{p,\infty}(X_0)}.$$

Global reconstruction of γ via a coupled elliptic system. While the preceding approach required a certain number of additional solutions, we now show how one can setup an alternate reconstruction procedure with only $m = 2$ additional solutions satisfying Hyp. 3.1.4. A microlocal study of linearized current densities functionals shows that this is the minimum number of functionals necessary to reconstruct all of γ.

The present approach consists is eliminating γ from the equations and writing an elliptic system of equations for the solutions u_j; see [9, 44, 45] for similar approaches in the setting of power density functionals. The method goes as follows. Assume that Hypothesis 3.1.2 holds for some $(u_1, \dots, u_n)$ over $X_0 = X$ and denote $[\nabla U] = [\nabla u_1, \cdots, \nabla u_n]$ as well as $H = [H_1, \cdots, H_n]$. Since $H = \gamma[\nabla U]$, we can thus reconstruct γ by $\gamma = [\nabla U]^{-1} H$ once $[\nabla U]$ is known. We now show that we may reconstruct $[\nabla U]$ by solving a second-order elliptic system of partial differential equations.

When Hyp. 3.1.4.A is satisfied for some u_{n+1} and considering an additional solution u_{n+2} and its corresponding current density, we first derive a system of coupled partial differential equations for $(u_1, \dots, u_n)$, whose coefficients only depend on measured quantities.

Proposition 3.1.8. *Suppose $n+2$ solutions $(u_1, \dots, u_{n+2})$ satisfy Hypotheses 3.1.2 and 3.1.4.A and consider their corresponding measurements $H_I = \{H_i\}_{i=1}^{n+2}$. Then the solutions $(u_1, \cdots, u_n)$ satisfy the coupled system of PDE's*

$$\begin{aligned} Z_2^\star Z_1^T(\mathbf{e}_p \otimes \mathbf{e}_q - \mathbf{e}_q \otimes \mathbf{e}_p) : \nabla^2 u_j + v_{ij}^{pq} \cdot \nabla u_i = 0, & \\ H Z_1^T(\mathbf{e}_p \otimes \mathbf{e}_q - \mathbf{e}_q \otimes \mathbf{e}_p) : \nabla^2 u_j + \tilde{v}_{ij}^{pq} \cdot \nabla u_i = 0, & \qquad u_j|_{\partial X} = g_j, \end{aligned} \tag{3.17}$$

for $1 \le j \le n$ and $1 \le p < q \le n$, and where the vector fields $\{v_{ij}^{pq}, \tilde{v}_{ij}^{pq}\}$ only depend on the current densities H_I.

If additionally, u_{n+2} is such that Hyp. 3.1.4.B is satisfied, we can deduce a strongly coupled elliptic system for $(u_1, \dots, u_n)$ from (3.17).

Theorem 3.1.9. *With the hypotheses of Proposition 3.1.8, assume further that Hypothesis*

3.1.4.B holds for some $A_n(\mathbb{R})$-valued functions

$$\Omega_i(x) = \sum_{1\le p<q\le n} \omega^i_{pq}(x)(\mathbf{e}_p \otimes \mathbf{e}_q - \mathbf{e}_q \otimes \mathbf{e}_q), \quad i = 1, 2.$$

Then $(u_1, \cdots, u_n)$ can be reconstructed via the strongly coupled elliptic system

$$-\nabla \cdot (S\nabla u_j) + W_{ij} \cdot \nabla u_i = 0, \quad u_j|_{\partial X} = g_j, \quad 1 \le j \le n, \tag{3.18}$$

where $S = \left(Z_2^\star Z_1^T \Omega_1(x) + HZ_1^T\Omega_2(x)\right)^{sym}$ as in (3.10) and where we have defined

$$W_{ij} := \nabla \cdot S - \sum_{1\le p<q\le n} \omega^1_{pq}(x)v^{pq}_{ij} + \omega^2_{pq}(x)\tilde{v}^{pq}_{ij}, \quad 1 \le i, j \le n. \tag{3.19}$$

Moreover, if system (3.18) with trivial boundary conditions has only the trivial solution, $u_1, \ldots, u_n$ are uniquely reconstructed. Subsequently, γ reconstructed as $\gamma = H[\nabla U]^{-1}$ satisfies the stability estimate

$$\|\gamma - \gamma'\|_{L^2(X)} + \|\nabla \times (\gamma^{-1} - \gamma'^{-1})\|_{L^2(X)} \le C\|H_I - H_I'\|_{H^1(X)}, \tag{3.20}$$

for data sets H_I, H_I close enough in H^1-norm.

3.1.3 What tensors are reconstructible ?

We now conclude with a discussion regarding what tensors are reconstructible from current densities, based on the extent to which Hypotheses 3.1.1-3.1.4 can be fulfilled, so that the above reconstruction algorithms can be implemented.

Proposition 3.1.10. *For any smooth domain $X \subset \mathbb{R}^n$ and considering a constant conductivity tensor γ_0, there exists a non-empty $\mathcal{C}^{2,\alpha}$-open subset of $[H^{\frac{1}{2}}(\partial X)]^{n+2}$ of boundary conditions fulfilling Hypotheses 3.1.1-3.1.4 throughout X.*

The second test case regards isotropic smooth tensors of the form $\gamma = \beta\mathbb{I}_n$, where we show that the scalar coefficient β can be reconstructed globally by using the real and imaginary parts of the same complex geometrical optics (CGO) solution. The use of CGOs for fulfilling internal conditions was previously used in [8, 15, 47].

Proposition 3.1.11. *For an isotropic tensor $\gamma = \beta\mathbb{I}_n$ with $\beta \in H^{\frac{n}{2}+3+\varepsilon}(X)$ for some $\varepsilon > 0$, there exists a non-empty $\mathcal{C}^{2,\alpha}$-open subset of $[H^{\frac{1}{2}}(\partial X)]^2$ fulfilling Hypothesis 3.1.1 thoughout X.*

Thanks to Proposition 3.1.5, we can also formulate the following without proof.

Corollary 3.1.12. *Suppose γ is a tensor as in either Proposition 3.1.10 or 3.1.11. Then, for any $0 < \alpha < 1$, there exists a $\mathcal{C}^{1,\alpha}$-neighborhood of γ for which the conclusion of the same proposition remains valid.*

Push-forwards by diffeomorphisms. Recall that for $\Psi : X \to \Psi(X)$ a $W^{1,2}$-diffeomorphism and $\gamma \in \Sigma(X)$, we define $\Psi_\star\gamma$ the conductivity tensor push-forwarded by Ψ from γ defined over $\Psi(X)$, by

$$\Psi_\star\gamma := (|J_\Psi|^{-1} D\Psi \cdot \gamma \cdot D\Psi) \circ \Psi^{-1}, \qquad J_\Psi := \det D\Psi. \tag{3.21}$$

We now show that, whenever a tensor is being push-forwarded from another by a diffeormorphism, then the local or global reconstructibility of one is equivalent to that of the other, in the sense of the Proposition below. While the existence of $\Psi_\star\gamma$ in $\Sigma(\Psi(X))$ merely requires that Ψ be a $W^{1,2}$-diffeomorphism, our results below will require that Ψ be smoother and that it satisfies the following uniform condition over X

$$C_\Psi^{-1} \le |J_\Psi| \le C_\Psi \quad \text{for some } C_\Psi \ge 1. \tag{3.22}$$

Proposition 3.1.13. *Assume that Hypothesis 3.1.1, 4.1.1, 3.1.3 or 3.1.4 holds over some* $X_0 \subseteq X$ *for a given number* m *of solutions of* (3.1) *with boundary conditions* $g_1, \dots, g_m$*. For* $\Psi : X \to \Psi(X)$ *a smooth diffeomorphism satisfying* (3.22)*, the same hypothesis holds true over* $\Psi(X_0)$ *for the conductivity tensor* $\Psi_\star\gamma$ *with boundary conditions* $(g_1 \circ \Psi^{-1}, \dots, g_m \circ \Psi^{-1})$*. In the case of Hyp. 3.1.4.B, it holds with the following* $A_n(\mathbb{R})$*-valued functions defined over* $\Psi(X)$*:*

$$\Psi_\star\Omega_1 := [D\Psi \cdot \Omega_1 \cdot D\Psi^t] \circ \Psi^{-1} \quad \text{and} \quad \Psi_\star\Omega_2 := [|J_\Psi| D\Psi \cdot \Omega_2 \cdot D\Psi^t] \circ \Psi^{-1}. \tag{3.23}$$

In contrast to inverse conductivity problems from boundary data, where the diffeomorphisms above are a well-known obstruction to injectivity, Proposition 3.1.13 precisely states the opposite: if a given tensor γ is reconstructible in some sense, then so is $\Psi_\star\gamma$, and the boundary conditions making the inversion valid are explicitely given in terms of the ones that allow to reconstruct γ.

Corollary 3.1.14. *Suppose* γ *is a tensor as in either Proposition 3.1.10 or 3.1.11 and* $\Psi : X \to \Psi(X)$ *is a diffeomorphism satisfying* (3.22)*. Then the conclusion of the same proposition holds for the tensor* $\Psi_\star\gamma$ *over* $\Psi(X)$ *and boundary conditions defined over* $\partial(\Psi(X))$*.*

Generic reconstructibility. We finally state that any $\mathcal{C}^{1,\alpha}$ smooth tensor is, in principle, reconstructible from current densities in the sense of the following proposition. This result uses the Runge approximation property, a property equivalent to the unique continuation principle, valid for Lipschitz-continuous tensors.

Proposition 3.1.15. *Let* $X \subset \mathbb{R}^n$ *a* $\mathcal{C}^{2,\alpha}$ *domain and* $\gamma \in \mathcal{C}^{1,\alpha}_\Sigma(X)$*. Then for any* $x_0 \in X$*, there exists a neighborhood* $X_0 \subset X$ *of* x_0 *and* $n+2$ *solutions of* (3.1) *fulfilling hypotheses 3.1.2 and 3.1.3 over* X_0*.*

3.2 Preliminaries

In this section, we briefly recall elliptic regularity results, the mapping properties of the current density operator and we conclude with the proof of Proposition 3.1.5.

Properties of the forward mapping. In the following, we will make use of the following result, based on Schauder estimates for elliptic equations. It is for instance stated in [33].

Proposition 3.2.1. *For $k \geq 2$ an integer and $0 < \alpha < 1$, if X is a $\mathcal{C}^{k+1,\alpha}$-smooth domain, then the mapping $(g, \gamma) \mapsto u$, solution of* (3.1)*, is continuous in the functional setting*

$$\mathcal{C}^{k,\alpha}(\partial X) \times \mathcal{C}^{k-1,\alpha}_{\Sigma}(X) \to \mathcal{C}^{k,\alpha}(X).$$

As a consequence, we can claim that, with the same k, α as above, the current density operator $(g, \gamma) \mapsto \gamma \nabla u$ is continuous in the functional setting

$$\mathcal{C}^{k,\alpha}(\partial X) \times \mathcal{C}^{k-1,\alpha}_{\Sigma}(X) \to \mathcal{C}^{k-1,\alpha}(X).$$

Moreover, this fact allows us to prove Proposition 3.1.5.

Proof of Proposition 3.1.5. Fixing some domain $X_0 \subset X$ and using Proposition 3.2.1, it is clear that the mappings

$$f_1 : (\mathcal{C}^{2,\alpha}(\partial X))^2 \times \mathcal{C}^{1,\alpha}_{\Sigma}(X) \ni (g_1, g_2, \gamma) \mapsto \inf_{X_0} \mathcal{F}_1(u_1, u_2),$$
$$f_2 : (\mathcal{C}^{2,\alpha}(\partial X))^n \times \mathcal{C}^{1,\alpha}_{\Sigma}(X) \ni (g_1, \dots, g_n, \gamma) \mapsto \inf_{X_0} \mathcal{F}_2(u_1, \dots, u_n),$$

with $\mathcal{F}_1, \mathcal{F}_2$ defined in (3.4),(3.5), are continuous, so $f_1^{-1}\big((0,\infty)\big)$ and $f_2^{-1}\big((0,\infty)\big)$ are open, which takes care of Hypotheses 3.1.1 and 4.1.1. Further, Hypothesis 3.1.3 is fulfilled if and only if condition 3.30 holds. Again, using Prop. 3.2.1, the mapping $f_3 := \inf_{X_0} \mathcal{B}$ with $\mathcal{B}$

defined in (3.30) is a continuous function of $(g_1, \dots, g_{n+m}, \gamma) \in (\mathcal{C}^{2,\alpha}(\partial X))^{n+m} \times \mathcal{C}^{1,\alpha}_\Sigma(X)$ so that $f_3^{-1}\big((0,\infty)\big)$ is open.

Along the same lines, Hypothesis 3.1.4.A is stable under such perturbations because the mapping

$$(\mathcal{C}^{2,\alpha}(\partial X))^{n+1} \times \mathcal{C}^{1,\alpha}_\Sigma(X) \ni (g_1, \dots, g_{n+1}, \gamma) \mapsto \inf_X \det Z_1,$$

is continuous whenever $u_1, \dots, u_n$ satisfy (3.5) over X. Finally, fixing two $A_n(\mathbb{R})$-valued functions $\Omega_1(x)$ and $\Omega_2(x)$, Hypothesis 3.1.4.B is fulfilled whenever

$$(g_1, \dots, g_{n+2}, \gamma) \in \bigcap_{i=1}^n s_i^{-1}\big((0,\infty)\big), \qquad (3.24)$$

where we have defined the functionals, for $1 \le i \le n$

$$s_i : (\mathcal{C}^{2,\alpha}(\partial X))^{n+2} \times \mathcal{C}^{1,\alpha}_\Sigma(X) \ni (g_1, \dots, g_{n+2}, \gamma) \mapsto \inf_X \det\{S_{pq}\}_{1\le p,q\le i},$$

with $S = \{S_{p,q}\}_{1\le p,q\le n}$ defined as in (3.10). Such functionals are, again, continuous, in particular the set in the right-hand side of (3.24) is open. This concludes the proof. □

3.3 Reconstruction approaches

3.3.1 Local reconstruction of β

In this section, we assume that $\tilde{\gamma}$ is known and with $W^{1,\infty}$ components. Assuming Hypothesis 3.1.1 is fulfilled for two solutions u_1, u_2 over an open set $X_0 \subset X$, we now prove equation (3.11).

Proof of equation (3.11). Rewriting (3.2) as $\frac{1}{\beta}\tilde{\gamma}^{-1}H_j = \nabla u_j$ and applying the operator $d(\cdot)$.

Using identities (3.48) and (3.49), we arrive at the following equation for $\log\beta$:

$$\nabla \log\beta \wedge (\tilde{\gamma}^{-1} H_j) = d(\tilde{\gamma}^{-1} H_j), \quad j = 1, 2. \tag{3.25}$$

Let us first notice the following equality of vector fields

$$\nabla \log\beta \wedge (\tilde{\gamma}^{-1}) H_1(\tilde{\gamma} H_1, \cdot) = (\nabla \log\beta \cdot \tilde{\gamma} H_1)(\tilde{\gamma}^{-1} H_1) - |H_1|^2 \nabla \log\beta,$$

so that

$$\begin{aligned} \nabla \log\beta &= \frac{1}{|H_1|^2}(\nabla \log\beta \cdot \tilde{\gamma} H_1)\tilde{\gamma}^{-1} H_1 - \frac{1}{|H_1|^2}\nabla \log\beta \wedge (\tilde{\gamma}^{-1} H_1)(\tilde{\gamma} H_1, \cdot) \\ &= \frac{1}{|H_1|^2}(\nabla \log\beta \cdot \tilde{\gamma} H_1)\tilde{\gamma}^{-1} H_1 - \frac{1}{|H_1|^2} d(\tilde{\gamma}^{-1} H_1)(\tilde{\gamma} H_1, \cdot). \end{aligned}$$

It remains thus to prove that

$$(\nabla \log\beta \cdot \tilde{\gamma} H_1) = \frac{1}{D}\left(|H_1|^2 d(\tilde{\gamma}^{-1} H_1) - (H_1 \cdot H_2) d(\tilde{\gamma}^{-1} H_2)\right)(\tilde{\gamma} H_1, \tilde{\gamma} H_2),$$

which may be checked directly by computing, for $j = 1, 2$

$$\begin{aligned} d(\tilde{\gamma}^{-1} H_j)(\tilde{\gamma} H_1, \tilde{\gamma} H_2) &= d\log\beta \wedge (\tilde{\gamma}^{-1} H_j)(\tilde{\gamma} H_1, \tilde{\gamma} H_2) \\ &= (\nabla \log\beta \cdot \tilde{\gamma} H_1) H_j \cdot H_2 - (\nabla \log\beta \cdot \tilde{\gamma} H_2)(H_j \cdot H_1). \end{aligned}$$

Taking the appropriate weighted sum of the above equations allows to extract $(\nabla \log\beta \cdot \tilde{\gamma} H_1)$, and hence (3.11). □

Reconstruction procedures for β, uniqueness and stability. Suppose equation (3.11) holds over some convex set $X_0 \subset X$ and fix $x_0 \in X_0$. Equation (3.11) is a gradient equation $\nabla \log\beta = F$ with known right-hand side F. For any $x \in X_0$, one may thus construct $\beta(x)$ by integrating (3.11) over the segment $[x_0, x]$, leading to one possible formula

$$\beta(x) = \beta(x_0) \exp\left(\int_0^1 (x - x_0) \cdot F((1-t)x_0 + tx)\ dt\right), \quad x \in X_0. \tag{3.26}$$

Proof of Proposition 3.1.6. Since $\det \tilde{\gamma} = 1$, the entries of $\tilde{\gamma}^{-1}$ are polynomials of the entries of $\tilde{\gamma}$, so that the entries of the right-hand side of (3.11) are polynomials of the entries of $H_1, H_2, \tilde{\gamma}$ and their derivatives, with bounded coefficients. It is thus straightforward to establish that

$$\|\nabla \log \beta - \nabla \log \beta'\|_{L^\infty(X_0)} \le C(\|H - H'\|_{W^{1,\infty}(X)} + \|\tilde{\gamma} - \tilde{\gamma}'\|_{W^{1,\infty}(X)}) \tag{3.27}$$

for some constant C. The stability estimate for β then follows from the fact that

$$\|\log \beta - \log \beta'\|_{L^\infty(X_0)} \le |\log \beta(x_0) - \log \beta'(x_0)| + \Delta(X)\|\nabla \log \beta - \nabla \log \beta'\|_{L^\infty(X_0)},$$

where $\Delta(X)$ denotes the diameter of X. □

One could use another integration curve than the segment $[x_0, x]$ to compute $\beta(x)$. In order for this integration to not depend on the choice of curve, the right-hand side F of (3.11) should satisfy the integrability condition $dF = 0$, a condition on the measurements which characterizes partially the range of the measurement operator.

When measurements are noisy, said right-hand side may no longer satisfy this requirement, in which case the solution to (3.11) no longer exists. One way to remedy this issue is to solve the *normal* equation to (3.11) over X_0 (whose boundary can be made smooth) with, for instance, Neuman boundary conditions:

$$-\Delta \log \beta = -\nabla \cdot F \quad (X_0), \qquad \partial_\nu \log \beta|_{\partial X_0} = F \cdot \nu,$$

where ν denotes the outward unit normal to X_0. This approach salvages existence while projecting the data onto the range of the measurement operator, with a stability estimate similar to (3.12) on the H^s Sobolev scale instead of the $W^{s,\infty}$ one.

3.3.2 Local reconstruction of $\tilde{\gamma}$

We now turn to the local reconstruction algorithm of $\tilde{\gamma}$. In this case, the reconstruction is algebraic, i.e. no longer involves integration of a gradient equation. In the sequel, we work with $n+m$ solutions of (3.1) denoted $\{u_i\}_{i=1}^{n+m}$, whose current densities $\{H_i = \gamma\nabla u_i\}_{i=1}^{n+m}$ are assumed to be measured.

Derivation of the space of linear constraints (3.8)**.** Apply the operator $d(\gamma^{-1}\cdot)$ to the relation of linear dependence

$$H_{n+k} = \mu_k^i H_i, \quad \text{where} \quad \mu_k^i := -\frac{\det(H_1, \dots, \overbrace{H_{n+k}}^{i}, \dots, H_n)}{\det(H_1, \dots, H_n)}, \quad 1 \le i \le n.$$

Using the fact that $d(\gamma^{-1}H_i) = d(\nabla u_i) = 0$, we arrive at the following relation,

$$Z_{k,i} \wedge \tilde{\gamma}^{-1} H_i = 0, \quad \text{where} \quad Z_{k,i} := \nabla \mu_k^i, \quad k = 1, 2, \dots$$

Since the 2-form vanishes, by applying two vector fields $\tilde{\gamma}\mathbf{e}_p$, $\tilde{\gamma}\mathbf{e}_p$, $1 \le p < q \le n$, we obtain,

$$H_{qi} Z_{k,i} \cdot \tilde{\gamma}\mathbf{e}_p = H_{pi} Z_{k,i} \cdot \tilde{\gamma}\mathbf{e}_q.$$

Notice that the above equation means $(\tilde{\gamma}Z_k)_{pi}H_{qi} = (\tilde{\gamma}Z_k)_{qi}H_{pi}$, which amounts to the fact that $\tilde{\gamma}Z_kH^T$ is symmetric. This means in particular that $\tilde{\gamma}Z_kH^T$ is orthogonal to $A_n(\mathbb{R})$, and for any $\Omega \in A_n(\mathbb{R})$, we can rewrite this orthogonality condition as

$$0 = \text{tr}\ (\tilde{\gamma} Z_k H^T \Omega) = \text{tr}\ (\tilde{\gamma}^T Z_k H^T \Omega) = \tilde{\gamma} : Z_k H^T \Omega = \tilde{\gamma} : (Z_k H^T \Omega)^{sym}, \tag{3.28}$$

where the last part comes from the fact that $\tilde{\gamma}$ is itself symmetric. Each matrix Z_k thus generates a subspace of $S_n(\mathbb{R})$ of linear contraints for $\tilde{\gamma}$. Considering m additional solutions, we arrive at the space of constraints defined in (3.8).

Algebraic inversion of $\tilde{\gamma}$ via cross-product. We now show how to reconstruct $\tilde{\gamma}$ explicitely at any point where the space $\mathcal{W}$ defined in (3.8) has codimension one. We define the generalized cross product as follows. Over an N-dimensional space $\mathcal{V}$ with a basis $(\mathbf{e}_1, \cdots, \mathbf{e}_N)$, we define the alternating $N-1$-linear mapping $\mathcal{N} : \mathcal{V}^{N-1} \to \mathcal{V}$ as the formal vector-valued determinant below, to be expanded along the last row

$$\mathcal{N}(V_1, \cdots, V_{N-1}) := \frac{1}{\det(\mathbf{e}_1, \cdots, \mathbf{e}_N)} \begin{vmatrix} \langle V_1, \mathbf{e}_1 \rangle & \dots & \langle V_1, \mathbf{e}_N \rangle \\ \vdots & \ddots & \vdots \\ \langle V_{N-1}, \mathbf{e}_1 \rangle & \dots & \langle V_{N-1}, \mathbf{e}_N \rangle \\ \mathbf{e}_1 & \dots & \mathbf{e}_N \end{vmatrix} \tag{3.29}$$

$\mathcal{N}(V_1, \cdots, V_{N-1})$ is orthogonal to $V_1, \cdots, V_{N-1}$. Moreover, $\mathcal{N}(V_1, \cdots, V_{N-1})$ vanishes if and only if $(V_1, \cdots, V_{N-1})$ are linearly dependent.

With this notion of cross-product in the case $\mathcal{V} \equiv S_n(\mathbb{R})$, we derive the following reconstruction algorithm for $\tilde{\gamma}$. Adding m additional solutions, we find that $\mathcal{W}$ can be spanned by $\sharp\mathcal{W} := \frac{n(n-1)}{2} m$ matrices whose expressions are given in (3.8), picking for instance $\{\mathbf{e}_i \otimes \mathbf{e}_j - \mathbf{e}_j \otimes \mathbf{e}_i\}_{1 \le i < j \le n}$ as a basis for $A_n(\mathbb{R})$. The condition that $\mathcal{W}$ is of codimension one over X_0 can be formulated as:

$$\inf_{x\in X_0} \mathcal{B}(x) > c_1 > 0, \qquad \mathcal{B} := \sum_{I\in\sigma(n_S-1,\sharp\mathcal{W})} |\det \mathcal{N}(I)|^{\frac{1}{n}}, \tag{3.30}$$

where $\sigma(n_S-1,\sharp\mathcal{W})$ denotes the sets of increasing injections from $[1, n_S-1]$ to $[1,\sharp\mathcal{W}]$, and where we have defined $\mathcal{N}(I) = \mathcal{N}(M_{I_1},\cdots,M_{I_{n_S-1}})$, where $\mathcal{N}$ is defined by (3.29) with $\mathcal{V}\equiv S_n(\mathbb{R})$. Then under condition (3.30), $\mathcal{W}$ is of rank n_S-1 in $S_n(\mathbb{R})$.

Whenever $(M_1,\ldots,M_{n_S-1})$ are picked in $\mathcal{W}$, their cross-product must be proportional to $\tilde\gamma$. The constant of proportionality can be deduced, up to sign, from the condition $\det\tilde\gamma = 1$ so we arrive at $\pm|\det \mathcal{N}(M_1,\cdots,M_{n_S-1})|^{\frac{1}{n}}\tilde\gamma = \mathcal{N}(M_1,\cdots,M_{n_S-1})$. The sign ambiguity is removed by ensuring that $\tilde\gamma$ must be symmetric definite positive, in particular its first coefficient on the diagonal should be positive. As a conclusion, we obtain the relation

$$|\det \mathcal{N}(I)|^{\frac{1}{n}}\tilde\gamma = \mathrm{sign}(\mathcal{N}_{11}(I))\mathcal{N}(I), \quad I\in\sigma(n_S-1,\sharp\mathcal{W}). \tag{3.31}$$

This relation is nontrivial (and allows to reconstruct $\tilde\gamma$) only if $(M_1,\ldots,M_{n_S-1})$ are linearly independent. When codim $\mathcal{W}=1$ but $\sharp\mathcal{W} > n_S-1$, we do not know *a priori* which n_S-1 -subfamily of $\mathcal{W}$ has maximal rank, so we sum over all possibilities. Equation (3.31) then becomes

$$\sum_{I\in\sigma(n_S-1,\sharp\mathcal{W})} \mathrm{sign}(\mathcal{N}_{11}(I))\mathcal{N}(I) = \mathcal{B}\tilde\gamma, \tag{3.32}$$

with $\mathcal{B}$ defined in (3.30). Since $\mathcal{B} > c_1 > 0$ over X_0, $\tilde\gamma$ can be algebraically reconstructed on X_0 by formula (3.32), where $\mathcal{N}$ is defined by (3.29) with $\mathcal{V} = S_n(\mathbb{R})$.

Uniqueness and stability. Formula (3.32) has no ambiguity provided condition (3.30), hence the uniqueness. Regarding stability, we briefly justify Proposition 3.1.7.

Proof of Proposition 3.1.7. In formula (3.32), the components of the cross-products $\mathcal{N}(I)$ are smooth (polynomial) functions of the components of the matrices $Z_k H$, which in turn are smooth functions of the components of $\{H_i\}_{i=1}^{n+m}$ and their first derivatives, and where the only term appearing as denominator is $\det(H_1, \dots, H_n)$, which is bounded away from zero by virtue of Hypothesis 3.1.2. Thus (3.13) holds for $p = 0$. That it holds for any $p \geq 1$ is obtained by taking partial derivatives of the reconstruction formula of order p and bounding accordingly. $\square$

3.3.3 Joint reconstruction of $(\tilde{\gamma}, \beta)$ and stability improvement

In this section, we justify equation (3.15), which allows to justify the stability claim (3.16). Starting from n solutions satisfying Hypothesis 3.1.2 over $X_0 \subseteq X$ and denote $H = \{H_{ij}\}_{i,j=1}^n = [H_1|\dots|H_n]$ as well as $H^{pq} := (H^{-1})_{pq}$. Applying the operator $d(\gamma^{-1}\cdot)$ to both sides of (3.2) yields $d(\gamma^{-1}H_j) = d(\nabla u_j) = 0$ due to (3.48). Rewritten in scalar components for $1 \leq j \leq n$ and $1 \leq p < q \leq n$

$$0 = \partial_q(\gamma^{pl}H_{lj}) - \partial_p(\gamma^{ql}H_{lj}) = (\partial_q\gamma^{pl} - \partial_p\gamma^{ql})H_{lj} + \gamma^{pl}\partial_q H_{lj} - \gamma^{ql}\partial_p H_{lj}.$$

Thus (3.15) is obtained after multiplying the last right-hand side by H^{ji}, summing over j and using the property that $\sum_{j=1}^n H_{lj}H^{ji} = \delta_{il}$.

3.3.4 Reconstruction of γ via an elliptic system

In this section, we will construct a second order system for $(u_1, \cdots, u_n)$ with $n+2$ measurements, assuming Hypotheses 3.1.2 and 3.1.4.A hold with $X_0 = X$. For the proof below, we shall recall the definition of the Lie Bracket of two vector fields in the euclidean setting:

$$[X, Y] := (X \cdot \nabla)Y - (Y \cdot \nabla)X = (X^i\partial_i)Y^j\mathbf{e}_j - (Y^i\partial_i)X^j\mathbf{e}_j.$$

Proof of Proposition 3.1.8. As is shown by (3.28), $\gamma Z_k H^T$ is symmetric. Multiplying both sides by γ^{-1} and using $\gamma^{-1}H = \nabla U$, we see that $Z_k[\nabla U]^T$ is symmetric. More explicitly, we have

$$Z_{k,pi}\partial_q u_i = Z_{k,qi}\partial_p u_i, \quad k = 1, 2, \tag{3.33}$$

or simply $Z_k[\nabla U]^T = [\nabla U]Z_k^T$. Assume Hypothesis 3.1.4.A holds with Z_2 invertible so that $(Z_{2,1}, \cdots, Z_{2,n})$ form a basis in $\mathbb{R}^n$. We define its dual frame such that $Z^\star_{2,j} \cdot Z_{2,i} = \delta_{ij}$. Denote $Z_2^\star = [Z^\star_{2,1}, \cdots, Z^\star_{2,n}]$ and $Z_2^\star = Z_2^{-T}$. Then the symmetry of $Z_2[\nabla U]^T$ reads,

$$Z^\star_{2,j} \cdot \nabla u_i = Z^\star_{2,i} \cdot \nabla u_j, \quad 1 \le i \le j \le n. \tag{3.34}$$

Pick v a scalar function, we have the following commutation relation:

$$(X \cdot \nabla)(Y \cdot \nabla)v = (Y \cdot \nabla)(X \cdot \nabla)v + [X, Y] \cdot \nabla v.$$

Rewrite $Z_{1,pi}\partial_q = Z_{1,pi}\mathbf{e}_q \cdot \nabla$ and apply $Z^\star_{2,j} \cdot \nabla$ to both sides of (3.33), we have the following equation by the above relations in Lie Bracket,

$$\left[Z^\star_{2,j}, Z_{1,pi}\mathbf{e}_q\right] \cdot \nabla u_i + (Z_{1,pi}\mathbf{e}_q \cdot \nabla)(Z^\star_{2,j} \cdot \nabla)u_i = \left[Z^\star_{2,j}, Z_{1,qi}\mathbf{e}_p\right] \cdot \nabla u_i + (Z_{1,qi}\mathbf{e}_p \cdot \nabla)(Z^\star_{2,j} \cdot \nabla)u_i \tag{3.35}$$

where $Z_{k,ij} = Z_k : \mathbf{e}_i \otimes \mathbf{e}_j$. Plugging (3.34) to the above equation gives,

$$(Z_{1,pi}\mathbf{e}_q \cdot \nabla)(Z^\star_{2,i} \cdot \nabla)u_j + \left[Z^\star_{2,j}, Z_{1,pi}\mathbf{e}_q\right] \cdot \nabla u_i = (Z_{1,qi}\mathbf{e}_p \cdot \nabla)(Z^\star_{2,i} \cdot \nabla)u_j + \left[Z^\star_{2,j}, Z_{1,qi}\mathbf{e}_p\right] \cdot \nabla u_i.$$

Looking at the principal part, the first term of the LHS reads

$$(Z_{1,pi}\mathbf{e}_q \cdot \nabla)(Z^\star_{2,i} \cdot \nabla)u_j = (Z^\star_2 Z_1^T \mathbf{e}_p \otimes \mathbf{e}_q) : \nabla^2 u_j + (Z_{1,pi}\mathbf{e}_q \cdot \nabla) Z^\star_{2,i} \cdot \nabla u_j.$$

Therefore, (3.35) amounts to the following coupled system,

$$Z^\star_2 Z_1^T (\mathbf{e}_p \otimes \mathbf{e}_q - \mathbf{e}_q \otimes \mathbf{e}_p) : \nabla^2 u_j + v^{pq}_{ij} \cdot \nabla u_i = 0, \quad u_j|_{\partial X} = g_j, \quad 1 \le p \le q \le n, \qquad (3.36)$$

where

$$v^{pq}_{ij} := \delta_{ij} \left[(Z_{1,pl}\mathbf{e}_q - Z_{1,ql}\mathbf{e}_p) \cdot \nabla \right] Z^\star_{2,l} + \left[Z^\star_{2,j}, Z_{1,pi}\mathbf{e}_q - Z_{1,qi}\mathbf{e}_p \right]. \qquad (3.37)$$

Notice that $H = \gamma[\nabla U]$ implies that $H^{-T}[\nabla U]^T$ is symmetric. Compared with equation (3.33), we can see that the same proof holds if we replace Z_2 by H^{-T}. In this case, the dual frame of H^{-T} is simply H. So (3.36) and (3.37) hold by replacing $Z_2^\star$ by H and defining $\tilde{v}^{pq}_{ij}$ accordingly. □

We now suppose that Hypothesis 3.1.4.B is satisfied and proceed to the proof of Theorem 3.1.9.

Proof. Starting from Hypothesis 3.1.4.B with $A_n(\mathbb{R})$-valued functions of the form

$$\Omega_i(x) = \sum_{1 \le p < q \le n} \omega^i_{pq}(x)(\mathbf{e}_p \otimes \mathbf{e}_q - \mathbf{e}_q \otimes \mathbf{e}_q), \quad i = 1, 2,$$

we take the weighted sum of equations (3.17) with weights $\omega^1_{pq}, \omega^2_{pq}$. The principal part becomes $S : \nabla^2 u_i$, which upon rewritting it as $\nabla \cdot (S \nabla u_i) - (\nabla \cdot S) \cdot \nabla u_i$ yields system (3.18).

On to the proof of stability, pick another set of data $H'_I := \{H'_i\}_{i=1}^{n+2}$ close enough to H_I

in $W^{1,\infty}$ norm, and write the corresponding system for $u'_1, \dots, u'_n$

$$-\nabla \cdot S' \nabla u'_j + W'_{ij} \cdot \nabla u'_i = 0, \quad 1 \le j \le n, \tag{3.38}$$

where S' and W'_{ij} are defined by replacing H_I in (3.19) by H'_I. Subtracting (3.38) from (3.18), we have the following coupled elliptic system for $v_j = u_j - u'_j$:

$$-\nabla \cdot S \nabla v_j + W_{ij} \cdot \nabla v_i = \nabla \cdot (S - S') \nabla u'_j + (W'_{ij} - W_{ij}) \cdot \nabla u'_i, \quad v_j|_{\partial X} = 0. \tag{3.39}$$

The proof is now a consequence of the Fredholm alternative (as in [9, Theorem 2.9]). We recast (3.39) as an integral equation. Denote the operator $L_0 = -\nabla \cdot (S\nabla)$ and define $L_0^{-1} : H^{-1}(X) \ni f \mapsto v \in H_0^1(X)$, where v is the unique solution to the equation

$$-\nabla \cdot (S \nabla v) = f \quad (X), \quad v|_{\partial X} = 0.$$

By the Lax-Milgram theorem, we have $\|v\|_{H_0^1(X)} \le C \|f\|_{H^{-1}(X)}$, where C only depends on X and S. Thus $L_0^{-1} : H^{-1}(X) \to H_0^1(X)$ is continuous, and by Rellich imbedding, $L_0^{-1} : L^2(X) \to H_0^1(X)$ is compact. Define the vector space $\mathcal{H} = (H_0^1(X))^n$, $\mathbf{v} = (v_1, \dots, v_n)$, $\mathbf{h} = (L_0^{-1} f_1, \dots, L_0^{-1} f_n)$, where $f_j = \nabla \cdot (S - S') \nabla u'_j + (W'_{ij} - W_{ij}) \cdot \nabla u'_i$, and the operator $\mathbf{P} : \mathcal{H} \to \mathcal{H}$ by,

$$\mathbf{P} : \mathcal{H} \ni \mathbf{v} \to \mathbf{P}\mathbf{v} := (L_0^{-1}(W_{i1} \cdot \nabla v_i), \cdots, L_0^{-1}(W_{in} \cdot \nabla v_i)) \in \mathcal{H}.$$

Since the W_{ij} are bounded, the differential operators $W_{ij} \cdot \nabla : H_0^1 \to L^2$ are continuous. Together with the fact that $L_0^{-1} : L^2 \to H_0^1$ is compact, we get that $\mathbf{P} : \mathcal{H} \to \mathcal{H}$ is compact. After applying the operator L_0^{-1} to (3.18), the elliptic system is reduced to the following

Fredholm equation:

$$(\mathbf{I}+\mathbf{P})\mathbf{v} = \mathbf{h}.$$

By the Fredholm alternative, if -1 is not an eigenvalue of $\mathbf{P}$, then $\mathbf{I}+\mathbf{P}$ is invertible and bounded $\|\mathbf{v}\|_{\mathcal{H}} \le \|(\mathbf{I}+\mathbf{P})^{-1}\|_{\mathcal{L}(\mathcal{H})}\|\mathbf{h}\|_{\mathcal{H}}$. Since $L_0^{-1} : H^{-1}(X) \to H_0^1(X)$ is continuous, $\mathbf{h}$ in $(H_0^1(X))^n$ is bounded by $\mathbf{f} = (f_1, \cdots, f_n)$ in $(H^{-1}(X))^n$.

$$\|\mathbf{h}\|_{\mathcal{H}} \le \|L_0^{-1}\|_{\mathcal{L}(H^{-1},H_0^1)}\|\mathbf{f}\|_{H^{-1}(X)}.$$

Then we have the estimate,

$$\|\mathbf{v}\|_{\mathcal{H}} \le \|(\mathbf{I}+\mathbf{P})^{-1}\|_{\mathcal{L}(\mathcal{H})}\|L_0^{-1}\|_{\mathcal{L}(H^{-1},H_0^1)}\|\mathbf{f}\|_{H^{-1}(X)}.$$

Noting that L_0^{-1} is continuous and the RHS of (3.39) is expressed by $H_I - H_I'$ and their derivatives up to second order, we have the stability estimate

$$\|\mathbf{u} - \mathbf{u}'\|_{H_0^1(X)} \le C\|H_I - H_I'\|_{H^1(X)},$$

where C depends on H_I but can be chosen uniform for H_I and H_I' sufficiently close. Then γ is reconstructed by $\gamma = H[\nabla U]^{-1}$ and $\nabla \times \gamma^{-1}$ by (3.15), with a stability of the form

$$\|\gamma - \gamma'\|_{L^2(X)} + \|\nabla \times (\gamma^{-1} - \gamma'^{-1}))\|_{L^2(X)} \le C\|H_I - H_I'\|_{H^1(X)}.$$

$\square$

3.4 What tensors are reconstructible ?

3.4.1 Test cases

Constant tensors. We first prove that Hypotheses 3.1.1-3.1.4 can be fulfilled with explicit constructions in the case of constant coefficients.

Proof of Proposition 3.1.10. Hypotheses 3.1.2 is trivially satisfied throughout X by choosing the collection of solutions $u_i(x) = x_i$ for $1 \le i \le n$, then Hypothesis 3.1.1 is fulfilled by picking any two distinct solutions of the above family.

Fulfilling Hypothesis 3.1.3. Let us pick

$$
\begin{aligned}
&u_i(x) := x_i, \quad 1 \le i \le n, \\
&u_{n+1}(x) := \frac{1}{2} x^T \gamma_0^{-\frac{1}{2}} \sum_{j=1}^n t_j (\mathbf{e}_j \otimes \mathbf{e}_j) \gamma_0^{-\frac{1}{2}} x, \quad \sum_{j=1}^n t_j = 0, \quad t_p \ne t_q \quad \text{if} \quad p \ne q, \\
&u_{n+2}(x) := \frac{1}{2} x^T \gamma_0^{-\frac{1}{2}} \sum_{j=1}^{n-1} (\mathbf{e}_j \otimes \mathbf{e}_{j+1} + \mathbf{e}_{j+1} \otimes \mathbf{e}_j) \gamma_0^{-\frac{1}{2}} x.
\end{aligned} \tag{3.40}
$$

In particular, $H = \gamma_0$ and $Z_i = \nabla^2 u_{n+i}$ for $i = 1, 2$, do not depend on x and admit the expression

$$
Z_1 = \gamma_0^{-\frac{1}{2}} \sum_{j=1}^n t_j (\mathbf{e}_j \otimes \mathbf{e}_j) \gamma_0^{-\frac{1}{2}} \quad \text{and} \quad Z_2 = \gamma_0^{-\frac{1}{2}} \sum_{j=1}^{n-1} (\mathbf{e}_j \otimes \mathbf{e}_{j+1} + \mathbf{e}_{j+1} \otimes \mathbf{e}_j) \gamma_0^{-\frac{1}{2}}.
$$

We will show that the (x-independent) space

$$
\mathcal{W} = \text{span} \left\{ (Z_1 H^T \Omega)^{sym}, (Z_2 H^T \Omega)^{sym}, \Omega \in A_n(\mathbb{R}) \right\}
$$

has codimension one in $S_n(\mathbb{R})$ by showing that $\mathcal{W}^\perp \subset \mathbb{R}\gamma_0$, the other inclusion $\supset$ being evident.

Let $A \in S_n(\mathbb{R})$ and suppose that $A \perp \mathcal{W}$, we aim to show that A is proportional to γ_0. The symmetry of AZ_1H^T implies that $\sum_{j=1}^n t_j \mathbf{e}_j \otimes \mathbf{e}_j \gamma_0^{-\frac{1}{2}} A \gamma_0^{-\frac{1}{2}}$ is symmetric. Denote $B = \gamma_0^{-\frac{1}{2}} A \gamma_0^{-\frac{1}{2}} \in S_n(\mathbb{R})$, we deduce that

$$t_i B_{ij} = t_j B_{ji}, \quad \text{for} \quad 1 \leq i, j \leq n.$$

Since B is symmetric and $t_i \neq t_j$ if $i \neq j$, the above equation gives that $B_{ij} = 0$ for $i \neq j$, thus B is a diagonal matrix, i.e. $B = \sum_{i=1}^n B_{ii} \mathbf{e}_i \otimes \mathbf{e}_i$. The symmetry of AZ_2H^T implies that $\sum_{j=1}^{n-1} (\mathbf{e}_j \otimes \mathbf{e}_{j+1} + \mathbf{e}_{j+1} \otimes \mathbf{e}_j) \gamma_0^{-\frac{1}{2}} A \gamma_0^{-\frac{1}{2}}$ is symmetric, which means that

$$\sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n-1}} B_{ii} (\mathbf{e}_j \otimes \mathbf{e}_{j+1} + \mathbf{e}_{j+1} \otimes \mathbf{e}_j)(\mathbf{e}_i \otimes \mathbf{e}_i) = \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n-1}} B_{ii} (\mathbf{e}_i \otimes \mathbf{e}_i)(\mathbf{e}_j \otimes \mathbf{e}_{j+1} + \mathbf{e}_{j+1} \otimes \mathbf{e}_j).$$

Write the above equation explicitly, we get

$$\sum_{j=1}^{n-1} B_{j+1,j+1} \mathbf{e}_j \otimes \mathbf{e}_{j+1} + B_{jj} \mathbf{e}_{j+1} \otimes \mathbf{e}_j = \sum_{j=1}^{n-1} B_{jj} \mathbf{e}_j \otimes \mathbf{e}_{j+1} + B_{j+1,j+1} \mathbf{e}_{j+1} \otimes \mathbf{e}_j$$

Which amounts to

$$\sum_{j=1}^{n-1} (B_{j+1,j+1} - B_{jj})(\mathbf{e}_{j+1} \otimes \mathbf{e}_j - \mathbf{e}_{j+1} \otimes \mathbf{e}_j) = 0.$$

Notice that $\{\mathbf{e}_{j+1} \otimes \mathbf{e}_j - \mathbf{e}_{j+1} \otimes \mathbf{e}_j\}_{1 \leq j \leq n-1}$ are linearly independent in $A_n(\mathbb{R})$, so $B_{j+1,j+1} = B_{jj}$ for $1 \leq j \leq n-1$, i.e. B is proportional to the identity matrix. This means that A must be proportional to γ_0 and thus $\mathcal{W}^\perp \subset \mathbb{R}\gamma_0$. Hypothesis 3.1.3 is fulfilled throughout X.

Fulfilling Hypothesis 3.1.4 with $\gamma = \mathbb{I}_n$. We split the proof according to dimension.

Even case $n = 2m$**.** Suppose that $n = 2m$, pick $u_i = x_i$ for $1 \le i \le n$, $u_{n+1} = \sum_{i=1}^{m} x_{2i-1}x_{2i}$ and $u_{n+2} = \sum_{i=1}^{m} \frac{(x_{2i-1}^2 - x_{2i}^2)}{2}$. Then simple calculations show that

$$Z_1 = \sum_{i=1}^{m} (\mathbf{e}_{2i-1} \otimes \mathbf{e}_{2i} + \mathbf{e}_{2i} \otimes \mathbf{e}_{2i-1}) \quad \text{and} \quad Z_2 = \sum_{i=1}^{m} (\mathbf{e}_{2i-1} \otimes \mathbf{e}_{2i-1} - \mathbf{e}_{2i} \otimes \mathbf{e}_{2i}).$$

We have $\det Z_1 = (-1)^m \neq 0$ so 3.1.4.A is fulfilled. Let us choose

$$\Omega_1 := \sum_{p=1}^{m} (\mathbf{e}_{2p} \otimes \mathbf{e}_{2p-1} - \mathbf{e}_{2p-1} \otimes \mathbf{e}_{2p}) \quad \text{and} \quad \Omega_2 = 0,$$

then direct calculations show that $S = (Z_2^\star Z_1^T \Omega_1 + H Z_1^T \Omega_2)^{sym} = \mathbb{I}_n$, which is clearly uniformly elliptic, hence 3.1.4.B is fulfilled.

Odd case $n = 3$**.** Pick $u_i' = x_i$ for $1 \le i \le 3$, $u_{3+1}' = x_1x_2 + x_2x_3$ and $u_{3+2}' = \frac{1}{2t_1}x_1^2 + \frac{1}{2t_2}x_2^2 + \frac{1}{2t_3}x_3^2$, where t_1, t_2, t_3 are to be chosen. In this case, $H' = \mathbb{I}_3$, $Z_1' = 2(\mathbf{e}_1 \odot \mathbf{e}_2 + \mathbf{e}_2 \odot \mathbf{e}_3)$ and $(Z_2')^\star = \sum_{i=1}^{3} t_i \mathbf{e}_i \otimes \mathbf{e}_i$ (note that Z_2' fulfills 3.1.4.A). Pick $\Omega_1'(x) = \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2$, $\Omega_2'(x) = \mathbf{e}_2 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_2$, simply calculations show that,

$$S' = \left((Z_2')^\star (Z_1')^T \Omega_1'(x) + H'(Z_1')^T \Omega_2'(x)\right)^{sym} = \begin{bmatrix} t_1 & 0 & \frac{t_3+1}{2} \\ 0 & -t_2 - 1 & 0 \\ \frac{t_3+1}{2} & 0 & 1 \end{bmatrix}. \tag{3.41}$$

(t_1, t_2, t_3) must be such that S' is positive definite and $\operatorname{tr}(Z_2') = 0$ (because u_2' solves (3.1)). This entails the conditions

$$t_1 > 0, \quad t_1(t_2 + 1) < 0, \quad -(t_2 + 1)\left(t_1 - \left(\frac{t_3 + 1}{2}\right)^2\right) > 0 \quad \text{and} \quad t_1 = -\frac{t_2 t_3}{t_2 + t_3}.$$

These conditions can be jointly satisfied for instance by picking $t_1 = 6$, $t_2 = -2$ and $t_3 = 3$, thus Hypothesis 3.1.4.B is fulfilled in the case $n = 3$.

Odd case $n = 2m + 3$. When $n = 2m + 3$ for $m \geq 0$, we build solutions based on the previous two cases. Let us pick

$$u_i = x_i, \quad 1 \leq i \leq n,$$
$$u_{n+1} = \sum_{i=1}^{m} x_{2i-1}x_{2i} + x_{2m+1}x_{2m+2} + x_{2m+2}x_{2m+3}$$
$$u_{n+2} = \sum_{i=1}^{m} \frac{(x_{2i-1}^2 - x_{2i}^2)}{2} + \frac{1}{12}x_{2m+1}^2 - \frac{1}{4}x_{2m+2}^2 + \frac{1}{6}x_{2m+3}^2.$$

Then one can simply check that $\tilde{Z}_j$ is of the form

$$\tilde{Z}_j = \left[\begin{array}{c|c} Z_j & 0_{2m\times 3} \\ \hline 0_{3\times 2m} & Z'_j \end{array}\right], \quad j = 1, 2,$$

where Z_j/Z'_j are constructed as in the case $n = 2m/n = 3$, respectively. Accordingly, let us construct $\Omega_{1,2}$ by block using the previous two cases,

$$\tilde{\Omega}_j = \left[\begin{array}{c|c} \Omega_j & 0_{2m\times 3} \\ \hline 0_{3\times 2m} & \Omega'_j \end{array}\right],$$

and the S matrix so obtained becomes

$$\tilde{S} = \left(\tilde{Z}_2^\star \tilde{Z}_1^T \tilde{\Omega}_1 + H \tilde{Z}_1^T \tilde{\Omega}_2\right)^{sym} = \left[\begin{array}{c|c} \mathbb{I}_{2m} & 0_{2m\times 3} \\ \hline 0_{3\times 2m} & S' \end{array}\right],$$

where S' is the definite positive matrix constructed in the case $n = 3$. Again, Hypothesis 3.1.4.B is fulfilled.

Fulfilling Hypothesis 3.1.4 with γ constant. Let $\{v_i\}_{i=1}^{n+2}$ denote the harmonic polynomials constructed in any case above (i.e. n even or odd) with $\gamma = \mathbb{I}_n$, and denote

$Z_1^0, Z_2^0, H^0, \Omega_1^0, \Omega_2^0$ and $S^0 = (Z_2^{0\star} Z_1^{0T}\Omega_1^0 + H^0 Z_1^{0T}\Omega_2^0)^{sym}$ the corresponding matrices. Define here, for $1 \le i \le n$, $u_i(x) := v_i(x)$ and for $i = n+1, n+2$, $u_i(x) = v_i(\gamma^{-\frac{1}{2}} x)$, all solutions of (3.1) with constant γ. Then we have that $Z_i = \gamma^{-\frac{1}{2}} Z_i^0 \gamma^{-\frac{1}{2}}$ for $i = 1, 2$ and $H = \gamma$. Upon defining $\Omega_i := \gamma^{\frac{1}{2}} \Omega_i^0 \gamma^{\frac{1}{2}} \in A_n(\mathbb{R})$ for $i = 1, 2$, direct calculations show that

$$S = (Z_2^\star Z_1^T \Omega_1 + H Z_1^T \Omega_2)^{sym} = \gamma^{\frac{1}{2}} S^0 \gamma^{\frac{1}{2}}.$$

Whenever Z_1^0 is non-singular, so is Z_1 and whenever S_0 is symmetric definite positive, so is S. The proof is complete. $\square$

Isotropic tensors. As a second test case, we show that, based on the construction of complex geometrical optics (CGO) solutions, Hypothesis 3.1.1 can be satisfied globally for an isotropic tensor $\gamma = \beta \mathbb{I}_n$ when β is smooth enough. CGO solutions find many applications in inverse conductivity/diffusion problems, and more recently in problems with internal functionals [8, 47, 15]. As established in [14], when $\beta \in H^{\frac{n}{2}+3+\varepsilon}(X)$, one is able to construct a complex-valued solution of (3.1) of the form

$$u_{\boldsymbol{\rho}} = \frac{1}{\sqrt{\beta}} e^{\boldsymbol{\rho}\cdot x}(1 + \psi_{\boldsymbol{\rho}}), \tag{3.42}$$

where $\boldsymbol{\rho} \in \mathbb{C}^n$ is a complex frequency satisfying $\boldsymbol{\rho} \cdot \boldsymbol{\rho} = 0$, which is equivalent to taking $\boldsymbol{\rho} = \rho(\mathbf{k} + i\mathbf{k}^\perp)$ for some unit orthogonal vectors $\mathbf{k}, \mathbf{k}^\perp$ and $\rho = |\boldsymbol{\rho}|/\sqrt{2} > 0$. The remainder $\psi_{\boldsymbol{\rho}}$ satisfies an estimate of the form $\rho\psi_{\boldsymbol{\rho}} = \mathcal{O}(1)$ in $\mathcal{C}^1(\overline{X})$. The real and imaginary parts of $\nabla u_{\boldsymbol{\rho}}$ are almost orthogonal, modulo an error term that is small (uniformly over X) when ρ is large. We use this property here to fulfill Hypothesis 3.1.1.

Proof of Proposition 3.1.11. Pick two unit orthogonal vectors $\mathbf{k}$ and $\mathbf{k}^\perp$, and consider the CGO solution $u_{\boldsymbol{\rho}}$ with $\boldsymbol{\rho} = \rho(\mathbf{k} + i\mathbf{k}^\perp)$ for some $\rho > 0$ which will be chosen large enough later. Computing the gradient of $u_{\boldsymbol{\rho}}$, we arrive at

$$\nabla u_{\boldsymbol{\rho}} = e^{\boldsymbol{\rho}\cdot x}(\boldsymbol{\rho} + \boldsymbol{\varphi}_{\boldsymbol{\rho}}), \quad \text{with} \quad \boldsymbol{\varphi}_{\boldsymbol{\rho}} := \nabla \psi_{\boldsymbol{\rho}} - \psi_{\boldsymbol{\rho}} \nabla \log \sqrt{\beta},$$

with $\sup_{\overline{X}} |\boldsymbol{\varphi}_{\boldsymbol{\rho}}| \le C$ independent of $\boldsymbol{\rho}$. Splitting into real and imaginary parts, each of which is a real-valued solution of (3.1), we obtain the expression

$$\nabla u^{\Re}_{\boldsymbol{\rho}} = \frac{\rho e^{\rho \mathbf{k}\cdot x}}{\sqrt{\beta}} \left((\mathbf{k} + \rho^{-1}\boldsymbol{\varphi}^{\Re}_{\boldsymbol{\rho}}) \cos(\rho \mathbf{k}^\perp \cdot x) - (\mathbf{k}^\perp + \rho^{-1}\boldsymbol{\varphi}^{\Im}_{\boldsymbol{\rho}}) \sin(\rho \mathbf{k}^\perp \cdot x) \right),$$
$$\nabla u^{\Im}_{\boldsymbol{\rho}} = \frac{\rho e^{\rho \mathbf{k}\cdot x}}{\sqrt{\beta}} \left((\mathbf{k}^\perp + \rho^{-1}\boldsymbol{\varphi}^{\Im}_{\boldsymbol{\rho}}) \cos(\rho \mathbf{k}^\perp \cdot x) + (\mathbf{k} + \rho^{-1}\boldsymbol{\varphi}^{\Re}_{\boldsymbol{\rho}}) \sin(\rho \mathbf{k}^\perp \cdot x) \right),$$

from which we compute directly that

$$|\nabla u^{\Re}_{\boldsymbol{\rho}}|^2 |\nabla u^{\Im}_{\boldsymbol{\rho}}|^2 - (\nabla u^{\Re}_{\boldsymbol{\rho}} \cdot \nabla u^{\Im}_{\boldsymbol{\rho}})^2 = \frac{\rho^2 e^{2\rho \mathbf{k}\cdot x}}{\beta} (1 + o(\rho^{-1})).$$

Therefore, for ρ large enough, the quantity in the left-hand side above remains bounded away from zero throughout X, and the proof is complete. ☐

3.4.2 Push-forward by diffeomorphism

Let $\Psi : X \to \Psi(X)$ be a $W^{1,2}$-diffeomorphism where X has smooth boundary. Then for $\gamma \in \Sigma(X)$, the push-forwarded tensor $\Psi_\star \gamma$ defined in (3.21) belongs to $\Sigma(\Psi(X))$ and Ψ pushes forward a solution u of (3.1) to a function $v = u \circ \Psi^{-1}$ satisfying the conductivity equation

$$-\nabla_y \cdot (\Psi_\star \gamma \nabla_y v) = 0 \quad (\Psi(X)), \quad v|_{\partial(\Psi(X))} = g \circ \Psi^{-1},$$

moreover Ψ and $\Psi|_{\partial X}$ induce respective isomorphisms of $H^1(X)$ and $H^{\frac{1}{2}}(\partial X)$ onto $H^1(\Psi(X))$ and $H^{\frac{1}{2}}(\partial(\Psi(X)))$.

Proof of Proposition 3.1.13. The hypotheses of interest all fomulate the linear independence of some functionals in some sense. We must see first how these functionals are push-forwarded via the diffeomorphism Ψ. For $1 \le i \le m$, we denote $v_i := \Psi_\star u_i = u_i \circ \Psi^{-1}$ as well as $\Psi_\star H_i := [\Psi_\star \gamma] \nabla_y v_i$ where y denotes the variable in $\Psi(X)$. Direct use of the chain rule allows to establish the following properties, true for any $x \in X$:

$$\begin{aligned} \nabla u_i(x) &= [D\Psi]^T(x) \nabla_y v_i(\Psi(x)), \\ H_i(x) &= \gamma \nabla u_i(x) = |J_\Psi|(x) [D\Psi]^{-1} \Psi_\star H(\Psi(x)), \\ Z_i(x) &= [D\Psi]^T(x) \Psi_\star Z_i(\Psi(x)), \end{aligned} \tag{3.43}$$

where we have defined $\Psi_\star Z_i$ the matrix with columns

$$[\Psi_\star Z_i]_{,j} = -\nabla_y \frac{\det(\nabla_y v_1, \dots, \overbrace{\nabla_y v_{n+i}}^{j}, \dots, \nabla_y v_n)}{\det(\nabla_y v_1, \dots, \nabla_y v_n)}, \quad 1 \le j \le n.$$

Hypotheses 3.1.1 and 3.1.2. Since $[D\Psi]$ is never singular over X, relations (3.43) show that for any $1 \le k \le n$, the vectors fields $(\nabla u_1, \dots, \nabla u_k)$ are linearly dependent at x if and only if the vectors fields $(\nabla_y v_1, \dots, \nabla_y v_k)$ are linearly dependent at $\Psi(x)$. The case $k = 2$ takes care of Hyp. 3.1.1 while the case $k = n$ takes care of Hyp. 3.1.2.

Hypothesis 3.1.3. If we denote

$$\Psi_\star \mathcal{W}(\Psi(x)) = \text{span} \left\{ (\Psi_\star Z_k (\Psi_\star H)^T \Omega)^{sym}, \quad \Omega \in A_n(\mathbb{R}), 1 \le k \le m \right\},$$

direct computations show that

$$\mathcal{W}(x) = [D\Psi(x)]^T \cdot \Psi_\star \mathcal{W}(\Psi(x)) \cdot [D\Psi(x)],$$

thus since $D\Psi(x)$ is non-singular, we have that $\dim \mathcal{W}(x) = \dim \Psi_\star \mathcal{W}(\Psi(x))$, so the statement of Proposition holds for Hyp. 3.1.3.

Hypothesis 3.1.4. The transformation rules (3.43) show that Z_1 is nonsingular at x iff $\Psi_\star Z_1$ is nonsingular at $\Psi(x)$, so the statement of the proposition holds for Hyp. 3.1.4.A.

Second, for two $A_n(\mathbb{R})$-valued functions $\Omega_1(x)$ and $\Omega_2(x)$, and upon defining $\Psi_\star\Omega_1$, $\Psi_\star\Omega_2$ as in (3.23), as well as

$$\Psi_\star S := \left([\Psi_\star Z_2]^{-T}[\Psi_\star Z_1]^T\Psi_\star\Omega_1 + [\Psi_\star H][\Psi_\star Z_1]^T\Psi_\star\Omega_2\right)^{sym},$$

direct use of relations (3.43) yield the relation

$$S(x) = [D\Psi(x)]^{-1}\cdot\Psi_\star S(\Psi(x))\cdot[D\Psi(x)]^{-T}, \quad x \in X,$$

and since $D\Psi$ is uniformly non-singular, S is uniformly elliptic if and only if $\Psi_\star S$ is, so the statement of the proposition holds for Hyp. 3.1.4.B. $\square$

3.4.3 Generic reconstructibility

We now show that, in principle, any $\mathcal{C}^{1,\alpha}$-smooth conductivity tensor is locally reconstructible from current densities. The proof relies on the Runge approximation for elliptic equations, which is equivalent to the unique continuation principle, valid for conductivity tensors with Lipschitz-continuous components.

This scheme of proof was recently used in the context of other inverse problems with internal functionals [15, 45], and the interested reader is invited to find more detailed proofs there.

Proof of Proposition 3.1.15. Let $x_0 \in X$ and denote $\gamma_0 := \gamma(x_0)$. We first construct solutions of the constant-coefficient problem by picking the functions defined in (3.40) (call them $v_1, \dots, v_{n+2}$) and by defining, for $1 \le i \le n+2$, $u_i^0(x) := v_i(x) - v_i(x_0)$. These solutions satisfy $\nabla \cdot (\gamma_0 \nabla u_i) = 0$ everywhere and fulfill Hypotheses 3.1.2 and 3.1.3 globally.

Second, from solutions $\{u_i^0\}_{i=1}^{n+2}$, we construct a second family of solutions $\{u_i^r\}_{i=1}^{n+2}$ via the following equation

$$\nabla \cdot (\gamma \nabla u_i^r) = 0 \quad (B_{3r}), \quad u_i^r|_{\partial B_{3r}} = u_i^0, \qquad 1 \le i \le n+2, \tag{3.44}$$

where B_{3r} is the ball centered at x_0 and of radius $3r$, r being tuned at the end. The maximum principle as well as interior regularity results for elliptic equations allow to deduce the fact that

$$\lim_{r\to 0} \max_{1\le i\le n+2} \|u_i^r - u_i^0\|_{\mathcal{C}^2(B_{3r})} = 0. \tag{3.45}$$

Third, assuming that r has been fixed at this stage, the Runge approximation property allows to claim that for every $\varepsilon > 0$ and $1 \le i \le n+2$, there exists $g_i^\varepsilon \in H^{\frac{1}{2}}(\partial X)$ such that

$$\|u_i^\varepsilon - u_i^r\|_{L^2(B_{3r})} \le \varepsilon, \quad \text{ where } \; u_i^\varepsilon \text{ solves (3.1) with } u_i^\varepsilon|_{\partial X} = g_i^\varepsilon, \tag{3.46}$$

which, combined with interior elliptic estimates, yields the estimate

$$\|u_i^\varepsilon - u_i^r\|_{\mathcal{C}^2(\overline{B_r})} \le \frac{C}{r^2} \|u_i^\varepsilon - u_i^r\|_{L^\infty(B_{2r})} \le \frac{C}{r^2}\varepsilon,$$

Since r is fixed at this stage, we deduce that

$$\lim_{\varepsilon\to 0} \max_{1\le i\le n+2} \|u_i^\varepsilon - u_i^r\|_{\mathcal{C}^2(B_r)} = 0. \tag{3.47}$$

Completing the argument, we recall that Hypotheses 3.1.2 and 3.1.3 are characterized by continuous functionals (say f_2 and f_3) in the topology of $\mathcal{C}^{2,\alpha}$ boundary conditions. While the first step established that $f_2 > 0$ and $f_3 > 0$ for the constant-coefficient solutions, limits (3.45) and (3.47) tell us that there exists a small $r > 0$, then a small $\varepsilon > 0$ such that $\max_{1\le i\le n+2} \|u_i^\varepsilon - u_i^0\|_{\mathcal{C}^2(B_r(x_0))}$ is so small that, by the continuity of f_2 and f_3, these functionals remain positive. Hypotheses 3.1.2 and 3.1.3 are thus satisfied over B_r by the family $\{u_i^\varepsilon\}_{i=1}^{n+2}$ which is controlled by boundary conditions. The proof is complete. $\square$

Remark 3.4.1 (On generic global reconstructibility). *Let us mention that from the local reconstructibility statement above, one can establish a global reconstructibility one. Heuristically, by compactness of $\overline{X}$, one can cover the domain with a finite number of either neighborhoods as above or subdomains diffeomorphic to a half-ball if the point x_0 is close to ∂X, over each of which γ is reconstructible. One can then patch together the local reconstructions using for instance a partition of unity, and obtain a globally reconstructed γ. The additional technicalities that this proof incurs may be found in [15].*

As a conclusion, for any $\mathcal{C}^{1,\alpha}$-smooth tensor γ, there exists a finite N and non-empty open set $\mathcal{O} \subset (\mathcal{C}^{2,\alpha}(\partial X))^N$ such that any $\{g_i\}_{i=1}^N \in \mathcal{O}$ generates current densities that reconstruct γ uniquely and stably (in the sense of estimate (3.16)*) throughout X.*

3.5 Appendix: Exterior calculus and notations

Throughout this chapter, we use the following convention regarding exterior calculus. Because we are in the Euclidean setting, we will avoid the flat operator notation by identifying vector fields with one-forms via the identification $\mathbf{e}_i \equiv \mathbf{e}^i$ where $\{\mathbf{e}_i\}_{i=1}^n$ and $\{\mathbf{e}^i\}_{i=1}^n$ denote bases of $\mathbb{R}^n$ and its dual, respectively. In this setting, if $V = V^i\mathbf{e}_i$ is a vector field, dV denotes the two-vector field

$$dV = \sum_{1\le i<j\le n} (\partial_i V^j - \partial_j V^i)\mathbf{e}_i \wedge \mathbf{e}_j.$$

A two-vector field can be paired with two other vector fields via the formula

$$A \wedge B(C, D) = (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C),$$

which allows to make sense of expressions of the form

$$dV(A, \cdot) = \sum_{1 \leq i < j \leq n} (\partial_i V^j - \partial_j V^i)((A \cdot \mathbf{e}_i)\mathbf{e}_j - (A \cdot \mathbf{e}_j)\mathbf{e}_i).$$

Note also the following well-known identities for f a smooth function and V a smooth vector field, rewritten with the notation above:

$$d(\nabla f) = 0, \qquad f \in \mathcal{C}^2(X), \tag{3.48}$$

$$d(fV) = \nabla f \wedge V + f dV. \tag{3.49}$$

Chapter 4

Inverse problems in the Maxwell system

In this chapter, we study the hybrid inverse problem proposed in Section 1.2. Let X be a bounded domain with smooth boundary in $\mathbb{R}^3$. The smooth anisotropic electric permittivity, conductivity, and the constant isotropic magnetic permeability are respectively described by $\epsilon(x)$, $\sigma(x)$ and μ_0, where $\epsilon(x)$, $\sigma(x)$ are tensors and μ_0 is a constant scalar, known, coefficient. Let E and H denote the electric and magnetic fields inside the domain X with a harmonic time dependence. Thus E and H solve the following system of Maxwell's equations:

$$\begin{cases} \nabla \times E + \imath\omega\mu_0 H = 0 \\ \nabla \times H - \gamma E = 0 \end{cases} \quad \text{on} \quad X, \qquad \nu \times E|_{\partial X} = f. \tag{4.1}$$

Here, $\gamma = \sigma + \imath\omega\varepsilon$ in X, ν is the exterior unit normal vector on the boundary ∂X, with the frequency $\omega > 0$ fixed. We assume that $\varepsilon(x)$ and $\sigma(x)$ satisfy the uniform ellipticity condition. We present an explicit (stable) reconstruction procedure for the anisotropic, complex-valued tensor γ from knowledge of a set of (at least 6) magnetic fields H_j for $1 \leq j \leq J$, where H_j solves (4.1) with prescribed boundary conditions f_j.

4.1 Main hypotheses and stability results

We first introduce the solution space,

$$H^s_{\mathrm{Div}}(X) := \{u \in (H^s(X))^3 | \mathrm{Div}(\nu \times u) \in H^{s-\frac{1}{2}}(\partial X)\}$$

where Div denotes the surface divergence (see, e.g., [19] for the definition). Let $TH^s_{\mathrm{Div}}(\partial X)$ denotes the Sobolev space through the tangential trace mapping acting on $H^s_{\mathrm{Div}}(X)$,

$$TH^s_{\mathrm{Div}}(\partial X) = \{f \in (H^s(\partial X))^3 | \mathrm{Div} f \in H^s(\partial X)\}$$

They are Hilbert spaces for the norms

$$\begin{aligned} \|u\|_{H^s_{\mathrm{Div}}(X)} &= \|u\|_{(H^s(X))^3} + \|\mathrm{Div}(\nu \times u)\|_{H^{s-\frac{1}{2}}(\partial X)} \\ \|f\|_{TH^s_{\mathrm{Div}}(\partial X)} &= \|f\|_{(H^s(\partial X))^3} + \|\mathrm{Div}(f)\|_{H^s(\partial X)}. \end{aligned}$$

The boundary value problem (4.1) admits a unique solution $(E, H) \in H^k_{\mathrm{Div}}(X) \times H^k_{\mathrm{Div}}(X)$ with imposed boundary electric condition $\nu \times E|_{\partial X} = f \in TH^{k-\frac{1}{2}}_{\mathrm{Div}}(\partial X)$ except for a discrete set of magnetic resonance frequencies $\{\omega\}$ when $\sigma = 0$; see [35]. The solution satisfies

$$\|E\|_{H^s_{\mathrm{Div}}(X)} + \|H\|_{H^s_{\mathrm{Div}}(X)} \leq C\|f\|_{TH^{s-\frac{1}{2}}_{\mathrm{Div}}(\partial X)}. \tag{4.2}$$

We assume that ω is not a resonance frequency.

4.1.1 Main hypotheses

We now list the main hypotheses, which allow us to set up our reconstruction formulas, which are *local* in nature: the reconstruction of γ at $x_0 \in X$ requires the knowledge of $\{H_j(x)\}_{1\leq j\leq J}$ for x only in the vicinity of x_0.

The first hypothesis requires the existence of a basis of electric fields which satisfy (4.1).

Hypothesis 4.1.1. *Given Maxwell's equations in form of* (4.1) *with* ε *and* σ *uniformly elliptic, there exist* $(f_1, f_2, f_3) \in TH^{\frac{1}{2}}_{Div}(\partial X)^3$ *and a sub-domain* $X_0 \subset X$*, such that the corresponding solutions* E_1, E_2, E_3 *satisfy*

$$\inf_{x\in X_0} |\det(E_1, E_2, E_3) \geq c_0| > 0.$$

Assuming that E_1, E_2, E_3 solutions to (4.1) satisfy the Hypothesis 4.1.1, we consider additional solutions $\{E_{3+k}\}_{k=1}^m$ and obtain the linear dependence relations for each additional solution,

$$E_{3+k} = \sum_{i=1}^{3} \lambda_i^k E_i, \quad 1 \leq k \leq m. \tag{4.3}$$

As shown in [9, 26], the coefficients λ_i^k can be computed as follows:

$$\lambda_i^k = -\frac{\det(E_1, \overbrace{E_{3+k}}^{i}, E_3)}{\det(E_1, E_2, E_3)} = -\frac{\det(\nabla \times H_1, \overbrace{\nabla \times H_{3+k}}^{i}, \nabla \times H_3)}{\det(\nabla \times H_1, \nabla \times H_2, \nabla \times H_3)}.$$

Therefore these coefficients are computable from magnetic fields. The reconstruction procedures will make use of the matrices Z_k defined by

$$Z_k = [Z_{k,1}, Z_{k,2}, Z_{k,3}], \quad \text{where} \quad Z_{k,i} = \nabla\lambda_i^k, \quad 1 \leq k \leq m. \tag{4.4}$$

These matrices are also uniquely determined from the known magnetic fields.

The next hypothesis which gives a sufficient condition for a local reconstruction of the anisotropic tensor γ, is that a sufficiently large number of matrices Z_k satisfy a full-rank condition.

Hypothesis 4.1.2. *Assume that Hypothesis 4.1.1 holds for (E_1, E_2, E_3) over $X_0 \subset X$. We denote Y as the matrix with columns Y_1, Y_2, Y_3, where $Y_i = \nabla \times H_i$, $1 \leq i \leq 3$. Then there exist $E_1, \cdots, E_{J=3+m}$ solutions of Maxwell equations (4.1) and some $X' \subseteq X_0$ such that the space,*

$$\mathcal{W} = \{(\Omega Z_k Y^T)^{sym} | \Omega \in A_3(\mathbb{R}), 1 \leq k \leq m\}, \tag{4.5}$$

has full rank in $S_3(\mathbb{C})$ for all $x \in X'$, where S_3 and A_3 denote the space of 3×3 symmetric and anti-symmetric matrices, respectively.

Remark 4.1.3. *Hypotheses 4.1.1 and 4.1.2 can be both fulfilled for well-chosen boundary conditions $\{f_i\}_{1 \leq i \leq 6}$ when γ is close to a constant tensor γ_0. The proof of such a statement can be found in Section 4.2.3. For a arbitrary tensor γ, Hypothesis 4.1.1 can be fulfilled locally. If we suppose additionally that γ is the $C^{1,\alpha}$ vicinity of $\gamma(x_0)$ on some open domain of x_0, then Hypothesis 4.1.2 also holds locally, see Section 4.2.6.*

4.1.2 Uniqueness and stability results

We denote by $M_n(\mathbb{C})$ the space of $n \times n$ matrices with inner product $\langle A, B \rangle := \text{tr}\,(A^*B)$. We assume that Hypotheses 4.1.1 and 4.1.2 hold over some $X_0 \subset X$ with $J = 3 + m$ solutions $(E_1, \cdots, E_{3+m})$. In particular, the linear space $\mathcal{W} \subset S_3(\mathbb{C})$ defined in (4.5) is of full rank in $S_3(\mathbb{C})$. We will see that the inner products of $(\gamma^{-1})^*$ with all elements in $\mathcal{W}$ can be calculated from knowledge of $(H_1, \cdots, H_{3+m})$. Together with the fact that $\mathcal{W}$ is also constructed by the measurements, γ can be completely determined by $H_1, \cdots, H_{3+m}$. The reconstruction formulas can be found in Theorem 4.2.2. This algorithm leads to a unique and stable reconstruction in the sense of the following theorem.

Theorem 4.1.4. *Suppose that Hypotheses 4.1.1 and 4.1.2 hold over some $X_0 \subset X$ for two $3+m$-tuples $\{E_i\}_{i=1}^{3+m}$ and $\{E_i'\}_{i=1}^{3+m}$, solutions of the Maxwell system* (4.1) *with the complex tensors γ and γ' satisfying the uniform ellipticity condition* (2.2)*. Then γ can be uniquely reconstructed in X_0 with the following stability estimate,*

$$\|\gamma - \gamma'\|_{W^{s,\infty}(X_0)} \leq C \sum_{i=1}^{3+m} \|H_i - H_i'\|_{W^{s+2,\infty}(X)} \tag{4.6}$$

for any integer $s > 0$. If γ is isotropic or in the vicinity of a constant tensor γ_0, then γ can be reconstructed with 6 *measurements and the above estimate holds on $X_0 = X$.*

Remark 4.1.5. *For the case γ is isotropic, it can be reconstructed via a redundant elliptic equation which is based on the construction of Complex Geometrical Optics solutions(CGOs). The algorithms will be given in Section 4.2.4.*

4.2 Reconstruction approaches

4.2.1 Preliminary

Exterior calculus and notations: Throughout this chapter, we will identify vector fields with one-forms via the identification $\mathbf{e}_i \equiv \mathbf{e}^i$ where $\{\mathbf{e}_i\}_{i=1}^n$ and $\{\mathbf{e}^i\}_{i=1}^n$ denote bases of $\mathbb{R}^n$ and its dual, respectively. In this setting, if $V = V^i\mathbf{e}_i$ is a vector field, dV denotes the two-vector field

$$dV = \sum_{1\leq i<j\leq n} (\partial_i V^j - \partial_j V^i)\mathbf{e}_i \wedge \mathbf{e}_j.$$

A two-vector field can be paired with two other vector fields via the formula

$$A \wedge B(C, D) = (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C),$$

Note also the following well-known identities for f a smooth function and V a smooth vector field, rewritten with the notation above:

$$d(\nabla f) = 0, \qquad f \in \mathcal{C}^2(X),$$
$$d(fV) = \nabla f \wedge V + f dV.$$

Hodge star operator: For $x \in \mathbb{R}^n$, let $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ and $\{\mathbf{e}^1, \cdots, \mathbf{e}^n\}$ denote the canonical bases of $T_x\mathbb{R}^n$ and its dual $T_x^*\mathbb{R}^n$. The Hodge star operator on an l-form is defined as the linear extension of

$$\star(\mathbf{e}^{\alpha_1} \wedge \cdots \wedge \mathbf{e}^{\alpha_l})|_x = (\mathbf{e}^{\beta_1} \wedge \cdots \wedge \mathbf{e}^{\beta_{n-l}})|_x, \tag{4.7}$$

where $(\beta_1, \cdots, \beta_{n-l}) \in \{1, \cdots, n\}^{n-l}$ is chosen such that

$$\{\mathbf{e}^{\alpha_1}, \cdots, \mathbf{e}^{\alpha_l}, \mathbf{e}^{\beta_1}, \cdots, \mathbf{e}^{\beta_{n-l}}\} \tag{4.8}$$

is a positive base of $T_x^*\mathbb{R}^n$. For a l-form η, the Hodge star operator follows,

$$\star \star \eta = (-1)^{l(n-l)}\eta. \tag{4.9}$$

4.2.2 Reconstruction algorithms

For some matrices $A, B \in M_n(\mathbb{C})$, we denote their product $A : B$ by,

$$A : B = \text{tr } (AB^T) = \text{tr } (A^T B). \tag{4.10}$$

Starting with 3 solutions (E_1, E_2, E_3) satisfying Hypothesis 4.1.1, we then pick additional magnetic fields H_{3+k}. The corresponding electric fields E_{3+k} and E_1, E_2, E_3 satisfy the linear dependence relations defined in (4.3). We recall the 3×3 matrices,

$$Y = [Y_1, Y_2, Y_3], \quad Y_i = \nabla \times H_i.$$

$\nabla \times H_{3+k}$ satisfies the same linear dependence with Y_1, Y_2, Y_3 as E_{3+k}, E_1, E_2, E_3. Thus λ_i^k defined in (4.3) are computable from only knowledge of the magnetic fields(we use implicit summation notation),

$$\nabla \times H_{3+k} = \lambda_i^k \nabla \times H_i; \quad \nabla \lambda_i^k := -\nabla \frac{\det(\nabla \times H_1, \overbrace{\nabla \times H_{3+k}}^{i}, \nabla \times H_3)}{\det Y}. \tag{4.11}$$

Now we construct the subspace $\mathcal{W}$ of $S_3(\mathbb{C})$ as denoted in Hypothesis 4.1.2,

$$\mathcal{W} = \{(\Omega Z_k Y^T)^{sym} | \Omega \in A_3(\mathbb{R}), 1 \le k \le m\}. \tag{4.12}$$

Denote $(\mathbf{w}_1, \cdots, \mathbf{w}_6)$ as the natural basis of the 6 dimensional space $S_3(\mathbb{C})$. Given 6 vectors $W_1, \cdots, W_6$ in $\mathcal{W}$, for any vector $W \in S_3(\mathbb{C})$, we define a $(7, 1)$ type tensor $\mathcal{N}$ dealing with inner products $\langle W, W_p\rangle$:

$$\mathcal{N}(W, W_1, \cdots, W_6) := \sum_{p=1}^{6} \langle W, W_p\rangle \begin{vmatrix} \langle W_1, \mathbf{w}_1\rangle & \dots & \langle W_1, \mathbf{w}_6\rangle \\ \vdots & & \vdots \\ \overbrace{\mathbf{w}_1}^{i} & \dots & \overbrace{\mathbf{w}_6}^{i} \\ \vdots & & \vdots \\ \langle W_6, \mathbf{w}_1\rangle & \dots & \langle W_6, \mathbf{w}_6\rangle \end{vmatrix} = F(W_1, \cdots, W_6)W, \tag{4.13}$$

where $F(W_1, \cdots, W_6) := \det\{\langle W_p, \mathbf{w}_q\rangle\}_{1\leq p,q\leq 6}$. Obviously, $\det\{\langle W_p, \mathbf{w}_q\rangle\}_{1\leq p,q\leq 6} W = 0$ if and only if $W_1, \cdots, W_6$ are linearly dependent. In other words, $\mathcal{N}(W, W_1, \cdots, W_6) = 0$ never vanishes if $W_1, \cdots, W_6$ are linearly independent and $W \neq 0$.

We summarize the reconstruction algorithms in the following theorem and show that γ can be algebraically reconstructed via Gram-Schmidt procedure and the explicit expression (4.13).

Theorem 4.2.1. *Assume that Hypothesis 4.1.1 and 4.1.2 are fulfilled on a sub-domain $X_0 \subset X$, then γ can be reconstructed on X_0 as follows*

$$\gamma = \overline{\det}\{\langle W_p, \mathbf{w}_q\rangle\}_{1\leq p,q\leq 6}(\mathcal{N}^{-1}(\bar{\gamma}^{-1}, W_1, \cdots, W_6))^*. \tag{4.14}$$

Here, $(\mathbf{w}_1, \cdots, \mathbf{w}_6)$ denotes the natural basis of $S_3(\mathbb{C})$ and $\{W_p\}_{1\leq p\leq 6}$ are linearly independent matrices, which can be constructed from the matrices $\{(\Omega Z_k Y^T)^{sym}\}_{1\leq k\leq m}$ in $\mathcal{W}$ by the Gram-Schmidt procedure. The inner product of $\bar{\gamma}^{-1}$ with matrices in $\mathcal{W}$ are given by:

$$\langle \bar{\gamma}^{-1}, (\Omega Z_k Y^T)^{sym}\rangle = tr\ (\Omega M_k^T), \tag{4.15}$$

where $M_k := \frac{\iota}{2}\omega\mu_0 \star (\lambda_i^k H_i - H_{3+k})(\mathbf{e}_p, \mathbf{e}_q)\mathbf{e}_q \otimes \mathbf{e}_p$ for $1 \leq k \leq m$ and $\star$ denotes the Hodge *star operator. Moreover, for any other γ' satisfying* (2.2) *and Maxwell system* (4.1)*, we have the following stability estimate,*

$$\|\gamma - \gamma'\|_{W^{s,\infty}(X_0)} \leq C \sum_{i=1}^{3+m} \|H_i - H_i'\|_{W^{s+2,\infty}(X)}, \tag{4.16}$$

where C is a constant and s is any integer.

Proof. We rewrite the time-harmonic Maxwell's equations (4.1) in terms of differential forms,

$$\begin{cases} \star dE_i = -\iota\omega\mu_0 H_i \\ \star dH_i = \gamma E_i. \end{cases} \tag{4.17}$$

Here d is the exterior derivative and $\star$ denotes the Hodge star operator. Applying the exterior derivative d to (4.3) gives,

$$d(\sum_{i=1}^{3} \lambda_i^k E_i - E_{3+k}) = 0. \tag{4.18}$$

Using the formula $d(fV) = df \wedge V + f dV$ for a scalar function f and a vector field V, we have

$$d\lambda_i^k \wedge E_i + \lambda_i^k dE_i = dE_{3+k}. \tag{4.19}$$

Applying the Hodge operator to (4.17) and using the fact that $E_i = \gamma^{-1}\nabla \times H_i$, we obtain the following equation,

$$d\lambda_i^k \wedge \gamma^{-1}\nabla \times H_i = \iota\omega\mu_0 \star (\lambda_i^k H_i - H_{3+k}). \tag{4.20}$$

By applying two vector fields $\mathbf{e}_p$, $\mathbf{e}_p$, $1 \leq p < q \leq 3$ to the above 2-form, we obtain,

$$(\nabla \lambda_i^k \cdot \mathbf{e}_p)(\gamma^{-1} Y_i \cdot \mathbf{e}_q) - (\nabla \lambda_i^k \cdot \mathbf{e}_q)(\gamma^{-1} Y_i \cdot \mathbf{e}_p) = \boldsymbol{\iota} \omega \mu_0 \star (\lambda_i^k H_i - H_{3+k})(\mathbf{e}_p, \mathbf{e}_q), \quad (4.21)$$

where $Y_i = \nabla \times H_i$ for $1 \leq i \leq 3$. The above equation reads explicitly,

$$(\gamma^{-1} Y)_{qi} Z_{k,pi} - Z_{k,qi} (\gamma^{-1} Y)_{pi} = \boldsymbol{\iota} \omega \mu_0 \star (\lambda_i^k H_i - H_{3+k})(\mathbf{e}_p, \mathbf{e}_q) \quad (4.22)$$

which amounts to the following matrix equation,

$$\gamma^{-1} Y Z^T - (\gamma^{-1} Y Z^T)^T = \boldsymbol{\iota} \omega \mu_0 \star (\lambda_i^k H_i - H_{3+k})(\mathbf{e}_p, \mathbf{e}_q) \mathbf{e}_q \otimes \mathbf{e}_p. \quad (4.23)$$

Since γ is symmetric, we pick $\Omega \in A_3(\mathbb{R})$ and calculate its ':' product with both sides of the above equation,

$$\langle \bar{\gamma}^{-1}, (\Omega Z_k Y^T)^{sym} \rangle = \gamma^{-1} : (\Omega Z_k Y^T)^{sym} = \text{tr } (\Omega M_k^T), \quad (4.24)$$

where $M_k := \frac{\boldsymbol{\iota}}{2} \omega \mu_0 \star (\lambda_i^k H_i - H_{3+k})(\mathbf{e}_p, \mathbf{e}_q) \mathbf{e}_q \otimes \mathbf{e}_p$. The stability estimate is clear by inspection of the reconstruction procedure. Two derivatives on $\{H_k\}_{1 \leq k \leq 3+m}$ are taken in the reconstructions of the matrices Z_k and one derivative is taken for the reconstructiongs of M_k. The Gram-Schmidt procedure preserves errors in the uniform norm. Therefore, we have a total loss of 2 derivatives in the reconstruction of γ as indicated in Theorem 4.1.4. □

4.2.3 Global reconstructions close to constant tensor

In this section, we assume that γ is in the vicinity of a diagonalizable constant tensor γ_0. We will construct special solutions, namely plane waves, of the Maxwell's equations (4.1) and demonstrate that Hypothesis 4.1.1 and 4.1.2 are fulfilled with these solutions. The following lemma shows that Hypothesis 4.1.1 is satisfied in the homogeneous media.

Lemma 4.2.2. *Suppose that the admittivity γ is sufficiently close to a constant tensor γ_0, where the real and imaginary parts of γ_0 satisfy the uniform ellipticity condition* (2.2)*. Then Hypothesis 4.1.1 holds on X.*

Proof. Decompose the tensor $\gamma_0 = Q\Lambda Q^T$ for a diagonal $\Lambda \in M_3$ and $Q^TQ = I$. This decomposition is possible since a symmetric matrix is diagonalizable if and only if it is complex orthogonally diagonalizable, see [30, Theorem 4.4.13]. We write $Q = [\beta_1, \beta_2, \beta_3]$ and k_1, k_2, k_3 the components on the diagonal of Λ, such that $\gamma_0\beta_j = k_j\beta_j$, $j = 1, 2, 3$. We choose plane waves as possible solutions to Maxwell's equations (4.1),

$$E_j = \beta_j e^{i\zeta_j\cdot x}, \quad 1 \le j \le 3, \tag{4.25}$$

with some ζ_j to be chosen in $\mathbb{C}^3$. Applying the curl operator to the first equation in (4.1), we get the vector Helmholtz equation,

$$\nabla \times \nabla \times E_j + \boldsymbol{\iota}\omega\mu_0\gamma_0 E_j = 0, \tag{4.26}$$

where $\gamma_0 = \sigma_0 + \boldsymbol{\iota}\omega\varepsilon_0$. Using the fact that $\nabla\times\nabla\times = -\Delta + \nabla\nabla\cdot$, the above equation amounts to

$$(\zeta_j \cdot \zeta_j)e^{\boldsymbol{\iota}\zeta_j\cdot x}\beta_j - (\beta_j \cdot \zeta_j)e^{\boldsymbol{\iota}\zeta_j\cdot x}\zeta_j + \boldsymbol{\iota}\omega\mu_0 e^{\boldsymbol{\iota}\zeta_j\cdot x}\gamma_0\beta_j = 0. \tag{4.27}$$

Since $e^{\boldsymbol{\iota}\zeta_j\cdot x}$ is never zero, the above equation reduces to,

$$(\beta_j \cdot \zeta_j)\zeta_j - (\zeta_j \cdot \zeta_j)\beta_j = \boldsymbol{\iota}\omega\mu_0\gamma_0\beta_j. \tag{4.28}$$

By choosing ζ_j to be orthogonal to β_j and $\zeta_j \cdot \zeta_j = -\iota\omega\mu_0 k_j$, equation (4.28) obviously holds by noticing that $\gamma_0 \beta_j = k_j \beta_j$. From the above analysis, the solutions can be chosen as follows,

$$\begin{cases} E_1 = \beta_1 e^{it_1 \beta_2 \cdot x} \\ E_2 = \beta_2 e^{it_2 \beta_3 \cdot x} \\ E_3 = \beta_3 e^{it_3 \beta_1 \cdot x} \end{cases}, \tag{4.29}$$

where t_i are chosen such that $t_i^2 = -\iota\omega\mu_0 k_i$ for $1 \leq i \leq 3$. Then E_1, E_2, E_3 are solutions to Maxwell's equations (4.1) and are obviously independent. □

The next proposition states that, some proper linear combinations of the solutions chosen in Hypothesis 4.1.1 also satisfy the Maxwell system (4.1).

Proposition 4.2.3. *Let us choose the electric fields* $E_{3+k} = \sum_{i=1}^{3} \lambda_i^k \beta_i e^{\iota\zeta_i \cdot x}$ *such that* λ^k *has a constant gradient verifying that* $\nabla \lambda_i^k \perp \{\beta_i, \zeta_i\}$*, where* β_i, ζ_i *are chosen in* (4.29). *Then* E_{3+k} *solves Maxwell's equations* (4.1) *for* $\gamma = \gamma_0$.

Proof. Assume that Hypothesis 4.1.1 holds and pick $E_i = \beta_i e^{\iota\zeta_i \cdot x}$ defined in (4.29) for $i = 1, 2, 3$. We pick addition electric fields as indicated in (4.3),

$$E_{3+k} = \sum_{i=1}^{3} \lambda_i^k E_i, \quad k = 1, 2, \ldots \tag{4.30}$$

where λ_i^k are to be determined. Inserting E_{3+k} into the vector Helmholtz equation (4.26), we get,

$$\begin{aligned} \nabla \times \nabla \times E_{3+k} =& \nabla \times \nabla \times (\lambda_i^k E_i) \\ =& (\nabla \cdot E_i + E_i \cdot \nabla)\nabla\lambda_i^k - (\nabla \cdot \nabla\lambda_i^k + \nabla\lambda_i^k \cdot \nabla)E_i + \nabla\lambda_i^k \times \nabla \times E_i + \lambda_i^k \nabla \times \nabla \times E_i \\ =& -\iota\omega\mu_0\gamma_0\lambda_i^k E_i. \end{aligned}$$

Here we choose $\nabla\lambda_i^k$ to be constant and $\nabla\lambda_i^k \perp \beta_i$. Using the fact that $\nabla \cdot E_i = 0$ for the special solutions in (4.29) and E_i satisfies the Helmholtz equation (4.26), the above equation reads

$$-(\nabla\lambda_i^k \cdot \nabla)E_i + \nabla\lambda_i^k \times (\nabla \times E_i) = 0. \tag{4.31}$$

Let ∇_{E_i} denotes the subscripted gradient operator on the factor E_i, the basic formulas for curl operator give that,

$$\begin{aligned}\nabla\lambda_i^k \times (\nabla \times E_i) =& \nabla_{E_i}(\nabla\lambda_i^k \cdot E_i) - (\nabla\lambda_i^k \cdot \nabla)E_i \\ =& \boldsymbol{\iota}(\nabla\lambda_i^k \cdot \beta_i)(\nabla\lambda_i^k \cdot \mathbf{e}_p)e^{\boldsymbol{\iota}\zeta_i \cdot x}\mathbf{e}_p - (\nabla\lambda_i^k \cdot \nabla)E_i.\end{aligned}$$

By choosing $\nabla\lambda_i^k \perp \beta_i$, equation (4.31) reduces to,

$$(\nabla\lambda_i^k \cdot \nabla)E_i = \boldsymbol{\iota}(\nabla\lambda_i^k \cdot \zeta_i)E_i = 0. \tag{4.32}$$

Since E_1, E_2, E_3 are independent, the above equation holds if and only if $\nabla\lambda_i^k \cdot \zeta_i = 0$, for $i = 1, 2, 3$. Therefore, $E_{3+k} = \sum_{i=1}^{3} \lambda_i^k \beta_i e^{\boldsymbol{\iota}\zeta_i \cdot x}$ solves the Maxwell's equation (4.1), with $\nabla\lambda_i^k, \beta_i, \zeta_i$ an orthogonal basis in $\mathbb{C}^3$. □

Thanks to Proposition 4.2.3, we can choose 3 additional solutions as follows:

$$\begin{cases} E_{3+1} = \lambda_1 E_1 = \lambda_1 \beta_1 e^{it_1 \beta_2 \cdot x} \\ E_{3+2} = \lambda_2 E_2 = \lambda_2 \beta_2 e^{it_2 \beta_3 \cdot x} \\ E_{3+3} = \lambda_3 E_3 = \lambda_3 \beta_3 e^{it_3 \beta_1 \cdot x} \end{cases}, \tag{4.33}$$

where E_1, E_2, E_3 are chosen in (4.29) and $\nabla\lambda_1, \nabla\lambda_2, \nabla\lambda_3$ are chosen to be $\beta_3, \beta_1, \beta_2$, respectively.

The following lemma proves that $\mathcal{W}$ is of full rank in $S_3(\mathbb{C})$ in homogeneous media.

Lemma 4.2.4. *Suppose that the admittivity γ is sufficiently close to a constant tensor γ_0. Then Hypothesis 4.1.2 is fulfilled by choosing a minimum number of 6 electric fields as indicated in* (4.29) *and* (4.33)*.*

Proof. As indicated in Proposition 4.2.3, we pick additional solutions $E_{3+k} = \lambda_k E_k$, for $k = 1, 2, 3$, where $\nabla\lambda_1 = \beta_3, \nabla\lambda_2 = \beta_1$ and $\nabla\lambda_3 = \beta_2$. Let $A \in S_3(\mathbb{C})$ and suppose that $A \perp \mathcal{W}$, we aim to show that A vanishes. Decompose A in terms of $\beta_i \otimes \beta_j$,

$$A = A_{ij}\beta_i \otimes \beta_j, \quad \text{where} \quad A_{ij} = A_{ji}. \tag{4.34}$$

Here and below, we use the implicit summation notation for the index i and j. Thus,

$$\begin{aligned} Z_k Y^T = Z_k(\gamma E)^T &= -\frac{1}{\iota\omega\mu_0} Z_k[(\zeta_1 \cdot \zeta_1)E_1, (\zeta_2 \cdot \zeta_2)E_2, (\zeta_3 \cdot \zeta_3)E_3]^T \\ &= -\frac{1}{\iota\omega\mu_0}(\zeta_k \cdot \zeta_k)\nabla\lambda_k \otimes E_k \end{aligned}$$

for $k = 1, 2, 3$. Since $A \perp \mathcal{W}$ implies that $Z_k Y^T A$ is symmetric, we deduce the following equation,

$$A_{ij}(\nabla\lambda_k \otimes E_k)(\beta_i \otimes \beta_j) = A_{ij}(\beta_i \otimes \beta_j)(E_k \otimes \nabla\lambda_k). \tag{4.35}$$

By definition $E_k = \beta_k e^{\iota\zeta_k \cdot x}$ and the orthogonality of $\{\beta_i\}_{1\le i\le 3}$, the above equation reduces to

$$A_{i,k+1}(\beta_k \otimes \beta_i - \beta_i \otimes \beta_k) = 0, \quad \text{for} \quad k = 1, 2, 3, \tag{4.36}$$

where we identify $k+1 := 1$, for $k = 3$. Notice that $\{\beta_k \otimes \beta_i - \beta_i \otimes \beta_k\}_{i,k=1,2,3}$ form a basis in $A_3(\mathbb{C})$, so obviously $A_{i,k+1} = 0$, for any $i \neq k$, which implies that $A_{ij} = 0$, for any i, j. Thus $\mathcal{W}$ is of full rank in $S_3(\mathbb{C})$. $\square$

Remark 4.2.5. *Since the Maxwell system can be written in the sense of differential forms as in* (4.17) *for an arbitrary n dimension space, the above reconstruction formulas can thus be generalized to the n dimensional case. The proof of Lemmas 4.2.2 and 4.2.4 in n dimensions is analogous to the* 3 *dimensional case.*

4.2.4 Global reconstructions for isotropic tensor

In this section, we suppose that the admittivity γ is scalar. We will show that γ can be reconstructed via a redundant elliptic system by constructing 6 Complex Geometrical Optics solutions. CGO solutions are constructed in [20] and their properties can be extended to higher order Sobolev spaces, see [18]. The approach in [18] can be used to reconstruct the scalar γ.

Theorem 4.2.6. *Let $\gamma(x)$ be a smooth scalar function. Then there exist* 6 *internal magnetic fields $\{H_i\}_{1 \leq i \leq 6}$ such that γ is uniquely reconstructed via the following redundant elliptic equation,*

$$\nabla\gamma + \beta(x)\gamma = 0, \tag{4.37}$$

where $\beta(x)$ is an invertible matrix, which is uniquely determined by the measurements. Moreover, the stability result (4.6) *holds for $X_0 = X$.*

Proof. The system (4.1) can be rewritten as the Helmholtz equation,

$$\nabla \times \nabla \times E - k^2 nE = 0, \tag{4.38}$$

where the wave number k is given by $k = \omega\sqrt{\epsilon_0\mu_0}$ with ϵ_0 the dielectric constant, and the refractive index $n = \frac{1}{\epsilon_0}(\epsilon(x) - \boldsymbol{\iota}\frac{\sigma(x)}{\omega})$. The proof is based on the construction of complex geometrical optics solutions of the form,

$$E(x) = e^{\boldsymbol{\iota}\zeta\cdot x}(\eta + R_\zeta(x)), \tag{4.39}$$

where $\zeta, \eta \in \mathbb{C}^3$, $\zeta \cdot \zeta = k^2$ and $\zeta \cdot \eta = 0$. The existence of R_ζ in $\mathcal{C}^2(X)$ was proved in [20] and can be generalized to an arbitrary regular space $\mathcal{C}^d(X)$, see [18]. Now picking two CGO solutions E_1, E_2 as defined in (4.39), we derive the following equation from (4.38),

$$\nabla\times\nabla\times E_1 \cdot E_2 + \nabla\times\nabla\times E_2 \cdot E_1 = 0. \tag{4.40}$$

Substituting the measurements $Y_j = \nabla \times H_j = \gamma E_j$ into the above equation gives the following transport equation,

$$\theta \cdot \nabla\gamma + \vartheta\gamma = 0, \tag{4.41}$$

where

$$\begin{aligned}
&\theta = \chi[(Y_2\cdot\nabla)Y_1 + (\nabla\cdot Y_1)Y_2 + 2\nabla_{Y_2}(Y_1\cdot Y_2) - (Y_1\cdot\nabla)Y_2 - (\nabla\cdot Y_2)Y_1 - 2\nabla_{Y_1}(Y_1\cdot Y_2)],\\
&\vartheta = \chi(\nabla\times\nabla\times Y_1\cdot Y_2 - \nabla\times\nabla\times Y_2\cdot Y_1).
\end{aligned} \tag{4.42}$$

We choose two specific sets of vectors ζ, η as in [20]. Define ζ_1, ζ_2 and η_1, η_2 in terms of a large real parameter c and an arbitrary real number a,

$$\begin{cases} \zeta_1 &= (a/2, \boldsymbol{\iota}\sqrt{c^2 + a^2/4 - k^2}, c),\\ \zeta_2 &= (a/2, -\boldsymbol{\iota}\sqrt{c^2 + a^2/4 - k^2}, -c) \end{cases}, \quad \begin{cases} \eta_1 &= \frac{1}{\sqrt{c^2+a^2}}(c, 0, -a/2)\\ \eta_2 &= \frac{1}{\sqrt{c^2+a^2}}(c, 0, a/2). \end{cases} \tag{4.43}$$

Note that

$$\lim_{c\to\infty} \eta_j = \eta_0 := (1,0,0), \quad j = 1,2,$$
$$\lim_{c\to\infty} \frac{\zeta_1}{|\zeta_1|} = -\lim_{c\to\infty} \frac{\zeta_2}{|\zeta_2|} = \zeta_0 := \frac{1}{\sqrt{2}}(0, \boldsymbol{\iota}, 1), \tag{4.44}$$

and

$$\zeta_1 + \zeta_2 = (a, 0, 0), \quad \zeta_0 \cdot \zeta_0 = 0, \quad \eta_0 \cdot \zeta_0 = 0. \tag{4.45}$$

By choosing $\chi(x) = -e^{-\boldsymbol{\iota}(\zeta_1+\zeta_2)\cdot x} \frac{1}{4\sqrt{2}c}$, θ and ζ_0 have approximately the same direction when $|\zeta|$, the length of ζ_1, ζ_2, tends to infinity (see [18, Proposition 3.6]),

$$\|\theta - \gamma^2 \zeta_0\|_{\mathcal{C}^d(X)} \leq \frac{C}{|\zeta|}. \tag{4.46}$$

Now we choose 3 independent unit vectors ζ_0^j and η_0^j, such that $\zeta_0^j \cdot \zeta_0^j = \zeta_0^j \cdot \eta_0^j = 0$, $j = 1, 2, 3$. Similarly to (4.43), we choose (ζ_1^j, ζ_2^j) and (η_1^j, η_2^j) such that, $|\zeta| := |\zeta_1^j| = |\zeta_2^j|$, and also,

$$\lim_{|\zeta|\to\infty} \frac{\zeta_1^j}{|\zeta|} = -\lim_{|\zeta|\to\infty} \frac{\zeta_2^j}{|\zeta|} = \zeta_0^j \quad \text{and} \quad \lim_{|\zeta|\to\infty} \eta^j = \eta_0^j. \tag{4.47}$$

We pick 3 pairs of CGO solutions $\{E_1^j, E_2^j\}_{1\leq j\leq 3}$ as defined in (4.39) and define the corresponding $\{\theta_j, \vartheta_j\}_{1\leq j\leq 3}$ by (4.42). From the estimate (4.46), we deduce that $[\theta_1, \theta_2, \theta_3]$ is invertible for $|\zeta|$ sufficiently large. Therefore equation (4.41) amounts to a redundant elliptic equation,

$$\nabla\gamma + \beta(x)\gamma = 0, \tag{4.48}$$

where $\beta = [\theta_1, \theta_2, \theta_3]^{-1}[\vartheta_1, \vartheta_2, \vartheta_3]$. Then γ can be reconstructed using (4.48) if it is known at one point on the boundary. Since we have to differentiate the measurements twice for the construction of β, there is a loss of two derivatives compared to H for the reconstruction of γ via (4.48). The stability estimate (4.6) obviously follows. $\square$

4.2.5 Runge approximation for the anisotropic Maxwell system

To derive local reconstruction formulas for a more general γ, we need to control the local behavior of solutions by well-chosen boundary conditions. This is done by means of a Runge approximation. In this section, we will prove the Runge approximation for an anisotropic Maxwell system using the unique continuation property. For UCP and Runge approximation in our context, we refer the readers to, e.g., [41, 50].

Unique continuation property

Unique continuation property for an anisotropic Maxwell system with only real magnetic permeability ϵ has been proved in [22]. We generalize the result to the case of a complex tensor $\gamma = \sigma + \iota\omega\epsilon$ in (4.1). We recall the div-curl system as follows,

$$\begin{aligned} &\gamma(x)E(x) - \nabla \times H(x) = 0, \quad \iota\omega\mu(x)H(x) + \nabla \times E(x) = 0, \\ &\nabla \cdot (\gamma(x)E(x)) = 0, \qquad \nabla \cdot (\mu(x)H(x)) = 0. \end{aligned} \tag{4.49}$$

We will use the Calderón approach to derive a Carleman estimate which implies the unique continuation property across every $\mathcal{C}^2$-surface. For Calderón approach, we refer the readers to [16, 52].

Lemma 4.2.7 (Basic Carleman inequality). *Let $(u(x,t), v(x,t)) \in \mathcal{C}^1(B_r(x_0))^3$ with support contained in $|x| \le r$, $0 \le t \le T$. There is a constant C independent of (u, v) such that for r, T and k^{-1} sufficiently small, the following inequality holds*

$$\int_0^T \|u,v\| w(t)dt \le C(k^{-1} + T^2) \int_0^T \|P(u,v)\| w(t)dt. \tag{4.50}$$

where P denotes the div-curl operator,

$$P(u,v) = (\iota\omega\mu v + \nabla \times u, \gamma u - \nabla \times v, \nabla \cdot (\gamma u), \nabla \cdot (\mu v)). \tag{4.51}$$

Here $\|\cdot\|$ denotes the L^2 norm with respect to x-variable, $w(t) = e^{k(T-t)^2}$ with k a positive constant. Then if (E,H) is a solution of the system (4.49) in a neighborhood of the origin, vanishing identically for $t < 0$, then $(E,H) = 0$ in a full neighborhood of the origin.

Proof. We first introduce the div-curl system,

$$L(x,D) = (\nabla \times u, \nabla \cdot (\gamma u)), \tag{4.52}$$

where the principle symbol of L is

$$L(x,\xi) = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & \xi_1 \\ -\xi_2 & \xi_1 & 0 \\ \sum_{j=1}^3 \gamma_{1j}\xi_j & \sum_{j=1}^3 \gamma_{2j}\xi_j & \sum_{j=1}^3 \gamma_{3j}\xi_j \end{pmatrix}. \tag{4.53}$$

Notice that the third curl equation does not involve any derivatives in x_3 direction, thus it can be dropped. Then we derive a square system,

$$\tilde{L}(x,\xi) = \begin{pmatrix} \xi_3 & 0 & \xi_1 \\ 0 & \xi_3 & -\xi_2 \\ \gamma_{1j}\xi_j & \gamma_{2j}\xi_j & \gamma_{3j}\xi_j \end{pmatrix}. \tag{4.54}$$

We rewrite the principal part of (4.52) in the form $l(x, e_3)D_3u + \bar{L}(x, D')u$, where

$$l(x, e_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma_{13} & \gamma_{23} & \gamma_{33} \end{pmatrix} \tag{4.55}$$

is invertible and $\bar{L}(x, D')u$ contains only the derivatives with respect to x_1 and x_2. Hence the equation (4.52) can be rewritten as follows,

$$D_3u + l^{-1}(x, e_3)\bar{L}(x, D')u = l^{-1}(x, e_3)\tilde{L}(x, D)u. \tag{4.56}$$

We then calculate the eigenvalues of $l^{-1}(x, e_3)\bar{L}(x, \xi')u$, namely the roots of $p(x, \xi', \alpha) = \det(\alpha\mathbb{I} + l^{-1}(x, e_3)\bar{L}(x, \xi'))$. We first list the standard hypotheses in Calderón's approach: For (x, t) in a neighborhood of the origin, and for every unit vector ξ' in $\mathbb{R}^n$:

- $p(x, \xi', \alpha)$ has at most simple real roots α and at most double complex roots,
- distinct roots α_1, α_2 satisfy $\|\alpha_1 - \alpha_2\| \geqq \epsilon > 0$
- nonreal roots α satisfy $\|\Im\alpha\| \geqq \epsilon$

Here ϵ is some fixed positive constant. In the following, the summations will be from 1 to 2.

$$\begin{aligned} p(x, \xi', \alpha) &= \det(l(x, e_3))^{-1} \det \tilde{L}(x, \xi', \alpha) \\ &= \frac{1}{\gamma_{33}}\alpha(\gamma_{jk}\xi_j\xi_k + 2\alpha\gamma_{3j}\xi_j + \gamma_{33}\alpha^2) \\ &= \frac{1}{\gamma_{33}}\alpha(\xi', \alpha)\gamma(\xi', \alpha)^T. \end{aligned}$$

Hence the three roots of $p(x, \xi', \alpha)$ are:

$$\alpha_1 = 0, \qquad \alpha_{2,3} = -\frac{\gamma_{3j}\xi_j}{\gamma_{33}} \pm \sqrt{(\frac{\gamma_{3j}\xi_j}{\gamma_{33}})^2 - \frac{\gamma_{jk}\xi_j\xi_k}{\gamma_{33}}}. \tag{4.57}$$

α_2 and α_3 satisfy the above hypothesis and the prove is essentially given in [28, Lemma 17.2.5]. Since $\Re\gamma$ and $\Im\gamma$ are both positive definite, the roots $\alpha_{2,3}$ are non-real, by noticing that $(\xi', \alpha)\gamma(\xi', \alpha)^T \neq 0$ for real α. Then $|\alpha_1 - \alpha_2|^2 = 4|(\frac{\gamma_{3j}\xi_j}{\gamma_{33}})^2 - \frac{\gamma_{jk}\xi_j\xi_k}{\gamma_{33}}| = \frac{4}{|\gamma_{33}|^2}|(\gamma_{3j}\xi_j)^2 - \gamma_{33}\gamma_{jk}\xi_j\xi_k|$. A simple calculation shows that,

$$|(\gamma_{3j}\xi_j)^2 - \gamma_{33}\gamma_{jk}\xi_j\xi_k| \geqq |\Im((\gamma_{3j}\xi_j)^2 - \gamma_{33}\gamma_{jk}\xi_j\xi_k))| \tag{4.58}$$

$$= |e_3^T\tau e_3 \cdot \xi'^T\epsilon\xi' + e_3^T\epsilon e_3 \cdot \xi'^T\tau\xi' - 2e_3^T\tau\xi' \cdot e_3^T\epsilon\xi'|, \tag{4.59}$$

which is obviously strictly positive for a unit vector ξ' by the Cauchy-Schwarz inequality and the fact that $e_3 = (0, 0, 1)$ and $\xi' = (\xi_1, \xi_2, 0)$ are not collinear. Then we obtain a Carleman type inequality, see [52, Page 33],

$$\int_0^T \|u\|w(t)dt \leq C(k^{-1} + T^2)\int_0^T (\|\nabla \times u)\| + \|\nabla \cdot (\gamma u)\|)w(t)dt. \tag{4.60}$$

Here u is compactly support in a neighborhood of the origin and k^{-1} and T are sufficiently small. Applying the same analysis to v and using Cauchy-Schwarz, we have the following estimate,

$$\int_0^T \|u, v\|w(t)dt \leq C(k^{-1} + T^2)[\int_0^T (\|\iota\omega\mu v + \nabla \times u)\| + \|\gamma u - \nabla \times v\|)w(t)dt \tag{4.61}$$

$$+ \int_0^T (\|u, v\| + \|\nabla \cdot (\gamma u)\| + \|\nabla \cdot (\mu v)\|)w(t)dt]. \tag{4.62}$$

The term $\|u, v\|$ can be moved to the RHS by choosing k^{-1} and T sufficiently small. We thus get the Carleman estimate,

$$\int_0^T \|u,v\| w(t)dt \le C(k^{-1}+T^2)\int_0^T \|P(u,v)\| w(t)dt, \tag{4.63}$$

where P denote the div-curl operator in (4.51). Now suppose $z=(E,H)$ satisfies $Pz=0$. Let $\zeta(t)$ be a nonnegative smooth function defined in $t \geqq 0$ equal to 1 for $t \le 2T/3$ and 0 for $t \geqq T$. By applying (4.50) to $(u,v)=\zeta z$, we obtain that,

$$\int_0^{\frac{2T}{3}} \|z\|^2 w dt \le C(k^{-1}+T^2)\int_{\frac{2T}{3}}^T \|P(\zeta z)\|^2 w dt \le C'(k^{-1}+T^2)\int_{\frac{2T}{3}}^T w dt, \tag{4.64}$$

with some fixed constant T and C' independent of k. Thus, we obtain,

$$e^{kT^2/4}\int_0^{\frac{T}{2}} \|z\|^2 dt \le C'(k^{-1}+T^2)Te^{kT^2/9}. \tag{4.65}$$

Letting $k \to \infty$, we see that $z=0$ for $t \le T/2$. □

Due to the above lemma, we may generalize the unique continuation property to the Maxwell system with a complex tensor γ, which is a more general case of [22, Corollary 1.2] but requires more smoothness of the coefficients in order to apply the Calderón machinery. We formulate it as the following theorem.

Theorem 4.2.8. *Let $(E,H) \in H^1(X)$ satisfying the Maxwell's equation* (4.1) *and let $S = \{\Phi(x) = \Phi(x_0)\}$ be a level surface of the function $\Phi \in \mathcal{C}^2(\bar{X})$ near $x_0 \in X$ such that $\nabla\Phi(x_0) \neq 0$. If (E,H) vanish on one side of S, then $(E,H)=0$ in a full neighborhood of $x_0 \in X$.*

Proof. The proof is analogue to Lemma 4.2.7 by introducting new coordinates $x_3 = \Phi(x) - \Phi(x_0)$, in which the level surfaces of Φ becomes $\{x_3 = 0\}$. By the ellipticity property of Maxwell's equations, the analysis of the new system can be returned to the original one.

See, for example, [22] for details. ☐

Proof of Runge approximation property

The Runge approximation can be proved with the unique continuation property of Maxwell's system, since we have the uniqueness of the Cauchy problem near every direction. The prove of the following theorem follows the idea in [50].

Theorem 4.2.9 (Runge approximation)**.** *Let X_0 and X be two bounded domains with smooth boundary such that $\bar{X}_0 \subset X$. Let $(E_0, H_0) \in H^1(X_0)$ locally satisfy the Maxwell's equations* (4.49)*,*

$$P(E, H) = 0 \quad (X_0). \tag{4.66}$$

Then for each $\epsilon > 0$, there is a function $f_\epsilon \in TH^{\frac{1}{2}}_{Div}(\partial X)$ such that the solutions $(E_\epsilon, H_\epsilon) \in H^1(X)^3$ satisfy,

$$P(E_\epsilon, H_\epsilon) = 0 \quad (X), \quad \nu \wedge E_\epsilon|_{\partial X} = f_\epsilon. \tag{4.67}$$

Moreover, for a compact subset $K \subset X_0$,

$$\|E_\epsilon - E_0\|_{H^1(K)} \leq \epsilon. \tag{4.68}$$

Proof. We rewrite Maxwell's equations (4.49) into the following Helmholtz-type equation,

$$L(E) := \nabla \times \mu^{-1} \nabla \times E + \imath\omega\gamma E = 0. \tag{4.69}$$

Applying the interior estimate to solutions of Maxwell's equations (see [61]), we get the local estimate,

$$\|E_\epsilon - E_0\|_{H^1(K)} \leq C\|E_\epsilon - E_0\|_{L^2(\tilde{K})} \tag{4.70}$$

for some constant $C > 0$ and where $\tilde{K} \subset X_0$ is a compact containing K. Therefore, we wish to prove that,

$$M = \{w : w = u|_{\tilde{K}}, u \in H^1(X), Lu = 0 \text{ in } X\} \tag{4.71}$$

is dense in

$$N = \{w : w = u|_{\tilde{K}}, u \in H^1(X_0), Lu = 0 \text{ in } X_0\} \tag{4.72}$$

for the strong L^2 topology. By Hahn-Banach theorem, this means that for all $f \in L^2(\tilde{K})$ such that,

$$(f, w)_{L^2(\tilde{K})} = 0 \quad \text{for all } w \text{ in } M. \tag{4.73}$$

This implies that

$$(f, w)_{L^2(\tilde{K})} = 0 \quad \text{for all } w \text{ in } N. \tag{4.74}$$

We extend f ouside $\tilde{K}$ and still call it f as the extension on X_0. Define then

$$L^*E = f \text{ on } X, \quad n \wedge E = 0 \text{ on } \partial X, \tag{4.75}$$

where $L^* = \nabla \times \mu^{-1}\nabla \times +\iota\omega\gamma^*$ denotes the adjoint to L. For any $u \in H^1(X)$ satisfying

$Lu = 0$ on X, integrations by parts show that,

$$(f, u)_{L^2(\tilde{K})} = \int_{\tilde{K}} f \cdot u^* d\sigma = \int_X L^* E \cdot u^* dx = \int_{\partial X} n \wedge (\mu^{-1} \nabla \times E) \cdot u^* d\sigma = 0. \quad (4.76)$$

Then we deduce that $\nu \wedge (\mu^{-1} \nabla \times E) = 0$ on ∂X. Combining with equation (4.75), we obtain,

$$L^* E = 0 \text{ on } X \backslash \tilde{K}, \quad \nu \wedge E = \nu \wedge (\mu^{-1} \nabla \times E) = 0 \text{ on } \partial X. \quad (4.77)$$

Recalling that $H = \frac{\iota}{\omega} \mu^{-1} \nabla \times E$, we will prove that (E, H) together with all their first order derivatives vanish on ∂X, so that the solution can be extended by 0 outside the domain X. With a local diffeomorphism, we restrict ∂X on a neighborhood of the plan $x_3 = 0$ for simplicity. In this particular case, $\nu = e_3$ and $\nu \wedge E = 0$ means that,

$$E^1 = E^2 = 0 \quad \text{on } \partial X, \quad (4.78)$$

where E^i denotes the i-th component of E. Moreover, the third component of $\nabla \times E$ vanishes on ∂X,

$$\nu \cdot \nabla \times E = \partial_1 E^2 - \partial_2 E^1 = 0 \quad \text{on } \partial X \quad (4.79)$$

by the fact that $\partial_1 E^2 - \partial_2 E^1$ concerns only the tangential derivatives of E^1, E^2, which vanish on the boundary. As for (4.78), $\nu \wedge (\mu^{-1} \nabla \times E) = 0$ implies that the first and second components of $\mu^{-1} \nabla \times E$ are both zero. Together with (4.79) and the fact that μ^{-1} is positive definite, we infer that,

$$\nabla \times E = 0 \quad \text{on } \partial X. \quad (4.80)$$

Therefore $H = \frac{\iota}{\omega}\mu^{-1}\nabla \times E = 0$ on ∂X. Recalling that $L^*E = \nabla \times \mu^{-1}\nabla \times E + \iota\omega\gamma^* E = 0$ on $X \backslash \tilde{K}$, we obviously have,

$$\nabla \times H = \gamma^* E \quad \text{on } \partial X. \tag{4.81}$$

Since the third component of $\nabla \times H$ only concerns the tangential derivatives, it has to vanish. Then by (4.78) and the fact that $\gamma_{33} \neq 0$, we have the following equality,

$$\nabla \times H = E = 0 \quad \text{on } \partial X. \tag{4.82}$$

Since the tangential derivatives of H are both zero on the boundary,

$$\nabla \times H = (\partial_2 H^3 - \partial_3 H^2, \partial_3 H^1 - \partial_1 H^3, \partial_1 H^2 - \partial_2 H^1) = 0 \quad \text{on } \partial X \tag{4.83}$$

implies that

$$\partial_3 H^1 = \partial_3 H^2 = 0 \quad \text{on } \partial X. \tag{4.84}$$

Noticing that $\nabla \cdot (\mu H) = 0$ and $H = 0$ on the boundary ∂X, we get,

$$\sum_{1 \leq i,j \leq 3} \partial_i(\mu_{ij} H^j) = \mu_{33}\partial_3 H^3 = 0 \quad \text{on } \partial X. \tag{4.85}$$

Together with (4.84) and $\mu_{33} \neq 0$, this implies that $\partial_3 H^1 = \partial_3 H^2 = \partial_3 H^3 = 0$. Applying the same calculations for E and its first order derivatives as above, we have

$$\nabla \times E = \nabla \cdot (\gamma^* E) = 0 \quad \text{on } \partial X. \tag{4.86}$$

We deduce that all first-order derivatives of E and E itself vanish on ∂X. Thus (E, H) can be extended to 0 outside ∂X. By the unique continuation property in Theorem 4.2.8, we conclude that $E = 0$ on $X \backslash \tilde{K}$. So for any $u \in H^1(X_0)$ with $Lu = 0$ in X_0, we have,

$$\int_{\tilde{K}} f \cdot u^* dx = \int_{X_0} L^* E \cdot u^* = 0, \tag{4.87}$$

which completes the proof. □

Remark 4.2.10. *In the above analysis of UCP and Runge approximation, the magnetic permeability μ in the Maxwell system* (4.49) *can be any uniformly elliptic tensor, but not necessarily a constant scalar μ_0 as imposed at the beginning of this chapter.*

The next corollary shows that the Runge approximation can be applied to more regular spaces, such as Hölder space.

Corollary 4.2.11. *Let $X_0 \subset X$ be a bounded domain with smooth boundary. With same hypotheses as Theorem 4.2.9, there is a open subset $X' \subset X_0$ such that for any ϵ*

$$\|E_\epsilon - E_0\|_{\mathcal{C}^{1,\alpha}(X')} \leq \epsilon, \tag{4.88}$$

where E_0, E_ϵ satisfy the Maxwell equations (4.1) *on X_0 and X, respectively.*

Proof. Recall that E^ϵ and E_0 satisfy the equations,

$$\nabla \times \nabla \times E^\epsilon + \boldsymbol{\iota}\omega\mu_0\gamma E^\epsilon = \nabla \times \nabla \times E_0 + \boldsymbol{\iota}\omega\mu_0\gamma E_0 = 0 \quad (X_0). \tag{4.89}$$

Let $v = E_\epsilon - E_0$, then v also satisfy the equation

$$\nabla \times \nabla \times v + \boldsymbol{\iota}\omega\mu_0\gamma v = 0 \quad (X_0). \tag{4.90}$$

Differentiating (4.90) with respect to x_j for $1 \leq j \leq 3$, we obtain,

$$\nabla \times \nabla \times \partial_j v + \iota\omega\mu_0\gamma\partial_j v = -\iota\omega\mu_0\partial_j\gamma v \quad (X_0), \tag{4.91}$$

where the operator ∂_j denotes the x_j-derivative applied on each component of v and γ. Recalling the local estimate $\|v\|_{H^1(K)} \leq \epsilon$ in Theorem 4.2.9, with the interior estimate and the smoothness of γ, we deduce

$$\|\partial_j v\|_{H^1(X')} \leq C\|\partial_j\gamma v\|_{H^1(X')} \leq C'\epsilon, \tag{4.92}$$

where X' is contained in K. We iterate the above procedure such that $s > \frac{5}{2}$. By applying Sobolev embedding theorem, we obtain the following estimate,

$$\|v\|_{\mathcal{C}^{1,\alpha}(X')} \leq C\|v\|_{H^s(X')} \leq C''\epsilon, \tag{4.93}$$

which completes the proof. □

4.2.6 Local reconstructions with redundant measurements

In this section, we will show that local reconstructions are possible for a more general γ than presented in earlier sections. The linear independence of the matrices in Hypothesis 4.1.1 becomes local. If in addition, γ is in the $C^{1,\alpha}(X)$ vicinity of a constant tensor γ_0 on some open domain $X' \subset X$, Hypothesis 4.1.2 also holds locally. We thus need to use potentially more than 6 internal magnetic fields, although we do not expect this large number of measurements to be necessary in practice. The control of linear independence from the boundary relies on the Runge approximation in Theorem 4.2.9. This scheme was used in [26, 15].

Theorem 4.2.12. *Let $X \subset \mathbb{R}^n$ a smooth domain and γ a smooth tensor. Then for any*

$x_0 \in X$, there exists a neighborhood $X' \subset X$ of x_0 and 6 solutions of (4.1) such that Hypothesis 4.1.1 holds. Moreover, if γ is in the $C^{1,\alpha}$ vicinity of $\gamma(x_0)$, then Hypothesis 4.1.2 also holds locally on some open domain $X_0 \subset X$.

Proof. We denote $\gamma_0 := \gamma(x_0)$. We first construct solutions of the constant-coefficient problem by picking the functions $\{E_i^0\}_{1\leq i\leq 6}$ defined in (4.29) and (4.33). These solutions satisfy $\nabla\times\nabla\times E + \iota\omega\mu_0\gamma_0 E = 0$ and fulfill Hypothesis 4.1.1 and 4.1.2 globally. Second, we look for solutions of the form,

$$\nabla\times\nabla\times E_i^r + \iota\omega\mu_0\gamma E_i^r = 0 \quad \text{in } B_r, \quad \nu\times E_i^r = \nu\times E_i^0 \quad \text{on } \partial B_r, \quad 1\leq i\leq 6, \tag{4.94}$$

where B_r is the ball centered at x_0 with r to be chosen. Let $w = E_i^r - E_i^0$,

$$\nabla\times\nabla\times w + \iota\omega\mu_0\gamma w = \iota\omega\mu_0(\gamma_0-\gamma)E_i^0 \quad \text{in } B_r, \quad \nu\times w = 0 \quad \text{on } \partial B_r. \tag{4.95}$$

By the smoothness of γ as well as interior regularity results for elliptic equations, we deduce that,

$$\lim_{r\to\infty}\|E_i^r - E_i^0\|_{\mathcal{C}^{0,\alpha}(B_r)} \leq C\lim_{r\to\infty}\|(\gamma_0-\gamma)E_i^0\|_{\mathcal{C}^{0,\alpha}(B_r)} = 0. \tag{4.96}$$

Thus we can fix r sufficiently small such that $\|E_i^r - E_i^0\|_{\mathcal{C}^{0,\alpha}(B_r)} \leq \epsilon$ for ϵ sufficiently small. Finally, by the Runge Approximation property, we claims that for every $\epsilon > 0$ and $1\leq i\leq 6$, there exists $f_\epsilon \in TH^{\frac{1}{2}}_{\mathrm{Div}}(\partial X)$ such that the corresponding solution E_i^ϵ to (4.1) satisfy,

$$\|E_i^\epsilon - E_i^r\|_{\mathcal{C}^{1,\alpha}(B_r)} \leq \epsilon, \quad \text{where } \nu\times E_i^\epsilon = f_\epsilon \quad \text{on } \partial X. \tag{4.97}$$

Combined with equation (4.96), we deduce that,

$$\|E_i^\epsilon - E_i^0\|_{\mathcal{C}^{0,\alpha}(B_r)} \leq 2\epsilon. \tag{4.98}$$

By choosing a sufficiently small ϵ, Hypothesis 4.1.1 obviously holds by continuity of the determinant. In addition, if γ is in the $\mathcal{C}^{1,\alpha}$ vicinity of γ_0, we can choose a sufficiently small r, such that (4.96) holds in $\mathcal{C}^{1,\alpha}(B_r)$,

$$\|E_i^r - E_i^0\|_{\mathcal{C}^{1,\alpha}(B_r)} \leq C\|(\gamma_0 - \gamma)E_i^0\|_{\mathcal{C}^{1,\alpha}(B_r)} \leq \epsilon. \tag{4.99}$$

Then together with (4.97), we derive the estimate as following,

$$\|E_i^\epsilon - E_i^0\|_{\mathcal{C}^{1,\alpha}(B_r)} \leq 2\epsilon. \tag{4.100}$$

Notice that the space $\mathcal{W}$ constructed in (4.5) contains up to first derivatives of E_i. Again by choosing a sufficient small ϵ, the full rank property of $\mathcal{W}$ in Hypothesis 4.1.2 is satisfied by $\{E_i^\epsilon\}_{1\leq i\leq 6}$. □

Chapter 5

Imaging of conductivities from current densities

In this chapter, we consider the problem of reconstructing an anisotropic conductivity γ in a domain $X \in \mathbb{R}^2$ from measurement of internal current densities H. The explicit inversion procedure is presented in several numerical simulations, which demonstrate the influence of the choice of boundary conditions on the stability of the reconstruction.

5.1 Modeling of the problem

Let $X \subset \mathbb{R}^2$ be a bounded domain with a $C^{2,\alpha}$ boundary ∂X. Although most of the following results generalize to arbitrary spatial dimensions, we restrict the setting to $\mathbb{R}^2$; see [10] for results in higher dimensions. We consider the inverse problem of reconstructing an anisotropic conductivity tensor in the second-order elliptic equation,

$$\nabla \cdot (\gamma \nabla u) = 0 \quad (X), \qquad u|_{\partial X} = g, \tag{5.1}$$

from knowledge of internal current densities of the form $H = \gamma\nabla u$, where u solves (5.1). The above equation has real-valued coefficients and $\gamma = (\gamma_{ij})_{1\le i,j\le 2}$ is a symmetric (real-valued) tensor satisfying the uniform ellipticity condition

$$\kappa^{-1}\|\xi\|^2 \le \xi\cdot\gamma\xi \le \kappa\|\xi\|^2, \quad \xi\in\mathbb{R}^2, \quad \text{for some } \kappa\ge 1, \tag{5.2}$$

so that (5.1) admits a unique solution in $H^1(X)$ for $g \in H^{\frac{1}{2}}(\partial X)$.

5.1.1 Global reconstructibility condition

We start by selecting 4 boundary conditions (g_1, g_2, g_3, g_4) and the corresponding current densities

$$H_i = \gamma\nabla u_i, \quad 1\le i\le 4, \tag{5.3}$$

where the function u_i solves (5.1). Assuming that over X, the two solutions u_1, u_2 satisfy the following positivity condition

$$\inf_{x\in X} |\det(\nabla u_1, \nabla u_2)| \ge c_0 > 0, \tag{5.4}$$

then the gradients of additional solutions $\nabla u_3, \nabla u_4$ can be decomposed as linear combinations in the basis $(\nabla u_1, \nabla u_2)$,

$$\begin{cases} \nabla u_3 = \mu_1\nabla u_1 + \mu_2\nabla u_2 \\ \nabla u_4 = \lambda_1\nabla u_1 + \lambda_2\nabla u_2 \end{cases}, \tag{5.5}$$

where the coefficients $\{\mu_i\}_{1\le i\le 2}$ can be computed by Cramer's rule as

$$(\mu_1,\mu_2) = (\frac{\det(\nabla u_3,\nabla u_2)}{\det(\nabla u_1,\nabla u_2)}, \frac{\det(\nabla u_1,\nabla u_3)}{\det(\nabla u_1,\nabla u_2)}) = (\frac{\det(H_3,H_2)}{\det(H_1,H_2)}, \frac{\det(H_1,H_3)}{\det(H_1,H_2)}). \quad (5.6)$$

The same expression holds for $\{\lambda_i\}_{1\le i\le 2}$ by replacing u_3 by u_4 in the above equation. Therefore these coefficients are computable from the available current densities. The reconstruction procedures will make use of the matrices Z_k defined by

$$Z_k = [Z_{k,1}, Z_{k,2}], \quad \text{where} \quad Z_{1,i} = \nabla\mu_i \quad Z_{2,i} = \nabla\lambda_i, \quad 1\le i,k\le 2. \quad (5.7)$$

These matrices are also uniquely determined by the known current densities. Denoting the matrix $H = [H_1, H_2]$ and the skew-symmetric matrix $J = \mathbf{e}_2\otimes\mathbf{e}_1 - \mathbf{e}_1\otimes\mathbf{e}_2$, we construct two matrices as follows,

$$M_k = (Z_k H^T J)^{sym}, \quad \text{for} \quad k = 1, 2. \quad (5.8)$$

The calculations in the following section show that condition (5.4) and the independence of $M_1, M_2 \in S_2(\mathbb{R})$ give a sufficient condition for a global reconstruction of γ. Condition (5.4) may be fulfilled using [1, Theorem 4] which guarantees that (5.4) holds if the map $\partial X \ni x \to (g_1(x), g_2(x))$ is a homeomorphism onto its image. That all required conditions are met for some boundary conditions is provided in the following lemma.

Lemma 5.1.1. *Let $\gamma(x) \in H^{5+\epsilon}(X)$ for some $\epsilon > 0$ and satisfy the uniform elliptic condition* (2.2)*. Then there exists a set of illuminations $\{g_i\}_{1\le i\le 4}$, such that the solutions $\{u_i\}_{1\le i\le 4}$ satisfy the following conditions:*

A. $\inf_{x\in X} |\det(H_1,H_2)| \ge c_0 > 0$ *holds on X.*

B. The two matrices M_1, M_2 constructed by (5.8) *are independent in $S_2(\mathbb{R})$ throughout*

X.

Since γ is uniformly elliptic on X, condition A is completely equivalent to equation (5.4), though it is expressed in terms of measured quantities, and as such can be checked directly during experiments.

The proof of Lemma 5.1.1 is based on the construction of Complex Geometrical Optics(CGO) solutions and will be given in Section 5.2.3.

Remark 5.1.2. *For the general n dimensional case, Lemma 5.1.1 does not necessarily hold globally. However, it holds locally with* 4 *well-chosen illuminations. The proof is based on the Runge approximation; see [10] for details.*

5.1.2 Uniqueness and stability results

We denote by $M_2(\mathbb{R})$ the space of 2×2 matrices with inner product $\langle A, B\rangle := \mathrm{tr}\,(A^T B)$. Assuming that there exist 4 illuminations $\{g_i\}_{1\le i\le 4}$ with their corresponding solutions $(u_i)_{1\le i\le 4}$ satisfying the conditions in Lemma 5.1.1. Then the isotropic part β can be reconstructed via a redundant elliptic system with a prior knowledge of the anisotropic part $\tilde{\gamma}$. In particular, the matrices M_1, M_2 constructed by (5.8) are independent and of codimensiton 1 in $S_2(\mathbb{R})$. We will see that $\tilde{\gamma}$ is orthogonal to M_1, M_2 which can be calculated from knowledge of $\{H_i\}_{1\le i\le 4}$. Together with the fact that $\det\tilde{\gamma} = 1$ and $\tilde{\gamma}$ is positive, $\tilde{\gamma}$ can be completely determined by $(H_i)_{1\le i\le 4}$. The algorithm is based on an appropriate generalization of the cross-product. The reconstruction formulas can be found in Section 5.2.1 and 5.2.2. This algorithm leads to a unique and stable reconstruction in the sense of the following theorem.

Theorem 5.1.3. *Suppose that Lemma 5.1.1.A holds over* X *for two couples* $\{u_i\}_{i=1}^2$ *and* $\{u'_i\}_{i=1}^2$*, solutions of the conductivity equation* (5.1) *with the tensors* $\gamma = \beta\tilde{\gamma}$ *and* $\gamma' = \beta'\tilde{\gamma}'$ *satisfying the uniform ellipticity condition* (2.2)*, where* $\tilde{\gamma}, \tilde{\gamma}' \in W^{1,\infty}(X)$ *are known. Then*

β can be uniquely reconstructed in X with the following stability estimate,

$$\|\log\beta - \log\beta'\|_{W^{p,\infty}(X)} \le \epsilon_0 + C\left(\sum_{i=1,2}\|H_i - H_i'\|_{W^{p,\infty}(X)} + \|\tilde{\gamma} - \tilde{\gamma}'\|_{W^{p,\infty}(X)}\right). \quad (5.9)$$

Here, $\epsilon_0 = |\log\beta(x_0) - \log\beta'(x_0)|$ is the error committed at some fixed $x_0 \in X$. If in addition Lemma 5.1.1.B holds for the two sets $\{u_i\}_{i=1}^4$ and $\{u_i'\}_{i=1}^4$ as above, then $\tilde{\gamma}$ can be reconstructed with the stability as follows,

$$\|\tilde{\gamma} - \tilde{\gamma}'\|_{W^{p,\infty}(X)} \le C\sum_{i=1}^{4}\|H_i - H_i'\|_{W^{p+1,\infty}(X)}. \quad (5.10)$$

Remark 5.1.4. *From Theorem 5.1.3, with a prior knowledge of the anisotropic part $\tilde{\gamma}$, the reconstruction of the scalar β has a better stability estimate than $\tilde{\gamma}$. This will be demonstrated by the numerical experiments in Section 5.3.1.*

5.2 Reconstruction approaches

The reconstruction approaches were presented in [10] for a general n dimensional case. To make this chapter self-contained, we briefly list the algorithm for the 2 dimensional case and prove the *global* reconstructibility condition in Lemma 5.1.1. We first present the reconstruction formula for β, assuming that the anisotropic part $\tilde{\gamma}$ is known from prior informations or reconstructed by current densities.

5.2.1 Reconstruction of β

Denoting the curl operator in $\mathbb{R}^2$ by $J\nabla\cdot$, where $J = \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2$. We rewrite (5.3) as $\frac{1}{\beta}\tilde{\gamma}^{-1}H_i = \nabla u_i$ for $i = 1, 2$ and apply the curl operator to both sides. Using the fact that

∇u_i is curl free, we get the following equation,

$$\nabla \log \beta \cdot (J\tilde{\gamma}^{-1} H_i) = -J\nabla \cdot (\tilde{\gamma}^{-1} H_i).$$

Considering both $j = 1, 2$, simple calculations lead to

$$\nabla \log \beta = -J\tilde{\gamma} H^{-T} \begin{pmatrix} J\nabla \cdot (\tilde{\gamma}^{-1} H_1) \\ J\nabla \cdot (\tilde{\gamma}^{-1} H_2) \end{pmatrix}. \tag{5.11}$$

Since both first order derivatives of $\log \beta$ can be reconstructed by (5.11), together with the boundary condition, the above equation leads to an over-determined elliptic system for β.

5.2.2 Reconstruction of $\tilde{\gamma}$

We now develop the reconstruction algorithm for $\tilde{\gamma}$. This reconstruction is algebraic and *local* in nature: the reconstruction of γ at $x_0 \in X$ requires the knowledge of current densities for x only in the vicinity of x_0. In addition to H_1, H_2, we pick 2 more measurements H_3, H_4 satisfying Lemma 5.1.1.B. We apply the curl operator $J\nabla\cdot$ to the linear combinations in (5.5). Again, using the fact that $\nabla u_i = \gamma^{-1} H_i$ is curl free, we obtain the following equations,

$$\sum_{i=1,2} Z_{k,i} \cdot (J\tilde{\gamma}^{-1} H_i) = 0 \quad \text{where} \quad k = 1, 2.$$

Using the fact that tr $(A) =$ tr $(S^{-1}AS)$ and γ is symmetric, the above equation amounts to

$$0 = \tilde{\gamma} : Z_k H^T J = \tilde{\gamma} : (Z_k H^T J)^{sym} = \tilde{\gamma} : M_k.$$

Since $\{M_1, M_2\}$ are of codimension 1 in $S_2(\mathbb{R})$, the above equation leads to the fact that $\tilde{\gamma}$ must be parallel to the following matrix constructed with M_1, M_2,

$$B = \begin{pmatrix} 2M_1^{22}M_2^{12} - 2M_1^{12}M_2^{22} & M_1^{11}M_2^{22} - M_1^{22}M_2^{11} \\ M_1^{11}M_2^{22} - M_1^{22}M_2^{11} & 2M_1^{12}M_2^{11} - 2M_1^{11}M_2^{12} \end{pmatrix}. \tag{5.12}$$

Here, M_k^{ij} denotes the ij element of the symmetric matrix M_k. Notice that B vanishes only if M_1 and M_2 are linearly dependent. Together with the fact that $\det \tilde{\gamma} = 1$ and $\tilde{\gamma}$ is positive, we obtain the following explicit reconstruction,

$$\tilde{\gamma} = \mathrm{sign}(B^{11})|B|^{-\frac{1}{2}}B. \tag{5.13}$$

Proof of Theorem 5.1.3: The proof is straightforward by noticing that one derivative is taken in the reconstruction procedure for $\tilde{\gamma}$. The stability for β is a direct result from the standard regularity estimate for elliptic operators. See [10] for details.

5.2.3 Proof of Lemma 5.1.1

Isotropic tensors $\gamma = \beta\mathbb{I}_2$. The proof is based on the construction of complex geometrical optics (CGO) solutions. As shown in [14], letting $\beta \in H^{5+\varepsilon}(X)$, one is able to construct a complex-valued solution of (5.1) of the form

$$u_{\boldsymbol{\rho}} = \frac{1}{\sqrt{\beta}} e^{\boldsymbol{\rho}\cdot x}(1 + \psi_{\boldsymbol{\rho}}), \tag{5.14}$$

where $\boldsymbol{\rho} \in \mathbb{C}^2$ is of form $\boldsymbol{\rho} = \rho(\mathbf{k} + i\mathbf{k}^\perp)$ with $\mathbf{k} \in \mathbb{S}^1$ and $\mathbf{k}\cdot\mathbf{k}^\perp = 0$. Thus $e^{\boldsymbol{\rho}\cdot x}$ is a harmonic complex plane wave with $\boldsymbol{\rho}\cdot\boldsymbol{\rho} = 0$. With the assumed regularity, we have the following estimate (see [14, Proposition 3,3]),

$$\lim_{\rho\to\infty} \|\psi_{\boldsymbol{\rho}}\|_{\mathcal{C}^2(\bar{X})} = 0.$$

Computing the gradient of $u_{\boldsymbol{\rho}}$ and rearranging terms, we obtain that

$$\nabla u_{\boldsymbol{\rho}} = e^{\boldsymbol{\rho}\cdot x}(\boldsymbol{\rho} + \boldsymbol{\varphi}_{\boldsymbol{\rho}}), \quad \text{with} \quad \boldsymbol{\varphi}_{\boldsymbol{\rho}} := \nabla \psi_{\boldsymbol{\rho}} + \psi_{\boldsymbol{\rho}}\boldsymbol{\rho} - (1+\psi_{\boldsymbol{\rho}})\nabla \log \sqrt{\beta},$$

where $\|\boldsymbol{\varphi}_{\boldsymbol{\rho}}\|_{\mathcal{C}^1(\bar{X})}$ is uniformly bounded independent of $\boldsymbol{\rho}$. Since β is real-valued, both the real and imaginary parts of $u_{\boldsymbol{\rho}}$ are real-valued solutions of (5.1) and we obtain the following expression

$$\begin{aligned}
\nabla u^{\Re}_{\boldsymbol{\rho}} &= \frac{\rho e^{\rho \mathbf{k}\cdot x}}{\sqrt{\beta}} \left((\mathbf{k} + \rho^{-1}\boldsymbol{\varphi}^{\Re}_{\boldsymbol{\rho}}) \cos(\rho \mathbf{k}^\perp \cdot x) - (\mathbf{k}^\perp + \rho^{-1}\boldsymbol{\varphi}^{\Im}_{\boldsymbol{\rho}}) \sin(\rho \mathbf{k}^\perp \cdot x) \right), \\
\nabla u^{\Im}_{\boldsymbol{\rho}} &= \frac{\rho e^{\rho \mathbf{k}\cdot x}}{\sqrt{\beta}} \left((\mathbf{k}^\perp + \rho^{-1}\boldsymbol{\varphi}^{\Im}_{\boldsymbol{\rho}}) \cos(\rho \mathbf{k}^\perp \cdot x) + (\mathbf{k} + \rho^{-1}\boldsymbol{\varphi}^{\Re}_{\boldsymbol{\rho}}) \sin(\rho \mathbf{k}^\perp \cdot x) \right).
\end{aligned}$$

Straightforward computations lead to

$$\det(\nabla u^{\Re}_{\boldsymbol{\rho}}, \nabla u^{\Im}_{\boldsymbol{\rho}}) = \frac{\rho^2 e^{2\rho \mathbf{k}\cdot x}}{\beta}(1 + f_{\boldsymbol{\rho}}), \quad \text{where} \quad \lim_{\rho\to\infty} \|f_{\boldsymbol{\rho}}\|_{\mathcal{C}^1(\bar{X})} = 0.$$

Now we identify $\mathbf{k} = \mathbf{e}_1$ and define $\mathbf{k}_1 = \mathbf{k}$, $\mathbf{k}_2 = -\mathbf{k}$. For $j = 1, 2$, define $\boldsymbol{\rho}_j := \rho(\mathbf{k}_j + i\mathbf{k}_j^\perp)$. Considering the solutions $(u^{\Re}_{\boldsymbol{\rho}_1}, u^{\Im}_{\boldsymbol{\rho}_1}, u^{\Re}_{\boldsymbol{\rho}_2}, u^{\Im}_{\boldsymbol{\rho}_2})$, the previous calculations show that

$$\inf_{x\in X} |\det(\nabla u^{\Re}_{\boldsymbol{\rho}_1}, \nabla u^{\Im}_{\boldsymbol{\rho}_1})| \geq c_0 > 0, \tag{5.15}$$

Together with the uniform ellipticity of γ, the above inequality implies condition A . Then using Cramer's rule in (5.6), simple algebra shows that

$$\mu_1 = \frac{\sin(2\rho\mathbf{k}^\perp \cdot x) + g_{\mu_1}}{e^{2\rho\mathbf{k}\cdot x}(1 + f_{\boldsymbol{\rho}_1})}, \quad \mu_2 = \frac{-\cos(2\rho\mathbf{k}^\perp \cdot x) + g_{\mu_2}}{e^{2\rho\mathbf{k}\cdot x}(1 + f_{\boldsymbol{\rho}_1})}$$

and similarly,

$$\lambda_1 = \frac{-\cos(2\rho\mathbf{k}^\perp \cdot x) + g_{\lambda_1}}{e^{2\rho\mathbf{k}\cdot x}(1 + f_{\boldsymbol{\rho}_1})}, \quad \lambda_2 = \frac{-\sin(2\rho\mathbf{k}^\perp \cdot x) + g_{\lambda_2}}{e^{2\rho\mathbf{k}\cdot x}(1 + f_{\boldsymbol{\rho}_1})},$$

where $\|g_{\mu_i}\|_{\mathcal{C}^1(\bar{X})}, \|g_{\lambda_i}\|_{\mathcal{C}^1(\bar{X})}$ are bounded for $i = 1, 2$. Then by the definition of Z_k in (5.7), we obtain the following expression,

$$Z_1 = 2\rho e^{-2\rho\mathbf{k}\cdot x}[(-\mathbf{k}\sin(2\rho\mathbf{k}^\perp \cdot x) + \mathbf{k}^\perp\cos(2\rho\mathbf{k}^\perp \cdot x), \mathbf{k}\cos(2\rho\mathbf{k}^\perp \cdot x) + \mathbf{k}^\perp\sin(2\rho\mathbf{k}^\perp \cdot x)) + o(\rho^{-1})]$$

$$Z_2 = 2\rho e^{-2\rho\mathbf{k}\cdot x}[(\mathbf{k}\cos(2\rho\mathbf{k}^\perp \cdot x) + \mathbf{k}^\perp\sin(2\rho\mathbf{k}^\perp \cdot x), \mathbf{k}\sin(2\rho\mathbf{k}^\perp \cdot x) - \mathbf{k}^\perp\cos(2\rho\mathbf{k}^\perp \cdot x)) + o(\rho^{-1})].$$

Together with $\mathbf{k} = \mathbf{e}_1$ and $H = \beta(\nabla u^{\Re}_{\boldsymbol{\rho}_1}, \nabla u^{\Im}_{\boldsymbol{\rho}_1})$, we obtain that,

$$(Z_1 H^T J)^{sym} = 2\rho^2\sqrt{\beta}e^{-\rho\mathbf{k}\cdot x}\left[\begin{pmatrix} \cos(\rho\mathbf{k}^\perp \cdot x) & \sin(\rho\mathbf{k}^\perp \cdot x) \\ \sin(\rho\mathbf{k}^\perp \cdot x) & -\cos(\rho\mathbf{k}^\perp \cdot x) \end{pmatrix} + o(\rho^{-1})\right]$$

$$(Z_2 H^T J)^{sym} = 2\rho^2\sqrt{\beta}e^{-\rho\mathbf{k}\cdot x}\left[\begin{pmatrix} \sin(\rho\mathbf{k}^\perp \cdot x) & -\cos(\rho\mathbf{k}^\perp \cdot x) \\ -\cos(\rho\mathbf{k}^\perp \cdot x) & -\sin(\rho\mathbf{k}^\perp \cdot x) \end{pmatrix} + o(\rho^{-1})\right].$$

Since

$$\frac{M_1 : M_2}{\|M_1\|\|M_2\|} = o(\rho^{-1}),$$

M_1, M_2 are almost orthogonal as ρ is large enough, which implies the independence. This proves condition B.

General case: Following the idea in [15, Theorem 4.4], we extend γ to a smooth tensor on $\mathbb{R}^2 \simeq \mathbb{C}$, which remains uniformly positive definite and equal to $\mathbb{I}_2$ outside of a compact domain. For $\varphi : \mathbb{R}^2 \ni x \mapsto \varphi(x) = y \in \mathbb{R}^2$ a diffeomorphism, we denote the push-forward of γ by the φ as follows,

$$\varphi_*\gamma(y) = \frac{D\varphi(x)\gamma(x)D\varphi^t(x)}{|\det(D\varphi)|}|_{x=\varphi^{-1}(y)}. \tag{5.16}$$

The theory of quasi-conformal mappings [5] implies that there exists a unique such diffeomorphism φ satisfying the *Beltrami* system,

$$\varphi_*\gamma(y) = |\gamma|^{\frac{1}{2}} \circ \varphi^{-1}(y), \quad \varphi(z) = z + \mathcal{O}(z^{-1}) \quad \text{as} \quad z \to \infty,$$

which means that the conductivity γ is conformal to the Euclidean conductivity $\mathbb{I}_2$. As in the isotropic case, we can construct CGOs of the form,

$$v_{\boldsymbol{\rho}} = \frac{1}{\sqrt{\varphi_*\gamma(y)}} e^{\boldsymbol{\rho}\cdot y}(1 + \psi_{\boldsymbol{\rho}}(y)), \tag{5.17}$$

where $\lim_{\rho\to\infty} \|\psi_{\boldsymbol{\rho}}\|_{\mathcal{C}^2(\varphi(X))} = 0$. Using the change of variables, we construct $u = v \circ \varphi$,

$$u_{\boldsymbol{\rho}} = \frac{1}{\sqrt{\varphi_*\gamma \circ \varphi(x)}} e^{\boldsymbol{\rho}\cdot\varphi(x)}(1 + \phi_{\boldsymbol{\rho}}(x)), \tag{5.18}$$

where $\lim_{\rho\to\infty} \|\phi_{\boldsymbol{\rho}}\|_{\mathcal{C}^2(X)} = 0$. By the method in the isotropic case, we construct the solutions $(v_1, v_2, v_3, v_4) = (v^{\Re}_{\boldsymbol{\rho}_1}, v^{\Im}_{\boldsymbol{\rho}_1}, v^{\Re}_{\boldsymbol{\rho}_2}, v^{\Im}_{\boldsymbol{\rho}_2})$, with $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2$ defined as before. Then for $1 \le i \le 4$, the functions $u_i = v_i \circ \varphi$ satisfy the conductivity equation (5.1). Using the chain rule $\nabla u_i = \nabla(v_i \circ \varphi) = D\varphi^t \nabla v_i \circ \varphi$, condition A is satisfied with ρ sufficiently large since $\nabla v_1, \nabla v_2$ are linearly independent as indicated in the isotropic case. Denote the skew-symmetric matrice $J' = D\varphi^t J D\varphi$ and $Z'_k(y) = Z_k(x)|_{x=\varphi^{-1}(y)}$. Again by the chain rule, the following relation holds for every $x \in X$,

$$\begin{aligned}(Z_k H^T J')^{sym} &= (D\varphi^t Z_k'(\nabla v_1, \nabla v_2)^t D\varphi\gamma D\varphi^t J D\varphi)^{sym} \\ &= \det(D\varphi) D\varphi^t((Z_k'\beta(\nabla v_1, \nabla v_2)^t J)^{sym} \circ \varphi) D\varphi,\end{aligned}$$

where $J' = D\varphi^t J D\varphi$ is skew-symmetric. As in the proof in the isotropic case, we see that $(Z_k'\beta(\nabla v_1, \nabla v_2)^t J)^{sym}$ are linearly independent over $\varphi(X)$ for $k = 1, 2$. Thus, M_1, M_2 are linearly independent thoughout X, which proves condition B.

5.3 Numerical experiments

To demonstrate the computational feasibility of the reconstruction algorithm, we performed some numerical experiments to validate the reconstruction algorithms from the previous section, assess their robustness to noisy measurements and determine how reconstructions are affected by boundary conditions limited to a part of the domain.

5.3.1 Preliminary facts on the numerical implementation

Recall that we decompose γ into the following form with three unknown coefficients $\{\xi, \zeta, \beta\}$,

$$\gamma = \beta\tilde{\gamma} = \beta \begin{bmatrix} \xi & \zeta \\ \zeta & \frac{1+\zeta^2}{\xi} \end{bmatrix}, \quad \xi > 0, \tag{5.19}$$

where $\beta = |\gamma|^{\frac{1}{2}}$ and $|\tilde{\gamma}| = 1$. The full reconstruction is a two-step procedure, starting with the reconstruction of the anisotropy $\tilde{\gamma}(\xi, \zeta)$ via formula (5.13). This requires implementing the formula

$$\tilde{\gamma} = \frac{\sum_{i=1}^m \operatorname{sign}(B_i^{11}) B_i}{\sum_{i=1}^m |\det B_i|^{\frac{1}{2}}}, \tag{5.20}$$

where each B_i is constructed via (5.12) by choosing two additional current densities. Once $\tilde{\gamma}$ is reconstructed, β is in turn reconstructed via the redundant elliptic system (5.11).

Regularized inversion. Since we have explicit reconstruction formulas for γ, we use a total variation method as the denoising procedure by minimizing the following functional,

$$f = \arg\min_g \frac{1}{2}\|g - f_{\mathrm{rc}}\|_2^2 + \rho\|Mg\|_{\mathrm{TV}}, \tag{5.21}$$

where f_{rc} denotes the explicit reconstructions of the coefficients of γ and M is the discretized version of the gradient operator. We choose the l^1-norm as the regularization TV norm for discontinuous, piecewise constant, coefficients. In this case, the minimization problem can be solved using the split Bregman method presented in [25]. To recover smooth coefficients, we minimize the following least square problem,

$$f = \arg\min_g \frac{1}{2}\|g - f_{\mathrm{rc}}\|_2^2 + \rho\|Mg\|_2^2,$$

where the Tikhonov regularization functional admits an explicit solution $f = (\mathbb{I}+\rho M^*M)^{-1}M^*f_{\mathrm{rc}}$.

5.3.2 Experiment with control over the full boundary

In the numerical experiments below, we take the domain of interest to be the square $X = [-1,1]^2$ and use the notation $\mathbf{x} = (x,y)$. We use a $\mathsf{N}+1 \times \mathsf{N}+1$ square grid with $\mathsf{N} = 80$, the tensor product of the equi-spaced subdivision $\mathsf{x} = -1 : \mathsf{h} : 1$ with $\mathsf{h} = 2/\mathsf{N}$. The internal current densities $H(x)$ used are synthetic data that are constructed by solving the conductivity equation (5.1) using a finite difference method implemented with `MatLab`. Although the data constructed this way may contain some noise, we refer to these data as the "noise-free" or "clean" data.

We also perform the reconstructions with noisy data by perturbing the internal functionals $H(\mathbf{x})$ so that,

$$\widetilde{H}(\mathbf{x}) = H(\mathbf{x}). * (1 + \alpha * \text{random}(\mathbf{x})),$$

where random($\mathbf{x}$) is a $\mathsf{N}+1 \times \mathsf{N}+1$ random matrix taking uniformly distributed values in $[-1, 1]$ and α is the noise level. We then run a de-noising process on the random matrix, which we chose as a low-pass filter constructed by a 5-point sliding averaging process.

We use the relative L^2 error between reconstructed and true coefficients to measure the quality of the reconstructions. $\mathcal{E}_\xi^C, \mathcal{E}_\xi^N, \mathcal{E}_\zeta^C, \mathcal{E}_\zeta^N, \mathcal{E}_\beta^C, \mathcal{E}_\beta^N$ denote the relative L^2 error in the reconstructions from clean and noisy data for ξ, ζ and β, respectively.

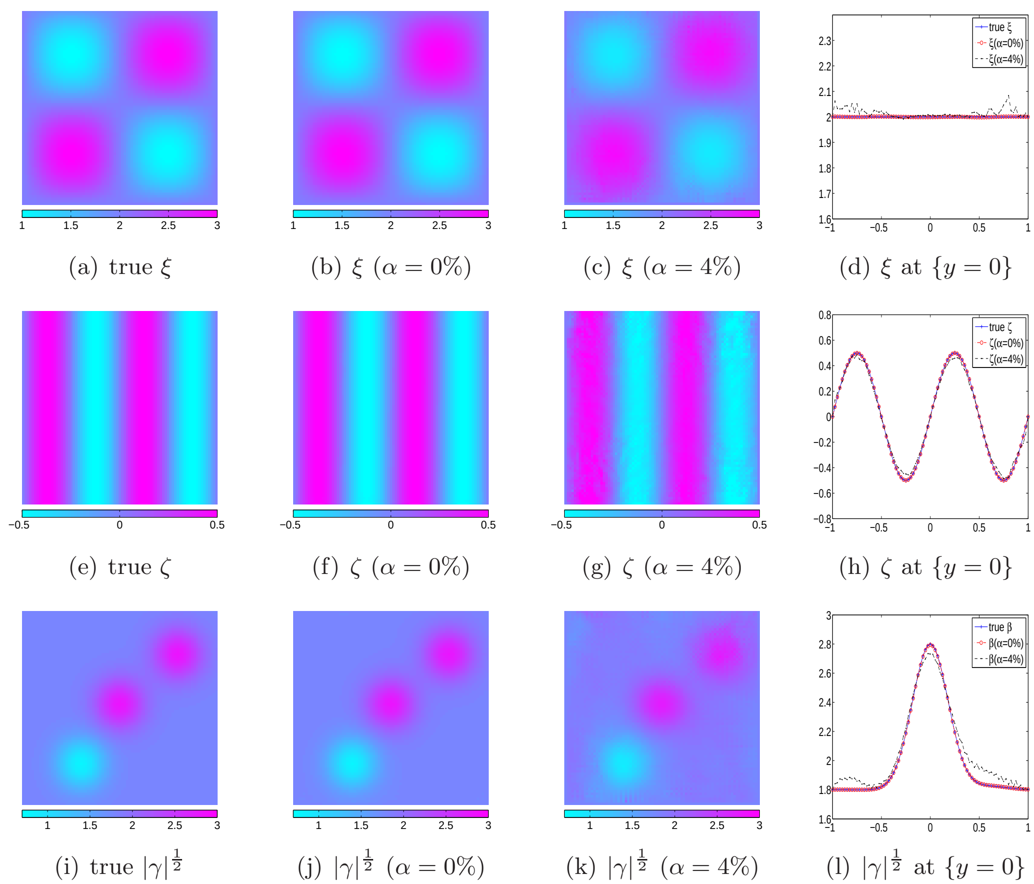
Experiment 1. In the first experiment, we intend to reconstruct the smooth coefficients ξ, ζ and β defined in (5.19) and given by,

$$\begin{cases} \xi = 2 + \sin(\pi x)\sin(\pi y) \\ \zeta = 0.5\sin(2\pi x) \\ \beta = 1.8 + e^{-15(x^2+y^2)} + e^{-15((x-0.6)^2+(y-0.5)^2)} - e^{-15((x+0.4)^2+(y+0.6)^2)}. \end{cases} \tag{5.22}$$

We consider five different illuminations $(g_1, g_2, g_3, g_4, g_5)$ that are defined as follows,

$$(g_1, g_2, g_3, g_4, g_5)(\mathbf{x}) = (x + y, y + 0.1y^2, 3x^2 + 2y^2, x^2 - 0.5y^2, xy) \quad \mathbf{x} \in \partial[-1, 1]^2 \tag{5.23}$$

where g_1, g_2 are used generating the solutions satisfying Lemma 5.1.1.A. We performed two sets of reconstructions using clean and noisy synthetic data respectively. The l_2-regularization procedure is used in this simulation. For the noisy data, the noise level $\alpha = 4\%$. The results of the numerical experiment are shown in Figure 5.1. The relative L^2 errors in the reconstructions are $\mathcal{E}_\xi^C = 0.1\%$, $\mathcal{E}_\xi^N = 4.0\%$, $\mathcal{E}_\zeta^C = 0.6\%$, $\mathcal{E}_\zeta^N = 11.8\%$, $\mathcal{E}_\beta^C = 0.2\%$ and $\mathcal{E}_\beta^N = 3.7\%$.



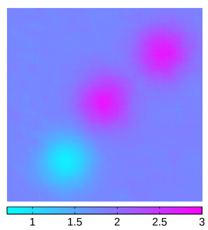
(a) true ξ (b) ξ $(\alpha = 0\%)$ (c) ξ $(\alpha = 4\%)$ (d) ξ at $\{y = 0\}$

(e) true ζ (f) ζ $(\alpha = 0\%)$ (g) ζ $(\alpha = 4\%)$ (h) ζ at $\{y = 0\}$

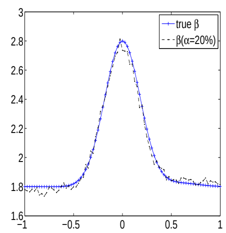
(i) true $|\gamma|^{\frac{1}{2}}$ (j) $|\gamma|^{\frac{1}{2}}$ $(\alpha = 0\%)$ (k) $|\gamma|^{\frac{1}{2}}$ $(\alpha = 4\%)$ (l) $|\gamma|^{\frac{1}{2}}$ at $\{y = 0\}$

Figure 5.1: Experiment 1. (a)&(e)&(i): true values of (ξ, ζ, β). (b)&(f)&(j): reconstructions with noiseless data. (c)&(g)&(k): reconstructions with noisy data($\alpha = 4\%$). (d)&(h)&(l): cross sections along $\{y = 0\}$.

Reconstruction of β with (known) true anisotropic part $\tilde{\gamma}$. We now use the true ξ and ζ to reconstruct β with noisy data ($\alpha = 20\%$). Figure 5.2 displays the numerical results. The reconstruction is quite robust to noise when the anisotropy is known: the L^2 relative error is 1.6%. Comparing Fig.5.1(k)&(l) with Fig.5.2(a)&(b), it is clear that the reconstruction of the isotropy β is more stable than that of the anisotropy $\tilde{\gamma}$. This is consistent with the better stability estimates obtained in Theorem 5.1.3.



(a) $|\gamma|^{\frac{1}{2}}$ ($\alpha = 20\%$)



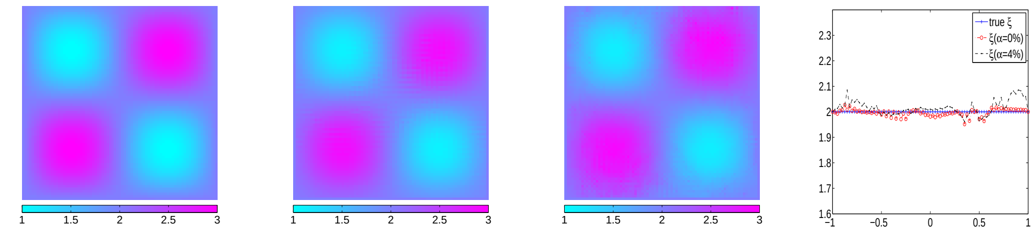
(b) $|\gamma|^{\frac{1}{2}}$ at $\{y = 0\}$

Figure 5.2: Reconstruction of β with true anisotropy. (a): reconstructed β using true anisotropy and noisy data($\alpha = 20\%$). (b): cross-section along $y = 0$.

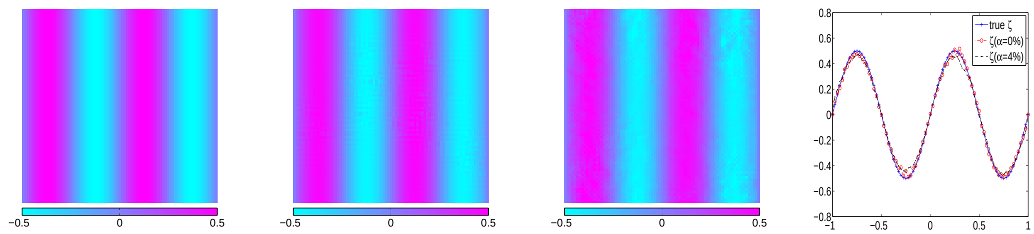
Experiment 2. In this experiment, we intend to reconstruct the isotropy given by

$$\beta(\mathbf{x}) = \begin{cases} 1 + (\text{sign}(\text{random}) + 1), & \mathbf{x} \in X_{ij}, \;\; 1 \le i, j \le 10 \\ 1 + (\text{sign}(\text{random}) + 1), & \mathbf{x} \in X'_{ij} \cup X''_{ij}, \;\; 1 \le i \le 3, 1 \le j \le 5 \\ 1, & \text{otherwise} \end{cases}, \tag{5.24}$$

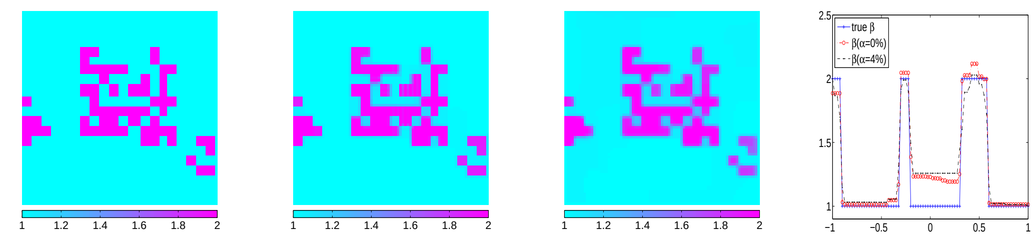
where random is a random number in $[-1, 1]$, $X_{ij} = [0.1(i-1) - 0.4, 0.1i - 0.4] \times [0.1(j-1) - 0.4, 0.1j - 0.4]$, $X'_{ij} = [0.1(i-1) - 1, 0.1i - 1] \times [0.1(j-1) - 0.4, 0.1j - 0.4]$ and $X''_{ij} = [0.1(i-1) + 0.7, 0.1i + 0.7] \times [0.1(j-1) - 0.8, 0.1j - 0.8]$. The anisotropy characterized by (ξ, ζ) is the same as Experiment 1. The measurements are constructed with the 5 illuminations given by (5.23). Reconstructions with noise-free and noisy data are performed with a l_2 regularization for the anisotropy and l_1 regularization using the split Bregman



(a) true ξ (b) ξ $(\alpha = 0\%)$ (c) ξ $(\alpha = 4\%)$ (d) ξ at $\{y = 0\}$



(e) true ζ (f) ζ $(\alpha = 0\%)$ (g) ζ $(\alpha = 4\%)$ (h) ζ at $\{y = 0\}$

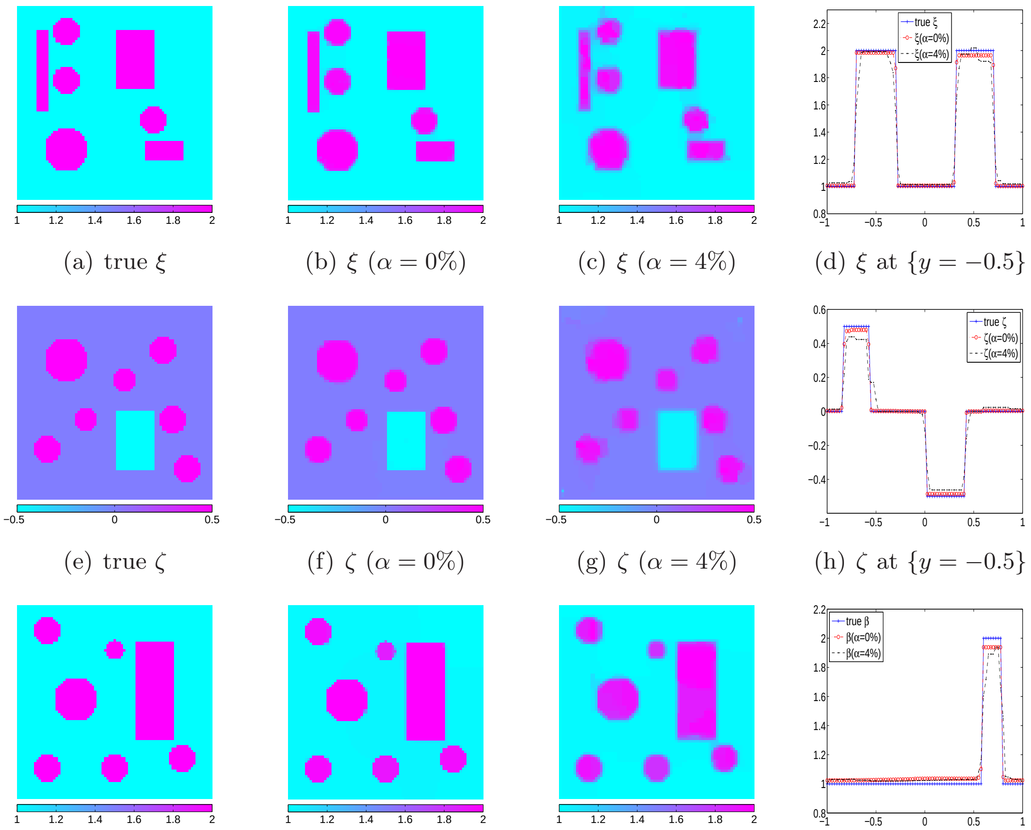


(i) true $|\gamma|^{\frac{1}{2}}$ (j) $|\gamma|^{\frac{1}{2}}$ $(\alpha = 0\%)$ (k) $|\gamma|^{\frac{1}{2}}$ $(\alpha = 4\%)$ (l) $|\gamma|^{\frac{1}{2}}$ at $\{y = 0\}$

Figure 5.3: Experiment 2. (a)&(e)&(i): true values of (ξ, ζ, β). (b)&(f)&(j): reconstructions with noiseless data. (c)&(g)&(k): reconstructions with noisy data($\alpha = 4\%$). (d)&(h)&(l): cross sections along $\{y = 0\}$.

iteration method for the isotropic component. The noise level $\alpha = 4\%$. The numerical results of the numerical experiment are shown in Figure 5.3. The relative L^2 errors in the reconstructions are $\mathcal{E}_\xi^C = 2.8\%$, $\mathcal{E}_\xi^N = 3.7\%$, $\mathcal{E}_\zeta^C = 6.9\%$, $\mathcal{E}_\zeta^N = 11.8\%$, $\mathcal{E}_\beta^C = 5.1\%$ and $\mathcal{E}_\beta^N = 8.2\%$, respectively.

Experiment 3. In this experiment, we attempt to reconstruct coefficients with discontinuities. To simplify the implementation, we only consider piecewise constant coefficients. Here we use the same illuminations as in Experiment 1. Reconstructions with both noise-



(a) true ξ (b) ξ $(\alpha = 0\%)$ (c) ξ $(\alpha = 4\%)$ (d) ξ at $\{y = -0.5\}$

(e) true ζ (f) ζ $(\alpha = 0\%)$ (g) ζ $(\alpha = 4\%)$ (h) ζ at $\{y = -0.5\}$

(i) true $|\gamma|^{\frac{1}{2}}$ (j) $|\gamma|^{\frac{1}{2}}$ $(\alpha = 0\%)$ (k) $|\gamma|^{\frac{1}{2}}$ $(\alpha = 4\%)$ (l) $|\gamma|^{\frac{1}{2}}$ at $\{y = -0.5\}$

Figure 5.4: Experiment 3. (a)&(e)&(i): true values of (ξ, ζ, β). (b)&(f)&(j): reconstructions with noiseless data. (c)&(g)&(k): reconstructions with noisy data($\alpha = 4\%$). (d)&(h)&(l): cross sections along $\{y = -0.5\}$.

less and noisy data are performed with l_1 regularization using the split Bregman iteration method for both the anisotropic and isotropic components. The noise level $\alpha = 4\%$. The results of the numerical experiment are shown in Figure 5.4. From the figures, we observe that the singularities of the coefficients create minor artifacts on the reconstructions and the error in the reconstruction is larger at the discontinuities than in the rest of the domain. The relative L^2 errors in the reconstructions are $\mathcal{E}^C_\xi = 3.9\%$, $\mathcal{E}^N_\xi = 9.6\%$, $\mathcal{E}^C_\zeta = 13.4\%$, $\mathcal{E}^N_\zeta = 31.9\%$, $\mathcal{E}^C_\beta = 3.7\%$ and $\mathcal{E}^N_\beta = 8.2\%$.

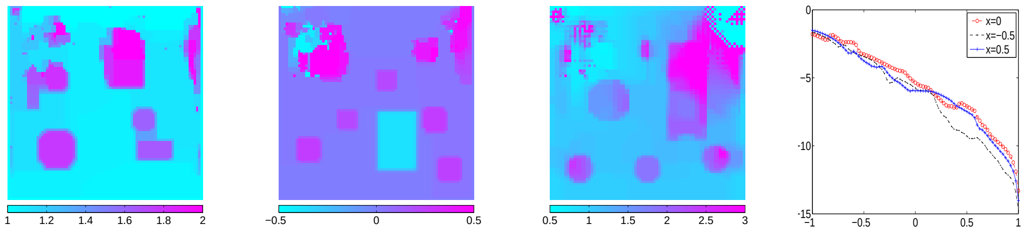
5.3.3 Experiments with control over part of the boundary

The previous experiments show that the reconstruction of both smooth and discontinuous coefficients is very accurate and robust to noise when one can fully prescribe boundary conditions ensuring conditions $A\&B$ of Lemma 5.1.1. In practice, one does not always have access to the whole boundary, and instead may have to prescribe boundary conditions on only a small part of the domain. In the next series of experiments, we assume to only have control over the bottom boundary of the square domain X, call it $\partial X_B = [-1,1] \times \{-1\}$. Over the rest of the boundary, we successively impose homogeneous Dirichlet boundary conditions (Experiment 4), then homogeneous Neumann boundary conditions (Experiment 5). In two spatial dimensions, either case forces all conductivity solutions to have their gradients to be pairwise collinear (normal to the boundary for Dirichlet conditions, tangential to the boundary for Neumann conditions). This violates both conditions of Lemma 5.1.1, and we expect reconstructions to do poorly near the uncontrolled part of the boundary. However, conditions $A\&B$ in Lemma 5.1.1 could be verified by calculating the determinant of measurements. Thus, the quality of reconstructions could be predicted by the measurements: if the determinant of two measurements are somewhere too small, i.e., they are not linearly independent, one can pick 2 other measurements and apply the reconstruction formulas there again.

Note that in higher spatial dimensions, the practically more relevant homogeneous Neumann conditions should lead to better reconstructions as these conditions impose less constraints on gradients than homogeneous Dirichlet conditions.

Experiment 4. We now repeat Experiment 3 using illuminations that are only non-zero on the bottom boundary of the domain.

Reconstructions of the anisotropy $\tilde{\gamma}$ in $[-1,1]^2$. We first perform the reconstructions of ξ



(a) ξ $(\alpha = 0\%)$ (b) ζ $(\alpha = 0\%)$ (c) ζ $(\alpha = 0\%)$ (d) $\log|\det(\nabla u_i, \nabla u_j)|$

Figure 5.5: Simulations on X. (a)&(b)&(c): reconstructions with noiseless data. (d): cross section of $\max_{1\le i<j\le 5} \log|\det(\nabla u_i, \nabla u_j)|$ along $\{x = 0\}$, $\{x = -0.5\}$ and $\{x = 0.5\}$.

and ζ. We use five illuminations given by Gaussian functions as follows,

$$g_i(\mathbf{x}) = \begin{cases} (2\pi \cdot 0.2^2)^{-\frac{1}{2}} \exp\{-\frac{1}{2\cdot 0.2^2}(x + x_i)^2\}, & \mathbf{x} \in \partial X_B \\ 0, & \mathbf{x} \in \partial X \setminus \partial X_B \end{cases} \qquad 1 \le i \le 5, \tag{5.25}$$

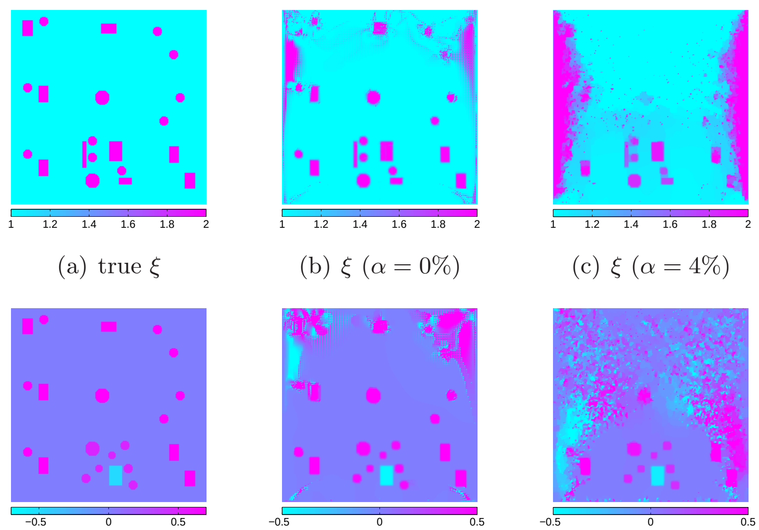
where $\{\mathbf{x}_i\}_{1\le i\le 5} = \{-0.8, -0.4, 0, 0.4, 0.8\}$. Reconstructions with noise-free data are shown in Figure 5.5. From this simulation, we can see that even with noise-free data, the reconstruction degrades as one gets farther away from the controlled boundary ∂X_B, while it remains accurate near ∂X_B.

Reconstructions of γ in an extended domain. From the numerical simulation in Figure 5.5, it is clear that the reconstruction procedure does not perform well for $\mathbf{x}$ far from ∂X_B. From Fig.5.5(d), we can see that $\det(\nabla u_i, \nabla u_j)$ decays very rapidly, which means that Lemma 5.1.1.A is not fulfilled.

A way to scan a deeper part of the domain with conductivity solutions of linearly independent gradients is obtained by spreading out the various boundary conditions along the x-axis. To this end, we now extend the domain X to $X' = [-3, 3] \times [-1.2, 4.8]$ and use a $\mathsf{N}' + 1 \times \mathsf{N}' + 1$ square grid with $\mathsf{N}' = 240$. We use the following five Gaussian functions as illuminations,

$$g_i(\mathbf{x}) = \begin{cases} (2\pi \cdot 0.2^2)^{-\frac{1}{2}} \exp\{-\frac{1}{2\cdot 0.2^2}(x+x_i)^2\}, & \mathbf{x} \in \partial X'_B \\ 0, & \mathbf{x} \in \partial X' \setminus \partial X'_B \end{cases} \quad 1 \le i \le 5, \tag{5.26}$$

where $\{\mathbf{x}_i\}_{1 \le i \le 5} = \{-2.8, -1.5, 0, 1.5, 2.8\}$. The reconstruction of the anisotropy $\tilde{\gamma}$ with noise free data is shown in Figure 5.6. In this setting, we see that the domain X is now fully covered by conductivity solutions whose gradients fulfill condition A from Lemma 5.1.1, and the reconstruction performs well everywhere on X. On the other hand, as expected, the reconstruction does not perform well outside X.



(a) true ξ (b) ξ ($\alpha = 0\%$) (c) ξ ($\alpha = 4\%$)

(d) true ζ (e) ζ ($\alpha = 0\%$) (f) ζ ($\alpha = 4\%$)

Figure 5.6: Simulations on extended domain X'. (a)&(d): true anisotropy (ξ, ζ). (b)&(e): reconstructions with noiseless data. (c)&(f): reconstructions with noisy data($\alpha = 4\%$).

We then use the reconstructions restricted on X to present the desired anisotropy. In the next step, β can be recovered on X by using the reconstructed $\tilde{\gamma}$ in the first step. Figure 5.7 displays the numerical results with noiseless data and noisy data($\alpha = 1\%, 4\%$). A l_1 regularization using the split Bregman iteration method is used for both the anisotropic and isotropic components in this simulation. The relative L^2 errors in the reconstructions are $\mathcal{E}_\xi^C = 9.4\%$, $\mathcal{E}_\zeta^C = 27.6\%$, $\mathcal{E}_\beta^C = 7.2\%$; $\mathcal{E}_\xi^N = 9.6\%$, $\mathcal{E}_\zeta^N = 28.1\%$, $\mathcal{E}_\beta^N = 7.6\%$ when $\alpha = 1\%$; $\mathcal{E}_\xi^N = 15.8\%$, $\mathcal{E}_\zeta^N = 38.3\%$ $\mathcal{E}_\beta^N = 13.7\%$ when $\alpha = 4\%$.

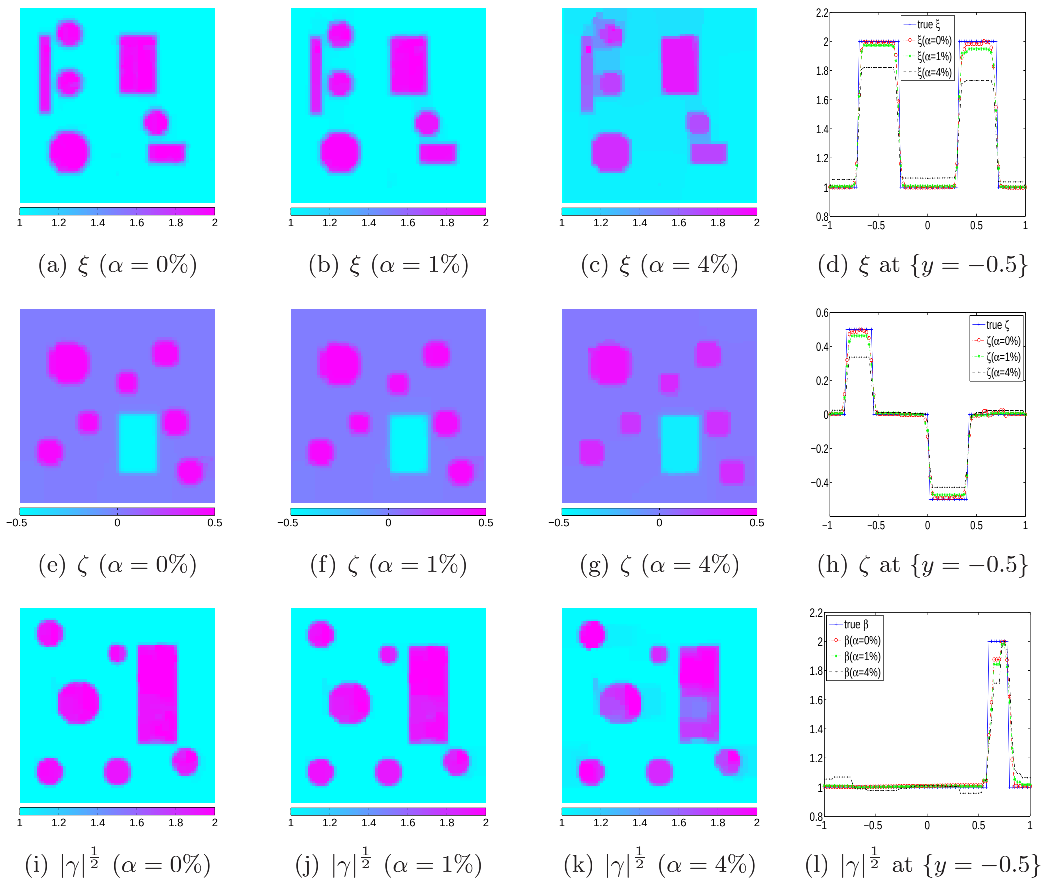


Figure 5.7: Experiment 4. (a)&(e)&(i): reconstructions with noiseless data. (b)&(f)&(j): reconstructions with noisy data ($\alpha = 1\%$). (c)&(g)&(k): reconstructions with noisy data ($\alpha = 4\%$). (d)&(h)&(l): cross sections along $y = -0.5$.

Experiment 5. In this experiment, we repeat Experiment 4 on the extended domain X', replacing homogeneous Dirichlet boundary conditions on the left, top and right edges, by homogeneous Neumann conditions on the three other edges. The same (Dirichlet) boundary conditions are used on the bottom edge of the domain,

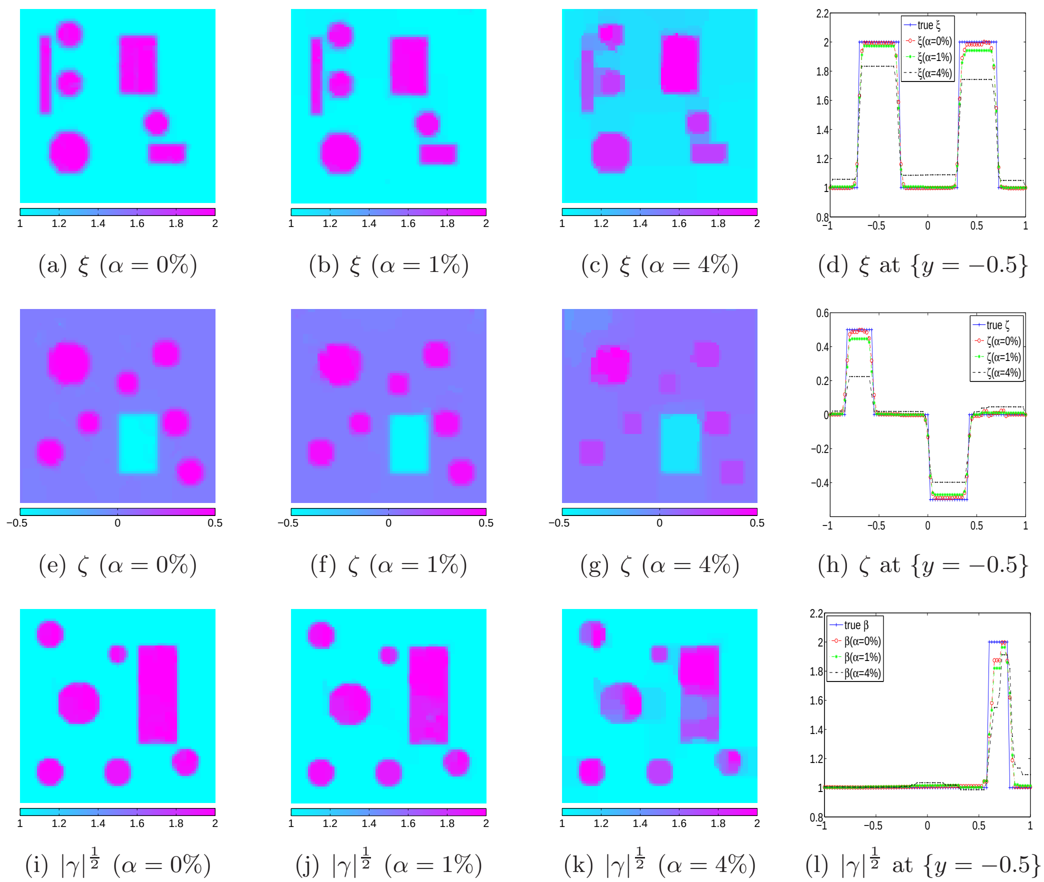
$$\begin{cases} u(\mathbf{x}) = (2\pi \cdot 0.2^2)^{-\frac{1}{2}} \exp\{-\frac{1}{2\cdot 0.2^2}(x+x_i)^2\}, & \mathbf{x} \in \partial X'_B \\ \frac{\partial u}{\partial n}(\mathbf{x}) = 0, & \mathbf{x} \in \partial X' \setminus \partial X'_B \end{cases} \quad 1 \le i \le 5, \qquad (5.27)$$

where $\{\mathbf{x}_i\}_{1\le i\le 5} = \{-2.8, -1.5, 0, 1.5, 2.8\}$. As in the last experiment, we first apply the reconstruction algorithm of $\tilde{\gamma}$ on X' and present its restriction on X. Then β can be recovered on X by using the reconstructed $\tilde{\gamma}$. Figure 5.8 displays the numerical results with noiseless data and noisy data ($\alpha = 1\%, 4\%$). An l_1 regularization procedure is again used in this simulation. The relative L^2 errors in the reconstructions are $\mathcal{E}^C_\xi = 9.4\%$, $\mathcal{E}^C_\zeta = 26.9\%$, $\mathcal{E}^C_\beta = 6.8\%$; $\mathcal{E}^N_\xi = 9.5\%$, $\mathcal{E}^N_\zeta = 28.7\%$, $\mathcal{E}^N_\beta = 7.7\%$ when $\alpha = 1\%$; $\mathcal{E}^N_\xi = 14.3\%$, $\mathcal{E}^N_\zeta = 52.1\%$, $\mathcal{E}^N_\beta = 13.5\%$ when $\alpha = 4\%$.

5.4 Conclusion

This work presents an explicit reconstruction procedure for an anisotropic conductivity tensor $\gamma = (\gamma_{ij})_{1\le i,j\le 2}$ from knowledge of current densities of the form $H = \gamma\nabla u$.

As explained in Theorem 5.1.3, these reconstruction algorithms, displaying local reconstruction formulas with Lipschitz stability (with the loss of one derivative from the measurements to the reconstructed quantities) for the anisotropic part of γ and Lipschitz stability (with no loss of derivatives) for $\det\gamma$, rely heavily on the ability to construct families of solutions of the conductivity equation with linearly independent gradients (i.e. conditions A and B in Lemma 5.1.1). As the experimenter pilots these solutions from the boundary, it is then necessary to find appropriate boundary conditions ensuring the linear independence criterion. These linear independence conditions could be checked by the measurements. If they are violated somewhere, one can pick more measurements and reconstruct over again. This method was used in Experiments 4 and 5.



(a) ξ ($\alpha = 0\%$) (b) ξ ($\alpha = 1\%$) (c) ξ ($\alpha = 4\%$) (d) ξ at $\{y = -0.5\}$

(e) ζ ($\alpha = 0\%$) (f) ζ ($\alpha = 1\%$) (g) ζ ($\alpha = 4\%$) (h) ζ at $\{y = -0.5\}$

(i) $|\gamma|^{\frac{1}{2}}$ ($\alpha = 0\%$) (j) $|\gamma|^{\frac{1}{2}}$ ($\alpha = 1\%$) (k) $|\gamma|^{\frac{1}{2}}$ ($\alpha = 4\%$) (l) $|\gamma|^{\frac{1}{2}}$ at $\{y = -0.5\}$

Figure 5.8: Experiment 5. (a)&(e)&(i): reconstructions with noiseless data. (b)&(f)&(j): reconstructions with noisy data ($\alpha = 1\%$). (c)&(g)&(k): reconstructions with noisy data ($\alpha = 4\%$). (d)&(h)&(l): cross sections along $y = -0.5$.

We first prove in Lemma 5.1.1 that, if one can control the entire boundary, then boundary conditions close to traces of Complex Geometrical Optics solutions will generate solutions satisfying conditions A and B throughout the domain. In fact, these conditions can be verified numerically for quite a large class of boundary conditions, such as for instance traces of well-chosen polynomials, and Experiments 1-3 in the numerics section illustrate the success of the method on full reconstruction of both smooth and discontinuous coefficients, as well as its robustness to noise.

On the other hand, when one has control over only part of the boundary, there will inherently be a breakdown in the reconstruction near the part of the boundary that is not controlled, as homogeneous boundary conditions there will automatically violate the linear independence criterion. On the controlled part of the boundary, using solutions generated with peaked Gaussian profiles at various positions yields satisfactory reconstructions up to a certain depth. As seen numerically on Experiments 4 and 5, the region where reconstructions are stable can be improved by increasing the spacing between the Gaussian profiles.

Chapter 6

Imaging of tensors for Maxwell's equations

In this chapter we are interested in the hybrid inverse problem of reconstructing (σ, ε) in the Maxwell's system in $\mathbb{R}^2$ from the internal magnetic fields H. The reconstructibility hypothesis is proved in Section 6.2. The numerical implementations of the algorithm with synthetic data are shown in Section 6.3.

6.1 Modeling of the problem

Let X be a simply connected, bounded domain of $\mathbb{R}^2$ with smooth boundary. The smooth anisotropic electric permittivity, conductivity, and the constant isotropic magnetic permeability are respectively described by $\varepsilon(x)$, $\sigma(x)$ and μ_0, where $\varepsilon(x)$, $\sigma(x)$ are tensors and μ_0 is a constant scalar, known, coefficient. We denote $\gamma = \sigma + \imath\omega\varepsilon$, where $\omega > 0$ is the frequency of the electromagnetic wave. We assume that $\varepsilon(x)$ and $\sigma(x)$ are uniformly bounded from below and above, i.e., there exist constants $\kappa_\varepsilon, \kappa_\sigma > 1$ such that for all $\xi \in \mathbb{R}^2$,

$$\begin{aligned} \kappa_\varepsilon^{-1}\|\xi\|^2 \le \xi \cdot \varepsilon\xi \le \kappa_\varepsilon\|\xi\|^2 \quad \text{in } X \\ \kappa_\sigma^{-1}\|\xi\|^2 \le \xi \cdot \sigma\xi \le \kappa_\sigma\|\xi\|^2 \quad \text{in } X. \end{aligned} \tag{6.1}$$

Let $\boldsymbol{E} = (E^1, E^2)' \in \mathbb{C}^2$ and $H \in \mathbb{C}$ denote the electric and magnetic fields inside the domain X. Thus $\boldsymbol{E}$ and H solve the following time-harmonic Maxwell's equations:

$$\begin{cases} \nabla \times \boldsymbol{E} + \boldsymbol{\iota}\omega\mu_0 H = 0 \\ \boldsymbol{\nabla} \times H - \gamma\boldsymbol{E} = 0, \end{cases} \tag{6.2}$$

with the boundary condition

$$\boldsymbol{\nu} \times \boldsymbol{E} := \nu_1 E^2 - \nu_2 E^1 = f, \quad \text{on } \partial X, \tag{6.3}$$

where $\boldsymbol{\nu} = (\nu_1, \nu_2)$ is the exterior unit normal vector on the boundary ∂X. The standard well-posedness theory for Maxwells equations [21] states that given $f \in H^{\frac{1}{2}}(\partial X)$, the equation (6.2)-(6.3) admits a unique solution in the Sobolev space $H^1(X)$. In this chapter, the notations ∇ and $\boldsymbol{\nabla}$ distinguish between the scalar and vector curl operators:

$$\nabla \times \boldsymbol{E} = \frac{\partial E^2}{\partial x_1} - \frac{\partial E^1}{\partial x_2} \quad \text{and} \quad \boldsymbol{\nabla} \times H = (-\frac{\partial H}{\partial x_2}, \frac{\partial H}{\partial x_1})'. \tag{6.4}$$

Although (6.2) can be reduced to a scalar Laplace equation for H, we treat it as a system. The reconstruction method holds for the full 3 dimensional case. In this chapter, we assume that the conductivity σ and the permeability ε are independent of the third component in $\mathbb{R}^3$ and give the numerical simulations in two dimension to validate the reconstruction method.

6.1.1 Local reconstructibility condition

We select 5 boundary conditions $\{f_i\}_{1\leq i\leq 5}$ such that the corresponding electric and magnetic fields $\{\boldsymbol{E}_i, H_i\}_{1\leq i\leq 5}$ satisfy the Maxwell's equations (6.2). Assuming that over a sub-domain $X_0 \subset X$, the two electric fields $\boldsymbol{E}_1$, $\boldsymbol{E}_2$ satisfy the following positive condition,

$$\inf_{x\in X_0} |\det(\boldsymbol{E}_1, \boldsymbol{E}_2)| \geq c_0 > 0. \tag{6.5}$$

Thus the 3 additional solutions $\{\boldsymbol{E}_{2+j}\}_{1\leq j\leq 3}$ can be decomposed as linear combinations in the basis $(\boldsymbol{E}_1, \boldsymbol{E}_2)$,

$$\boldsymbol{E}_{2+j} = \lambda_1^j \boldsymbol{E}_1 + \lambda_2^j \boldsymbol{E}_2, \quad 1 \leq j \leq 3, \tag{6.6}$$

where the coefficients $\{\lambda_1^j, \lambda_2^j\}_{1\leq j\leq 3}$ can be computed by Cramer's rule as follows:

$$\begin{aligned} \lambda_1^j &= \frac{\det(\boldsymbol{E}_{2+j}, \boldsymbol{E}_2)}{\det(\boldsymbol{E}_1, \boldsymbol{E}_2)} = \frac{\det(\boldsymbol{\nabla} \times H_{2+j}, \boldsymbol{\nabla} \times H_2)}{\det(\boldsymbol{\nabla} \times H_1, \boldsymbol{\nabla} \times H_2)}, \\ \lambda_2^j &= \frac{\det(\boldsymbol{E}_1, \boldsymbol{E}_{2+j})}{\det(\boldsymbol{E}_1, \boldsymbol{E}_2)} = \frac{\det(\boldsymbol{\nabla} \times H_1, \boldsymbol{\nabla} \times H_{2+j})}{\det(\boldsymbol{\nabla} \times H_1, \boldsymbol{\nabla} \times H_2)}. \end{aligned} \tag{6.7}$$

Thus these coefficients can be computed from the available magnetic fields. The reconstruction procedures will make use of the matrices Z_j defined by

$$Z_j = \left[\boldsymbol{\nabla} \times \lambda_1^j | \boldsymbol{\nabla} \times \lambda_2^j\right], \quad \text{where} \quad 1 \leq j \leq 3. \tag{6.8}$$

These matrices are also uniquely determined from the known magnetic fields. Denoting the matrix $H := [\boldsymbol{\nabla} \times H_1 | \boldsymbol{\nabla} \times H_2]$ and the skew-symmetric matrix $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, we construct three matrices as follows,

$$M_j = (Z_j H^T)^{sym}, \quad 1 \leq j \leq 3, \tag{6.9}$$

where A^T denotes the transpose of a matrix A and $A^{sym} := (A+A^T)/2$. The calculations in the following section show that condition (6.5) and the linear independence of $\{M_j\}_{1\le j\le 3}$ in $S_2(\mathbb{C})$ are sufficient to guarantee local reconstruction of γ. The required conditions, which allow us to set up our reconstruction formulas, are listed in the following hypotheses. The reconstructions are *local* in nature: the reconstruction of γ at $x_0 \in X$ requires the knowledge of $\{H_j(x)\}_{1\le j\le J}$ for x only in the vicinity of x_0.

Hypothesis 6.1.1. *Given Maxwell's equations* (6.2) *with smooth* ε *and* σ *satisfying the uniform ellipticity conditions* (6.1)*, there exist a set of illuminations* $\{f_i\}_{1\le i\le 5}$ *such that the corresponding electric fields* $\{\boldsymbol{E}_i\}_{1\le i\le 5}$ *satisfy the following conditions:*

1. $\inf_{x\in X_0} |\det(\boldsymbol{E}_1, \boldsymbol{E}_2)| \ge c_0 > 0$ *holds on a sub-domain* $X_0 \subset X$,
2. *The matrices* $\{M_j\}_{1\le j\le 3}$ *constructed in* (6.9) *are linearly independent in* $S_2(\mathbb{C})$ *on* X_0*, where* $S_2(\mathbb{C})$ *denotes the space of* 2×2 *symmetric matrices.*

Remark 6.1.2. *Note that both conditions in Hypothesis 6.1.1 can be expressed in terms of the measurements* $\{H_j\}_j$*, and thus can be checked during the experiments. When the above constant* c_0 *is deemed too small, or the matrices* M_j *are not sufficiently independent, then additional measurements might be required. For the* 3 *dimensional case, Hypothesis 6.1.1 holds locally, under some smoothness assumptions on* γ*, with* 6 *well-chosen boundary conditions. The proof is based on the Runge approximation, see [26] for details.*

6.1.2 Reconstruction approaches and stability results

The reconstruction approaches were presented in [26] for a 3 dimensional case. To make this chapter self-contained, we briefly list the algorithm for the two-dimensional case. Denote $M_2(\mathbb{C})$ as the space of 2×2 matrices with inner product $\langle A, B\rangle := \text{tr}\,(A^*B)$. We assume that Hypothesis 6.1.1 holds over some $X_0 \subset X$ with 5 electric fields $\{\boldsymbol{E}_i\}_{1\le i\le 5}$. In particular, the matrices $\{M_j\}_{1\le j\le 3}$ constructed in (6.9) are linearly independent in $S_2(\mathbb{C})$. We will see that the inner products of $(\gamma^{-1})^*$ with all M_j can be calculated from knowledge of $\{H_j\}_{1\leq j\leq 5}$. Then γ can be explicitly reconstructed by least-square method. The reconstruction formulas can be found in Section 6.1.2. This algorithm leads to a unique and stable reconstruction and the stability estimate will be given in Section 6.1.2.

Reconstruction algorithms

We apply the curl operator to both sides of (6.6). Using the product rule, we get the following equation,

$$\sum_{i=1,2} \lambda_i^j \nabla \times \boldsymbol{E}_i + \boldsymbol{E}_i \cdot \boldsymbol{\nabla} \times \lambda_i^j = \nabla \times \boldsymbol{E}_{2+j}, \quad \text{for } j \geq 3. \tag{6.10}$$

Substituting H_i into $\boldsymbol{E}_i$ in the above equation, we obtain the following equation after rearranging terms,

$$\sum_{i=1,2} \boldsymbol{\nabla} \times \lambda_i^j \cdot (\gamma^{-1} \boldsymbol{\nabla} \times H_i) = \sum_{i=1,2} \boldsymbol{\iota}\omega\mu_0(\lambda_i^j H_i - H_{2+j}). \tag{6.11}$$

Recalling the definition of Z_j by (6.8), the above equation leads to

$$\gamma^{-1} : (Z_j H^t)^{sym} = \sum_{i=1,2} \boldsymbol{\iota}\omega\mu_0(\lambda_i^j H_i - H_{2+j}), \tag{6.12}$$

where the matrix $H = [\boldsymbol{\nabla} \times H_1 | \boldsymbol{\nabla} \times H_2]$. Note that $M_j = (Z_j H^T)^{sym}$ and the RHS of the above equation are computable from the measurements, thus γ can be explicitly reconstructed by (6.12) provided that $\{M_j\}_{1\leq j\leq 3}$ are of full rank in $S_2(\mathbb{C})$.

Remark 6.1.3. *The reconstruction formulae is local. In practice, we add more measurements and get additional M_j such that $\{M_j\}_j$ is of full rank in $S_2(\mathbb{C})$. The system* (6.12) *becomes overdetermined and γ can be reconstructed by solving* (6.12) *using least-square* *method.*

Uniqueness and stability results

The algorithm derived in the above section leads to a unique and stable reconstruction in the sense of the following theorem:

Theorem 6.1.4. *Suppose that Hypotheses 6.1.1 hold over some $X_0 \subset X$ for two sets of electric fields $\{\boldsymbol{E}_i\}_{1\le i\le 5}$ and $\{\boldsymbol{E}'_i\}_{1\le i\le 5}$, which solve the Maxwell's equations* (6.2) *with the complex tensors γ and γ' satisfying the uniform ellipticity condition* (6.1)*. Then γ can be uniquely reconstructed in X_0 with the following stability estimate,*

$$\|\gamma - \gamma'\|_{W^{s,\infty}(X_0)} \le C \sum_{i=1}^{5} \|H_i - H'_i\|_{W^{s+2,\infty}(X)}, \tag{6.13}$$

for any integer $s > 0$ and some constant $C = C(s)$.

Proof. The above estimate is straightforward by noticing that two derivatives are taken in the reconstruction procedure for γ. □

6.2 Fulfilling Hypothesis

In this section, we assume that γ is a diagonalizable constant tensor. We will take special CGO-like solutions of the Maxwell's equations (6.2) and demonstrate that Hypothesis 6.1.1 can be fulfilled with these solutions. By definition of the curl operator, it suffices to show that

$$\tilde{M}_j = (\tilde{Z}_j \tilde{H}^T)^{sym}, \quad 1 \le j \le 3, \tag{6.14}$$

are linearly independent in $S_2(\mathbb{C})$, where $\tilde{Z}_j = \left[\nabla\lambda_1^j|\nabla\lambda_2^j\right]$ and $\tilde{H} = [\nabla H_1|\nabla H_2]$. We derive the following Helmholtz-type equation from (6.2),

$$-\nabla\cdot\tilde{\gamma}^{-1}\nabla H_i + H_i = 0, \quad \text{for } 1 \le i \le 5, \tag{6.15}$$

where $\tilde{\gamma} = -\iota\omega\mu J^T\gamma J$ and admits a decomposition $\tilde{\gamma} = QQ^T$ with Q invertible. We take special CGO-like solutions of the form

$$H_i = e^{x\cdot Qu_i}, \tag{6.16}$$

where the u_i are vectors of unit length. Obviously, u_i defined in (6.16) satisfy (6.15) and

$$\tilde{H} = QU\begin{bmatrix} e^{x\cdot Qu_1} & 0 \\ 0 & e^{x\cdot Qu_2} \end{bmatrix}, \tag{6.17}$$

where $U = [u_1|u_2]$. Therefore, Hypothesis 6.1.1.1 can be easily fulfilled by choosing independent unit vectors $u_1 = \mathbf{e}_1$, $u_2 = \mathbf{e}_2$. Using the corresponding additional electric fields $\{\boldsymbol{E}_{2+j}\}_{1\le j\le 3}$, Cramer's rule as in (6.7) yields the decompositions

$$\boldsymbol{E}_{2+j} = \lambda_1^j\boldsymbol{E}_1 + \lambda_2^j\boldsymbol{E}_2, \quad \text{with } \lambda_1^j = e^{x\cdot Q(u_{2+j}-u_1)}\det(u_{2+j}, u_2),\ \ \lambda_2^j = e^{x\cdot Q(u_{2+j}-u_2)}\det(u_1, u_{2+j}).$$

Then by definition of $\tilde{Z}_j$, we get the following expression,

$$\tilde{Z}_j = Q[\frac{H_{2+j}}{H_1}\det(u_{2+j}, u_2)(u_{2+j}-u_1), \frac{H_{2+j}}{H_2}\det(u_1, u_{2+j})(u_{2+j}-u_2)]. \tag{6.18}$$

Together with (6.17), straightforward calculations lead to

$$\tilde{Z}_j\tilde{H}^T = H_{2+j}Q[\det(u_{2+j}, u_2)(u_{2+j}-u_1), \det(u_1, u_{2+j})(u_{2+j}-u_2]Q^T. \tag{6.19}$$

Using the fact that $u_{2+j} = (u_{2+j} \cdot u_1)u_1 + (u_{2+j} \cdot u_2)u_2$, the above equation leads to

$$\tilde{M}_j = H_{2+j} Q \begin{bmatrix} (u_{2+j} \cdot u_1)((u_{2+j} \cdot u_1) - 1) & (u_{2+j} \cdot u_1)(u_{2+j} \cdot u_2) \\ (u_{2+j} \cdot u_1)(u_{2+j} \cdot u_2) & (u_{2+j} \cdot u_2)((u_{2+j} \cdot u_2) - 1) \end{bmatrix} Q^T, \qquad (6.20)$$

where $u_1 = \mathbf{e}_1$, $u_2 = \mathbf{e}_2$. Therefore, it is easy to find u_{2+j} vectors of unit length such that $\tilde{M}_j$ are linearly independent in $S_2(\mathbb{C})$.

Remark 6.2.1. *To derive local reconstruction formulas for more general tensors (e.g. $\mathcal{C}^{1,\alpha}(X)$), we need local independence conditions of $\{M_j\}_j$ and we need to control the local behavior of solutions by well-chosen boundary conditions. This is done by means of a Runge approximation. For details, we refer the reader to [10],[15] and [26].*

6.3 Numerical experiments

In this section we present some numerical simulations based on synthetic data to validate the reconstruction algorithms from the previous section.

6.3.1 Preliminary

We decompose $\gamma = \sigma + \iota\omega\varepsilon$ into the following form with six unknown coefficients $\{\sigma_i\}_{1\leq i\leq 3}$, $\{\varepsilon_i\}_{1\leq i\leq 3}$ respectively for σ and ϵ,

$$\gamma = \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{bmatrix} + \iota\omega \begin{bmatrix} \varepsilon_1 & \varepsilon_2 \\ \varepsilon_2 & \varepsilon_3 \end{bmatrix}, \qquad (6.21)$$

where each coefficient can be explicitly reconstructed by solving the overdetermined linear system (6.12) using least-square method.

In the numerical experiments below, we take the domain of reconstruction to be the square $X = [-1, 1]^2$ and use the notation $\mathbf{x} = (x, y)$. We use a $\mathsf{N}+1 \times \mathsf{N}+1$ square grid with $\mathsf{N} = 80$, the tensor product of the equi-spaced subdivision $\mathsf{x} = -1 : \mathsf{h} : 1$ with $\mathsf{h} = 2/\mathsf{N}$. The synthetic data H are generated by solving the Maxwell's equations (6.2) for *known* conductivity σ and electric permittivity ε, using a finite difference method implemented with `MatLab`. We refer to these data as the "noiseless" data. To simulate noisy data, the internal magnetic fields H are perturbed by adding Gaussian random matrices with zero means. The standard derivations α are chosen to be 0.1% of the average value of $|H|$.

We use the relative L^2 error to measure the quality of the reconstructions. This error is defined as the L^2-norm of the difference between the reconstructed coefficient and the true coefficient, divided by the L^2-norm of the true coefficient. $\mathcal{E}^C_{\sigma_i}$, $\mathcal{E}^N_{\sigma_i}$, $\mathcal{E}^C_{\varepsilon_i}$, $\mathcal{E}^N_{\varepsilon_i}$ with $1 \le i \le 3$ denote respectively the relative L^2 error in the reconstructions from clean and noisy data for σ_i and ε_i.

Regularization procedure. We use a total variation method as the denoising procedure by minimizing the following functional,

$$\mathcal{O}(\mathbf{f}) = \frac{1}{2}\|\mathbf{f} - \mathbf{f}_{\mathrm{rc}}\|_2^2 + \rho\|\Gamma\mathbf{f}\|_{\mathrm{TV}}, \tag{6.22}$$

where $\mathbf{f}_{\mathrm{rc}}$ denotes the explicit reconstructions of the coefficients of σ and ε, Γ denotes discretized version of the gradient operator. We choose the l^1-norm as the regularization TV norm for discontinuous, piecewise constant, coefficients. In this case, the minimization problem can be solved using the split Bregman method presented in [25]. To recover smooth coefficients, we minimize the following least square problem with the l^2-norm for the regularization term,

$$\mathcal{O}(\mathbf{f}) = \frac{1}{2}\|\mathbf{f} - \mathbf{f}_{\mathrm{rc}}\|_2^2 + \rho\|\Gamma\mathbf{f}\|_2^2, \tag{6.23}$$

where the Tikhonov regularization functional admits an explicit solution $\mathbf{f} = (\mathbb{I}+\rho\Gamma^*\Gamma)^{-1}\mathbf{f}_{\mathrm{rc}}$. The regularization methods are used when the data are differentiated.

6.3.2 Simulation results

Simulation 1. In the first experiment, we intend to reconstruct the smooth coefficients $\{\sigma_i, \varepsilon_i\}_{1\leq i\leq 3}$ defined in (6.21). The coefficients are given by,

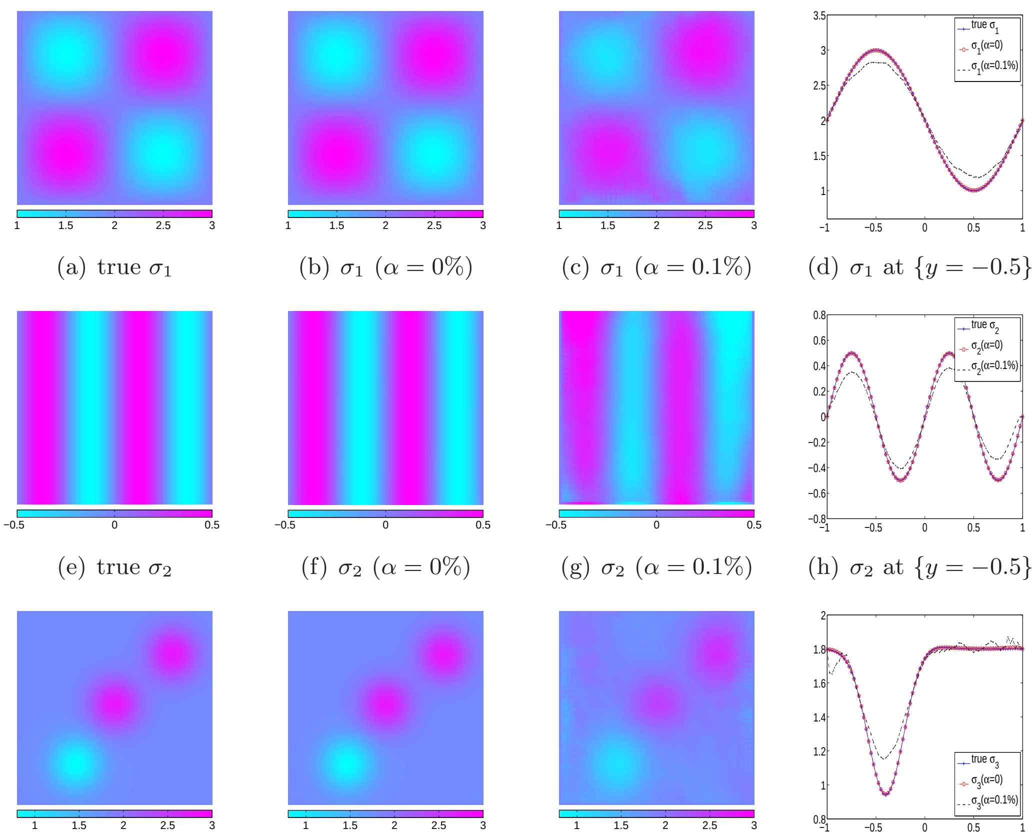
$$\begin{cases} \sigma_1 = 2 + \sin(\pi x)\sin(\pi y) \\ \sigma_2 = 0.5\sin(2\pi x) \\ \sigma_3 = 1.8 + e^{-15(x^2+y^2)} + e^{-15((x-0.6)^2+(y-0.5)^2)} - e^{-15((x+0.4)^2+(y+0.6)^2)} \end{cases}$$

and

$$\begin{cases} \varepsilon_1 = 2 - \sin(\pi x)\sin(\pi y) \\ \varepsilon_2 = 0.5\sin(2\pi y) \\ \varepsilon_3 = 1.8 + e^{-12(x^2+y^2)} + e^{-12((x+0.6)^2+(y-0.5)^2)} - e^{-12((x-0.4)^2+(y+0.6)^2)}. \end{cases}$$

We performed two sets of reconstructions using clean and noisy synthetic data respectively. The l_2-regularization procedure is used in this simulation. For the noisy data, the noise level is $\alpha = 0.1\%$. The results of the numerical experiment are shown in Figure 6.1 and Figure 6.2. The relative L^2 errors in the reconstructions are $\mathcal{E}^C_{\sigma_1} = 0.3\%$, $\mathcal{E}^N_{\sigma_1} = 5.1\%$, $\mathcal{E}^C_{\sigma_2} = 0.8\%$, $\mathcal{E}^N_{\sigma_2} = 33.4\%$, $\mathcal{E}^C_{\sigma_3} = 0.2\%$, $\mathcal{E}^N_{\sigma_3} = 4.9\%$; $\mathcal{E}^C_{\varepsilon_1} = 0.1\%$, $\mathcal{E}^N_{\varepsilon_1} = 5.8\%$, $\mathcal{E}^C_{\varepsilon_2} = 0.5\%$, $\mathcal{E}^N_{\varepsilon_2} = 30.0\%$, $\mathcal{E}^C_{\varepsilon_3} = 0.1\%$, $\mathcal{E}^N_{\varepsilon_3} = 4.8\%$.

Simulation 2. In this experiment, we attempt to reconstruct piecewise constant coefficients. Reconstructions with both noiseless and noisy data are performed with l_1-regularization using the split Bregman iteration method. The noise level is $\alpha = 0.1\%$. The results of the numerical experiment are shown in Figure 6.3 and 6.4. From the figures,

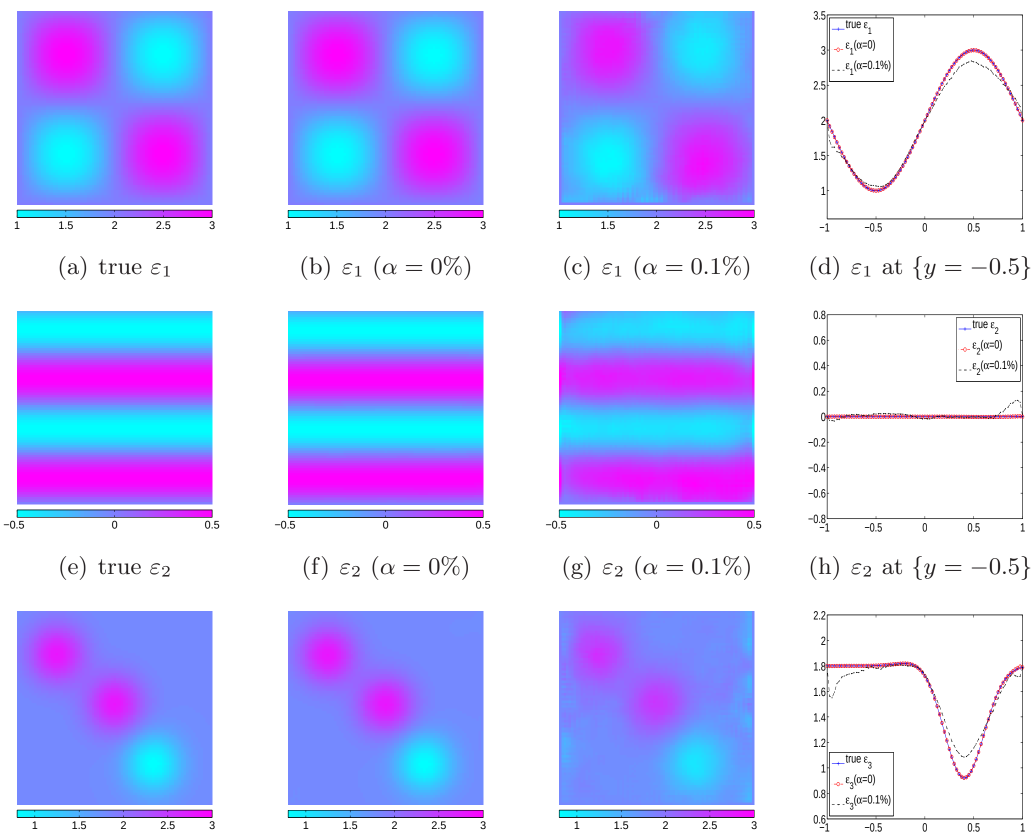


(a) true σ_1 (b) σ_1 ($\alpha = 0\%$) (c) σ_1 ($\alpha = 0.1\%$) (d) σ_1 at $\{y = -0.5\}$

(e) true σ_2 (f) σ_2 ($\alpha = 0\%$) (g) σ_2 ($\alpha = 0.1\%$) (h) σ_2 at $\{y = -0.5\}$

(i) true σ_3 (j) σ_3 ($\alpha = 0\%$) (k) σ_3 ($\alpha = 0.1\%$) (l) σ_3 at $\{y = 0\}$

Figure 6.1: σ in Simulation 1. (a)&(e)&(i): true values of $(\sigma_1, \sigma_2, \sigma_3)$. (b)&(f)&(j): reconstructions with noiseless data. (c)&(g)&(k): reconstructions with noisy data($\alpha = 0.1\%$). (d)&(h)&(l): cross sections along $\{y = -0.5\}$.



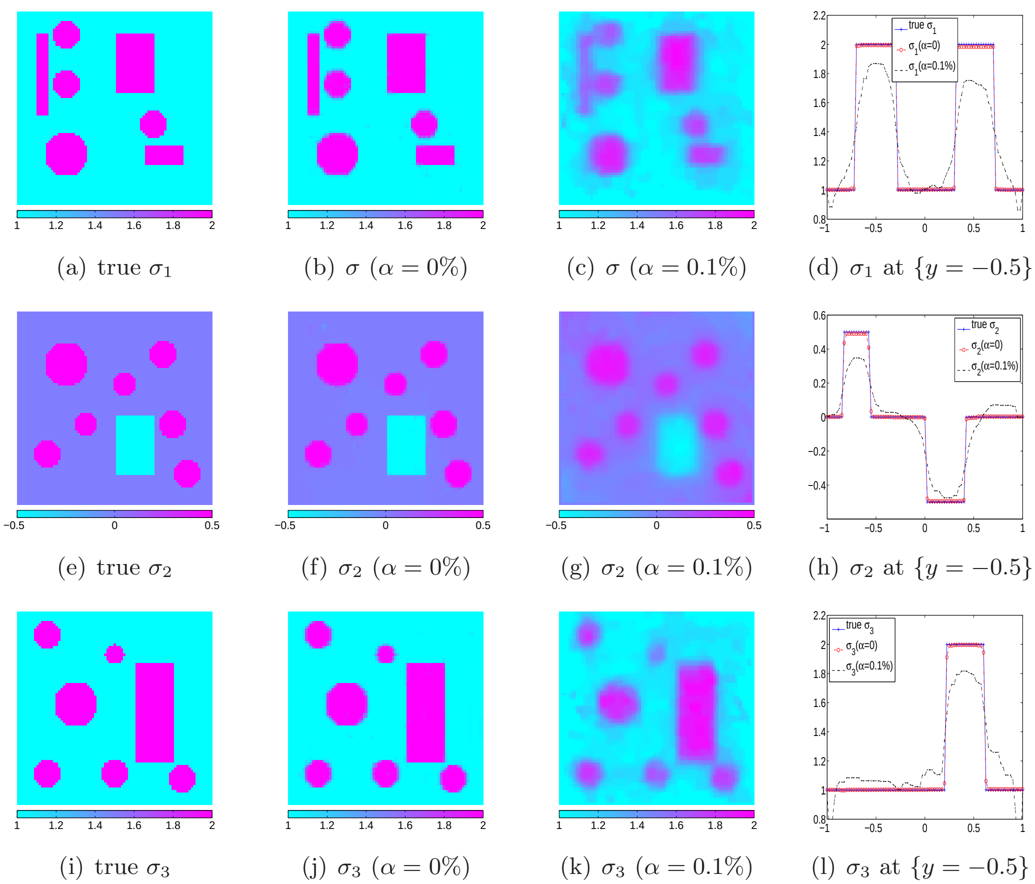
(a) true ε_1 (b) ε_1 ($\alpha = 0\%$) (c) ε_1 ($\alpha = 0.1\%$) (d) ε_1 at $\{y = -0.5\}$

(e) true ε_2 (f) ε_2 ($\alpha = 0\%$) (g) ε_2 ($\alpha = 0.1\%$) (h) ε_2 at $\{y = -0.5\}$

(i) true ε_3 (j) ε_3 ($\alpha = 0\%$) (k) ε_3 ($\alpha = 0.1\%$) (l) ε_3 at $\{y = -0.5\}$

Figure 6.2: ε in Simulation 1. (a)&(e)&(i): true values of $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. (b)&(f)&(j): reconstructions with noiseless data. (c)&(g)&(k): reconstructions with noisy data($\alpha = 0.1\%$). (d)&(h)&(l): cross sections along $\{y = -0.5\}$.

we observe that the singularities of the coefficients create minor artifacts on the reconstructions and the error in the reconstruction is larger at the discontinuities than in the rest of the domain. The relative L^2 errors in the reconstructions are $\mathcal{E}^C_{\sigma_1} = 4.0\%$, $\mathcal{E}^N_{\sigma_1} = 17.6\%$, $\mathcal{E}^C_{\sigma_2} = 12.8\%$, $\mathcal{E}^N_{\sigma_2} = 48.1\%$, $\mathcal{E}^C_{\sigma_3} = 4.5\%$, $\mathcal{E}^N_{\sigma_3} = 16.5\%$; $\mathcal{E}^C_{\varepsilon_1} = 0.1\%$, $\mathcal{E}^N_{\varepsilon_1} = 16.3\%$, $\mathcal{E}^C_{\varepsilon_2} = 0.5\%$, $\mathcal{E}^N_{\varepsilon_2} = 35.2\%$, $\mathcal{E}^C_{\varepsilon_3} = 0.1\%$, $\mathcal{E}^N_{\varepsilon_3} = 16.2\%$.

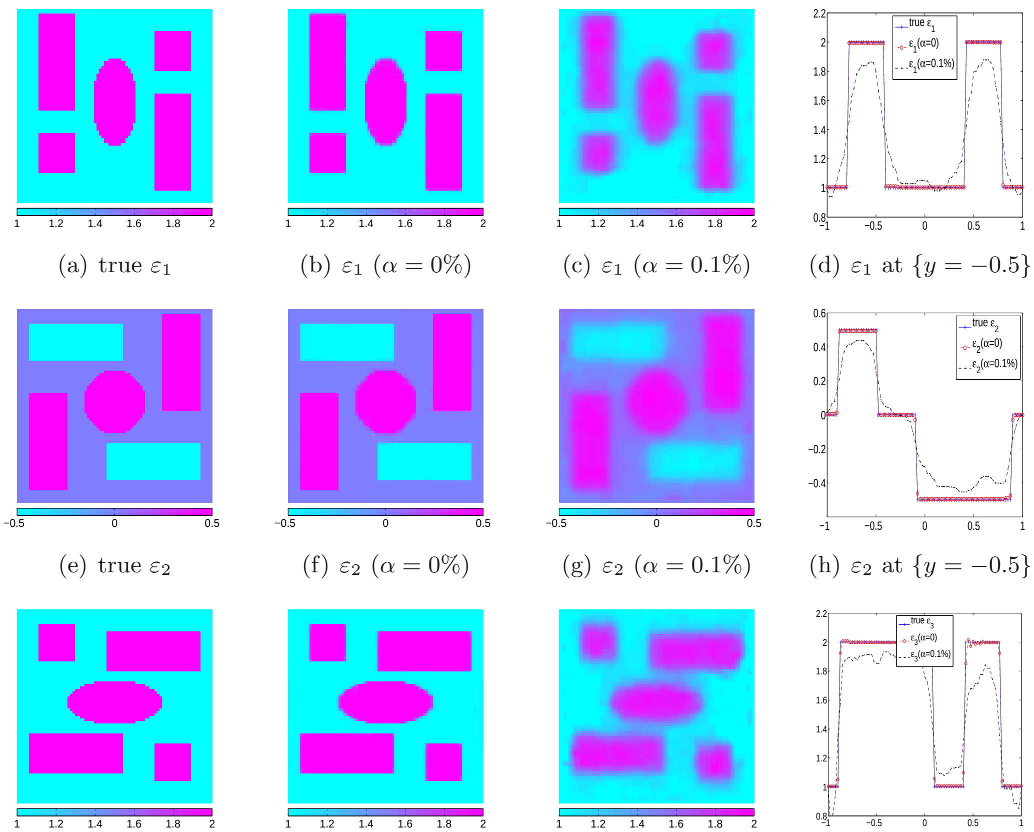


(a) true σ_1 (b) σ ($\alpha = 0\%$) (c) σ ($\alpha = 0.1\%$) (d) σ_1 at $\{y = -0.5\}$

(e) true σ_2 (f) σ_2 ($\alpha = 0\%$) (g) σ_2 ($\alpha = 0.1\%$) (h) σ_2 at $\{y = -0.5\}$

(i) true σ_3 (j) σ_3 ($\alpha = 0\%$) (k) σ_3 ($\alpha = 0.1\%$) (l) σ_3 at $\{y = -0.5\}$

Figure 6.3: σ in Simulation 2. (a)&(e)&(i): true values of $(\sigma_1, \sigma_2, \sigma_3)$. (b)&(f)&(j): reconstructions with noiseless data. (c)&(g)&(k): reconstructions with noisy data($\alpha = 0.1\%$). (d)&(h)&(l): cross sections along $\{y = -0.5\}$.



(a) true ε_1 (b) ε_1 ($\alpha = 0\%$) (c) ε_1 ($\alpha = 0.1\%$) (d) ε_1 at $\{y = -0.5\}$

(e) true ε_2 (f) ε_2 ($\alpha = 0\%$) (g) ε_2 ($\alpha = 0.1\%$) (h) ε_2 at $\{y = -0.5\}$

(i) true ε_3 (j) ε_3 ($\alpha = 0\%$) (k) ε_3 ($\alpha = 0.1\%$) (l) ε_3 at $\{y = 0\}$

Figure 6.4: ε in Simulation 2. (a)&(e)&(i): true values of $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. (b)&(f)&(j): reconstructions with noiseless data. (c)&(g)&(k): reconstructions with noisy data($\alpha = 0.1\%$). (d)&(h)&(l): cross sections along $\{y = -0.5\}$.

6.4 Conclusion

We presented in this chapter the reconstruction of (σ, ε) from knowledge of several magnetic fields H_j, where the measurements H_j solve the Maxwell's equations (6.2) with prescribed illuminations $f = f_j$ on ∂X.

The reconstruction algorithms rely heavily on the linear independence of electric fields and the families of $\{M_j\}_j$ constructed in Hypothesis 6.1.1. These linear independence conditions can be checked by the available magnetic fields $\{H_j\}_j$ and additional measurements could be added if necessary. This method was used in the numerical simulations. We proved in Section 6.2 that these linear independence conditions could be satisfied by constructing CGO-like solutions for constant tensors. In fact, these conditions can be verified numerically for a large class of illuminations and more general tensors. The numerical simulations illustrate that both smooth and rough coefficients could be well reconstructed, assuming that the interior magnetic fields H_j are accurate enough. However, the reconstructions are very sensitive to the additional noise on the functionals H_j. This fact is consistent with the stability estimate (with the loss of two derivatives from the measurements to the reconstructed quantities) in Theorem 6.1.4.

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