# Essays in macroeconomics and corporate finance

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ABSTRACT

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This dissertation consists of three essays in macroeconomics and corporate finance. In the first chapter, we study firms' inventory investment choices in order to shed light on a perennial question in macroeconomics: what are the sources of business-cycle fluctuations? We focus on the importance of "news" shocks, which change agents' expectations about future fundamentals without changin current fundamentals. The existing literature has provided a wide range of estimates of the importance of these shocks, suggesting that they may accountfor anywhere between 10 percent and 60 percent of the volatility of output. We show that looking at the dynamics of inventories, so far neglected in this literature, cleanly isolates the role of news shocks in driving business cycles. In particular, inventory dynamics provide an upper bound on the explanatory power of news shocks. We show, for a broad class of theoretical models, that finished-good inventories must fall when there is an increase in consumption and investment induced by news shocks. When good news about future fundamentals lowers expected future marginal costs, firms delay current production and satisfy the increase in demand by selling from existing inventories. This result is holds regardless of the nature of the news, and is robust to the introduction of different types of adjustment costs. We therefore propose a novel empirical identification strategy for news shocks: negative comovement between inventories and components of private spending. Estimating a structural VAR with sign restrictions on inventories, consumption and investment, our identified shock explains at most 20 percent of output variations. Intuitively, since inventories are procyclical in the data, shocks that generate negative comovement between inventories and sales cannot account for the bulk of business-cycle fluctuations.

The second and third chapters focus on firms' choice between intermediated and disintermediated finance. In the second chapter, I propose a static model in which firms endogenously choose the scale and the composition of their debt structure, between bank (intermediated) lending and market (disintermediated) lending. An entrepreneur finances a project of variable size using internal funds and external borrowing from two types of creditors: banks and public debt markets. The key distinction between the two is that, when liquidation looms, bank loans are easier to restructure than market debt. Absent deadweight losses in liquidation, debt structure is irrelevant to the investment choices of the entrepreneur, and projects are financed by whichever lender has the lowest marginal lending cost. With liquidation losses, I show that investment is financed by a combination of bank and market finance so long as 1) banks have higher marginal lending costs than markets and 2) entrepreneurs' internal resources are sufficiently small. In that case, the share of bank finance in total investment depends non-monotonically on internal resources: firms with very limited internal resources are increasingly reliant on bank finance to expand investment, while medium-sized firms reduce the contribution of bank finance for each additional marginal unit of equity. I show that the model's predictions are well supported in data on the debt structure of US manufacturing firms.

Finally, chapter 3 builds on the results of chapter 2 in order to propose a macroeconomic model with a rich financial intermediation sector. The composition of corporate borrowing between bank loans and market debt varies substantially, both across countries and over the business cycle. The model of chapter 3 sheds light on the causes and aggregate implications of this variation. Banks are assumed to differ from markets in their ability to restructure debt payments when a firm is in financial distress; however, banks' flexibility in distress comes at the expense of tougher lending standards. The steady-state of the model is consistent with key cross-sectional facts about debt composition: in particular, firms simultaneously use bank loans and market debt, and the share of bank loans in total debt is negatively related to firms' net worth. Over the business cycle, asymmetric shocks to banks' lending costs generate substitution from bank loans to market debt, as in the US during the 2007–2009 recession. However, debt substitution is accompanied by a precautionary reduction in total borrowing, as firms that replace bank loans with public debt issuance internalize the fact that this type of borrowing will be harder to restructure in bad times. The contribution of this new "substitution" channel to the decline in aggregate investment following the banking shock can be substantial. Finally, I study the macroeconomic effects of corporate finance policies aimed at encouraging market debt financing by small and medium-sized firms. While these policies stimulate investment of small firms, they also induce mid-sized firms to adopt a debt structure that increases their vulnerability to financial distress.

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*À la mémoire de Jean Hubac*

# Chapter 1

# What do inventories tell us about news-driven business cycles?

Nicolas Crouzet and Hyunseung Oh

## 1.1 Introduction

The sources of business cycles are an enduring subject of debate among macroeconomists. Recently, the literature has focused on news shocks — shocks that change agents' expectations about future economic fundamentals, without affecting current fundamentals — as a potential driving force of aggregate fluctuations. Starting with Beaudry and Portier (2006), this literature has argued that news shocks may provide a good account of expansions and recessions, stressing episodes such as the US and Asian investment booms and busts of the late 1990s as examples.

In the news view of business cycles, booms and busts come through changes in expectations and investment (Beaudry and Portier, 2013). For example, when productivity is expected to increase in the future, investment increases to build up the capital stock to take advantage of the lower marginal costs in the future. This boom in investment raises wages and hours worked, and the additional income leads to a consumption boom. Hence good news about future productivity leads to a current boom in output, and investment is a key channel. Recent theories of the business cycle based on news shocks are successful in capturing this mechanism. A prominent example is Jaimovich and Rebelo (2009) where they show that, in a neoclassical growth model with investment adjustment costs, variable capacity utilization, and weak wealth effects on hours worked, an expected rise in the marginal product of capital leads to a boom in investment today. Adding variable capacity and weak wealth effects on hours worked allows output to rise on impact and satisfy current demand, while investment adjustment costs lead firms to smooth the desired increase in the stock of capital over time and start investing today.

However, the empirical literature on estimating the role of news shocks over the business cycle has yet to come to a consensus. While some estimate that news shocks account for as high as 60 percent of output variations, others with equally plausible methods end up with as low numbers as 10 percent.[1] This indicates that the literature is still in need of additional information to precisely characterize the importance of news shocks. The goal of this paper is to bring in new insight that could improve on our empirical characterization of news shocks over the business cycle. To be specific, we focus on a variable that is highly informative about news shocks, but so far has been neglected in the literature: investment in finished-good inventories.

Investment in finished-good inventories is informative in the context of news shocks for the following reasons. First, finished-good inventories are a forward-looking variable that responds to changes in expectations about future economic conditions. For instance, Kesavan, Gaur, and Raman (2010) find that finished-good inventory data are valuable for forecasting sales. Since expectations and investment behavior are at the center of the economic mechanism for how news shocks work, investment in finished-good inventories should be a good source of identification. Second, finished-good inventories provide a clear differentiation between shocks that happen today and shocks that are expected to happen in the future. A straightforward illustration is when the economy faces temporary changes in productivity. When productivity increases today, then higher income today will raise sales. Firms at the same time will bunch production to make the most out of the productivity increase and finished-good inventories will also rise. Hence with a change in productivity today, there will be positive comovement between inventories and sales.[2] When productivity is expected to increase tomorrow, then higher income in the future will also raise sales today. How-

[1]For example, Beaudry and Portier (2006) and Schmitt-Grohé and Uribe (2012) estimate that the contribution of news shocks to the business-cycle volatility of output is above 50 percent, while Barsky and Sims (2011) and Khan and Tsoukalas (2012) find much smaller numbers. This will be discussed in detail in later sections.

[2]From now on, we will use the term "inventories" to indicate finished-good inventories when there is no confusion.

ever, since firms expect future production to be cheaper than current production, they will satisfy this increase in sales by depleting inventories. Hence with a change in productivity tomorrow, there will be negative comovement between inventories and sales. The sign of the comovement between sales and inventories should therefore help us distinguish between current and future expected changes in productivity.

In section 1.2, we start our analysis by introducing inventories as in Bils and Kahn (2000) into a news-driven business-cycle model. In section 1.3, we use this model to show that good news about the future leads to a boom in consumption and investment, but a fall in inventories. The intuition at the heart of our result is that news shocks lead to strong intertemporal substitution in production. With good news about the future, marginal cost is expected to be lower in the future than today. Optimal inventory investment behavior then dictates that firms should delay production, and satisfy current demand by drawing down on existing inventories. Thus, in the context of this model, news shocks indeed generate negative comovement between inventories and sales.

In section 1.4, we show that our result survives a number of extensions of the baseline model. First, we show that the fall in inventories after a positive news shock is deep and protracted. Second, we establish that our result holds for other types of news, especially news on demand. Third, we introduce various types of adjustment costs to check whether our result is robust. In section 1.5, we show that our result also holds in alternative inventory models, such as the stockout-avoidance model of Kahn (1992), Kryvtsov and Midrigan (2013) and Wen (2011) or the (S,s) inventory model of Khan and Thomas (2007b). Although each class of models introduce inventories for different reasons, it is important to note that the strong intertemporal substitution channel is a general feature.

Having established that the negative comovement of inventories and sales is a solid outcome of news shocks, we propose to use this prediction as a means to identify news shocks.

In section 1.6, we describe an empirical strategy based on this idea, a structural VAR with sign restrictions. We show that a range of shocks identified in this manner explain less than 20 percent of output variations over the business cycle. The reason we get a small and precise contribution of news shocks is because inventories are procyclical in the data. Any shock that generates negative comovement between inventories and sales must have limited importance over the business cycle. In section 1.7, we show that our results also hold in an estimated DSGE model with inventories. Using a stock-elastic demand inventory model and including a wide range of shocks studied in the literature, we estimate that news shocks account for less than 20 percent of output growth variations. Section 1.8 concludes.

Our work relates to a number of papers that examine the behavior of investment with news shocks. Jaimovich and Rebelo (2009), Christiano, Ilut, Motto, and Rostagno (2008), as well as Schmitt-Grohé and Uribe (2012) document the importance of investment adjustment cost for news shocks to generate an immediate boom in investment and output. However, inventory investment has been mostly neglected in this literature. One exception is Vukotic (2013) where inventories are introduced as a factor of production in the durable sector. Our approach is quite different from hers since we examine inventories that are stored as finished goods. These type of inventories do not enter the production function, and therefore the previous channels through which investment operates under news shocks no longer applies. Our contribution to the news literature then is to illustrate a new channel through which news shocks operate by focusing on the investment behavior of finished-good inventories that is distinct from capital investment.

Our work also relates to the recent literature on inventories that matches the stylized business-cycle facts of inventories with micro foundations at the firm level. The main difference across these models is how they generate a positive level of inventories at the steady state. To be specific, one branch of the literature argues that inventories exist in order to facilitate sales, either by their use for displaying and advertising purposes (Bils and Kahn, 2000), or by their use as a buffer against stockouts (Wen, 2011; Kryvtsov and Midrigan, 2013). Another branch of the literature argues that inventories exist due to bunching behavior induced by fixed ordering costs (Fisher and Hornstein, 2000; Khan and Thomas, 2007b). Since our focus is on finished-good inventories, we fit better into the former approach.[3] Nevertheless, our result also applies to the latter approach, since a common feature of all these models is that inventories are producers' means of intertemporal substitution. Our contribution to this literature is highlighting this common mechanism of a wide range of inventory models when business cycles are driven by news shocks.

Lastly, our empirical approach is based on the sign restriction literature in a vector autoregression (VAR) framework. These approaches have been applied in identifying monetary policy shocks (Faust, 1998; Uhlig, 2005), fiscal policy shocks (Mountford and Uhlig, 2008; Caldara and Kamps, 2012) and also news shocks (Beaudry, Nam, and Wang, 2011).

## 1.2 A finished-good inventory model

In this section, we lay out a general equilibrium model of inventory dynamics based on the work of Pindyck (1994), Bils and Kahn (2000), and Jung and Yun (2006). The tractability of this model delivers a clear intuition on how inventories work in the economy in response to news shocks. Other models will be discussed in later sections.

The key feature of the so-called "stock-elastic" demand model of this section is the assumption that sales of a firm are elastic to the amount of goods available for sale, which we term "on-shelf goods." This assumption finds empirical support for many categories of goods, as documented by Pindyck (1994) or Copeland, Dunn, and Hall (2011). The

[3] See Khan and Thomas (2007a) for a comparison of the two approaches.

positive elasticity of sales to on-shelf goods captures the idea that with more on-shelf goods, customers are more likely to find a good match and purchase the product. This may arise either because of greater availability of goods, or because more on-shelf goods may provide a wider variety within the same product. For example, a shoe store with more colors and size of all kinds are likely to attract more customers and sell more goods.

### 1.2.1 Description of the stock-elastic demand model

The economy consists of a representative household and monopolistically competitive firms. The output of the firms are storable goods, of which they keep a positive inventory. We start with the household problem.

**Household problem** A representative household maximizes the following expected sum of discounted utility,

$$E_0\left[\sum_{t=0}^{\infty}\beta^t U(c_t, n_t; \psi_t)\right], \tag{1.1}$$

where $c_t$ is the consumption of the final good, $n_t$ denotes the supply of labor services, and $\psi_t$ is a exogenous variable that introduces a wedge between the marginal rate of substitution between consumption and leisure, and the real wage, and which we call a "labor wedge" shock. We assume that the household's period utility function takes the form proposed by Greenwood, Hercowitz, and Huffman (1988, henceforth GHH):

$$U(c, n; \psi) = \frac{1}{1-\sigma}\left(c - \psi\frac{n^{1+\xi^{-1}}}{1+\xi^{-1}}\right)^{1-\sigma},$$

where $\xi$ is the Frisch elasticity of labor supply and $\sigma$ denotes the inverse of the elasticity of the household's intertemporal substitution. This preference specification has been widely used in the literature on news shocks, and it implies zero wealth effects on labor supply.

The household's maximization problem is subject to the following constraints:

$$\int_0^1 p_t(j)s_t(j)dj + \mathbb{E}_t\left[Q_{t,t+1}B_{t+1}\right] \leq W_t n_t + R_t k_t + \int_0^1 \pi_t(j)dj + B_t, \tag{1.2}$$

$$k_{t+1} = i_t\left[1 - \phi\left(\frac{i_t}{i_{t-1}}\right)\right] + (1-\delta_k)k_t, \tag{1.3}$$

$$c_t + i_t \leq x_t. \tag{1.4}$$

Equation (1.2) is the household budget constraint. The household earns income each period by providing labor $n_t$ at a given wage $W_t$, lending capital $k_t$ at a rate $R_t$, claiming the profit $\pi_t(j)$ from each firm $j \in [0,1]$, and receiving bond payments $B_t$. It spends its income in purchases of each variety in the amount $s_t(j)$ at a price $p_t(j)$, and in purchases of the state-contingent one-period bonds $B_{t+1}$. The probability-adjusted price of each of these bonds is $Q_{t,t+1}$, for each state in period $t+1$.

Equation (1.3) is the accumulation rule of capital with adjustment costs to investment. The adjustment cost function $\phi(\cdot)$ is twice-differentiable with $\phi(1) = \phi'(1) = 0$, and $\phi''(1) > 0$. Adjustment costs of this form generate an immediate build-up motive for capital when the desired level of capital is high in the future.

Equation (1.4) states that the household's consumption and investment cannot exceed its total absorption of final goods, $x_t$, which is constructed by aggregating their purchase of intermediate goods $\{s_t(j)\}_{j\in[0,1]}$. The aggregation of the intermediate goods $\{s_t(j)\}_{j\in[0,1]}$ into $x_t$ is given by a Dixit-Stiglitz type aggregator of the form:

$$x_t = \left(\int_0^1 v_t(j)^{\frac{1}{\theta}} s_t(j)^{\frac{\theta-1}{\theta}} dj\right)^{\frac{\theta}{\theta-1}}, \tag{1.5}$$

where $v_t(j)$ is the taste-shifter for each product $j$ and $\theta$ is the elasticity of substitution across intermediate goods. It follows from expenditure minimization that the demand function for each good and the aggregate price level take the following forms:

$$s_t(j) = v_t(j) \left( \frac{p_t(j)}{P_t} \right)^{-\theta} x_t, \qquad P_t = \left( \int_0^1 v_t(j) p_t(j)^{1-\theta} dj \right)^{\frac{1}{1-\theta}}.$$

In the stock-elastic demand model, the taste shifter for variety $j$ is assumed to depend on the amounts of goods on shelf proposed by the firm producing variety $j$, $a_t(j)$, in the following fashion:

$$v_t(j) = \left( \frac{a_t(j)}{a_t} \right)^{\zeta}, \tag{1.6}$$

where the normalization by $a_t$, defined as the the economy-wide average of on-shelf goods, ensures that the mean of $\nu_t(j)$ across goods is equal to 1. The parameter $\zeta > 0$ controls the degree of the shift in taste due to the relative amount of goods on-shelf.

Finally, the household is given an initial level of capital $k_0$ and bonds $B_0$, and its optimization problem is subject to a no-Ponzi condition for both capital and stage-contingent bond holdings.

**Firm problem** Each monopolistically competitive firm $j \in [0, 1]$ maximizes the expected discounted sum of profits

$$E_0 \left[ \sum_{t=0}^{\infty} Q_{0,t} \pi_t(j) \right], \tag{1.7}$$

where

$$\pi_t(j) = p_t(j) s_t(j) - W_t n_t(j) - R_t k_t(j). \tag{1.8}$$

Note that profit in each period is the revenue from sales net of the cost from hiring labor $n_t(j)$ and renting capital $k_t(j)$ at their respective prices $W_t$ and $R_t$. The term $Q_{0,t}$ is the discount factor of bonds between period 0 and $t$, so that $Q_{0,t} = \prod_{T=0}^{t-1} Q_{T,T+1}$. This discount factor is consistent with households being the final owners of firms. The firm faces the following constraints:

$$a_t(j) = (1 - \delta_i)inv_{t-1}(j) + y_t(j), \tag{1.9}$$

$$inv_t(j) = a_t(j) - s_t(j), \tag{1.10}$$

$$y_t(j) = z_t k_t^{1-\alpha}(j) n_t^{\alpha}(j), \tag{1.11}$$

$$s_t(j) = \left(\frac{a_t(j)}{a_t}\right)^{\zeta} \left(\frac{p_t(j)}{P_t}\right)^{-\theta} x_t. \tag{1.12}$$

Equation (1.9) is the stock accumulation equation. The stock (on-shelf goods) of the firm, $a_t(j)$, consists of the undepreciated stock of inventories from the previous period $(1 - \delta_i)inv_{t-1}(j)$ and current production $y_t(j)$. The parameter $\delta_i$ denotes the depreciation rate of inventories. Equation (1.10) states that on-shelf goods that are unsold are accounted as inventories.[4] Equation (1.11) is the production function. Firms use a constant returns to scale production function, with capital and labor as inputs. The variable $z_t$ represents total factor productivity and is exogenous. Finally, monopolistically competitive firms face the demand function (1.12) stemming from the household problem.

**Market clearing** Labor and capital markets clear, and net bond holdings is zero:

$$n_t = \int_0^1 n_t(j)dj, \tag{1.13}$$

[4] In the data, this is recorded as the end-of-period inventory stock in each period.

$$k_t = \int_0^1 k_t(j)dj, \tag{1.14}$$

$$B_{t+1} = 0. \tag{1.15}$$

Sales of goods for each variety $j$ also clears by the demand function described above. The average on-shelf goods in the economy $a_t$ is defined by:

$$a_t = \int_0^1 a_t(j)dj. \tag{1.16}$$

### 1.2.2 Equilibrium

A market equilibrium of this economy is defined as follows.

**Definition 1.1** (Market equilibrium of the stock-elastic demand model)**.** *A market equilibrium of the stock-elastic demand model is a set of stochastic processes for aggregate variables*

$$c_t, n_t, k_{t+1}, i_t, B_{t+1}, x_t, a_t, W_t, R_t, P_t, Q_{t,t+1},$$

*and firm-level variables*

$$\{a_t(j)\}, \{n_t(j)\}, \{k_t(j)\}, \{v_t(j)\}, \{s_t(j)\}, \{y_t(j)\}, \{inv_t(j)\}, \{p_t(j)\},$$

*such that, given the exogenous stochastic processes $z_t$, $\psi_t$, as well as initial conditions $k_0$, $B_0$ and $\{inv_{-1}(j)\}$:*

- *households maximize* (1.1) *subject to* (1.2) - (1.6) *and a no-Ponzi condition,*
- *each firm $j \in [0,1]$ maximizes* (1.7) *subject to* (1.8) - (1.12)*,*
- *markets clear according to* (1.13) - (1.16)*.*

The two exogenous processes in our economy are total factor productivity $z_t$ and the labor wedge $\psi_t$. The news component to these two shocks are the primary contributors to aggregate fluctuations in Schmitt-Grohé and Uribe (2012).[5]

### 1.2.3 The optimal choice of inventories

The full set of equilibrium conditions are provided in appendix A.4. As we show there, a market equilibrium of the stock-elastic demand model is symmetric, so that $a_t(j) = a_t$, $s_t(j) = s_t$, $inv_t(j) = inv_t$, $y_t(j) = y_t$, and $p_t(j) = p_t$ for all $j$. Here, we discuss the optimal stock choice of firms.

In the market equilibrium, marginal cost is the real wage divided by the marginal product of labor:

$$mc_t = \frac{W_t/P_t}{\alpha z_t (k_t/n_t)^{1-\alpha}}. \tag{1.17}$$

Using this, the optimal stock choice of firms is governed by the equation:[6]

$$mc_t = \frac{\partial s_t}{\partial a_t} + \left(1 - \frac{\partial s_t}{\partial a_t}\right) \mathbb{E}_t[q_{t,t+1}(1-\delta_i)mc_{t+1}]. \tag{1.18}$$

The left hand side of this equation represents the cost of adding an extra unit of goods to the stock of goods on sale, $a_t$, which equals the current marginal cost of production. The right hand side represents the two benefits of adding this extra unit. First, by producing and stocking an extra unit, the firm is able generate an additional fraction $(\partial s_t/\partial a_t)$ of sales. Second, since some of these additional stock of goods will not be sold and will be stored as

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[5] Other types of shocks will be discussed in later sections.

[6] Here, $q_{t,t+1} = Q_{t,t+1}P_{t+1}/P_t$ denotes the real stochastic discount factor of the household.

inventories for the next period, future production costs are reduced.

It is important to notice that at the nonstochastic steady state of the economy, the stock of inventories ais positive. Since the real interest rate and the inventory depreciation rate are both positive at the steady state, holding inventories over time is a loss. However, consistent with the first term on the right hand side of (1.18), there is a convenience yield in holding a positive amount of inventories in each period. In the model, the convenience yield is the additional sales created by holding a positive level of stock. Therefore, even with the cost over time, the economy will hold a positive level of inventories at the steady state to maintain their level of sales.

Rearranging, (1.18) can be expressed as:

$$\frac{\partial s_t}{\partial a_t} = \frac{\gamma_t^{-1} - 1}{\mu_t - 1}, \tag{1.19}$$

where:

$$\mu_t \equiv \frac{1}{(1-\delta_i)\mathbb{E}_t[q_{t,t+1} mc_{t+1}]}, \quad \gamma_t \equiv (1-\delta_i)\mathbb{E}_t\left[\frac{q_{t,t+1} mc_{t+1}}{mc_t}\right].$$

The variable $\mu_t$ is the markup of price over expected discounted marginal cost. This is the relevant markup concept in an economy where firms produce to stock: indeed, the true cost of sales is not current but future marginal cost, since selling an extra unit reduces tomorrow's stock of goods. The variable $\gamma_t$ is the expected discounted growth rate of marginal cost, which summarizes the firm's opportunity cost of producing today. The optimal stocking behavior of a firm balances these 3 margins: markup, discounted growth rate of marginal cost, and the benefit of stocking in generating sales.

In equilibrium, the optimal choice of inventories expressed in a first-order log-linear approximation around its steady state is:[7]

$$\widehat{inv}_t = \hat{s}_t + \eta\hat{\gamma}_t,$$

where hatted variables represent log-deviations from its steady-state. This condition states that two factors determine the dynamics of inventories.

First, $\hat{s}_t$ represents the demand channel: firms in this economy build in their inventories when sales are high. For example, when there is an increase in aggregate demand, firms make the most out of it by stocking more goods on shelf to generate additional sales. However, since the additional unit on stock will not lead to a full amount of realized sales, (end-of-period) inventories also increase.

Second, $\eta\hat{\gamma}_t$ represents the intertemporal substitution channel, where $\eta > 0$ is a combination of structural parameters that will be specified in proposition 1.2. Intuitively, $\eta$ represents the degree of intertemporal substitution of production in this economy. For example, when there is an increase in future expected discounted marginal cost relative to current marginal cost, then $\hat{\gamma}_t$ is positive and firms will increase their inventories. This happens because firms realize that it is cheaper to produce today than in the future and they now bunch their production today and store more inventories. When the value of $\eta$ is infinitely large, then the degree of intertemporal substitution is so large that even a small change in the perception of the marginal cost will result in a massive change in inventories.

Hence the optimal decision of inventories in our model depends on the relative strength between the demand channel and the intertemporal substitution channel.

[7]This equation is derived by combining (1.10), (1.19) and the optimal pricing condition $\hat{\mu}_t = 0$.

## 1.3 The impact effect of news shocks

We now turn to studying the effect of news shocks in this model economy. In this section, we focus on impact responses. We derive analytical conditions under which news shocks result in positive comovement on impact between sales and inventories, assess whether those conditions are likely to hold in reasonable calibrations of the model, and inspect the mechanisms underpinning them.

### 1.3.1 A log-linearized framework

We analyze a first-order log-linear approximation of the model around its steady-state. The following framework summarizes the equilibrium conditions needed for the purpose of our analysis on inventories and news shocks.

**Proposition 1.2** (Stock-elastic demand model)**.** *On impact and with only news shocks, so that $\hat{z}_t = 0$ and $\hat{\psi}_t = 0$, the following equations represent the log-linearized market equilibrium of definition 1.1:*

$$\widehat{mc}_t = \omega \hat{y}_t, \tag{1.20}$$

$$\kappa \hat{y}_t = \hat{s}_t + \frac{\kappa - 1}{\delta_i}[\widehat{inv}_t - (1-\delta_i)\widehat{inv}_{t-1}], \tag{1.21}$$

$$\widehat{inv}_t = \hat{s}_t + \tau \hat{\mu}_t + \eta \hat{\gamma}_t, \tag{1.22}$$

$$\hat{\mu}_t = 0, \tag{1.23}$$

$$\hat{\mu}_t + \hat{\gamma}_t + \widehat{mc}_t = 0. \tag{1.24}$$

*The mapping from the structural model parameters to the parameters of the reduced-form* *equations is given by:*

$$\omega = \frac{1+(1-\alpha)\xi}{\alpha\xi}, \tag{1.25}$$

$$\kappa = 1 + \delta_i IS, \tag{1.26}$$

$$\eta = \frac{1+IS}{IS}\frac{1}{1-\beta(1-\delta_i)}, \tag{1.27}$$

$$\tau = \theta\frac{1+IS}{IS},$$

*where $IS$ is the steady-state inventory-sales ratio, given by*

$$IS = \frac{(\theta-1)(1-\beta(1-\delta_i))}{\zeta\beta(1-\delta_i)-(\theta-1)(1-\beta(1-\delta_i))}.$$

Equation (1.20) relates marginal cost to output, which is derived by combining the labor supply and demand conditions, and the production function. Importantly, this equation is not connected to the introduction of inventories in our model. With $\omega > 0$, the equation states that real marginal cost increases with output. The parameter $\omega$ is the elasticity of marginal cost with respect to output, keeping constant total factor productivity. In other words, $\omega$ represents the degree of decreasing returns in the economy due to predetermined capital in the short run represented by $\alpha$ and the disutility of labor supply represented by $\xi$. The value of $\omega$ itself has been at the center of debate in the monetary economics literature, and there is disagreement about its magnitude. In fact, Woodford (2003) contrasts two values of $\omega$: 1.25, from Chari, Kehoe, and McGrattan (2000), and 0.47, from Rotemberg and Woodford (1997). Moreover, Dotsey and King (2006) suggest a lower bound of 0.33 for $\omega$. A conservative lower bound for $\omega$ is thus:

$$\omega \geq 0.3.$$

Equation (1.21) is the law of motion for the stock of inventories, obtained from combining equations (1.9) and (1.10). This law of motion states that output should equal sales plus inventory investment. In its log-linearized form, $\kappa$ in (1.21) denotes the steady-state output to sales ratio. In NIPA, the time series average of inventory investment over output is around 0.5 percent, so that:

$$\kappa = 1.005.$$

Equations (1.22) and (1.23) are the optimal stocking and pricing conditions, respectively. Combining these two equations, we see that inventories are determined by the demand channel ($\hat{s}_t$) and the intertemporal substitution channel ($\eta\hat{\gamma}_t$), as we discussed in section 1.2. Here we focus on the numerical value of $\eta$, the degree of intertemporal substitution in production. Equation (1.27) indicates that a lower bound of $\eta$ is $(1-\beta(1-\delta_i))^{-1}$. The lower bound depends on two parameters $\beta$ and $\delta_i$. First, the household discount factor $\beta$ governs the opportunity cost of holding inventories. In the extreme case where $\beta = 1$, there is no opportunity cost of holding inventories since the real interest rate $1/\beta - 1$ is 0. Second, the depreciation rate of inventories $\delta_i$ represent the physical cost of holding inventories. Therefore, the value $1-\beta(1-\delta_i)$ represents the overall intertemporal cost of adjusting inventories. In the extreme case when both the opportunity cost and the physical cost of inventories are zero, then the lower bound of $\eta$ is infinity. At quarterly frequency, we set $\beta = 0.99$, which is standard. For $\delta_i$, the logistics literature estimates the carrying cost to be around 12–15 percent in annual terms.[8] With a rather high value of $\delta_i = 0.04$, the lower bound is

$$\eta > 20.$$

[8]The overall carrying cost suggested in the literature is on average 25 percent in annual terms (Stock and Lambert, 2001). However, these include interest payments and clerical costs of managing inventories. Excluding those costs gives our numbers.

Lastly, equation (1.24) follows from the definition of $\mu_t$ and $\gamma_t$ in section 1.2.

### 1.3.2 The impact response of inventories to good news about the future

Given sales $\hat{s}_t$, equations (1.20) - (1.24) relate the following four variables: output $\hat{y}_t$, inventories $\hat{inv}_t$, the discounted growth rate of marginal cost $\hat{\gamma}_t$, and markups $\hat{\mu}_t$. We adopt the following definition of a news shock in the context of this reduced-form framework: a news shock has no impact on current fundamentals ($\hat{z}_t = 0$ and $\hat{\psi}_t = 0$), but future fundamentals are expected to change ($\mathbb{E}_t \hat{z}_{t+k} \neq 0$ or $\mathbb{E}_t \hat{\psi}_{t+k} \neq 0$ for some $k > 0$).

**Proposition 1.3** (The impact response of inventories to a good news about the future). *When news arrives, inventories and sales positively comove on impact if and only if:*

$$\eta < \frac{\kappa}{\omega}.$$

This proposition indicates that the positive comovement between inventories and sales depend on three parameters: $\kappa$, $\omega$ and $\eta$. With $\kappa = 1.005$, the two parameters $\omega$ and $\eta$ need to be sufficiently small for positive comovement between inventories and sales. Following our previous discussion on numerical values, a conservative upper bound on $\kappa/\omega$ is 3.3. However, given that our lower bound of $\eta$ with a large carrying cost of inventories is still 20, the condition of proposition 1.3 is not met and in fact fails by an order of magnitude. Thus, our framework indicates that following the arrival of good news about the future, the boom in sales associated to a news shock is accompanied by a fall in inventories. In other words, there is negative comovement between inventories and sales in response to news shocks.

### 1.3.3 Discussion

The numerical discussion of proposition 1.3 concludes that inventories must fall when good news about the future generates a current boom in sales. The two key parameters that drive this result are $\omega$ and $\eta$.

First, when $\omega$ is small, then a sales boom will also correspond to an inventory boom. This is because with a small $\omega$, marginal cost barely responds to changes in production of the firm. Therefore, intertemporal substitution in production is less attractive for firms. In this case, inventories are mostly used to affect demand, and with a sufficient increase in demand, firms will optimally accumulate inventories.

Second, when $\eta$ is small, the intertemporal substitution channel itself becomes weak. This is the case when the firm faces large costs in storing goods for the future. When the interest rate is high or the depreciation of inventories are high, then it is costly for firms to hold inventories. In this economy, even though marginal cost may respond sensitively to production, firms will be less willing to smooth this out by adjusting inventories. Therefore a sufficient increase in demand will also lead to an accumulation of inventories.

To be more precise on this connection between $\eta$ and the cost of storing goods, recall that the lower bound of $\eta$ is negatively related to the intertemporal cost of adjusting inventories, $1-\beta(1-\delta_i)$. In fact, we also find that the value of $\eta$ itself is negatively related with the intertemporal cost. In figure 1.1, we fix the other structural parameters and change the value of $1-\beta(1-\delta_i)$ to show this relation.[9] In the extreme case with zero intertemporal cost of adjusting inventories, we see that the degree of intertemporal substitution, $\eta$, reaches infinity. With higher intertemporal cost imposed, the value of $\eta$ becomes smaller, but far from satisfying the positive comovement condition of proposition 1.3 even for the upper

[9]The value of $\eta$ is a function of $\beta$ and $\delta_i$ only in the form of $1-\beta(1-\delta_i)$. Hence there is no need to consider $\beta$ and $\delta_i$ separately.

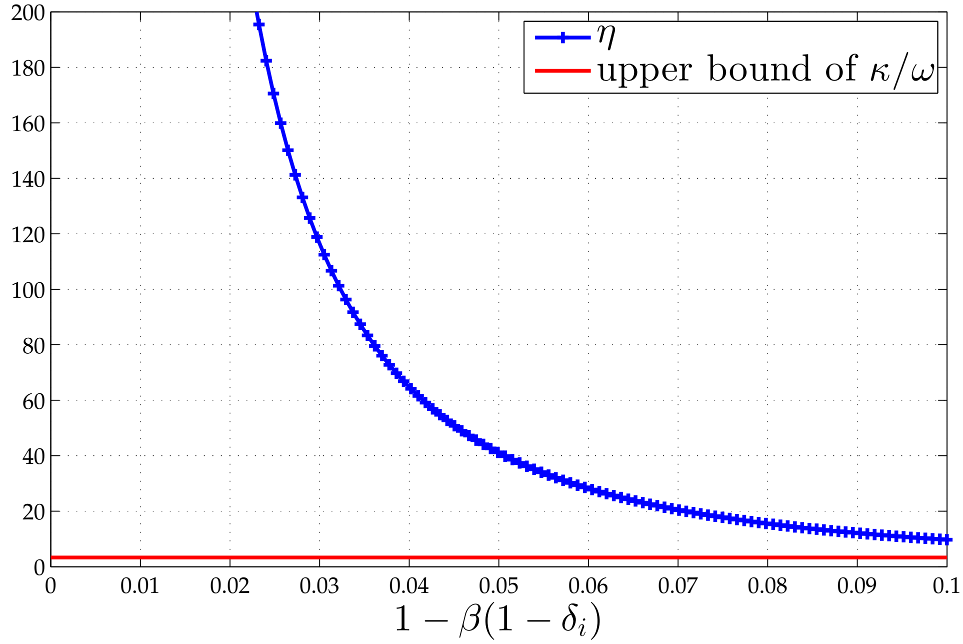


*Figure 1.1: Value of $\eta$ as a function of $1-\beta(1-\delta_i)$. Stock-elastic demand model; holding fixed all the other structural parameters.*

bound of $\kappa/\omega$, which is 3.3 (the horizontal line on the graph).

## 1.4 Dynamic analysis

The analysis of the previous section focused on the impact responses to news shocks, in an effort to understand forces underlying the joint response of inventories and sales. We found that news shocks generate negative comovement between inventories and sales. We now turn to several extensions of this result. We first show that the negative comovement between inventories and sales holds beyond impact, and we study whether allowing variable capacity utilization changes our result. Second, we study inventory behavior with surprise shocks to confirm that the negative comovement property is an identifying feature of news shocks. Third, we study the comovement property with other types of news shocks. Fourth,

| Parameter | Value | Description |
|---|---|---|
| $\beta$ | 0.99 | Subjective discount factor |
| $\sigma$ | 1 | Inverse elasticity of household intertemporal substitution |
| $\delta_k$ | 0.025 | Depreciation rate of capital |
| $\phi''(1)$ | 9.11 | Investment adjustment costs |
| $\xi$ | 2.5 | Frisch elasticity of labor supply |
| $\psi$ | 6.72 | Steady-state hours worked = 0.2 |
| $\alpha$ | 0.67 | Labor elasticity of production function |
| $\theta$ | 5 | Elasticity of substitution across intermediate goods |
| $\delta_i$ | 0.025 | Depreciation rate of inventories |
| $\zeta$ | 0.25 | Steady-state inventory-sales ratio = 0.75 |
| $\rho_z$ | 0.99 | Persistence of the productivity process |
| $\rho_\psi$ | 0.95 | Persistence of the labor wedge process |

*Table 1.1: Calibration of the stock-elastic demand model.*

we check the robustness of our result by introducing different types of adjustment costs.

Since the analysis will be numerical, we start with a brief discussion on the calibration of parameters.

### 1.4.1 Calibration

The numerical values for the parameters are summarized in table 1.1. Standard model parameters are calibrated using estimates from the business-cycle literature. Parameters specific to the inventory blocks of the models are calibrated to match sample averages of the inventory-sales ratio. For the exogenous variables we assume that the realization of these shocks follow AR(1) processes. For the persistence of each shocks, $\rho_z = 0.99$ and $\rho_\psi = 0.95$ are assumed.

Our calibration implies that $\eta = 67.15$, $\omega = 1.09$ and $\kappa = 1.02$, so that applying proposition 1.3, inventories respond negatively to news shocks on impact.

### 1.4.2 Impulse response to news shocks and variable capacity utilization

We first study the impulse responses of output, sales and inventories to 4-period positive news shocks to productivity and labor wedge. That is, at period 0, agents get signals that future productivity ($\mathbb{E}_0 z_4$) will increase or future labor wedge ($\mathbb{E}_0 \psi_4$) will decrease.[10]

Figure 1.2 reports the impulse responses. Note first that consumption and investment, which are components of sales, increase immediately, and throughout the realization of the shock. Consumption increases because of the wealth effect associated and investment increases because of the presence of investment adjustment costs.

In line with our discussion of the previous sections, inventories fall. The fall is large and persistent, and reaches its trough in the period preceding the realization of the shock. At the same time, output remains mostly unchanged until period 4, when the shock realizes. That is, the increase in sales during the news period is almost entirely met by inventory disinvestment. To build further intuition for the responses of inventories, note that labor market clearing implies that:

$$\psi_t n_t^{\frac{1}{\xi}} = \alpha mc_t z_t k_t^{1-\alpha} n_t^{\alpha-1}, \tag{1.28}$$

so that marginal cost is given by:

$$\widehat{mc}_t = \omega \hat{y}_t - \hat{z}_t + \hat{\psi}_t - (\omega+1)(1-\alpha)\hat{k}_t. \tag{1.29}$$

This marginal cost equation tells us that both news about an increase in future productivity

[10]We define a positive news shock as a shock that generates an increase in sales. When the labor wedge is expected to decrease, then households expect to face less disutility of working in the future. This will also boost current sales.

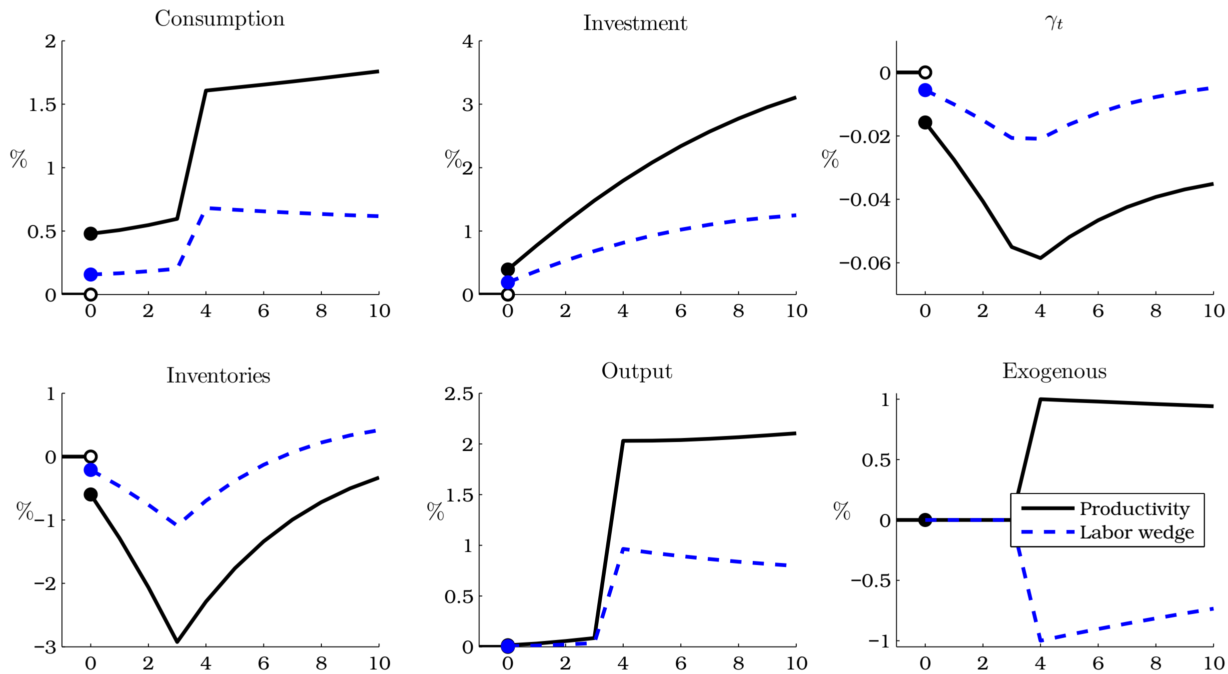


*Figure 1.2: Impulse responses to news shocks in the stock-elastic demand model. Solid line: 4 period news on productivity; dashed line: 4 period news on labor wedge. The time unit is a quarter. Impulse responses are reported in terms of percent deviation from steady-state values.*

and news about a decrease in future labor wedge would contribute to a decline in future marginal cost. In general equilibrium, this downward pressure in the marginal cost profile is reflected in the negative impulse response of the expected discounted marginal cost $\gamma_t$, which we report in the upper right panel of figure 1.2. Since inventories are used to smooth out the difference in marginal cost of production over time, this fall in the expected discounted marginal cost leads to a fall in inventories which is sufficient to overcome the effect of the increase in sales, as we see from equation (1.22).

Note that we are not forcing output to be fixed during the news period and that there still is a small increase in output for the first four periods. Although capital is fixed in the short run, and both productivity and the labor wedge are unchanged during the news period, the labor demand schedule of firms may still shift with changes in marginal cost, as we see from the right hand side of equation (1.28). Indeed, in contrast to models without inventories, the optimal pricing policy of firms does not imply that marginal cost is fixed — instead, it is the expected discounted marginal cost that is constant. Through equation (1.28), the increase in demand is associated to a rise in marginal cost which shifts out the labor demand curve, resulting in a small increase in hours worked. However, since the marginal cost is effectively smoothed out by the strong inventory substitution channel in our economy, the actual movement in marginal cost is small and therefore labor only slightly increases in equilibrium. Therefore the small change in output is an optimal response of the economy with inventories.

To make this point more clear, we allow capacity utilization to vary and see whether our result remains. Denoting $u_t$ as the utilization of capital at period $t$, the production function and the captial accumulation function are modified respectively as follows:

$$\begin{aligned} y_t &= z_t(u_t k_t)^{1-\alpha} n_t^{\alpha}, \\ k_{t+1} &= (1-\delta(u_t))k_t + \left[1 - \phi\left(\frac{i_t}{i_{t-1}}\right)\right] i_t, \end{aligned}$$

where $\delta'(\cdot) > 0$ and $\delta''(\cdot) > 0$. In words, higher utilization of capital increases output, but this comes at a cost of higher rate of depreciation of capital. In a model without inventories, such as Jaimovich and Rebelo (2009), capacity utilization increases with news about a future rise in productivity. This is because with a future rise in productivity, the presence of investment adjustment costs leads to an increase in capital investment today. The increase in capital investment generates a fall in the value of installed capital. At the same time, the positive income effect from the household generates a fall in the marginal value of income due to the concavity of the utility function. Overall, the fall in the value of installed capital is steeper than the fall in the marginal value of income, and therefore capacity is utilized more to

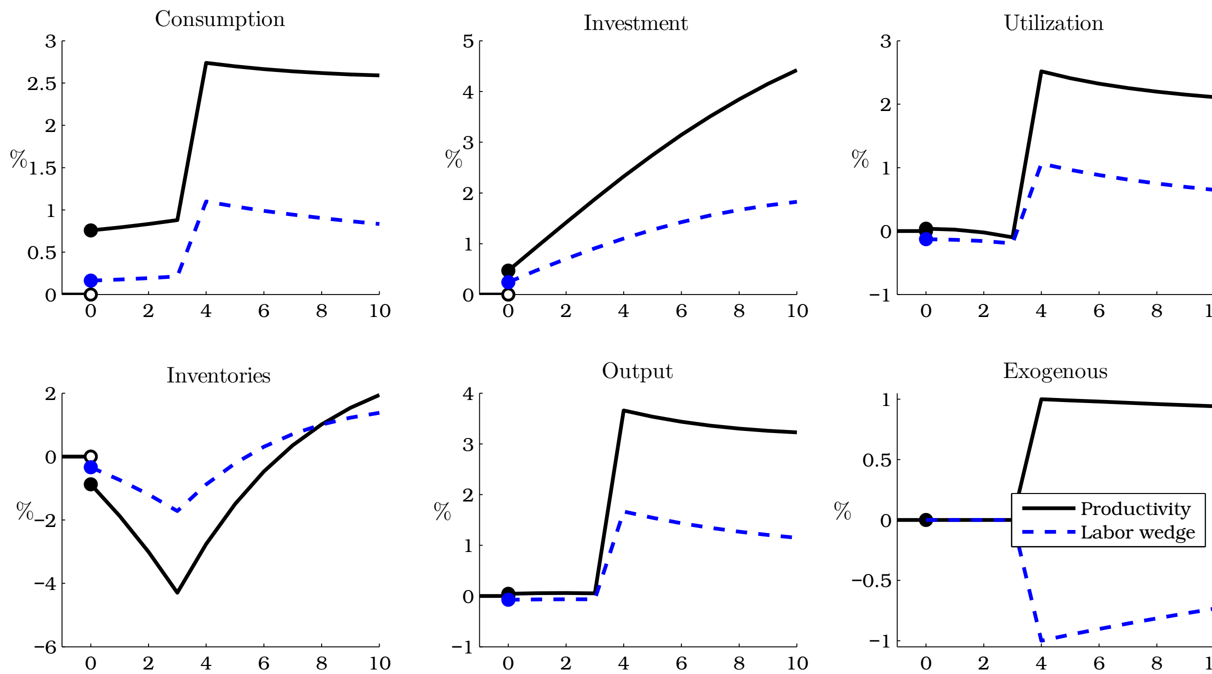


*Figure 1.3: Impulse responses to news shocks in the stock-elastic demand model with variable capacity utilization. Utilization parameter: $\delta_k''(1) = 0.34$; solid line: 4 period news on productivity; dashed line: 4 period news on labor wedge. The time unit is a quarter. Impulse responses are reported in terms of percent deviation from steady-state values.*

satisfy the additional demand.

In figure 1.3, we plot the impulse responses for the inventory model with variable capacity utilization. As we see, the quantitative response of capacity utilization during the news period is modest. Utilization significantly increases only after the shock realizes.

The small response of capacity utilization during the news period comes directly from the household preference and the role of inventories in the economy. The marginal value of income $\lambda$ in our model with GHH preference is the following:

$$\lambda = \left( c - \psi \frac{n^{1+\xi^{-1}}}{1+\xi^{-1}} \right)^{-\sigma}.$$

With inventories, the increase in consumption and investment can be matched by depleting inventories rather than working more. Therefore, $n$ does not go up with an increase in $c$, which generates a steeper fall in the marginal value of income. Hence even with capacity utilization, the economy does not ask for more production at the expense of depreciating installed capital, since the lack of adjustment of labor means that utility is already high. Again, we confirm that our negative comovement between inventories and sales in response to news shocks is an equilibrium outcome even when we allow for production to increase in the short-run.

### 1.4.3 Do surprise shocks generate positive comovement?

While news shocks generate a persistent negative comovement between inventories and sales, one may wonder whether this also occurs after surprise innovations to fundamentals. The impulse responses reported in figure 1.4 show that this is not the case. Inventories, consumption, investment and output all increase in response to surprise innovations to productivity and the labor wedge. The short-run response of the inventory-sales ratio is also consistent with its observed countercyclicality at business-cycle frequencies, in line with the findings of Khan and Thomas (2007a) and Wen (2011).[11] The model prediction is thus broadly consistent with the observed behavior of inventories and sales over the business cycle. Thus, the negative comovement of inventories and sales is an identifying feature of news shocks to fundamentals.

[11] The countercyclicality of the inventory-sales ratio is not completely robust to the calibration of the shock, as it depends partly on the magnitude of the initial increase in sales. For a smaller persistence of productivity shocks of $\rho_z = 0.8$, for example, the response of sales is more muted, and the IS ratio becomes procyclical. This behavior of the inventory-sales ratio has motivated Kryvtsov and Midrigan (2013) to investigate the ability of countercyclical markup movements to mute inventory increases in response to demand-side shocks, since in the data, the inventory-sales ratio is countercyclical. However, in response to both productivity and demand shocks, the procyclicality of inventories holds regardless of the values of the persistence parameters $\rho_z$ and $\rho_\psi$.

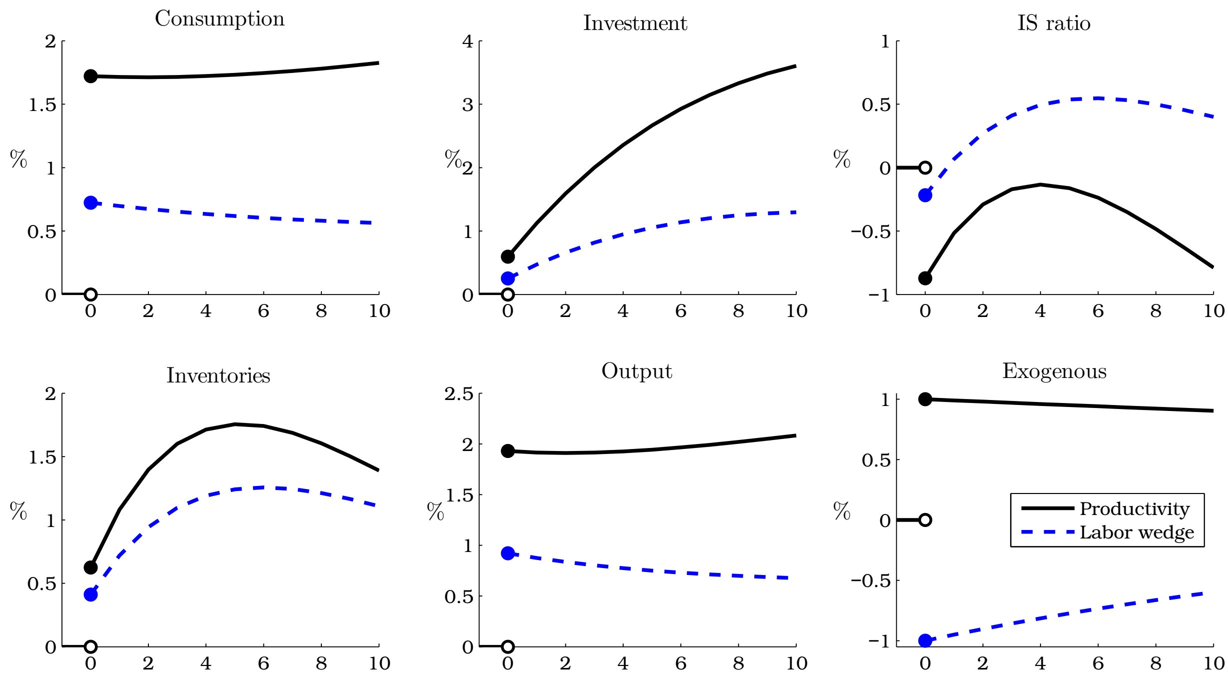


*Figure 1.4: Impulse responses to surprise shocks in the stock-elastic demand model. Solid line: productivity; dashed line: labor wedge. The time unit is a quarter. Impulse responses are reported in terms of percent deviation from steady-state values.*

### 1.4.4 Other types of news shocks

Although the two types of shocks we have considered up to now have been found to be significant sources of news in the literature (Schmitt-Grohé and Uribe, 2012), we do not need to limit our result to these shocks. In fact proposition, 1.3 implies that the negative comovement holds for any type of news shocks, since on impact, all news shocks share the feature that no fundamentals change.

In this section, we consider two other types of news shocks: discount factor shocks and government spending shocks. First, consider a news shock to the discount factor. When the discount factor is expected to increase in the future, then households expect that in the future they will consume more and save less. Then they will consume less today since they now discount the future less. Moreover since savings and hence investment will decrease in the future, with investment adjustment costs, investment will also start decreasing today. Therefore, news about an increase in the future discount factor generates a fall in sales. At the same time, the fall in investment leads to an decrease in future capital, which generates an increase in the future marginal cost. Therefore, inventories will increase, confirming that the negative comovement property holds with this type of news shock.

Second, when there is a future increase in government spending, then inventories will increase to build up for the demand from government spending, since marginal cost is expected to rise in the future with the additional demand from the government. At the same time, since the households in the end take the burden of this spending, consumption and investment falls. Again, there is negative comovement between inventories and sales with this type of news shock as well.

Figure 1.5 shows the impulse responses to these two shocks.[12] As discussed, the negative comovement property is also holds, and even after impact.

### 1.4.5 Adding adjustment costs

Adding adjustment costs to capital investment has been a key element for generating an investment boom with news shocks (Jaimovich and Rebelo, 2009). Capital is slow to adjust, and with this form of adjustment cost, investment decisions depend solely on the discounted sum of future marginal values of capital, or future Tobin's Q. News shocks affect the marginal productivity of future capital, and thus raise future Tobin's Q, which directly translates into an increase in current investment.

The rationale for imposing adjustment on inventory investment is less clear, in particular in the case of finished-good inventories. First, whereas building a factory or machinery

[12]The persistence of each process are 0.17 for the discount factor and 0.95 for the spending. These values come from Schmitt-Grohé and Uribe (2012).

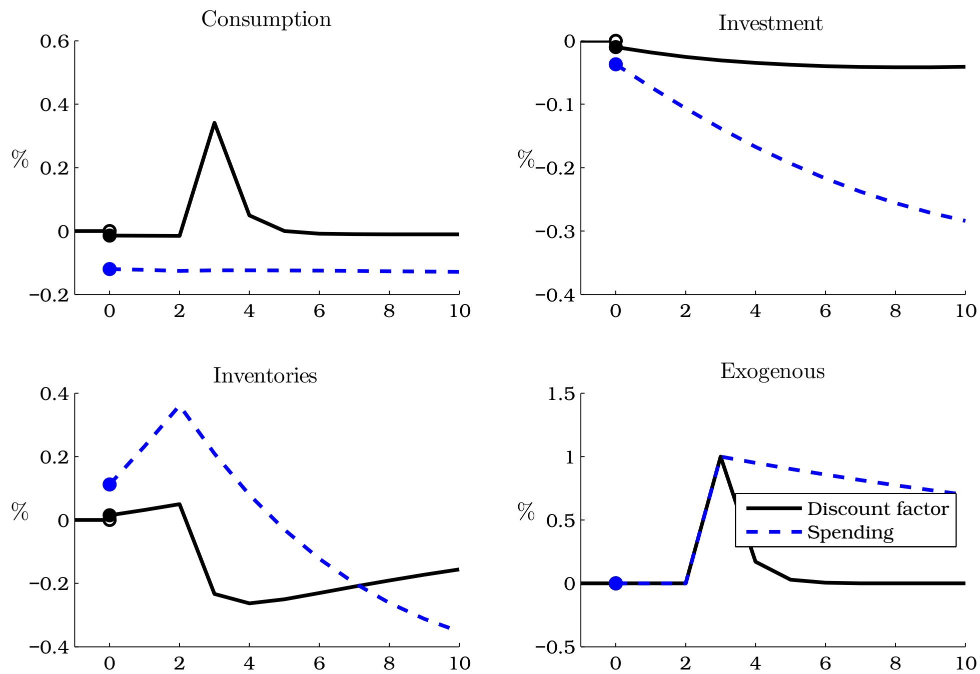


*Figure 1.5: Impulse responses to other news shocks in the stock-elastic demand model. Solid line: 4 period discount factor; dashed line: 4 period government spending. The time unit is a quarter. Impulse responses are reported in terms of percent deviation from steady-state values.*

takes time and hence requires adjustment periods, stocking or depleting an already existing product should be the most flexible adjustment that firms can take. Second, as we discussed in the previous sections, it is not the level, but the growth rate of marginal cost that is important for finished-good inventory investment decisions. Therefore, adding adjustment costs to finished-good inventory investment is a less appealing approach.

However, adjustment costs to the stock of inventories may have a better justification: the total stock of inventories do seem large and slowly moving. Moreover, our intuition tells us that with a positive news shock, we need additional channels for production to increase and adjustment costs may be helpful in that respect. We consider three possible types of adjustment costs: adjustment costs to inventories, output and on-shelf goods. Adjustment cost to inventories penalizes immediate inventory depletion and thus weakens the intertemporal substitution motive. Adjustment cost to output force firms to smooth out the response of output to the shock, and in turn reduce the incentive to deplete inventories to satisfy sales. Finally, adjustment cost to goods on shelf are the sum of output and past inventories. Making adjustment costs bear on this variable might have effects that combine both types of adjustment costs described above.

These adjustment costs are introduced by assuming that the law of motion for inventories are modified as follows:

$$inv_t = (1 - \delta_i)inv_{t-1} + y_t - s_t - ADJ_t,$$

where $ADJ_t$ is the adjustment cost of each type. We assume the following form:

$$ADJ_t = \phi_x \left( \frac{x_t}{x_{t-1}} \right) x_t, \qquad x \in \{inv, y, a\},$$

where $\phi_x(1) = \phi_x'(1) = 0$ and $\phi_x''(1) > 0$. In figure 1.6, we show the responses of the model with and without adjustment costs. We experiment with different levels of adjustment costs, and for all values, we observe that the initial fall in inventories are smaller in both models with adjustment costs, but not close to being positive. We conclude that adjustment costs to inventories and output are not sufficient to generate a procyclical response of inventories.

The logic behind this result is that with adjustment costs to inventories or production, firms are now more willing to smoothly adjust their stock of inventories, and hence produce more today when there is good news. However, to make this happen, wages must increase to induce households to work more. With an increase in wages, households have more income, and consumers will increase their current consumption level not only to compensate for the current loss of utility by working more, but also to increase their level of utility with their higher income.

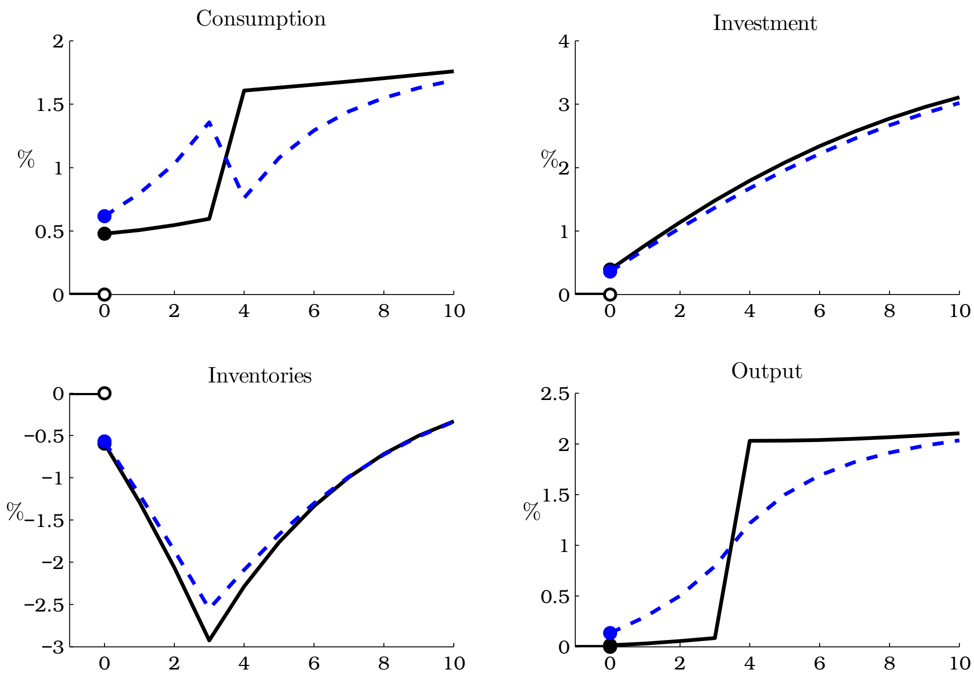


*Figure 1.6: Robustness of impulse responses to output adjustment cost. Impulse responses to 4 period productivity news shock. Solid line: without output adjustment cost; dashed line: with output adjustment cost. The time unit is a quarter. Impulse responses are reported in terms of percent deviation from steady-state values.*

## 1.5 Robustness: Other inventory models

A natural question is whether our result is specific to the particular inventory model we have chosen to analyze. In this section, we discuss other models that illustrate alternative margins of inventory adjustment discussed in the business-cycle literature. In the leading business-cycle models, inventories are introduced either as buffers to uncertainties in demand at the firm level (stockout-avoidance models), or as economies of scale due to nonconvex delivery costs at the firm level (Ss inventory models). We will focus more on the first approach since they fit better for finished-good inventories (Khan and Thomas, 2007a). Nevertheless, we also discuss the second approach for completeness.

A summary of the discussion is that our result remains for all other models as well. This is because one important role for inventories in all of these models is the intertemporal substitution channel. With inventories, producers are allowed to flexibly change their production schedule based on their perception on the marginal cost profile. Since news shocks directly affect this perception, the other margins which differ across models matter less, in particular close to the moment when the news shock is expected to realize in the next period.

### 1.5.1 Stockout-avoidance model

One branch of the literature on finished-good inventories motivates inventories by introducing a lag between production and the realization of sales. In these so-called "stockout avoidance" model assume that there is a lag between production and sales, and that firms have imperfect information on the demand schedule for their variety at the time they make production decisions. When realized demand for their product is unusually high, firms may run out of available product — a "stockout" — and lose potential sales. This motivates firms to put, on average, more on-shelf goods than they expect to sell, and carry over excess goods as inventory into the next period.[13]

In appendix A.3, we study the effects of news shocks in this class of models in detail. We show that a reduced-form framework similar to that of proposition 1.2 obtains, and moreover that our main result carries through: in response to good news about the future, under standard calibrations of the model, sales increase while inventories fall. This follows from obtaining analytical restrictions on reduced-form parameters to precisely quantify the conditions under which this result holds. Additionally, we argue that, as in the stock-elastic

[13]This mechanism is consistent with existing evidence that stockouts occur relatively frequently at the firm level. Bils (2004) uses data from the BLS survey underlying the CPI and estimates that stockout probabilities in this dataset are roughly 5 percent. More recently, using supermarket-level data for a large retailer, Matsa (2011) suggests that stockout probabilities are in the range of $5-10$ percent. See Kahn (1987, 1992), Kryvtsov and Midrigan (2010, 2013), and Wen (2011) for detailed analysis of the properties of this class of models.

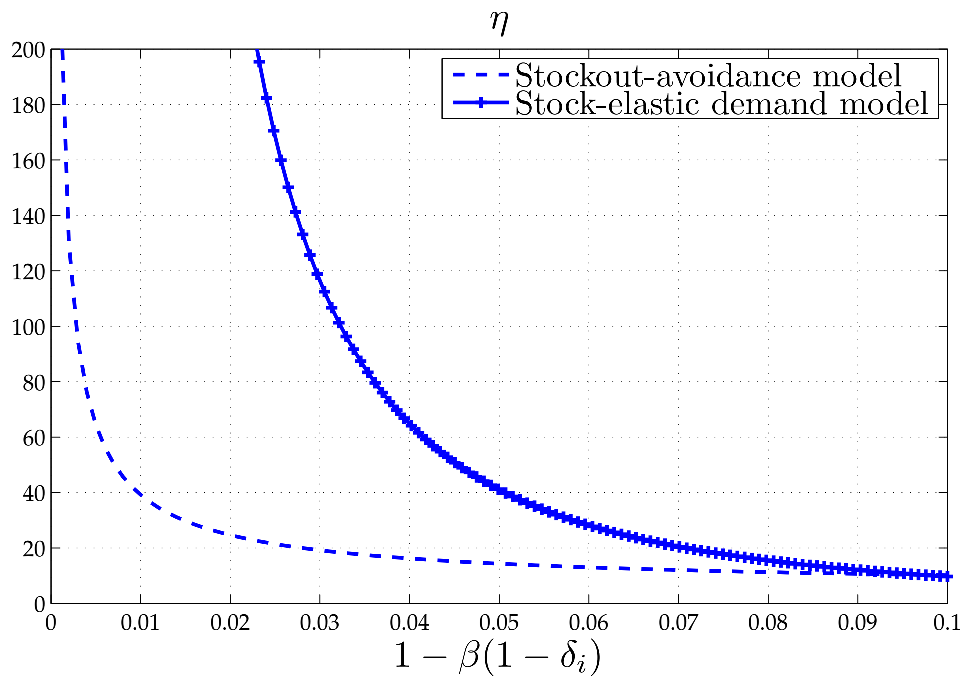


*Figure 1.7: Value of $\eta$ as a function of $1-\beta(1-\delta_i)$. Stockout-avoidance model. Holding fixed all the other structural parameters. For comparison, $\eta$ for the stock-elastic demand model, same as figure 1.1, is also plotted.*

demand model, the main mechanism dominating the response of inventories to news shocks is intertemporal substitution in production. In figure 1.7, we plot the value of $\eta$, the degree of intertemporal substitution, as a function of the intertemporal cost. Again, we see that even with large intertemporal cost, the degree of intertemporal substitution is strong.

The similarity of the two classes of models comes from the fact that the optimal stocking condition (1.18) also holds in the stockout-avoidance model. The cost of stocking is the marginal cost. The benefits of stocking are twofold: (i) in case sales turn out to be higher than expected, the firm can increase its sales by producing additional product; (ii) in case sales turn out lower than expected, the firm can lower future production costs by stocking output as inventories. It turns out that even in this class of models, the intertemporal substitution motive is quantitatively stronger for news shocks.

### 1.5.2 (S,s) inventory model

The existence of nonconvex delivery costs at the firm level has also been claimed as an important reason for the presence of inventories, albeit mainly input inventories, as opposed to finished-good inventories on which our analysis has so far focused. In the model of Khan and Thomas (2007b), the firm pays a fixed cost when placing an order for inputs. This cost comes is randomly distribute across firms, and generates a steady-state distribution of firms with different levels of inventories. In this model, the optimal stocking condition for stock-adjusting firms also balances the cost and benefit of ordering goods, along the lines of the discussion of equation (1.18). To be precise, the cost of stocking is the total cost of goods and a fixed delivery cost. The benefits of stocking are twofold: (i) in case sales turn out to be higher than expected, firms will not order at that time. The total production capacity of the firm is then constrained by the amount of input inventories it holds. Hence, more input inventories allow the firm to produce more goods when demand is high but delivery cost becomes too high. (ii) In case sales turn out lower than expected, firms can order at that time as well. In this case, the firm will reduce its total cost if it expects that the unit cost of goods will be high in the future.

In response to news about an increase in future productivity, firms understand that future demand will increase. At the same time, they understand that future unit costs of input inventories are also cheaper. We solved for the perfect foresight transition dynamics with a news shock to productivity in Khan and Thomas (2007b).[14] Along this path, as in the case of the other models, inventories fall in the short-run, while sales increase.

Overll, these alternative models also have the feature that, in response to good news about the future, inventories fall while sales increase, especially right before the realization

[14] Refer to Khan and Thomas (2007b) for the solution algorithm.

of the shock. Therefore, we conclude that the strong intertemporal substitution channel with news shocks is a common feature of all inventories models.

## 1.6 Estimating the importance of news shocks I: SVAR approach

Our analysis of inventory models suggests that the negative comovement of inventories and sales is a defining feature of news shocks. Indeed, as we have discussed at length, it holds for all plausible calibrations of the models. In this section, we use this structural restriction to estimate the importance of news shocks.

The approach we take in this section is estimating a structural VAR with sign restrictions. Since the robust prediction of our theoretical analysis is that news shocks generate negative comovement between inventories and sales, we will use this prediction directly to estimate the explanatory power of news shocks. The appealing aspect of our sign restriction VAR approach is that we can remain agnostic as to other identifying features, and therefore robustly identify shocks without misspecification concerns. On the other hand, a drawback of this approach is that identification of shocks is weaker, in the sense that we may be capturing non-news shocks that could also drive negative comovement between inventories and sales.

### 1.6.1 Data

We use four observables in our exercise: inventories, consumption, investment and output. Consumption includes nondurables and services, investment includes fixed investment and durables, and output is GDP. For inventories, we used nonfarm private inventories as a whole, or only retail trade inventories to focus on finished-good inventories. However, our results are not sensitive to the type of inventories used for estimation. Therefore, in this section, we present results for nonfarm private inventories. All data are seasonally adjusted, and expressed in real per capita terms. Our sample period is 1955Q1–2006Q4.[15]

### 1.6.2 Baseline specification and estimation

Our baseline identification strategy imposes that on impact, there is disinvestment in inventories, whereas consumption and fixed investment increase.[16] The VAR model we estimate is the following:

$$X_t = A + B(L)X_{t-1} + U_t.$$

For $X_t$, we use log levels of each variable, in order for the estimation to be robust to the presence of cointegrating relations. We estimate the model with a constant term and four lags.[17] We use Bayesian methods, with a diffuse prior for both the coefficients of the autoregressive structure and the variance-covariance matrix of the error terms. Each draw from the posterior identifies a set of possible impulse responses satisfying our impact restriction, and we use a uniform conditional prior on the identified set to draw from the posterior of the impulse responses, following Moon, Schorfheide, and Granziera (2013). Using 20000 draws, the posterior distribution of the forecast error variance (FEV) of output accounted for by

[15] The source of the data is NIPA table 1.1.5 and 5.7.6.

[16] On impact, a fall in inventories is equivalent to a fall in inventory investment, since the impulse response is from the steady state. The joint restriction on consumption and investment is not restrictive since in the data, the two series are highly correlated.

[17] The Schwartz information criterion suggests two lags but our results are not sensitive to the number of lags.

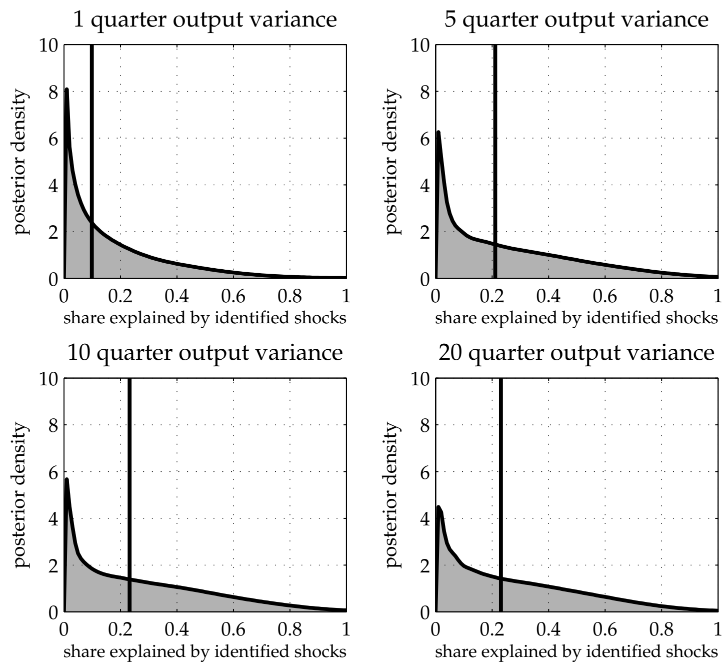


*Figure 1.8: Output variation accounted for by identified shocks with impact restriction. Posterior probability density and the median (vertical line) for the share of forecast error variance at each horizon.*

these identified shocks is computed.[18]

### 1.6.3 Baseline result

Figure 1.8 reports the posterior distribution of the FEV of our identified shocks on output, for different horizons.[19] The posterior has a sharp mode close to zero, and the median is close to 20 percent in most horizons. In figure 1.9, we plot the set of identified impulse responses. The median response to our identified shock generates a persistent boom in consumption and investment, and a moderate boom in output. The fall in inventories is short lived; on average, inventory investment occurs immediately after the initial disinvestment, and the

[18]Our result to follow is not sensitive to adding more draws.

[19]As noted above, we plot the case for nonfarm private inventories but the plot is similar with retail trade inventories as well.

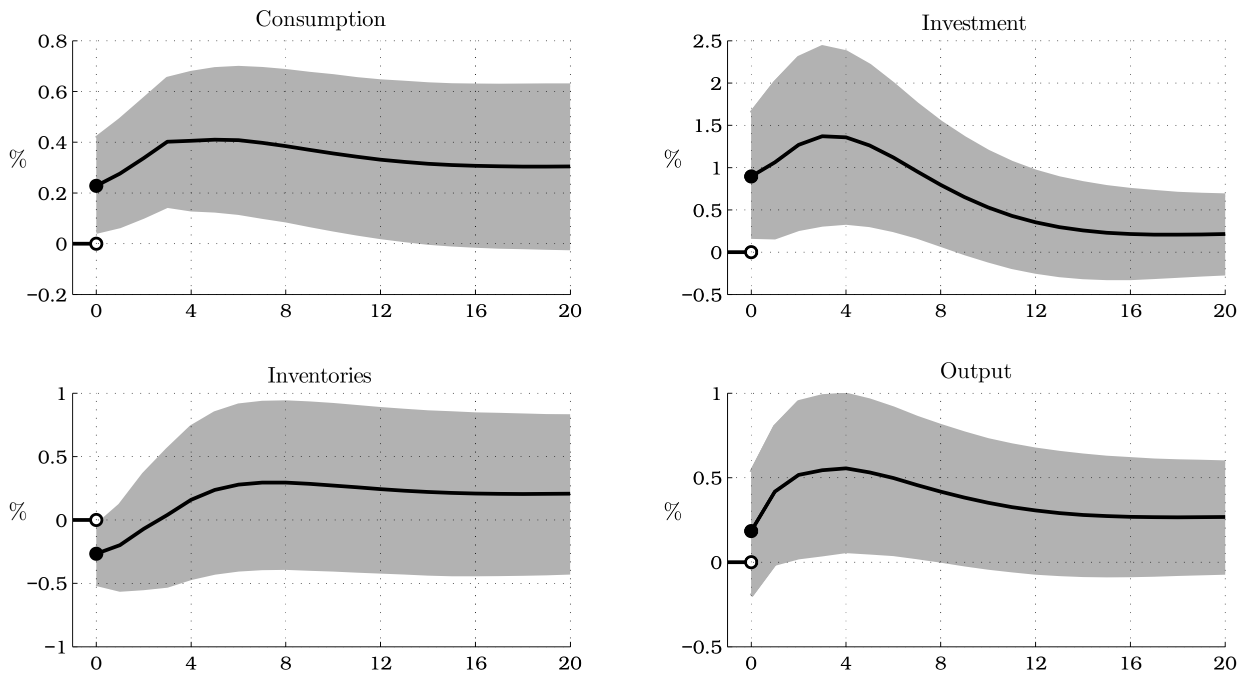


*Figure 1.9: Impulse responses of identified shock with impact restriction. Median (solid line) and* 80% *credible set.*

stock inventories become positive after 3 quarters. Note that in our model, this is also the case when good news is expected to materialize in the near future. Therefore, our identified shock resembles short-horizon news, with news lasting for only 1 period.

Our identification strategy only imposes impact restrictions, and therefore we are not able to distinguish among short and long-horizon news shocks. Since the focus of the news literature is not on one or two quarter news shocks, but rather longer horizons, our next step is to impose restrictions beyond impact.

### 1.6.4 Extension: Dynamic restriction

An immediate extension from our identification strategy is imposing that inventory investment falls for two periods whereas consumption and investment increase for two periods. In

| Parameter | Distribution | Median | 95% | 5% |
|---|---|---|---|---|
| $\delta_i$ | Beta | 0.01 | 0.05 | 0.00 |
| $\tau(\zeta)$ | Gamma | 0.7 | 3 | 0.05 |
| $\phi_y$ | Gamma | 3 | 6 | 0.60 |

*Table 1.2: Distribution for the inventory parameters. The parameter $\tau$ is the steady-state inventory-sales ratio.*

doing so, we claim that short-horizon news shocks are excluded from our identification and hence we will be able to focus on long-horizon news shocks.

To verify this claim, we test our identification strategy by simulated data from an estimated medium-scale DSGE model.[20] In particular, we add inventories to the model estimated in Schmitt-Grohé and Uribe (2012), and simulate the impulse responses for different horizons of news shocks. For each horizon of the news shocks, we test whether our identification strategy is satisfied or not. We take a probabilistic approach since the newly introduced parameters related to inventories are not estimated in the model.

In table 1.2, we specify the distribution of the three new parameters. These are $\delta_i$, the depreciation rate of inventories, $\zeta$, the elasticity of sales to stock of goods, and $\phi_y$, the output adjustment cost. For $\zeta$, specifying a distribution directly on this parameter is difficult since the value has a theoretical lower bound at

$$\underline{\zeta} = \frac{1-\beta(1-\delta_i)}{\beta(1-\delta_i)}(\theta - 1),$$

so that the lower bound changes with different draws of $\delta_i$. Rather than directly forming a distribution on $\zeta$, we specify a distribution of the transformed parameter $\tau = (\zeta - \underline{\zeta})/\underline{\zeta}$, which is the steady-state inventory-sales ratio.

In table 1.3, we show the success probability of our identification approach with different

[20]This part may be skipped if the reader finds the claim to be straightforward.

| 2 period restriction | 4Q news | 2Q news | surprise |
|---|---|---|---|
| productivity | 1 | 1 | 0 |
| labor wedge | 1 | 1 | 0 |
| discount factor | 1 | 1 | 0 |
| spending | 1 | 1 | 0 |
| 3 period restriction | 4Q news | 2Q news | surprise |
| productivity | 1 | 0 | 0 |
| labor wedge | 1 | 0 | 0 |
| discount factor | 1 | 0 | 0 |
| spending | 1 | 0 | 0 |

*Table 1.3: Success probability of identifying assumption.*

horizons of news shocks.[21] For the shocks we consider, our dynamic restriction is successful in identifying longer horizon news shocks.

### 1.6.5 Estimation result and discussion

Figure 1.10 reports the posterior distribution of the FEV of output attributable to our identified shocks, when the followin restriction holds: inventory disinvestment occurs for 2 periods, and at the same time both consumption and investment increase for 2 periods. We see that the posterior has a sharp mode close to zero, and the median is now close to 10 percent at all horizons, about half smaller than the result with impact restrictions only. To get a sense of the information that inventories deliver, figure 1.10 also plots the posterior distribution of the FEV when only consumption and investment are above the steady state for 2 periods. As we see, without the inventory restriction, the distribution is disperse and the median share of FEV for the set of shocks that drive positive comovement of consumption and investment is 30 percent overall. Hence with inventories, the posterior density becomes

[21]We focus on the stationary shocks in Schmitt-Grohé and Uribe (2012). We exclude investment specific shock since with our model has two types of investment, and the meaning of this shock is less clear. For example, one important change in productivity specific to inventory investment is the introduction of just-in-time technology, which should also have an effect on the accumulation of capital goods.

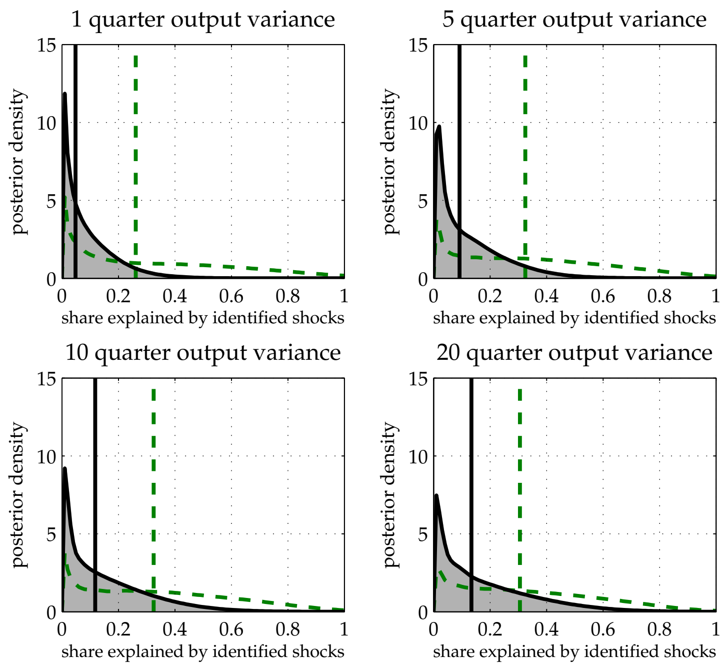


*Figure 1.10: Output variation accounted for by identified shocks with 2 period restriction. Posterior probability density and the median (vertical line) for the share of forecast error variance at each horizon. Solid line: 2 period negative comovement between $\Delta inv_t$ and $(c_t, i_t)$. Dashed line: 2 period positive comovement between $c_t$ and $i_t$.*

much tighter, and the median share of the shock falls by about 67 percent.

Figure 1.11 reports the impulse responses to the identified shock with 2 period restrictions. Inventory disinvestment occurs for 2 periods, but after that, there is again investment in inventories. Consumption and investment increases, but the increase in output is now modest.

We also extend our dynamic restriction to 3 periods, that is 3 periods of inventory disinvestment accompanied by increases in consumption and fixed investment. As in figure 1.12, the median share of FEV explained by the identified shock is now below 5 percent in most horizons, and tight with basically no probability assigned above 20 percent. Therefore, our news shocks identified with 3 period restrictions *at most* account for 20 percent of output variations. Figure 1.13 reports the impulse responses of the identified shock with 3 period

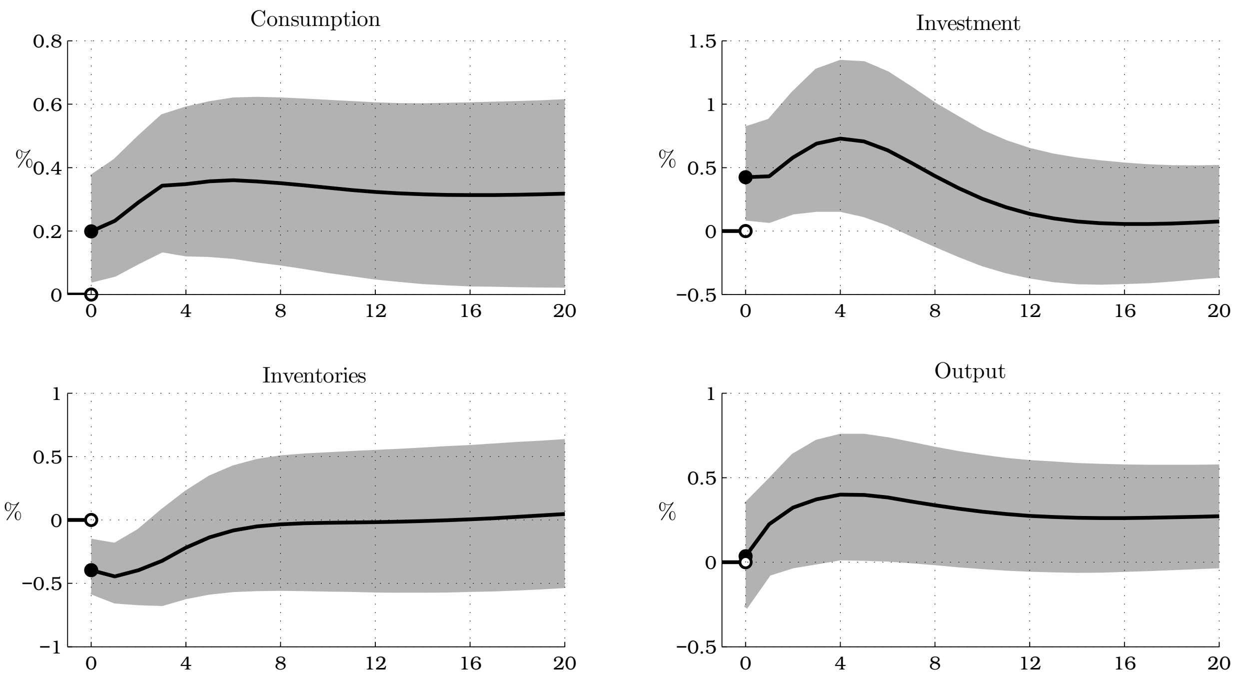


*Figure 1.11: Impulse responses of identified shock with 2 period restriction. Median (solid line) and* 80% *credible set.*

restrictions. Although the movement in output is modest, it actually declines on impact.

We summarize the key points of our empirical results as follows: (i) the identified impulse response with impact restrictions suggest that most news shocks are short-lived, with an immediate investment in inventories after the impact disinvestment; (ii) the identified news shock based on impact restrictions explain on average 20 percent of output variations at all horizons; (iii) restrictions beyond impact generate a tighter posterior distribution of output variations; (iv) long-horizon news shocks explain on average 5 percent, and at most 20 percent of output variations at all horizons.

The reason why the FEV turns out to be small is that inventories are a procyclical variable. In the data, the unconditional contemporaneous correlation between inventories

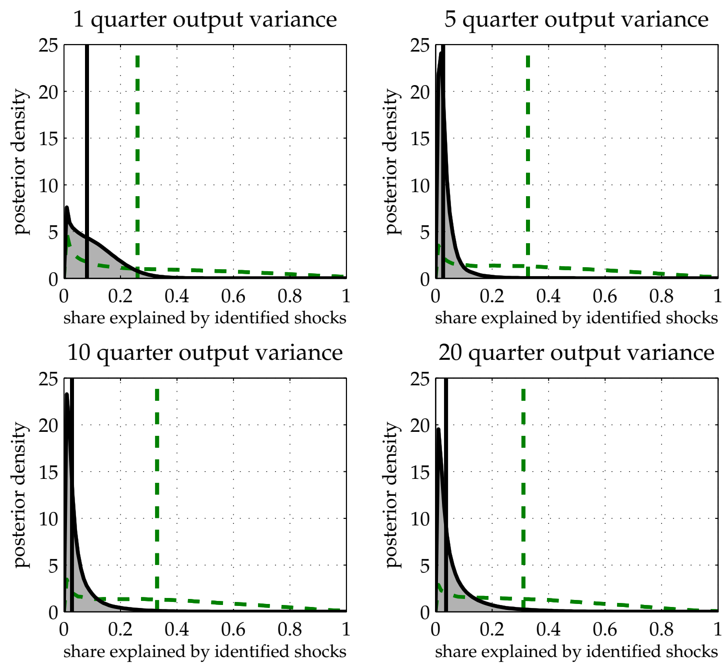


*Figure 1.12: Output variation accounted for by identified shocks with 3 period restriction. Posterior probability density and the median (vertical line) for the share of forecast error variance at each horizon. Solid line: 3 period negative comovement between $\Delta inv_t$ and $(c_t, i_t)$. Dashed line: 3 period positive comovement between $c_t$ and $i_t$.*

and sales (consumption plut investment) is 0.50.[22] Since our identification is based on negative comovement of these generally positively comoving variables, it would come as a surprise if the identified shocks were able to generate the bulk of business cycles.

### 1.6.6 Robustness

Since our identifying assumption only relates to the sign of the responses of inventories and components of sales, it is robust to changes in specification. We have nevertheless performed several robustness checks. First, we used different priors for the coefficients, such as the Minnesota prior or the Normal-Wishart prior. None of these priors alter significantly the

[22]This is based on HP filtered data but the result is not sensitive to filtering methods.

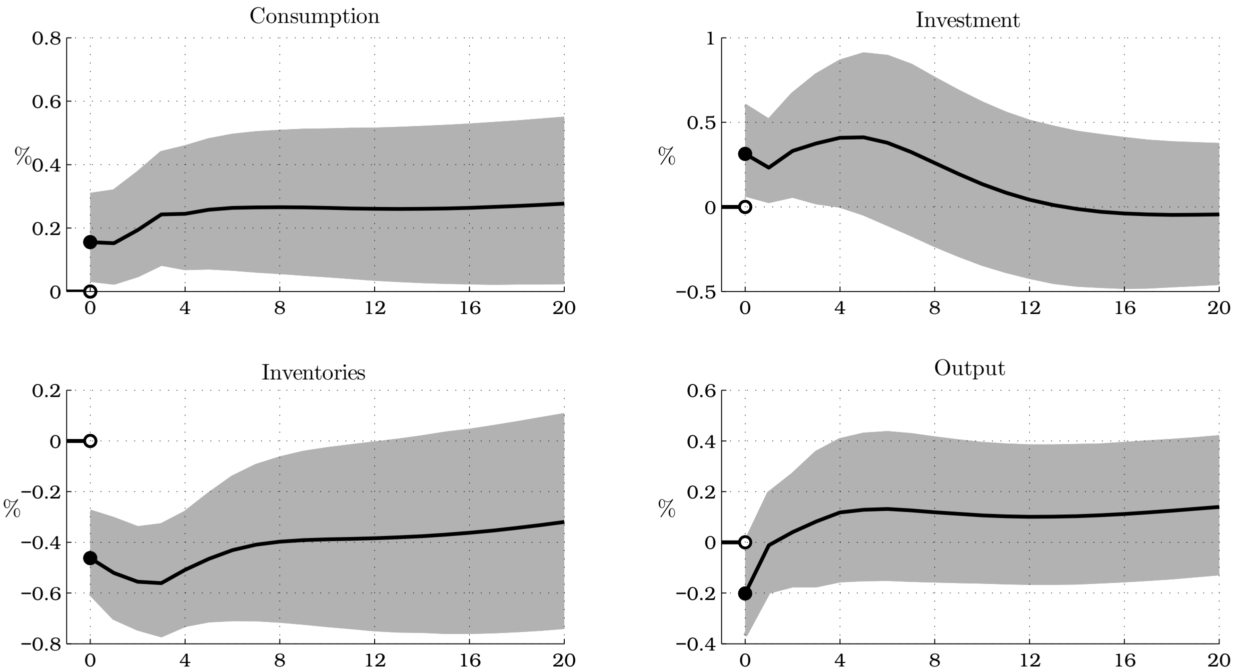


*Figure 1.13: Impulse responses of identified shock with 3 period restriction. Median (solid line) and* 80% *credible set.*

results.[23] Second, when imposing our dynamic restriction, we also tried to be less restrictive by not imposing the negative comovement on impact or second period, in order to control for any demand effects that may remain in the short run with long-horizon news shocks. The result is not sensitive to this change since the stock of inventories move in a persistent manner. For example, by imposing that inventories are below average only in the third period, it mostly follows that inventories are below average for the first and second period as well. Third, as we mentioned above, our result is not sensitive to using different types of inventory data. Fourth, as studied in detail by McCarthy and Zakrajšek (2007), inventory

[23]Since our focus is mainly on FEV decomposition, it might be more desirable to set a uniform prior on this moment. However, forecast error variance is a highly nonlinear transformation of the VAR coefficients, and existing methodologies do not allow us to easily adress this "inverse" problem. As a way to overcome this issue, we are showing our result with and without the negative comovement assumption in order to control for the role of the prior.

dynamics have changed since the 1980s: while the procyclicality of inventories remains, the volatility of total inventory investment has fallen, possibly because of improvements in inventory management, contributing to the fall in output volatility. To address this issue, we take into account the possibility of different "inventory regimes" in the data by creating two separate samples, before and after 1984, and conduct our empirical exercise on each of the sub-samples. Our results are not sensitive to these various sample splits. This suggests that the nature of the comovement between inventories and private sales did not change substantially around this period.

### 1.6.7 Other VAR approaches

Existing methods of identifying news shocks in a VAR setup have typically used data on productivity (Barsky and Sims, 2011), sometimes combining them with data on stock prices (Beaudry and Portier, 2006; Beaudry and Lucke, 2010). Our new piece of information could also be incorporated into these existing approaches. For example, one standard approach in identifying news shocks is to look at movements in stock prices orthogonal to any changes in current productivity. To include the movements of inventories in this estimation strategy, we ran a 3 variable VAR with utilization-adjusted productivity, the S&P 500 index as stock prices, and inventories. We imposed impact zero restriction on productivity, and traced out the dynamics of inventories followin an increase in stock prices. We found that the sign of the impact and short-run responses of inventories are inconclusive.[24] This suggests that existing methods are not fully incorporating the information inventories provide in response to news shocks.[25] This is linked to the fact that the existing literature provides a wide range

[24]This plot is in the appendix.

[25]A similar point is made in Arias, Rubio-Ramirez, and Waggoner (2013) with regards to the penalty function approach in Beaudry, Nam, and Wang (2011). Our information could add to this debate as well.

of numbers for the contribution of news shocks to output volatility. For instance, while Beaudry and Portier (2006), Beaudry and Lucke (2010) all find that news shocks contribute to 50–60 percent of output variation, a similar approach by Barsky and Sims (2011) finds that news shocks only contribute to 10 percent of output volatility in the short run (1–4 quarters), and about 40 percent in the long run. Our finding is closer to the latter approach, although we find that news shocks should explain less than 20 percent of output volatility even in the long run.

## 1.7 Estimating the importance of news shocks II: DSGE approach

In this section, we estimate a structural DSGE model incorporating inventories by Bayesian methods, and use the estimates to assess whether news shocks are important contributors to business cycle fluctuations. The purpose of this section is twofold. First, while the agnostic VAR method is robust to misspecification, it is still a partial identification strategy. Using additional information based on the structure of our economy is in principle helpful in identifying news shocks more precisely. Second, our discussion is so far limited to shocks that are stationary. However, an important component of news shocks may be nonstationary and the importance of these nonstationary components are better understood when we directly model them.

It is however important to keep in mind that estimating a structural DSGE model has its own limitations. Our theoretical analysis did not require us to take a stand on a specific view of the structure of the economy. However, to estimate a DSGE model, we need to select a specific model to estimate, so that the results we obtain are potentially subject to misspecification issues.

### 1.7.1 Model specification

The model we estimate in this section is an extended version of Schmitt-Grohé and Uribe (2012) with inventories introduced as in Bils and Kahn (2000). The model we estimate is similar to that of section 1.2, and its details are described in appendix A.1. However, there are several differences that are worthwhile to mention here.

First, we allow for two sources of nonstationary shocks in the model: nonstationary productivity and nonstationary investment-specific productivity shocks. By allowing for these shocks, we will be able to separately estimate the importance of stationary versus nonstationary news shocks.

Second, we allow for the price markup to change over time. That is, the demand function in (1.12) is now written as

$$s_t(j) = \left(\frac{a_t(j)}{a_t}\right)^{\zeta} \left(\frac{p_t(j)}{P_t}\right)^{-\theta_t} x_t,$$

where $\theta_t$ are assumed to be AR(1) processes.[26]

Third, on top of the seven observables used in Schmitt-Grohé and Uribe (2012), we also use the inventory series described in the previous section as an additional observable in our estimation procedure.

### 1.7.2 Estimation results

Table 1.4 summarizes the variance decomposition of the estimated model. While the prior median parameter values imply that the contribution of news shocks to account for 37 percent of output variation, we find that the median posterior estimate drops to 17 percent. This

[26]For $\theta_t$, we transform it into the markup $\mu_t = \theta_t/(\theta_t - 1)$, and assume that it follows an AR(1) process.

| Innovation | Y | C | I | N | G | INV |
|---|---|---|---|---|---|---|
| Prior News Total | 37 | 47 | 48 | 39 | 49 | 40 |
| Posterior News Total | 17 | 49 | 10 | 38 | 52 | 14 |
| Stationary Productivity Shock | | | | | | |
| News | 1 | 0 | 0 | 1 | 0 | 1 |
| Current | 16 | 3 | 8 | 10 | 0 | 14 |
| Nonstationary Productivity Shock | | | | | | |
| News | 1 | 1 | 0 | 0 | 0 | 0 |
| Current | 15 | 10 | 6 | 3 | 4 | 7 |
| Stationary Investment-Specific Shock | | | | | | |
| News | 1 | 1 | 5 | 1 | 0 | 2 |
| Current | 22 | 4 | 63 | 9 | 0 | 8 |
| Nonstationary Investment-Specific Shock | | | | | | |
| News | 0 | 0 | 0 | 0 | 0 | 0 |
| Current | 0 | 0 | 1 | 0 | 0 | 0 |
| Government Spending Shock | | | | | | |
| News | 1 | 0 | 0 | 1 | 51 | 0 |
| Current | 1 | 0 | 0 | 1 | 44 | 0 |
| Household Preference Shock | | | | | | |
| News | 4 | 41 | 0 | 6 | 0 | 1 |
| Current | 3 | 28 | 0 | 5 | 0 | 1 |
| Labor Wedge Shock | | | | | | |
| News | 8 | 6 | 3 | 27 | 0 | 4 |
| Current | 7 | 5 | 3 | 28 | 0 | 4 |
| Price Markup Shock | | | | | | |
| News | 1 | 0 | 1 | 0 | 0 | 6 |
| Current | 18 | 1 | 9 | 7 | 0 | 47 |

*Table 1.4: Variance decomposition from estimated model. All values are rounded and are in percentage terms. Y, C, I, N, G, INV refer to the growth rates of output, consumption, fixed investment, hours worked, government spending and inventories, respectively.*

contrasts the result in a model without inventories where 41 percent of output variation could be accounted for by news shocks (Schmitt-Grohé and Uribe, 2012). Therefore, when firms are allowed to adjust inventories in the model, news shocks now play a smaller role. This small contribution of news shocks also holds for fixed investment and inventory investment. For all these variables, news shocks now account for around 10 percent of total variation.

However, for other variables such as consumption, government spending, and hours, we still large role for news shocks, consistently with Schmitt-Grohé and Uribe (2012).

To summarize, structural estimation of a DSGE model including inventories indicates that news shocks account for less than 20 percent of output variation over the business cycle, consistently with the results obtained using the VAR approach.

## 1.8 Conclusion

In this paper, we studied the response of inventories to news shocks. We established conditions on model parameters under which inventories and sales will positively comove in response to news shocks. We showed that these conditions are violated by standard calibrations of the classes of models we study, resulting in negative comovement between inventories and sales in response to news shocks. Our analysis highlighted the key mechanism behind this result: news shocks generate a strong intertemporal substitution motive in production. Moreover, we showed that this mechanism persists during the "news period", even after introducing various frictions analyzed by the news literature, such as variable capacity utilization and adjustment costs. Lastly, we used the negative comovement between inventories and sales to identify news shocks in postwar US data. We find that news shocks play a small role in aggregate fluctuations, for two reasons: the identified "news period" is short, on average 1 quarter; and the long-horizon shock contributes less than 20 percent of output variation . The insight behind this result is that inventories are procyclical at business-cycle frequencies.

Our work suggests two future directions for progress. First, one contribution of our analysis was to highlight that a key parameter governing the response of inventories to news shocks is the elasticity of inventories to the discounted growth rate of marginal cost. The approach we have taken in this paper is to compute the elasticity implied by existing models of finished-good inventories. An alternative approach is to obtain empirical estimates of this elasticity, and explore modifications of existing models that may match those estimates. Second, we proposed a new way of identifying news shocks, using aggregate data on inventories and sales. An interesting question is whether our theoretical and empirical results could be modified if we were to take a more disaggregated view of inventories, with different sectors having different inventory intensities (Chang, Hornstein, and Sarte, 2009). Theoretically, news shocks in one particular sector may lead to negative comovement of inventories and sales in that sector, but this need not be so in the aggregate. Empirically, differences in the comovement of sales and inventories across sectors, using industry-level data, could be used to identify these sectoral news shocks. We leave this to future research.

# Chapter 2

# Firm investment and the composition of debt

Nicolas Crouzet

## 2.1 Introduction

The most prominent feature of the corporate debt structure in the US is how dramatically it varies, both in the cross-section of firms and over time. Firms use various types of debt instruments, issued to different types of lenders, with diverse covenants and organized within rich priority structures. In the cross-section, recent work by Rauh and Sufi (2010) analyzes the holdings of bank debt, bonds, program debt (such as commerical paper) and mortgage debt of a sample of publicly traded firms. They find that 70% of firm-year observations hold at least two different types of debt instruments on their balance sheets; moreover, the two most prominently used types of debt are bonds (convertible and non-convertible), and bank debt.[1] Over time, aggregates of corporate debt also display dramatic variation. Adrian, Colla, and Shin (2012), echoing the findings of Kashyap, Stein, and Wilcox (1993) on bank debt and commercial paper, document the fact that over the course of the 2007-2009 recession, the fall in aggregate bank debt outstanding was mirrored by a rise in the issuance of corporate bonds. This suggests that the overall debt structure of firms also changes with business cycle conditions.

Little attention has however been paid to the interaction between debt heterogeneity and the real decisions of firms. On the one hand, the litterature on the link between financial constraints and firm-level investment typically treats all corporate debt as homogeneous. Empirical work on the topic (Fazzari, Hubbard, and Petersen (1988), Hoshi, Kashyap, and Scharfstein (1991), Whited (1992), Kaplan and Zingales (1997), Hubbard (1998)) focuses on measures of total leverage. Likewise, theoretical work on financial constraints and firm-level investment (Cooley and Quadrini (2001), Albuquerque and Hopenhayn (2004), Clementi and Hopenhayn (2006)) explores single borrower-lender relationships, thereby also abstain-

[1]See the first column of their table 1.

ing from issues of debt heterogeneity. On the other hand, there is an extensive literature on debt heterogeneity (Diamond (1991), Diamond (1993), Rajan (1992), Bolton and Scharfstein (1996), Bolton and Freixas (2000), Park (2000), DeMarzo and Fishman (2007)). This literature explores optimal capital structure with multiple potential lenders, but typically in the context of a firm that must finance a project of fixed size. Each of these strands of the literature is silent on the question of whether, or how, the scale of borrowing and investment could be related to the forms of debt financing that are available to a firm.

The goal of this paper is to provide a simple theoretical framework in which to analyze this question. Specifically, I ask how access to different forms of debt financing affects the borrowing and investment choices of firms. I study a static model in which an entrepreneur can finance investment by borrowing from two potential sources, which I label "bank" and "market" lenders. The entrepreneur uses her own internal finance, in combination with these sources of debt, to finance investment in fixed capital. Production possibilities are standard. Capital is the input into a decreasing returns to scale technology. This technology is subject to idiosyncratic productivity risk, which is realized only after the entrepreneur has raised debt. The entrepreneur has limited liabity; she can default on any debt claims, in which case the project is liquidated. Borrowing is constrained by the fact that liquidation entails inefficient losses.

The model postulates two main differences between bank and market debt. On the one hand, I assume that bank debt can be easily renegotiated in times of financial distress, while market debt cannot. The view that bank debt is more flexible than market debt underlies much of the literature on debt heterogeneity. Gertner and Scharfstein (1991) and Bolton and Scharfstein (1996) provide microfoundations for this view, building on the idea that dispersed holding of market debt creates a hold-out problem in which individual debt holders have limited incentives to agree to a debt restructuring plan.[2] In the model, this flexibility implies that bank debt typically commands a smaller liquidation risk premium than market debt. On the other hand, I assume that the marginal cost of debt issuance is larger for banks that for market lenders. The debt structure chosen by the entrepreneur balances the benefit of bank debt flexibility with the difference in marginal cost of debt issuance between banks and market lenders.

The first prediction of the model is that the optimal debt structure of most firms features a mix of bank and market finance. This is consistent with the fact documented by Rauh and Sufi (2010) that firms simultaneously use different types of debt instruments, but it stands in contrast with much of theoretical literature on debt heterogeneity, which typically finds that the optimal debt structure involves borrowing either from one source or from the other, depending on firms' characteristics. Here, the fact that investment and scale are endogenous is crucial. Indeed, the very largest firms, which have sufficient internal finance relative to the optimal investment scale and face little liquidation risk, do not find it beneficial to combine bank with market debt, and instead prefer to use only market debt. For the other firms, with more limited internal resources and higher liquidation risk, the marginal benefit of bank flexbility is not negligible relative to the "spread" between bank and market marginal lending costs. This leads to an interior solution for the debt structure.

The second prediction of the model is that the share of bank debt as a fraction of total debt is a non-monotonic function of firms' internal ressources. Among those firms that use a mixed debt structure, the share of bank debt is increasing when internal finance is very small, but starts decreasing when internal finance becomes sufficiently large (and eventually drops to 0 for the largest firms). This non-monotonic pattern arises because the very smallest firms

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[2] Empirical evidence in support of the view that bank debt is easier to restructure than bonds is discussed in detail in section 2.2.

choose an investment policy which exhausts their bank borrowing capacity. For these firms, any marginal increase in internal finance expands their bank borrowing capacity, and leads to an expansion in bank debt issuance. On the other hand, when firms have accumulated sufficient internal finance, their bank borrowing constraint need not bind. These firms choose an interior debt structure. In that case, a marginal increase in internal finance results in a substitution away from bank debt, and into market debt.

The third prediction relates to comparative statics. Specifically, the model helps to distinguish between parameters that leave that affect overall borrowing, but leave the composition of borrowing unchanged, from parameters that affect borrowing composition. For example, an increase in the spread between bank and market lending costs causes firms with large internal finance to substitute market debt for bank debt. These firms therefore increase market debt issuance, but reduce bank debt issuance, when the spread increases. For smaller firms, with less internal finance, the increase in the spread leads to a decline in issuance of both market and bank debt. By contrast, a fall in average productivity lowers the overall level of borrowing of all firms, small and large, and leaves the composition of borrowing mostly unchanged. This provides a rationale for the intuition underlying much of the empirical work on the identification of monetary policy shocks through their differential effects on debt choices of small vs. large firms, such as Kashyap, Stein, and Wilcox (1993) and Gertler and Gilchrist (1994).

Finally, I use evidence from the balance sheet of US manufacturing firms, both in the cross-section and over time, to shed light the key predictions of the model.[3] In the cross-section of this sample of firms, much as in the model, the bank share of total debt is non-

[3]This evidence, which I document in a sample including both private and publicly traded firms, echoes and supplements the findings of Rauh and Sufi (2010) and Adrian, Colla, and Shin (2012), who focus on publicly traded, and therefore large firms.

monotonic in firm size: it increases for the very smallest firms, and decreasing thereafter.[4] Additionally, over the last three recessions, and especially over the most recent one, the level and composition of debt of small and large firms evolved in very different ways. The stock of outstanding debt of small firms declined substantially, mostly due to a decline in the stock of oustanding bank debt, while large firms substituted market debt for bank debt and did not experience a significant fall in their overall liabilities. From the standpoint of the model, this pattern is consistent with the view that asymmetric shocks afffecting banks' lending costs played an important role in the in the early stages of the 2007-2009 recession.

This paper contributes to the theoretical literature on debt heterogeneity in two ways. First, while it builds on the "trade-off" theory of debt structure (Gertner and Scharfstein (1991), Rajan (1992), Bolton and Scharfstein (1996)), the paper extends it by endogenizing the choice of investment scale. In this respect, the closest setup to this paper is Hackbarth, Hennessy, and Leland (2007), who likewise rely on the assumption that bank debt is more flexible than market debt in restructuring, but do not allow for an endogenous determination of investment scale. Second, it shows that debt heterogeneity can help explain why credit supply shocks have different effects on the investment behavior of small and large firms, as documented by Kashyap, Stein, and Wilcox (1993) or Gertler and Gilchrist (1994).

The rest of the paper is organized as follows. Section 2.2 describes the model and derives the set of feasible debt structures for a firm with a given level of internal resources. Section 2.3 studies the optimal debt structure. Section 2.4 studies some comparative statics of the model. Section 2.5 relates the findings of section 2.3 and 2.4 to data on the debt composition of a sample of US manufacturing firms during the last three recessions. Finally, section 2.6 concludes.

[4]The data I use focuses on total assets as a measure of size, rather than internal finance; I discuss this issue in more detail in section 2.3.

## 2.2 A model of bank and market financing

This section describes a static model in which an entrepreneur with internal funds $e$ finances a project by borrowing from two lenders: a bank, and the market. The only friction of the model is that there is limited liability; the entrepreneur can choose to default on her debt obligations. However, doing may involve output losses, if the project is liquidated. This motivates the key distinction between bank and market lenders: bank loans can be restructured in times of financial distress, in order to avoide inefficient liquidation losses. Market debt, on the other hand, cannot. I come back to the interpretation of the model's assumptions below after describing the key elements of the model.

### 2.2.1 Production structure and timing

The entrepreneur owns the firm and operates a technology which takes physical assets $k$ as an input, and produces output:

$$y = \phi k^{\zeta}$$

Here, $\phi$, the productivity of the technology employed by the entrepreneur, is a random variable, the realization of which is unknown to both the entrepreneur and the lenders at the time when investment in physical assets is carried out. In what follows, I denote the CDF of $\phi$ by $F(.)$. $\zeta$ governs the degree of returns to scale of the technology operated by the entrepreneur. Assets depreciate at rate $\delta \in [0, 1]$. After production has been carried out and depreciation has taken place, the following ressources are available to the entrepreneur to either consume or repay creditors:

$$\pi(\phi) = \phi k^{\zeta} + (1 - \delta)k \tag{2.1}$$

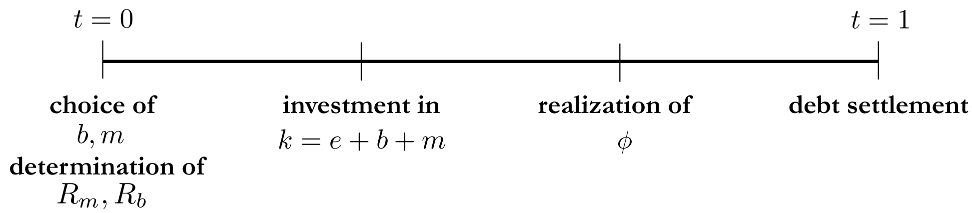


*Figure 2.1: Timing of model.*

I make the following assumptions about the production structure:

**Assumption 2.1.** *The firm's production technology has the following characteristics:*

- *Production has decreasing returns to scale:* $\zeta < 1$*;*

- *The productivity shock* $\phi$ *is a positive, continuous random variable with density* $f$*. Moreover,* $f(0) = 0$ *and the hazard rate of* $\phi$ *is strictly increasing.*

The first part of the assumption is standard in models of firm investment, and guarantees that firms have a finite optimal scale of operation. The second part of the assumption consists in restrictions on the distribution of productivity shocks. Restricting the shock $\phi$ to be a postive random variable implies that there is a positive lower bound on ressources, $(1 - \delta)k$, so that riskless lending may occur, to the extent that $\delta < 1$. The increasing hazard rate is a technical assumption which guarantees the unicity of lending contracts.[5]

The entrepreneur finances investment in physical assets, $k$, from three sources: internal finance $e$, with which it is initially endowed; bank debt, $b$, and market debt $m$. The resulting balance sheet constraint is thus simply:

$$k = e + b + m.$$

[5]See appendix B.1 for details.

The timing of actions and events, for an entrepreneur with internal finance $e$, is summarized in figure 2.1. The model has two periods. At $t = 0$, the entrepreneur, the bank lender and the market lender agree about a debt structure $(b, m)$, and promised repayments, $R_b$ to the bank, and $R_m$ for the market lender. Investment in $k$ then takes place, and the productivity of the firm, $\phi$, is realized. At time $t = 1$, debt payments are settled; that is, the firm can choose to make good on promised repayments, restructure its debt, or proceed to bankruptcy.

Finally, in this section, I only assume that all agents are utility maximizers and have preferences that are weakly increasing in their monetary payoffs. In the next section, I will focus on optimal choices in the case of a risk-neutral entrepreneur; however, all the results presented in this section on the settlement of debt are entirely independent of the assumption of risk-neutrality, and hold for general preference specifications so long as preferences are increasing in payoffs. In particular, the set of feasible debt structures characterized in this section is identical across preference specifications.[6]

### 2.2.2 Debt settlement

The debt settlement stage takes place once the productivity of the firm has been observed by all parties. I model the debt settlement process as a two-stage game. In the first stage, the entrepreneur can choose between three alternatives, summarized in figure 2.2: repay in full both its bank and market creditors; make a restructuring offer to the bank; or file for bankruptcy. If the entrepreneur chooses to repay in full both its creditors, her payoff is:

$$\pi_P(\phi) = \pi(\phi) - R_m - R_b \tag{2.2}$$

[6] In particular, this allows for the characterization of the set of feasible contracts developed below to be used in the dynamic version of this model, which I study in chapter 3.

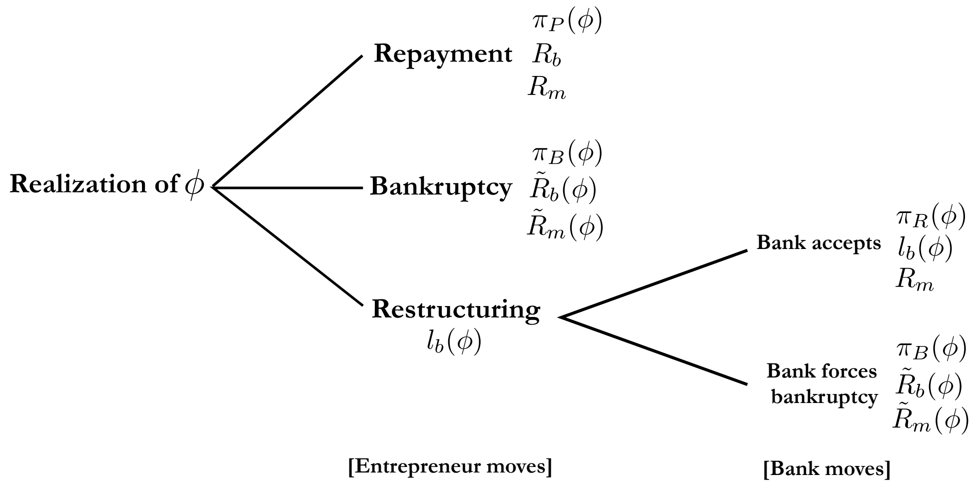


*Figure 2.2: Debt settlement.*

while the payoff to the bank and market lender are, respectively, $R_b$ and $R_m$, as initially promised. I next turn to describing each party's payoff under the two other alternatives, bankruptcy and restructuring.

### Bankruptcy

If the entrepreneur chooses to file for bankruptcy, the project is terminated and liquidated, and the proceeds from liquidation are distributed to creditors. Once bankruptcy is declared, the entrepreneur has no claim to liquidation proceeds; that is, her liquidation payments are assumed to be 0, so that the monetary payoff to the entrepreneur in bankruptcy is:[7]

$$\pi_B(\phi) = 0. \tag{2.3}$$

[7] This is without loss of generality. Allowing for the entrepreneur to be a residual claimant in bankruptcy would not alter the results, since in the debt settlement stage, bankruptcy would never be declared in states in which the entrepreneur has sufficient resources to repay both lenders. I omit this possibility to alleviate notation.

I make the following assumption about the impact of liquidation on output:

**Assumption 2.2** (Liquidation losses)**.** *Under bankruptcy, the proceeds $\tilde{\pi}(\phi)$ to be distributed to creditors and the entrepreneur are a fraction $\chi$ of the project's value:*

$$\tilde{\pi}(\phi) = \chi\pi(\phi) \quad , \quad 0 \leq \chi < 1.$$

Liquidation leads to inefficient losses of output; that is, the liquidation value of the project is strictly smaller than the value of the project under restructuring or repayment. Specifically, liquidation losses are equal to $(1-\chi)\pi(\phi)$. Consistent with the evidence in Bris, Welch, and Zhu (2006) discussed below, this assumption captures fact that bankruptcy proceedings are typically costly, both administratively and because they halt production activities. Moreover, asset values of firms after cash auction proceedings are typically only a fraction of pre-bankruptcy values. This is the key friction in the static model with risk-neutrality: absent bankruptcy losses, lending would be unconstrained, as I will discuss below.

I assume that bankruptcy proceeds are distributed among creditors according to an agreed-upon priority structure, in line with the Absolute Priority Rule (APR) that governs chapter 7 proceedings in the US.[8] In this section, I assume that bank debt is senior to market debt.[9] Under this priority structure, the payoff to bank lenders and market lenders, are:

$$\tilde{R}_b^K(\phi) = \min\left(R_b, \chi\pi(\phi)\right), \tag{2.4}$$

$$\tilde{R}_m^K(\phi) = \max\left(\chi\pi(\phi) - R_b, 0\right). \tag{2.5}$$

[8] See White (1989) for institutional details on the APR.

[9] I come back to the issue of the optimality of bank seniority in conclusion. It is likely that, in this model, bank seniority is the optimal priority structure from the standpoint of the firm; numerically, it is the case for all the versions of the model which I have explored.

The first line states that the bank's payoff in bankruptcy is at most equal to its promised repayment $R_b$. The second payoff states that market lenders are residual claimants. Note that this formulation does not, a priori, preclude cases in which $\tilde{R}_m(\phi) \geq R_m$, that is, market lenders receiving a residual payment larger than their initial claim. I will however show that this never occurs in the equilibrium of the debt settlement game.

### Restructuring

Instead of filing for bankruptcy, I assume that the entrepreneur can enter a private workout process with her creditors. Because going bankrupt implies losses of output, it may sometimes be in the interest of creditors and the entrepreneur to arrive at a compromise. I make the following restriction to the workout process.

**Assumption 2.3** (Bank debt flexibility)**.** *The entrepreneur may only restructure debt payments owed to the bank, $R_b$; payments to the market lender, $R_m$, cannot be renegotiated.*

This is the key distinction between bank and market lending in the model; I delay its discussion to the next paragraph, and first describe its implications for the debt settlement process. I assume that the private workout operates as follows: the entrepreneur makes a one-time offer to the bank which takes the form of a reduction in promised repayments $l_b \leq R_b$. In case the offer is accepted, the bank obtains $l_b$, and the entrepreneur obtains:

$$\pi_R(\phi) = \pi(\phi) - R_m - l_b(\phi). \tag{2.6}$$

If, on the other hand, the bank refuses the entrepreneur's offer, the private workout fails, and the project is liquidated. In this case, the payoff to the bank is given by equation (2.4).

The participation constraint of the bank is thus:

$$l_b \geq \tilde{R}_b^K(\phi).$$

The entrepreneur will choose her restructuring offer to maximize her net payoff under restructuring, subject to the participation constraint of the bank. Formally:

$$\begin{aligned} \pi_R(\phi) &= \max_{l_b} \quad \pi(\phi) - R_m - l_b \quad \text{s.t.} \quad l_b \geq \tilde{R}_b^K(\phi) \\ &= \begin{cases} \pi(\phi) - R_b - R_m & \text{if } R_b \leq \chi\pi(\phi) \\ (1-\chi)\pi(\phi) - R_m & \text{if } R_b > \chi\pi(\phi) \end{cases} \end{aligned} \tag{2.7}$$

Intuitively, this result indicates that the entrepreneur will choose to make a restructuring offer only when its cash on hand is small enough, relative to promised repayments to the bank. Note that the larger the value of $\chi$, the higher the restructuring threshold; that is, potential bankruptcy losses effectively allow the entrepreneur to extract concessions from the bank.

#### Debt settlement equilibria

Given the realization of $\phi$, the entrepreneur chooses whether to repay, restructure or file for bankrucpy, by comparing her payoffs $\pi_P(\phi)$, $\pi_B(\phi)$ and $\pi_R(\phi)$ under each option. The following proposition describes the resulting perfect equilibria in pure strategies of the debt settlement game described in figure 2.2. There is a unique equilibrium for each realization of $\phi$; however, the set of possible equilibria, parametrized by $\phi$, depends on the terms of the debt contracts.

**Proposition 2.4** (Debt settlement equilibria).

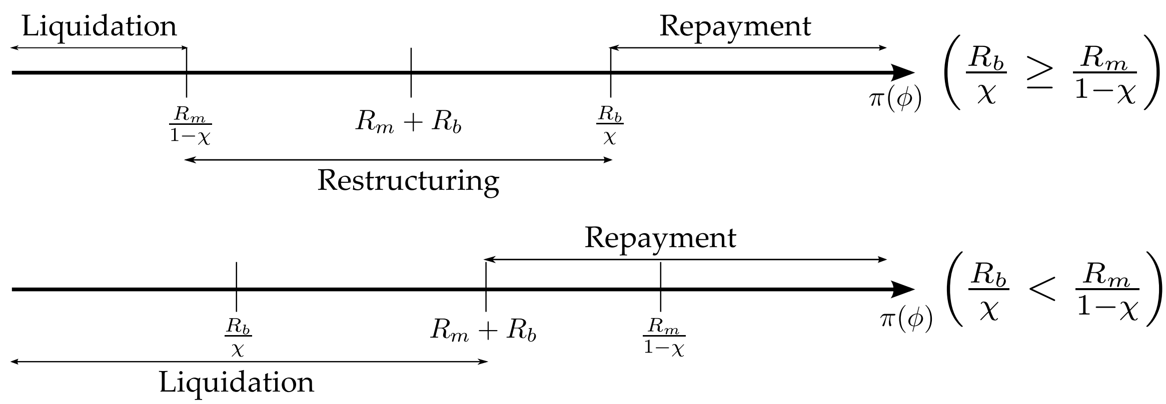


*Figure 2.3: Debt settlement equilibria.*

*If $\frac{R_m}{1-\chi} < \frac{R_b}{\chi}$ (**R-contracts**), there are some realizations of $\phi$ for which the entepreneur chooses to use her restructuring option. Specifically, the entrepreneur chooses to repay her creditors when $\pi(\phi) \geq \frac{R_b}{\chi}$; to restructure debt when $\frac{R_m}{1-\chi} \leq \pi(\phi) < \frac{R_b}{\chi}$; and to file for bankruptcy when $\pi(\phi) < \frac{R_m}{1-\chi}$ .*

*If $\frac{R_m}{1-\chi} \geq \frac{R_b}{\chi}$ (**K-contracts**), there are no realizations of $\phi$ such that the entrepreneur attempts to restructure debt with the bank. Instead, she chooses to repay her creditors when $\pi(\phi) \geq R_m + R_b$, and otherwise, she files for bankruptcy.*

*Moreover, in bankruptcy or restructuring, market and bank lenders never obtain more than their promised repayments: $\tilde{R}_m(\phi) \leq R_m$ and $\tilde{R}_b(\phi) \leq R_b$, regardless of whether the debt contract is an $R-contract$ or a $K-contract$.*

The proof for this and all following propositions are reported in appendix B.1. Figure 2.3 illustrates sets of equilibria for each type of contract. In the case of a K-contract ($\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$), no restructuring ever occurs, and bankruptcy losses cannot be avoided when the cash on hand of the firm, $\pi(\phi)$, falls below the threshold at which the firm prefers declaring bankruptcy over repayment, $R_m + R_b$. This occurs because the stake of the flexible creditors, $R_b$, is too small for restructuring to bring about sufficient gains for the entrepreneur to avoid default on market debt.

On the other hand, in the case of an R-contract, ($\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$), the flexibility of bank debt sometimes allows the entrepreneur to make good on its payments on market debt (this corresponds to restructuring region below $R_m + R_b$ in figure 2.3). Some R-equilibria will see the entrepreneur exert a degree of bargaining power over the bank: indeed, the bank will be forced to accept a settlement, even though the firm has enough cash on hand to make good on both its promises (this corresponds to the restructuring region above $R_m + R_b$ in figure 2.3). This region corresponds to strategic restructurings on the part of the entrepreneur, who takes advantage of the fact that the bank can never extract from her more than its reservation value under restructuring, $\chi\pi(\phi)$, in any private workout.

### 2.2.3 Discussion

The model's fundamental distinction between bank lending and market lending is that bank lenders are capable of flexibility in times when the firm is not able (or willing) to repay her debt. A natural question is then whether this assumption is borne out in the data. Gilson, Kose, and Lang (1990) examine a sample of 169 financially distressed firms. They show that about half (80 or 47% of the total) firms successfully restructure outside of formal judicial proceedings, while the other half (89, or 53% of the total) file for bankruptcy. They show that successful restructurings out of court involve, in 90% of cases, a firm that has outstanding bank loans, while only 37.5% of successful restructuring involves firms with oustanding public debt. Moreover, they show that the existence of bank loans is the single most important determinant of whether firms successfully restructure out of court, more so than other firm characteristics such as firm size, age or leverage. A theoretical justification of the observation that bank debt is easier to restructure is developped by Gertner and Scharfstein (1991), who argue that when coordination problems among dispersed holders of public debt lead to a failure to efficiently restructure that debt. Bolton and Scharfstein (1996) also study how free-riding problems can lead to inefficiencies during debt restructuring involving a large number of creditors. The assumption of bank debt flexibility can thus be thought of as a reduced-form manner of capturing the coordination and free-riding problems discussed elsewhere in the literature.

Additionally, the model makes the assumption that the formal liquidation of a project – which occurs only if restructuring has failed – leads to inefficient losses. In the model, this occurs when $\chi < 1$; when $\chi = 1$, liquidation leads to a transfer of ownership but not losses in values. Bris, Welch, and Zhu (2006) study a sample of 61 chapter 7 and 225 chapter 11 filings between 1995 and 2001. In particular, they report measures of the ratio of pre to post-bankruptcy asset values (excluding legal fees). For chapter 11 proceedings, which provides a legal framework for debt restructuring but does not involve liquidation, the mean of the ratio of pre to post-bankruptcy asset values in their sample is 106.5%; that is, on average, this ratio increases after chapter 11 proceedings. For chapter 7 proceedings, this is not the case: asset values decline as a result after the bankruptcy. Here, Bris, Welch, and Zhu (2006) offer two measures of post-bankruptcy asset values. The first measure is liquidation value of the firm after collateralized lenders have seized the assets to which they had a lien outside of bankruptcy proceedings. With this measure, asset values post-bankruptcy are 17.2% of pre-bankruptcy asset values, on average. The second measure tries attempts to include the value of collateralized assets; average post-bankruptcy values are then estimated to represent 80.0% of pre-bankruptcy values; the median ratio is 38.0%.

While the model's debt settlement stage does not clearly distinguish between private and formal (chapter 11) workouts, it assumes that they are costless, whereas liquidation (chapter 7) is assumed to be costly. In this respect, the model's assumptions are thus consistent with the results of Bris, Welch, and Zhu (2006). In general, assuming that debt renegotiation (private or under a chapter 11 filing) is costly should not alter the key qualitative results of the model. So long as renegotiation costs are strictly smaller than those associated to liquidation, renegotiation will be beneficial to the firm provided that promised repayment to bank lenders are sufficiently large, as in figure 2.3.

### 2.2.4 Debt pricing and the lending menu

I now turn to describing the pricing of bank and market debt contracts at $t = 0$, before the realization of the shock $\phi$ and the ressources $\pi(\phi)$.

#### Expected lending returns

Let:

$$\mathbb{E}\left[\Pi_i\left(e, b, m, R_b, R_m\right)\right] = \int_0^{+\infty} \tilde{R}_i(\phi) dF(\phi), \quad i = b, m$$

denote the gross expected returns for each lender (market or bank) at time $t = 0$. Appendix B.1 details the expressions of lenders' payoff functions $\tilde{R}_b(\phi)$ and $\tilde{R}_m(\phi)$ for the debt settlement equilibrium associated to each realization of $\phi$. They depend on $\pi(\phi) = \phi k^\zeta + (1-\delta)k = \phi(e+b+m)^\zeta + (1-\delta)(e+b+m)$. Moreover, their expression also depends on whether the debt contract is an R- or a K-contract. For example, assume that the contract is a K-contract, that is, $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$. In this case, the bank will face three possible outcomes, conditional on the realization of $\pi(\phi)$:

- If $\phi \geq \overline{\phi} \equiv \frac{R_b + R_m - (1-\delta)(e+b+m)}{(e+b+m)^\zeta}$, that is, above the bankruptcy threshold, the bank will be repayed in full;
- if $\overline{\phi} > \phi \geq \underline{\phi} \equiv \frac{R_b - (1-\delta)(e+b+m)}{\chi(e+b+m)^\zeta}$, the entrepreneur will file for bankruptcy; but the bank, because of its seniority in the priority structure of the bank, will still be repayed in full (market lenders will only receive partial repayments);

- if $\underline{\phi} > \phi$, the bank is only partially repayed, and the market lenders receive no payments.

Thus, the expected return function of the bank lender will be given by:

$$\mathbb{E}\left[\Pi_b\left(e, b, m, R_b, R_m\right)\right] = \underbrace{\left(1 - F\left(\overline{\phi}\right)\right) R_b}_{\text{no bankruptcy}} \quad + \quad \underbrace{R_b \int_{\underline{\phi}}^{\overline{\phi}} dF(\phi)}_{\text{bankruptcy, full repayment}} \quad + \quad \underbrace{\chi \int_0^{\underline{\phi}} \pi(\phi) dF(\phi)}_{\text{bankruptcy, partial repayment}}$$

Expressions of the expected returns functions of both types of lenders for each type of contract are also reported in appendix B.1. Importantly, expected lending returns for the bank are independent of $R_m$, and thus independent of whether the contract is an R- or a K- contract. This result follows from the assumption that the bank may only accept or reject the offer of the firm. Because of this, the bank always accepts its reservation value, the bankruptcy payoff, as restructuring settlement. When the bank is senior in the priority structure, the bankruptcy payoff is therefore independent of $R_m$, so that the expected returns from bank lending are independent of the value of $R_m$, and therefore of the condition $\frac{R_m}{1-\chi} \gtrless \frac{R_b}{\chi}$.

**The lending menu**

I make the assumption that both kinds of financial intermediaries are perfectly competitive, so that debt is priced by equating the gross expected return from lending to the lenders' gross cost of funds.

I assume that lenders have constant marginal costs of funds, and I denote lender $i$'s cost of funds by $1 + r_i$, with $r_i > 0$. For now, I make no assumptions on which type lender has the higher cost of funds; I come back to this issue in the next section.

A contract $(R_m, R_b)$ corresponding to debt structure $(b, m)$ will be available to a firm with internal finance $e$ if it satisfies the zero profit condition of both lenders.

**Definition 2.5** (Lending contracts)**.** *The set of lending contracts proposed to a firm with internal finance $e$ that desires to implement a debt structure $(b, m)$ is the set:*

$$\mathcal{L}(b,m,e) \equiv \left\{ (R_b, R_m) \in \mathbb{R}_+^2 \;\middle|\; \begin{array}{l} \mathbb{E}\left[\Pi_b\left(e,b,m,R_b,R_m\right)\right] = (1+r_b)b \\ \mathbb{E}\left[\Pi_m\left(e,b,m,R_b,R_m\right)\right] = (1+r_m)m \end{array} \right\}$$

Since $\mathcal{L}(b,m,e)$ is a subset of $\mathbb{R}^2$, it is a partially ordered set when endowed by the product order $\leq_\times$.[10] $(\mathcal{L}(b,m,e), \leq_\times)$ therefore has at most one least element.[11]. This justifies the following definition:

**Definition 2.6** (The dominating contract)**.** *The dominating contract for debt structure $(b, m)$ and internal finance $e$ is the least element of the partially ordered set $(\mathcal{L}(b,m,e), \leq_\times)$ (if it exists).*

Finally, the lending menu for a firm with internal finance $e$ is the set of all debt structures $(b, m)$ such that there exists a dominating contract at $(b, m)$ for equity e:

**Definition 2.7** (The lending menu)**.** *The lending menu for a firm with internal finance $e$ is the set:*

$$\mathcal{S}(e) \equiv \left\{ (b,m) \in \mathbb{R}_+^2 \;\middle|\; \begin{array}{l} \mathcal{L}(b,m,e) \neq \emptyset \\ (\mathcal{L}(b,m,e), \leq_\times) \text{ has a least element} \end{array} \right\}$$

Several intuitive elements of these formal definitions are worth emphasizing. First, the lending menu $\mathcal{S}(e)$ of definition 2.7 is the set of feasible debt structures for the entrepreneur with internal finance $e$. There are two requirements for a debt structure to be part of the lending menu: first, there must exist lending contracts for that debt structure; second,

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[10]This is the partial order defined by $(x_1, y_1) \leq_\times (x_2, y_2) \iff x_1 \leq y_1$ and $x_2 \leq y_2$.

[11]This is $(R_b, R_m) \in \mathcal{L}(b,m,e)$ such that $\forall (\tilde{R}_b, \tilde{R}_m) \in \mathcal{L}(b,m,e)$, $(R_b, R_m) \leq_\times (\tilde{R}_b, \tilde{R}_m)$.

one of them must be a dominating contract, in the sense of definition 2.6. Intuitively, the dominating contract has the property that it is (weakly) cheaper, in both dimensions $(R_b, R_m)$, than any other contract that satisfies the lenders' zero-profit conditions.

Note that, a priori, there is no reason to think that $(\mathcal{L}(b,m,e), \leq_\times)$ generically has a least element. It could well be that, for a certain level of internal finance $e$ and a certain debt structure $(b,m)$, the set of lending contracts contains two elements $(R_b, R_m)$ and $(R_b', R_m')$ such that $R_b > R_b'$ but $R_m < R_m'$, which cannot be ordered by $\leq_\times$. In that case, $(\mathcal{L}(b,m,e), \leq_\times)$ would have no least element. Definition 2.7 would then exclude $(b,m)$ from the firms' feasible set, the lending menu $\mathcal{S}(e)$, despite the fact that $\mathcal{L}(b,m,e)$ would be non-empty. This would seem to arbitrarily restrict the set of feasible contracts.

Fortunately, this is never the case: $(\mathcal{L}(b,m,e), \leq_\times)$ always has a least element when $\mathcal{L}(b,m,e) \neq \emptyset$, so that the lending menu never contains debt structures associated to several possible contracts.[12] I discuss this further in the next subsection.

Finally, while the definition of competitive contracts only require promised repayments to be positive, they in fact also satisfy $R_m \geq (1+r_m)m$ and $R_b \geq (1+r_b)b$; that is, lenders never ask for promised repayments which imply a yield below their marginal cost of funds. Intuitively, since the expected value of total (bankruptcy and non-bankruptcy) claims has to equal the cost of funds for each lenders, it cannot be the case that both are strictly smaller or strictly larger that that cost of funds. As bankruptcy claims are strictly smaller than non-bankruptcy claims, it must therefore be the case that non-bankruptcy claims exceed the costs of funds.

[12]This result is proved in appendix B.1, as part of the proof of proposition 2.8.

**Some general properties of the lending menu**

The following proposition describes in more detail the structure of the lending menu, emphasizing the fact that it can be explicitly partitioned betwen contracts leading to $K$-equilibria and contracts leading to $R$-equilibria.

**Proposition 2.8** (A partition of the lending menu in the general case)**.** *The lending menu $\mathcal{S}(e)$ can be partitioned as:*

$$\mathcal{S}(e) = \mathcal{S}_R(e) \cup \mathcal{S}_K(e) \quad , \quad \mathcal{S}_R(e) \cap \mathcal{S}_K(e) = \emptyset,$$

*where:*

$$\mathcal{S}_R(e) = \tilde{\mathcal{S}}_R(e) \quad , \quad \mathcal{S}_K(e) = \tilde{\mathcal{S}}_K(e) \setminus \left(\tilde{\mathcal{S}}_R(e) \cap \tilde{\mathcal{S}}_K(e)\right),$$

*and*

$$\tilde{\mathcal{S}}_K(e) \equiv \left\{ (b,m) \in \mathbb{R}_+^2 \;\middle|\; \begin{array}{lr} 0 \leq \frac{(1+r_b)b+(1+r_m)m}{(e+b+m)^\zeta} \leq \hat{M}(e+b+m) + (1-\delta)(e+b+m)^{1-\zeta} & (\mathrm{c-joint}) \\ R_l(b,m,e) > \frac{R_b(b,m,e)}{\chi} & (\mathrm{frontier-K}) \end{array} \right\},$$

$$\tilde{\mathcal{S}}_R(e) \equiv \left\{ (b,m) \in \mathbb{R}_+^2 \;\middle|\; \begin{array}{lr} 0 \leq \frac{(1+r_b)b}{\chi(e+b+m)^\zeta} \leq \mathbb{E}(\phi) + (1-\delta)(e+b+m)^{1-\zeta} & (\mathrm{c-bank}) \\ 0 \leq \frac{(1+r_m)m}{(1-\chi)(e+b+m)^\zeta} \leq \hat{I}(e+b+m) + (1-\delta)(e+b+m)^{1-\zeta} & (\mathrm{c-market}) \\ \frac{R_{m,l}(b,m,e)}{1-\chi} \leq \frac{R_b(b,m,e)}{\chi} & (\mathrm{frontier-R}) \end{array} \right\}.$$

*The sets $\mathcal{S}_R(e)$ and $\mathcal{S}_K(e)$ are non-empty, compact and connected subsets of $\mathbb{R}_+^2$. Moreover, for a firm with internal finance $e$:*

- *The dominating contract associated to $(b,m)$ is an $R$-contract, if and only if, $(b,m) \in \mathcal{S}_R(e)$;*

- *The dominating contract associated to $(b,m)$ is a $K$-contract, if and only if, $(b,m) \in$*

*$\mathcal{S}_K(e)$;*

*Expressions for the functions $\hat{M}(e+b+m)$, $\hat{I}(e+b+m)$, $R_l(b,m,e)$, $R_{m,l}(b,m,e)$ and $R_b(b,m,e)$ are given in appendix B.1.*

There are two important elements in this proposition. First, the lending menu is the union of two subsets, $\tilde{S}_K(e)$ and $\tilde{S}_R(e)$, which correspond, respectively, with the set of debt structures for which there exists a K-contract (the set $\tilde{S}_K(e)$) and the set of debt structures consistent for which there exists an R-contract (the set $\tilde{S}_R(e)$). The intersection of these two sets is however not empty; that is, there are debt structures for which there exists both a K- and an R-contract. However, the proposition establishes that for such debt structures, the R-contract is always the dominating contract. The intuition for this result is straightforward. Imagine that there are two contracts, $(R_b, R_m)$ and $(\tilde{R}_b, \tilde{R}_m)$, associated to a debt structure $(b,m)$: an R-contract, $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$, and a K-contract, $\frac{\tilde{R}_b}{\chi} < \frac{\tilde{R}_m}{1-\chi}$. As discussed previously, because of bank debt seniority, banks' gross expected returns from lending are the same for both contracts, so that so $R_b = \tilde{R}_b$. Therefore, $\frac{\tilde{R}_m}{1-\chi} > \frac{\tilde{R}_b}{\chi} = \frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$, so that the two contracts can be ordered, and the R-contract dominates.

Second, each of the two subsets $\tilde{S}_K(e)$ and $\tilde{S}_R(e)$ can be described as the intersection of the sets defined by three inequality constraints. The inequalities $(\mathrm{c}-\mathrm{bank})$, $(\mathrm{c}-\mathrm{market})$ and $(\mathrm{c}-\mathrm{joint})$ correspond, respectively, to borrowing constraints with respect to market, bank lending and total lending. The borrowing constraint with respect to bank lending, for example, can be rewritten as:

$$0 \leq (1+r_b)b \leq \chi\left(\mathbb{E}(\phi)(e+b+m)^{\zeta} + (1-\delta)(e+b+m)\right) = \chi \int_0^{+\infty} \pi(\phi) dF(\phi).$$

The first inequality states that firms are note allowed to lend. The second inequality states that gross lending costs $(1+r_b)b$ cannot exceed the maximum expected return which the bank can achieve by lending to the firm. This maximum is attained when the threshold for $\phi$ below which the firm renegotiates, which is implicitly defined by $\frac{R_b}{\chi} = \pi(\phi)$ as per proposition 2.4, goes to infinity. This implies that when the bank borrowing constraint is binding (that is, when the debt structure is such that (c − bank) holds with equality), the entrepreneur always renegotiates her loan with the bank, so that the banks' effective repayment is $\chi\pi(\phi)$, for all $\phi$. The banks' expected repayment is then a fraction $\chi$ of total expected output. Thus, when the firm is at her bank borrowing constraint, the bank effectively holds a claim to a constant fraction of the project's value – an equity-like stake.

This is in contrast to the market borrowing constraint (c − market). This borrowing constraint also states that gross lending costs of markets cannot exceed the maximum expected repayment which market lenders can achieve. Contrary to the banks' case, this expected repayment is not monotonic in the threshold for $\phi$ below which the entrepreneur goes bankrupt, which is implicitly defined by $\frac{R_m}{1-\chi} = \pi(\phi)$. This is a direct consequence of the fact that there are losses associated with going bankrupt, that is, $\chi < 1$. These losses (along with the technical part of assumption 2.1) imply that expected returns of market lenders are increasing in $R_m$ when $R_m$ is small (so that bankruptcy probability and bankruptcy losses are small), but decreasing in $R_m$ when $R_m$ is large (so that bankruptcy probabilities and bankruptcy losses are large).[13] The value $\tilde{I}(e + b + m) < \mathbb{E}(\phi)$ correspond to the bankrucpy threshold which maximizes expected returns for market lenders, which trades off bankruptcy losses with higher promised repayments. If, instead, there were no bankruptcy losses, $\chi = 1$, expected returns would never decrease in promised repayments $R_m$, and the market borrowing constraint would take the same form as the banks'; that is, we would have $\hat{I}(e + b + m) = \mathbb{E}(\phi)$. Likewise, in the case of a K-contract, renegotiation is never used for any realization $\phi$; bankruptcy losses are therefore unavoidable. For the same reasons as in

[13]Appendix B.1 contains a detailed proof of this statement.

the case of the R-contract, this implies that the total surplus from lending has an interior maximum in the bankruptcy threshold, which corresponds to the value $\hat{M}(e+b+m) < \mathbb{E}(\phi)$. Thus, the ability to renegotiate debt is a crucial determinant of borrowing constraints in this model.

Proposition 2.8 does not provide a full characterization of the lending menu. However, some general results can be established about the type of debt structures which are to be found in each of the subsets $\mathcal{S}_K(e)$ and $\mathcal{S}_R(e)$. This is the object of the following corollary to proposition 2.8.

**Corollary 2.9** (Debt structure and the lending menu)**.** *For a given level of internal finance $e$ and a debt structure $(b, m) \in \mathcal{S}(e)$, let $s = \frac{b}{b+m}$ denote bank debt as a fraction of total debt. Define:*

$$\underline{s}_R = \frac{1}{1 + \frac{1-\chi}{\chi}\frac{1+r_b}{1+r_m}},$$

*and let $\overline{s}_K(e)$, $1 \geq \overline{s}_K(e) \geq \underline{s}_R$, be defined as in appendix B.1. Then:*

- *if $1 \geq s \geq \overline{s}_K(e)$, then $(b, m) \in \mathcal{S}_R(e)$;*
- *if $\underline{s}_R > s \geq 0$, then $(b, m) \in \mathcal{S}_K(e)$.*

This corollary indicates that the set of debt structures $\mathcal{S}_R(e)$ contains no debt structures $(b, m)$ with an excessively low fraction of bank debt $s = \frac{b}{b+m}$: there can indeed be no debt structures such that $s < \underline{s}_R$ in that set. In particular, when $s = 0$ (a pure market contract), the associated contract must be a K-contract. This is intuitive: in that case, there can be no renegotiation from the part of the entrepreneur, since no bank has taken part in lending. The interesting result, however, is that the debt structure of firms must contains a *sufficient fraction* of bank debt for renegotiation to be a possibility in the equilibrium of the debt settlement game. Otherwise, the gains associated with the renegotiation of bank liabilities are never sufficient to repay in full the market lenders in case of bankruptcy, so that bankruptcy can never be avoided. Accordingly, the threshold $s_R$ below which the debt structure ceases to allow for renegotiation is increasing in $\chi$: the larger the bankruptcy losses (the smaller $\chi$), the larger the renegotiation gains, and the more market debt the firm can take on as a fraction of total debt. On the contrary, in the limit where $\chi = 1$, there are no renegotiation gains, and $s_R = 1$, so that there are no debt structures associated with R-contracts.

Likewise, the corollary also indicates that the set $S_K(e)$ contains no debt structures particularly tilted towards bank debt; a pure bank contract, $s = 1$, must be associated to an R-contract, since in the absence of market debt, it is always beneficial for the entrepreneur to renegotiate down debt payments (provided her productivity $\phi$ is sufficiently low). Thus, the model indicates that debt structures tilted towards bank debt tend to be associated with contracts leading to debt renegotiations, whereas debt structures tilted towards market debt tend to be associated with contracts where debt renegotiations are not an equilibrium outcome.

**An analytical characterization of the lending menu when $\delta = 1$**

I next turn to a particular case in which the lending menu can be characterized analytically.

**Proposition 2.10.** *Let $(d, s) \in \mathbb{R}^+ \times [0, 1]$ denote the debt structure, with $d = b + m$ and $s = \frac{b}{b+m}$. The sets $\mathcal{S}_R(e)$ and $\mathcal{S}_K(e)$ can be parametrized as:*

$$\mathcal{S}_R(e) = \left\{ (d, s) \in \mathbb{R}_+ \times [\underline{s}_R, 1] \;\middle|\; 0 \leq d \leq \overline{d}_R(s, e) \right\},$$

$$\mathcal{S}_K(e) = \left\{ (d, s) \in \mathbb{R}_+ \times [0, \overline{s}_K] \;\middle|\; \underline{d}_K(s, e) \leq d \leq \overline{d}_K(s, e) \right\}.$$

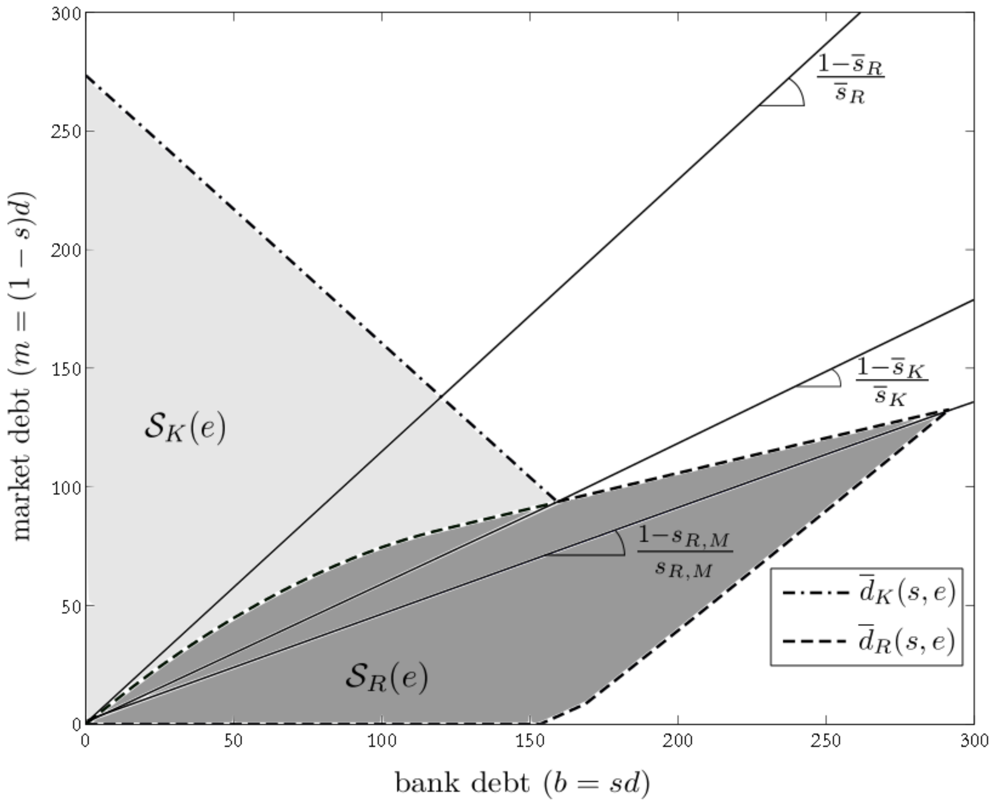



*Figure 2.4: The lending menu $\mathcal{S}(e)$ when $\delta = 1$.*

*Here, $\underline{s}_R$ is defined as in corollary 2.9, and:*

$$\overline{s}_B = \frac{1}{1 + \left(\frac{1}{\chi}\upsilon - 1\right)\frac{1+r_b}{1+r_m}}$$

*Expression for the borrowing limits $\overline{d}_R(s,e)$, $\overline{d}_K(s,e)$ and $\underline{d}_K(s,e)$ and the constant $\upsilon > \chi$ are given in appendix B.1.*

*Moreover, $\frac{\partial \bar{d}_K}{\partial s}(s,e) < 0$, while $\frac{\partial \bar{d}_R}{\partial s}(s,e) \geq 0$, if and only if, $\underline{s}_R \leq s \leq s_{R,M} < 1$, where:*

$$s_{R,M} = \frac{1}{1 + \frac{1-\chi}{\chi}\frac{1+r_b}{1+r_m}\frac{\hat{I}}{\mathbb{E}(\phi)}}.$$

Figure 2.3 depicts the lending menu $\mathcal{S}(e)$ when $\delta = 1$, using the results of proposition

2.16.[14] The lending menu is plotted in $(b, m)$ rather than $(d, s)$ space, but there is a simple correspondence between the two: each value of $s$ corresponds to a straight line passing through 0 and with slope $\frac{1-s}{s}$, and along each of these lines, total debt $d$ increases.

Note first that the bank share of external liabilities $s = \frac{b}{b+m}$ spans the lending menu, as it varies from $s = 0$ (which corresponds to the the vertical axis) to $s = 1$ (which corresponds to the horizontal axis). As emphasized in the general case, the lending menu contains only elements in $\mathcal{S}_K(e)$ if $s \leq \overline{s}_R$, that is, to the left of the solid black line with slope $\frac{1-\overline{s}_R}{\overline{s}_R}$); and only elements in $\mathcal{S}_R(e)$ is $s > \overline{s}_K(e)$, that is, to the right of the solid black line with slope $\frac{1-\overline{s}_K}{\overline{s}_K}$).

The set $\mathcal{S}_K(e)$ corresponds to the light gray area of the graph. This set comprises debt structures such that the bank share of external liabilities $s = \frac{b}{b+m}$ ranges from 0 to $\overline{s}_K$ (corresponding to the solid black line with slope $\frac{1-\overline{s}_K}{\overline{s}_K}$). For this range of bank shares, the upper bound on borrowing using a K-contract, $\overline{d}_R(s, e)$, is the dotted line at the boundary of $\mathcal{S}_K(e)$. This frontier corresponds to the pairs $(b, m)$ such that condition $(\mathrm{c-joint})$ is binding, and it is downward sloping. Intuitively, this indicates that bank and market borrowing, for this type of contract, are "substitutes", in the sense that a marginal increase in bank borrowing tightens the market borrowing constraint (and conversely). This is because for K-contracts, given the seniority of bank debt, a marginally larger amount of bank liabilities makes it less likely that market liabilities will be repayed. Note that for $s < \underline{s}_R$, the lower bound on total debt for debt structures belonging to $\mathcal{S}_K(e)$ is 0, while for $\underline{s}_R \leq s \leq \overline{s}_K$, it coincides with the upper bound on total debt of the other set of debt structures, $\mathcal{S}_R(e)$. Appendix $B.1$ shows that this boundary corresponds to the conditions $(\mathrm{frontier-R})$ and

[14] Other model parameters, and in particular the shock distribution, are identical to those of the baseline calibration of the model, discussed in section 2.4. In this example, $e = 10$; model parameters are choosen in such a way that a firm with internal finance $e = 100$ is indifferent between borrowing from market lenders and having no debt.

(frontier – K), which exactly coincide in the range $\underline{s}_R \leq s \leq \overline{s}_K$. For this range of debt structures, a particular bank share $s$ can therefore correspond to either a K-contract (if total borrowing is small enough, ie below $\overline{d}_R(s, e)$), or to an R-contract (if total borrowing is large enough, ie above $\overline{d}_R(s, e)$).

The set $\mathcal{S}_R(e)$ corresponds to the dark gray area of the graph. In this region, the upper bound on total borrowing is associated to the dashed blacked line $\overline{d}_R(s, e)$. For debt structures such that $\overline{s}_K \leq s \leq s_{R,M}$, this dotted line corresponds to cases in which the constraint (c – market) is binding, while for debt structures such that $s \geq s_{R,M}$, this line corresponds to the bank borrowing constraint (c – bank). The slope of the frontiers thus indicate that and increase in the bank share loosens the market borrowing constraint so long as the bank borrowing constraint is not binding, and thus leads to higher total borrowing limits. In that region, the lending menu thus exhibits "complementarity", contrary to the part of the lending menu associated with K-contracts.

In this section, I have proposed a model of market and bank debt pricing based on the view that bank debt is easier to renegotiate than market debt, and I have derived the set of feasible debt contracts for an entrepreneur with internal finance $e$ implied by the model. This set was derived under the relatively weak assumptions that credit markets were perfectly competitive and that entrepreneurs' utility was increasing in monetary payoffs. The key insight from the analysis of feasible debt structures is that, regardless of entrepreneur's internal ressources $e$, renegotiation of debt in times of financial distress is only desirable for firms that choose to hold a sufficient amount of bank debt relative to market debt. I next turn to drawing the implications of these findings for the optimal choice of debt structure.

## 2.3 The optimal choice of debt structure

This section adresses the question of which debt structure, among those that are feasible, an entrepreneur with internal finance $e$ will choose to finance investment. Furthermore, I explore how both the optimal share of borrowing financed by banks $s = \frac{b}{b+m}$ as well as the optimal total amount of borrowing $b + m$ vary with own equity $e$.

Throughout the section, I maintain the two following assumptions:

**Assumption 2.11.** *The enterpreneur is risk-neutral, and her assets completely depreciate after productivity is realized and output is produced: $\delta = 1$.*

I first assume that assets fully depreciate at the end of period 1, that is, $\delta = 1$. Given the static nature of the model, this is a natural assumption, and it furthermore simplifies the analytical characterization of the optimal debt structure. It is however is not crucial to any of the results below. The second assumption I maintain in this section is that the entrepreneur is risk-neutral. While risk-neutrality can be viewed as a benchmark case, it influences strongly the results, by linking closely profit maximization and maximization of total surplus from investment, as I analyze below.

### 2.3.1 The entrepreneur's profit function

Under assumption 2.11, the optimal debt structure of an enterpreneur with own equity $e$ solves:

$$\hat{\pi}(e) = \max_{b,m \in \mathcal{S}(e)} \mathbb{E}\left[\tilde{\pi}(\phi; e, b, m)\right],$$

where $\tilde{\pi}(\phi; e, b, m)$ denotes the profits accruing to the entrepreneur, conditional on the debt structure $(b, m)$ and therefore the associated contract $(R_b, R_m)$, and the realization of the shock $\phi$.

**Proposition 2.12.** *For $(b, m) \in \mathcal{S}(e)$, the objective function of the entrepreneur is given by:*

$$\mathbb{E}\left[\tilde{\pi}(\phi; e, b, m)\right] = \underbrace{\mathbb{E}(\pi(\phi)) - (1+r_b)b - (1+r_m)m}_{\textit{total expected surplus from investment}} \underbrace{- (1-\chi)\int_0^{\underline{\phi}(e,b,m)} \pi(\phi) dF(\phi)}_{\textit{expected liquidation losses}}, \quad (2.8)$$

*where:*

$$\underline{\phi}(e, b, m) = \begin{cases} \frac{R_K(b,m,e)}{(e+b+m)^\zeta} & \textit{if } (b, m) \in \mathcal{S}_K(e) \\ \frac{R_{m,l}(b,m,e)}{(1-\chi)(e+b+m)^\zeta} & \textit{if } (b, m) \in \mathcal{S}_R(e) \end{cases}$$

This result is a consequence of risk-neutrality of the lenders and the entrepreneur: under risk-neutrality, profit maximization for the entrepreneur is equivalent to the maximization of total expected surplus, net of the losses incurred in case liquidation is carried out. In particular, in the absence of bankruptcy costs (that is, when $\chi = 1$), profit maximization for the firm is equivalent to maximization of total surplus. In this case, it is clear that the optimal debt structure is always a corner solution, with the entrepreneur borrowing only from the lender with the smallest cost of funds, as described in the lemma below.

**Lemma 2.13** (The optimal debt structure in the absence of liquidation losses)**.** *Assume there are no liquidation losses, that is: $\chi = 1$. Then, the solution to the entrepreneur's problem is given by:*

$$\forall e, \quad \hat{k}_i \equiv \left(\frac{\mathbb{E}(\phi)}{1+r_i}\right)^{\frac{1}{1-\zeta}}, \quad \hat{\pi}(e) = (1+r_{min})\left(\frac{1-\zeta}{\zeta}k^* + e\right),$$

*where $r_{min} = \min(r_b, r_m)$. Moreover:*

- *If $r_b < r_m$, the optimal debt structure is entirely bank-financed: $\forall e$, $\hat{b}(e) = \hat{k}_b - e$, $\hat{m}(e) = 0$.*

- *If $r_b > r_m$, the optimal debt structure is entirely market-financed:* $\forall e$, $\hat{b}(e) = 0$, $\hat{m}(e) = \hat{k}_m - e$.

- *If $r_b = r_m = r$, the entrepreneur is indifferent between all debt structures such that* $\forall e$, $\hat{b}(e) + \hat{m}(e) + e = \hat{k}_m = \hat{k}_b$.

### 2.3.2 Lenders' cost of funds

Lemma 2.13 emphasizes that the relative cost of funds of lenders is a crucial determinant of the optimal debt structure; when $\chi = 1$, the entrepreneur only borrows from the lender with the smallest cost of funds.

When $\chi < 1$, the entrepreneur's expected profits depend not only on the relative cost of funds, but also on the magnitude of expected liquidation losses, the second term in the right hand side of equation (2.8). Debt structure plays a role in determining the magnitude of these losses, because it affects how often the project must be liquidated at the debt settlement stage. Generically, for a comparable total amount of lending $b+m$, debt structures $(b, m) \in \mathcal{S}_R(e)$ will be liquidated less often. Indeed, the very point of an R-contract is that, by restructuring bank debt, it offers the entrepreneur a means of avoiding liquidation even when her ressources fall below "natural" liquidation threshold, that is, when productivity is such that $\pi(\phi) < R_m + R_b$ (see figure 2.1). Thus, by choosing an R-contract, the entrepreneur will reduce expected liquidation losses. At the same time, corollary 2.9 indicates that debt structures $(b, m) \in \mathcal{S}_R(e)$ typically feature more bank debt; that is, $s = \frac{b}{b+m}$ is closer to 1. By borrowing more from bank lenders, the entrepreneur thus reduces her expected liquidation losses.

Thus, intuitively, the optimal debt structure should limit expected liquidation losses, while at the same time it should feature as much debt as possible from the lender that has the smallest cost of funds, as suggested by lemma 2.13. It is clear that these objectives need not conflict with one another. If $r_b \leq r_m$, borrowing more from bank lenders allows the entrepreneur both to minimize her liquidation losses, and to gain from the smaller cost of funds of banks. Given a particular scale of total borrowing $b + m$, when $r_b \leq r_m$, the entrepreneur should always try to use as much bank finance as possible. This intuition is formalized in the following proposition.

**Proposition 2.14** ( The optimal debt structure when $r_b \leq r_m$ )**.** *Assume that banks have a lower marginal cost of funds than markets, that is, $r_b \leq r_m$. Then, the optimal debt structure of an entrepreneur with internal finance $e$ either features no borrowing from markets, $m^*(e) = 0$, or, is such that the bank borrowing constraint,* (c − bank)*, is binding.*

Thus, when $r_b \leq r_m$, the *only* reason for which an entrepreneur would want to issue liabilities to market lenders is that she has already exhausted her borrowing capacity from bank lenders. In that case, only firms with little internal finance $e$ will issue market liabilities, since for large levels of $e$, the bank borrowing constraint is less likely to bind. In turn, this would imply that larger firms are less likely to issue market debt; in particular, the very largest firms would issue only bank debt. When $r_b \leq r_m$, the model thus leads to a counterfactually negative relationship between internal finance and size, on the one hand, and bank debt's fraction of total external liabilities, on the other. This motivates the following assumption.

**Assumption 2.15** (Lending costs)**.** *Bank lenders have a larger marginal cost of funds:*

$$r_b > r_m.$$

A possible interpretation for this assumption is the following. Lenders both have the same marginal cost of funds, $1 + r$, but banks incur an additional costs $c(b)$ per dollar lent.

This cost could arise for two reasons. First, there might be due diligence costs associated to obtaining information that allows the bank to restructure the entrepreneur's liability when profits are low, as in Rajan (1992). Although this information acquisition, and the problems of information revelation associated with it, are not modelled here, the cost function $c(b)$ could be seen as a reduced-form way of modelling them. Second, the bank may face different balance sheet restrictions than market lenders; in particular, banks may face tighter capital requirement. The function $c(b)$ may then stand for the costs of issuing additional bank equity in order to meet those capital requirements when making a additional loan in the amount $b$. With these additional lending costs $c(b)$, the bank's cost of funds becomes $(1+r)b + c(b)b$. In particular, if $c(b) = c$, this formulation of the model is equivalent to the one developed above, with $r_m = r$ and $r_b = r + c$. Overall, assumption 2.15 is not only necessary for the model to deliver relevant comparative statics; it is also consistent with the view that bank loans is more costly process, per unit of debt issue, than the issuance of market debt.

### 2.3.3 Which type of debt structure does the entrepreneur choose?

With the assumption that $r_b > r_m$, there is tension between the two objectives that an optimally chosen debt structure should pursue, that is, limiting liquidation losses while at the same time borrowing as much as possible from lenders with the smallest possible cost of funds. The optimal debt structure chosen by an entrepreneur with internal finance $e$ reflects this tension.

The trade-off between flexibility and cost does not necessarily lead to a debt structure in which the entrepreneur borrows both from market lenders and from bank lenders. The following proposition emphasizes that this depends on the extent of her internal funds. When the entrepreneur has sufficiently large $e$, the gains associated to bank debt flexibility are always outweighed by the gains from borrowing from the cheaper source of funds, the market.

**Proposition 2.16** (Market finance vs. mixed finance)**.** *Assume that banks have a larger marginal cost of funds than markets, that is, $r_b > r_m$. Let $(\hat{b}(e), \hat{m}(e))$ denote the optimal debt structure of an entrepreneur with equity $e$. There exists $e^* > 0$ such that:*

- *if $e > e^*$, $(\hat{b}(e), \hat{m}(e)) \in \mathcal{S}_K(e)$; moreover, the optimal debt structure features "pure market finance":*

$$\hat{m}(e) > 0 \quad , \quad \hat{b}(e) = 0;$$

- *if $e < e^*$, $(\hat{b}(e), \hat{m}(e)) \in \mathcal{S}_R(e)$; moreover, the optimal debt structure features "mixed finance":*

$$\hat{m}(e) \geq 0 \quad , \quad \hat{b}(e) > 0.$$

This proposition states that there exists a switching point $e^*$ in equity $e$, which dictates the type of debt structure that the entrepreneur will choose. For those entrepreneurs with $e < e^*$, it is optimal to choose a debt structure in $\mathcal{S}_R(e)$, which allows for restructuring. On the other hand, for entrepreneurs with $e \geq e^*$, it is optimal to choose a debt structure in $\mathcal{S}_K(e)$; and in fact, it is optimal to choose a debt structure with only market debt.

There are two intuitions behind this proposition. First, among debt structures $(b, m) \in \mathcal{S}_K(e)$, the entrepreneur always prefers those with no bank debt at all. This is because, under a K-contract, restructuring never occurs; so the additional costs of borrowing from banks come with no added benefits of flexibility in restructuring. The two types of debt are equivalent in terms of their effect on liquidation losses, so that part of the trade-off is irrelevant, and the entrepreneur simply issues all her liabilities with whichever lender has the smallest marginal cost of funds. Under assumption 2.15, market lenders have the cheapest cost of funds, so that the optimal debt structure always has $b = 0$, that is, lies on the y-axis in the lending menu depicted in figure 2.3. If instead $r_b \leq r_m$, then the optimal debt structure within $\mathcal{S}_K(e)$ would have been located at the frontier between the two sets $\mathcal{S}_K(e)$ and $\mathcal{S}_R(e)$, and would be locally dominated by some debt structure $(b, m) \in \mathcal{S}_R(e)$ allowing for restructuring.

The second aspect of the proposition is the fact that entrepreneurs with small $e$ prefer debt structures which allow for restructuring, that is, prefer debt structure $(b, m) \in \mathcal{S}_R(e)$. Here, the assumption of decreasing returns to scale plays an important role. To understand this, it is useful to think of the case of no liquidation costs. In that case, decreasing returns imply that projects have an optimal size $e + b + m = \hat{k}_m$, using the notation of lemma B.1. In that case, leverage ratios $\frac{b+m}{e} = \frac{\hat{k}_m - e}{e}$ are decreasing with $e$. In turn, a higher leverage ratio implies a higher probability of liquidation. This is because, for two different levels of equity $e$ and $e' > e$, the distribution of profits is identical (and given by the distribution of $\phi(\hat{k}_m)^\zeta$), but promised repayment $R_b$ and $R_m$ are larger for $e$ than for $e'$, because more total borrowing is needed to with less internal funds, that is, $\hat{k}_m - e > \hat{k}_m - e'$. Thus, absent liquidation costs, the smaller an entrepreneur's internal funds, the higher her probability of being liquidated.

With liquidation costs, the optimal investment scale $\hat{k}_m$ cannot necessarily be reached for any level of $e$, so that two entrepreneurs with different levels of $e$ need not choose the same total scale $k = e + b + m$; in particular, entrepreneurs with smaller $e$ may be limited to debt structures allowing only total a maximum total scale smaller than $\hat{k}_m$. However, to the extent that the elasticity of total borrowing to internal funds $e$ is sufficiently small (in particular, strictly smaller than 1), it will still be the case that leverage ratios decrease with $e$. Thus, a logic similar to the case without liquidation costs applies, and an entrepreneur with little internal funds has a higher probability of being liquidated at the debt settlement stage.

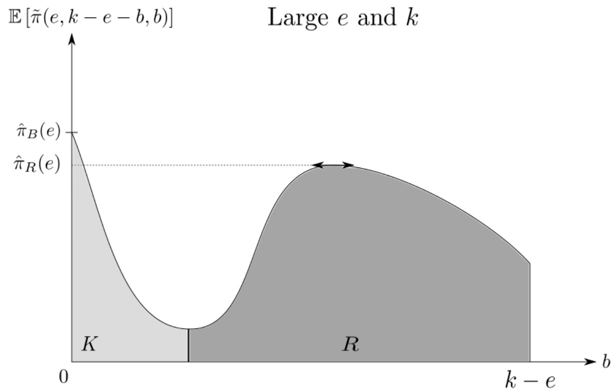



*Figure 2.5: Cross section of the profit function. Two different levels of internal funds e and total investment k are represented: small e and k (left), large e and k (right).*

Thus, for entrepreneurs with small $e$, the gains from choosing a debt structure that allows for debt restructuring, that is, a debt structure which is an element of $\mathcal{S}_R(e)$, are large, relative to the costs of using this type of debt structure (which arise because $r_b > r_m$). On the contrary, with large $e$, the entrepreneur needs little ouside funding; lending is therefore close to risk-free, the benefits of debt flexibility are negligible, and debt structures allowing for restructuring become unappealing to the firm. The switching point $e^*$ thus corresponds to the level of internal funds such that the optimal debt structure within $\mathcal{S}_R(e)$ (which allows for restructuring) and the optimal debt structure within $\mathcal{S}_K(e)$ (which does not) leave the entrepreneur indifferent.

This logic is further illustrated in figure 2.5. The left panel represents a section of the profit function of the entrepreneur along a line corresponding to the optimal total borrowing $\hat{d}(e) = \hat{b}(e) + \hat{m}(e)$ associated with equity level $e$. This corresponds, graphically, to a line with slope $-1$ in the lending menu depicted in figure 2.3. The region marked with a $K$ corresponds to debt structures in $\mathcal{S}_K(e)$. In this region, debt structures are tilted towards market debt, and the optimum $\hat{\pi}_B(e)$ within that region is attained for $b = 0$, the leftmost point on the graph; this corresponds to the first point discussed above. However, the global maximum of the entrepreneur's profit function, $\hat{\pi}_B(e)$ corresponds to the local maximum in region marked with an R, which contains the debt structures in $\mathcal{S}_R(e)$ along the line $b+m=\hat{d}(e)$. In this case, $e$ is sufficiently small that the benefits of flexiblity outweigh the costs of borrowing from banks rather than markets; the resulting debt structure is mixed, between bank and market finance. The right panel of the figure looks at a similar cross-section, for a value of internal funds $e' > e$. In this case, the leftmost point in the region of K-contracts is the global optimum; the firm then chooses a pure market debt structure. The switching point $e^*$ corresponds to the case in which the two local maxima of the regions associated with K and R contracts are equal: $\hat{\pi}_K(e) = \hat{\pi}_R(e)$.

Summarizing, the first important prediction of the model is that firms with large amounts of internal funds $e$ will choose to finance themselves through market debt, while firms with small amounts of internal finance will rely on a mix of market and bank debt dominated by bank debt.

### 2.3.4 The optimal debt structure under R-contracts

Next, I turn to characterizing more precisely the optimal debt structure when $e < e^*$. In that case, following proposition 2.16, the optimal debt structure $(\hat{b}(e), \hat{m}(e))$ is an element of $\mathcal{S}_R(e)$, and is therefore associated to an R-contract. The key results are summarized in the following proposition.

**Proposition 2.17** (The optimal debt structure when $e \leq e^*$)**.** *Assume $r_b > r_m$. Consider an entrepreneur with internal funds $e < e^*$ and let $\hat{s}(e) = \frac{\hat{b}(e)}{\hat{b}(e)+\hat{m}(e)}$ denote the fraction of total liabilities that are bank debt in her optimal debt structure, and let $\hat{d}(e) = \hat{b}(e) + \hat{m}(e)$ denote total borrowing. Then, there exists $\tilde{e} < e^*$ such that:*

- *For $0 \leq e < \tilde{e}$, the bank borrowing constraint is binding at the optimal debt structure, $\frac{\partial \hat{s}}{\partial e} > 0$ and $\frac{\partial \hat{d}}{\partial e}$;*

- *For $\tilde{e} \leq e \leq e^*$, the optimal debt structure of the firm satisfies:*

$$\hat{s}(e) = 1 - \frac{\Gamma}{1 + r_m} \frac{(\hat{k}_{int})^{\zeta}}{\hat{k}_{int} - e}, \quad \hat{d}(e) = \hat{k}_{int} - e \tag{2.9}$$

*where the expression of $\Gamma$ and $\hat{k}_{int}$ are given in appendix B.2. In particular, $\frac{\partial \hat{s}}{\partial e} \leq 0$ and $\frac{\partial \hat{d}}{\partial e} \leq 0$.*

**The bank share**

Proposition 2.17 states that the optimal debt structure is such that the optimal bank share $\hat{s}(e)$ is non-monotonic in the entrepreneur's own equity, $e$. This is illustrated in figure 2.6, which plots $\hat{s}(e)$ as a function of $e$.[15]

To understand why the banking share of liabilities has a non-monotonic relationship with internal funds, it is useful to note that the derivative of the objective function of the entrepreneur with respect to the share of bank debt, $s = \frac{b}{b+m}$, using equation (2.8), is given by:

$$\frac{\partial \mathbb{E}(\tilde{\pi})}{\partial s} = (1 - \chi)\pi \left(\underline{\phi}\right) f(\underline{\phi}) \left(-\frac{\partial \underline{\phi}}{\partial s}\right) - (r_b - r_m)$$

The first term in this expression, $(1 - \chi)\pi \left(\underline{\phi}\right) f(\underline{\phi}) \left(-\frac{\partial \underline{\phi}}{\partial s}\right)$, represents the marginal decrease in expected liquidation losses associated with an increase in the share of bank borrowing, keeping total borrowing fixed. Note that $\frac{\partial \underline{\phi}}{\partial s} < 0$, that is, an increase in bank borrowing keeping total borrowing fixed leads to a reduction in expected liquidation losses, as previously discussed. The second term in this expression, $(r_b - r_m)$, represents the increase in lending costs associated with a marginal increase in bank borrowing, again keeping total borrowing fixed. By borrowing one dollar more from banks and one less from markets, the firm incurs

[15] Parameter values corresponding to the solution depicted in figure 2.6 are identical to those of the baseline calibration explored in the next section; see table B.1 for details.

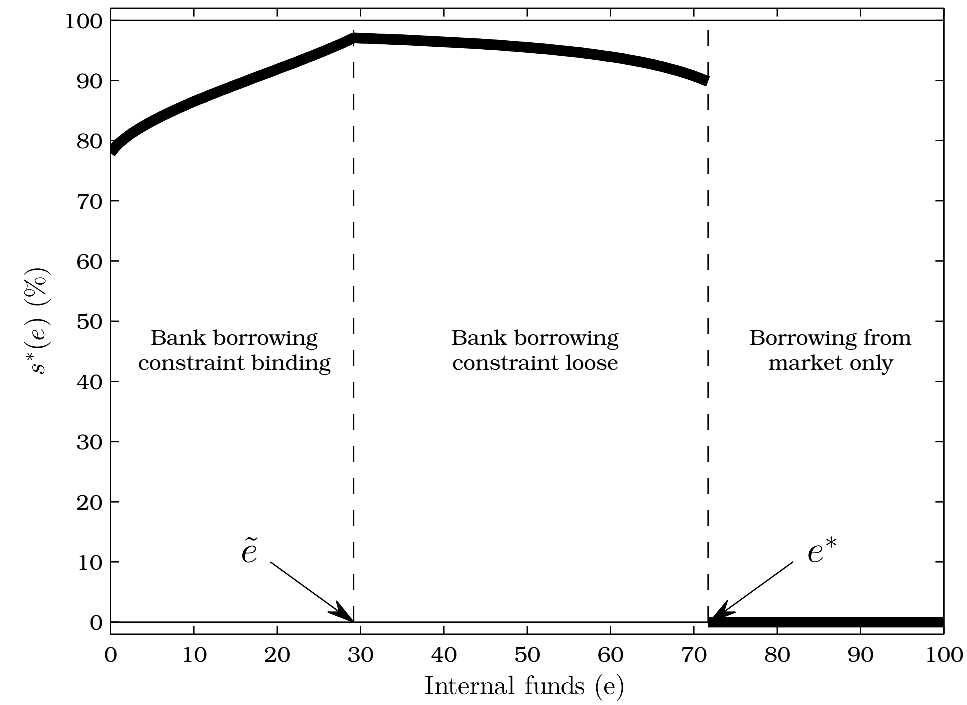



*Figure 2.6: Optimal bank share of total debt.*

an additional cost of $(r_b - r_m) > 0$.

When internal funds are sufficiently small ($e < \tilde{e}$), the bank share is increasing in $e$ because the bank borrowing constraint binds at the optimal debt structure; an increase in $e$, by loosening the bank borrowing constraint, will necessarily lead to more bank borrowing. The bank borrowing constraint itself is binding because, when $e$ is small, marginal reductions in expected liquidation losses associated to more bank borrowing are large. In particular, when $e < \tilde{e}$, any debt structure $(b, m) \in \mathcal{S}_R(e)$ is such that:

$$(1 - \chi)\pi\left(\underline{\phi}\right) f(\underline{\phi}) \left(-\frac{\partial \underline{\phi}}{\partial s}\right) > r_b - r_m.$$

Under this condition, at any debt structure, the marginal value of more bank debt is strictly larger than the associated costs, $r_b - r_m$.The internal funds level $\tilde{e}$ in fact solves $(r_b - r_m) =$

$-(1-\chi)\pi\left(\underline{\phi}\right) f(\underline{\phi})\frac{\partial\underline{\phi}}{\partial s}$, with the right hand side is evaluated at $s = s_{R,M}$ and $d = \overline{d}_R(s_{R,M}, \tilde{e})$; see appendix B.2 for details. However, note that this need not imply that $s = 1$; even though the firm exhausts it bank borrowing capacity, it may still find it profitable to borrow from the market.

Note that, as discussed previously, when the bank borrowing constraint is binding, bank debt is always renegotiated by the firm, regardless of the realization of the productivity $\phi$. In this case, the bank always receives a fraction $\chi$ of the output produced by the firm. Thus, the optimal debt structure, when $e$ is sufficiently small, is such that the bank contract essentially has the feature of an equity contract. Namely, the bank effectively agrees to receive a constant fraction of the firms' output. This contract is beneficial to the entrepreneur because it allows her to avoid very frequent liquidation, given her high leverage.

For a sufficiently large level of internal funds, $e \geq \tilde{e}$, the bank borrowing constraint becomes loose at the optimal debt structure. This is the second case described in the proposition. In that case, the optimal debt structure of the firm satisfies:

$$(1-\chi)\pi\left(\underline{\phi}\right) f(\underline{\phi})\left(-\frac{\partial\underline{\phi}}{\partial s}\right) = r_b - r_m$$

The marginal benefits of bank and market lending are exactly equalized, and the optimal debt structure is an interior point of the set $\mathcal{S}_K(e)$. Expression (2.9) is the analytical solution to this first order condition; in particular, it indicates that bank's share of total debt is decreasing, as a function of internal funds. This result comes from the fact that an increase in internal funds reduces the marginal impact of bank lending on total expected liquidation losses; formally, $\frac{\partial^2\underline{\phi}}{\partial e\partial s} < 0$. As internal funds increase, gains from bank borrowing relative to market borrowing thin out, as the optimal leverage ratio falls and liquidation becomes less likely. Note, additionally, that for a given level of internal funds $e$, the bank's share of total

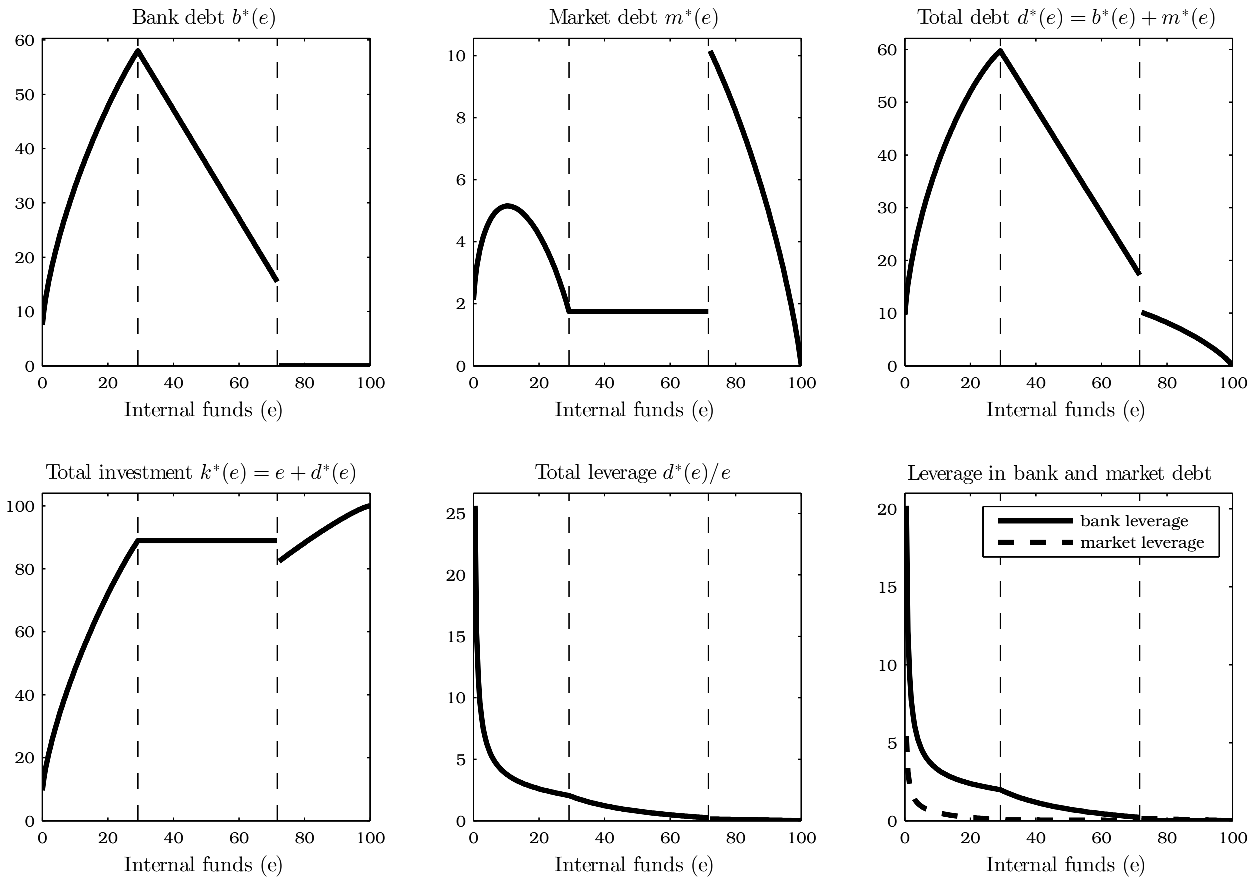



*Figure 2.7: Optimal borrowing and investment policies.*

borrowing is increasing in $r_m$.

### Total borrowing

The second important result from proposition 2.17 is that total borrowing, $\hat{d}(e)$, is non-monotonic in own equity $e$. This is reported in the two top panels of figure 2.7, which plot optimal bank borrowing $\hat{b}(e)$ and optimal market borrowing $\hat{m}(e)$. The two middle panels of figure 2.7 plot total borrowing $\hat{d}(e) = \hat{b}(e) + \hat{m}(e)$ and total investment $\hat{k}(e) = e + \hat{b}(e) + \hat{m}(e)$ as a function of $e$, while the two bottom ones report leverage ratios.

For small values of $e$, total borrowing is increasing, so that total assets are also increasing.

Like in the case of the bank share, total borrowing is increasing for small values of internal funds because in that range, the bank borrowing constraint binds. Total borrowing increases mostly as a result of the increase in bank borrowing associated to the loosening of the borrowing constraint. In that range of $e$, most borrowing originates from the bank. Moreover, since total borrowing increases with $e$, total assets also do. In that region, the leverage ratio of the firm is increasing with $e$.

On the other hand, for $\tilde{e} \leq e \leq e^*$, borrowing falls one for one with equity, so total assets are constant and equal to $\hat{k}_{int}$. On this range, for every dollar increase in own equity, the firm chooses to retire one dollar of bank debt; the amount of borrowing from market lenders, on the other hand, is unchanged. The leverage ratio of the firm is now decreasing with internal funds. Using the results of proposition 2.17, the amount of market borrowing is constant, and given by:

$$\forall e, \quad \hat{m}(e) = \frac{\Gamma(\chi)}{1+r_m}(\hat{k}_{int})^{\zeta}$$

This result, as others reported in this section, does not depend on whether there is full depreciation of assets or not; that is, when $\delta = 1$, total borrowing is still constant over a certain range of equity. However, there is no simple analytical characterization of borrowing and the debt structure in that case.

Summarizing, I have shown that the model outlined in section 2.2, in which debt structure is the result of a trade-off between bank flexibility and the relative cost of funds of lenders, predicts a broadly decreasing relationship between a entrepreneur's internal funds $e$ and the share of bank debt employed in her optimal debt structure. The bank share is in fact increasing for the very smallest firms, after which the entrepreneur retires bank debt as her internal ressources increase, first progressively, then – as $e$ reaches the threshold $e^*$ – abruptly switching to market debt only.

I next turn to analyzing the dependence of debt structure and borrowing on other factors than internal funds, and in particular, the cost of funds of borrowers, and the distribution of entrepreneur's productivity shocks, $F(.)$.

## 2.4 Comparative statics of the debt structure

In this section, I ask how the optimal debt structure of firms changes in response to changes in fundamentals. I focus, in particular, on the effects of changes in the cost of funds of lenders, $r_b$ and $r_m$, and of changes in the distribution of productivity shocks $F(.)$.

While I provide some general results on these comparative statics, most of the discussion focuses on the comparison between numerical solutions to the model. In all calibrations, the baseline case is as follows. First, I assume full depreciation, $\delta = 1$. Second, I choose a degree of returns to scale of $\zeta = 0.92$, in line with the gross output estimates of Basu and Fernald (1997) for the US manufacturing sector. Third, I use the estimates of Bris, Welch, and Zhu (2006) for the gap between pre and post-bankruptcy asset values. Their mean and median estimates of post- to pre-bankruptcy asset values, are, respectively, 38.0% and 80.0%; I choose an intermediate value and set $\chi = 0.60$, implying a post- to pre-bankruptcy value of assets of 60.0%. Finally, I assume that $\phi$ follows a Weibull distribution.[16] In the baseline calibration, the location parameter of the distribution is normalized so that a firm with internal finance $e = 100$ is indifferent between borrowing from the market and using only its internal funds to invest in physical assets. Finally, in the baseline calibration, the cost of funds of bank and market lenders are set to $r_m = 5\%$ and $r_b = 6\%$, respectively. This calibration is summarized in B.1, along with the other calibrations discussed in this section.

[16]Unlike the log-normal distribution commonly used in the literature, this distribution has the advantage of having an increasing hazard rate, in accordance with assumption 2.1.

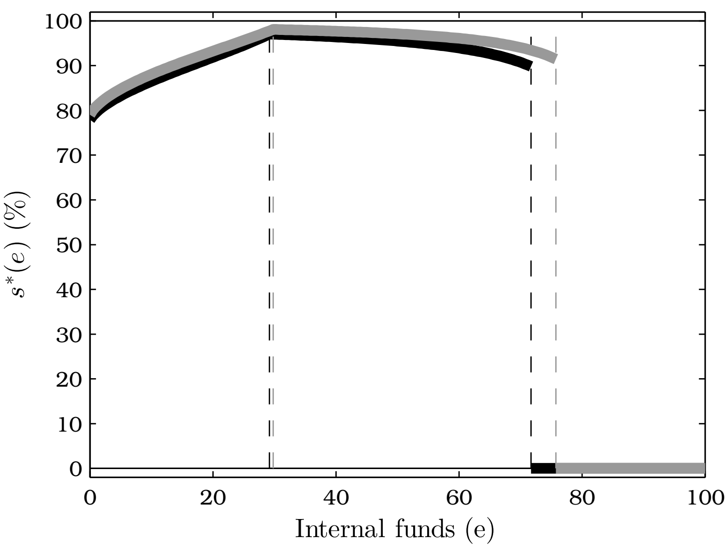



*Figure 2.8: The effect of an increase in the spread $r_b - r_m$ on bank share's of total debt. The black line is the bank share in the baseline calibration; the grey line is the bank share after an increase in the spread $r_b - r_m$.*

### 2.4.1 The effect of an increase in banks' cost of lending

I first ask what patterns of change in debt structure the model predicts, when banks' cost of funds increase relative to market lenders'. The following proposition provides a general result on the effect of an increase in the spread $r_b - r_m$.

**Proposition 2.18** (The threshold between mixed and market finance)**.** *The threshold $e^*$ is a decreasing function of the lending cost of banks $r_b$ and of the spread $r_b - r_m$.*

This proposition implies that any reduction in the spread between bank and market will induce a switch from mixed to market finance for firms with sufficiently high $e$. Specifically, for any $(r_b, r_m)$ and $(\hat{r}_b, \hat{r}_m)$ such that $r_b - r_m < \hat{r}_b - \hat{r}_m$, all firms with $e^*_{\hat{r}_b - \hat{r}_m} < e < e^*_{r_b - r_m}$ will choose a mixed debt structure under the spread $r_b - r_m$, but will move to borrowing only from market lenders under the higher spread $\hat{r}_b - \hat{r}_m$.

To illustrate this, figure 2.8 reports the optimal bank share as a function of internal finance, in the baseline case $r_m = 5\%$ and $r_b = 6\%$, and in a case where $\hat{r}_m = r_m = 5\%$

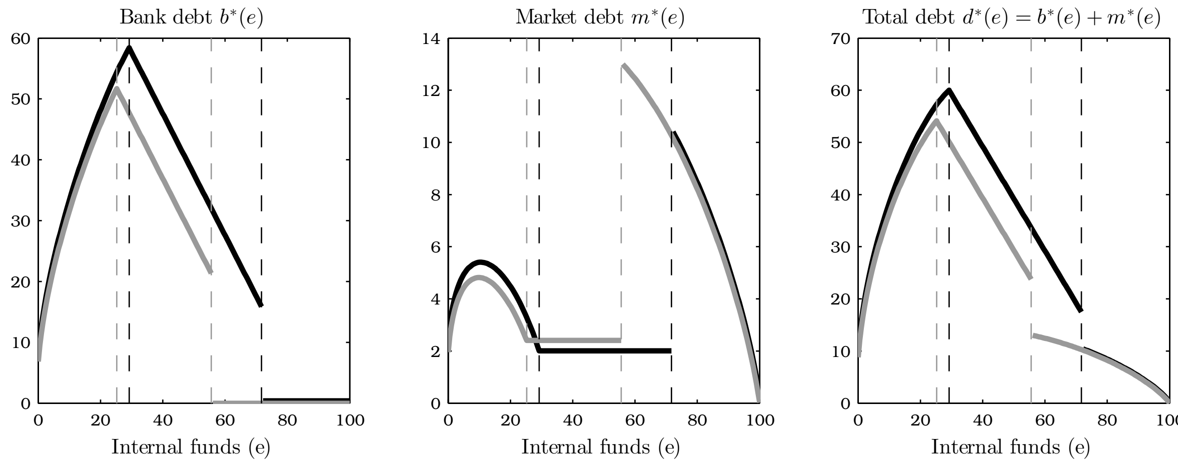



*Figure 2.9: The effect of an increase in the spread $r_b - r_m$ on borrowing. The black line corresponds to the baseline calibration; the grey line corresponds to a higher lending spread (see table B.1 for details on the calibrations used).*

but $\hat{r}_b = 7\%$. The red line corresponds to the optimal debt structure under the high spread, while the green line corresponds to the optimal debt structure under the low spread. In this case, the increase in spread occurs because of an increase in banks' cost of funds; the markets' cost of funds, however is unchanged. For this reason, any firm with $e > e^*_{r_b - r_m}$ is unaffected by the change in borrowing costs. For firms with $e < e^*_{r_b - r_m}$, several features of the change in optimal debt structure are noticeable.

First, as indicated by proposition 2.18, the threshold for switching to bank finance decreases and consequently, firms with a sufficently large internal funds switch to a debt structure with only market debt. Second, for firms with internal finance below $e_{\hat{r}_b - \hat{r}_m}$, the main effect of the increase in spreads is a large reduction in bank borrowing. This is visible in figure 2.9, which reports the borrowing policy functions of firms under the low and high spread calibrations. Firms that are not at their bank borrowing constraint reduce significantly there is very little change in market debt. In fact, market borrowing somewhat falls. Constrained firms, however, experience a smaller reduction in bank credit. Market borrowing changes very little. It increases slightly for firms that are not at their bank borrowing constraint, while, for firms whose borrowing constraint tightens as a result of the increase in banks' borrowing costs, it falls.

It is natural to ask whether the model predicts that a joint increase in the lending costs of both types of lenders has similar consequences. Figures B.1 and B.2 in appendix B.3 adress this question. There, I document the effect of an increase of $r_m$ from 5% to 6% and of $r_b$ from 6% to 7%. Figure B.1 corresponds to the calibration reported under "High cost levels (1)" in table B.1. In this calibration, to keep things comparable to the baseline case, I also reduce the location parameter of the productivity distribution in such a way that the largest internal finance level is unchanged under the new calibration.[17] In this case, the joint increase in the level of borrowing costs leaves borrowing and the bank share almost completely unchanged. By contrast, figure B.2 reports the debt structure when the location parameter of productivity is kept fixed, according to the calibration reported under "High cost levels (2)" in table B.1. In this case, the maximum level of internal finance falls substantially, as the unconstrained size of the entrepreneur's project declines (see lemma 2.13). As a result, borrowing in both types of debt falls. Particularly, overall borrowing falls for the largest firms – in contrast with both fact 3 and with the model's prediction in the case of an increase in the spread $r_b - r_m$.

This exercise suggests that while joint increases in lenders' costs of fund depresses the level of overall borrowing, it does not alter its composition. This is furthermore true across the spectrum of internal finance levels. Some degree of asymmetry in the increase in lending costs, that is, some variation in the spread $r_b - r_m$, is thus needed to account for changes in the composition as well as the scale of borrowing in the cross-section of firms.

[17]That is, the level of internal finance at which firms are indifferent between borrowing from the market and lending risk-free.

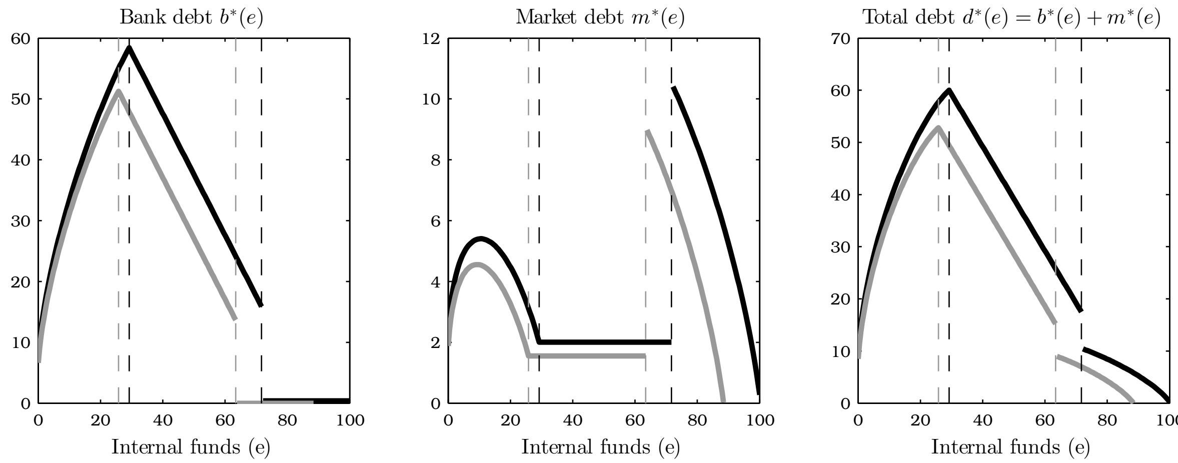



*Figure 2.10: The effect of a fall in average productivity on borrowing. The black line corresponds to the baseline calibration; the grey line corresponds to lower average productivity (see table B.1 for details on the calibrations used).*

### 2.4.2 The effect of changes in the distribution of firms' productivity

I next turn to discussing the relationship between the distribution of productivity shocks, $F(.)$, and the optimal debt structure.

#### A fall in average productivity

I first look at the impact of a fall in the average productivity of firms, $\mathbb{E}(\phi)$. Specifically, I consider a change in the location and scale parameters of Weibull distribution such that average productivity, $\mathbb{E}(\phi)$, falls by 1% relative to the baseline calibration, while the second moment of productivity is unchanged.[18]

The resulting changing in the optimal debt structure is reported in figure 2.10.[19] Much as a joint increase in lenders' cost of funds, the fall in average productivity has similar effects

[18]The corresponding calibration is reported in table B.1, under the column "Low average productivity".

[19]Figure B.3 in appendix B.3 reports changes in borrowing in more detail.

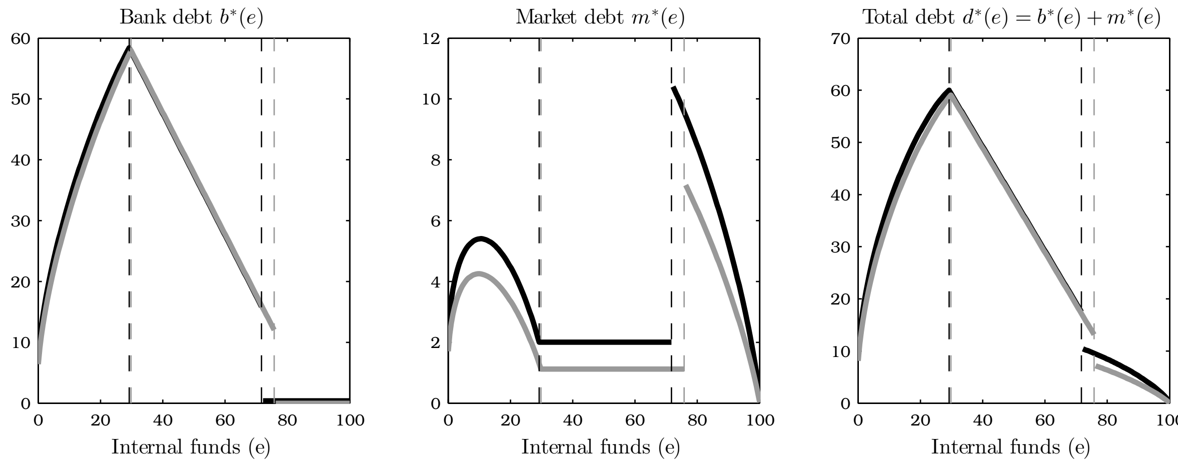



*Figure 2.11: The effect of an increase in the dispersion of productivity. The black line corresponds to the baseline calibration; the grey line corresponds to higher productivity dispersion (see table B.1 for details on the calibrations used).*

on borrowing both from banks and from the markets: optimal borrowing in both types of debt is strictly smaller, at all levels of internal funds. Note, additionally, that the switching threshold $e^*$ moves to the left, since the maximum size of the firm also falls. As a result, the change in average productivity has a pure scale effect, but does not alter the composition of debt across firms, similarly to joint changes in lending costs.[20]

### An increase in the dispersion of productivity

Next, I look at the impact of an increase in the second central moment of the distribution of productivity, keeping the first moment constant. In figure 2.11, I report the result of increasing the coefficient of variation of the distribution by 10 %, relative to the baseline calibration, while keeping the first moment constant. The corresponding calibration is reported in table B.1, under the column "High productivity dispersion".

Note that, contrary to the change in the first moment of the distribution, this change does

[20]This point is made even clearer in figure B.3, which reports the optimal bank share as a function of the level of internal funds. This schedule shifts left, but its shape is unchanged.

not affect the maximum size of the firm, so that the range of relevant levels of internal funds for the firm is unchanged. The result of the increase in dispersion is different for market and bank lending: market lending decreases uniformly, across values of $e$, while bank debt is relatively unchanged. As higher dispersion generically increases expected liqudation losses, the threshold for switching towards market finance increases. The model thus suggests that an increase in the dispersion of productivity should lead to a shift toward more bank-financing for firms with small and intermediate levels of internal finance, mirroring the results on debt composition obtained for the spread between bank and market finance. Note, however, that in both cases, total borrowing and total investment of firms is lower than in the baseline calibration.

Summarizing, I have shown the comparative statics of the optimal debt structure are markedly different depending on whether one focuses on parameters that affect symmetrically the two types of debt (common lending costs, average productivity), or on parameters that have asymmetric effects on their costs or on their return to the firm (the lending spread, productivity dispersion). With a higher spread or higher productivity dispersion, small firms borrow less but remain bank-dependent. Medium-sized firms are more likely to use purely market-financed debt structures with a higher spread, while they are *less* likely to do to so with higher productivity dispersion. These parameters thus have effects on both borrowing scale *and* borrowing composition. By contrast, higher lending costs and lower average productivity are associated with changes in the scale of borrowing, but not in borrowing composition.

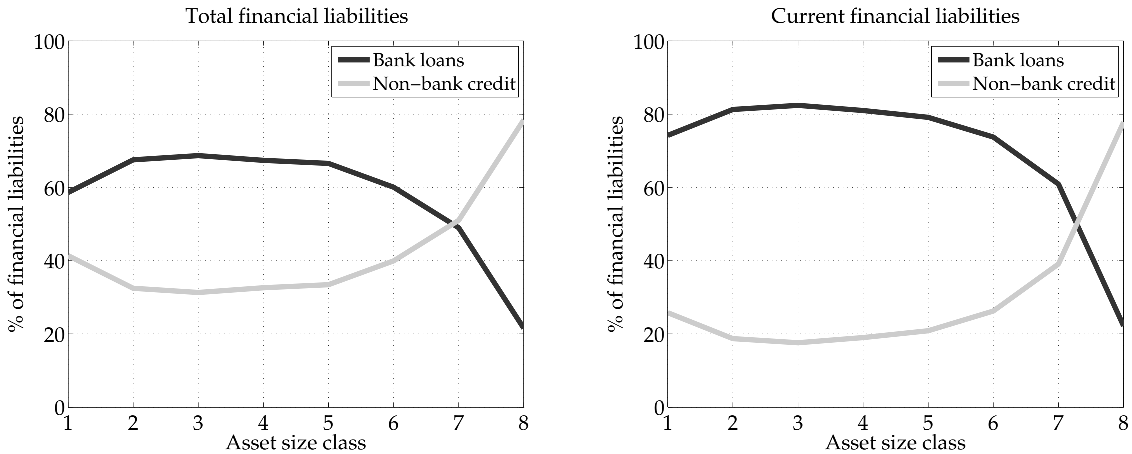
## 2.5 An empirical assessment of the model

I now relate the findings of sections 2.4 and 2.5 with empirical evidence on the debt structure of US firms. In particular, I ask to what extent the cross-sectional relationship between size and debt composition described in section 2.4 is borne out in the data, and explore whether the comparative statics of section 2.5 can help shed light on changes in debt structure of US firms in recent recessions.

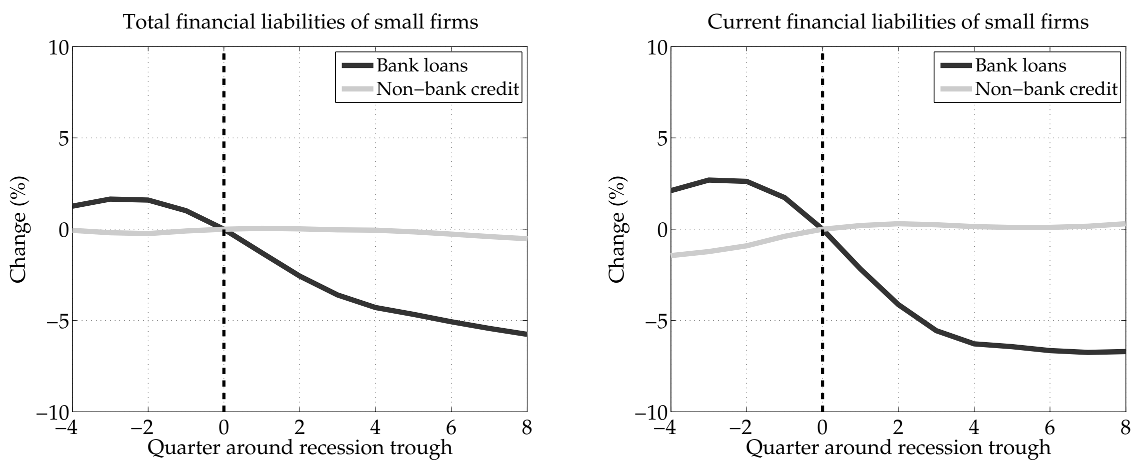
### 2.5.1 Data

I focus on data from the Quarterly Financial Report of Manufacturing firms (QFR). The QFR is a survey of manufacturing firms that reports aggregates of balance sheets and income statements across size bins. The QFR's measure of size is assets at book value. One important advantage of the QFR is that it contains information on firms of all sizes, and in particular smaller, privately owned companies. Moreover, it is a quarterly dataset. The drawback is that data is only available in semi-aggregated form. An alternative is to use firm-level data, such as Compustat. The Compustat sample contains more detailed information on all debt instruments used by firms. It is also not restricted to manufacturing firms. However, it contains only annual balance sheet information for publicly traded companies. Thus, it excludes most smaller, non-traded companies, and has lower frequency of observation.[21] More information on the QFR, as well as a description of the construction of the data used below, is given in appendix B.4.2.

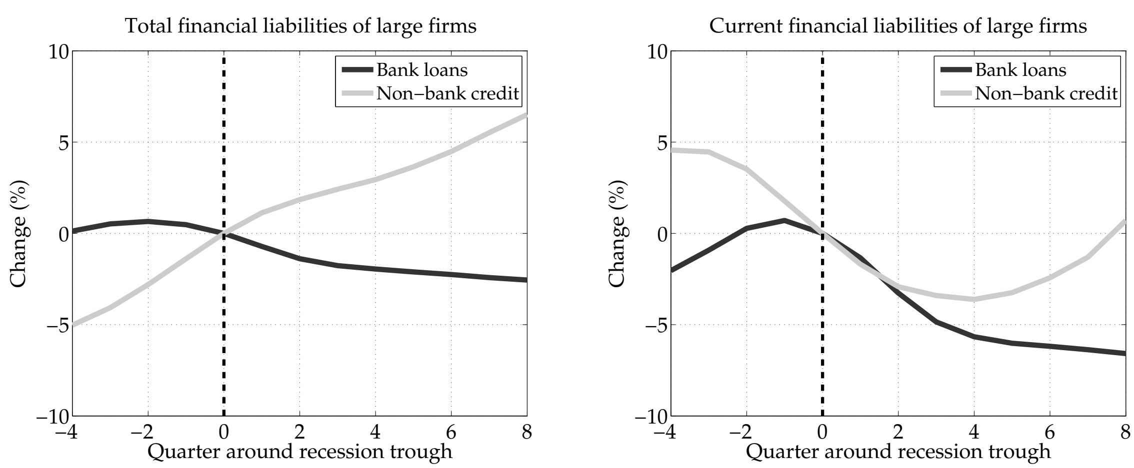
[21] The Compustat sample is focus of the empirical work of Rauh and Sufi (2010), for cross-sectional variation, and Adrian, Colla, and Shin (2012) for business-cycle changes. My results are in line with their findings, and extend them to the sample of private and smaller firms included in the QFR.




(a) The composition of debt in the cross-section




(b) The debt structure of small firms during recent recessions




(c) The debt structure of large firms during recent recessions

*Figure 2.12: Key features of the debt structure of firms in the QFR. In figure 2.12(a), the left panel graphs current liabilities: the ratios $CB/CFIN$ (bank loans, green line) and $CNB/CFIN$ (non-bank credit, red line). The right panel plots total liabilities: the ratios $TB/TFIN$ and $TNB/TFIN$. In figures 2.12(b) and 2.12(c), the right panels plot current liabilities and the left panels plot total liabilities; see text for details. Series in figures 2.12(b) and 2.12(c) are smoothed with a 2 by 4 MA smoother before computing growth rates.*

### 2.5.2 The composition of debt in the cross-section

Figure 2.12(a) illustrates the differences in debt structure between small and large firms, by reporting the average bank share of total (bank and non-bank) debt across different asset size bins. Roughly 70% of the debt of small firms is held in the form of bank loans, while 80% of the debt of the largest firms is held in the form of non-bank credit. Of these 80%, on average over the last 5 years, 60% were held in the form of commercial paper and bonds. This holds whether one focuses on total or current debt. Thus, the majority of the debt of small firms are bank loans, while the majority of the debt of large firms are non-bank credit instruments. Note, however, that the relationship between size and the bank share is not strictly monotonic; for the very smallest firms, the bank share is slightly increasing in size.

In the model, the bank share is, broadly speaking, high for entrepreneurs with little internal funds, and low for firms with large internal funds. However, for firms with internal funds below $e \leq e^*$, the bank share is non-monotonic, but overall relatively large, and in fact sufficiently so for the bank borrowing constraint to bind for entrepreneurs with very small $e$. When $e > e^*$, on the other hand, any debt issuance takes the form of market borrowing, as entrepreneurs find that the costs associated to borrowing from banks completely outweigh benefits of debt flexibility.

This paints a picture which is, at first glance, consistent with the cross-sectional debt structure reported in figure 2.12(a). However, the model predicts a relationship between internal finance $e$ and the bank share of liabilities, while that figure relates total assets to the bank share of liabilities. As discussed previously, the model does not predict a strictly monotonic relationship between size $k(e) = e + \hat{b}(e) + \hat{m}(e)$ and internal funds $e$. One cannot therefore infer from the predicted relationship between internal funds and the bank share, that a similar link between total assets and the bank share exists in the model.

This need not be viewed as a shortcoming of the model. First, in this model $b$ and $m$

represent new debt issuance, which are to be retired after a single period. In particular, the entrepreneur does not inherit legacy assets or long-lived debt that needs to be serviced. In reality, much of the balance sheet of the firms in the sample analyzed in section 2.5 is made up of liabilities maturing over long periods of time. It is also this lack of history-dependence which accounts for the fact that above the level $e^*$, the firm switches to a debt structure entirely composed of market borrowing. On the face of it, this is not a realistic feature of the model, but it should be interpreted as indicating that new debt issuances will take the form of market debt once internal funds have reached the threshold $e^*$.

Second, the assumption of risk neutrality is crucial to obtain the result that total borrowing is constant. With risk aversion, a marginal increase in internal funds would not necessarily need to a marginal fall in bank borrowing, to the extent that bank lending not only reduces expected liquidation losses, but also reduces the variance of the payoff to the entrepreneur. In a general version of the model with risk aversion, the fact that borrowing is constant on a range of levels is internal funds is likely to disappear.[22]

Third, the patter of figure 2.12(a) concerns the complete distribution of firms across asset sizes. The fact that firms are distributed across different asset sizes may originate from other reasons than initial differences in internal funds. Besides history-dependence, one source of firm size heterogeneity that has received particular attention in the literature are differences in the (average) marginal productivity of capital, $\mathbb{E}(\phi)$. Firms with different long-run productivity levels both operate at different scales, and face different borrowing constraints. Taking these long-run differences into account in the model may help shed better light on the asset size / debt structure link, as opposed to the internal funds / debt structure link.

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[22]Results for the dynamic model of chapter 3 illustrate the role of risk aversion.

### 2.5.3 Understanding cyclical changes in debt structure

The comparative statics reported in section 2.4 emphasizes that size is a key determinant of the relationship between debt structure and aggregate conditions. In studying how debt structure changed during the most recent recessions, I will therefore focus on the differences between small and large firms. For creating aggregates of small and large firms, I directly use the fixed bins provided by the QFR. I define small firms as those with $1bn\$$ in assets or less (the first seven size bins of the QFR sample; see table B.2), and large firms as those with more than $1bn\$$ in assets (the last bin of the QFR sample).[23]

The series reported in figures 2.12(b) and 2.12(c) show averages of cumulative growth rates of total financial liabilities of small and large firms, broken down between bank and non-bank debt, over the three recessions in the sample. Cumulative growth rates are computed around NBER recession troughs. I focus on a window of 1 year before the recession trough to 2 years after the recession trough.[24]

Figure 2.12(b) graphs the growth rate in bank loans and non-bank credit for small firms.[25] For these firms, outstanding bank debt falls and non-bank debt does not change, resulting in a fall in total liabilities. This holds regardless of whether one focuses on total liabilities

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[23]This definition of small and large firm categories is transparent, but does not adress the fact that, because of trend growth in the book value of assets, large firm bins become more populated over time. In appendix B.4.3, I consider an alternative definition of small and large firm groups, proposed by Gertler and Gilchrist (1994), which keeps the fraction of total sales accounted for by the small firm group category constant. I show that these facts, which contrast changes in debt structure of small and large firms, are robust to this alternative definition of size classes.

[24]Appendix B.4.4 reports the levels of all the series used.

[25]For each type of liability, the growth rate reported is weighted by its share as a fraction of total liabilities at the recession trough. For example, for total bank loans, the series reported is the average of $\gamma_{TB,t} = \frac{TB_0}{TB_0+TNB_0}\left(\frac{TB_t}{TB_0}-1\right)$ over the three recessions, where the time index is 0 for the quarter corresponding to the trough of a recession. Weighting the growth rates in this fashion conserves additivity: for example, for small firms, $\gamma_{TB,t}+\gamma_{TNB,t}=\gamma_{TFIN,t}$, where $\gamma_{TFIN,t}$ is the growth rate of total financial liabilities of small firms around the recession trough.

(left panel) or only current liabilities (right panel).

Figure 2.12(c) reports the same series as figure 2.12(b), for "large" firms (the last asset size bin of the QFR). The left panel of figure 2.12(c) shows that for large firms, total outstanding banks debt falls, while non-bank debt increases.[26] Note, importantly, that this pattern is driven by new and/or long-dated liabilities. Indeed, no increase in non-bank current liabilities (those maturing in one year or less) is visible, as reported in the right panel of figure 2.12(c).

Since this sample contains only three recessions, one of which, 2007-2009, was substantially larger in magnitude than the others, it is natural to ask whether the patterns documented here are driven mostly by the Great Recession. This question is adressed in appendix B.4.3, where I report figures similar to 2.12(b)-2.12(c), excluding the last recession. There are two differences: first, the overall reduction in bank lending is smaller over these recessions, both for small and large firms; second, substitution towards non-bank credit for large firms is more muted than when including the last recession. It is during the 2007-2009 recession that the changes in the debt structure of large firms were most different from changes in the debt structure of small firms.

The comparative statics of section 2.6 indicate that only variation in the lending spread and in productivity dispersion can account for changes in both the scale and composition of borrowing. Moreover, increases in productivity dispersion are associated with a substitution toward bank borrowing. The model thus suggests that the patterns documented in figures 2.12(b)-2.12(c) are a telltale sign of an asymmetric change in banks' lending cost, an result implicitly underlying much of the empirical work on the bank lending channel, such as for example Kashyap, Stein, and Wilcox (1993) and Gertler and Gilchrist (1994).

[26]Note that, since the increase in non-bank debt is larger than that of bank debt, total financial liabilities are in fact slightly increasing. The increase in total assets may be the artifact of the accumulation of firms in the larger size bins over time; in fact, as reported in appedix B.4.3, total liabilities for large firms do not increase for alternative definitions of the large firm category, but the debt substitution pattern remains.

There is some tension between the model and the patterns of debt substitution for large firms documented in figure 2.12(c). In the model, in response to an increase in the spread, total borrowing for these firms falls; the substitution between bank and market debt is less than one for one. Figure 2.12(c), on the other hand, suggests that the substitution of bank for market credit is at least one for one. As discussed section 2.5, it is likely that, in the data, the magnitude of debt substitution for large firms is overstated by upwards reclassification. Additionally, the model omits important alternative forms of financing than could also serve as substitutes to bank debt, in particular short-term instruments such as commercial paper, which is an important channel of adjustement for firms in times of tight money (see Acharya and Schnabl (2010)), and are therefore likely to account for a share of substitution away from bank debt.[27]

## 2.6 Conclusion

In this paper, I proposed a static model of the joint determination of investment and debt structure for a firm with access to both bank debt and market debt. The model builds on the trade-off theory of the debt structure, according to which a firm's debt structure reflects a trade-off between the flexibility afforded by bank debt in times of financial distress, and the higher marginal costs of lending of banks. The model extends existing studies by allowing the entrepreneur to choose the scale of the project she operates, and thus explicitly modelling the link between investment and debt structure choices.

I showed that this model is consistent with the fact that small firms borrow mostly from

[27]Because commercial paper is no separated from other non-bank liabilities for all asset size classes in the QFR, it was included in non-bank liabilities in the construction of the previous graphs. Excluding commercial paper from non-bank liabilities does not alter these graphs substantially; see appendix B.4.3 for details.

banks, while the debt structure of large firms consists predominantly of market debt. Additionally, the model predicts that in response to an increase in banks' cost of funds, relative to market lenders', firms with sufficient internal finance should optimally substitute their bank liabilities for market debt, while firms with less access to internal finance reduce bank borrowing without increasing market liabilities. This pattern is consistent with changes in the debt structure of US firms, small and large, over the last three recessions, and particularly after the 2007-2009 recession.

This paper suggests three avenues for future research. First, the model emphasizes the importance of lenders' costs of funds $r_b$ and $r_m$ as determinants of the debt structure in the cross-section, and suggests that an increase in the spread $r_b - r_m$ may account well for changes in the debt structure of US firms. An important question is then whether it is possible to construct an empirical counterpart to this spread, and whether it changes substantially over the business cycle. As emphasized by the model, interest rates on loans or bond yields are clearly only upper bounds on these costs since they incorporate premia (liquidation risk premia, in the case of the model). Second, the model postulates that bank debt is senior to market debt. While, empirically, bank debt tends to be placed on top of firms' priority structures, the rationale for this a subject of debate in much of the literature on debt heterogeneity; in fact, in some setups, such as Rajan (1992), the optimal debt structure makes bank debt junior to non-monitored debt. In the context of the model developed in this paper, numerical results suggest the alternative priority structure, whereby bank debt is junior, would not be preferred by the firm, if it were it be given the ability to choose. An important question is whether this result can be generally established, and in particular whether it always holds for any level of internal funds.[28] Third, because of the generality of

[28]Hackbarth, Hennessy, and Leland (2007), in the context of their model with fixed investment size, prove that bank debt seniority is generally preferable.

the results on set of feasible debt structures, the model I developed in this paper can be used to study the dynamic implications of debt heterogeneity for firm investment and growth, as well as to trace out the aggregate effects of shocks on borrowing and investment in an economy with different types of financial institutions.[29] I leave all three topics to future research.

[29] See Reis (2010) for an example of such a model, focusing on the implications of securitization, rather than on the bank/market debt choice.

# Chapter 3

# Corporate debt structure and the macroeconomy

Nicolas Crouzet

## 3.1 Introduction

How do financial frictions affect aggregate activity? Since the onset of the Great Recession, this question has attracted renewed attention among macroeconomists. A central hypothesis is that financial market imperfections limit firms' ability to issue debt in order to finance investment. To formalize this mechanism, much of the literature has focused on total debt issued by firms, and narrowed down the analysis of financial frictions to a situation where firms face a single borrowing constraint.

An important limitation of this approach is that it ignores the fact that, in reality, firms use a broad array of debt contracts to finance investment. This heterogeneity is reflected in the balance sheet of the corporate sector: in the US, for example, bank loans only accounted for 27.5% of the stock of oustanding debt of non-financial corporations on the eve of the Great Recession.[1] Moreover, there is substantial evidence that the use of these different debt instruments varies with aggregate conditions, and in turn affects firms' investment possibilities. Becker and Ivashina (2014) find evidence of substitution between loans and bonds when lending standards are tight, and show that this substitution is a strong predictor of reductions in investment by small firms. Analogously, Adrian, Colla, and Shin (2012) study debt issued for investment purposes, and show that during the 2007-2009 recession, bank-dependent firms experienced a larger decline in debt issuance than firms that with access to markets. This evidence echoes previous empirical work, such as Kashyap, Lamont, and Stein (1994), who show that bank-dependent firms experience larger declines in inventory investment during recessions.

Motivated by this evidence, this paper studies the relationship between aggregate outcomes and debt heterogeneity, defined as the coexistence and availability of different cor-

[1]The remaining 72.5% are mostly accounted for by corporate bonds (63%). Data is from the Flow of Funds for 2007Q3; see appendix C.1 for details.

porate debt instruments. I adress three specific questions. First, what drives long-run differences in the aggregate composition of corporate debt across countries, and how do these differences in financial structure relate to long-run differences in output and investment?[2] Second, how does debt heterogeneity affect the transmission of aggregate shocks? Third, can policy affect debt composition, and if so, what are the effects on aggregate investment and output?

In order to answer these questions, I develop a macroeconomic model with heterogeneous firms in which both the scale and composition of borrowing are endogenous. Section 3.2 describes this model. It is the first model of firm dynamics with financial frictions to endogenize jointly the borrowing composition, investment choices, and firm growth. In this economy, firms can finance investment either internally (through the accumulation of retained earnings), or by issuing two types of debt: bank loans and market debt. Credit is constrained by the fact that firms have limited liability, and default entails deadweight losses of output. The central assumption of the model is that banks and market lenders differ in their ability to deal with financial distress. Specifically, I assume that bank loans can be restructured when firms' revenues are low, whereas market liabilities cannot be reorganized. In this sense, banks offer more flexiblity than market lenders when a firm is in financial distress. On the other hand, outside of financial distress, I assume that bank lending is more restrictive than market lending. In the model, this difference is captured through differences in intermediation costs between bank and market lenders. The higher intermediation costs of banks are reflected in the equilibrium terms of lending contracts, which firms must honor outside of financial distress. In choosing the scale and composition of borrowing, firms therefore tradeoff the higher flexibility of bank debt in financial distress, with the lower costs associated

[2]By contrast to the 27.5% bank share in the US, in Italy bank loans accounted for 65.7% of outstanding corporate debt in 2007Q3; see appendix C.1 for details on the data.

with market financing in normal times.[3]

Section 3.3 analyzes firms' financial policies in steady-state. The model endogenously generates a distribution of firms across levels of internal finance. Two main results characterize variation in the composition and scale of borrowing across this distribution. First, some firms choose to borrow simultaneously from bank and market lenders. This is a key empirical finding of Rauh and Sufi (2010), who nevertheless stress that few models of debt structure have this feature. Intuitively, simultaneous borrowing occurs in the model because issuing liabilities with market lenders may sometimes relax the bank borrowing constraint. Crucially, this complementarity arises because investment and debt structure are jointly determined: borrowing more from markets helps firms increase their scale of operation; this in turn raises the liquidation value of the firm, and therefore relaxes its bank borrowing constraint. Second, a firm's stock of internal finance is negatively related to its bank share, defined as the ratio of bank loans to total debt. As a firm grows, it will therefore reduce its reliance on bank debt. I show that this is consistent with cross-sectional evidence from a number of advanced and developing countries.[4] In the model, this arises because of the interaction between decreasing returns, leverage, and bank flexibility. Because of decreasing returns, firms have an ex-ante optimal investment scale, irrespective of internal ressources. For firms that have large internal ressources, reaching that scale requires less leverage. Those firms will therefore be unlikely to face financial distress, and for them the flexibility gains

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[3]In section 3.2, I discuss at length the empirical support for the view that banks are more flexible creditors than markets in financial distress. I also provide different interpretations of the assumption that banks provide a more constraining form of debt in normal times than markets. For additional discussions of these assumptions, see chapter 2.

[4]Because internal finance is closely related to default probability in the model, this result is also consistent with the findings of Denis and Mihov (2003) and Rauh and Sufi (2010), who both show that credit quality is negatively related to the bank share.

associated with bank debt are of little value.[5]

The remainder of the paper uses this model to study the aggregate implications of debt heterogeneity. Section 3.4 focuses on long-run differences in aggregate debt composition. I show that this composition only depends on structural parameters that affect the trade-off between flexibility in distress and costs outside of distress. In particular, a lower wedge between the intermediation costs of banks and markets increases the aggregate bank share, by making bank borrowing relatively less constraining outside of bankruptcy. Likewise, higher idiosyncratic productivity risk increases the aggregate bank share, because it exposes firms to financial distress more frequently. By contrast, average productivity and average lending costs have no effect on aggregate debt composition. I calibrate the model using US and Italian data on idiosyncratic productivity risk and intermediation costs, and show that these two sources of exogenous variation alone can account for between one half and two thirds of the gap between the US bank share (25.5%) and the Italian bank share (65.7%). In this sense, the flexibility/cost trade-off gives a simple, yet powerful account of cross-country differences in aggregate debt structure. At the same time, the model suggests that these differences are manifestations of costly inefficiency in the intermediation of credit. For example, in the model, lowering Italian market lenders' costs to a level comparable to the US would generate, ceteris paribus, increases in steady-state output and investment in the order of 5%.

Section 3.5 focuses on the response of the model to aggregate shocks. In order to understand whether differences in debt structure lead to different sensitivites to aggregate shocks, I compare the baseline US calibration of the model to a calibration which matches more closely the higher average bank share of European countries. In the US calibration, a shock that increases the cost of lending of banks relative to those of markets leads to a persistent

[5] In fact, the model indicates that firms with sufficient internal finance will only borrow from markets.

fall in the aggregate bank share, as observed in the US since the onset of the Great Recession. This shock also generates a large and persistent fall in output, for two reasons. First, the shock increases financial costs for firms outside of distress, which results in lower borrowing and investment – a traditional intensive margin effect. Second, because the shock only affects the supply of credit and not firms' riskiness, it induces medium-sized firms to switch to an entirely market-financed debt structure. This is optimal from the standpoint of profitability; however, these switching firms also renounce the flexibility associated with bank borrowing. The debt structure these firms adopt is effectively more fragile, as it exposes them more frequently to inefficient liquidation. As a result, market lenders charge them high liquidation premia. This in turn has an impact on firms' investment choices. Most switching firms in fact choose to operate at a smaller scale than they did before the shock. This extensive margin effect of banking shocks on investment quantitatively accounts for roughly as much of the fall in total output than the intensive margin effect. Moreover, the same shock, in the European calibration the model, also generates a fall in output, but the fall is deeper, by 15-30%, than in the baseline US calibration. Extensive margin effects are particularly strong in the European calibration, because firms in that case have larger idiosyncratic volatility than in the baseline, so that they must drastically reduce their leverage when switching to the more fragile, market-only debt structures.

The long-run and business-cycle analyses of sections 3.4 and 3.5 point to complementary reasons for encouraging the development of alternatives to bank lending: long-run gains in aggregate investment and output, and a reduction of cylcical sensitivity to banking shocks. However, as suggested in the previous discussion, a shift toward disintermediation may have the adverse effect of pushing firms into choosing more fragile debt structures. In section 3.6, I use this insight to analyze the effects of two policies aimed at promoting disintermediation in Europe: German efforts to develop a bond market specifically targeted for small and medium-sized firms, and an Italian fiscal reform extending tax deductibility of interest payments to bond issues by private firms. In the model, these reforms have analogous effects: while they boost aggregate investment through their intensive margin effects on small and large firms, they also induce medium-sized firms that were previously partially bank-financed to switch entirely to market finance. As a result of their increased fragility, these firms borrow less, and their output and investment falls. The net effect of the policy on aggregate investment and output in steady-state is in general positive, but this comes at the expense of a precautionary reduction in leverage and activity of medium-sized firms.

**Related literature** This paper builds on the extensive literature on corporate debt structure, following the seminal contributions of Diamond (1991), Rajan (1992), Besanko and Kanatas (1993) and Bolton and Scharfstein (1996). While I do not model it explicitly, the assumption that bank and market lending differ in their degree of flexibility in times of financial distress builds on the insight of Bolton and Scharfstein (1996) that the dispersion of market creditors reduces individual incentives to renegotiate debt payments. The closest model to the one I develop in this paper is Hackbarth, Hennessy, and Leland (2007). This paper, along with most of theoretical literature on the topic, focuses on the structure of financing, given a fixed scale of investment; by contrast, in the model I propose, the scale of investment is also determined endogenously, which allows me to draw cross-sectional and aggregate implications of debt structure for output and investment.

This paper is also related to the literature on firm growth and financial frictions. My model's key friction is limited liability, as in Cooley and Quadrini (2001), Clementi and Hopenhayn (2006) or Hennessy and Whited (2007). In particular, the connection between the firms' optimal financial policies and their steady-state growth dynamics follows closely Cooley and Quadrini (2001). I contribute to this literature by introducing an endogenous debt structure choice, and illustrating its implications for firm growth and the distribution of firms across levels of internal finance in steady-state.

The macroeconomic implications of debt heterogeneity have been adressed by relatively few papers. Bolton and Freixas (2006), in the context of a static model, show that, by affecting the spread of bank loans over corporate bonds, monetary policy can lower banks' equity-capital base, in turn leading to a contraction in corporate credit. This channel is separate from the traditional "bank lending channel", which operates through reductions in bank reserves. My model does not allow me to distinguish between causes of contractions bank lending; its focus is purely and squarely on their consequences for firm-level and aggregate investment. De Fiore and Uhlig (2011) study an asymmetric information model of bond and bank borrowing, and show that the model accounts well for long-run differences between the Euro Area and the US to the extent that information availability on corporate risk is lower in Europe. Their findings suggest a complementary channel through which productivity risk could affect the aggregate bank share, separate from that of bank flexibility which I study in this paper. My model additionally has richer cross-sectional predictions; in particular, it is consistent with the fact that firms use mutliple types of debt instruments simultaneously, whereas, in their model, individual firms use a single type of liability.

Finally, this paper draws from the results of chapter 2. In particular, the characterization of firms' feasible set of debt structures, as a function of their internal ressources, is similar in the static setup of that paper, and the dynamic model considered here.

## 3.2 A model of debt composition

In this section, I describe a dynamic model in which heterogeneous firms interact with heterogeneous financial intermediaries. Firms have access to a decreasing returns to scale technology which takes capital as an input. Investment in capital is financed using three sources of funds: bank debt ($b_t$), market debt ($m_t$) and internal finance ($e_t$).[6] Firms are infinitely-lived, but both bank and market debt take the form of one-period contracts.

In this economy, firm investment is constrained for two reasons. First, firms have limited liability. A firm can choose to default on its debt obligations, and this default may entail the liquidation of the firm and the transfer of its resources to its creditors. Liquidation is inefficient: it involves deadweight losses of output. The second source of frictions in this economy is that firms cannot issue equity, and instead only accumulate internal finance through retained earnings.[7] Absent either friction, the firm would be able to finance investment to its optimal scale. With frictions to debt and equity issuance, the firm will, over time, accumulate retained earnings in order to fund investment internally, and limit its dependence on debts. In this sense, internal finance $e_t$ also represents the net worth of the firm; I will use the two terms interchangeably.

### 3.2.1 Overview of firms' problem

The productive sector is composed of a continuum of firms characterized by $e_t \geq 0$. All firms discount dividends using the same discount rate $\beta$. At the beginning of period $t$, the present discounted value of a firm with internal funds $e_t$ is denoted by $V(e_t)$. Total investment in capital is given by $k_t = e_t + b_t + m_t$. Capital is the sole input in a decreasing returns to

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[6] I describe a recursive competitive equilibrium of this economy in the absence of aggregate shocks; however, I use time subscripts for firm-level endogenous variables in order to emphasize the fact that firms solve a dynamic problem.

[7] The assumption that firms cannot issue equity *at all* is not crucial. It is sufficient to assume that equity issuance is costly for the main results of the paper to hold. This can be done by introducing a fixed marginal issuance cost of equity strictly larger than the marginal cost of lending of banks. However, the assumption of infinite equity issuance costs, effectively maintained here, allows the paper to focus on the relationship between debt composition and investment. Relative to debt and retained earnings, equity issuance accounts for a small fraction of funds used to finance investment of US firms.

scale production function, and depreciates at rate $\delta$. Given the firms' current productivity $\phi_t$, total resources after production are given by:

$$\pi_t = \pi(\phi_t, k_t) = \phi_t k_t^\zeta + (1-\delta)k_t,$$

where $0 < \zeta < 1$ denotes the degree of returns to scale. However, current productivity $\phi_t$ is realized only after the firm has borrowed and invested in productive capital $k_t$. I assume that $\phi_t$ is drawn from a distribution $F(.)$ with mean $\mathbb{E}(\phi)$ and standard deviation $\sigma(\phi)$.[8]

The terms of the borrowing contracts for $b_t$ and $m_t$ are given by $R_{b,t}$ and $R_{m,t}$. They represent the gross promised repayment from the firm to each type of entrepreneur, conditional on the firm not restructuring or being liquidated. These contract terms are agreed upon before the realization of $\phi_t$, and cannot be indexed to $\phi_t$. Financial intermediaries, however, observe the equity $e_t$ of the firm as well as its borrowing from the other financial intermediary ($m_t$ or $b_t$), so that lending terms will depend on $(e_t, b_t, m_t)$. In order to alleviate notation, I omit this dependence in the exposition of the model. I come back to the assumptions governing financial intermediaries' behavior after I discuss the firms' problem.

Given the realization of current productivity $\phi_t$, the firm can choose to either pay $(R_{b,t}, R_{m,t})$ to lenders, or try to restructure its debt; when the realization of $\phi_t$ is sufficiently bad, it may also be liquidated. Each firm's value, at this stage, is denoted by $V^s(\pi_t; R_{b,t}, R_{m,t})$.

After debt settlement, a firm with cash on hand $n_t$ can choose between issuing dividends and accumulating $n_t$ towards internal finance for the following period. After this dividend issuance choice, only a fraction $(1-\eta)$ of firms survive; the rest are destroyed, and their

[8]Productivity is i.i.d. across firms; in this section, it is also identically distributed over time. In section 3.4, I discuss the effects of time variation in $\mathbb{E}(\phi)$ and $\sigma(\phi)$. Additionally, to ensure unicity of lending contracts, $F(.)$ must have a strictly increasing hazard rate; see chapter 2 for more details on this assumption.

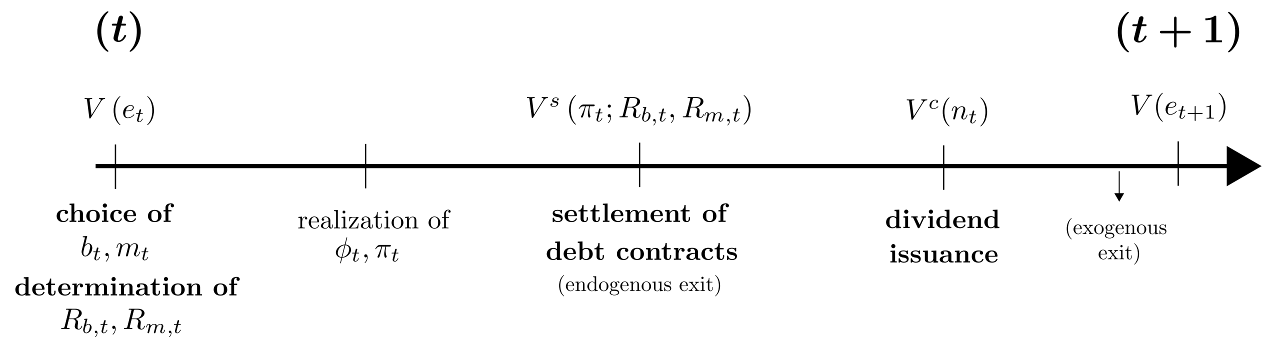


*Figure 3.1: Timing.*

stock of internal finance is lost.[9] The value of the firm, at this stage, is denoted by $V^c(n_t)$. The timing of firms' problem is summarized in figure 3.1.

I next turn in more detail to the different stages of the firm's problem. I start from the determination of the dividend issuance policy of the firm, and move backwards towards the initial choice of total borrowing and borrowing composition.

### 3.2.2 Dividend issuance

Given the value of the firm in period $t+1$, the dividend issuance problem of a firm that was not liqudiated at time $t$ is given by:

$$\begin{aligned} V^c(n_t) \quad &= \quad \max_{div_t,\ e_{t+1}} div_t + (1-\eta)\beta V(e_{t+1}) \\ \text{s.t.} \quad & div_t + e_{t+1} \leq n_t \\ & div_t \geq 0 \end{aligned} \tag{3.1}$$

[9]The assumption of exogenous exit is not necessary to guarantee the existence of a solution to each firm's individual problem, but it guarantees the existence of a stationary distribution of firms across levels of $e_t$. See the proof of proposition 3.8 for details.

Here, $div_t$ denotes dividends issued, $n_t$ is the cash on hand of a continuing firm after the debt settlement stage, and $\eta$ denotes the probability of exogenous exit.[10] The dividend policy of the firm is given by the following lemma, which is analogous to results in Cooley and Quadrini (2001).

**Lemma 3.1** (Dividend policy)**.** *If $V$ is continuous on $\mathbb{R}_+$ and satistfies $V(0) \geq 0$, then $V^c$ is continuous and strictly increasing on $\mathbb{R}_+$, and satisfies $V^c(0) \geq 0$. Moreover, if $V$ is strictly increasing, $V'$ is strictly decreasing except at a finite number of points, and if there exists a unique $\bar{e} > 0$ such that:*

$$(1-\eta)\beta\frac{dV}{de}(\bar{e}) = 1, \tag{3.2}$$

*then, the firm's optimal dividend policy is given by:*

$$\hat{div}(n_t) = \begin{cases} 0 & \textit{if} \quad 0 \leq n_t < \bar{e} \\ n_t - \bar{e} & \textit{if} \quad n_t \geq \bar{e} \end{cases}.$$

Intuitively, if the continuation value function $V(.)$ is concave, a firm with cash below $\bar{e}$ will find that the marginal value of keeping this cash in the form of internal finance for the next periods exceeds the marginal value of dividend issuance, which equals to 1.[11] The firm will therefore choose the corner solution $div_t = 0$, and accumulate all its cash as internal funds towards period $t+1$. On the other hand, a firm with cash above $\bar{e}$ will be able to choose the interior solution $e_{t+1} = \bar{e}$, and issue dividends. The relevant state-space of the firm's problem is therefore $[0, \bar{e}]$. [12]

[10]Note that $n_t$ depends on the realization of $\phi_t$; I omit this to simplify notation.

[11]The properties of the value function discussed in this proposition will hold as a result of the existence of decreasing returns at the firm level. This will be formally established as part of the proof of proposition 3.8.

[12]The key constraint in this problem is that firms are not allowed to issue equity: $div_t \geq 0$. This is a

### 3.2.3 Debt settlement

The structure of the debt settlement stage follows closely the static model of chapter 2. Given the realization of the firms' productivity $\phi_t$ and therefore its pre-settlement resources $\pi_t$, the firm has three options: liquidation; debt restructuring; or full payment of its liabilities. Let $V_t^L$, $V_t^R$ and $V_t^P$ denote the respective values of liquidation, restructuring and payment to the firm, at the debt settlement stage. The firm faces the discrete choice problem:

$$V^s\left(\pi_t, R_{b,t}, R_{m,t}\right) = \max_{L,R,P}\left(V_t^L, V_t^R, V_t^P\right).^{13} \tag{3.3}$$

I next describe the determination of the value of the firm under these three options. I delay the discussion of key assumptions embodied in this description to the next paragraph.

**Liquidation** In liquidation, the firm is shut down and its resources $\pi_t$ are seized by its creditors. I assume that the process of liquidating the firm may involve a loss of resources.

**Assumption 3.2** (Liquidation losses)**.** *The resources effectively available to be split among creditors are given by:*

$$\tilde{\pi}_t = \chi\pi_t \quad , \quad 0 \leq \chi \leq 1.$$

When $\chi < 1$, the transfer of the firm's ressources to creditors involves strictly positive losses of output. In that case, rational lenders charge the firm a liquidation risk premium. Absent these losses (when $\chi = 1$), if all participants are risk-neutral, lenders charge the

sufficient assumption to guarantee a non-degenerate solution to the problem. In general, following Cooley and Quadrini (2001), it is necessary that equity issuance be costly, that is, that the marginal cost of equity issuance be strictly larger than 1. Indeed, imagine that the firm were allowed to issue negative dividends at a marginal cost of 1. In that case, the firm can always achieve the interior optimum, by using the dividend policy $div_t = n_t - \bar{e}$, even when $\bar{e} > n_t$. All firms would then operate under identical debt structures and scales, and would only differ in the interim period, after idiosyncratic productivity has been realized.

[13] $V_t^L$, $V_t^R$ and $V_t^P$ are functions of $\pi_t, R_{b,t}$ and $R_{m,t}$, but I omit this to simplify notation.

risk-free rate, and firms can reach their ex-ante optimal size.

When there are multiple creditors, one must also describe how liquidation ressources are allocated among them in the event that the firm is liquidated. I assume that the split follows a rule similar to the Absolute Priority Rule (APR) which governs chapter 7 corporate bankruptcies in the US. That is, a claim by a stakeholder to liquidation ressources $\tilde{\pi}_t$ can be activated only if all stakeholders placed higher in the priority structure have been made whole. In this model, there are three stakeholders: bank lenders; market lenders; and the firm itslef. The firm is a residual claimant, and will receive a share of the liquidation resources only if both bank lenders and market lenders have received full payment. I furthermore assume that bank lenders are senior to market lenders in the priority structure. Payoffs to stakeholders are then given by:

$$\begin{aligned} \tilde{R}_{b,t} &= \min\left(R_{b,t}, \chi\pi_t\right) && \text{(bank lenders)} \\ \tilde{R}_{m,t} &= \min\left(\max(0, \chi\pi_t - R_{b,t}), R_{m,t}\right) && \text{(market lenders)} \\ V_t^L &= \max\left(0, \chi\pi_t - R_{b,t} - R_{m,t}\right) && \text{(firm)} \end{aligned} \tag{3.4}$$

In particular, note that the definition of $V_t^L$ implies that when a firm is liquidated, its entrepreneurs are left to consume residual liquidation ressources, and cannot reinvest these ressources into future internal finance; that is, the firm is shut down after liquidation.

**Restructuring** Restructuring debt is useful to both the firm and its creditors, because it offers a way to continue the firm, and avoid the losses that its liquidation would involve. The crucial distinction between bank and market lenders relates to their ability to offer this flexibility to the firm.

**Assumption 3.3** (Debt flexibility)**.** *Only bank debt can be restructured; market debt is not flexible.*

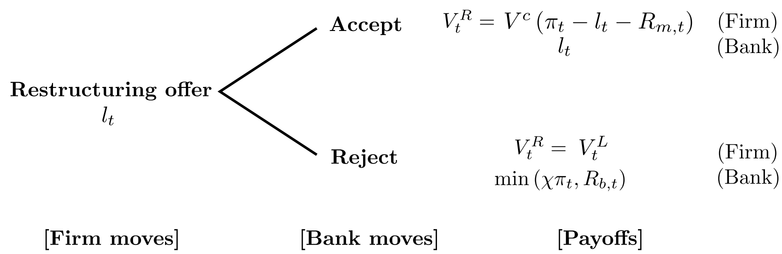


*Figure 3.2: Two-stage game for debt restructuring*

Debt restructuring takes the form of an offer of reduction of principal and interest payments to the bank. Specifically, I model the restructuring process as a two-stage game between the firm and the bank. This game is summarized in figure 3.2. In this game, the firm moves first, and makes a repayment offer $l_t$ to the bank. The bank can choose to accept or reject the offer. In case the offer is rejected, liquidation ensues, and bank lenders and the firm receive the payoffs described above.

The optimal action of the bank is to accept the offer, if and only if it exceeds the banks' reservation value, that is, if and only if $l_t \geq \min(R_{b,t}, \chi\pi_t)$. The value of an offer $l_t$ to the firm is therefore:

$$\tilde{V}^R(l_t; \pi_t, R_{b,t}, R_{m,t}) = \begin{cases} V^c(\pi_t - l_t - R_{m,t}) & \text{if} \quad l_t \geq \min(R_{b,t}, \chi\pi_t) \\ V_t^L & \text{if} \quad l_t < \min(R_{b,t}, \chi\pi_t) \end{cases}$$

The firm chooses its restructuring offer, $l_t$, in order to maximize this value, subject to the constraint that its net resources, after the restructuring offer, must be positive:[14]

$$\begin{aligned} V_t^R \quad = \quad & \max_{l_t} \quad \tilde{V}^R(l_t; \pi_t, R_{b,t}, R_{m,t}). \\ & \text{s.t.} \quad \pi_t - l_t - R_{m,t} \geq 0 \end{aligned} \tag{3.5}$$

[14]Formally, the solutions to problem (3.5) are the subgame perfect equilibria in pure strategies of the game of figure 3.2.

**Payment** Finally, the firm may choose to pay its creditors in full. I again impose the restriction that the firm's payments cannot exceed its current resources $\pi_t$; otherwise, the firm is liquidated. The value of paying its creditors, for the firm, is therefore given by:

$$V_t^P = \begin{cases} V^c\left(\pi_t - R_{m,t} - R_{b,t}\right) & \text{if } \pi_t \geq R_{b,t} + R_{m,t} \\ V_t^L & \text{if } \pi_t < R_{b,t} + R_{m,t} \end{cases} \tag{3.6}$$

**Debt settlement outcomes** Given values of $\pi_t$, $R_{m,t}$ and $R_{b,t}$, I refer to a solution to problem (3.3), subject to (3.4), (3.5) and (3.6) as a *debt settlement outcome*. The following proposition describes the optimal choice of the firm between liquidation, restructuring and repayment.

**Proposition 3.4** (Debt settlement outcomes)**.** *Assume that $V^c(.)$ is increasing, and $V^c(0) \geq 0$. Then, there are two types of debt settlement outcomes:*

- ***When** $\frac{R_{b,t}}{\chi} \geq \frac{R_{m,t}}{1-\chi}$, the firm chooses to repay its creditors in full, if and only, $\pi_t \geq \frac{R_{b,t}}{\chi}$. It successfully restructures its debt, if and only if, $\frac{R_{m,t}}{1-\chi} \leq \pi_t < \frac{R_{b,t}}{\chi}$, and it is liquidated when $\pi_t < \frac{R_{m,t}}{1-\chi}$.*

- ***When** $\frac{R_{b,t}}{\chi} < \frac{R_{m,t}}{1-\chi}$, the firm repays its creditors in full if and only if $\pi_t \geq R_{b,t} + R_{m,t}$, and it is liquidated otherwise.*

*Moreover, in any successful restructuring offer, the bank obtains its reservation value $\chi\pi_t$, and in all debt settlement outcomes resulting in liquidations, $V_t^L = 0$.*

Figure 3.3 offers a graphical representation of the two situations. Two points are worth noting. First, the firm does not necessarily exert its ability to restructure bank debt. This is because the value of restructuring to the firm depends on the reservation value of the bank, which itself is related to the firms' ressources. For restructuring to be worth the firm's while,

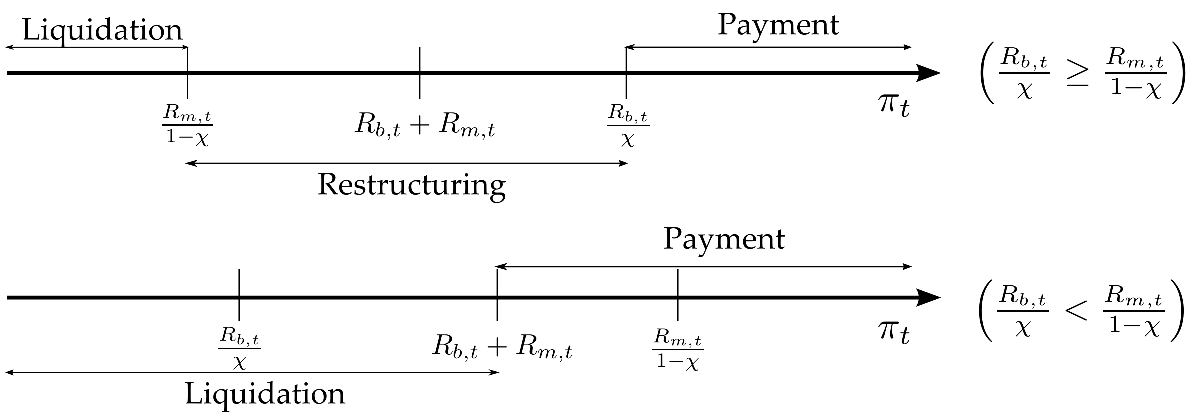


*Figure 3.3: Debt settlement outcomes*

it must be the case that those resources fall below the threshold $\frac{R_{b,t}}{\chi}$, so that the reservation value of the bank, $\chi\pi_t$, is smaller than what the firm owes to the bank. Thus, the option to force liquidation for the bank limits the firm's incentives to restructure too frequently.

Second, the option to restructure is not always sufficient to avoid liquidation. This is the case when $\frac{R_{b,t}}{\chi} < \frac{R_{m,t}}{1-\chi}$. In those cases, even if the option to restructure becomes profitable to the firm $\left(\pi_t < \frac{R_{b,t}}{\chi}\right)$, the liabilities of the firm to market lenders are too large to avoid liquidation. This is not the case when $\frac{R_{b,t}}{\chi} \geq \frac{R_{m,t}}{1-\chi}$. In these cases, there is a region of realization of $\pi_t$ in which restructuring is preferable for the firm. Note that part of that region corresponds to "opportunistic" restructurings, in which the firm, although it could pay in full both creditors, decides to exert its bargaining power and reduce its bank liabilities; this corresponds to cases when $R_{b,t} + R_{m,t} \leq \pi_t \leq \frac{R_{b,t}}{\chi}$.

Additionally, proposition 3.4 indicates that liquidation never involves a strictly positive payment to the firm ($V_t^L = 0$). This result is intuitive. Imagine indeed that $V_t^L > 0$. This is possible only if both bank and market lenders have been repayed, so that necessarily $\chi\pi_t - R_{b,t} - R_{m,t} > 0$. But it is then also the case that $\pi_t - R_{b,t} - R_{m,t} \geq \chi\pi_t - R_{b,t} - R_{m,t} > 0$. In that case, the firm would be better off by simply paying its creditors.[15]

[15]Note that the decision to liquidate, repay or restructure does not depend on the shape of the value

### 3.2.4 Discussion of debt settlement assumptions

**Liquidation** The first key assumption is that liquidation involves losses of output: $\chi < 1$. This assumption embodies the notion that bankruptcy and liquidation may be costly processes, both in terms of time and resources devoted to bankruptcy proceedings, and in terms of changes in asset values after bankruptcies. I discuss the empirical relevance of this assumption, as well as the case $\chi = 1$, in more detail in chapter 2.

The second key assumption is the seniority of bank lenders. This is motivated by two considerations. First, empirically, bank loans tend to be either senior or secured by liens on firm assets, as documented by Rauh and Sufi (2010). Second, in this model, putting bank debt ahead in the priority structure improves the firm's ability to issue bank debt because it increases the banks' claim to liquidation ressources. Bank debt issuance is moreover efficient because it reduces firms' liquidation losses. Although I have not managed to establish this analytically, in numerical calibrations of the model, given the choice of seniority structure, a firm would always prefer to put let bank lenders be senior.[16]

**Debt restructuring** The assumption that banks are more flexible in distress than markets receives considerable support in the data. Gilson, Kose, and Lang (1990) show that, in a

---

function $V$, but simply on whether it is increasing. Intuitively, this is because the decision of whether to restructure or repay creditors depends on the value of each option in continutation, which is given by $V^c(.)$. Since, by lemma 3.1, this continuation value is increasing, the decision to restructure or repay takes the form of a simple threshold rule for $\pi_t$, or equivalently, for $\phi_t$. Analogously, because firms are liquidated whenever they have negative net worth after repayment or settlement, the liquidation decision also takes the form of a simple threshold rule.

[16]The fact that bank seniority is ex-ante preferable for firms is obtained by Hackbarth, Hennessy, and Leland (2007) in a similar setup as the one studied here. The optimality of bank seniority also obtains in other models of debt, in which banks' role is to provide ex-ante monitoring of projects, such as for example Besanko and Kanatas (1993), Park (2000) and DeMarzo and Fishman (2007). The rationale for bank seniority, in these models, is that it raises banks' return on monitoring, by allowing them to seize more output in liquidation. This is distinct from the model I consider here, where seniority relaxes bank borrowing constraints.

sample of 169 financially distressed firms, the single best predictor of restructuring success is the existence of bank loans in the firm's debt structure. Denis and Mihov (2003), in a sample of 1560 new debt financings by 1480 public companies, show that bank debt issuances have more flexibility in the timing of borrowing and payment, and that firms with higher revenue volatility tend to issue more bank debt. Bolton and Scharfstein (1996) provide a theoretical rationale for bank flexibility, by noting that ownership of market debt tends to be more dispersed than ownership of bank debt. This creates a free-rider problem, as market creditors have little individual incentive to particpate in debt renegotiations. More evidence in support of the view that bank lenders are more flexible with financially distressed firms is discussed in chapter 2.

### 3.2.5 Debt pricing and feasible debt structures

Given a capital structure and lending terms $R_{b,t}$ and $R_{m,t}$, the debt settlement equilibria described in proposition 3.4 determine gross lending return functions, conditional on the realization of the idiosyncratic productivity shocks $\phi_t$. I denote these functions by $\tilde{R}_b(\phi_t, k_t, R_{b,t}, R_{m,t})$ and $\tilde{R}_m(\phi_t, k_t, R_{b,t}, R_{m,t})$, respectively.[17]

Banks and markets are perfectly competitive financial intermediaries, and have constant marginal lending costs $r_b$ (for banks) and $r_m$ (for markets). I come back below to the equilibrium determination of these lending costs. Perfect competition implies that lenders

---

[17]For example, when $\frac{R_{m,t}}{\chi} \leq \frac{R_{b,t}}{1-\chi}$, the gross lending return function for market lenders is given by:

$$\tilde{R}_m(\phi_t, k_t, R_{b,t}, R_{m,t}) = \begin{cases} 0 & \text{if } \pi(\phi_t, k_t) \leq \frac{R_{m,t}}{\chi} \\ R_{m,t} & \text{if } \pi(\phi_t, k_t) > \frac{R_{m,t}}{\chi} \end{cases}$$

The first case corresponds to realization of the productivity shock sufficiently low that the firm is forced into liquidation. The second case corresponds to realizations of the productivity shock such that the firm will either choose to pay its creditors in full, or will be able to successfully restructure debt payments with the bank.

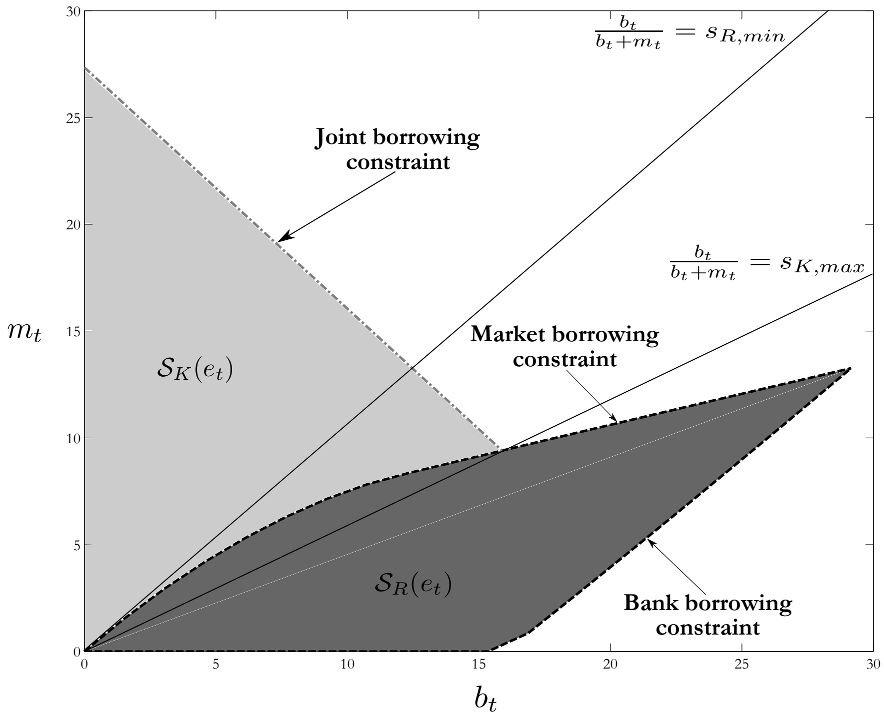


*Figure 3.4: The lending menu $\mathcal{S}(e_t)$.*

will make zero expected profits, in equilibrium, on each loan.[18] Therefore, the equilibrium lending terms $R_{b,t}$ and $R_{m,t}$ must satisfy:

$$\begin{aligned} \int_{\phi_t \geq 0} \tilde{R}_b(\phi_t, e_t + b_t + m_t, R_{b,t}, R_{m,t}) dF(\phi_t) &= (1 + r_b) b_t \\ \int_{\phi_t \geq 0} \tilde{R}_m(\phi_t, e_t + b_t + m_t, R_{b,t}, R_{m,t}) dF(\phi_t) &= (1 + r_m) m_t \end{aligned} \tag{3.7}$$

I define the lending menu $\mathcal{S}(e_t)$ as the set of all debt structures $(b_t, m_t) \in \mathbb{R}^2_+$ for which there exists a solution to (3.7); this is the set of feasible contracts, for a firm with internal finance $e_t$.[19]

**Proposition 3.5.** *The lending menu $\mathcal{S}(e_t)$ is non-empty and compact for all $e_t \geq 0$. Moreover, $S(e_t)$ can be partitioned into two non-empty, compact and convex subsets $\mathcal{S}_K(e_t)$ and*

[18]In particular, perfect competition precludes financial intermediaries from imposing tougher lending terms on certain firms in order to subsidize lending to others.

[19]In chapter 2, I show that there are at most two solutions to (3.7), and that one of the two solutions is strictly smaller, component-wise, than the other. Because of perfect competition in lending, in equilibrium the the contract with the tougher lending terms would not be offered to firms. This rules out multiplicity.

*$\mathcal{S}_R(e_t)$, such that:*

- *The lending terms $(R_{b,t}, R_{m,t})$ satisfy $\frac{R_{b,t}}{\chi} \geq \frac{R_{m,t}}{1-\chi}$, if and only if, $(b_t, m_t) \in \mathcal{S}_R(e_t)$;*

- *The lending terms $(R_{b,t}, R_{m,t})$ satisfy $\frac{R_{b,t}}{\chi} < \frac{R_{m,t}}{1-\chi}$, if and only if, $(b_t, m_t) \in \mathcal{S}_K(e_t)$.*

The set $\mathcal{S}(e_t)$ and its partition into the subsets $\mathcal{S}_K(e_t)$ and $\mathcal{S}_R(e_t)$ is depicted in figure 3.4.[20] The first subset, $\mathcal{S}_K(e_t)$, contains debt structures $(b_t, m_t)$ associated with liabilities $(R_{b,t}, R_{m,t})$ such that $\frac{R_{b,t}}{\chi} < \frac{R_{m,t}}{1-\chi}$. If a firm chooses these debt structures, following proposition 3.4, it will never restructure its bank liabilities once $\phi_t$ is realized. Visually, this corresponds to debt structures in the upper left part of figure 3.4, where borrowing from market lenders is larger than from bank lenders, and therefore restructuring gains are less likely to allow the firm to avoid liquidation.[21] On the other hand, if the firm chooses a debt structure in the second subset, $\mathcal{S}_R(e_t)$, it will sometimes use its restructuring option to reduce its bank liabilities and avoid liquidation. Debt structures in this subset are in the lower right part of figure 3.4, and therefore feature more bank than market debt.[22] The structure of the set of feasible contracts therefore reflects the intuition from proposition 3.4 that restructuring occurs, in equilibrium, only if bank borrowing is sufficiently important relative to market borrowing.[23]

[20]This notation reflects the fact that the lending menu does not depend on the value function of the firm, but simply on its level of internal finance $e_t$. This follows from the results in proposition 3.4, which indicates that the thresholds for restructuring and liquidating are themselves independent of the value function of the firm, so long as the value function is increasing.

[21]As indicated in the graph, all the debt structures in that set have a ratio of bank to total liabilities of at least $s_t^{K,max}$, where $\frac{1-s_t^{K,max}}{s_t^{K,max}}$ is the slope of the lower solid line in figure 3.4.

[22]Specifically, they have a ratio of bank to total liabilities of at least $s_t^{R,min}$, where $\frac{1-s_t^{R,min}}{s_t^{R,min}}$ is the slope of the frontier of $S_R(e_t)$ at $b_t = 0$, $m_t = 0$.

[23]The emphasis here is on *borrowing* $b_t$ and $m_t$, which are the quantitites effectively entering the production funciton; proposition 9 instead focused on *liabilities* $R_{b,t}$ and $R_{m,t}$, which also reflect liquidation risk premia, and need not be paid in full to intermediaries after the realization of productivity.

### 3.2.6 The firm's dynamic debt structure problem

The different elements of the firm's problem can now be stringed together to obtain its recursive formulation:

$$
\begin{aligned}
V(e_t) \quad &= \max_{(b_t,m_t)\in\mathcal{S}(e_t)} \int_{\max\left(n_t^P,n_t^R\right)\geq 0} V^s\left(\pi(\phi_t,k_t),R_{b,t},R_{m,t}\right)dF(\phi_t) && (3.8)\\
\text{s.t.} \qquad k_t &= e_t+b_t+m_t && \text{(Capital structure)}\\
\pi(\phi_t,k_t) &= \phi_t k_t^{\zeta}+(1-\delta)k_t && \text{(Production)}\\
(1+r_b)b_t &= \int_{\phi_t}\tilde{R}_b(\phi_t,k_t,R_{b,t},R_{m,t})dF(\phi_t) && \text{(Debt pricing, bank)}\\
(1+r_m)m_t &= \int_{\phi_t}\tilde{R}_m(\phi_t,k_t,R_{b,t},R_{m,t})dF(\phi_t) && \text{(Debt pricing, market)}\\
V^s\left(\pi(\phi,k_t),R_{b,t},R_{m,t}\right) &= V^c\left(\max\left(n_t^P,n_t^R\right)\right) && \text{(Debt settlement)}\\
n_t^P &= \pi(\phi_t,k_t)-R_{b,t}-R_{m,t} && \text{(Repayment)}\\
n_t^R &= \pi(\phi_t,k_t)-\chi\pi(\phi_t,k_t)-R_{m,t} && \text{(Restructuring)}\\
V^c(n_t) &= \max_{0\leq e_{t+1}\leq n_t} n_t-e_{t+1}+(1-\eta)\beta V(e_{t+1}) && \text{(Dividend issuance)}
\end{aligned}
$$

The formulation of problem (3.8) incorporates results from the stages of the firm's problem discussed in the previous paragraphs. The debt structure chosen by the firm must be feasible: $(b_t,m_t)\in\mathcal{S}(e_t)$. The expression for the value of the firm at the debt settlement stage, $V^s\left(\pi(\phi_t,k_t),R_{m,t},R_{b,t}\right)$, uses the results of proposition 3.4: between repayment and restructuring, the firm chooses the options which ensures it has the cash on hand after debt settlement. The firm is liquidated if and only if its cash on hand under both restructuring and payment are strictly negative, so that the integrand in the objective function of the firm is truncated at $max(n_t^P,n_t^R)=0$. (For all other realizations of productivity, the firm is liquidated, so that its value is $V_t^L=0$). Finally, the value of the firm at the dividend issuance is the sum of the current value of dividends issued and the discounted value of future dividends.

### 3.2.7 Entry and exit

The are two sources of firm exit in this economy. The first is that some firms are endogenously liquidated at the debt settlement stage. Given that the realization of $\phi_t$ is independent of $e_t$, by the law of large numbers, the fraction of existing firms with internal finance $e_t$ that are liquidated is given by $F\left(\underline{\phi}\left(e_t, \hat{b}(e_t), \hat{m}(e_t)\right)\right)$. Here, $\hat{b}(.)$ and $\hat{m}(.)$ denote the policy functions associated to problem 3.8. $\underline{\phi}(e_t, b_t, m_t)$ denotes the threshold such that firms with a productivity realization $\phi_t \leq \underline{\phi}(e_t, b_t, m_t)$ are liquidated.[24]

The second source of exit is that a fraction $\eta$ of all continuing firms exogenously exits after the debt issuance stage. Let $\mu_t$ the measure of firms on evels of internal finance $[0, \bar{e}]$ at time $t$. The total mass of exiting firms after during period $t$ is given by:

$$\begin{aligned}
\delta^e(\mu_t) &= \int_{e\in[0,\bar{e}]} d\mu_t(e_t) \left( \overbrace{F\left(\underline{\phi}\left(e_t, \hat{b}(e_t), \hat{m}(e_t)\right)\right)}^{\text{liquidations}} + \underbrace{\left((1 - F\left(\underline{\phi}\left(e_t, \hat{b}(e_t), \hat{m}(e_t)\right)\right)\right)\eta}_{\text{exogenous exits}} \right) \\
&= \eta + (1-\eta) \int_{e\in[0,\bar{e}]} d\mu_t(e_t) F\left(\underline{\phi}\left(e_t, \hat{b}(e_t), \hat{m}(e_t)\right)\right).
\end{aligned}$$

The fraction $\delta^e(\mu_t)$ of exiting firms is replaced by an identical number of entering firms at the beginning of the following period. Entry involves two costs: the amount of internal finance $e^e_t$ with which entering firms start; and a fixed entry cost $\kappa$. The surplus associated to entering with internal finance $e^e_t$ is given by $\beta V(e^e_t) - (\kappa + e^e_t)$. There is free entry, so that the surplus associated with entering is 0 and $e^e_t$ must solve:

$$\beta V(e^e_t) = \kappa + e^e_t. \tag{3.9}$$

[24]Note that these thresholds also depend on the lending terms $R_{b,t}$ and $R_{m,t}$, as indicated in proposition 3.4. I omit this depends because, in equilibrium, the lending terms will themselves depend only on $(e_t, b_t, m_t)$. A complete characterization of the restructuring and liquidation thresholds is given in the proof of proposition 5 reported in appendix C.2.

Given the entry scale, the mass of exiting firms, and the firm's optimal policy functions along with their dividend issuance policies, the law of motion for the distribution of firms across levels of internal finance can be expressed as:

$$\mu_{t+1} = M(\mu_t) \tag{3.10}$$

where $M : \mathcal{M}(\bar{e}) \rightarrow \mathcal{M}(\bar{e})$ is a transition mapping over firm measures, and $\mathcal{M}(\bar{e})$ denotes the set of measures on $[0, \bar{e}]$ that are absolutely continuous with respect to the Lebesgue measure.

### 3.2.8 Financial intermediation

I conclude the exposition of the model by describing the determination of the lending costs $r_b$ and $r_m$ of financial intermediaries. Intermediaries raise funds to extend credit to firms. I assume that the opportunity cost of funds of intermediaries, $1 + r$, equals the inverse of the discount rate, $\frac{1}{\beta}$. This restriction can be thought of as a general equilibrium outcome. Indeed, it would hold in a model in which intermediaries raise deposits from a representative, risk-neutral household. In such a model, perfect competition in the market for deposits would impose that $\beta(1 + r) = 1$.[25]. Alternatively, the restriction $\beta(1 + r) = 1$ would hold if both financial intermediaries and firms had access to a risk-free technology offering a rate of a rate of return $r$.

In the model, I distinguish between this opportunity cost, which common to all lenders, and intermediation costs, which differ across lender types. I make the following assumption about intermediation costs.

[25] In appendix C.4.1, I spell out in detail such a model. This model is notationally more burdensome, but leads to identical equilibrium outcomes.

**Assumption 3.6** (Financial intermediation costs)**.** *Banks and market lenders face exogenous lending costs per unit of credit extended $\gamma_b$ and $\gamma_m$. The wedge between bank and market-specific intermediation costs is strictly positive: $\theta = \gamma_b - \gamma_m \geq 0$.*

This implies that equilibrium financial intermediation costs for banks and markets are given by:

$$r_m = r + \gamma_m, \quad r_b = r + \gamma_b, \quad r = \frac{1}{\beta} - 1. \tag{3.11}$$

The motivations underlying assumption 3.6 are the following. First, the fact that financial intermediation is costly ($\gamma_b, \gamma_m \geq 0$) is not controversial: Philippon (2012) provides recent and comprehensive evidence that overall intermediation costs in the US financial sector have averaged approximately 2% between 1870 and 2012. The assumption specific to this model is that these intermediation costs are larger for banks than for markets. This assumption captures three key differences between bank and market lending:

1. Bank lenders place more stringent requirements on lenders outside of financial distress than markets, in particular, tighter loan covenants, as documented by Demiroglu and James (2010) and Rauh and Sufi (2010). The positive lending wedge is a reduced-form way of capturing tighter bank lending requirements: indeed, in the model, the wedge will be reflected in higher equilibrium lending terms for banks loans ($R_{b,t}$) outside of financial distress.

2. Banks specialize in costly activities related to lending and which markets typically shun. In particular, banks engage in screening and monitoring of borrowers, as documented in, for example, Berger and Udell (1995), Houston and James (1996) or Mester, Nakamura, and Renault (2007). The positive lending wedge then captures costs associated to these bank-specific activities.

3. Banks face specific regulatory environments that have an impact on their lending costs. In particular, capital requirements require firms to issue additional equity in order to expand their deposit and lending base. However, banks typically finds it costly to adjust their equity base (see Adrian and Shin (2011) for evidence on this topic). This mechanism contributes to making marginal loan issuance more costly for banks.

### 3.2.9 Equilibrium

**Definition 3.7** (Recursive competitive equilibrium). *Given intermediation costs $\gamma_m$ and $\gamma_b$, a recursive competitive equilibrium of this economy are value functions $V$, $V^s$ and $V^c$, an upper bound on internal finance $\bar{e}$, policy functions $\hat{div}$, $\hat{b}$, $\hat{m}$, $\bar{\phi}$, equilibrium lending costs $r_b$ and $r_m$, and functions for equilibrium lending terms $R_b$ and $R_m$, an entry size $e^e$, a distribution of firm size $\mu$ and a transition mapping $M$ for the distribution of firms across levels of internal finance, such that:*

- *given $\bar{e}$, the value functions solve problem (3.8), and $\hat{div}$, $\hat{b}$, $\hat{m}$, $\bar{\phi}$ are the associated policies;*
- *given the value function $V$, the upper bound $\bar{e}$ satisfies condition (3.2);*
- *equilibrium lending costs satisfy (3.11);*
- *functions for equilibrium lending terms satisfy the zero profit conditions of intermediaries (3.7);*
- *the entry scale $e^e$ satisfies the free-entry condition (3.9);*
- *the transition mapping $M$ is consistent with firms' policies and with the entry scale of firms $e^e$;*
- *the distribution $\mu$ is a fixed point of $T$.*

**Proposition 3.8** (Existence of a recursive competitive equilibrium). *There exists a recursive competitive equilibrium of this economy.*

There are two key steps to proving proposition 3.8.[26] First, one must show that problem (3.8) has a unique solution. In general, problem (3.8) is a triple fixed point problem, where the value function $V(.)$, the upper bound $\bar{e}$, and the correspondence $\mathcal{S}(.) : [0, \bar{e}] \to [0, \bar{e}]$ must be simultaneously determined. However, an insight of the exposition of the model is that $\mathcal{S}(.)$ can be characterized independently from $V(.)$ and $\bar{e}$, as implied by proposition C.2. Thus, problem (3.8) can be reduced to a double fixed point problem in $\bar{e}$ and $V(.)$, analogously to Cooley and Quadrini (2001). However, unlike that paper, standard approaches do not directly apply, for two reasons. First, the fact that the problem features a discrete choice between liqudiation, restructuring and repayment introduces non-convexities in the interim value function of the firm. Second, for each $e_t \in [0, \bar{e}]$, the set of feasible contracts $\mathcal{S}(e_t)$ is not convex (a fact clearly visible in figure 3.4). Instead, the set of feasible contract is the union of two sets, $\mathcal{S}_K(e_t)$ and $\mathcal{S}_R(e_t)$, on which interim value functions of the firm are continuous. These difficulties can be overcome showing that problem (3.8) is equivalent to $V(e_t) = \max_{K,R}(V_K(e_t), V_R(e_t))$, where $V_K(e_t)$ denotes the continuation value of a firm restricted to use debt structures that are in $\mathcal{S}_K(e_t)$, and $V_R(e_t)$ is analogously defined. The existence of a fixed point can then be established by studying separately each of the two value functions $V_R$ and $V_K$, and the two associated constraints correspondences $e_t \to \mathcal{S}_R(e_t)$ and $e_t \to \mathcal{S}_K(e_t)$.

The second step to prove proposition 3.8 is to use is to derive the expression of the transition mapping $M$, and show that it has a fixed point. This part of the proof also uses the insight that the firm's problem needs to be decomposed into to sub-problems with different feasible sets. This is because the probability that a firm with current internal finance $e_t$ will find itself with an internal finance $e_{t+1} \leq e'$, for some value of $e'$, in the following period, depends on whether the current optimal debt structure of the firm is $\mathcal{S}_R(e_t)$ or $\mathcal{S}_K(e_t)$, since

[26]See appendix C.2 for the details of the proof.

this determines, in particular, the likelihood with which it will it be liquidated. However, given the expression for $M$ provided in appendix C.2, standard approaches, such as those described in Stokey, Lucas, and Prescott (1989), are sufficient to prove that $M$ has a fixed point.[27]

## 3.3 Financial policies in steady-state

I now turn to the cross-sectional properties of the stationary competitive equilibrium described in the previous section. The model has no closed form solution. Instead, the discussion of this section focuses on a baseline calibration of the model, summarized in table 3.1. The numerical solution procedure is standard and described in appendix C.3.

### 3.3.1 A baseline calibration of the model

Wherever possible, I use existing empirical estimates of structural parameters of the model. I choose a degree of returns to scale of $\zeta = 0.90$. This is in line with the mean empirical estimates of Burnside (1996) for manufacturing industries.[28] The frequency of the model is annual, and I assume that capital depreciates at a rate $\delta = 0.10$ per year. This value is in the range of the estimates of Epstein and Denny (1980).[29] I set the discount rate to $\beta = 1/1.04$,

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[27]The issue of unicity never arises numerically, but I have not yet managed to establish that the sufficient conditions of monotonicity of the transition kernel developed by Stokey, Lucas, and Prescott (1989), or the weaker conditions of Hopenhayn and Prescott (1987) hold for every point in the state-space of the firm's.

[28]Structural estimates of the degree of returns to scale, obtained using firm- or plant-level evidence, suggest lower values for the degree of returns to scale. For example, Cooper and Haltiwanger (2006) estimate that $\zeta = 0.556$, using plant-level data. A lower degree of returns to scale accentuates the concavity of the value function of entrepreneurs and tends to increase bank borrowing, but it does not alter the qualitative properties of the model.

[29]This is somewhat higher than values obtained by matching steady-state ratios of investment to capital using aggregate data. In the context of the model, the rate of depreciation of capital governs the lower bound on firms' output, which is equal to $(1-\delta)k$. A relatively high value of depreciation implies that firms' lower

| Parameter | Description | Value | Parameter | Description | Value |
|---|---|---|---|---|---|
| $\zeta$ | degree of returns to scale | 0.900 | $\mathbb{E}(\phi)$ | average productivity | 0.264 |
| $\delta$ | depreciation rate | 0.100 | $\sigma(\phi)$ | s.d. of productivity | 0.181 |
| $\beta$ | discount rate | 1/1.040 | $\chi$ | liquidation efficiency | 0.380 |
| $\eta$ | exogenous exit rate | 0.0095 | $\gamma_m$ | market intermediation costs | 0.010 |
| $\kappa$ | entry cost | 29.151 | $\gamma_b$ | bank intermediation costs | 0.023 |

*Table 3.1: Baseline calibration of the model.*

so that the annual risk-free rate is $r = 0.04$, in line with the evidence in Gomme, Ravikumar, and Rupert (2011). I set the exogenous exit probability to $\eta = 0.01$, which generates an average exit rate of 4.20%, in line with the estimates of exit rates of Bartelsman, Haltiwanger, and Scarpetta (2009) for US manufacturing firms with more than 20 employees.[30] Finally, given all other parameters of the model, I choose the fixed entry cost $\kappa$ so that the size of entering firms is twenty percent of average size of incumbents in the economy, again in line with the figures of Bartelsman, Haltiwanger, and Scarpetta (2009) for the US manufacturing sector .

I assume that idiosyncratic productivity shocks $\phi_t$ follow a Weibull distribution with location parameter $\lambda$ and scale parameter $\xi$.[31] I choose these parameters to match two targets. First, given values for $\delta$, $\zeta$, $r$ and $\gamma_m$, I choose $\lambda$ and $\xi$ so that $\mathbb{E}(\phi)$ satisfies $\bar{k} = \left(\frac{\mathbb{E}(\phi)\zeta}{\delta+r+\gamma_m}\right)^{\frac{1}{1-\zeta}}$. This normalization ensures that firms' total size is smaller than or equal to $\bar{k}$; I set $\bar{k} = 100$. Second, $\lambda$ and $\xi$ are also chosen to match estimates of the cross-

bound on output is not too large, and that lending is risky for most small firms.

[30] A necessary condition for the existence of a solution to the firm's problem is that $\beta(1-\eta)(1+r+\gamma_m) \leq 1$. Large firms are risk-free borrowers and have a cost of debt of $1+r_m = 1+r+\gamma_m$, so that the marginal value of an extra unit of internal funds is $\frac{\partial V}{\partial e} = (1+r+\gamma_m)$. If $\beta(1-\eta)(1+r+\gamma_m) > 1$, these large firms will never find it optimal to issue dividends; that is, the condition of lemma 3.1 will never be met for any value of $\bar{e}$. When $\gamma_m = 0$, the condition $\beta(1-\eta)(1+r_m) \leq 1$ holds for any $\eta \geq 0$; when $\gamma_m > 0$, $\eta$ must satisfy $\eta \geq 1 - \frac{1}{\beta(1+r+\gamma_m)} = \frac{\gamma_m}{1+r+\gamma_m}$. The choice of $\eta$ in the baseline calibration satisfies this condition.

[31] I use the Weibull distribution because it has an increasing hazard rate. This condition is sufficient to ensure unicity of lending terms. I discuss this in more details in chapter 2.

sectional standard deviation of firm-level total factor productivity reported in the database on job flows and productivity constructed by Bartelsman, Haltiwanger, and Scarpetta (2009). These authors compute output-based measures of the dispersion of TFP across firms, for a sample of developed and developing countries. The model studied here assumes a constant prices of goods, so that output-based measures of productivity dispersion are more directly relevant that revenue-based measures. The standard deviation for the US estimated by Bartelsman, Haltiwanger, and Scarpetta (2009), $sd(\log(\phi)) = 0.620$, is in line with other evidence for output-based productivity measures for the US, for example Hsieh and Klenow (2009).[32]

The three remaining parameters directly reflect inefficiencies in financial intermediation: $\chi$, $\gamma_m$ and $\gamma_b$. Bris, Welch, and Zhu (2006) analyze a sample of 61 chapter 7 liquidations in Arizona and New York between 1995 and 2001. Their median estimate of the change in asset values pre- to post- chapter 7 liquidation is 38%.[33] I therefore use $\chi = 0.38$.

As a proxy for market-specific lending costs $\gamma_m$, I use existing estimates of underwriting fees for corporate bond issuances. Fang (2005) studies a sample of bond issuances in the US, and finds an average underwriting fee of 0.85%, while Altinkilic and Hansen (2000), in a sample including lower-quality issuances, find a an average underwriting fee of 1.09%. Given this evidence, I set market-specific intermediation costs to $\gamma_m = 0.0100$.

As discussed previously, differences between bank and market intermediation costs can arise, in particular monitoring and screening costs, and bank-specific opportunity costs asso-

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[32]Specifically, I match the coefficient of variation of the model's productivity distribution, to the number reported by Bartelsman, Haltiwanger, and Scarpetta (2009), assuming that their underlying productivity distribution is log-normal with mean 1. I make these assumptions because the numbers reported by Bartelsman, Haltiwanger, and Scarpetta (2009) are themselves normalized, and reported for log-TFP. This implies values of $\lambda = 0.2923$ and $\xi = 1.4865$ for the underlying parameters of the Weibull distribution.

[33]See their table III, p. 1265. This number adjusts for the value of collateralized assets that creditors may have seized outside of the formal bankruptcy proceedings.

ciated to capital and liquidity requirements. It is not straightforward to measure these costs from, for example, income statements of commercial banks, for two reasons. First, operating expenses of banks reflects expenses associated with a number of non-lending activities. Second, opportunity costs associated with capital or liquidity requirements are not directly incurred by the bank, and do not appear in income statements. Therefore, instead of trying to construct a direct measure of $\gamma_b$, I choose it to match the aggregate bank share of US nonfinancial corporations reported in the Flow of Funds in 2007Q3. Loans and advances from banks and bank-like institutions accounted for 27.5% of credit market liabilites of US non-financial corporations at that date. Given other parameters of the model, matching this aggregate share requires bank-specific intermediation costs of $\gamma_b = 0.02255$, or equivalently a lending wedge of $\theta = 0.01255$.[34]

### 3.3.2 Optimal debt structure

The key properties of the firm's optimal debt structure are reported in figures 3.5 and 3.6, and summarized in the following result. This result has no analytical proof, but holds for all numerical calibrations under which I have solved the model, provided that $\theta = \gamma_b - \gamma_m \geq 0$.

**Numerical Result 3.9** (The firm's optimal debt structure)**.** *Let $\hat{b}(e_t)$ and $\hat{m}(e_t)$ denote the policy functions associated to the solution to problem (3.8). There exists a unique value of internal finance $e^* \in [0, \bar{e}]$ such that:*

- *For $e_t \in [0, e^*]$, $\hat{b}(e_t) > 0$, $\hat{m}(e_t) \geq 0$, and $\left(\hat{b}(e_t), \hat{m}(e_t)\right) \in \mathcal{S}_R(e_t)$;*
- *For $e_t \in ]e^*, \bar{e}]$, $\hat{b}(e_t) = 0$, $\hat{m}(e_t) \geq 0$, and $\left(\hat{b}(e_t), \hat{m}(e_t)\right) \in \mathcal{S}_K(e_t)$.*

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[34] Note that combining a risk-free rate of $r = 4.00\%$ with the value of bank intermediation costs $\gamma_b$, one arrives at a risk-free bank lending rates of 6.25%. This aligns relatively well with the average of the monthly Bank Prime Loan Rate during 2007Q3 (8.19%), once inflation is taken into account. In the model, firms borrowing from banks are never entirely risk-free, so that this number cannot be used to directly match the lending wedge.

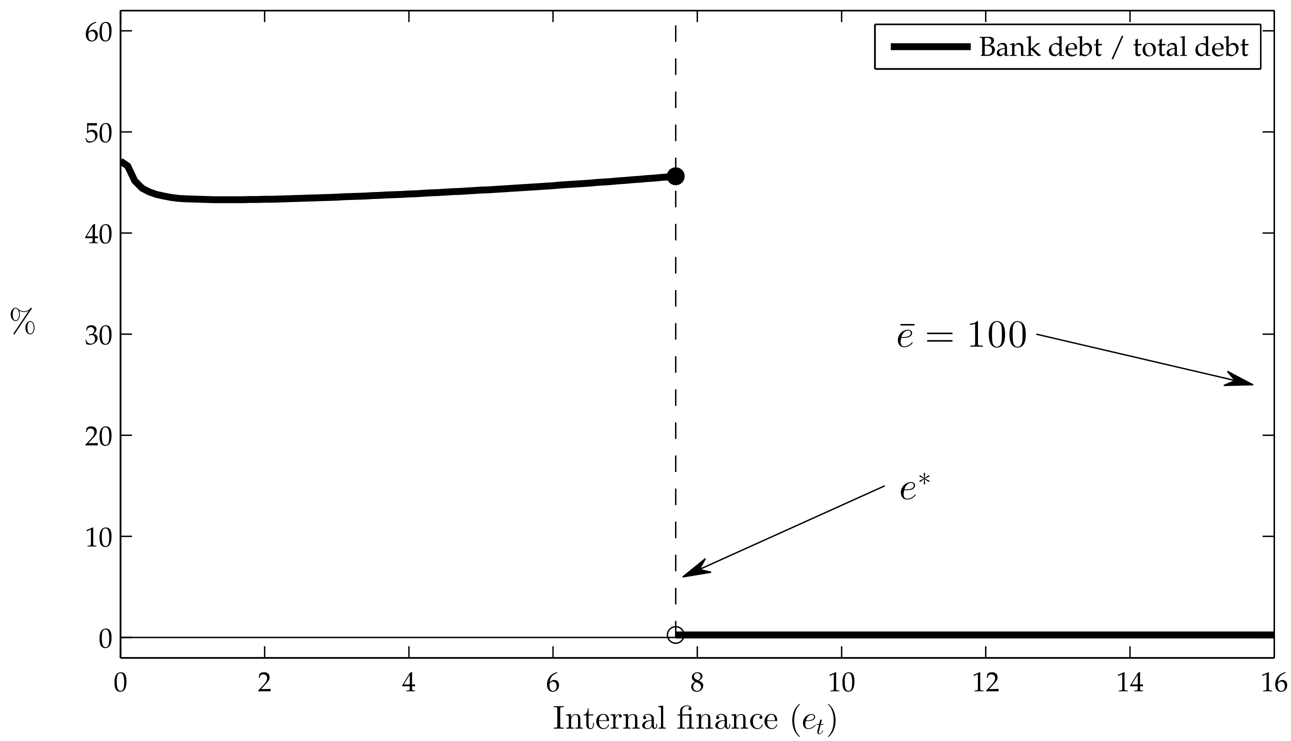



*Figure 3.5: Optimal composition of debt.*

*Moreover, the policy functions $\hat{b}(.)$ and $\hat{m}(.)$ are continuous on $[0, e^*]$ and $]e^*, \bar{e}]$, but piecewise continous on $[0, \bar{e}]$.*

This result first indicates that firms adopt two broad types of debt structure, as illustrated in figure 3.5. When a firm's internal funds iare below $e^*$, it will choose a debt structure which involves both bank and market debt.[35] On the other hand, large firms – those firms with internal funds strictly above $e^*$ – choose not to borrow at all from banks, but only from markets. As firms grow by accumulating internal funds from retained earnings, they will therefore switch from a mixed debt structure to a market-only debt structure.

The intuition for this is as follows. The trade-off between flexibility in distress and cost out of distress changes with the level of the firm's internal resources $e_t$. Because of decreasing

[35] The result indicates that $\left(\hat{b}(e_t), \hat{m}(e_t)\right) \in \mathcal{S}_R(e_t)$, so the share of bank debt in that mix is bounded from below by $s_{R,min}$ (see figure 3.4). These firms therefore always borrow in strictly positive amounts from banks.

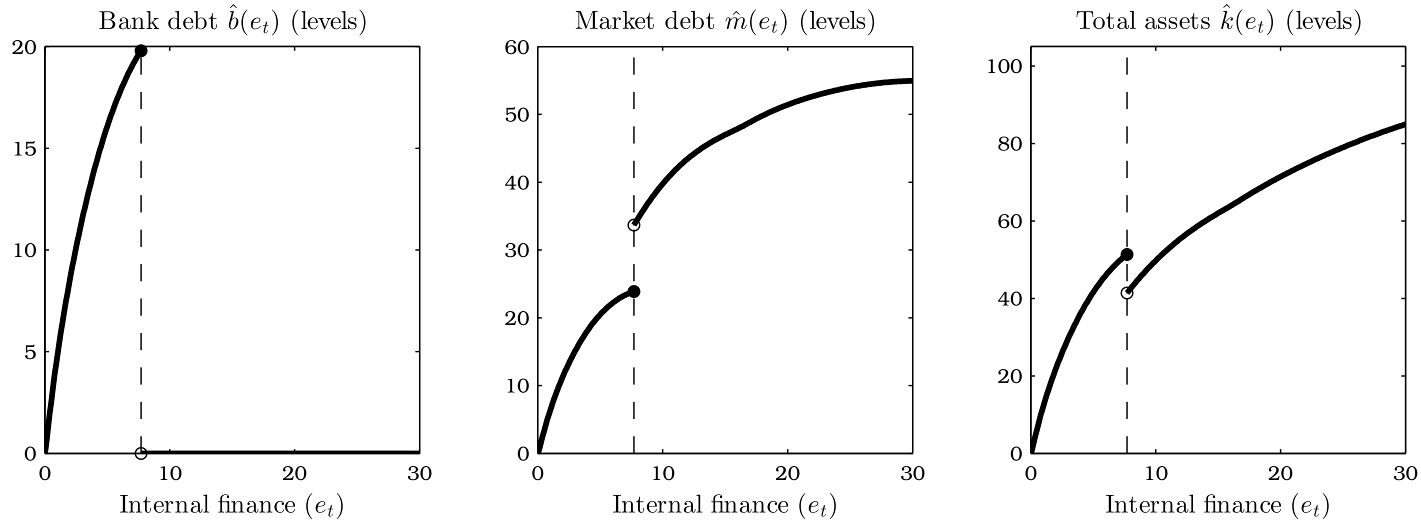



*Figure 3.6: Optimal level of bank and market debt, and total assets.*

returns, firms with small $e_t$ tend to borrow more, relative to their internal resources, than firms firms with large $e_t$. But this also implies a higher probability of financial distress. Since, all other things equal, borrowing more from banks reduces the expected losses associated with financial distress, firms with small $e_t$, seeking high leverage, have a strong incentive to use bank debt. In general, one should therefore expect the composition of small firms' debt to be more tilted towards bank debt.

This general intuition does not account for the fact that firms completely switch to market finance when $e_t \geq e^*$. To understand this, it is useful to think back to the results of proposition 3.4. This proposition indicates that, if a firm is sufficiently market-financed, it never exterts its option to restructure bank debt when it faces financial distress. Intuitively, this occurs because even if the firm were to restructure its bank liabilities in bad times, it would not be able to extract sufficient surplus from the bank in order to honor its market liabilities and avoid liquidation.[36] In that case, the flexibility associated with bank debt is

[36]In terms of the sets described in figure 3.4, when firms become sufficiently large, their optimal debt structure $\left(\hat{b}(e_t), \hat{m}(e_t)\right) \in \mathcal{S}_K(e_t)$.

irrelevant to the firm since, in equilibrium, the firm never uses that flexibility. But borrowing from banks results in large liabilities $R_{b,t}$ outside of financial distress, since the lending wedge $\theta = \gamma_b - \gamma_m$ is strictly positive. The net benefit of substituting a unit of bank debt for a unit of market debt, for these firms who do not restructure ex-post, is therefore always strictly positive. As a result, these firms choose the corner solution $\hat{b}(e_t) = 0$, $\hat{m}(e_t) > 0$.

Appart from the declining bank share, the second key aspect of firms' optimal debt choices is that they impliy non-monotonicities in the borrowing and investment policies of firms. This is illustrated in figure 3.6, which reports bank borrowing $\hat{b}(e_t)$ (left panel), market borrowing $\hat{m}(e_t)$ (middle panel), and total assets $\hat{k}(e_t) = e_t + \hat{b}(e_t) + \hat{m}(e_t)$ (right panel), as a function of internal funds $e_t$. In each of the two regions $e_t \leq e^*$ and $e_t > e^*$, the amounts borrowed from banks and markets are increasing (or equal to 0 for bank borrowing when $e_t > e^*$). However, when a firm crosses the threshold $e^*$, its total assets fall. Bank borrowing is not replaced one for one by market liabilities. This feature of the optimal debt structure is an example of the real implications of imperfect substituability between types of debt. At the point $e_t = e^*$, the firm is exactly indifferent between a mixed debt structure, and a market-only debt structure. However, it would operate at a smaller scale (and therefore, produce on average less output) under the latter debt structure than under the former. This is because a similar level of leverage exposes the firm to a higher likelihood of liquidation if it is entirely financed through market debt, than if it is financed partially through bank borrowing. As a result, firms display a "precautionary" borrowing behavior, and reduce total borrowing and investment as they switch to market lending.

The fact that firms "switch" to pure market finance when $e_t \geq e^*$ is a stark prediction of the model. One way to obtain a smoother transition of firms towards market finance would be to introduce a convex cost of market debt issuance for firms, that is sunk before productivity is realized. This cost would reduce firms' surplus from using a purely market-financed debt structure, while not affecting the pricing of debt contracts. However, the underlying mechanisms just discussed would not disappear. In particular, sufficiently large firms would still choose a debt structure that does not allow them to successfully restructure their liabilities. A threshold for switching to these debt structures would still exist, and as firms cross this threshold, firms would still reduce total borrowing, as a result of their increased fragility.

Finally, the dynamic nature of firms' problem also has a strong influence on the choice of debt structure, as it accentuates the concavity of firms' value functions. This concavity is particularly pronounced for small firms. As a result, small firms borrow more from banks, and the switching threshold $e^*$ is higher in the dynamic version of the model than in a similarly calibrated static version.[37]

### 3.3.3 Debt structure in the cross-section

Are the model's predictions about variation in debt structure consistent with the data? Figure 3.7 reports, on the left panel, the steady-state distribution of firms across levels of internal finance in the model. The key property of this distribution is that it is strongly skewed to the left. This occurs both because the entry size of firms is relatively small, and because small firms are much more frequently liquidated than large firms. Because of this left-skewness, the share of bank loans as a fraction of total borrowing is high for most firms, and only declines for the largest firms in the economy. This is reported in the right panel of figure 3.7. Each point in this graph corresponds to the median bank share of firms with a particular decile of the distribution of internal finance. The bottom 70% of firms use both bank and market debt, while the top 20% use only market debt.

[37] In appendix C.4, I report the composition of debt in a static version of the model with the same calibration.

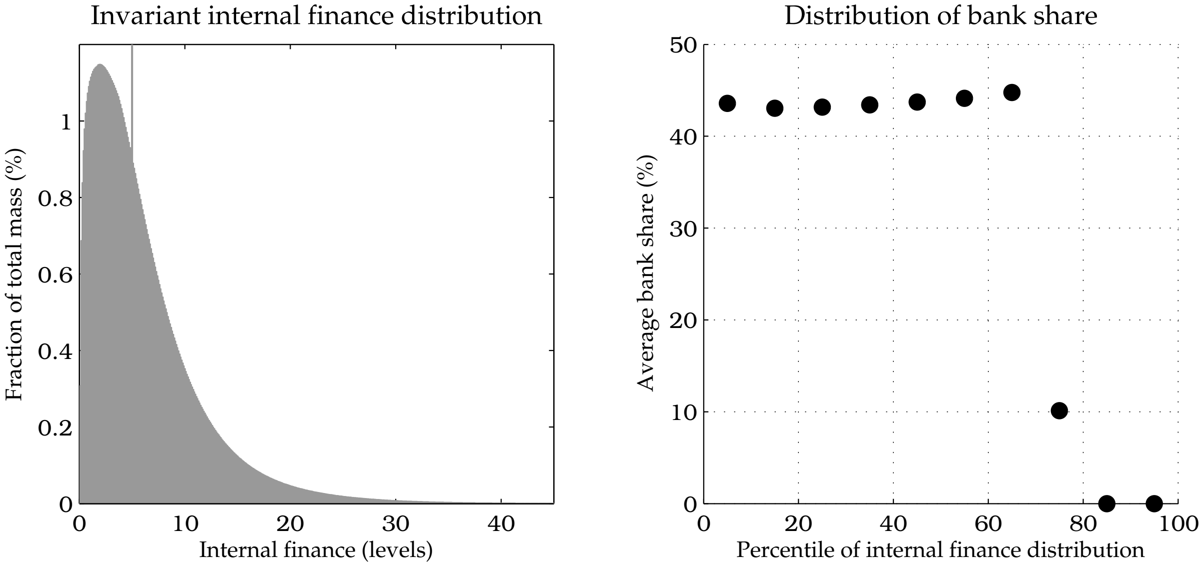



*Figure 3.7: Bank share and internal finance in the cross-section of the model.*

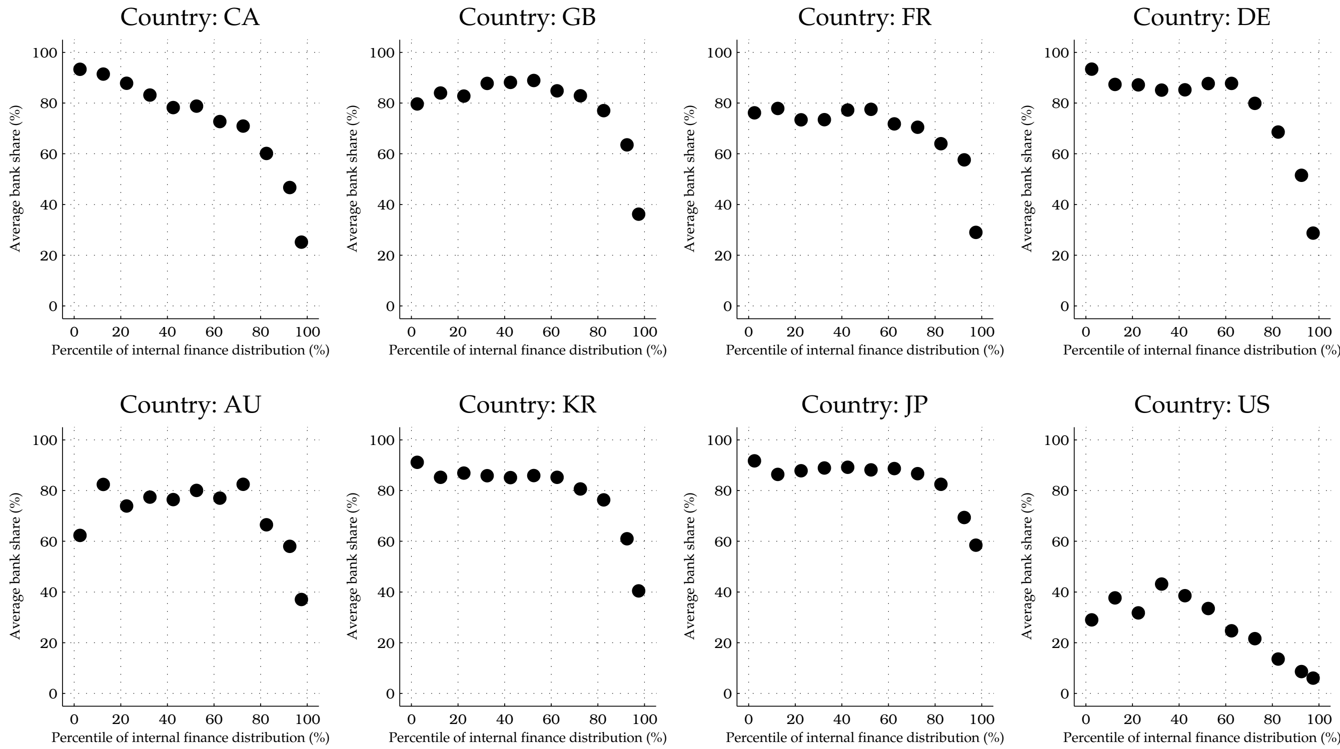
This negative relationship between internal finance and the bank share is also a prominent feature of the data. Figure 3.8 reports the same plot as the right panel of figure 3.7, for a set of publicly traded firms from different countries.[38] Figure 3.7 distinguishes between OECD and non-OECD countries. In OECD countries, as in the model, the median bank share is high in the lower deciles of the internal finance distribution, and rapidly declines for the top 3 deciles. The pattern for some non-OECD countries, such as Chile and Taiwan, is suprisingly close to that of OECD countries. The model thus accounts well for cross-sectional variation in debt structure in advanced economies, and certain developing economies in which market lending is well-established.[39]

The model's cross-sectional predictions differ from the data in two main ways. First, the overall level of the bank share is somewhat smaller in this calibration of the model than

[38]The definition of internal finance adopted in this graph follows closely that of the model: namely, it is measured as the difference between the book value of firms' assets and liabilities. Details on the definition of variables are given in appendix C.1.

[39]The model also predicts that a firms' total assets and its bank share are negatively related, in the cross-section. In appendix , I report the distribution of firms across asset levels, and the associated cross-sectional link between bank share and total assets. The model predicts a downward sloping relationship between total assets and the location of the firm in the asset distribution. This relationship is also a prominent feature of the data; the appendix documents this for the same sample of firms used in the construction of figure 3.10.

(a) **Internal finance and bank share in OECD-countries**




(b) **Internal finance and bank share in non-OECD-countries**

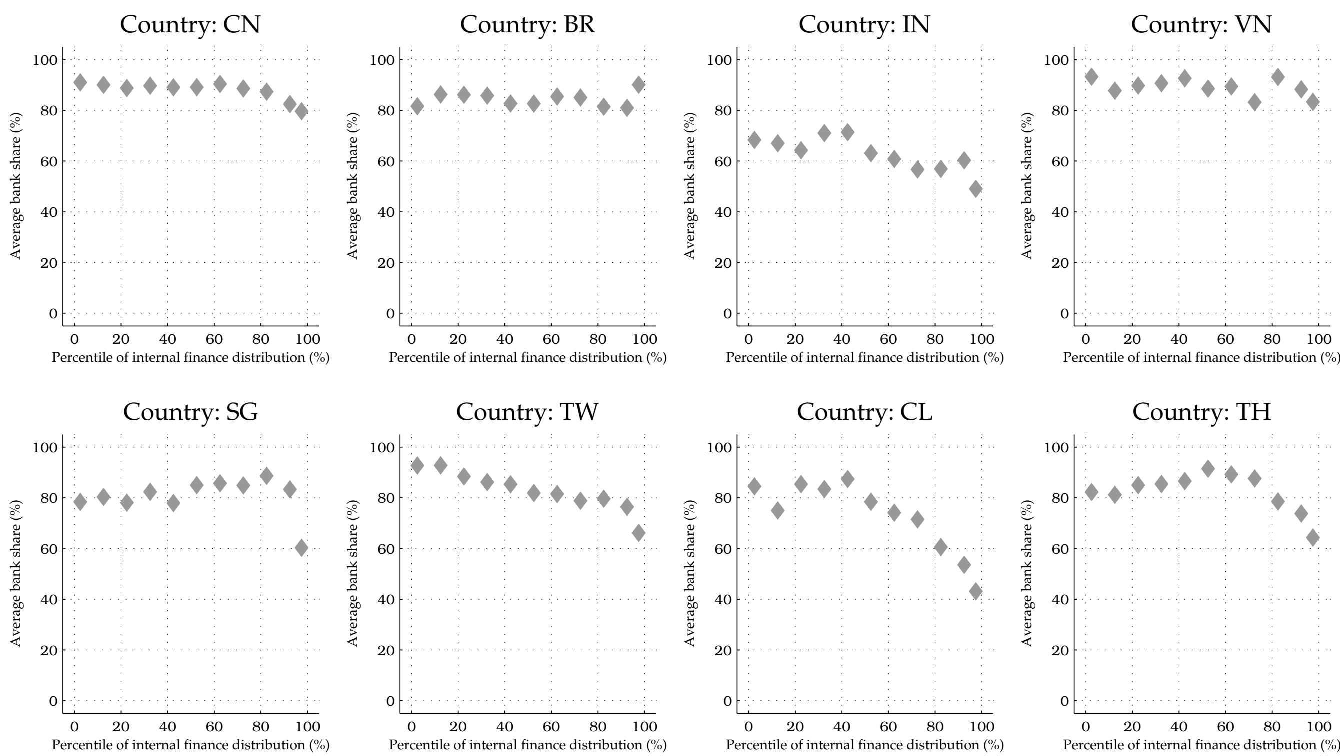



*Figure 3.8: Bank share and internal finance in the cross-section. Each graph reports, for a particular country, the median ratio of bank loans to total firm liabilities, in each decile of the distribution of internal finance. For the US, data from taken from Rauh and Sufi (2010); for other countries, data from Bureau Van Djik. See online appendix for details on the definitions of variables.*

that in the data. Second, there seem to be too many market-financed firms, relative to the data. Both caveats can be partially adressed by allowing for firms to have persistenly different average levels of productivities. Indeed, if the majority of firms in the economy have persistenly lower productivity, but coexist with a minority of very productive firms, then the distribution of internal funds will be more skewed to the left (resulting in even fewer firms adopting a market-only debt policy), while the mix of credit to smaller firms will be more tilted toward bank debt (as a result of firms' lower average productivities).[40]

### 3.3.4 Other aspect of firms' policies

The previous discussion indicates that the model has cross-sectional predictions on the composition of debt that align well with the data. Does this come at the expense other predictions, in particular for firm dynamics? Figure 3.9 reports adresses this question. Besides the fact that total assets increases with internal resources (see figure 3.6), with respect to financial policies, the model predicts that (1) firms with small internal resources are generally more leveraged (top row, left panel) and (2) firms with small internal resources have a higher rate of profit, but distribute more dividends (top row, middle panel). These features of firms' financial policies are broadly similar to the results of the Cooley and Quadrini (2001) model, and consistent with empirical facts on financial behavior of firms documented in, e.g., Fazzari, Hubbard, and Petersen (1988) and Gilchrist and Himmelberg (1998). On the other

[40]The evidence reported in this section is also consistent with previous empirical work on the cross-sectional variation of debt structure; in both the samples of Denis and Mihov (2003) and Rauh and Sufi (2010), for example, size (typically measured by total assets, rather than net worth) is negatively related to the bank share. These studies also document that credit quality negatively is the strongest empirical determinant of the choice and the composition of debt. This is consistent with the equilibrium choices of firms, in this model, in the sense that firms with smaller net worth have, ex-ante, the highest likelihood of restructuring or being liquidated. However, in the model, this link does not arise as a result of exogenous differences in risk, but rather, of endogenous choices of leverage of heterogeneous firms. The relationship between the model's predictions, and the findings of Rauh and Sufi (2010) are discussed in more detail in chapter 2.

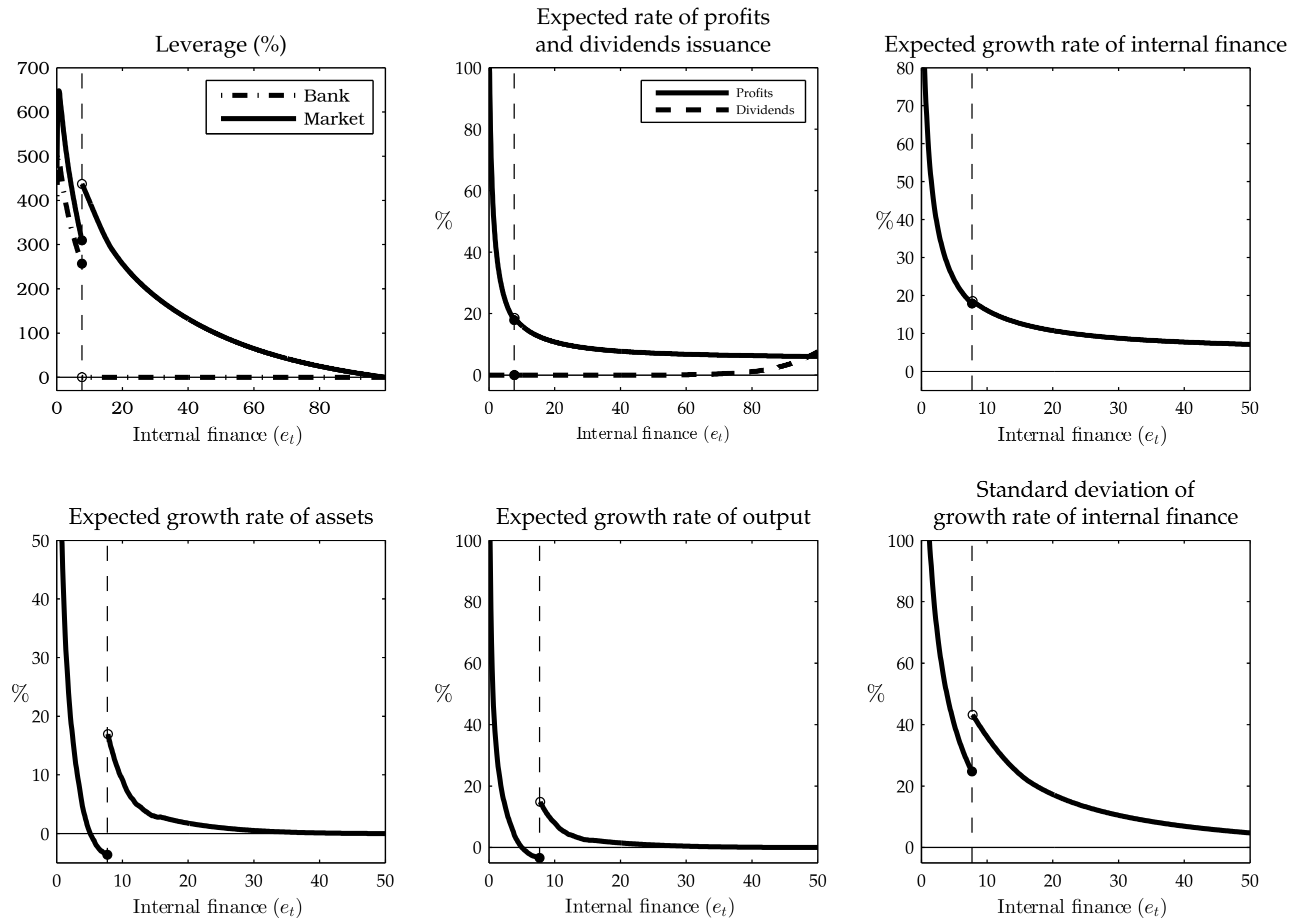



*Figure 3.9: Other aspects of firms' financial policies and implied growth dynamics. Definitions of the variables reported on this graph are given in appendix C.4.4.*

hand, the key properties of firm dynamics in the model can be summarized as follows: (1) small firms experience a higher rate of growth, either measured in terms of internal funds, output, or total assets (top row, right graph; bottom row, left and middle graphs); (2) small firms have a higher volatility of growth (bottow row, right graph). These facts also align well with empirical evidence on the relationship between size and growth dynamics.[41] Note that, as was the case for financial policies, the switch between financial regimes is associated with changes in the growth dynamics of firms. Namely, firms immediately to the left of the switching threshold experience negative expected growth rates, as they anticipate the

[41]See for example Evans (1987), or more recently Davis, Haltiwanger, Jarmin, and Miranda (2006).

fact that reaching the switching threshold will imply a reduction in the scale of their operations. The volatility of growth of firms that switch to market-only debt also increases, in line with the intuition that the debt structure adopted by these firms increases their exposure to liquidation risk.

Summarizing, in this section I showed that a baseline calibration of the model has cross-sectional predictions that are consistent with the data, both for the composition of debt, and for other aspects of firm's financial policies and firm growth. Additionally, I emphasized that firms in the model adopt two qualitatively different types of debt structures: when they are small, they combine bank and market lending; when they become large, they switch to market finance. Moreover, this change in debt structure has implications for firm-level borrowing and investment, with firms displaying a precautionary drop in investment as they switch as they switch from one financial regime to the other. I now turn to the aggregate implications of the model.

## 3.4 Long-run differences in debt structure

Across countries, there is considerable variation in aggregate bank shares (defined as the fraction of the total stock of non-financial corporate debt that are bank loans). Figure 3.10 illustrates this for a sample of non-OECD and OECD countries, on average between 2000 and 2007. In this sample, output per capita and the aggregate bank share are negatively related; on average, the bank share is higher for non-OECD countries than for OECD countries. However, even within these two groups, there is considerable variation in the bank share. In fact, in some developing countries, such as Malaysia and Chile, markets seem to play, by this measure, a larger role in credit intermediation than in certain developed countries.

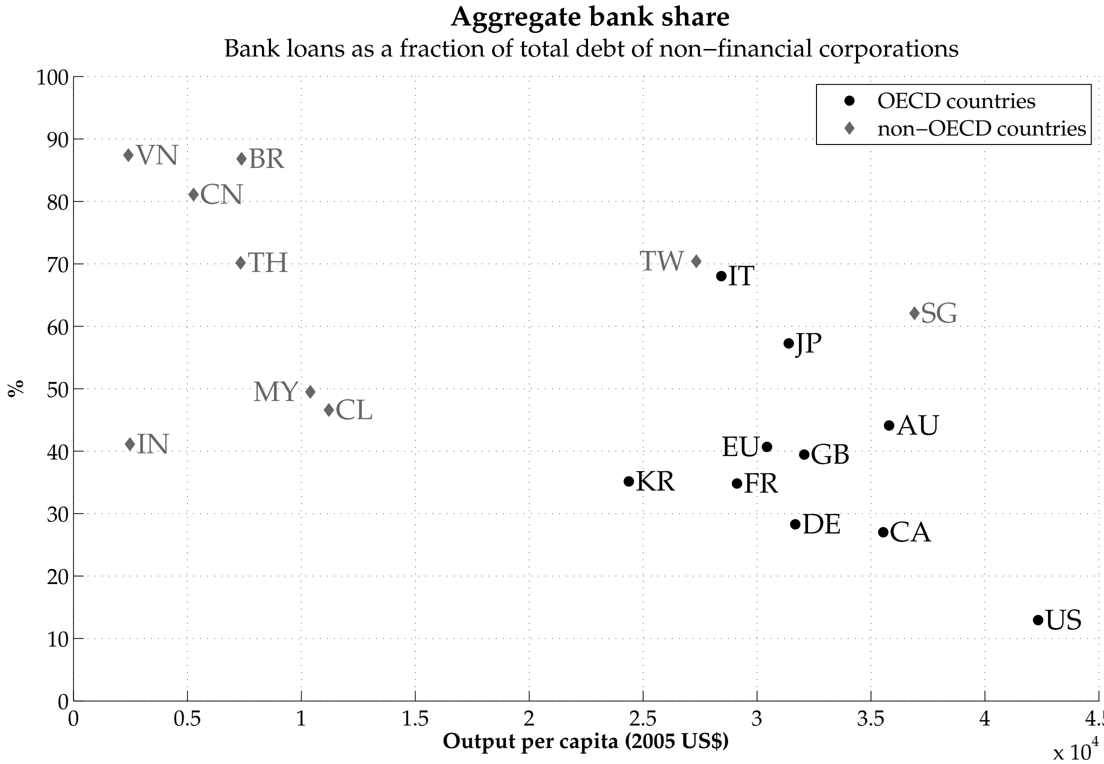



*Figure 3.10: Aggregate bank shares across countries. See appendix C.1 for details on the data used to construct this graph.*

Can the model help shed light on the causes and implications of this variation? To answer this question, I start by establishing which structural parameters can account for variation in the aggregate bank share. I then ask to what extent evidence on cross-country variation in these parameters can account for observed differences in the case of two specific countries, the US and Italy.

### 3.4.1 Comparative statics of the aggregate debt structure

The aggregate bank share, in the model, is given by:

$$S = \frac{\int_{e_t \in [0,\bar{e}]} \hat{b}(e_t) d\mu(e_t)}{\int_{e_t \in [0,\bar{e}]} (\hat{b}(e_t) + \hat{m}(e_t)) d\mu(e_t)}.$$

Variation in any structural parameter of this economy always affect the composition of debt at the firm level, that is, the schedules $\hat{b}(e_t)$ and $\hat{m}(e_t)$. At the aggregate level, these changes may however wash out, or be compensated by changes in the distribution of firms across levels internal finance. The comparative statics of the model indeed reveal that the aggregate bank share $S$ only depends on a subset of structural parameters.

**What leaves aggregate debt structure unchanged?** Figure 3.11 **(1)** looks at economies where mean productivity $\mathbb{E}(\phi)$ is lower than in the baseline economy of table 3.1. Naturally, in these lower-productivity economies, total output is lower, as reported in the right panel of figure 3.11 **(1)**.[42] The aggregate debt structure, however, is identical to the baseline economy, as reported in the left panel. Lower aggregate productivity is associated with lower total borrowing by all firms, and also reduces the maximum operation size of operating firms $\bar{e}$, but it does not alter the overall composition of borrowing.[43] In an analogous manner, economies in which the risk-free rate $r$ is higher (or equivalently, the discount factor $\beta$ is lower) have lower total output, but an identical aggregate debt structure as the baseline economy. This is reported in figure 3.11 **(2)**. At the firm level, a lower risk-free is also associated with lower borrowing and lower maximum scales of operation, but no changes in the composition of borrowing. Thus, both mean productivity and the risk-free rate have scale effects on total borrowing and investment, but not effect on aggregate debt composition.

**What affects aggregate debt structure?** The left panel of figure 3.11 **(3)** plots the aggregate bank share in economies where productivity dispersion is larger than in the baseline

[42] Aggregate output, in the steady-state of the model, is defined as $Y = \int_{\phi_t \geq 0, e_t \in [0,\bar{e}]} \phi_t (e_t + \hat{b}(e_t) + \hat{m}(e_t))^{\zeta} d\mu(e_t) dF(\phi_t) = \mathbb{E}(\phi) \int_{e_t \in [0,\bar{e}]} (e_t + \hat{b}(e_t) + \hat{m}(e_t))^{\zeta} d\mu(e_t)$.

[43] Firm-level borrowing policies for the baseline case and alternative calibrations, for all comparative statics considered in this section, are reported in figure C.4.

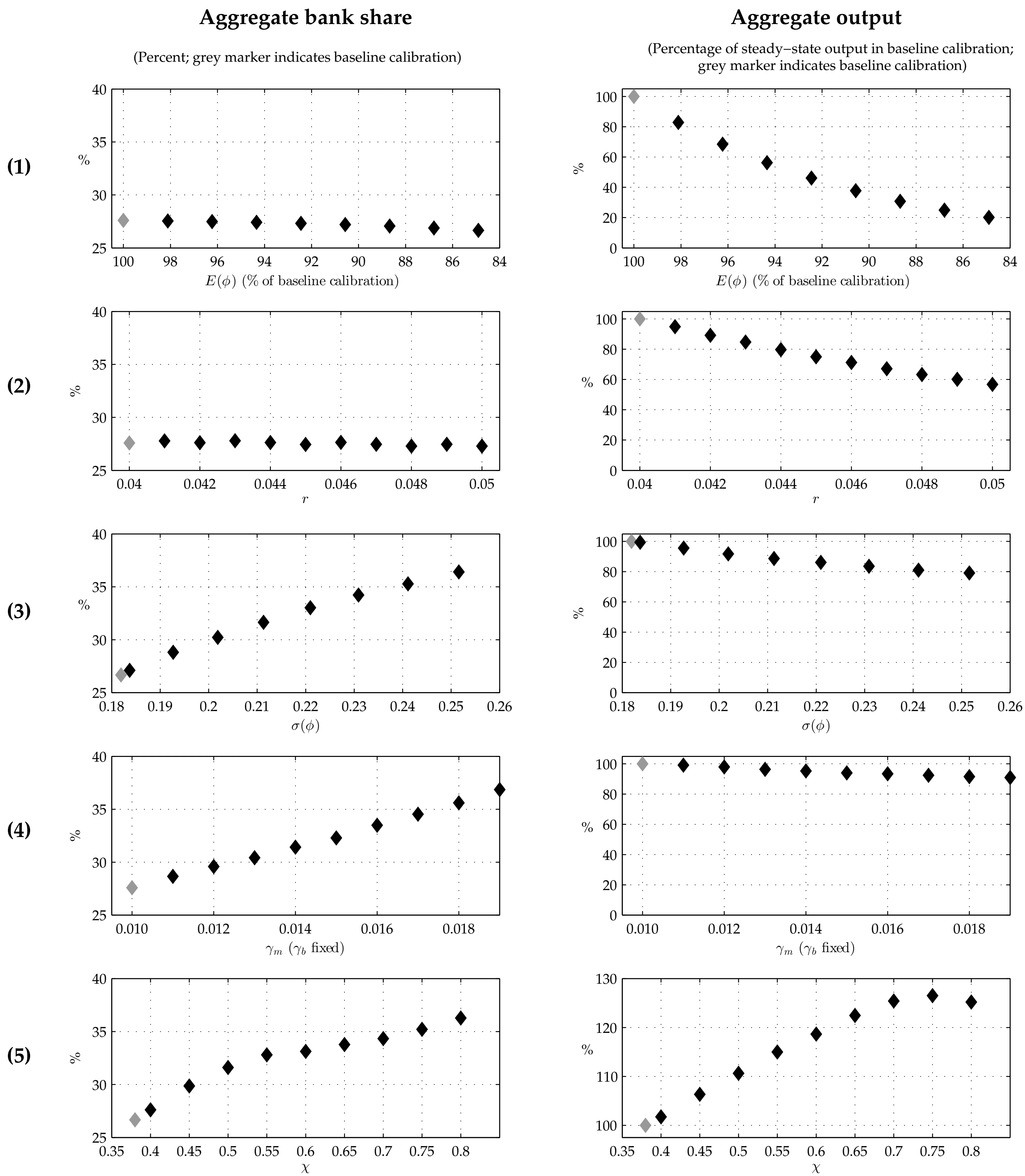



*Figure 3.11: Comparative statics for aggregate bank share and aggregate output. The left column reports the ratio $S$. The right column reports aggregate ouput $Y$ in different calibrations, expressed as a percentage of the baseline calibration reported in table 3.1. The grey point in each graph corresponds to that baseline calibration. Each line corresponds to a particular comparative static exercise; see text for details.*

calibration. Greater productivity dispersion is associated with a higher aggregate bank share. Economies with higher productivity differ from the baseline calibration in two main ways. First, among firms that choose a mixed debt structure, the amount of bank borrowing is higher. Higher productivity dispersion indeed increases the likelihood of financial distress, and makes bank borrowing a more attractive mode of financing for firms. Second, there are more firms using a mixed debt structure. This is because, as liquidation is more frequent, the steady-state distribution of firm size has more mass to the left, and fewer firms reach the maximum size $\bar{e}$. Note that aggregate output is also lower in high-dispersion economies. This reflects lower borrowing by small and medium-sized firms, which display the highest degree of risk-aversion. Borrowing policies of large firms, relative to the baseline, are almost unchanged.[44]

Differences in the lending wedge $\theta = \gamma_b - \gamma_m$ also generate differences in $S$. In figure 3.11 **(4)**, I compare economies with identical bank intermediation costs $\gamma_b$ but higher market intermediation costs $\gamma_m$. This experiment can be thought of as reducing the efficiency of market lending, or tightening market lending requirements outside of financial distress. This increases the aggregate bank share. Interestingly, the resulting change in aggregate debt composition arises because more medium-sized firms choose to adopt a mixed debt structure, and not so much because of existing market-financed firms borrowing more from markets. Economies with higher market intermediation costs $\gamma_m$ also have lower aggregate output. This occurs in large part because the scale of firms is smaller in such an economy.

Finally, differences in liquidation efficiency $\chi$ also generates variation in $S$, as reported in figure 3.11 **(5)**. With a higher liquidation efficiency, more medium-sized firms choose to

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[44] This is visible in figure C.4 **(4)**, which reports the optimal borrowing policies of firms in the baseline calibration, and in alternative calibration with higher productivity dispresion

use only market debt.[45] At the same time, firms using a mixed debt structure borrow more, and more of it comes from banks. This is because, with a higher liquidation efficiency $\chi$, banks' restructuring payoff $\chi\pi_t$ is larger, which relaxes bank lending terms outside of financial distress. Because the bulk of firms in the model are small, their increased borrowing and investment translates into higher output, as depicted in the right panel of figure 3.11 **(5)**.

These aggregate comparative statics also suggest that, apart from their effects on $S$, the two key financial frictions of the model – liquidation costs and the lending wedge – have large effects on aggregate output. A 10% increase in liquidation efficiency, for example, leads to an increase in aggregate output of 9.1%. Policies that alleviate these frictions can therefore lead to large output gains. I come back to this issue in section 3.6.

### 3.4.2 A quantitative example: the US and Italy

In the model, the aggregate bank share only varies with specific structural parameters, namely those that affect the trade-off between flexibility in distress and cost outside of distress: idiosyncratic productivity risk $\sigma(\phi)$, the intermediation wedge $\theta$ and liquidation efficiency $\chi$. To what extent can these parameters account, quantitatively, for observed differences in aggregate debt structures across countries? To answer this question, I study the case of two specific countries: Italy and the US. Italy is of particular interest because, among OECD economies, it is one of the most bank-oriented economies. In 2007Q3, bank loans accounted for 65.7% of financial liabilities of Italian non-financial corporations, a high ratio even among European countries.[46]

To evaluate the potential contribution of $\sigma(\phi)$, $\theta$ and $\chi$ to the differences between Italy

[45] the threshold for switching is smaller, as documented in the left panel of figure C.4 **(5)**.

[46] See appendix C.1 for data sources.

and the US, I compute the aggregate bank share, along with aggregate output and the aggregate debt/assest ratio, under alternative values of each of these three parameters. Results of these alternative calibrations, along with the corresponding moments in the data, and in the baseline calibration, are reported in table 3.2.

| | Bank share | Debt/Assets | Output |
|---|---|---|---|
| Data (US) | 27.5% | 71.9% | 100 |
| Data (Italy) | 65.7% | 56.3% | 68.7 |
| Baseline calibration (US) | 27.3% | 83.7% | 100 |
| Alternative calibrations (Italy) | | | |
| High market intermediation costs | | | |
| $\gamma_{m,IT} = 0.016, \gamma_{b,IT} = \gamma_{b,US}$ | 32.4% | 83.1% | 91.0 |
| $\gamma_{m,IT} = \gamma_{b,US}, \gamma_{b,IT} = \gamma_{b,US}$ | 36.8% | 82.1% | 85.8 |
| High productivity dispersion | | | |
| $\sigma_{IT} = 1.14\sigma_{US}$ | 32.8% | 82.6% | 88.0 |
| $\sigma_{IT} = 1.30\sigma_{US}$ | 37.5% | 80.7% | 78.6 |
| High liquidation efficiency | | | |
| $\chi_{IT} = 0.75$ | 34.8% | 86.2% | 121.0 |
| High productivity dispersion and high market costs | | | |
| $\gamma_{m,IT} = 0.016, \sigma_{IT} = 1.14\sigma_{US}$ | 47.6% | 78.3% | 75.1 |
| $\gamma_{m,IT} = \gamma_{b,IT}, \sigma_{IT} = 1.30\sigma_{US}$ | 52.1% | 76.2% | 72.1 |

*Table 3.2: Sources of differences in aggregate debt structure between the US and Italy. Output is expressed relative to US output; data is output-side real GDP at current PPPs for 2007, from World Penn Tables. For other the data used to computed other ratios, see appendix C.1.*

The bottom panel of table 3.2 focuses on alternative calibrations. The first lines of the panel focus on calibrations in which bank intermediation costs are identical to the US ($\gamma_{b,IT} = \gamma_{b,US}$), but market intermediation costs are higher. The lending wedge $\theta_{IT}$ is therefore lower. This is motivated by the following evidence. First, OECD data on income statements of Italian and US banks indicates that, in 2007, the ratio of their total income to total expenses were close (12.3% for Italy and 13.8% for the US). This suggests that bank-specific intermediation costs in the two countries could be similar.[47] For market lending costs, no direct evidence is available on Italian underwriting costs. However, Santos and Tsatsaronis (2003) estimate that the average difference in underwriting fees between the Euro-area and the US, between 1996 and 2001, stood at 56bps.[48] This motivates choosing higher values for $\gamma_{b,IT}$. I first set $\gamma_{m,IT} = 0.006 + \gamma_{m,US} = 0.016$, resulting in a lending wedge of $\theta_{IT} = 0.010$. In that alternative calibration, the bank share rises to 32.4%. I also consider the limit case in which market intermediation costs are as large as banks': $\gamma_{m,IT} = \gamma_{b,IT}$. In that case, the bank share rises to 36.8%. In those calibrations, consistently with the data, both the Italian debt to assets ratio and output are smaller than in the baseline (US) calibration.

The following lines of table 3.2 looks at whether productivity dispersion can account for differences in the aggregate bank share between the US and Italy. The US calibration uses direct evidence on the dispersion of quantity-based total factor productivity to calibrate the standard deviation of log productivity, $\sigma_{US}$, which are not available for Italy. Michelacci and Schivardi (2013) estimate proxies for idiosyncratic business risk across a panel of countries, using data on firm-level stock price volatility; Italian idiosyncratic business risk is, according to this measure, 14% larger than the US's.[49] Accordingly, I first look at a calibration in which productivity dispersion in Italy is 14% larger than in the US; this increases the bank share to 32.8%. I also look at a more dramatic case, under which productivity dispersion in Italy is 30% higher, approximately corresponding to measured differences between China

[47] As emphasized before, operating expenses do not only reflect monitoring and screening costs, as they include expenses associated with other, e.g. fee-based, activities of banks. However, for the underlying level of $\gamma_b$ to be substantially different from the US's, given similar operating expenses, the allocation of banks' expenses would need to be substantially different.

[48] See their table 3.

[49] See table 2 of their online appendix.

and the US in Hsieh and Klenow (2009). In that case, the bank share increases to 37.5%.

A higher liquidation efficiency in Italy could also account for the different aggregate bank shares. Accordingly, table 3.2 looks at a calibration with a liquidation efficiency of $\chi = 0.75$. This calibration leads to two counterfactual predictions: first, output is higher than in the baseline calibration; second, the ratio of debt to assets is also higher. By contrast, Italy's output per capita and aggregate debt to asset ratios are smaller than that of the US.[50]

The last lines of table 3.2 looks at the combined effects of high productivity dispersion ($\sigma_{IT} = 1.14\sigma_{US}$) and and high market lending costs ($\gamma_{m,IT} = 0.016$). Together, these parameter values lead to an aggregate bank share of 47.6%. This calibration accounts for 20.1% out of the 38.0%, or roughly half, of the difference between the bank shares of the US and Italy. I also look at a calibration in $\gamma_{m,IT} = 0.016$ and $\sigma_{IT} = 1.30\sigma_{US}$. This calibration can be viewed as an upper bound on the joint effects of productivity dispersion and the lending wedge on the aggregate bank share. It leads to an aggregate bank share of 52.1%, accounting for two thirds of the gap between the US and Italy. Both calibrations are also consistent with a lower debt to assets ratio and lower output per capita in Italy than in the US. Overall, these results indicate that productivity risk and lending wedge differentials, alone, can account for between one half and two thirds of the US-Italy difference in aggregate debt structure.

The results of this section suggest that long-run differences in countries' debt structures are closely linked to cross-country variation in parameters associated with financial frictions (intermediation costs $\gamma_b$ and $\gamma_m$, liquidation losses $\chi$), or parameters that amplify their effects (productivity dispersion $\sigma(\phi)$). Because of this, changes in debt structure can be associated

[50]Moreover, even the substantial difference in liquidation efficiency considered here only increases the bank share to 34.8%.

with large gains in output: lowering market-specific intermediation costs, in Italy, to levels comparable to those of the US, would lead to long-run output gains of 9.0% ($= \frac{78.6}{72.1} - 1$). In this sense, the model indicates that cross-country differences in aggregate debt structure are manifestations of potential long-run inefficiencies in financial intermediation.

## 3.5 Aggregate shocks and the corporate debt structure

This section focuses on the business cycle implications of debt heterogeneity. I study the response of the economy to aggregate shocks. For most of the section, I center the discussion on shocks to the lending wedge $\theta$, which I argue offer the best account of observed changes in debt structure in the US during the Great Recession. I come back to other aggregate shocks at the end of the section. Details on the computation of the perfect foresight response of the economy to aggregate shocks are discussed in appendix C.3.

The central finding of this section is that taking into account debt heterogeneity introduces a new channel of propagation of financial shocks, whic operates through substitution across debt instruments, and affects the composition, as well as the scale, of aggregate and firm-level borrowing.

### 3.5.1 Debt structure during the Great Recession

In the US, the Great Recession was accompanied by large and persistent changes in debt structure, both at the aggregate level and at the firm level. Figure 3.12 illustrates these changes. First, the aggregate US bank share fell substantially during the course of the crisis, from 27.5% in 2007Q3 to 19.5% in 2009Q2 (left panel). The pivotal moment is 2008Q3, indicated by a dotted line on the graph. Second, debt structure changes of small firms were different from those of large firms. The middle and right panels of figure 3.12 illustrate this

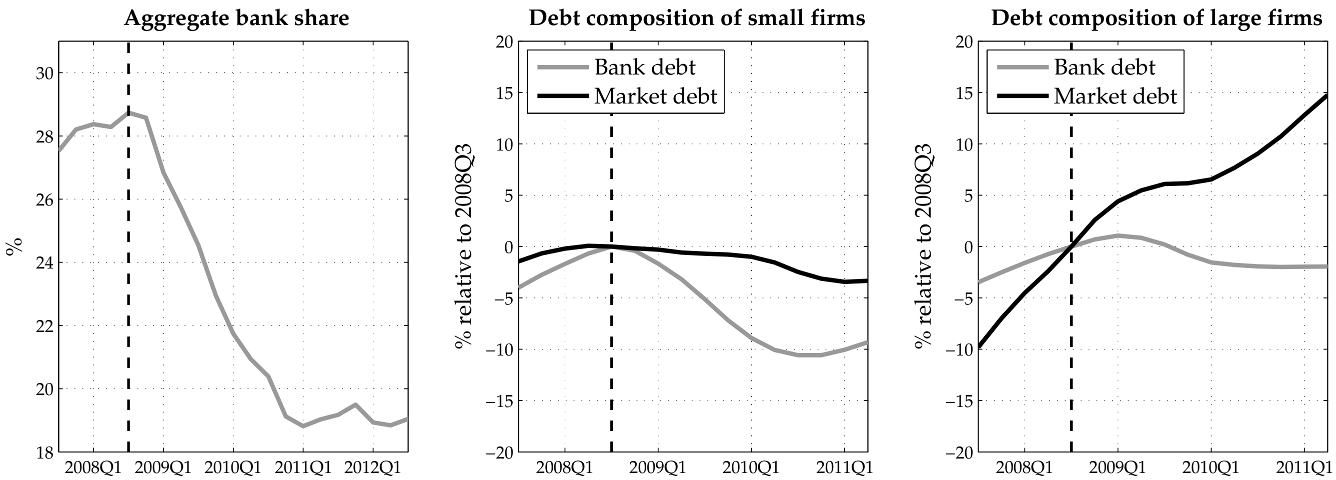



*Figure 3.12: Changes in debt structure, US 2007-2011. Data from the Flow of Funds and the Quarterly Financial Report of Manufacturing firms; small firms are firms with assets less than \$1bn, and large firms are all other firms. See appendix C.1 for details on the construction of the series.*

using data from the sample of US manufacturing firms surveyed by the Quarterly Financial Report (QFR).[51] The middle panel focuses on "small firms", those with less than \$1bn in assets. The "bank" and "market" debt series are scaled so that their sum is equal to the change in total borrowing relative to 2008Q3. Both bank and non-bank borrowing by small firms in this sample fell during the recession. The right panel of figure 3.12 shows that this pattern did not hold for larger firms (defined as the remaining firms surveyed by the QFR). The reduction of these firms' bank liabilities was accompanied by an increase in non-bank liabilities, so that their total borrowing did not change substantially.[52] Thus, small firms mainly reduced their overall level of borrowing during the recession, while large firms mainly changed the composition of their borrowing.

[51]Appendix C.1 discusses this data and the construction of the time series of figure 3.12 in more detail.

[52]The patterns documented here echo other evidence on the 2007-2009 recession, most notably, Adrian, Colla, and Shin (2012). They are also consistent with evidence from other periods or other countries on the effects of shocks to the supply of bank credit on debt composition, such as Becker and Ivashina (2014) and Baumann, Hoggarth, and Pain (2003).

### 3.5.2 A shock to the lending wedge

In the model, these patterns emerge naturally in response to a shock to the lending wedge $\theta$. To illustrate this, I compute the perfect foresight response of the model to an exogenous increase in the lending wedge, driven by an increase in bank lending costs costs $\gamma_{b,t}$. The economy starts from the steady-state described in section 3.2. I choose the path of $\gamma_{b,t}$ to match, quantitatively, the fall in the aggregate bank share documented in figure 3.12. The path of the aggregate bank share, and the implied path of the lending wedge, are reported on the left column of figure 3.13.

**The response of aggregate variables** The response of output and investment to the increase in the lending wedge are reported in the bottom row of figure 3.12. The shock to $\theta$ causes output to fall by 3.9% and investment to fall by 24.7% in the first three years, in line with thepeak to trough drop in output and private investment observed during the 2007-2009 recession. The response of output displays endogenous persistence, since output continues declining even after $\theta$ has reached its new long-run value. At that point, investment instead starts recovering, having someone overshot relative to its long-run value.[53] In the new long-run steady-state, output is 10.1%, and investment 11.3%, below the year 0 steady-state. The recession generated by this shock is thus large, and the responses of output and investment display endogenous persistence.

**The response of small and large firms** The effect of the shock on borrowing by small and large firms is in line with figure 3.12. To maintain comparability with that figure, where "small" and "large" firms are defined, respectively, as those with less and more than \$250m, I construct the "small" and "large" firms of the model as follows. I first determine a cutoff

[53]Aggregate investment is defined as: $I_t = K_t - (1-\delta)K_{t-1}$, with: $K_t \equiv \int_{e_t \in [0,\bar{e}]} (e_t + \hat{b}(e_t) + \hat{m}(e_t)) d\mu(e_t)$.

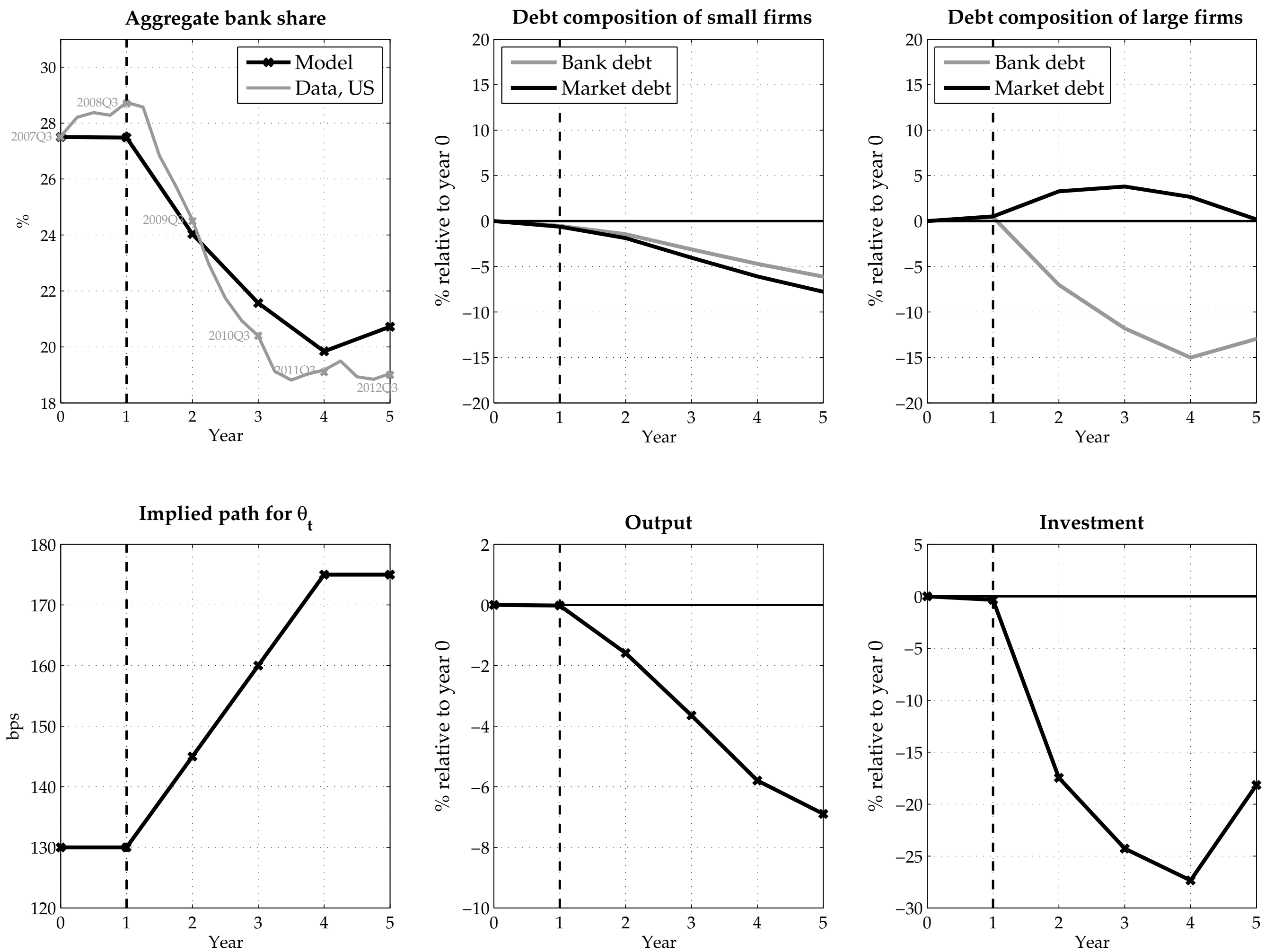



*Figure 3.13: Response of the model to an increase in the lending wedge.*

for internal finance, $e_{S/L}$, such that, in year 0, firms with $e_0 \leq e_{S/L}$ account for 20% of total assets. For $t \geq 1$, I define borrowing by small and large firms, in bank and market debt, as:

$$BS_t = \int_{0 \leq e_t \leq e_{S/L}} \hat{b}_t(e_t) d\mu_t(e_t) \quad , \quad MS_t = \int_{0 \leq e_t \leq e_{S/L}} \hat{m}_t(e_t) d\mu_t(e_t) \quad ;$$

$$BL_t = \int_{e_{S/L} \leq e_t \leq \bar{e}_t} \hat{b}_t(e_t) d\mu_t(e_t) \quad , \quad ML_t = \int_{e_{S/L} \leq e_t \leq \bar{e}_t} \hat{m}_t(e_t) d\mu_t(e_t) \quad ,$$

where, along the perfect foresight path, $\hat{b}_t(.)$, $\hat{m}_t(.)$ and $\bar{e}_t$ characterize firms' policies, and $\mu_t$ is the distribution of firms levels of internal finance.[54] The top row of 3.13 reports the changes in $BS_t, MS_t, BL_t$ and $ML_t$.[55] In the first 5 years, the response of these two groups of firms is qualitatively consistent with the data: both market and bank debt falls for "small" firms; "large" firms, however, substitute market debt for bank debt. Additionally, for large firms, changes in borrowing are quantitatively close to those documented in the QFR, with bank borrowing dropping by roughly 10% over the first three years of the recession.

### 3.5.3 Propagation mechanisms

In order to understand the results, it is useful to first analyze the short-run response of firms' policy functions; the distribution of firms across levels of internal finance is indeed fixed in the short run.

Figure 3.14 illustrates the two key mechanisms through which the shock affects firms' policy function for total assets in the initial period, $\hat{k}_0(e_0) = e_0 + \hat{b}_0(e_0) + \hat{m}_0(e_0)$.

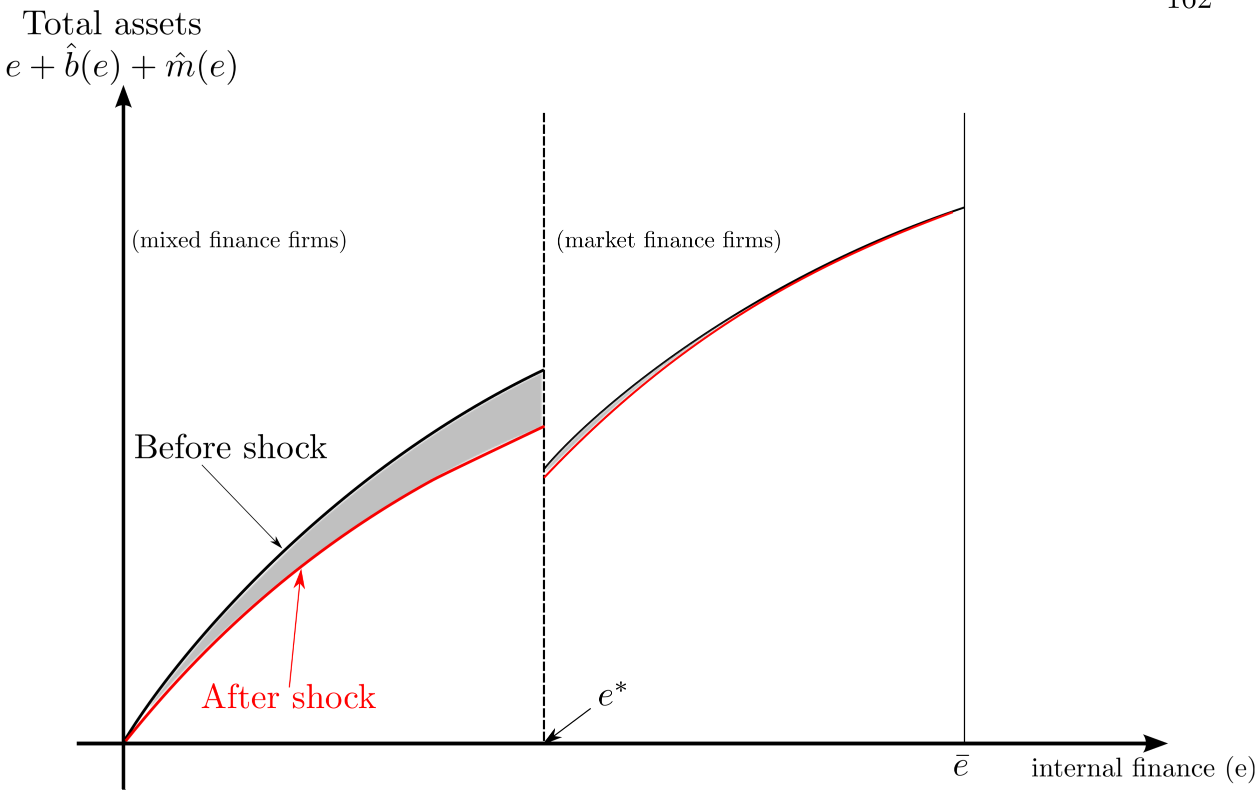
The first mechanism is straightforward: the increase in the lending wedge results in a higher cost of bank borrowing outside of financial distress, and makes bank lending less attractive for mixed-finance firms. Those firms issue less bank debt and reduce their scale of operation, as illustrated on the top panel of figure 3.14.

This is the traditional "bank credit" channel of financial shocks in models with a single borrowing cosntraint. The model however makes two more subtle predictions about the

[54]This definition is analogous to that of the data because it uses fixed cut-offs on successive cross-sections to characterize the small and large firm groups. Qualitative results on the borrowing of these aggregate "small" and "large" firms are similar if one uses a cutoff for assets, instead of internal finance. Because of the non-monotonicity of assets in equity, figures total borrowing $BS_t$, $MS_t$, $BL_t$ and $ML_t$ are however less precisely approximated.

[55]The series are normalized by their time 0 value, and weighted by their shares of total borrowing at year 0, so that they add up to the change in total borrowing, as in figure 3.12.

(a) **"Bank credit" channel**




(b) **"Substitution" channel**

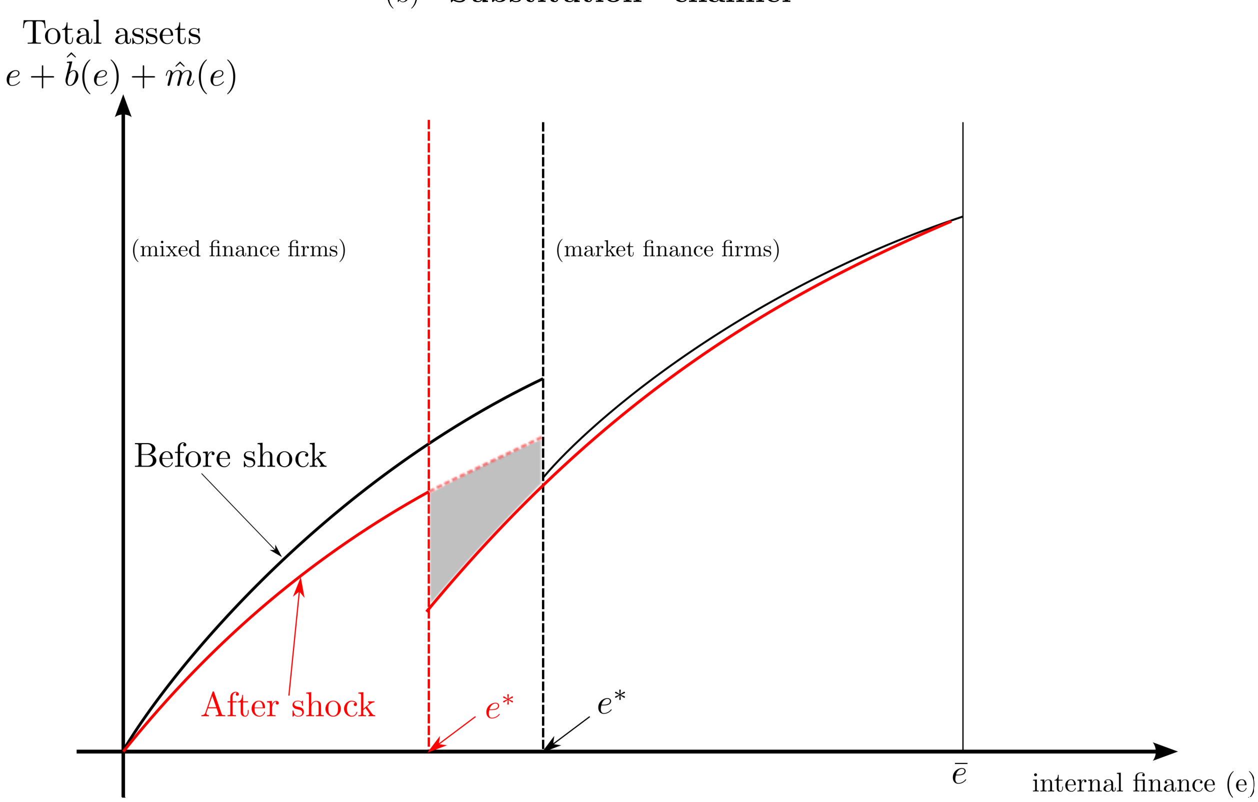



*Figure 3.14: "Credit" and "substitution" channels in the response of firms to the lending wedge shock.*

"bank credit" channel. First, for mixed-debt firms, the increase in the lending wedge actually leads to a fall in market borrowing. The complementarity between bank and market debt, for mixed-debt firms, intuitively comes from the fact that issuing market liabilities, for the firm, is partially a way of relaxing its bank borrowing constraint, since it increases the scale of operation of firms and therefore their value in liquidation. The second additional prediction of the model concerning the "bank credit" channel is that, although market-financed firms are not directly affected by the increase in the lending wedge (since, initially, these firms do not borrow from banks, and only the intermediation cost of bank $\gamma_{b,t}$ increases), they still somewhat reduce their borrowing from markets. Larger firms anticipate the fact that, given a sufficiently bad sequence of shocks $\phi_t$, they will have to revert to mixed debt structures and face the tighter lending terms of banks. They seek to avoid this by reducing their leverage and thus the volatility of their profits.

The second mechanism through which the shock affects output and investment is through its effect on the threshold at which firms switch from a mixed debt structure to a market-financed debt structure. This is illustrated in the bottom panel of figure 3.14. This threshold shifts to the left, and all firms between the pre- and post-shock threshold switch to a market-financed debt structure. A central prediction of the model is that this switch has real effects: it affects the scale at which switching firms choose to operate. Firms that switch loose the flexibility associated with bank finance, and become more exposed to liquidation risk. Put differently, these firms would face excessively high liquidation premia, under a market-financed structure, in order to continue operating at the same scale. As a result, their total borrowing, and therefore their output and investment, drop below what they would be under a mixed debt structure. The additional fall in borrowing due to this "substiution" channel corresponds to the shaded area in the bottom panel of figure 3.14. Quantitatively, this channel is large: in the experiment depicted in figure 3.13, it accounts for roughly half of the intial response of borrowing to the shock. It is this mechanism which is behind most of the increase in market lending of large firms reported in figure 3.13.

The previous discussion helps understand the short-run response of the economy to the shock. In the long-run, the shock has protracted effects on firms' ability to accumulate internal finance, both because lending terms of banks worsen, and because the shock induces firms to leverage less. As a result, the steady-state distribution of firms across levels of internal finance shifts to the left: in the long run, the median stock of internal finance of firms in the economy falls. The slow adjustment of firms' internal funds in the new environment with tougher lending bank lending standards is what drives the endogenous persistence in the response output and investment, since, after the lending wedge has stabilized to a new, higher level, the threshold $e^*$ does not change.

### 3.5.4 A counterfactual experiment

The recession triggerred by an increase in the lending wedge is large, and deepens even after the lending wedge has stabilized to higher levels. A natural question is whether this response depends on the initial aggregate debt structure of the economy. That is, do differences in debt structure alter or amplify the response to shocks?

To answer this question, I compute the response of an initially more bank-dependent economy to the increase in the lending wedge depicted in figure 3.13. The initial calibration of the alternative economy differs from the US calibration of section 3.2 in two ways: intermediation costs of market lenders are larger ($\gamma'_m = 0.016$, instead of $\gamma_m = 0.010$), and firm-level dispersion of productivity is larger ($\sigma(\phi)' = 1.08\sigma(\phi)$). These two differences imply that the model, in steady-state, has an aggregate bank share of 40.3%, in line with the EU average in the sample of firms analyzed in sections 3.3 and 3.4. I refer to this as the European calibration of the model.

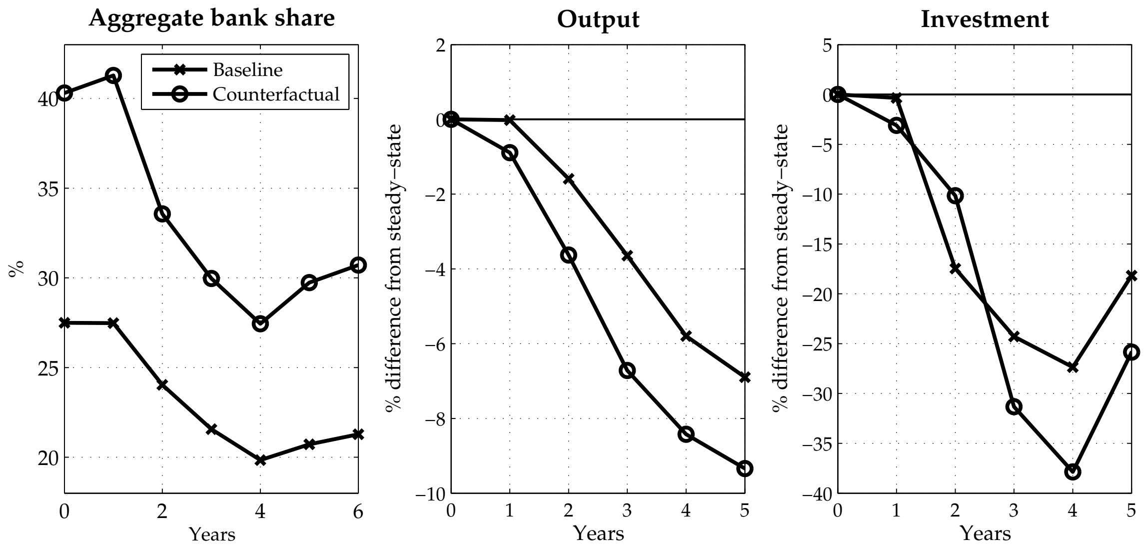



*Figure 3.15: Response to an increase in the lending wedge in the European calibration of the model.*

Figure 3.15 reports the response of the aggregate bank share and of output in this alternative economy. The shock to the lending wedge, in that economy, cause the aggregate bank share to fall, roughly by the same amount than in the baseline US calibration. However, the fall in output in the European calibration of the model is substantially larger than in the US calibration. Following the discussion of the effects of the shock in the US version of the model, the response of output can be understood as the combination of the effects of the "bank credit" and the "substitution" channel. Both effects are larger in this counterfactual experiment. In this European calibration, the threshold $e^*$ at which firms switch to market finance is higher than in the baseline calibration. More firms therefore experience the intensive margin response, but at the firm level this response has quantitatively similar effects as in the baseline calibration. The extensive margin response, however, is magnified. The extensive margin effect in general arises because firms' liquidation premia increase when they switch to market finance. This effect is amplified in this calibration, firm-level productivity risk is higher. The combined effects of the intensive and extensive margin effects leads to a recession that is, at different horizons, between 15 and 30% deeper than in the baseline calibration.

### 3.5.5 Can other aggregate shocks account for the behavior of the bank share?

The previous discussion focused on the effects of an increase in the lending wedge. The specificity of shocks to the lending wedge, however, is that they are the only shocks which, in the model, can generate recessions accompanied by a fall in the aggregate bank share. Shocks to $\mathbb{E}(\phi)$, and shocks that increase $\gamma_b$ and $\gamma_m$ jointly (and thus do not affect the lending wedge), do generate large recessions, but have very limited effects on the composition of aggregate borrowing, and have no substantial substitution at the firm level. A recession driven by an increase in $\sigma\left(\log(\phi)\right)$, on the other hand, is accompanied by an *increase* in the aggregate bank share. This reflects the greater demand for debt flexibility when firm-level uncertainty is high. The response of small and large firms features debt substitution, but in the *opposite* direction, as medium and large firms substitute away from market debt and into bank debt. Thus, in this model, dispersion-driven recessions are associated with changes in corporate debt structure that are at odds with the patterns documented in figure 3.12.

Summarizing, this section has shown that lending wedge shocks can generate large recessions accompanied by substitution toward market borrowing, both at the firm and aggregate levels. The recession generated by such a shock is deeper in an initially more bank-dependent economy, thus suggesting that the differences in aggregate debt structure can amplify the effects of aggregate shocks.

## 3.6 Corporate finance policies and their real effects

The comparison between the US and Italy in section 3.4 suggests that a more market-oriented intermediation sector can lead to higher long-run output and investment, independent of firms' underlying productivity. Moreover, section 3.5 has established that a more market-oriented intermediation sector is also associated with dampened responses to asymmetric financial shocks, such as shocks to the lending wedge. At first blush, these observations seem to suggest that policies that improve firms' access to market credit should be beneficial to aggregate output and investment.

In Europe, some countries indeed seem to have embraced market-based intermediation as a potential remedy to the slowdown in bank lending that has taken place since 2008.[56] This embrace has typically taken the form of corporate finance policies targeted at small and medium-sized firms.

This section discusses the effects of two of these size-dependent policies: a German effort to lower intermediation costs for SME bond issuances, and an Italian tax reform introducing tax deductibility of interest payments on bonds issued by private firms. The environment analyzed in this paper incorporates explicitly the determination of firms' size, so that it is a particularly convenient laboratory to study the effects of size-dependent policies.

### 3.6.1 German markets for SME bonds

Similar to the trends described for the US, the slowdown in bank loan issuance in Germany since 2009 was partially offset by an increase in corporate bond issuance. The increase in corporate bond issuance mostly originated from large companies; small and medium-sized firms did not contribute significantly.

The Bondm exchange was launched by Boerse Stuttgart in 2010 with the goal of changing this situation, and improving bond market access for SMEs. The exchange is restricted to

[56]The Euro Area Bank Lending Survey indicates that between 2008 and 2009, half of Euro area banks tightened their margins on "normal" loans; two thirds tightened their margins on "risky" loans. An additional tightening of bank lending standards also took place in 2010, and coincided with a pick-up in the issuance of new corporate bonds.

firms with less than €250m in assets, and to issuances between €50m and €150m. Crucially, it aims at making these issuances attractive by reducing intermediation costs. Bondm provides firms with a primary market for new issuances, and also operates a secondary market in which private and retail investors trade existing issues. Bondm issuances do not need to be individually rated, and do not need to be underwritten by a bank. Underwriting requirements are particularly costly for SMEs, as few European investment banks specialize in underwriting small issuers.[57]

In the context of the model described in section 3.2, a simple way to capture the advantages offered by Bondm to SME bond issuance is to let market intermediation costs depend on internal funds $e_t$:

$$\gamma_m(e_t) = \begin{cases} \tilde{\gamma}_m & \text{if } e_t \leq e_{sm} \\ \gamma_m & \text{if } e_t > e_{sm} \end{cases} , \quad \tilde{\gamma}_m < \gamma_m.$$

The introduction of the Bondm exchange corresponds to an economy in which $e_{sm} > 0$, as opposed to the baseline case where $e_{sm} = 0$. $\tilde{\gamma}_m \leq \gamma_m$ denotes intermediation costs for small firms ($e_t \leq e_{sm}$) that use Bondm for their issuances.

Figure 3.16 compares the baseline economy (black line), to the economy with low market intermediation costs for SMEs (grey line). The baseline economy with $e_{sm} = 0$ is identical to the European calibration discussed in section 3.5. In the alternative economy, the intermediation costs for Bondm-type issuances is set to $\tilde{\gamma}_m = 0.010 < \gamma_m = 0.016$.[58]

The left panel of figure 3.16 indicates that the policy has limited effects on the size

[57]For more details on the German corporate bond market during the recession, and the Bondm exchange, see the case study of the German mid-cap Dürr in Hillion, Wee, and White (2012).

[58]The threshold $e_{sm}$ is set to $e_{sm} = e^*$, where $e^*$ is the switching threshold in the baseline economy. The qualitative effects of the policy would be identical if $e_{sm} < e$, but this choice makes the graphical discussion that follows clearer.

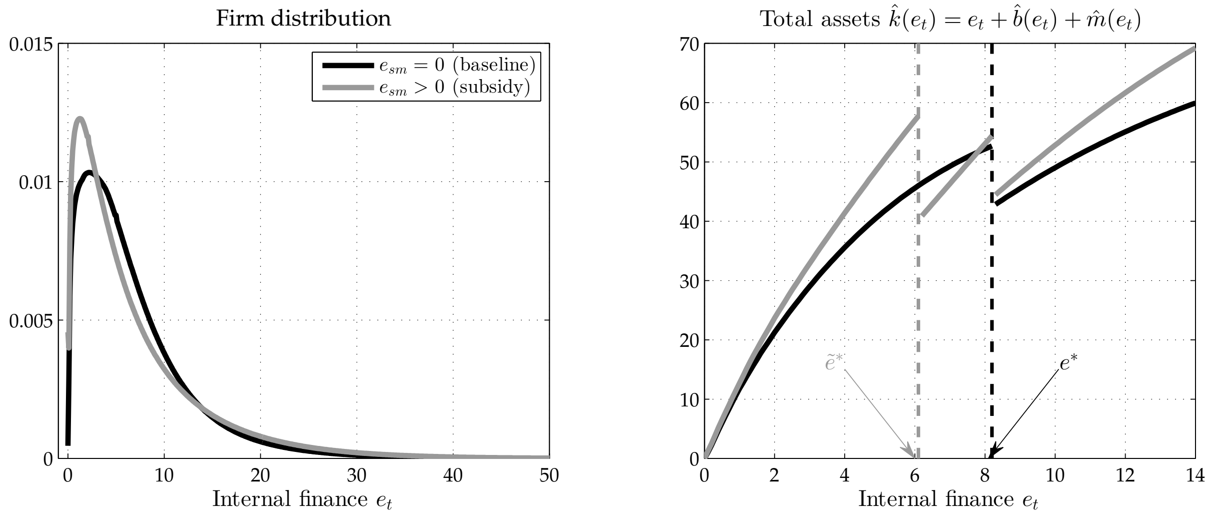



*Figure 3.16: The effect of lower intermediation costs on firms' choice of scale.*

distribution. Its effects on total output and investment are best understood by looking at the response of firms' total assets $\hat{k}(e_t) = e_t + \hat{b}(e_t) + \hat{m}(e_t)$, which are reported in the right panel of figure 3.16.

This response depends strongly on internal funds $e_t$. There are three cases. First, the lower intermediation costs create a new threshhold $\tilde{e}^*$. Above this threshold, firms switch to a market-financed debt structure, but nevertheless still benefit from lower intermediation costs. As one would expect, firms with $e_t \leq \tilde{e}^*$ respond to the policy by increasing total borrowing: lower intermediation costs indeed encourage these small firms to increase market debt issuance, and additionally relaxes their bank borrowing constraint, because of the complementarity between forms of borrowing for firms in the mixed-finance regime.

The second case is that of firms with internal funds $e_t > e^*$, where $e^*$ is the threshold above which firms do not benefit from the low market intermediation costs (i.e., cannot issue on the Bondm market anymore). Although these firms are not directly targeted by the policy, they nevertheless increase bond issuance in response to it. This is because they anticipate that, now that market borrowing is cheaper at small scales, a string of bad shocks will not reduce their profitability as much as before.

The third case relates to firms with $\tilde{e}^* < e_t \leq e^*$. These firms are large enough that the policy will induce them to switch to an entirely market-financed debt structure, but small enough that they will still benefit from the Bondm intermediation costs for their bond issuances. The switch fragilizes the debt structure of these firms, as they loose the ability to restructure debt in bad times. As a result, some of these firms deleverage: their total borrowing is *smaller* than in the world with higher intermediation costs, and they operate at a *smaller* scale.[59] Surprisingly, the lower intermediation costs therefore result in a precautionary reduction in total debt issuance and total investment by these firms.

How do these two effects – the "credit channel" effect that stimulates borrowing and investment by firms with $e_t \leq \tilde{e}^*$ and $e_t > e^*$, and the "substitition channel" effect that depresses borrowing and investment by firms with $\tilde{e}^* < e_t \leq e^*$ – measure up against each other? Overall, the effect of the policy $e_{sm} > 0$ on total output and investment is positive: they increase, respectively, by 6.3% and 7.5% under the policy of lower intermediation cost for SMEs. However, the negative effect on investment by medium-sized firms is sizeable, because the mass of firms that are in this region is large (see the left panel of figure 3.16). Specifically, these firms account for a $-3.8\%$ *decline* in aggregate investment, and a $-2.3\%$ *decline* in aggregate output, relative to the initial steady-state. This policy thus offers an example in which improving access to markets for SMEs may have unintended consequences, precisely because the of the way in which it affects investment behavior of firms.

### 3.6.2 An Italian tax reform

To the extent that they discriminate between bank and market liabilities, taxes are an alternative tool that can be used as an instrument to influence firms' choice of debt structure.

[59]Visually, this is indicated by the fact that the light grey line is below the black line on the right panel of figure 3.16.

A 2012 Italian tax reform meant to improve access to bond markets for private companies includes the tax treatment of interest payment to corporate bonds as a part of a larger array of policy tools.[60] Specifically, the reform allows private firms to deduct interest paid on bonds in the same way as interest paid on other debt, in line with the tax rules imposed on large firms.

The model of section 3.2 does not explicitly incorporate tax deductibility of debt. However, in appendix C.4.6, I show that a simple extension can accomodate this. Let cash on hand of the firm in repayment be given by:

$$n_t^P = (1-\tau)\pi_t - (1-\tau_b)R_{b,t} - (1-\tau_m)R_{m,t},$$

where $\tau$ denotes the marginal coporate tax rate. Then, $\tau_b \leq \tau$ and $\tau_m \leq \tau$ reflect preferential tax treatments of different types of debt instruments.

The two tax regimes that existed in Italy prior to the reform were, respectively, $\tau_b = \tau$ and $\tau_m = 0$ (only bank debt is interest-deductible); and which $\tau_b = \tau_m = \tau$ (both types of debt are treated indentically for tax purposes). The policy experiment then consists in comparing an economy where only sufficiently large firms enjoy preferential tax treatment for market debt ($\tau_b = \tau_m = \tau$ for $e_t > e^*$; $\tau_b = \tau$ and $\tau_m = 0$ for $e_t \leq e^*$), to an economy in which all firms enjoy the smae tax shields for market and bank debt ($\tau_m = \tau_b = \tau$ for all $e_t$).

The impact of this change in tax policy on borrowing and investment are qualitatively similar to the previous experiment.[61] Intuitively, a differential treatment of bank and market debt ($\tau_m < \tau_b$) directly affects the relative cost of market and bank debt outside of liquida-

[60] The reform also relaxes the requirements to find a sponsor to guarantee the issuance of the bond, and eliminates existing limits on indebtedness as a fraction of net worth for private firms. See http://www.paulhastings.com/Resources/Upload/Publications/2351.pdf for more details on the contents of the reform.

[61] Appendix C.4.6 reports these results.

tion. Introducing a tax shield on market debt issuance for SMEs will therefore have similar effects on borrowing as lowering market intermediation costs. Namely, the tax shield boosts total borrowing by the smallest and the largest firms, because of the "credit channel" effect discussed above. However, it reduces total borrowing by the mass of intermediate firms that switch entirely to market-financed debt structure in response to the introduction of the tax shield.

While there is a rationale for using corporate finance policies to encourage firms to rely more on markets as a source of funds, this section has shown that this these reforms may also have adverse effects on firm-level investment. The corporate finance policies may indeed create an incentive for firms to adopt more fragile, market-financed debt structures. Crucially, the model indicates the real implications of this fragility: firms that switch also these debt structures also reduce borrowing and investment. As the German experiment suggests, the effect of this "substitution" channel on aggregate investment can be quantitatively large.

## 3.7 Conclusion

The composition of corporate borrowing between bank loans and market lending exhibits substantial variation, both across countries, across firms, and over time. In this paper, I started from the simple view that banks provide more flexibility to firms, while markets have lower marginal lending costs. I showed that embedding this trade-off between flexibility and cost into a simple model of firm dynamics leads to cross-sectional predictions that align well with cross-sectional data on the composition of firm debt. The model is useful to parse some simple explanations of cross-country variation in debt structure; I argued that productivity dispersion and differences in marginal lending costs can account for a large share of US-Italy differences in debt structure. It is also useful to understand the propagation of financial shocks and the effects of financial policies when firms have access to different debt instruments. A central finding is that, aside from the traditional "credit" channel effects of financial shocks or policies on borrowing and investment, debt heterogeneity introduces a new "substitution" channel. Size distribution plays a key role in mediating the effects of these various channels. The "substitution" channel only operates medium-size firms, who respond to variation in aggregate conditions or to changes in policy by switching to market-financed debt structures and, in the process, deleveraging for precautionary reasons.

The results of this paper also suggest that the development of corporate bond markets can provide a tool for macroprudential policy, to the extent that they can help mitigate the effects of banking shocks. As discussed by Eichengreen and Luengnaruemitchai (2004), this is indeed the insight embraced by many Asian countries, whose active development of local currency corporate bond market came as a reaction to the large contraction in bank credit during the Asian crises of 1997-1998. As described in section 3.6, similar changes are taking place in Europe since 2010, in anticipation of the impact of the Solvency II and Basel III on bank financing to corporates. One potential drawback of this tool is that, to the extent that market debt is harder to restructure than bank loans, gains during banking crises may be offset by exacerbated business-cycle volatility in response to other shocks. I leave this topic to future research.

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# Appendices

# Appendix A

# Appendix for Chapter 1

## A.1 The inventory model for estimation

We describe the stock-elastic inventory model, allowing for trends and both stationary and nonstationary shocks as in Schmitt-Grohé and Uribe (2012). We start by defining the trend components of the model.

### A.1.1 Trends in the model

The two sources of nonstationarity in the model of Schmitt-Grohé and Uribe (2012) are neutral and investment-specific productivity. Aggregate sales $S_t$ can be written as

$$S_t = C_t + Z_t^I I_t + G_t,$$

where $Z_t^I$ is the nonstationary investment-specific productivity. From this equation and balanced growth path, we observe that $Z_t^I I_t / S_t$ is stationary. Letting the trend of aggregate sales to be $X_t^Y$ and the trend of $I_t$ to be $X_t^I$, the balanced-growth condition tells us that

$$X_t^Y = Z_t^I X_t^I. \tag{A.1}$$

Moreover, from the capital accumulation function, capital and investment should follow the same trend. Writing $X_t^K$ as the trend of capital, the second condition is

$$X_t^K = X_t^I. \tag{A.2}$$

Lastly, the production function is

$$Y_t = z_t(u_t K_t)^{\alpha_K}(X_t n_t)^{\alpha_N}(X_t L)^{1-\alpha_K-\alpha_N}.$$

Since the trend must also be consistent, we have the following equation

$$X_t^Y = (X_t^K)^{\alpha_K} X_t^{1-\alpha_K}. \tag{A.3}$$

From the three conditions (A.1), (A.2) and (A.3), we can solve for the trends $X_t^Y$, $X_t^I$, $X_t^K$ as

$$X_t^Y = X_t(Z_t^I)^{\frac{\alpha_K}{\alpha_K-1}}, \qquad X_t^K = X_t^I = X_t(Z_t^I)^{\frac{1}{\alpha_K-1}}.$$

We are now ready to write the stationary problem. It will be useful to write the stationary variables in lower cases as follows:

$$y_t = \frac{Y_t}{X_t^Y}, \quad c_t = \frac{C_t}{X_t^Y}, \quad i_t = \frac{I_t}{X_t^I}, \quad k_{t+1} = \frac{K_{t+1}}{X_t^K}, \quad g_t = \frac{G_t}{X_t^G}.$$

Note that the trend on government spending $X_t^G$ is defined as a smoothed version of $X_t^Y$:

$$X_t^G = (X_{t-1}^G)^{\rho_{xg}}(X_{t-1}^Y)^{1-\rho_{xg}}.$$

We can also express the two exogenous trends in stationary variables:

$$\mu_t^X = \frac{X_t}{X_{t-1}}, \quad \mu_t^A = \frac{Z_t^I}{Z_{t-1}^I}.$$

Using this, we get an expression for the endogenous trends:

$$\mu_t^Y = \mu_t^X(\mu_t^A)^{\frac{\alpha_K}{\alpha_K - 1}}, \qquad \mu_t^I = \mu_t^K = \frac{\mu_t^Y}{\mu_t^A}.$$

We also define $x_t^G$ as the relative trend of government spending:

$$x_t^G \equiv \frac{X_t^G}{X_t^Y} = \frac{(X_{t-1}^G)^{\rho_{xg}}(X_{t-1}^Y)^{1-\rho_{xg}}}{X_t^Y} = \frac{(x_{t-1}^G)^{\rho_{xg}}}{\mu_t^Y}.$$

With these stationary variables, we can express the problem in terms of stationary variables. We start with the household problem.

### A.1.2 Household problem

To write down all the equilibrium conditions, the household utility is defined as follows:

$$U = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \zeta_{h,t} \frac{M_t^{1-\sigma} - 1}{1-\sigma},$$

$$M_t = C_t - bC_{t-1} - \psi_t \frac{n_t^{1+\xi^{-1}}}{1+\xi^{-1}} H_t,$$

$$H_t = (C_t - bC_{t-1})^{\gamma_h} H_{t-1}^{1-\gamma_h}.$$

The household constraints are the following:

$$
\begin{aligned}
\int_0^1 \frac{p_t(j)}{P_t} S_t(j)dj + \mathbb{E}_t q_{t,t+1} B_{t+1} = W_t n_t + R_t u_t K_t + B_t + \Pi_t, \\
S_t = \left( \int_0^1 \nu_t(j)^{\frac{1}{\theta}} S_t(j)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}}, \\
C_t + Z_t^I I_t + G_t = S_t, \\
K_{t+1} = z_t^k I_t \left( 1 - \phi \left( \frac{I_t}{I_{t-1}} \right) \right) + (1 - \delta(u_t)) K_t.
\end{aligned}
$$

Notice that given the symmetry of the firm behavior, $\nu_t(j) = 1$ and $\int_0^1 \frac{p_t(j)}{P_t} S_t(j)dj = S_t$. Hence the household problem can be written as

$$
\begin{aligned}
\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \zeta_{h,t} \frac{M_t^{1-\sigma} - 1}{1-\sigma} + \Lambda_{m,t} \left[ C_t - bC_{t-1} - \psi_t \frac{n_t^{1+\xi^{-1}}}{1+\xi^{-1}} H_t - M_t \right] \\
+ \Lambda_{h,t} [H_t - (C_t - bC_{t-1})^{\gamma_h} H_{t-1}^{1-\gamma_h}] \\
+ \Lambda_t [W_t n_t + R_t u_t K_t + \Pi_t + B_t - C_t - Z_t^I I_t - G_t - \mathbb{E}_t q_{t,t+1} B_{t+1}] \\
+ \Lambda_{k,t} \left[ z_t^k I_t \left( 1 - \phi \left( \frac{I_t}{I_{t-1}} \right) \right) + (1 - \delta(u_t)) K_t - K_{t+1} \right] \Bigg\}.
\end{aligned}
$$

Hence the household first order conditions are characterized by the following:

$$[M_t] : \zeta_{h,t} M_t^{-\sigma} = \Lambda_{m,t}, \tag{A.4}$$

$$[H_t] : \Lambda_{h,t} - \Lambda_{m,t} \psi_t \frac{n_t^{1+\xi^{-1}}}{1+\xi^{-1}} = \beta \mathbb{E}_t \Lambda_{h,t+1} (C_{t+1} - bC_t)^{\gamma_h} H_t^{-\gamma_h} (1 - \gamma_h), \tag{A.5}$$

$$[C_t] : \Lambda_{m,t} - \Lambda_{h,t} \gamma_h (C_t - bC_{t-1})^{\gamma_h - 1} H_{t-1}^{1-\gamma_h} - \beta b \mathbb{E}_t [\Lambda_{m,t+1} - \Lambda_{h,t+1} \gamma_h (C_{t+1} - bC_t)^{\gamma_h - 1} H_t^{1-\gamma_h}] = \Lambda_t, \tag{A.6}$$

$$[n_t] : \Lambda_t W_t = \Lambda_{m,t} \psi_t n_t^{\xi^{-1}} H_t, \tag{A.7}$$

$$[I_t] : Z_t^I \Lambda_t - \Lambda_{k,t} z_t^k \left(1 - \phi\left(\frac{I_t}{I_{t-1}}\right) - \left(\frac{I_t}{I_{t-1}}\right)\phi'\left(\frac{I_t}{I_{t-1}}\right)\right) = \beta \mathbb{E}_t \Lambda_{k,t+1} z_{t+1}^k \left(\frac{I_{t+1}}{I_t}\right)^2 \phi'\left(\frac{I_{t+1}}{I_t}\right), \tag{A.8}$$

$$[K_{t+1}] : \Lambda_{k,t} = \beta \mathbb{E}_t[\Lambda_{t+1} R_{t+1} u_{t+1} + \Lambda_{k,t+1}(1 - \delta(u_{t+1}))], \tag{A.9}$$

$$[u_t] : \Lambda_t R_t = \Lambda_{k,t} \delta'(u_t), \qquad [u_t = 1 \text{ if not allowed to vary}], \tag{A.10}$$

$$[B_{t+1}] : q_{t,t+1} = \beta \frac{\Lambda_{t+1}}{\Lambda_t}, \tag{A.11}$$

$$[\Lambda_{m,t}] : M_t = C_t - bC_{t-1} - \psi_t \frac{n_t^{1+\xi^{-1}}}{1+\xi^{-1}} H_t, \tag{A.12}$$

$$[\Lambda_{h,t}] : H_t = (C_t - bC_{t-1})^{\gamma_h} H_{t-1}^{1-\gamma_h}, \tag{A.13}$$

$$[\Lambda_{k,t}] : K_{t+1} = z_t^k I_t \left(1 - \phi\left(\frac{I_t}{I_{t-1}}\right)\right) + (1 - \delta(u_t))K_t. \tag{A.14}$$

and the household budget constraint. We also want private spending $S_t^p$ and total absorption $S_t$ as

$$S_t^p = C_t + Z_t^I I_t, \tag{A.15}$$

$$S_t = C_t + Z_t^I I_t + G_t. \tag{A.16}$$

We define the following stationary variables:

$$\lambda_{m,t} = \frac{\Lambda_{m,t}}{(X_t^Y)^{-\sigma}}, \quad \lambda_{h,t} = \frac{\Lambda_{h,t}}{(X_t^Y)^{-\sigma}}, \quad \lambda_t = \frac{\Lambda_t}{(X_t^Y)^{-\sigma}}, \quad \lambda_{k,t} = \frac{\Lambda_{k,t}}{(X_t^Y)^{-\sigma} Z_t^I}, \quad w_t = \frac{W_t}{X_t^Y}, \quad r_t = \frac{R_t}{Z_t^I}.$$

Using these expressions as well as the ones defined in the previous section, we rewrite the household first order condition in terms of stationary variables:

$$[m_t] : \zeta_{h,t} m_t^{-\sigma} = \lambda_{m,t}, \tag{A.17}$$

$$[h_t] : \lambda_{h,t} - \lambda_{m,t} \psi_t \frac{n_t^{1+\xi^{-1}}}{1+\xi^{-1}} = \beta \mathbb{E}_t \lambda_{h,t+1} (\mu_{t+1}^Y)^{-\sigma} (c_{t+1} \mu_{t+1}^Y - bc_t)^{\gamma_h} h_t^{-\gamma_h} (1 - \gamma_h), \tag{A.18}$$

$$[c_t] : \lambda_{m,t} - \lambda_{h,t}\gamma_h \left(\frac{c_t\mu_t^Y - bc_{t-1}}{h_{t-1}}\right)^{\gamma_h - 1} - \beta b\mathbb{E}_t(\mu_{t+1}^Y)^{-\sigma}\left[\lambda_{m,t+1} - \lambda_{h,t+1}\gamma_h\left(\frac{c_{t+1}\mu_{t+1}^Y - bc_t}{h_t}\right)^{\gamma_h-1}\right] = \lambda_t, \tag{A.19}$$

$$[n_t] : \lambda_t w_t = \lambda_{m,t}\psi_t n_t^{\xi^{-1}} h_t, \tag{A.20}$$

$$\begin{aligned}[i_t] : \lambda_t - \lambda_{k,t} z_t^k \left(1 - \phi\left(\frac{i_t}{i_{t-1}}\mu_t^I\right) - \left(\frac{i_t}{i_{t-1}}\mu_t^I\right)\phi'\left(\frac{i_t}{i_{t-1}}\mu_t^I\right)\right) \\ = \beta\mathbb{E}_t\lambda_{k,t+1}\frac{\mu_{t+1}^A}{(\mu_{t+1}^Y)^\sigma} z_{t+1}^k\left(\frac{i_{t+1}}{i_t}\mu_{t+1}^I\right)^2\phi'\left(\frac{i_{t+1}}{i_t}\mu_{t+1}^I\right),\end{aligned} \tag{A.21}$$

$$[k_{t+1}] : \lambda_{k,t} = \beta\mathbb{E}_t(\mu_{t+1}^Y)^{-\sigma}\mu_{t+1}^A[\lambda_{t+1}r_{t+1}u_{t+1} + \lambda_{k,t+1}(1-\delta(u_{t+1}))], \tag{A.22}$$

$$[u_t] : \lambda_t r_t = \lambda_{k,t}\delta'(u_t), \qquad [u_t = 1 \text{ if not allowed to vary}], \tag{A.23}$$

$$[b_{t+1}] : q_{t,t+1} = \beta\frac{\lambda_{t+1}}{\lambda_t}(\mu_{t+1}^Y)^{-\sigma}, \tag{A.24}$$

$$[\lambda_{m,t}] : m_t = c_t - b\frac{c_{t-1}}{\mu_t^Y} - \psi_t\frac{n_t^{1+\xi^{-1}}}{1+\xi^{-1}}h_t, \tag{A.25}$$

$$[\lambda_{h,t}] : h_t = \left(c_t - b\frac{c_{t-1}}{\mu_t^Y}\right)^{\gamma_h}\left(\frac{h_{t-1}}{\mu_t^Y}\right)^{1-\gamma_h}, \tag{A.26}$$

$$[\lambda_{k,t}] : k_{t+1} = z_t^k i_t\left(1 - \phi\left(\frac{i_t}{i_{t-1}}\mu_t^I\right)\right) + (1-\delta(u_t))\frac{k_t}{\mu_t^I}, \tag{A.27}$$

$$[s_t^p] : s_t^p = c_t + i_t, \tag{A.28}$$

$$[s_t] : s_t = c_t + i_t + g_t x_t^G. \tag{A.29}$$

Now, in log-linearized form:

$$[m_t] : \hat{\zeta}_{h,t} - \sigma\hat{m}_t = \hat{\lambda}_{m,t}, \tag{A.30}$$

$$\begin{aligned}[h_t] : \hat{\lambda}_{h,t} - \left[1 - \beta(1-\gamma_h)(\mu^Y)^{1-\sigma}\right]\left[\hat{\lambda}_{m,t} + \hat{\psi}_t + (1+\xi^{-1})\hat{n}_t\right] \\ = \beta(1-\gamma_h)(\mu^Y)^{1-\sigma}\left[\mathbb{E}_t\hat{\lambda}_{h,t+1} + (1-\sigma)\mathbb{E}_t\hat{\mu}_{t+1}^Y + \mathbb{E}_t\hat{h}_{t+1} - \hat{h}_t\right],\end{aligned} \tag{A.31}$$

$$\begin{aligned}[c_t] : \lambda\hat{\lambda}_t = \lambda_m\hat{\lambda}_{m,t} - \lambda_h\gamma_h(\mu^Y)^{1-\frac{1}{\gamma_h}}\left[\hat{\lambda}_{h,t} + \hat{h}_t - \frac{\mu^Y}{\mu^Y - b}\hat{c}_t + \frac{b}{\mu^Y - b}\hat{c}_{t-1} - \frac{b}{\mu^Y - b}\hat{\mu}_t^Y\right] \\ + \sigma\beta b(\mu^Y)^{-\sigma}\left[\lambda_m - \lambda_h\gamma_h(\mu^Y)^{1-\frac{1}{\gamma_h}}\right]\mathbb{E}_t\hat{\mu}_{t+1}^Y - \beta b(\mu^Y)^{-\sigma}\lambda_m\mathbb{E}_t\hat{\lambda}_{m,t+1}\end{aligned}$$

$$+ \beta b(\mu^Y)^{-\sigma}\lambda_h\gamma_h(\mu^Y)^{1-\frac{1}{\gamma_h}}\left[\mathbb{E}_t\hat{\lambda}_{h,t+1} + \mathbb{E}_t\hat{h}_{t+1} - \frac{\mu^Y}{\mu^Y - b}\mathbb{E}_t\hat{c}_{t+1} + \frac{b}{\mu^Y - b}\hat{c}_t - \frac{b}{\mu^Y - b}\mathbb{E}_t\hat{\mu}^Y_{t+1}\right], \tag{A.32}$$

$$[n_t] : \hat{\lambda}_t + \hat{w}_t = \hat{\lambda}_{m,t} + \hat{\psi}_t + \frac{1}{\xi}\hat{n}_t + \hat{h}_t, \tag{A.33}$$

$$[i_t] : \hat{\lambda}_{k,t} = \hat{\lambda}_t - \hat{z}^k_t + \mu^I\phi''_I(\hat{i}_t - \hat{i}_{t-1} + \hat{\mu}^I_t) - \beta\frac{\mu^A}{(\mu^Y)^\sigma}(\mu^I)^3\phi''_I(\mathbb{E}_t\hat{i}_{t+1} - \hat{i}_t + \mathbb{E}_t\hat{\mu}^I_{t+1}), \tag{A.34}$$

$$[k_{t+1}] : \hat{\lambda}_{k,t} = \mathbb{E}_t\hat{\mu}^A_{t+1} - \sigma\mathbb{E}_t\hat{\mu}^Y_{t+1} + \beta(\mu^Y)^{-\sigma}\mu^A(1-\delta_k)\mathbb{E}_t\hat{\lambda}_{k,t+1}$$

$$+ \left[1 - \beta(\mu^Y)^{-\sigma}\mu^A(1-\delta_k)\right](\mathbb{E}_t\hat{\lambda}_{t+1} + \mathbb{E}_t\hat{r}_{t+1} + \mathbb{E}_t\hat{u}_{t+1}) - \beta(\mu^Y)^{-\sigma}\mu^A\delta'_k\mathbb{E}_t\hat{u}_{t+1}, \tag{A.35}$$

$$[u_t] : \hat{\lambda}_t + \hat{r}_t = \hat{\lambda}_{k,t} + \frac{\delta''_k}{\delta'_k}\hat{u}_t, \qquad [\hat{u}_t = 0 \text{ if not allowed to vary}], \tag{A.36}$$

$$[b_{t+1}] : \mathbb{E}_t\hat{q}_{t,t+1} = \mathbb{E}_t\hat{\lambda}_{t+1} - \hat{\lambda}_t - \sigma\mathbb{E}_t\hat{\mu}^Y_{t+1}, \tag{A.37}$$

$$[\lambda_{m,t}] : m\hat{m}_t = c\hat{c}_t - b\frac{c}{\mu^Y}\hat{c}_{t-1} + b\frac{c}{\mu^Y}\hat{\mu}^Y_t - \psi\frac{n^{1+\xi^{-1}}}{1+\xi^{-1}}h\left[\hat{\psi}_t + \hat{h}_t + (1+\xi^{-1})\hat{n}_t\right], \tag{A.38}$$

$$[\lambda_{h,t}] : \hat{h}_t = \frac{\gamma_h\mu^Y}{\mu^Y - b}\hat{c}_t - b\frac{\gamma_h}{\mu^Y - b}\hat{c}_{t-1} + b\frac{\gamma_h}{\mu^Y - b}\hat{\mu}^Y_t + (1-\gamma_h)\hat{h}_{t-1} - (1-\gamma_h)\hat{\mu}^Y_t, \tag{A.39}$$

$$[\lambda_{k,t}] : \hat{k}_{t+1} = \left(1 - \frac{1-\delta_k}{\mu^I}\right)\hat{z}^k_t + \left(1 - \frac{1-\delta_k}{\mu^I}\right)\hat{i}_t + \frac{1-\delta_k}{\mu^I}\hat{k}_t - \frac{1-\delta_k}{\mu^I}\hat{\mu}^I_t - \frac{\delta'_k}{\mu^I}\hat{u}_t, \tag{A.40}$$

$$[s^p_t] : \hat{s}^p_t = \frac{c}{c+i}\hat{c}_t + \frac{i}{c+i}\hat{i}_t, \tag{A.41}$$

$$[s_t] : \hat{s}_t = \frac{c}{s}\hat{c}_t + \frac{i}{s}\hat{i}_t + \frac{gx^G}{s}\hat{g}_t + \frac{gx^G}{s}\hat{x}^G_t, \tag{A.42}$$

$$[\mu^Y_t] : \hat{\mu}^Y_t = \hat{\mu}^X_t + \frac{\alpha_K}{\alpha_K - 1}\hat{\mu}^A_t, \tag{A.43}$$

$$[\mu^I_t] : \hat{\mu}^I_t = \hat{\mu}^Y_t - \hat{\mu}^A_t, \tag{A.44}$$

$$[x^G_t] : \hat{x}^G_t = \rho_{xg}\hat{x}^G_{t-1} - \hat{\mu}^Y_t. \tag{A.45}$$

### A.1.3 Firm problem without inventories

This section is only for completeness. The readers should skip this section and read the firm problem with stock-elastic inventories. The firm side is subject to monopolistic competition.

As you will see, this aspect itself will introduce no changes in the dynamics relative to the real model since no price rigidity is assumed. Firm $j \in [0,1]$ solves the following problem:

$$\max\ \mathbb{E}_0 q_{0,t}\left[\frac{p_t(j)}{P_t}S_t(j) - W_t n_t(j) - R_t u_t(j)K_t(j)\right],$$

subject to

$$\begin{aligned}
S_t(j) &= \left(\frac{p_t(j)}{P_t}\right)^{-\theta} S_t,\\
Y_t(j) &= z_t(u_t(j)K_t(j))^{\alpha_K} n_t(j)^{\alpha_N} l^{1-\alpha_K-\alpha_N} X_t^{1-\alpha_K},\\
Y_t(j) &= S_t(j).
\end{aligned}$$

As is well known, the last constraint is the demand constraint when no inventory adjustment is allowed. Letting the multiplier on this constraint to be the marginal cost, we can state the firm problem as the following:

$$\begin{aligned}
\max \mathbb{E}_0 q_{0,t} \Bigg[ & \frac{p_t(j)^{1-\theta}}{P_t^{1-\theta}} S_t - W_t n_t(j) - R_t u_t(j) K_t(j) \\
& + mc_t(j)\left\{ z_t(u_t(j)K_t(j))^{\alpha_K} n_t(j)^{\alpha_N} l^{1-\alpha_K-\alpha_N} X_t^{1-\alpha_K} - \left(\frac{p_t(j)}{P_t}\right)^{-\theta} S_t \right\} \Bigg].
\end{aligned}$$

Hence the first order conditions are:

$$\begin{aligned}
[p_t(j)] &: \frac{p_t(j)/P_t}{mc_t(j)} = \frac{\theta}{\theta-1},\\
[n_t(j)] &: \alpha_N mc_t(j)\frac{Y_t(j)}{n_t(j)} = W_t,\\
[u_t(j)K_t(j)] &: \alpha_K mc_t(j)\frac{Y_t(j)}{u_t(j)K_t(j)} = R_t,\\
[mc_t(j)] &: Y_t(j) = S_t(j),
\end{aligned}$$

and a technology constraint: $Y_t(j) = z_t(u_t(j)K_t(j))^{\alpha_K} n_t(j)^{\alpha_N} l^{1-\alpha_K-\alpha_N} X_t^{1-\alpha_K}$.

In a symmetric equilibrium the following conditions hold:

$$
\begin{aligned}
[p_t] &: \frac{1}{mc_t} = \frac{\theta}{\theta - 1}, \\
[n_t] &: \alpha_N mc_t \frac{Y_t}{n_t} = W_t, \\
[u_t k_t] &: \alpha_K mc_t \frac{Y_t}{u_t K_t} = R_t, \\
[mc_t] &: Y_t = S_t, \\
[tech] &: Y_t = z_t (u_t K_t)^{\alpha_K} n_t^{\alpha_N} l^{1-\alpha_K-\alpha_N} X_t^{1-\alpha_K}.
\end{aligned}
$$

Writing in terms of stationary variables, we have:

$$
\begin{aligned}
[p_t] &: \frac{1}{mc_t} = \frac{\theta}{\theta - 1}, \\
[n_t] &: \alpha_N mc_t \frac{y_t}{n_t} = w_t, \\
[u_t k_t] &: \alpha_K mc_t \frac{y_t}{u_t k_t} = \frac{r_t}{\mu_t^I}, \\
[mc_t] &: y_t = s_t, \\
[tech] &: y_t = z_t (u_t k_t)^{\alpha_K} n_t^{\alpha_N} l^{1-\alpha_K-\alpha_N} (\mu_t^I)^{-\alpha_K}.
\end{aligned}
$$

In a log-linear setup, we can rewrite these conditions as

$$[p_t] : \widehat{mc}_t = 0, \tag{A.46}$$

$$[n_t] : \widehat{mc}_t + \hat{y}_t - \hat{n}_t = \hat{w}_t, \tag{A.47}$$

$$[u_t k_t] : \widehat{mc}_t + \hat{y}_t - \hat{u}_t - \hat{k}_t = \hat{r}_t - \hat{\mu}_t^I, \tag{A.48}$$

$$[mc_t] : \hat{y}_t = \hat{s}_t, \tag{A.49}$$

$$[tech] : \hat{y}_t = \hat{z}_t + \alpha_K \hat{u}_t + \alpha_K \hat{k}_t + \alpha_N \hat{n}_t - \alpha_K \hat{\mu}_t^I. \tag{A.50}$$

### A.1.4 Computing the steady state in the no-inventory model

First of all, by targeting the markup $\mu$, we get $\theta = \mu/(\mu - 1)$. Also, $mc = 1/\mu$. The other targets we want to force are labor supply $n$, steady-state output growth rate $\mu^Y$, and steady-state investment growth rate $\mu^I$.

Now from the capital investment condition, we get that $\lambda = \lambda_k$. Hence the capital stock condition tells us that $r = (\mu^Y)^\sigma(\mu^A\beta)^{-1} - 1 + \delta_k$. With $u = 1$, the utilization condition forces the depreciation acceleration due to utilization to be $\delta'_k = r$. Using the capital rental condition at the firm side, we get the steady-state capital:

$$k = \mu^I \left[\alpha_K \frac{mc}{r} n^{\alpha_N} l^{1-\alpha_K-\alpha_N}\right]^{\frac{1}{1-\alpha_K}}.$$

Therefore, output is $y = k^{\alpha_K} n^{\alpha_N} l^{1-\alpha_K-\alpha_N}(\mu^I)^{-\alpha_K}$ and investment is $i = (1-(1-\delta_k)/\mu^I)k$. Real wage is $w = \alpha_N mcy/n$ and consumption is therefore $c = y - i - x^G g$.

With these pillars, we also get the household utility aspects. The stock of habit is $h = c(\mu^Y - b)(\mu^Y)^{-1/\gamma_h}$. We have the following steady-state conditions:

$$\begin{aligned}
m^{-\sigma} &= \lambda_m, \\
\lambda_h(1-\beta(\mu^Y)^{1-\sigma}(1-\gamma_h)) &= \lambda_m \psi \frac{n^{1+\xi^{-1}}}{1+\xi^{-1}}, \\
\frac{\lambda}{\lambda_m} &= \left(1 - \frac{\beta b}{(\mu^Y)^\sigma}\right)\left[1 - \gamma_h(\mu^Y)^{1-\frac{1}{\gamma_h}}\frac{\lambda_h}{\lambda_m}\right], \\
\frac{\lambda}{\lambda_m} &= \frac{\psi n^{\xi^{-1}} h}{w}, \\
m &= \left(1 - \frac{b}{\mu^Y}\right)c - \psi\frac{n^{1+\xi^{-1}}}{1+\xi^{-1}}h.
\end{aligned}$$

The first thing to pin down is $\psi$. Using the second to fourth conditions above, we can obtain

$\psi$:

$$\psi = (1 - \beta b(\mu^Y)^{-\sigma}) / \left[ \frac{n^{\xi^{-1}} h}{w} + \frac{(1 - \beta b(\mu^Y)^{-\sigma}) \gamma_h (\mu^Y)^{1-\frac{1}{\gamma_h}} n^{1+\xi^{-1}}}{(1 + \xi^{-1})(1 - \beta (\mu^Y)^{1-\sigma}(1 - \gamma_h))} \right].$$

Once you pin down $\psi$, you can also obtain $m$ as above. Then, from the first condition, you also get $\lambda_m$. Therefore $\lambda_h$ and $\lambda$ are also obtained and we are done.

### A.1.5 Writing down all the equilibrium conditions for the no-inventory model

The 21 endogenous variables are

$$m_t, \lambda_{m,t}, \lambda_{h,t}, n_t, c_t, h_t, \lambda_t, w_t, \lambda_{k,t}, i_t, r_t, u_t, r_t^f, k_{t+1}, s_t^p, s_t, mc_t, y_t, x_t^G, \mu_t^Y, \mu_t^I,$$

and the 7 exogenous processes are $\zeta_{h,t}, \psi_t, z_t, z_t^k, g_t, \mu_t^X, \mu_t^A$. The 21 endogenous equations are:

$$[m_t] : \hat{\zeta}_{h,t} - \sigma \hat{m}_t = \hat{\lambda}_{m,t}, \tag{A.51}$$

$$\begin{aligned} [h_t] : \hat{\lambda}_{h,t} - \left[1 - \beta(1 - \gamma_h)(\mu^Y)^{1-\sigma}\right] \left[\hat{\lambda}_{m,t} + \hat{\psi}_t + (1 + \xi^{-1})\hat{n}_t\right] \\ = \beta(1 - \gamma_h)(\mu^Y)^{1-\sigma} \left[\mathbb{E}_t \hat{\lambda}_{h,t+1} + (1 - \sigma)\mathbb{E}_t \hat{\mu}^Y_{t+1} + \mathbb{E}_t \hat{h}_{t+1} - \hat{h}_t\right], \end{aligned} \tag{A.52}$$

$$\begin{aligned} [c_t] : \lambda \hat{\lambda}_t = \lambda_m \hat{\lambda}_{m,t} - \lambda_h \gamma_h (\mu^Y)^{1-\frac{1}{\gamma_h}} \left[\hat{\lambda}_{h,t} + \hat{h}_t - \frac{\mu^Y}{\mu^Y - b}\hat{c}_t + \frac{b}{\mu^Y - b}\hat{c}_{t-1} - \frac{b}{\mu^Y - b}\hat{\mu}^Y_t\right] \\ + \sigma \beta b (\mu^Y)^{-\sigma} \left[\lambda_m - \lambda_h \gamma_h (\mu^Y)^{1-\frac{1}{\gamma_h}}\right] \mathbb{E}_t \hat{\mu}^Y_{t+1} - \beta b (\mu^Y)^{-\sigma} \lambda_m \mathbb{E}_t \hat{\lambda}_{m,t+1} \\ + \beta b (\mu^Y)^{-\sigma} \lambda_h \gamma_h (\mu^Y)^{1-\frac{1}{\gamma_h}} \left[\mathbb{E}_t \hat{\lambda}_{h,t+1} + \mathbb{E}_t \hat{h}_{t+1} - \frac{\mu^Y}{\mu^Y - b}\mathbb{E}_t \hat{c}_{t+1} + \frac{b}{\mu^Y - b}\hat{c}_t - \frac{b}{\mu^Y - b}\mathbb{E}_t \hat{\mu}^Y_{t+1}\right], \end{aligned} \tag{A.53}$$

$$[n_t] : \hat{\lambda}_t + \hat{w}_t = \hat{\lambda}_{m,t} + \hat{\psi}_t + \frac{1}{\xi}\hat{n}_t + \hat{h}_t, \tag{A.54}$$

$$[i_t] : \hat{\lambda}_{k,t} = \hat{\lambda}_t - \hat{z}_t^k + \mu^I \phi''(\mu^I)(\hat{i}_t - \hat{i}_{t-1} + \hat{\mu}_t^I) - \beta\frac{\mu^A}{(\mu^Y)^\sigma}(\mu^I)^3\phi''(\mu^I)(\mathbb{E}_t\hat{i}_{t+1} - \hat{i}_t + \mathbb{E}_t\hat{\mu}_{t+1}^I), \tag{A.55}$$

$$[k_{t+1}] : \hat{\lambda}_{k,t} = \mathbb{E}_t\hat{\mu}_{t+1}^A - \sigma\mathbb{E}_t\hat{\mu}_{t+1}^Y + \beta(\mu^Y)^{-\sigma}\mu^A(1-\delta_k)\mathbb{E}_t\hat{\lambda}_{k,t+1}$$

$$+ [1 - \beta(\mu^Y)^{-\sigma}\mu^A(1-\delta_k)](\mathbb{E}_t\hat{\lambda}_{t+1} + \mathbb{E}_t\hat{r}_{t+1} + \mathbb{E}_t\hat{u}_{t+1}) - \beta(\mu^Y)^{-\sigma}\mu^A\delta_k'\mathbb{E}_t\hat{u}_{t+1}, \tag{A.56}$$

$$[u_t] : \hat{\lambda}_t + \hat{r}_t = \hat{\lambda}_{k,t} + \frac{\delta_k''}{\delta_k'}\hat{u}_t, \qquad [\hat{u}_t = 0 \text{ if not allowed to vary}], \tag{A.57}$$

$$[b_{t+1}] : -\hat{r}_t^f = \mathbb{E}_t\hat{\lambda}_{t+1} - \hat{\lambda}_t - \sigma\mathbb{E}_t\hat{\mu}_{t+1}^Y, \qquad [\text{written in terms of the real interest rate}], \tag{A.58}$$

$$[\lambda_{m,t}] : m\hat{m}_t = c\hat{c}_t - b\frac{c}{\mu^Y}\hat{c}_{t-1} + b\frac{c}{\mu^Y}\hat{\mu}_t^Y - \psi\frac{n^{1+\xi^{-1}}}{1+\xi^{-1}}h\left[\hat{\psi}_t + \hat{h}_t + (1+\xi^{-1})\hat{n}_t\right], \tag{A.59}$$

$$[\lambda_{h,t}] : \hat{h}_t = \frac{\gamma_h\mu^Y}{\mu^Y - b}\hat{c}_t - b\frac{\gamma_h}{\mu^Y - b}\hat{c}_{t-1} + b\frac{\gamma_h}{\mu^Y - b}\hat{\mu}_t^Y + (1-\gamma_h)\hat{h}_{t-1} - (1-\gamma_h)\hat{\mu}_t^Y, \tag{A.60}$$

$$[\lambda_{k,t}] : \hat{k}_{t+1} = \left(1 - \frac{1-\delta_k}{\mu^I}\right)\hat{z}_t^k + \left(1 - \frac{1-\delta_k}{\mu^I}\right)\hat{i}_t + \frac{1-\delta_k}{\mu^I}\hat{k}_t - \frac{1-\delta_k}{\mu^I}\hat{\mu}_t^I - \frac{\delta_k'}{\mu^I}\hat{u}_t, \tag{A.61}$$

$$[s_t^p] : \hat{s}_t^p = \frac{c}{c+i}\hat{c}_t + \frac{i}{c+i}\hat{i}_t, \tag{A.62}$$

$$[s_t] : \hat{s}_t = \frac{c}{s}\hat{c}_t + \frac{i}{s}\hat{i}_t + \frac{gx^G}{s}\hat{g}_t + \frac{gx^G}{s}\hat{x}_t^G, \tag{A.63}$$

$$[\mu_t^Y] : \hat{\mu}_t^Y = \hat{\mu}_t^X + \frac{\alpha_K}{\alpha_K - 1}\hat{\mu}_t^A, \tag{A.64}$$

$$[\mu_t^I] : \hat{\mu}_t^I = \hat{\mu}_t^Y - \hat{\mu}_t^A, \tag{A.65}$$

$$[x_t^G] : \hat{x}_t^G = \rho_{xg}\hat{x}_{t-1}^G - \hat{\mu}_t^Y, \tag{A.66}$$

$$[p_t] : \widehat{mc}_t = 0, \tag{A.67}$$

$$[n_t] : \widehat{mc}_t + \hat{y}_t - \hat{n}_t = \hat{w}_t, \tag{A.68}$$

$$[u_t k_t] : \widehat{mc}_t + \hat{y}_t - \hat{u}_t - \hat{k}_t = \hat{r}_t - \hat{\mu}_t^I, \tag{A.69}$$

$$[mc_t] : \hat{y}_t = \hat{s}_t, \tag{A.70}$$

$$[tech] : \hat{y}_t = \hat{z}_t + \alpha_K\hat{u}_t + \alpha_K\hat{k}_t + \alpha_N\hat{n}_t - \alpha_K\hat{\mu}_t^I. \tag{A.71}$$

### A.1.6 Firm problem with stock-elastic inventories

Again, the firm side is subject to monopolistic competition. Firm $j \in [0,1]$ solves the following problem:

$$\max \mathbb{E}_0 q_{0,t} \left[ \frac{p_t(j)}{P_t} S_t(j) - W_t n_t(j) - R_t u_t(j) K_t(j) \right],$$

subject to

$$\begin{aligned}
S_t(j) &= \left(\frac{A_t(j)}{A_t}\right)^{\zeta_t} \left(\frac{p_t(j)}{P_t}\right)^{-\theta_t} S_t, \\
Y_t(j) &= z_t (u_t(j) K_t(j))^{\alpha_K} n_t(j)^{\alpha_N} l^{1-\alpha_K-\alpha_N} X_t^{1-\alpha_K}, \\
A_t(j) &= (1-\delta_i)(A_{t-1}(j) - S_{t-1}(j)) + Y_t(j) \\
&\quad - \phi_y \left(\frac{Y_t(j)}{Y_{t-1}(j)}\right) Y_t(j) - \phi_{inv} \left(\frac{INV_t(j)}{INV_{t-1}(j)}\right) INV_t(j) - \phi_a \left(\frac{A_t(j)}{A_{t-1}(j)}\right) A_t(j), \\
INV_t(j) &= A_t(j) - S_t(j).
\end{aligned}$$

The firm problem now has an active dynamic margin by storing more goods and selling in the future, at the same time by being able to create more demand by producing more goods.[1] We can state the firm problem as the following:

$$\max \mathbb{E}_0 q_{0,t} \left[ \frac{p_t(j)}{P_t} S_t(j) - W_t n_t(j) - R_t u_t(j) K_t(j) + \tau_t(j) \{ z_t (u_t(j) K_t(j))^{\alpha_K} n_t(j)^{\alpha_N} l^{1-\alpha_N-\alpha_K} X_t^{1-\alpha_K} - Y_t(j) \} \right.$$

[1]For quantitative issues on matching the smoothness of the aggregate stock of inventories, we also allow for adjustment costs for inventories. As we noted in the main paper, the smoothness of the stock of inventories relative to sales remains a challenge on inventory models. We leave this as future research and approximate that aspect by allowing for adjustment costs. However, we believe that the moment we focus on (which is the comovement property between inventories and components of sales) is not sensitive to the smoothness of the inventory series that we observe in the data.

$$+ mc_t(j)\left\{Y_t(j) + (1-\delta_i)(A_{t-1}(j) - S_{t-1}(j)) - A_t(j) - \phi_y\left(\frac{Y_t(j)}{Y_{t-1}(j)}\right)Y_t(j)\right.$$

$$\left. -\phi_{inv}\left(\frac{INV_t(j)}{INV_{t-1}(j)}\right)INV_t(j) - \phi_a\left(\frac{A_t(j)}{A_{t-1}(j)}\right)A_t(j)\right\} + \varsigma_t(j)\left\{\left(\frac{A_t(j)}{A_t}\right)^{\zeta_t}\left(\frac{p_t(j)}{P_t}\right)^{-\theta_t}S_t - S_t(j)\right\}\Bigg],$$

The first order conditions turn out to be the following:

$$\begin{aligned}
[p_t(j)] &: S_t(j) = \theta_t \varsigma_t(j)\left(\frac{A_t(j)}{A_t}\right)^{\zeta_t}\left(\frac{p_t(j)}{P_t}\right)^{-\theta_t-1}S_t,\\
[S_t(j)] &: \frac{p_t(j)}{P_t} + mc_t(j)\left(\phi_{inv}\left(\frac{INV_t(j)}{INV_{t-1}(j)}\right) + \frac{INV_t(j)}{INV_{t-1}(j)}\phi'_{inv}\left(\frac{INV_t(j)}{INV_{t-1}(j)}\right)\right)\\
&\quad = \varsigma_t(j) + \mathbb{E}_t q_{t,t+1} mc_{t+1}(j)(1-\delta_i) + \mathbb{E}_t q_{t,t+1} mc_{t+1}(j)\left(\frac{INV_{t+1}(j)}{INV_t(j)}\right)^2\phi'_{inv}\left(\frac{INV_{t+1}(j)}{INV_t(j)}\right),\\
[Y_t(j)] &: \tau_t(j) = mc_t(j)\left(1 - \phi_y\left(\frac{Y_t(j)}{Y_{t-1}(j)}\right) - \phi'_y\left(\frac{Y_t(j)}{Y_{t-1}(j)}\right)\right)\\
&\quad + \mathbb{E}_t q_{t,t+1} mc_{t+1}(j)\left(\frac{Y_{t+1}(j)}{Y_t(j)}\right)^2\phi'_y\left(\frac{Y_{t+1}(j)}{Y_t(j)}\right),\\
[n_t(j)] &: \alpha_N \tau_t(j)\frac{Y_t(j)}{N_t(j)} = W_t,\\
[u_t(j)K_t(j)] &: \alpha_k \tau_t(j)\frac{Y_t(j)}{u_t(j)K_t(j)} = R_t,
\end{aligned}$$

$$\begin{aligned}
[A_t(j)] &: mc_t(j)\left(1 + \phi_{inv}\left(\frac{INV_t(j)}{INV_{t-1}(j)}\right) + \frac{INV_t(j)}{INV_{t-1}(j)}\phi'_{inv}\left(\frac{INV_t(j)}{INV_{t-1}(j)}\right)\right.\\
&\quad \left. +\phi_a\left(\frac{A_t(j)}{A_{t-1}(j)}\right) + \frac{A_t(j)}{A_{t-1}(j)}\phi'_a\left(\frac{A_t(j)}{A_{t-1}(j)}\right)\right) = \varsigma_t(j)\zeta_t\left(\frac{A_t(j)}{A_t}\right)^{\zeta_t}\left(\frac{p_t(j)}{P_t}\right)^{-\theta_t}\frac{S_t}{A_t(j)}\\
&\quad + \mathbb{E}_t q_{t,t+1} mc_{t+1}(j)\left[(1-\delta_i) + \left(\frac{INV_{t+1}(j)}{INV_t(j)}\right)^2\phi'_{inv}\left(\frac{INV_{t+1}(j)}{INV_t(j)}\right) + \left(\frac{A_{t+1}(j)}{A_t(j)}\right)^2\phi'_a\left(\frac{A_{t+1}(j)}{A_t(j)}\right)\right],\\
[INV_t(j)] &: INV_t(j) = A_t(j) - S_t(j).
\end{aligned}$$

In a symmetric equilibrium, the following conditions hold:

$$\begin{aligned}
[\tau_t] &: Y_t = z_t(u_t K_t)^{\alpha_K} n_t^{\alpha_N} l^{1-\alpha_K-\alpha_N} X_t^{1-\alpha_K},\\
[mc_t] &: A_t = (1-\delta_i)(A_{t-1} - S_{t-1}) + Y_t - Y_t\phi_y\left(\frac{Y_t}{Y_{t-1}}\right) - INV_t\phi_{inv}\left(\frac{INV_t}{INV_{t-1}}\right) - A_t\phi_a\left(\frac{A_t}{A_{t-1}}\right),
\end{aligned}$$

$$
\begin{aligned}
[p_t] &: 1 = \theta_t \varsigma_t, \\
[S_t] &: 1 + mc_t \left( \phi_{inv} \left( \frac{INV_t}{INV_{t-1}} \right) + \frac{INV_t}{INV_{t-1}} \phi'_{inv} \left( \frac{INV_t}{INV_{t-1}} \right) \right) \\
&= \varsigma_t + \mathbb{E}_t q_{t,t+1} mc_{t+1} \left[ 1 - \delta_i + \left( \frac{INV_{t+1}}{INV_t} \right)^2 \phi'_{inv} \left( \frac{INV_{t+1}}{INV_t} \right) \right], \\
[Y_t] &: \tau_t = mc_t \left( 1 - \phi_y \left( \frac{Y_t}{Y_{t-1}} \right) - \phi'_y \left( \frac{Y_t}{Y_{t-1}} \right) \right) + \mathbb{E}_t q_{t,t+1} mc_{t+1} \left( \frac{Y_{t+1}}{Y_t} \right)^2 \phi'_y \left( \frac{Y_{t+1}}{Y_t} \right), \\
[n_t] &: \alpha_N \tau_t \frac{Y_t}{n_t} = W_t, \\
[u_t K_t] &: \alpha_K \tau_t \frac{Y_t}{u_t K_t} = R_t, \\
[A_t] &: mc_t \left( 1 + \phi_{inv} \left( \frac{INV_t}{INV_{t-1}} \right) + \frac{INV_t}{INV_{t-1}} \phi'_{inv} \left( \frac{INV_t}{INV_{t-1}} \right) + \phi_a \left( \frac{A_t}{A_{t-1}} \right) + \frac{A_t}{A_{t-1}} \phi'_a \left( \frac{A_t}{A_{t-1}} \right) \right) \\
&= \varsigma_t \zeta_t \frac{S_t}{A_t} + \mathbb{E}_t q_{t,t+1} mc_{t+1} \left[ (1 - \delta_i) + \left( \frac{INV_{t+1}}{INV_t} \right)^2 \phi'_{inv} \left( \frac{INV_{t+1}}{INV_t} \right) + \left( \frac{A_{t+1}}{A_t} \right)^2 \phi'_a \left( \frac{A_{t+1}}{A_t} \right) \right], \\
[INV_t] &: INV_t = A_t - S_t.
\end{aligned}
$$

Note that $\varsigma_t = 1/\theta_t$. Hence simplifying the above notation we get the following 8 conditions:

$$
[\tau_t] : Y_t = z_t (u_t K_t)^{\alpha_K} n_t^{\alpha_N} l^{1-\alpha_K-\alpha_N},
$$

$$
\begin{aligned}
[mc_t] &: A_t = (1-\delta_i)(A_{t-1} - S_{t-1}) + Y_t - Y_t \phi_y \left( \frac{Y_t}{Y_{t-1}} \right) - INV_t \phi_{inv} \left( \frac{INV_t}{INV_{t-1}} \right) - A_t \phi_a \left( \frac{A_t}{A_{t-1}} \right), \\
[S_t] &: \frac{\theta_t - 1}{\theta_t} + mc_t \left( \phi_{inv} \left( \frac{INV_t}{INV_{t-1}} \right) + \frac{INV_t}{INV_{t-1}} \phi'_{inv} \left( \frac{INV_t}{INV_{t-1}} \right) \right) \\
&= \mathbb{E}_t q_{t,t+1} mc_{t+1} \left[ (1 - \delta_i) + \left( \frac{INV_{t+1}}{INV_t} \right)^2 \phi'_{inv} \left( \frac{INV_{t+1}}{INV_t} \right) \right], \\
[Y_t] &: \tau_t = mc_t \left( 1 - \phi_y \left( \frac{Y_t}{Y_{t-1}} \right) - \phi'_y \left( \frac{Y_t}{Y_{t-1}} \right) \right) + \mathbb{E}_t q_{t,t+1} mc_{t+1} \left( \frac{Y_{t+1}}{Y_t} \right)^2 \phi'_y \left( \frac{Y_{t+1}}{Y_t} \right), \\
[n_t] &: \alpha_N \tau_t \frac{Y_t}{n_t} = W_t, \\
[u_t K_t] &: \alpha_K \tau_t \frac{Y_t}{u_t K_t} = R_t,
\end{aligned}
$$

$$[A_t] : mc_t \left(1 + \phi_{inv}\left(\frac{INV_t}{INV_{t-1}}\right) + \frac{INV_t}{INV_{t-1}}\phi'_{inv}\left(\frac{INV_t}{INV_{t-1}}\right) + \phi_a\left(\frac{A_t}{A_{t-1}}\right) + \frac{A_t}{A_{t-1}}\phi'_a\left(\frac{A_t}{A_{t-1}}\right)\right)$$
$$= \frac{\zeta_t}{\theta_t}\frac{S_t}{A_t} + \mathbb{E}_t q_{t,t+1} mc_{t+1}\left[(1-\delta_i) + \left(\frac{INV_{t+1}}{INV_t}\right)^2 \phi'_{inv}\left(\frac{INV_{t+1}}{INV_t}\right) + \left(\frac{A_{t+1}}{A_t}\right)^2 \phi'_a\left(\frac{A_{t+1}}{A_t}\right)\right],$$

$$[INV_t] : INV_t = A_t - S_t.$$

Expressing these into stationary variables (with $A_t = a_t X_t^Y$ and $INV_t = inv_t X_t^Y$):

$$[\tau_t] : y_t = z_t(u_t k_t)^{\alpha_K} n_t^{\alpha_N} l^{1-\alpha_K-\alpha_N}(\mu_t^I)^{-\alpha_K},$$

$$[mc_t] : a_t\mu_t^Y = (1-\delta_i)(a_{t-1} - s_{t-1}) + y_t\mu_t^Y - y_t\mu_t^Y\phi_y\left(\frac{y_t}{y_{t-1}}\mu_t^Y\right)$$
$$- inv_t\mu_t^Y\phi_{inv}\left(\frac{inv_t}{inv_{t-1}}\mu_t^Y\right) - a_t\mu_t^Y\phi_a\left(\frac{a_t}{a_{t-1}}\mu_t^Y\right),$$

$$[s_t] : \frac{\theta_t - 1}{\theta_t} + mc_t\left(\phi_{inv}\left(\frac{inv_t}{inv_{t-1}}\mu_t^Y\right) + \frac{inv_t}{inv_{t-1}}\mu_t^Y\phi'_{inv}\left(\frac{inv_t}{inv_{t-1}}\mu_t^Y\right)\right)$$
$$= \mathbb{E}_t q_{t,t+1} mc_{t+1}\left[(1-\delta_i) + \left(\frac{inv_{t+1}}{inv_t}\mu_{t+1}^Y\right)^2 \phi'_{inv}\left(\frac{inv_{t+1}}{inv_t}\mu_{t+1}^Y\right)\right],$$

$$[y_t] : \tau_t = mc_t\left(1 - \phi_y\left(\frac{y_t}{y_{t-1}}\mu_t^Y\right) - \phi'_y\left(\frac{y_t}{y_{t-1}}\mu_t^Y\right)\right) + \mathbb{E}_t q_{t,t+1} mc_{t+1}\left(\frac{y_{t+1}}{y_t}\mu_{t+1}^Y\right)^2 \phi'_y\left(\frac{y_{t+1}}{y_t}\mu_{t+1}^Y\right),$$

$$[n_t] : \alpha_N\tau_t\frac{y_t}{n_t} = w_t,$$

$$[u_t k_t] : \alpha_K\tau_t\frac{y_t}{u_t k_t} = \frac{r_t}{\mu_t^I},$$

$$[A_t] : mc_t\left[1 + \phi_{inv}\left(\frac{inv_t}{inv_{t-1}}\mu_t^Y\right) + \left(\frac{inv_t}{inv_{t-1}}\mu_t^Y\right)\phi'_{inv}\left(\frac{inv_t}{inv_{t-1}}\mu_t^Y\right) + \phi_a\left(\frac{a_t}{a_{t-1}}\mu_t^Y\right)\right.$$
$$\left. + \left(\frac{a_t}{a_{t-1}}\mu_t^Y\right)\phi'_a\left(\frac{a_t}{a_{t-1}}\mu_t^Y\right)\right] = \frac{\zeta_t}{\theta_t}\frac{s_t}{a_t}$$
$$+ \mathbb{E}_t q_{t,t+1} mc_{t+1}\left[(1-\delta_i) + \left(\frac{inv_{t+1}}{inv_t}\mu_{t+1}^Y\right)^2 \phi'_{inv}\left(\frac{inv_{t+1}}{inv_t}\mu_{t+1}^Y\right) + \left(\frac{a_{t+1}}{a_t}\mu_{t+1}^Y\right)^2 \phi'_a\left(\frac{a_{t+1}}{a_t}\mu_{t+1}^Y\right)\right],$$

$$[INV_t] : inv_t = a_t - s_t.$$

Writing $\mu_t = \theta_t/(\theta_t - 1)$, the 8 log-linearized conditions are the following:

$$[\tau_t] : \hat{y}_t = \hat{z}_t + \alpha_K \hat{u}_t + \alpha_K \hat{k}_t + \alpha_N \hat{n}_t - \alpha_K \hat{\mu}^I_t, \tag{A.72}$$

$$[mc_t] : a\mu^Y \hat{a}_t + a\mu^Y \hat{\mu}^Y_t = (1-\delta_i)a\hat{a}_{t-1} - (1-\delta_i)s\hat{s}_{t-1} + y\mu^Y \hat{y}_t + y\mu^Y \hat{\mu}^Y_t, \tag{A.73}$$

$$\begin{aligned}[s_t] : {} & (\mu^Y)^2 \phi''_{inv}(\widehat{inv}_t - \widehat{inv}_{t-1} + \hat{\mu}^Y_t) \\ & = \beta(\mu^Y)^{-\sigma}(1-\delta_i)[\hat{\mu}_t - \hat{r}^f_t + \mathbb{E}_t\widehat{mc}_{t+1}] + \beta(\mu^Y)^{3-\sigma}\phi''_{inv}[\mathbb{E}_t\widehat{inv}_{t+1} - \widehat{inv}_t + \mathbb{E}_t\hat{\mu}^Y_{t+1}],\end{aligned} \tag{A.74}$$

$$\begin{aligned}[y_t] : {} & \hat{\tau}_t = \widehat{mc}_t + \beta(\mu^Y)^{3-\sigma}\phi''_y\mathbb{E}_t\hat{y}_{t+1} - (\mu^Y + \beta(\mu^Y)^{3-\sigma})\phi''_y\hat{y}_t + \mu^Y\phi''_y\hat{y}_{t-1} \\ & + \beta(\mu^Y)^{3-\sigma}\phi''_y\mathbb{E}_t\hat{\mu}^Y_{t+1} - \mu^Y\phi''_y\hat{\mu}^Y_t,\end{aligned} \tag{A.75}$$

$$[n_t] : \hat{\tau}_t + \hat{y}_t - \hat{n}_t = \hat{w}_t, \tag{A.76}$$

$$[u_t k_t] : \hat{\tau}_t + \hat{y}_t - \hat{u}_t - \hat{k}_t = \hat{r}_t - \hat{\mu}^I_t, \tag{A.77}$$

$$\begin{aligned}[a_t] : {} & \widehat{mc}_t + (\mu^Y)^2\phi''_{inv}[\widehat{inv}_t - \widehat{inv}_{t-1} + \hat{\mu}^Y_t] + (\mu^Y)^2\phi''_a[\hat{a}_t - \hat{a}_{t-1} + \hat{\mu}^Y_t] \\ & = (1 - \beta(\mu^Y)^{-\sigma}(1-\delta_i))\left(\hat{\zeta}_t + \hat{s}_t - \hat{a}_t + \frac{1}{\mu - 1}\hat{\mu}_t\right) + \beta(\mu^Y)^{-\sigma}(1-\delta_i)(-\hat{r}^f_t + \mathbb{E}_t\widehat{mc}_{t+1}) \\ & + \beta(\mu^Y)^{3-\sigma}\phi''_{inv}[\mathbb{E}_t\widehat{inv}_{t+1} - \widehat{inv}_t + \mathbb{E}_t\hat{\mu}^Y_{t+1}] + \beta(\mu^Y)^{3-\sigma}\phi''_a[\mathbb{E}_t\hat{a}_{t+1} - \hat{a}_t + \mathbb{E}_t\hat{\mu}^Y_{t+1}],\end{aligned} \tag{A.78}$$

$$[inv_t] : inv\widehat{inv}_t = a\hat{a}_t - s\hat{s}_t. \tag{A.79}$$

### A.1.7 Computing the steady state in the stock-elastic inventory model

Again, we target directly the markup $\mu$ and in the inventory model, note that $mc = [\mu\beta(\mu^Y)^{-\sigma}(1-\delta_i)]^{-1}$. The values for n, $\mu^Y$, $\mu^I$, r, u, $\delta'_k$, k, y, i, and w are all obtained in the same manner as in the no-inventory model.

The new parameters and steady-state values we compute are $\zeta$, $\delta_i$, a, inv, $\tau$. First, $\delta_i$ is calibrated directly and $\tau = mc$. To obtain $\zeta$, we target the steady-state stock-sales ratio $a/s$

in the data. Using the two inventory conditions, we get

$$\zeta = \frac{1}{\mu - 1}\left(\frac{1-\beta(\mu^Y)^{-\sigma}(1-\delta_i)}{\beta(\mu^Y)^{-\sigma}(1-\delta_i)}\right)\frac{a}{s}.$$

From this, we also get

$$\begin{aligned} s &= \frac{\mu^Y y}{\mu^Y - 1 + \delta_i} / \left(\frac{a}{s} + \frac{1-\delta_i}{\mu^Y - 1 + \delta_i}\right), \\ a &= \frac{a}{s}s. \end{aligned}$$

Therefore, $c = s - i - x^G g$. The same procedure follows in getting the values for h, $\psi$, m, $\lambda_m$, $\lambda_h$, $\lambda$, $\lambda_k$.

### A.1.8 Writing down all the equilibrium conditions for the stock-elastic inventory model

The 24 endogenous variables are

$$m_t, \lambda_{m,t}, \lambda_{h,t}, n_t, c_t, h_t, \lambda_t, w_t, \lambda_{k,t}, i_t, r_t, u_t, r_t^f, k_{t+1}, s_t^p, s_t, mc_t, y_t, \tau_t, a_t, inv_t, x_t^G, \mu_t^Y, \mu_t^I.$$

The 3 endogenous variables $\tau_t, a_t, inv_t$ are newly added in the inventory model. The 9 exogenous processes are $\zeta_{h,t}, \psi_t, z_t, z_t^k, g_t, \mu_t^X, \mu_t^A, \zeta_t, \mu_t$. The 24 endogenous equations are:

$$[m_t] : \hat{\zeta}_{h,t} - \sigma\hat{m}_t = \hat{\lambda}_{m,t}, \tag{A.80}$$

$$\begin{aligned} [h_t] : \hat{\lambda}_{h,t} - [1-\beta(\mu^Y)^{1-\sigma}(1-\gamma_h)]\left[\hat{\lambda}_{m,t} + \hat{\psi}_t + (1+\xi^{-1})\hat{n}_t\right] \\ = \beta(\mu^Y)^{1-\sigma}(1-\gamma_h)[\mathbb{E}_t\lambda_{h,t+1} + (1-\sigma)\mathbb{E}_t\hat{\mu}_{t+1}^Y + \mathbb{E}_t\hat{h}_{t+1} - \hat{h}_t], \end{aligned} \tag{A.81}$$

$$[c_t] : \lambda\hat{\lambda}_t = \lambda_m\hat{\lambda}_{m,t} - \lambda_h\gamma_h(\mu^Y)^{1-\frac{1}{\gamma_h}}\left[\hat{\lambda}_{h,t} + \hat{h}_t - \frac{\mu^Y}{\mu^Y - b}\hat{c}_t + \frac{b}{\mu^Y - b}\hat{c}_{t-1} - \frac{b}{\mu^Y - b}\hat{\mu}_t^Y\right]$$

$$+\sigma\beta b(\mu^Y)^{-\sigma}\left[\lambda_m-\lambda_h\gamma_h(\mu^Y)^{1-\frac{1}{\gamma_h}}\right]\mathbb{E}_t\hat{\mu}^Y_{t+1}-\beta b(\mu^Y)^{-\sigma}\lambda_m\mathbb{E}_t\hat{\lambda}_{m,t+1}$$
$$+\beta b(\mu^Y)^{-\sigma}\lambda_h\gamma_h(\mu^Y)^{1-\frac{1}{\gamma_h}}\left[\mathbb{E}_t\hat{\lambda}_{h,t+1}+\mathbb{E}_t\hat{h}_{t+1}-\frac{\mu^Y}{\mu^Y-b}\mathbb{E}_t\hat{c}_{t+1}+\frac{b}{\mu^Y-b}\hat{c}_t-\frac{b}{\mu^Y-b}\mathbb{E}_t\hat{\mu}^Y_{t+1}\right], \tag{A.82}$$

$$[n_t]:\hat{\lambda}_t+\hat{w}_t=\hat{\lambda}_{m,t}+\hat{\psi}_t+\frac{1}{\xi}\hat{n}_t+\hat{h}_t, \tag{A.83}$$

$$[i_t]:\hat{\lambda}_{k,t}=\hat{\lambda}_t-\hat{z}^k_t+\mu^I\phi''(\mu^I)(\hat{i}_t-\hat{i}_{t-1}+\hat{\mu}^I_t)-\beta\frac{\mu^A}{(\mu^Y)^\sigma}(\mu^I)^3\phi''(\mu^I)(\mathbb{E}_t\hat{i}_{t+1}-\hat{i}_t+\mathbb{E}_t\hat{\mu}^I_{t+1}), \tag{A.84}$$

$$[k_{t+1}]:\hat{\lambda}_{k,t}=\mathbb{E}_t\hat{\mu}^A_{t+1}-\sigma\mathbb{E}_t\hat{\mu}^Y_{t+1}+\beta(\mu^Y)^{-\sigma}\mu^A(1-\delta_k)\mathbb{E}_t\hat{\lambda}_{k,t+1}$$
$$+[1-\beta(\mu^Y)^{-\sigma}\mu^A(1-\delta_k)](\mathbb{E}_t\hat{\lambda}_{t+1}+\mathbb{E}_t\hat{r}_{t+1}+\mathbb{E}_t\hat{u}_{t+1})-\beta(\mu^Y)^{-\sigma}\mu^A\delta'_k\mathbb{E}_t\hat{u}_{t+1}, \tag{A.85}$$

$$[u_t]:\hat{\lambda}_t+\hat{r}_t=\hat{\lambda}_{k,t}+\frac{\delta''_k}{\delta'_k}\hat{u}_t,\qquad[\hat{u}_t=0\text{ if not allowed to vary}], \tag{A.86}$$

$$[b_{t+1}]:-\hat{r}^f_t=\mathbb{E}_t\hat{\lambda}_{t+1}-\hat{\lambda}_t-\sigma\mathbb{E}_t\hat{\mu}^Y_{t+1},\qquad[\text{written in terms of the real interest rate}], \tag{A.87}$$

$$[\lambda_{m,t}]:m\hat{m}_t=c\hat{c}_t-b\frac{c}{\mu^Y}\hat{c}_{t-1}+b\frac{c}{\mu^Y}\hat{\mu}^Y_t-\psi\frac{n^{1+\xi^{-1}}}{1+\xi^{-1}}h\left[\hat{\psi}_t+\hat{h}_t+(1+\xi^{-1})\hat{n}_t\right], \tag{A.88}$$

$$[\lambda_{h,t}]:\hat{h}_t=\frac{\gamma_h\mu^Y}{\mu^Y-b}\hat{c}_t-b\frac{\gamma_h}{\mu^Y-b}\hat{c}_{t-1}+b\frac{\gamma_h}{\mu^Y-b}\hat{\mu}^Y_t+(1-\gamma_h)\hat{h}_{t-1}-(1-\gamma_h)\hat{\mu}^Y_t, \tag{A.89}$$

$$[\lambda_{k,t}]:\hat{k}_{t+1}=\left(1-\frac{1-\delta_k}{\mu^I}\right)\hat{z}^k_t+\left(1-\frac{1-\delta_k}{\mu^I}\right)\hat{i}_t+\frac{1-\delta_k}{\mu^I}\hat{k}_t-\frac{1-\delta_k}{\mu^I}\hat{\mu}^I_t-\frac{\delta'_k}{\mu^I}\hat{u}_t, \tag{A.90}$$

$$[s^p_t]:\hat{s}^p_t=\frac{c}{c+i}\hat{c}_t+\frac{i}{c+i}\hat{i}_t, \tag{A.91}$$

$$[s_t]:\hat{s}_t=\frac{c}{s}\hat{c}_t+\frac{i}{s}\hat{i}_t+\frac{gx^G}{s}\hat{g}_t+\frac{gx^G}{s}\hat{x}^G_t, \tag{A.92}$$

$$[\mu^Y_t]:\hat{\mu}^Y_t=\hat{\mu}^X_t+\frac{\alpha_K}{\alpha_K-1}\hat{\mu}^A_t, \tag{A.93}$$

$$[\mu^I_t]:\hat{\mu}^I_t=\hat{\mu}^Y_t-\hat{\mu}^A_t, \tag{A.94}$$

$$[x^G_t]:\hat{x}^G_t=\rho_{xg}\hat{x}^G_{t-1}-\hat{\mu}^Y_t, \tag{A.95}$$

$$[\tau_t]:\hat{y}_t=\hat{z}_t+\alpha_K\hat{u}_t+\alpha_K\hat{k}_t+\alpha_N\hat{n}_t-\alpha_K\hat{\mu}^I_t, \tag{A.96}$$

$$[mc_t]:a\mu^Y\hat{a}_t+a\mu^Y\hat{\mu}^Y_t=(1-\delta_i)a\hat{a}_{t-1}-(1-\delta_i)s\hat{s}_{t-1}+y\mu^Y\hat{y}_t+y\mu^Y\hat{\mu}^Y_t, \tag{A.97}$$

$$[s_t] : (\mu^Y)^2\phi''_{inv}(\widehat{inv}_t - \widehat{inv}_{t-1} + \hat{\mu}^Y_t)$$
$$= \beta(\mu^Y)^{-\sigma}(1-\delta_i)[\hat{\mu}_t - \hat{r}^f_t + \mathbb{E}_t\widehat{mc}_{t+1}] + \beta(\mu^Y)^{3-\sigma}\phi''_{inv}[\mathbb{E}_t\widehat{inv}_{t+1} - \widehat{inv}_t + \mathbb{E}_t\hat{\mu}^Y_{t+1}], \quad \text{(A.98)}$$
$$[y_t] : \hat{\tau}_t = \widehat{mc}_t + \beta(\mu^Y)^{3-\sigma}\phi''_y\mathbb{E}_t\hat{y}_{t+1} - (\mu^Y + \beta(\mu^Y)^{3-\sigma})\phi''_y\hat{y}_t + \mu^Y\phi''_y\hat{y}_{t-1}$$
$$+ \beta(\mu^Y)^{3-\sigma}\phi''_y\mathbb{E}_t\hat{\mu}^Y_{t+1} - \mu^Y\phi''_y\hat{\mu}^Y_t, \quad \text{(A.99)}$$
$$[n_t] : \hat{\tau}_t + \hat{y}_t - \hat{n}_t = \hat{w}_t, \quad \text{(A.100)}$$
$$[u_tk_t] : \hat{\tau}_t + \hat{y}_t - \hat{u}_t - \hat{k}_t = \hat{r}_t - \hat{\mu}^I_t, \quad \text{(A.101)}$$
$$[a_t] : \widehat{mc}_t + (\mu^Y)^2\phi''_{inv}[\widehat{inv}_t - \widehat{inv}_{t-1} + \hat{\mu}^Y_t] + (\mu^Y)^2\phi''_a[\hat{a}_t - \hat{a}_{t-1} + \hat{\mu}^Y_t]$$
$$= (1 - \beta(\mu^Y)^{-\sigma}(1-\delta_i))\left(\hat{\zeta}_t + \hat{s}_t - \hat{a}_t + \frac{1}{\mu - 1}\hat{\mu}_t\right) + \beta(\mu^Y)^{-\sigma}(1-\delta_i)(-\hat{r}^f_t + \mathbb{E}_t\widehat{mc}_{t+1})$$
$$+ \beta(\mu^Y)^{3-\sigma}\phi''_{inv}[\mathbb{E}_t\widehat{inv}_{t+1} - \widehat{inv}_t + \mathbb{E}_t\hat{\mu}^Y_{t+1}] + \beta(\mu^Y)^{3-\sigma}\phi''_a[\mathbb{E}_t\hat{a}_{t+1} - \hat{a}_t + \mathbb{E}_t\hat{\mu}^Y_{t+1}], \quad \text{(A.102)}$$
$$[inv_t] : inv\widehat{inv}_t = a\hat{a}_t - s\hat{s}_t. \quad \text{(A.103)}$$

## A.2 Model estimation

The estimation strategy will be Bayesian, and mostly follow section 5 of Schmitt-Grohé and Uribe (2012). Readers should refer to that section for a detailed discussion. In table A.1, we present the calibration in estimating the above illustrated stock-elastic inventory model as described in Schmitt-Grohé and Uribe (2012), with our own calibrations for inventories as discussed in the main paper.

The period of data we use are 1955Q2-2006Q4. For the measurement equation, we use the same 7 observables (output growth, consumption growth, investment growth, hours growth, government consumption growth, productivity growth, investment price growth) as in Schmitt-Grohé and Uribe (2012), where measurement errors are only allowed on output

| Parameter | Value | Description |
|---|---|---|
| $\beta$ | 0.99 | Subjective discount factor |
| $\sigma$ | 1 | Household intertemporal elasticity of substitution |
| $\alpha_K$ | 0.225 | Capital share |
| $\alpha_N$ | 0.675 | Labor share |
| $\delta_k$ | 0.025 | Capital depreciation rate |
| $u$ | 1 | Capacity utilization rate |
| $\mu^Y$ | 1.0045 | Gross per capita GDP growth rate |
| $\mu^A$ | 0.9957 | Gross investment price growth rate |
| $G/Y$ | 0.2 | Government consumption to GDP |
| $n$ | 0.2 | Hours |
| $\mu$ | 1.15 | Price markup |
| $\delta_i$ | 0.025 | Inventory depreciation rate |
| $IS$ | 0.75 | Inventory-sales ratio |

*Table A.1: Calibrated parameters.*

growth. On top of that, we also use the per capita real growth rate of inventories as an additional observable, with measurement errors also allowed on this series. The source of measurement error on inventories is due to different valuations in GDP computation and inventory measurement. That is, real stock of inventories in NIPA are computed by taking the average price during the period, using various valuation methods (FIFO, market value). On the other hand, inventory investment used to produce GDP is computed by the end-of-period price of inventories.[2] We allow for persistence in the measurement error for inventories.

It is important to notice that adding data on inventories as an observable is not crucial to our estimation purpose. Inventory investment is implicitly included in the existing observables used for estimation (output, consumption, investment and government spending) by the resource constraint (output net of consumption, investment, and government spending is inventory investment in a closed economy). However, in the actual output data, net exports are also included and may potentially mask the dynamics of inventories. By directly

[2]We thank Michael Cortez at the Bureau of Economic Analysis for clarifying this.

| | Bayesian Estimation | | | | | | |
|---|---|---|---|---|---|---|---|
| | Prior | | | | Posterior | | |
| Parameter | Distribution | Median | 5% | 95% | Median | 5% | 95% |
| $1/\xi$ | Gamma | 3.92 | 2.51 | 5.77 | 1.70 | 1.13 | 2.25 |
| $\gamma_h$ | Uniform | 0.50 | 0.05 | 0.95 | 0.00 | 0.00 | 0.01 |
| $\phi''_I$ | Gamma | 3.92 | 2.51 | 5.77 | 9.23 | 7.34 | 10.35 |
| $\delta''_k/\delta'_k$ | Igamma | 0.75 | 0.32 | 0.96 | 0.31 | 0.24 | 0.38 |
| $b$ | Beta | 0.50 | 0.17 | 0.83 | 0.92 | 0.91 | 0.94 |
| $\rho_{xg}$ | Beta | 0.73 | 0.32 | 0.96 | 0.79 | 0.64 | 0.91 |
| $\phi''_y$ | Gamma | 3.67 | 1.37 | 7.75 | 0.52 | 0.20 | 0.87 |
| $\phi''_{inv}$ | Gamma | 3.67 | 1.37 | 7.75 | 2.68 | 2.02 | 3.28 |
| $\phi''_a$ | Gamma | 3.67 | 1.37 | 7.75 | 0.65 | 0.21 | 1.13 |
| $\rho_{\tilde{z}}$ | Beta | 0.73 | 0.32 | 0.96 | 0.95 | 0.92 | 0.97 |
| $\sigma^0_{\tilde{z}}$ | Gamma | 1.15 | 0.27 | 3.06 | 0.64 | 0.54 | 0.74 |
| $\sigma^4_{\tilde{z}}$ | Gamma | 0.66 | 0.16 | 1.76 | 0.12 | 0.00 | 0.26 |
| $\sigma^8_{\tilde{z}}$ | Gamma | 0.66 | 0.16 | 1.76 | 0.09 | 0.00 | 0.21 |
| $\rho_{\mu^A}$ | Beta | 0.50 | 0.17 | 0.83 | 0.46 | 0.37 | 0.55 |
| $\sigma^0_{\mu^A}$ | Gamma | 0.24 | 0.06 | 0.64 | 0.18 | 0.01 | 0.33 |
| $\sigma^4_{\mu^A}$ | Gamma | 0.14 | 0.03 | 0.37 | 0.15 | 0.04 | 0.24 |
| $\sigma^8_{\mu^A}$ | Gamma | 0.14 | 0.03 | 0.37 | 0.24 | 0.10 | 0.32 |
| $\rho_g$ | Beta | 0.73 | 0.32 | 0.96 | 0.96 | 0.94 | 0.99 |
| $\sigma^0_g$ | Gamma | 0.81 | 0.19 | 2.15 | 0.71 | 0.24 | 1.01 |
| $\sigma^4_g$ | Gamma | 0.46 | 0.11 | 1.02 | 0.73 | 0.48 | 0.92 |
| $\sigma^8_g$ | Gamma | 0.46 | 0.11 | 1.02 | 0.25 | 0.00 | 0.70 |
| $\rho_{\mu^X}{}^*$ | Beta | 0.23 | –0.18 | 0.46 | 0.30 | 0.17 | 0.47 |
| $\sigma^0_{\mu^X}$ | Gamma | 0.35 | 0.08 | 0.94 | 0.47 | 0.30 | 0.63 |
| $\sigma^4_{\mu^X}$ | Gamma | 0.20 | 0.05 | 0.53 | 0.08 | 0.00 | 0.18 |
| $\sigma^8_{\mu^X}$ | Gamma | 0.20 | 0.05 | 0.53 | 0.09 | 0.00 | 0.18 |
| $\rho_\psi$ | Beta | 0.73 | 0.32 | 0.96 | 0.99 | 0.98 | 1.00 |
| $\sigma^0_\psi$ | Gamma | 0.92 | 0.22 | 2.45 | 1.42 | 0.90 | 1.80 |
| $\sigma^4_\psi$ | Gamma | 0.53 | 0.13 | 1.42 | 1.57 | 1.24 | 1.91 |
| $\sigma^8_\psi$ | Gamma | 0.53 | 0.13 | 1.42 | 0.31 | 0.00 | 0.84 |
| $\rho_{\zeta_h}$ | Beta | 0.50 | 0.17 | 0.83 | 0.19 | 0.11 | 0.30 |
| $\sigma^0_{\zeta_h}$ | Gamma | 4.82 | 1.15 | 12.87 | 6.03 | 1.15 | 9.13 |
| $\sigma^4_{\zeta_h}$ | Gamma | 2.78 | 0.66 | 7.43 | 6.10 | 1.08 | 8.51 |
| $\sigma^8_{\zeta_h}$ | Gamma | 2.78 | 0.66 | 7.43 | 3.77 | 1.17 | 6.10 |
| $\rho_{\tilde{z}^k}$ | Beta | 0.50 | 0.17 | 0.83 | 0.85 | 0.77 | 0.93 |
| $\sigma^0_{\tilde{z}^k}$ | Gamma | 13.14 | 3.14 | 35.07 | 6.41 | 4.25 | 7.89 |
| $\sigma^4_{\tilde{z}^k}$ | Gamma | 7.59 | 1.81 | 20.26 | 0.68 | 0.00 | 1.69 |
| $\sigma^8_{\tilde{z}^k}$ | Gamma | 7.59 | 1.81 | 20.26 | 1.62 | 0.01 | 3.30 |
| $\rho_\mu$ | Beta | 0.50 | 0.17 | 0.83 | 0.77 | 0.71 | 0.83 |
| $\sigma^0_\mu$ | Gamma | 0.86 | 0.20 | 2.29 | 2.85 | 2.34 | 3.31 |
| $\sigma^4_\mu$ | Gamma | 0.50 | 0.12 | 1.33 | 0.63 | 0.26 | 0.93 |
| $\sigma^8_\mu$ | Gamma | 0.50 | 0.12 | 1.33 | 0.22 | 0.00 | 0.47 |
| $\sigma^{\mathrm{me}}_{gy}$ | Uniform | 0.15 | 0.02 | 0.29 | 0.27 | 0.25 | 0.30 |
| $\rho^{\mathrm{me}}_{ginv}$ | Beta | 0.50 | 0.17 | 0.83 | 0.21 | 0.09 | 0.33 |
| $\sigma^{\mathrm{me}}_{ginv}$ | Uniform | 0.15 | 0.02 | 0.29 | 0.30 | 0.29 | 0.30 |

*Table A.2: Parameter Estimation on US Data. Posterior is the result of estimation with using inventories as an additional observable. Hence 8 observable series (output, consumption, fixed investment, government spending, hours worked, TFP, investment price, inventories) are used. All numbers are rounded. A transformed parameter $\rho_{\mu^X} + 0.5$ is estimated for $\rho_{\mu^X}$.*

| Statistic | Y | C | I | N | G | TFP | A | INV |
|---|---|---|---|---|---|---|---|---|
| | | | | Standard Deviations | | | | |
| Data | 0.91 | 0.51 | 2.28 | 0.84 | 1.14 | 0.75 | 0.41 | 0.88 |
| Model | 0.89 | 0.63 | 3.56 | 0.82 | 1.08 | 0.77 | 0.38 | 1.39 |
| | | | | Correlations With Output Growth | | | | |
| Data | 1.00 | 0.50 | 0.69 | 0.72 | 0.25 | 0.40 | –0.12 | 0.44 |
| Model | 1.00 | 0.45 | 0.59 | 0.53 | 0.20 | 0.47 | 0.01 | 0.20 |
| | | | | Autocorrelations | | | | |
| Data | 0.28 | 0.20 | 0.53 | 0.60 | 0.05 | –0.01 | 0.49 | 0.55 |
| Model | 0.39 | 0.39 | 0.75 | 0.21 | 0.02 | 0.06 | 0.46 | 0.80 |

*Table A.3: Model estimation result is based on posterior median estimates. The columns are output (Y), consumption (C), fixed investment (I), hours (N), government spending (G), total factor productivity (TFP), relative price of investment (A), and inventories (INV) all in growth rates.*

including the stock of inventories as an observable, the inventory adjustment mechanism is likely to be more precisely estimated.

Table A.2 summarizes the priors and posteriors in the model. Notice that for the priors on the standard deviations, we set the contemporaneous shock to account for 75 percent of the total variance of the shocks. That is, priors are set such that news shocks account for 25 percent of the total variance.

Table A.3 summarizes the prediction of the model. For standard deviations, most values are close to the data, but for fixed investment and inventories, the standard deviations are about 50 percent higher. Second, the model also predicts that inventories are positively correlated with output growth, with a correlation of 0.21. Lastly, we observe that the model autocorrelation is quite similar to the data, with hours (N) showing the most trouble, which is also discussed in Schmitt-Grohé and Uribe (2012).

# A.3 News shocks in the stockout-avoidance inventory model

In this appendix, we describe a Real Business Cycle version of the stockout-avoidance models of Kahn (1987) and Kryvtsov and Midrigan (2013), and analyze its impact response to news shocks.

## A.3.1 Model description

The economy consists of a representative household and monopolistically competitive firms, where again firms produce storable goods. We start with the household problem. Since many aspects of the model are similar to the stock-elastic model, we will frequently refer to chapter 1.

**Household problem** A representative household maximizes (1.1), subject to the household budget constraint (1.2), capital accumulation rule (1.3), and the resource constraint (1.4). The aggregation of goods $\{s_t(j)\}_{j\in[0,1]}$ into $x_t$ is given by (1.5), where $v_t(j)$ is the taste-shifter for product $j$ in period $t$.

In stockout-avoidance models, in contrast to the stock-elastic demand models, this taste-shifter is assumed to be exogenous. In particular, we assume it is identically distributed across firms and over time according to a cumulative distribution function $F(\cdot)$ with a support $\Omega(\cdot)$:

$$v_t(j) \sim F, \qquad v_t(j) \in \Omega. \tag{A.104}$$

For each product $j$, households cannot buy more than the goods on-shelf $a_t(j)$, which is chosen by firms:

$$s_t(j) \leq a_t(j), \qquad \forall j \in [0,1]. \tag{A.105}$$

Although (A.105) also holds for the stock-elastic model, it has not been mentioned since it was never binding. Households observe these shocks, and the amount of goods on shelf $a_t(j)$, before making their purchase decisions. Firms, however, do not observe the shock $v_t(j)$ when deciding upon the amount $a_t(j)$ of goods that are placed on shelf, so that (A.105) occasionally binds, resulting in a stockout.

Again, a demand function and a price aggregator can be obtained from the expenditure minimization problem of the household. The demand function for product $j$ becomes

$$s_t(j) = \min\left\{ v_t(j)\left(\frac{p_t(j)}{P_t}\right)^{-\theta} x_t,\ a_t(j)\right\}, \tag{A.106}$$

which states that when $v_t(j)$ is high enough so that demand is higher than the amount of on-shelf goods, a stockout occurs and demand is truncated at $a_t(j)$. The price aggregator $P_t$ is given by:

$$P_t = \left(\int_0^1 v_t(j)\tilde{p}_t(j)^{1-\theta} dj\right)^{\frac{1}{1-\theta}}. \tag{A.107}$$

The variable $\tilde{p}_t(j)$ is the Lagrange multiplier on constraint (A.105). It reflects the household's shadow valuation of goods of variety $j$. For varieties that do not stock out, $\tilde{p}_t(j) = p_t(j)$, whereas for varieties that do stock out, $\tilde{p}_t(j) > p_t(j)$.

**Firm problem** Each monopolistically competitive firm $j \in [0,1]$ maximizes (1.7) with $\pi_t(j)$ defined as

$$\pi_t(j) = p_t(j)\tilde{s}_t(j) - W_t n_t(j) - R_t k_t(j). \tag{A.108}$$

As explained before, firms do not observe the exogenous taste-shifter $v_t(j)$ and hence their demand $s_t(j)$ when making their price and quantity decisions. Therefore, they will have to form expectations on sales $s_t(j)$, conditional on all variables except $\nu_t(j)$. This conditional expectation is denoted by $\tilde{s}_t(j)$.

The constraints on the firm are (1.9), (1.10), (1.11) and the demand function (A.106) with a known distribution for the taste-shifter $v_t(j)$ in (A.104). Notice that this distribution is identical across all firms and invariant to aggregate conditions. By the law of large numbers, firms observe $P_t$ and $x_t$ in their demand function. Therefore, $\tilde{s}_t(j)$ in (A.108) is given by:

$$\tilde{s}_t(j) = \int_{v \in \Omega(v)} \min \left\{ v \left( \frac{p_t(j)}{P_t} \right)^{-\theta} x_t,\ a_t(j) \right\} dF(v). \tag{A.109}$$

**Market clearing** The market clearing conditions for labor, capital, and bond markets are identical to the stock-elastic model and are given by (1.13), (1.14) and (1.15). Sales of goods also clear by the demand function for each variety.

### A.3.2 Equilibrium

A market equilibrium of the stockout-avoidance model is defined as follows.

**Definition A.1** (Market equilibrium of the stockout-avoidance model)**.** *A market equilib-*

*rium in the stockout-avoidance model is a set of stochastic processes:*

$$c_t, n_t, k_{t+1}, i_t, B_{t+1}, x_t, \{a_t(j)\}, \{v_t(j)\}, \{s_t(j)\}, \{\tilde{s}_t(j)\}, \{y_t(j)\}, \{inv_t(j)\}, \{p_t(j)\}, W_t, R_t, P_t, Q_{t,t+1}$$

*such that, given the exogenous stochastic process $z_t$ and initial conditions $k_0$, $B_0$, and $\{inv_{-1}(j)\}$:*

- *households maximize (1.1) subject to (1.2) - (1.4),* (A.104) - (A.105)*, and a no-Ponzi condition,*
- *each firm $j \in [0,1]$ maximizes (1.7) subject to (1.9) - (1.11),* (A.108) - (A.109),
- *markets clear according to (1.13) - (1.15).*

In what follows, we use the following notation for aggregate output, sales, and inventories:

$$y_t = \int_0^1 y_t(j)dj, \qquad s_t = \int_0^1 s_t(j)dj, \qquad inv_t = \int_0^1 inv_t(j)dj. \tag{A.110}$$

In stockout-avoidance models, a market equilibrium is not symmetric across firms. Indeed, because of the idiosyncratic taste shocks $\{\nu_t(j)\}$, realized sales $\{s_t(j)\}$, and therefore end of period inventories $\{inv_t(j)\}$ differ across firms. However, it can be shown that all firms make identical *ex-ante* choices. That is, firms' choice of price $p_t(j)$ and amount of on-shelf goods $a_t(j)$ depends only on aggregate variables, and not on the inventory inherited from the past period $inv_{t-1}(j)$. We therefore denote $p_t = p_t(j)$ and $a_t = a_t(j)$. The *ex-ante* symmetric choices of price and on-shelf goods imply that there is a unique threshold of the taste shock, common across firms, above which firms stock out. Using (A.106), this threshold is given by:

$$\nu_t^*(j) = \nu_t^* = \left(\frac{p_t}{P_t}\right)^\theta \frac{a_t}{x_t}.$$

### A.3.3 The stockout wedge and firm-level markups

The fact that those firms with a taste shifter $\nu_t(j) \geq \nu_t^*$ run out of goods to sell implies that $p_t \neq P_t$. Indeed, as emphasized in (A.107), the aggregate price level $P_t$ depends on the household's marginal value of good $j$, $\tilde{p}_t(j)$. This marginal value equals the (symmetric) sales price $p_t$ for all varieties that do not stockout. However, for varieties that do stock out, firms would like to purchase more of the good than what is on sale. Therefore, the household's marginal value of the good is higher than their market price: $\tilde{p}_t(j) > p_t$. Thus, the standard aggregation relation $P_t = p_t$ fails to hold, and instead, $P_t > p_t$. In what follows, we denote:

$$d_t = \frac{p_t}{P_t}.$$

The relative price can be thought of as a stockout wedge. It is smaller when the household's valuation of the aggregate bundle of goods is large relative to the market price of varieties, that is, when stockouts are more likely. Formally, it can be shown that the wedge $d_t$ is a strictly increasing function of $\nu_t^*$, and therefore a decreasing function of the probability of stocking out, $1 - F(\nu_t^*)$.

Due to the stockout wedge, firm-level markup $\mu_t^F$ differs from the definition of aggregate markup $\mu_t$ defined in section 1.2. Indeed, since $\mu_t^F = \frac{p_t}{P_t}\mu_t$, so that:

$$\mu_t^F = d_t \mu_t. \tag{A.111}$$

### A.3.4 An alternative log-linearized framework

There are two important differences between stock-out avoidance models and the stock-elastic demand model described in section 1.2. The first difference is the occurence of stockouts, which implies the existence of the stockout wedge and hence the departure of firm-level and aggregate markups as described above. The second difference is that, even in our flexible-price environment, firm-level markups are not set at a constant rate over future marginal cost, as they did in the stock-elastic demand model. These two differences mean that unlike stock-elastic demand models, we cannot exactly map this class of models into the log-linearized framework we described in section 1.3. We need a more general framework, which we provide in the following lemma.

**Lemma A.2** (The log-linearized framework for the stockout-avoidance model)**.** *In an equilibrium of the stockout-avoidance model, if productivity $z_t$ is at its steady-state value, on impact, up to a first order approximation around the steady-state, equations (1.20) and (1.21) hold, along with:*

$$\hat{inv}_t - \hat{s}_t = \tau\hat{\mu}_t^F + \eta\hat{\gamma}_t, \tag{A.112}$$

$$\hat{\mu}_t^F = \hat{d}_t + \hat{\mu}_t, \tag{A.113}$$

$$\hat{d}_t = \epsilon_d\left(\hat{inv}_t - \hat{s}_t\right), \tag{A.114}$$

$$\hat{\mu}_t^F = \epsilon_\mu\left(\hat{inv}_t - \hat{s}_t\right). \tag{A.115}$$

*In this approximation, the parameters $\omega$ and $\kappa$ are given by (1.25) and (1.26), while the parameters $\eta > 0$, $\tau > 0$, $\epsilon_d > 0$, and $\epsilon_\mu$ differ and are given in section A.3.8.*

We discuss (A.112)-(A.115), which are new to this framework. First, the optimal choice of inventories (A.112) depends on the firm-level markup $\hat{\mu}_t^F$ that is not equal to the aggregate markup $\hat{\mu}_t$. The parameters expressed as $\tau$ and $\gamma$ also have a different expression that will be discussed later.

Second, in equation (A.113), aggregate markups and firm-level markups are linked with the stockout wedge $\hat{d}_t$. This follows from the definition of firm-level markup and stockout wedge given in (A.111).

Third, note that the framework of lemma A.2 now includes (A.114), an equation linking the stockout wedge to the aggregate IS ratio. As we argued previously, the stockout wedge is negatively related to the probability of stocking out. In turn, one can show that there is a strictly decreasing mapping between the stockout probability, or equivalently a strictly increasing mapping between $\nu_t^*$, and the ratio of average end of period inventory to average sales:

$$IS_t = \frac{inv_t}{s_t} = \frac{\int_0^1 inv_t(j)dj}{\int_0^1 s_t(j)dj}.$$

A lower probability of stocking out (a higher $\nu_t^*$) implies that firms will, on average, be left with a higher stock of inventories relative to the amount of goods sold. Combining these two mappings, we obtain that the stockout wedge is increasing in the aggregate IS ratio, so that $\epsilon_d > 0$.

Lastly, the framework of lemma A.2 includes variable firm-level markups, as described in equation (A.115). This is because in stockout-avoidance models, the desired firm-level markup is not constant. Instead, it depends on the ratio of goods on-shelf to expected demand, which itself is linked to the probability of stocking out. One can show that for log-normal and pareto-distributed idiosyncratic demand shocks, $\mu_t^F$ is a strictly decreasing function of $\nu_t^*$, and therefore an increasing function of the probability of stocking out. Thus, the elasticity $\epsilon_\mu$ is typically negative. Intuitively, this is because when firms are likely to stock out, the price-elasticity of demand is lower, and therefore markups are higher. Indeed, with a high stockout probability, demand is mostly constrained by the amount of goods available for sale, and does not vary much with price changes. The converse intuition holds when the stockout probability is low.

Before moving on, note that this framework reduces to the framework of section 1.3 when the stockout wedge is absent and firm-level markups are constant, so that $\hat{d}_t = \hat{\mu}_t^F = \hat{\mu}_t = 0$. Hence the framework is a generalized version of the basic framework given in section 1.3, nesting it as a particular case with $\epsilon_d = \epsilon_\mu = 0$.

### A.3.5 The impact response to news shocks

We now turn to discussing the effects of a news shock using our new log-linearized framework. We again maintain the assumption that the shock has the effect of increasing sales, $\hat{s}_t > 0$, while leaving current productivity unchanged, $\hat{z}_t = 0$, so that we can indeed used the log-linearized framework of lemma A.2. Combining the equations of lemma A.2, it is straightforward to rewrite the optimality condition for inventory choice as:

$$\hat{inv}_t = -\tilde{\eta}\omega\widehat{mc}_t + \hat{s}_t.$$

In this expression, the elasticity of inventories to relative marginal cost, $\tilde{\eta}$ is given by:

$$\tilde{\eta} = \frac{1}{1 - \eta\epsilon_d + (\eta - \tau)\epsilon_\mu}\eta \tag{A.116}$$

In contrast to the stock-elastic demand model, $\tilde{\eta}$ does not purely reflect the intertemporal substitution of production anymore. The relative marginal cost elasticity $\eta$ is now compensated for markup movements (the terms $\tau$ and $\epsilon_\mu$)and for movements in the stockout wedge (the term $\epsilon_d$).

Unlike in the stock-elastic demand model, the sign of $\tilde{\eta}$ cannot in general be established.[3]

[3] In the variant of this model considered by Wen (2011), it can however be proved that the analogous reduced-form parameter $\tilde{\eta}$ is strictly negative regardless of the shock distribution. The proof is available from the authors upon request.

| Value of $\tilde{\eta}$ | | | | | |
|---|---|---|---|---|---|
| $\sigma_d \downarrow \|\|\mu \rightarrow$ | **1.05** | **1.1** | **1.25** | **1.5** | **1.75** |
| **0.1** | -729.12 | -278.08 | -121.98 | -77.39 | -61.87 |
| **0.25** | -307.22 | -116.94 | -51.42 | -32.71 | -26.20 |
| **0.5** | -167.04 | -63.17 | -27.66 | -17.57 | -14.06 |
| **0.75** | -120.68 | -45.25 | -19.59 | -12.35 | -9.85 |
| **1** | -97.75 | -36.33 | -15.51 | -9.66 | -7.66 |
| Implied IS ratio | | | | | |
| $\sigma_d \downarrow \|\|\mu \rightarrow$ | **1.05** | **1.1** | **1.25** | **1.5** | **1.75** |
| **0.1** | 0.05 | 0.09 | 0.15 | 0.18 | 0.21 |
| **0.25** | 0.12 | 0.23 | 0.39 | 0.50 | 0.57 |
| **0.5** | 0.23 | 0.47 | 0.83 | 1.13 | 1.32 |
| **0.75** | 0.32 | 0.69 | 1.31 | 1.88 | 2.26 |
| **1** | 0.41 | 0.90 | 1.81 | 2.73 | 3.36 |

*Table A.4: Value of $\tilde{\eta}$ when idiosyncratic demand shocks follow a log-normal distribution with mean 1. Different lines correspond to different standard deviations of the associated normal distribution, and different columns to different steady-state markups. Values are for $\beta = 0.99$ and $\delta_i = 0.011$.*

This is because its sign depends on the distribution of the idiosyncratic taste shock. However, for a very wide range of calibrations and for the Pareto and Log-normal distributions, $\tilde{\eta}$ is negative. We document this in Table A.4. There, we compute different values of $\tilde{\eta}$, for different pairs of values of $\sigma_d$, the standard deviation of the shock, and different values of the steady-state markup. In all cases, we constraint the shock to have a mean equal to 1. The standard deviations we consider range from 0.1 to 1, and the markups range from 1.05 to 1.75. In all cases, $\tilde{\eta}$ is negative. In table A.5, we perform the same exercise for Pareto-distributed shocks, and results are similar.

These results can be understood using (A.116). First, as discussed before, since $\epsilon_\mu < 0$ for standard distributions, markups fall when the IS ratio increases. With a higher IS ratio, a stockout is less likely for a firm, so that its price elasticity of demand is high, and its charges low markups. Second, because $(\eta - \tau)\epsilon_\mu > 0$, markup movements tend to attenuate

| Value of $\tilde{\eta}$ | | | | | |
|---|---|---|---|---|---|
| $\sigma_d \downarrow \|\|\mu \rightarrow$ | **1.05** | **1.1** | **1.25** | **1.5** | **1.75** |
| **0.1** | -1959.13 | -297.78 | -62.89 | -27.02 | -18.16 |
| **0.25** | -926.82 | -142.18 | -30.44 | -13.30 | -9.06 |
| **0.5** | -598.66 | -92.85 | -20.20 | -8.98 | -6.20 |
| **0.75** | -499.86 | -78.04 | -17.14 | -7.69 | -5.35 |
| **1** | -456.12 | -71.51 | -15.80 | -7.13 | -4.97 |
| Implied IS ratio | | | | | |
| $\sigma_d \downarrow \|\|\mu \rightarrow$ | **1.05** | **1.1** | **1.25** | **1.5** | **1.75** |
| **0.1** | 0.03 | 0.07 | 0.15 | 0.22 | 0.26 |
| **0.25** | 0.05 | 0.15 | 0.34 | 0.51 | 0.63 |
| **0.5** | 0.09 | 0.25 | 0.57 | 0.90 | 1.13 |
| **0.75** | 0.10 | 0.30 | 0.71 | 1.15 | 1.48 |
| **1** | 0.11 | 0.33 | 0.80 | 1.31 | 1.70 |

*Table A.5: Value of $\tilde{\eta}$ when shock follow a Pareto distribution with mean 1. Different lines correspond to different standard deviations for the Pareto distribution, and different columns to different steady-state markups. Values are for $\beta = 0.99$ and $\delta_i = 0.011$.*

the intertemporal substitution channel; that is, if we were to set $\epsilon_d = 0$, then $\tilde{\eta} < \eta$. Lower markups signal a higher future marginal cost to the firm, thereby leading it to increase inventories (for fixed current marginal cost). At the same time, higher markups lead the firm to increase its sales relative to available goods, leaving it with fewer inventories at the end of the period. On net, the first effect dominates, leading to higher inventories at the end of the period, and reducing thus the inventory-depleting effects of the shock. Finally, $\eta\epsilon_d - (\eta - \tau)\epsilon_\mu > 1$, so that $\tilde{\eta} < 0$. Therefore, movements in the stockout wedge change the sign of the elasticity of inventories to marginal cost.

With $\tilde{\eta} < 0$, the following results hold for the impact response of news shocks in the stockout avoidance model.

**Proposition A.3** (The impact response to news shocks in the stockout-avoidance model)**.** *In the stockout-avoidance model with $\tilde{\eta} < 0$, after a news shock:*

1. *inventory-sales ratio and inventories move in the same direction;*

2. *inventories increase, if and only if:*

$$-\tilde{\eta} < \frac{\kappa}{\omega}\frac{\delta_i}{\kappa - 1}.$$

The first part of this proposition is by itself daunting to news shocks, since it implies a counterfactual positive comovement between the IS ratio and inventories in response to a news shock. The second part states the condition under which inventories could be procyclical. This condition is similar to that of proposition 1.3, with $-\tilde{\eta}$ taking place instead of $\eta$ on the left hand side, and $\kappa/\omega$ multiplied by $\delta_i/(\kappa - 1)$ on the right hand side. Again, inventories are procyclical if the degree of real rigidities represented by the inverse of $\omega$ is high compared to the absolute value of the elasticity of inventories to relative marginal cost $-\tilde{\eta}$. We turn to a discussion of the numerical values of the parameters for this condition to hold.

### A.3.6 When do inventories respond positively to news shocks?

The second part of proposition A.3 provides a condition under which inventories are procyclical. Much as in the case of the stock-elastic demand model, this condition for procyclicality of inventories implies a lower bound for the degree real rigidities (alternatively, an upper bound for $\omega$). We now provide a numerical illustration of this bound, by setting $\beta = 0.99$ and considering the same range of steady-state $IS$ ratios, 0.25, 0.5 and 0.75, as in section 1.3. Given these values and a depreciation rate of inventories $\delta_i$, the value $\overline{\omega}$ was uniquely pinned down in section 1.3. However, in the stockout-avoidance model considered above, the three variables are not sufficient to determine $\overline{\omega}$. Hence we also target the steady-state

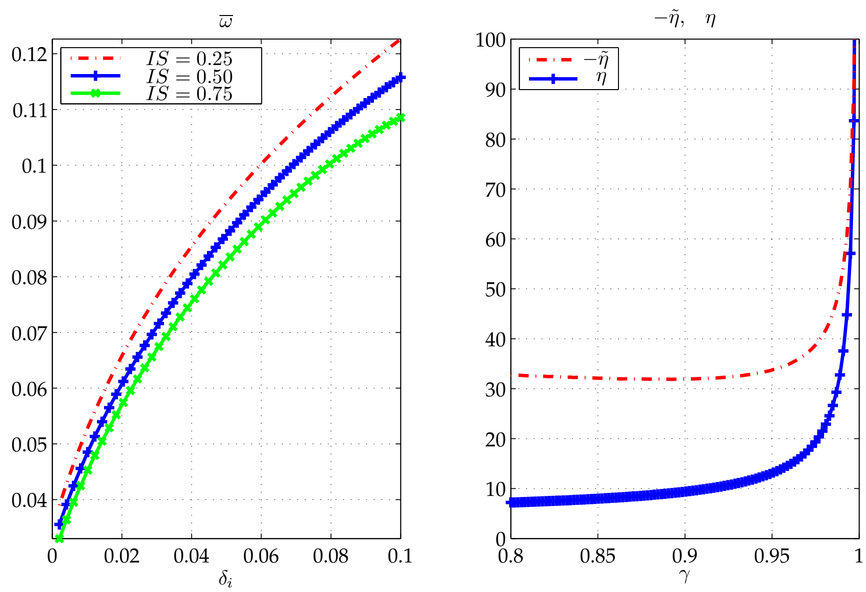



*Figure A.1: Implied parameter values for the stockout avoidance model. The left panel provides the upper bound on $\omega$ for procyclical inventories, derived from targeting the steady-state IS ratio and $\mu = 1.25$. The right panel provides the value of $-\tilde{\eta}$ and $\eta$ as a function of $\gamma(= \beta(1-\delta_i))$, holding fixed all the other structural parameters.*

gross markup $\mu$ at 1.25, which is within the range of estimates considered in the literature.[4]

In figure A.1, we plot the upper bound of $\omega$ for inventories to be procyclical, assuming a log-normal distribution for the taste-shifter. We observe that inventories are procyclical only with low levels of $\omega$. For a quarterly depreciation of 2 percent, the upper bound of $\omega$ is below 0.07, much lower than the existing measures. Hence with reasonable numerical values, the model still implies that inventories fall with regards to news shocks.

[4]It should be noted that with given values of the steady-state markup, the steady-state IS ratio, and the rate of depreciation of inventories, a unique steady-state stockout probability is implied. Indeed, in this model, a higher IS ratio implies a lower stockout probability, while at the same time, it is linked to a higher markup. The IS ratio and the markup thus cannot be targeted independently of the stockout probability.

### A.3.7 Is the response of inventories dominated by intertemporal substitution?

The inequality condition in proposition A.3 does not hold because $-\tilde{\eta}$ is large. An immediate question is whether this large value is due to the high intertemporal substitution, as was the case in section 1.3. Since the reduced-form parameter $\eta$ summarizes the intensity of the intertemporal substitution motive, we need to verify whether $\eta$ is large and positively related to $-\tilde{\eta}$.

First, the value $\eta$ in the stockout-avoidance model is determined by the following:

$$\eta = \underbrace{\frac{1}{1-\beta(1-\delta_i)}\frac{1+IS}{IS}}_{=\eta^{SE}}\left(1-\Gamma(1+IS)\right)\frac{1}{H(\Gamma)}.$$

Here, $\Gamma$ denotes the steady-state stockout probability. Note that this expression is similar to the relative marginal cost elasticity in the stock-elastic demand model, save for the two terms that depend on the stockout probability $\Gamma$. The function $H(\Gamma)$ is related to the hazard rate characterizing the cumulative distribution function of taste shocks $F$. For the type of distributions considered in the literature, $H(\Gamma)$ is typically larger than 1. Thus in general, $\eta \leq \eta^{SE}$, where $\eta^{SE}$ is the expression for $\eta$ in the stock-elastic demand model discussed in section 1.3. That is, the intertemporal substitution channel is weaker in these models than in the stock-elastic demand model. The fact that some firms stock out of their varieties prevents them altogether from smoothing production over time by storing goods or depleting inventories.

However, setting the targets at $IS = 0.5$ and $\mu = 1.25$, and assuming that the taste-shifter follows a log-normal distribution, $\eta$ is computed to be two thirds of the value in the stock-elastic demand model. Given that the lower bound for $\eta^{SE}$ was above 30, $\eta$ in the stockout-avoidance model is above 20, implying that a 1 percent increase in the present value of future marginal cost leads firms to adjust more than 20 percent of inventories relative to sales. Hence the intertemporal substitution motive remains large in the stockout-avoidance model.

Second, we need to verify whether a large $\eta$ implies a large $-\tilde{\eta}$. However, both parameters are in reduced form, and therefore the link between the two cannot be directly measured. Instead, we show whether the two values are positively correlated with $\gamma = \beta(1-\delta_i)$. Setting the benchmark targets at $IS = 0.5$ and $\mu = 1.25$, we fix the structural parameters, assuming that the taste-shifter follows a log-normal distribution. Given the structural parameters, we vary $\gamma$ and plot the implied value of $\eta$ and $-\tilde{\eta}$ on the right panel of figure A.1. Note that both values are increasing in $\gamma$ as $\gamma$ approaches 1. This suggests that the value of $-\tilde{\eta}$ is again dominated by the value of $\eta$ in (A.116), especially when $\gamma$ is close to 1.[5] In this sense, the strong intertemporal substitution channel again dominates the overall response of inventories to news shocks.

### A.3.8 Additional results for the stockout avoidance model

#### List of additional equilibrium conditions

The following equations are consitute an equilibrium of the stockout avoidance model:

$$1 - F(\nu_t^*) = \frac{\frac{1}{\gamma_t} - 1}{\mu_t^F - 1}, \tag{A.117}$$

$$\frac{\theta}{\theta - 1 - \frac{1 - F(\nu_t^*)}{\int_{\nu \le \nu_t^*} \frac{\nu}{\nu_t^*} dF(\nu)}} = \mu_t^F, \tag{A.118}$$

[5]The same result holds for a wide range of distributions for the taste-shifter.

$$\frac{\int_{\nu \leq \nu_t^*} \left(1 - \frac{\nu}{\nu_t^*}\right) dF(\nu)}{\int_{\nu \leq \nu_t^*} \frac{\nu}{\nu_t^*} dF(\nu) + 1 - F(\nu_t^*)} = \frac{inv_t}{s_t}, \tag{A.119}$$

$$\mu_t^F = d_t \mu_t \tag{A.120}$$

$$\left( \int_{\nu \leq \nu_t^*} \nu dF(\nu) + \nu_t^* \int_{\nu > \nu_t^*} \left(\frac{\nu}{\nu_t^*}\right)^{\frac{1}{\theta}} dF(\nu) \right)^{\frac{1}{\theta - 1}} = d_t, \tag{A.121}$$

$$\frac{\left( (\nu_t^*)^{\frac{1}{\theta}} \int_{\nu \leq \nu_t^*} \frac{\nu}{\nu_t^*} dF(\nu) + \int_{\nu > \nu_t^*} \nu^{\frac{1}{\theta}} dF(\nu) \right)^{\frac{\theta}{\theta - 1}}}{\int_{\nu \leq \nu_t^*} \frac{\nu}{\nu_t^*} dF(\nu) + 1 - F(\nu_t^*)} s_t = x_t. \tag{A.122}$$

Condition (A.117) determines the optimal choice of stock in the stockout avoidance model. Here, $\nu_t^*$ is related to the aggregate IS ratio through (A.119). Condition (A.118) is the optimal markup choice in the stockout avoidance model which also depends on the IS ratio through (A.119), reflecting the dependence of the price elasticity of demand on the stock of goods on sale in this (not iso-elastic) model. The firm markup $\mu_t^F$ and the aggregate markup $\mu_t$ are linked by the stockout wedge $d_t$ in equation (A.120). The stockout wedge itself is given by (A.121). Finally, condition (A.122) reflects market clearing when some varieties stock out.

**Equilibrium symmetry**

Because some firms stock out while others do not, the equilibrium of the stock-elastic demand model is not symmetric across firms. We define the aggregate variables $s_t$ and $inv_t$ as the aggregate sales and inventories, respectively:

$$inv_t \equiv \int_{j \in [0,1]} inv_t(j) dj \quad , \quad s_t = \equiv \int_{j \in [0,1]} s_t(j) dj.$$

However, the choices of price $p_t(j)$ and goods on shelf $a_t(j)$ are identical across firms. To see this, note first that for the same reason mentioned for the stock-elastic demand model, marginal cost is constant across firms. Second, the first-order conditions for optimal pricing and optimal choice of stock are given, respectively, by:

$$mc_t = \frac{\partial \tilde{s}_t(j)}{\partial a_t(j)} \frac{p_t(j)}{P_t} + \left(1 - \frac{\partial \tilde{s}_t(j)}{\partial a_t(j)}\right)(1-\delta_i)\mathbb{E}_t\left[q_{t,t+1} mc_{t+1}\right],$$

$$\frac{p_t(j)/P_t}{(1-\delta_i)\mathbb{E}_t\left[q_{t,t+1} mc_{t+1}\right]} = \frac{\theta}{\theta - 1 - \frac{\tilde{s}_t(j)}{p_t(j)} \frac{\partial \tilde{s}_t(j)}{\partial p_t(j)}}$$

where $mc_t$ denotes nominal marginal cost deflated by the CPI, $P_t$. Here, $\tilde{s}_t(j)$ denotes firm j's expected sales. Following equation (A.109), expected sales of firm $j$ depend only on price $p_t(j)$ and on-shelf goods $a_t(j)$, and aggregate variables. In turn, the above optimality conditions can be solved to obtain a decision rule for $a_t(j)$ and $p_t(j)$ as a function of current and expected values of aggregate values, so that the choices of individual firms for these variables are symmetric. This implies in turn that the stockout cutoff,

$$\nu_t^*(j) = \left(\frac{p_t(j)}{P_t}\right)^{\theta} \frac{a_t(j)}{x_t},$$

is also symmetric across firms.

**Expressions for the reduced-form coefficients of lemma A.2**

In what follows, we denote the steady-state stockout probability by:

$$\Gamma = 1 - F(\nu^*).$$

First, note that the log-linear approximation of equation (A.119) is:

$$\widehat{inv}_t - \hat{s}_t = (1 - \Gamma(1+IS))\frac{1+IS}{IS}\hat{\nu}_t^*$$

This implies that the IS ratio and the stockout threshold move in the same direction. Indeed, the restriction:

$$1 > \Gamma(1 + IS)$$

follows from the fact that in the steady state,

$$IS = \frac{\int_{\nu \leq \nu^*} \left(1 - \frac{\nu}{\nu^*}\right) dF(\nu)}{\int_{\nu \leq \nu^*} \frac{\nu}{\nu^*} dF(\nu) + \Gamma} \Leftrightarrow \frac{1}{1 + IS} - \Gamma = \int_{\nu \leq \nu^*} \frac{\nu}{\nu^*} dF(\nu) > 0.$$

Second, it can be shown that the log-linear approximations to equations (A.117), (A.118) and (A.121) are respectively given by:

$$\frac{\nu^* f(\nu^*)}{\Gamma} \hat{\nu}_t^* = \frac{\mu^F}{\mu^F - 1} \hat{\mu}_t^F + \frac{1}{1 - \gamma} \hat{\gamma}_t,$$

$$\hat{\mu}_t^F = (\mu^F - 1)\Gamma(1 + IS) \left(1 - \frac{\nu^* f(\nu^*)}{\Gamma} \frac{1}{1 - \Gamma(1 + IS)}\right) \nu_t^*,$$

$$\hat{d}_t = \frac{\mu^F - 1}{\mu^F} (1 - \Gamma(1 + IS)) \Delta \hat{\nu}_t^*.$$

Here, the coefficient $\Delta \in (0, 1]$ is defined as:

$$\Delta \equiv \frac{\int_{\nu > \nu^*} \left(\frac{\nu}{\nu^*}\right)^{\frac{1}{\theta}} dF(\nu)}{\int_{\nu \leq \nu^*} \frac{\nu}{\nu^*} dF(\nu) + \int_{\nu > \nu^*} \left(\frac{\nu}{\nu^*}\right)^{\frac{1}{\theta}} dF(\nu)},$$

where the relationship between the parameter $\theta$ and the steady-state markup is given by:

$$\theta = \frac{\mu^F}{\mu^F - 1} \frac{1}{1 - \Gamma(1 + IS)}.$$

Combining these equations, one arrives at the following expressions for the different reduced-form parameters defining the log-linear framework of lemma A.2:

$$\tau = \frac{\Gamma}{\nu^* f(\nu^*)}(1-\Gamma(1+IS))\frac{1+IS}{IS}\frac{\mu^F}{\mu^F-1} > 0, \tag{A.123}$$
$$\eta = \frac{\Gamma}{\nu^* f(\nu^*)}(1-\Gamma(1+IS))\frac{1+IS}{IS}\frac{1}{1-\gamma} > 0, \tag{A.124}$$
$$\epsilon_d = \frac{IS}{1+IS}\frac{1}{1-\Gamma(1+IS)}\frac{\mu^F-1}{\mu^F}(1-\Gamma(1+IS))\Delta > 0, \tag{A.125}$$
$$\epsilon_\mu = \frac{IS}{1+IS}\frac{1}{1-\Gamma(1+IS)}(\mu^F-1)\Gamma(1+IS)\left(1-\frac{\nu^* f(\nu^*)}{\Gamma}\frac{1}{1-\Gamma(1+IS)}\right). \tag{A.126}$$

## A.4 Detailed equilibrium conditions of the stock-elastic demand model

### A.4.1 List of equilibrium conditions

A market equilibrium of the stock-elastic demand model is characterized by the following set of equations:

$$\left(c_t - \psi_t \frac{n_t^{1+\xi^{-1}}}{1+\xi^{-1}}\right)^{-\sigma} = \lambda_t \tag{A.127}$$
$$w_t = \psi_t n_t^{\xi^{-1}} \tag{A.128}$$
$$\xi_t\left(1-\phi\left(\frac{i_t}{i_{t-1}}\right)-\left(\frac{i_t}{i_{t-1}}\right)\phi'\left(\frac{i_t}{i_{t-1}}\right)\right)+\beta\mathbb{E}_t\left[\xi_{t+1}\left(\frac{i_{t+1}}{i_t}\right)^2\phi'\left(\frac{i_{t+1}}{i_t}\right)\right] = \lambda_t \tag{A.129}$$
$$i_t\left(1-\phi\left(\frac{i_t}{i_{t-1}}\right)\right)+(1-\delta_k)k_t = k_{t+1} \tag{A.130}$$
$$\beta\mathbb{E}_t\left[(1-\delta_k)\xi_{t+1}+\lambda_{t+1}r_{t+1}\right] = \xi_t \tag{A.131}$$
$$c_t + i_t = x_t \tag{A.132}$$

$$z_t k_t^{1-\alpha} n_t^{\alpha} = y_t \tag{A.133}$$

$$mc_t \alpha \frac{y_t}{n_t} = w_t \tag{A.134}$$

$$mc_t (1-\alpha) \frac{y_t}{k_t} = r_t \tag{A.135}$$

$$(1-\delta_i) inv_{t-1} + y_t = s_t + inv_t \tag{A.136}$$

$$s_t + inv_t = a_t \tag{A.137}$$

$$\mathbb{E}_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} (1-\delta_i) \frac{mc_{t+1}}{mc_t} \right] = \gamma_t \tag{A.138}$$

$$\frac{1}{\mathbb{E}_t \left[ (1-\delta_i) q_{t,t+1} mc_{t+1} \right]} = \mu_t \tag{A.139}$$

$$\zeta \frac{1}{1 + \frac{inv_t}{s_t}} = \frac{\frac{1}{\gamma_t} - 1}{\mu_t - 1} \tag{A.140}$$

$$\frac{\theta}{\theta - 1} = \mu_t \tag{A.141}$$

$$s_t = x_t \tag{A.142}$$

Conditions (A.127)-(A.132) characterize the optimum of the household's problem, conditions (A.133)-(A.140) characterize that of the firm, and condition (A.142) reflects market clearing for goods. Condition (A.140) characterizes its optimal choice of inventory holdings, while conditions (A.139) and (A.141) characterize optimal pricing by monopolistic firms in this environment. Conditions (A.136) and (A.137) are the law of motion for inventories, and the definition of goods on shelf, respectively.

### A.4.2 Equilibrium symmetry

In the formulation of the first order conditions above, we dropped the subscript $j$, since a market equilibrium of the stock-elastic demand model is symmetric. One can see this as follows. First, combining the labor and capital demand schedules of firms, one sees that the capital-labor ratio is identical across firms. This in turn implies (using either the capital demand or labor demand schedule) that marginal cost is constant across firms. Using equation (A.140), the first order condition governing the optimal choice of inventories, the inventory to sales ratio, or equivalently the stock-to-sales ratio, is constant across firms. Furthermore, using the optimal pricing condition (A.141), price is constant across firms. Thus, substituting out price and the stock-to-sales ratio in the demand schedule for variety $j$, we have:

$$s_t(j) = \left(\frac{a_t(j)}{a_t}\right)^{\zeta} s_t = \left(\frac{\Xi_t s_t(j)}{a_t}\right)^{\zeta} s_t,$$

which in turn implies that sales are constant across firms, $s_t(j) = s_t$.

## A.5 Robustness of the empirial sign restriction VAR

First, we checked whether our result is sensitive to the long-run properties of the data. Towards that, we focus only on variations at the business-cycle frequency component by applying an HP filter to each series. In figure A.2, we observe that the impulse responses are quickly mean reverting. With this series, the same picture still remains. Moreover, our result on the forecast error variance is also similar to our benchmark since in the very short run, the shock accounts for 10 percent of output variation on average, and 30 percent of that in the long run. Comparing the result with no restrictions on inventories, we see that the short run (1 quarter) output variation becomes significantly more precise with a downward shift in the mean.

Second, in our benchmark estimation, we considered output as real GDP. To be consistent with our model definition of output $y = c + i + \delta inv$, we have also constructed an alternative output series which subtracts government spending and net exports from the GDP series.

That is, the alternative output measure is nominal GDP net of government spending and net exports, deflated by the GDP deflator, expressed in per capita terms. Figures A.3 and A.4 again confirm that our result is not sensitive to this extension. In figure A.5, we see that by imposing 2 period restrictions, the mean output variation explained by the identified shock shifts significantly downwards in all horizons.

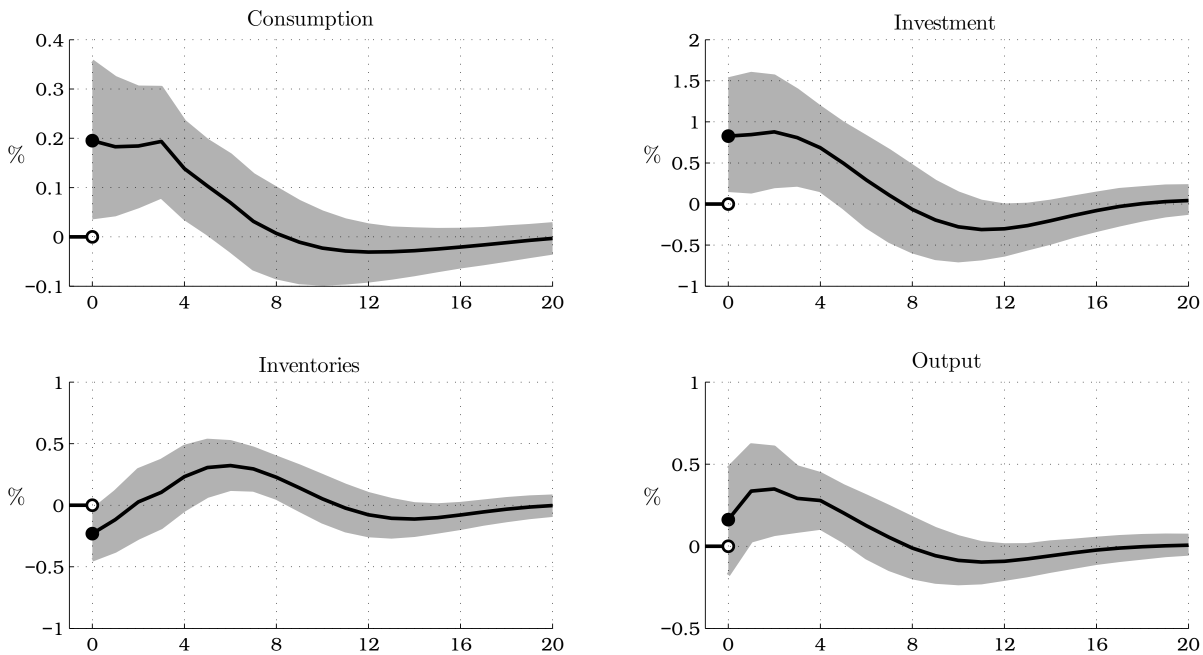



*Figure A.2: Robustness of impulse responses 1. Median and 80% credible set impulse responses of the identified shock with impact (1 period) restriction for the HP filtered series.*

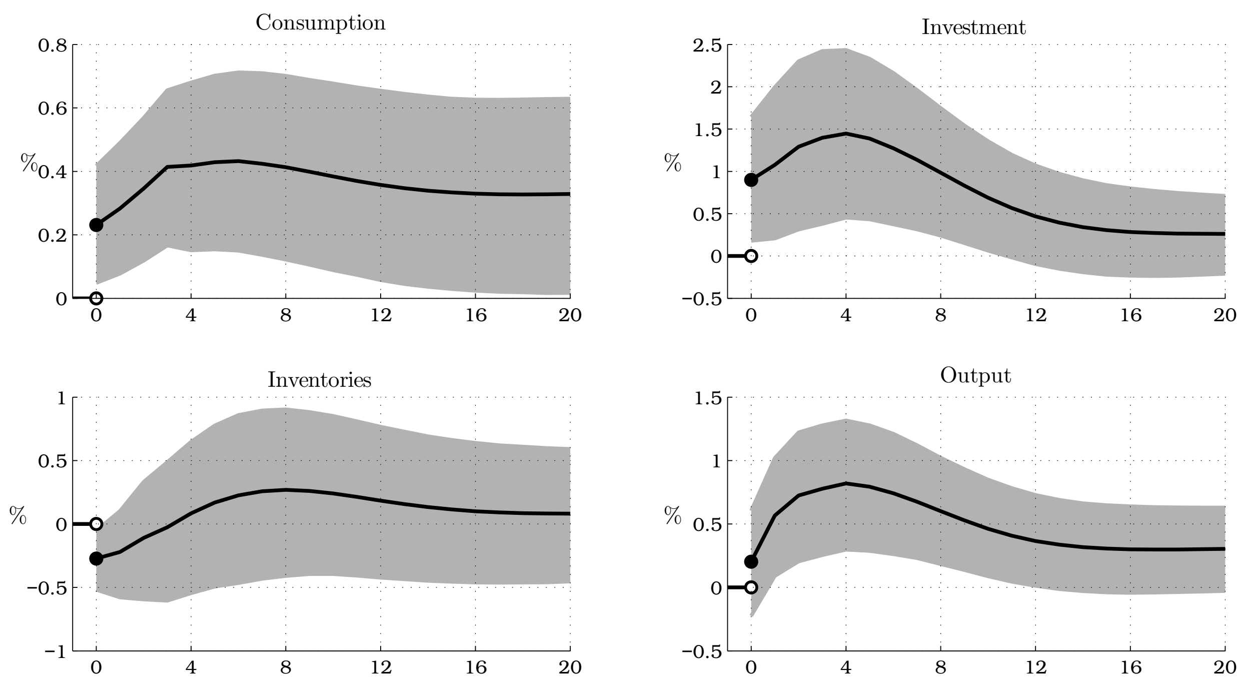



*Figure A.3: Robustness of impulse responses 2. Median and* 80% *credible set impulse responses of the identified shock with 1 period restriction for the alternative output series (without government spending and net exports), with 1 period restrictions applied on inventories, consumption and investment.*

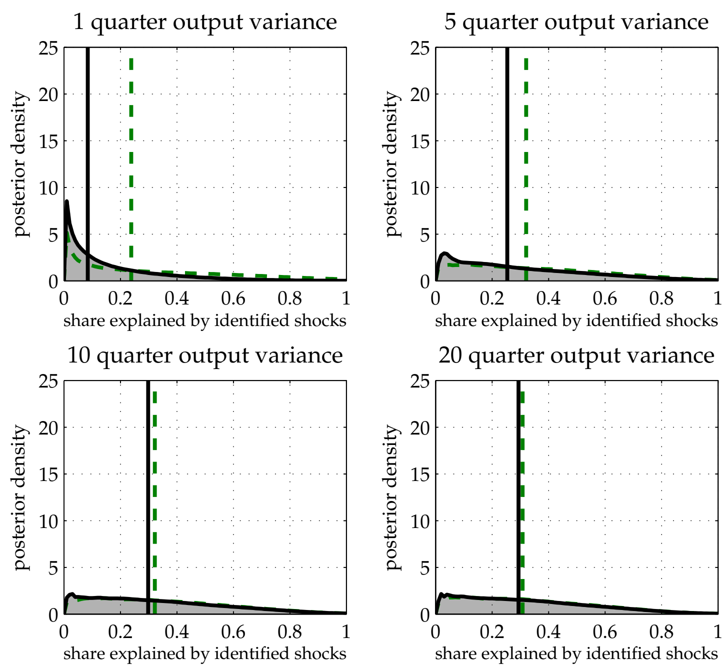



*Figure A.4: Robustness of forecast error variance 2. Posterior probability density and median (vertical line) for the share of forecast error variance of output at each horizon explained by identified shocks for the alternative output series (without government spending and net exports), with 1 period restriction. Solid line: 1 period negative comovement between $\Delta inv_t$ and $(c_t, i_t)$. Dashed line: 1 period positive comovement between $c_t$ and $i_t$.*

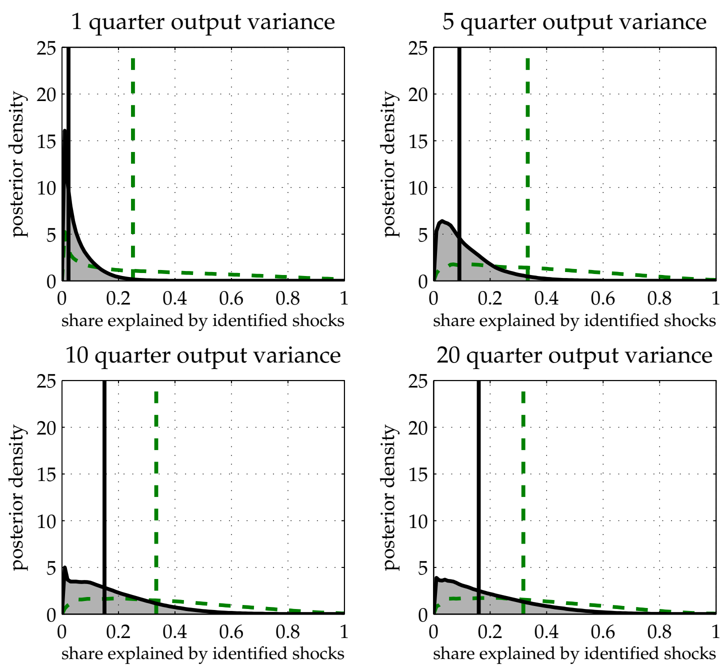



*Figure A.5: Robustness of forecast error variance 3. Posterior probability density and median (vertical line) for the share of forecast error variance of output at each horizon explained by identified shocks for the alternative output series (without government spending and net exports), with 2 period restriction. Solid line: 2 period negative comovement between $\Delta inv_t$ and $(c_t, i_t)$. Dashed line: 2 period positive comovement between $c_t$ and $i_t$.*

# Appendix B

# Appendix for Chapter 2

## B.1 Appendix to section 2.2

### B.1.1 Proof of proposition 1

*Proof of proposition 2.4.* Note first that the payoff to the entrepreneur under the optimal restructuring offer, given by equation (2.7), can be rewritten as:

$$\pi_R(\phi) = \pi(\phi) - min(\chi\pi(\phi), R_b) - R_m. \tag{B.1}$$

First, assume that the repayments promised to the bank and market lenders satisfy:

$$\frac{R_b}{\chi} > \frac{R_m}{1-\chi}. \tag{B.2}$$

Since:

$$R_m + R_b = \chi\frac{R_b}{\chi} + (1-\chi)\frac{R_m}{1-\chi},$$

condition (B.2) implies that:

$$\frac{R_b}{\chi} > R_m + R_b > \frac{R_m}{1-\chi}.$$

The threshold $\frac{R_b}{\chi}$ is the one where restructuring becomes profitable relative to repayment for the entrepreneur, while the threshold $R_m + R_b$ is the "natural" bankrupcy threshold – the one that would obtain absent the possibility of private workouts with the bank. Given condition (B.2), consider the three following possibilities:

- **if** $\pi(\phi) \geq \frac{R_b}{\chi}$**,** using equation (B.1), the payoff of the entrepreneur under restructuring is indentical to that under repayments; that is, the firm cannot lower its repayments in a private workout. Moreover, as $\pi(\phi) > \frac{R_b}{\chi} > R_m + R_b$, repayment is preferable to bankrupcy. Thus, the entrepreneur chooses full repayment.

- **if** $\frac{R_b}{\chi} > \pi(\phi) \geq \frac{R_m}{1-\chi}$**,** the entrepreneur prefers restructuring to repayment since:

  $$\pi_R(\phi) = \pi(\phi) - \chi\pi(\phi) - R_m > \pi(\phi) - R_b - R_m.$$

  Moreover, the entrepreneur also prefers restructuring to bankrupcy, since when $\pi(\phi) > \frac{R_m}{1-\chi}$,

  $$\pi_R(\phi) > 0 = \pi_B(\phi).$$

  Thus, the entrepreneur chooses to restructure her debt with the bank,

- **if** $\frac{R_m}{1-\chi} > \pi(\phi)$**,** the entrepreneur now prefers bankrupcy to restructuring, as the gains achieved under restructuring would still not be sufficient to repay market creditors in full and end up with a strictly positive amount of cash on hand. Furthermore, since under condition (B.2), $\pi(\phi) < \frac{R_m}{\chi} < R_m + R_b$, the entrepreneur also prefers bankrupcy to repayment. Thus, in this case, the entrepreneur chooses to file for bankrupcy.

This establishes the first part of the proposition on the structure of R-equilibria. Additionally, note that in this type of equilibrium, the payoff to market lenders under bankrupcy is always 0, because bankrupcy occurs only when $\pi(\phi) < \frac{R_m}{1-\chi} < \frac{R_b}{\chi}$.

Next, consider the case:

$$\frac{R_b}{\chi} \leq \frac{R_m}{1-\chi}, \tag{B.3}$$

which implies that:

$$\frac{R_b}{\chi} < R_m + R_b < \frac{R_m}{\chi}.$$

Under condition (B.3), consider the three following possibilities:

- **if** $\pi(\phi) \geq R_m + R_b$, repayment is preferable to bankrupcy. Moreover, because in this case $\pi(\phi) \geq R_m + R_b > \frac{R_b}{\chi}$, restructuring cannot lead to any gains for the entrepreneur. Thus, the entrepreneur chooses repayment.

- **if** $R_m + R_b > \pi(\phi) \geq \frac{R_b}{\chi}$, bankrupcy is preferable to repayment, and moreover, restructuring still cannot achieve any gains relative to repayment; thus, the entrepreneur chooses bankrupcy.

- **if** $\frac{R_b}{\chi} > \pi(\phi)$, restructuring can now achieve gains relative to repayment. However, because $\frac{R_m}{1-\chi} > \frac{R_b}{\chi} > \pi(\phi)$ the payoff under restructuring satisfies:

$$\pi_R(\phi) = (1-\chi)\pi(\phi) - R_m < 0 = \pi_B(\phi).$$

  Thus, the entrepreneur still chooses bankrupcy.

This establishes the second part of the proposition. Additionally, note that unde the seniority structure assumed, bankrupcy payments to market lenders are $\chi\pi(\phi) - R_b$ if $R_m + R_b > \pi(\phi) \geq \frac{R_b}{\chi}$, and 0 otherwise. In the latter case, this is obviously smaller than $R_m$. In the former case, since bankrucpy only occurs when $\pi(\phi) < R_m + R_b$, we have that

$$\pi(\phi) < R_m + R_b < \frac{R_m + R_b}{\chi},$$

so that:

$$\chi\pi(\phi) - R_b < R_m.$$

Therefore, under the assumed priority structure, market lenders never obtain a repayment under bankrupcy that exceeds the promised repayment $R_m$.

□

### B.1.2 Debt pricing

In this section, I describe the payoff and the expected gross return functions of lenders that were omitted from the main text.

#### Payoff functions

Given the description of the equilibria in proposition 2.4, the payoffs to the lenders and the entrepreneur can be expressed as a function of $\phi$. I denote them $\tilde{R}_b(\phi)$ and $\tilde{R}_b(\phi)$ for the bank and market lenders, respectively, and $\tilde{\pi}(\phi)$ for the entrepreneur.

In B-equilibria $\left(\frac{R_b}{\chi} < \frac{R_m}{1-\chi}\right)$, payoffs are given by:

$$\tilde{R}_b(\phi) = \begin{cases} R_b & \text{if} \quad \frac{R_b}{\chi} \leq \pi(\phi) \\ \chi\pi(\phi) & \text{if} \quad \pi(\phi) < \frac{R_b}{\chi} \end{cases}$$

$$\tilde{R}_m(\phi) = \begin{cases} R_m & \text{if} \quad R_m + R_b \leq \pi(\phi) \\ \chi\pi(\phi) - R_b & \text{if} \quad \frac{R_b}{\chi} \leq \pi(\phi) < R_m + R_b \\ 0 & \text{if} \quad \pi(\phi) < \frac{R_b}{\chi} \end{cases}$$

$$\tilde{\pi}(\phi) = \begin{cases} \pi(\phi) - R_b - R_m & \text{if} \quad R_m + R_b \leq \pi(\phi) \\ 0 & \text{if} \quad \pi(\phi) < R_m + R_b \end{cases}$$

In R-equilibria $\left(\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}\right)$, payoffs are given by:

$$\tilde{R}_b(\phi) = \begin{cases} R_b & \text{if} \quad \frac{R_b}{\chi} \leq \pi(\phi) \\ \chi\pi(\phi) & \text{if} \quad \pi(\phi) < \frac{R_b}{\chi} \end{cases}$$

$$\tilde{R}_m(\phi) = \begin{cases} R_m & \text{if} \quad \frac{R_m}{1-\chi} \leq \pi(\phi) \\ 0 & \text{if} \quad \pi(\phi) < \frac{R_m}{1-\chi} \end{cases}$$

$$\tilde{\pi}(\phi) = \begin{cases} \pi(\phi) - R_b - R_m & \text{if} \quad \frac{R_b}{\chi} \leq \pi(\phi) \\ (1-\chi)\pi(\phi) - R_m & \text{if} \quad \frac{R_m}{1-\chi} \leq \pi(\phi) < \frac{R_b}{\chi} \\ 0 & \text{if} \quad \pi(\phi) < \frac{R_m}{1-\chi} \end{cases}$$

**Gross expected returns**

I now turn to the gross expected return functions of lenders. Throughout, I use the same change of variables as in the text:

$$b = ds$$

$$m = d(1-s)$$

$d$ thus denotes the total amount borrowed, $s$ denotes the fraction borrowed from the bank, and $1-s$ denotes the fraction borrowed from market lenders; clearly $s \in [0,1]$ and $d \geq 0$. I use the notation:

$$\tilde{E}_i(R_b, R_m; d, s, e) = E\left[\Pi_i\left(e, ds, d(1-s), R_b, R_m\right)\right]$$

for the associated gross return of lenders.

**Lemma B.1** (Gross expected return functions)**.** *The gross expect return of bank lenders is given by:*

$$\tilde{\mathbb{E}}_b(R_b; e+d) = \begin{cases} R_b & \text{if} \quad \frac{R_b}{\chi} < (1-\delta)(e+d) \\ \chi\left((e+d)^{\zeta} G(\phi_b) + (1-\delta)(e+d)\right) & \text{if} \quad \frac{R_b}{\chi} \geq (1-\delta)(e+d) \end{cases}$$

*where:*

$$\phi_b \equiv \frac{R_b - \chi(1-\delta)(e+d)}{\chi(e+d)^{\zeta}} \quad \text{and} \quad G(x) \equiv x(1-F(x)) + \int_0^x \phi dF(\phi)$$

*When* $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$*, the gross expected return of market lenders is given by:*

$$\tilde{\mathbb{E}}_{m,R}(R_m; e+d) = \begin{cases} R_m & \text{if} \quad \frac{R_m}{1-\chi} < (1-\delta)(e+d) \\ (1-\chi)\left((e+d)^{\zeta} I(\phi_{m,R}; e+d) + (1-\delta)(e+d)\right) & \text{if} \quad \frac{R_m}{1-\chi} \geq (1-\delta)(e+d) \end{cases}$$

*where:*

$$\phi_{m,R} \equiv \frac{R_m - (1-\chi)(1-\delta)(e+d)}{(1-\chi)(e+d)^{\zeta}} \quad \text{and} \quad I(x; e+d) \equiv x(1-F(x)) - F(x)(1-\delta)(e+d)^{1-\zeta}.$$

*When $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$, the gross expected return of market lenders is given by:*

$$\tilde{\mathbb{E}}_{m,B}(R_b, R_m; e{+}d) = \begin{cases} R_m & \textit{if} \quad R_m + R_b < (1-\delta)(e+d) \\ (e+d)^{\zeta} M(\phi_{m,B}; e+d) + (1-\delta)(e+d) & \textit{if} \quad R_m + R_b \geq (1-\delta)(e+d) \\ \qquad\qquad -\tilde{\mathbb{E}}_b(R_b; e+d) & \end{cases}$$

*where:*

$$\phi_{m,B} \equiv \frac{R_b + R_m - (1-\delta)(e+d)}{(e+d)^{\zeta}} \quad \textit{and} \quad M(x; e+d) \equiv (1-\chi)I(x; e+d) + \chi G(x).$$

*Proof.* In the case $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$, the entrepreneur never defaults when $(1-\delta)(e+d) > R_m + R_b$, since in that case, the lower bound on her output is greater than the sum of her promised repayments. Expected repayments for the bank and market lenders are therefore equal to promised repayments $R_b$ and $R_m$. When $\frac{R_b}{\chi} < (1-\delta)(e+d) \leq R_m + R_b$, there may be default on market debt, but even in liquidation bank debt will be repayed in full, so that $\tilde{\mathbb{E}}_b = R_b$. Similarly, in the case $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$, there is no default when $(1-\delta)(e+d) > \frac{R_b}{\chi}$, and no default on market debt if $\frac{R_m}{1-\chi} < (1-\delta)(e+d) \leq \frac{R_b}{\chi}$. The rest of the expressions follow from the expressions of the payoff functions.

□

**Zero profit conditions**

Using the results of lemma B.1, the zero profit condition (ZPC) of bank lenders can be written as:

$$\tilde{G}(R_b; e+d) = \frac{(1+r_b)ds}{\chi(e+d)^{\zeta}} \quad , \quad \text{with} \quad \tilde{G}(R_b; e+d) \equiv \frac{\tilde{\mathbb{E}}_b(R_b; e+d)}{\chi(e+d)^{\zeta}}. \tag{B.4}$$

**Lemma B.2** (The zero profit condition of bank lenders)**.** *Let $(d,s,e)$ be given. Then, there exists a unique solution $R_b(d,s,e)$ to the ZPC of bank lenders, equation (B.4), if an only if,*

$$0 \leq \frac{(1+r_b)ds}{\chi(e+d)^{\zeta}} \leq \mathbb{E}(\phi) + (1-\delta)(e+d)^{1-\zeta}. \tag{B.5}$$

*Moreover, $R_b(d,s,e)$ is given by:*

$$R_b(d,s,e) = \begin{cases} (1+r_b)ds & \text{if} \quad 0 \leq \frac{(1+r_b)ds}{\chi(e+d)^{\zeta}} < (1-\delta)(e+d)^{1-\zeta} \\ \chi(1-\delta)(e+d) & \text{if} \quad (1-\delta)(e+d)^{1-\zeta} \leq \frac{(1+r_b)ds}{\chi(e+d)^{\zeta}} \\ \qquad +\chi(e+d)^{\zeta} G^{-1}\left(y_G(d,s,e)\right) & \qquad \leq \mathbb{E}(\phi) + (1-\delta)(e+d)^{1-\zeta} \end{cases}$$

*Here, $y_G(d,s,e) \equiv \frac{(1+r_b)ds - \chi(1-\delta)(e+d)}{\chi(e+d)^{\zeta}}$, and $G^{-1}(.)$ denotes the inverse of $G(.)$, defined on $[0, \mathbb{E}(\phi)]$ with the abuse of notation that $G^{-1}(\mathbb{E}(\phi)) = +\infty$.*

*Proof.* Note that $\tilde{G}(.;e{+}d)$ is strictly increasing on $\mathbb{R}_+$, that $\tilde{G}(0;e{+}d) = 0$ and $\lim_{R_m \to +\infty} \tilde{G}(R_m; e+d) = \mathbb{E}(\phi) + (1-\delta)(e+d)^{1-\zeta}$. Similarly, $G(.)$ is strictly increasing on $\mathbb{R}_+$, $G(0) = 0$, and $\lim_{x \to +\infty} G(x) = \mathbb{E}(\phi)$. $\square$

Likewise, in the case $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$, the zero profit condition of market lenders can be written as:

$$\tilde{I}(R_b; e+d) = \frac{(1+r_m)d(1-s)}{(1-\chi)(e+d)^{\zeta}} \quad , \quad \text{with} \quad \tilde{I}(R_b; e+d) \equiv \frac{\tilde{\mathbb{E}}_{m,R}(R_m; e+d)}{(1-\chi)(e+d)^{\zeta}}. \tag{B.6}$$

**Lemma B.3** (The zero profit condition of market lenders when $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$)**.** *Let $(d,s,e)$ be given. Then, there exist exactly two solutions, $R_{m,l}(d,s,e) \leq R_{m,L}(d,s,e)$, to equation (B.6), if and only if,*

$$0 \leq \frac{(1+r_m)d(1-s)}{(1-\chi)(e+d)^{\zeta}} \leq \hat{I}(e+d) + (1-\delta)(e+d)^{1-\zeta}, \tag{B.7}$$

*where* $\hat{I}(e+d) = I(\phi_I(e+d); e+d)$ *is the maximum of* $I(.; d+e)$*, attained at* $\phi_I(e+d)$*, the unique (strictly positive) solution to:*

$$1 - F(\phi_I(e+d)) - f(\phi_I(e+d))\left(\phi_I(e+d) + (1-\delta)(e+d)^{1-\zeta}\right) = 0.$$

*Moreover,* $R_{m,l}(d,s,e)$ *is given by:*

$$R_{m,l}(d,s,e) = \begin{cases} (1+r_m)d(1-s) & \text{if} \quad 0 \leq \frac{(1+r_m)d(1-s)}{(1-\chi)(e+d)^\zeta} < (1-\delta)(e+d)^{1-\zeta} \\ (1-\chi)(1-\delta)(e+d) & \text{if} \quad (1-\delta)(e+d)^{1-\zeta} \leq \frac{(1+r_m)d(1-s)}{(1-\chi)(e+d)^\zeta} \\ \qquad +(1-\chi)(e+d)^\zeta I^{-1}\left(y_I(d,s,e); e+d\right) & \qquad \leq \hat{I}(e+d) + (1-\delta)(e+d)^{1-\zeta} \end{cases}$$

*Here,* $y_I(d,s,e) \equiv \frac{(1+r_m)d(1-s)-(1-\chi)(1-\delta)(e+d)}{(1-\chi)(e+d)^\zeta}$*, and* $I^{-1}(.; e+d)$ *denotes the mapping from* $\left[0, \hat{I}(e+d)\right]$ *to* $[0, \phi_I(e+d)]$ *such that* $I(I^{-1}(y; e+d); e+d) = y$*,* $\forall y \in \left[0, \hat{I}(e+d)\right]$*.*

*Proof.* Note that $\tilde{I}(0; e+d) = 0$ and $\lim_{R\to+\infty} \tilde{I}(R; e+d) = 0$. Moreover, $\frac{\partial \tilde{I}}{\partial R_b} = \frac{1}{(1-\chi)(e+d)^\zeta} > 0$ when $R_b < (1-\chi)(1-\delta)(e+d)^{1-\zeta}$, so that $\tilde{I}(., e+d)$ is increasing in $R_b$ on that range. For $R_b \geq (1-\chi)(1-\delta)(e+d)^{1-\zeta}$, note that $\frac{\partial \tilde{I}}{\partial R_b} \geq 0$, if and only if, $h(\phi_{m,R}) = \frac{f(\phi_{m,R})}{1-F(\phi_{m,R})} \leq \frac{1}{\phi_{m,R}+(1-\delta)(e+d)^{1-\zeta}}$, with $\phi_{m,R}$ defined as in lemma B.1. Fix $e$ and $d$ and let $\Delta(x) = h(x) - \frac{1}{x+(1-\delta)(e+d)^{1-\zeta}}$. $h$ is strictly increasing by assumption 2.1. As the sum of two strictly increasing functions, $\Delta$ is therefore strictly increasing on $]0; +\infty[$. Moreover, $\Delta(0) = -\frac{1}{(1-\delta)(e+d)^{1-\zeta}}$ and $\lim_{x\to+\infty} \Delta(x) = \lim_{x\to+\infty} h(x)$. Since $h(0) = 0$ (as $f(0) = 0$ by assumption 2.1) and $h$ is strictly increasing, $\lim_{x\to+\infty} h(x) > 0$. Thus $\Delta(x)$ has a unique strictly positive root at $\phi_{m,R} = \phi_I(e+d)$. In turn, because $\phi_{m,R}$ is a strictly increasing function of $R_m$, $\hat{R}_I(e+d) \equiv (1-\chi)\left(\phi_I(e+d)(e+d)^\zeta + (1-\delta)(e+d)\right)$ is the global maximum of $\tilde{I}(.; e+d)$. There are moreover no local maxima, so that $\tilde{I}(.; d, e)$ is strictly increasing to the left of $\hat{R}_I(e+d)$ and strictly decreasing to the right. (Note however that $I(.; e+d)$ and $\tilde{I}(.; e+d)$ need not be concave). This proves that there are exactly two solutions $\tilde{R}_{m,l}(d,s,e) \leq \hat{R}_I(e+d) \leq \tilde{R}_{m,L}(d,s,e)$ to equation (B.6) under the conditions given in the lemma. The expression for $R_{m,l}(d,s,e)$ follows from recognizing that, for the same reasons as $\tilde{I}(.;e+d)$, $I(.;e+d)$ is strictly increasing on $[0,\phi_I(e+d)]$ and strictly decreasing thereafter. $\square$

Finally, when $\frac{R_m}{1-\chi} > \frac{R_b}{\chi}$, when the zero profit condition of the bank holds, the zero profit condition of market lenders can be rewritten as:

$$\tilde{M}(R_b + R_m; e+d) = \frac{(1 + r_m(1-s) + r_b)d}{(e+d)^{\zeta}}, \tag{B.8}$$

where:

$$\tilde{M}(R; e+d) = \begin{cases} \frac{R}{(e+d)^{\zeta}} & \text{if} \quad R < (1-\delta)(e+d) \\ M(\phi_{m,B}; e+d) + (1-\delta)(e+d)^{1-\zeta} & \text{if} \quad R \geq (1-\delta)(e+d) \end{cases}$$

and $\phi_{m,B} \equiv \frac{R-(1-\delta)(e+d)}{(e+d)^{\zeta}}$.

**Lemma B.4** (The zero profit condition of market lenders when $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$)**.** *Let $(d,s,e)$ be given. Then, there exist exactly two solutions, $R_l(d,s,e) \leq R_L(d,s,e)$, to equation (B.8), if and only if,*

$$\chi\mathbb{E}(\phi) + (1-\delta)(e+d)^{1-\zeta} \leq \frac{(1 + r_m(1-s) + r_b s)d}{(e+d)^{\zeta}} \leq \hat{M}(e+d) + (1-\delta)(e+d)^{1-\zeta}, \tag{B.9}$$

*where $\hat{M}(e+d) = M(\phi_M(e+d), e+d)$ is the maximum of $M(.;e+d)$, attained at $\phi_M(e+d) > \phi_I(e+d)$, which is the unique (strictly positive) solution to:*

$$1 - F(\phi_M(e+d) - (1-\chi)f(\phi_M(e+d))\left(\phi_M(e+d) + (1-\delta)(e+d)^{1-\zeta}\right) = 0.$$

*There exists exactly one solution $R_l(d,s,e)$ to equation (B.8), if and only if:*

$$0 \leq \frac{(1+r_m(1-s)+r_b s)d}{(e+d)^{\zeta}} < \chi\mathbb{E}(\phi) + (1-\delta)(e+d)^{1-\zeta},$$

*Moreover, $R_l(d,s,e)$ is given by:*

$$R_l(d,s,e) = \begin{cases} (1+r_m(1-s)+r_b s)d & \text{if} \quad 0 \leq \frac{(1+r_m(1-s)+r_b s)d}{(e+d)^{1-\zeta}} < (1-\delta)(e+d)^{1-\zeta} \\ +(1-\delta)(e+d) & \text{if} \quad (1-\delta)(e+d)^{1-\zeta} \leq \frac{(1+r_m(1-s)+r_b s)d}{(e+d)^{1-\zeta}} \\ \quad (e+d)^{\zeta} M^{-1}\left(y_M(d,s,e); e+d\right) & \qquad \leq \hat{M}(e+d) + (1-\delta)(e+d)^{1-\zeta} \end{cases}$$

*Here, $y_M(d,s,e) \equiv \frac{(1+r_m(1-s)+r_b s)d-(1-\delta)(e+d)}{(e+d)^{\zeta}}$, and $M^{-1}(.;e+d)$ denotes the mapping from $\left[0,\hat{M}(d+e)\right]$ to $[0,\phi_M(e+d)]$ such that $M(M^{-1}(y;e+d);e+d) = y \ \forall y \in \left[0,\hat{M}(e+d)\right]$.*

### B.1.3 The lending menu in the general case

This subsection contains the proof of proposition 2.8. This proposition states that the set $\mathcal{S}(e)$ can be separated into two subsets, $\tilde{S}_R(e)$ and $\tilde{S}_B(e)$. $\tilde{\mathcal{S}}_K(e)$ roughly correspond to debt structures associated with dominating contracts with $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$, while $\tilde{\mathcal{S}}_K(e)$ corresponds to debt structures associated with dominating contracts such that $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$. The main difficulty of this proof is that when a debt structure $(d,s)$ is such that $\mathcal{L}(d,s,e) \neq \emptyset$, $\mathcal{L}(d,s,e)$ will typically contain multiple pairs $(R_b,R_m)$ (up to four pairs), each of which satisfy the zero profit conditions of lenders. The multiplicity of contracts comes from two sources. First, zero profit conditions of lenders generically have two solutions, as established in lemmas B.3-B.4. Second, there are debt structures for which the zero profit conditions of lenders have solutions with $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$ as well as solutions with $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$ (that is, $\tilde{\mathcal{S}}_R(e) \cap \tilde{\mathcal{S}}_K(e) \neq \emptyset$). All this multiplicity creates the potential for contracts that cannot be ordered using the product order. The main contribution of the lemma is to show that these contracts can in all instances be ordered, so that there is always a unique dominating contract, even when the contract menu contains multiple elements.

*Proof.* The proof draws heavily upon the results of lemmas B.2-B.4 and proceeds in three steps:

**Step 1 :** Any $(d,s) \in \tilde{\mathcal{S}}(e)$ must be an element of either $\tilde{\mathcal{S}}_K(e)$ or $\tilde{\mathcal{S}}_R(e)$, so that $\tilde{\mathcal{S}}(e) \subset \left(\tilde{\mathcal{S}}_K(e) \cup \tilde{\mathcal{S}}_R(e)\right)$;

**Step 2-a :** $\forall (d,s) \in \tilde{\mathcal{S}}_R(e)$, $\mathcal{L}(d,s,e)$ has a least element $(R_b, R_m)$ (for the product order $\geq_\times$), so that $\tilde{\mathcal{S}}_R(e) \subset \tilde{\mathcal{S}}(e)$; moreover, this element satisfies $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$;

**Step 2-b :** $\forall (d,s) \in \tilde{\mathcal{S}}_K(e) \setminus \left(\tilde{\mathcal{S}}_R(e) \cap \tilde{\mathcal{S}}_K(e)\right)$, $\mathcal{L}(d,s,e)$ has a least element $(R_b, R_m)$ (for the product order $\geq_\times$), so that $\left(\tilde{\mathcal{S}}_K(e) \setminus \left(\tilde{\mathcal{S}}_R(e) \cap \tilde{\mathcal{S}}_K(e)\right)\right) \subset \tilde{\mathcal{S}}(e)$; moreover, the least element satisfies $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$.

**Step 1** Let $(d,s) \in \mathcal{S}(e)$ and let $(R_b, R_m)$ be the associated dominating contract, that is, the least element of $\mathcal{L}(d,s,e)$ for $\geq_\times$. It must be the case that either $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$ or $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$.

**Assume that $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$.** Then $(R_b, R_m) \in \mathcal{L}(d,s,e)$, so that $\mathcal{L}(d,s,e) \neq \emptyset$. Moreover, $(R_b, R_m)$ must solve the zero profit conditions (ZPC) of the lenders in the case $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$, equations (B.4) and (B.8). Using the results of lemmas B.2 and B.4, conditions (c − bank) and (c − joint) are necessary for the existence of solutions to (B.4) and (B.8). Thus, $(d,s,e)$ must satisfy these two conditions; moreover, by lemma B.2, $R_b = R_b(d,s,e)$.

If $\frac{(1+r_m(1-s)+r_b s)d}{(e+d)^\varsigma} < \chi\mathbb{E}(\phi)$, the solution to equation (B.8) is unique, according to lemma B.4. Therefore we must have $R_m = R_l(d,s,e) - R_b(d,s,e)$. Since, by assumption, $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$, we also have $\frac{R_b}{\chi} < R_m + R_b$. Therefore, it must also be the case that $\frac{R_b(d,s,e)}{\chi} < R_l(d,s,e)$, so that condition (frontier − B) must hold.

If, on the other hand, $\frac{(1+r_m(1-s)+r_b s)d}{(e+d)^\zeta} \geq \chi\mathbb{E}(\phi)$, there are exactly two solutions to equation (B.8), which satisfy $0 \leq R_l(d,s,e) \leq R_L(d,s,e)$. However, $(R_b(d,s,e), R_L(d,s,e) - R_b(d,s,e)) \geq_\times (R_b(d,s,e), R_l(d,s,e) - R_b(d,s,e))$. Since $(R_b, R_m)$ is the least element of $\mathcal{L}(d,s,e)$, it must be therefore be the case that $R_m = R_l(d,s,e) - R_b(d,s,e)$. The fact that condition (frontier − B) holds then follows from the same steps as above. This finishes the proof that if $\frac{R_m}{1-\chi} > \frac{R_b}{\chi}$, then $(d,s) \in \tilde{\mathcal{S}}_K(e)$.

**Assume that** $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$. Similar to the other case, this implies that $\mathcal{L}(d,s,e) \neq \emptyset$ and therefore that conditions (c − bank) and (c − market), which are necessary for the existence of a solution to the ZPC of lenders when $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$, must hold. Moreover, by lemma B.2, $R_b = R_b(d,s,e)$.

Under condition (c − market), there exist exactly two solutions $R_{m,l}(d,s,e) \leq R_{m,L}(d,s,e)$, once of which $R_m$ must be equal to. Since $(R_b, R_m)$ is the least element of $\mathcal{L}(d,s,e)$, it must be the case that $R_m = R_{m,l}(d,s,e)$. Since $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$, this in turn imply that condition (frontier − B) holds. This finishes the proof that if $\frac{R_m}{1-\chi} \leq \frac{R_b}{\chi}$, then $(d,s) \in \tilde{\mathcal{S}}_R(e)$.

Thus, $\mathcal{S}(e) \in \left(\tilde{\mathcal{S}}_K(e) \cup \tilde{\mathcal{S}}_R(e)\right)$.

**Step 2-a :** Let $(d,s) \in \tilde{S}_R(e)$. We need to prove that $(d,s) \in \mathcal{S}_e$, that is, that $\mathcal{L}(d,s,e)$ is non-empty and has a least element. First, under conditions (c − bank) and (c − market), $(R_b(d,s,e), R_{m,l}(d,s,e)) \in \mathcal{L}(d,s,e)$, so that $\mathcal{L}(d,s,e) \neq \emptyset$. If $\mathcal{L}(d,s,e)$ contains only $(R_b(d,s,e), R_m(d,s,e))$, then it is the least element and we are done. Otherwise, let $(R_b, R_m) \in \mathcal{L}(d,s,e)$, $(R_b, R_m) \neq (R_b(d,s,e), R_{m,l}(d,s,e))$. If $(R_b, R_m)$ is such that $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$, then, since $(R_b, R_m) \neq (R_b(d,s,e), R_{m,l}(d,s,e))$, it must be the case that $R_b = R_b(d,s,e)$ and $R_m = R_{m,L}(d,s,e)$; therefore, $(R_b, R_m) \geq_\times (R_b(d,s,e), R_{m,l}(d,s,e))$. If, on the other hand,

$\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$, it must still be the case that $R_b = R_b(d,s,e)$; hence,

$$\frac{R_m}{1-\chi} > \frac{R_b}{\chi} = \frac{R_b(d,s,e)}{\chi} \geq \frac{R_{m,l}(d,s,e)}{1-\chi}.$$

Therefore, $(R_b, R_m) \geq_\times (R_b(d,s,e), R_{m,l}(d,s,e))$ in the case $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$ as well. Thus, $\mathcal{L}(d,s,e)$ has a least element, $(R_b(d,s,e), R_{m,l}(d,s,e))$, so that $(d,s) \in \mathcal{S}(e)$. Hence, $\tilde{S}_R(e) \subset \tilde{S}(e)$. Note that since the least element is always unique, I have also established the proposition's claim that the dominating contract (least element) $(R_b = R_b(d,s,e), R_m = R_{m,l}(d,s,e))$ associated to $(d,s) \in \tilde{S}_R(e)$ satisfies $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$.

**Step 2-b :** Let $(d,s) \in \tilde{S}_B(e) \setminus \left(\tilde{S}_B(e) \cap \tilde{S}_R(e)\right)$. First, under conditions $(\mathrm{c}-\mathrm{bank})$ and $(\mathrm{c}-\mathrm{joint})$, we have that $(R_b(d,s,e), R_l(d,s,e) - R_b(d,s,e)) \in \mathcal{L}(d,s,e)$, so $\mathcal{L}(d,s,e) \neq \emptyset$. Consdier $(R_b, R_m) \in \mathcal{L}(d,s,e)$, $(R_b, R_m) \neq (R_b(d,s,e), R_l(d,s,e) - R_b(d,s,e))$. If $(R_b, R_m)$ are such that $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$, then, as in step 1, it must be that $(d,s) \in \tilde{S}_R(e)$; but we ruled this out by assumption. So, $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$. In this case, since $(R_b, R_m) \neq (R_b(d,s,e), R_l(d,s,e) - R_b(d,s,e))$, it must be that $R_b = R_b(d,s,e)$ and $R_m = R_L(d,s,e) - R_b(d,s,e)$ (and additionally that $\frac{(1+r_m(1-s)+r_b s)d}{(e+d)^\zeta} > \chi\mathbb{E}(\phi)$, since otherwise the solution to the ZPC of market lenders is unique). So, $(R_b, R_m) \geq_\times (R_b(d,s,e), R_l(d,s,e) - R_b(d,s,e))$, and so $(R_b(d,s,e), R_l(d,s,e) - R_b(d,s,e))$ is the least element of $\mathcal{L}(d,s,e)$. This proves that $(d,s) \in \tilde{S}(e)$, and therefore that:

$$\left(\tilde{S}_B(e) \setminus \left(\tilde{S}_B(e) \cap \tilde{S}_R(e)\right)\right) \subset \tilde{S}(e).$$

Note that again because of unicity of the least element, I have also established the lemma's claim that the dominating contract $(R_b = R_b(d,s,e), R_m = R_l(d,s,e) - R_b(d,s,e))$ associated to $(d,s) \in \tilde{S}_B(e) \setminus \left(\tilde{S}_B(e) \cap \tilde{S}_R(e)\right)$ satisfies $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$.

$\square$

### B.1.4 The lending menu when $\delta = 1$

I next turn to characterizing the set $\mathcal{S}_R(e) = \tilde{\mathcal{S}}_R(e)$ in more detail, in the special case $\delta = 1$, where an analytical characterization can be obtained.

**Proposition B.5** (A parametrization of the set $\tilde{\mathcal{S}}_R(e)$)**.** *The set $\tilde{\mathcal{S}}_R(e)$ can be described as:*

$$\tilde{\mathcal{S}}_R(e) = \left\{ (d,s) \in \mathbb{R}_+ \times [\underline{s}_R, 1] \;\middle|\; 0 \le d \le \overline{d}_R(s,e) \right\},$$

*where:*

$$\overline{d}_R(s,e) = \begin{cases} \overline{d}_{R,F}(s,e) & \text{if} \quad \underline{s}_R \le s < s_{R,1} \\ \overline{d}_{R,m}(s,e) & \text{if} \quad s_{R,1} \le s < s_{R,2} \\ \overline{d}_{R,b}(s,e) & \text{if} \quad s_{R,2} \le s \le 1 \end{cases}$$

*The thresholds $\underline{s}_R < s_{R,1} < s_{R,2} < 1$ are given by:*

$$\begin{aligned} \underline{s}_R &= \frac{1}{1 + \frac{1-\chi}{\chi}\frac{1+r_b}{1+r_m}} \\ s_{R,1} &= \frac{1}{1 + \frac{1-\chi}{\chi}\frac{1+r_b}{1+r_m}\frac{I(\phi_I)}{G(\phi_I)}} \\ s_{R,2} &= \frac{1}{1 + \frac{1-\chi}{\chi}\frac{1+r_b}{1+r_m}\frac{I(\phi_I)}{\mathbb{E}(\phi)}} \end{aligned}$$

*while the functions $\overline{d}_{R,F}(s,e)$, $\overline{d}_{R,m}(s,e)$ and $\overline{d}_{R,b}(s,e)$ are implicitly defined by:*

$$\begin{aligned} G^{-1}\left(\frac{(1+r_b)s\overline{d}_{R,F}(s,e)}{\chi(e+\overline{d}_{R,F}(s,e))^{\zeta}}\right) &= I^{-1}\left(\frac{(1+r_m)(1-s)\overline{d}_{R,F}(s,e)}{(1-\chi)(e+\overline{d}_{R,F}(s,e))^{\zeta}}\right) \\ \frac{(1+r_m)(1-s)\overline{d}_{R,m}(s,e)}{(1-\chi)(e+\overline{d}_R(s,e))^{\zeta}} &= I(\phi_I) \\ \frac{(1+r_b)s\overline{d}_{R,b}(s,e)}{\chi(e+\overline{d}_R(s,e))^{\zeta}} &= \mathbb{E}(\phi) \end{aligned}$$

*Proof.* For this proof, I proceed in four steps:

**Step 1:** If $(d, s) \in \tilde{\mathcal{S}}_R(e)$, then $\underline{s}_R \leq s \leq 1$;

**Step 2:** If $\underline{s}_R \leq s < s_{R,1}$, then, $(d, s) \in \tilde{\mathcal{S}}_R(e)$, if and only if, condition (frontier − R) is verified;

**Step 3:** If $s_{R,1} \leq s < s_{R,2}$, then, $(d, s) \in \tilde{\mathcal{S}}_R(e)$, if and only if, condition (c − market) is verified;

**Step 4:** If $s_{R,2} \leq s \leq 1$, then, $(d, s) \in \tilde{\mathcal{S}}_R(e)$, if and only if, condition (c − bank) is verified.

**Step 1:** First, I prove that when $s < \underline{s}_R$, condition (frontier − R) does not hold, so that $\tilde{\mathcal{S}}_R(e)$ cannot containt debt structures with $s < \underline{s}_R$. To see this, first note that $\forall 0 \leq x \leq \phi_I$, $G(x) \geq I(x)$, so that $\forall 0 \leq y \leq I(\phi_I)$, $G^{-1}(y) \leq I^{-1}(y)$ (with equality only at $y = 0$). Note moreover that:

$$s < \underline{s}_R \quad \Longrightarrow \quad \frac{(1 + r_b)ds}{\chi(e + d)^\zeta} < \frac{(1 + r_m)(1 - s)d}{(1 - \chi)(e + d)^\zeta}$$

Thus,

$$\begin{aligned} s < \underline{s}_R \quad &\Longrightarrow \quad G^{-1}\left(\frac{(1 + r_b)ds}{\chi(e + d)^\zeta}\right) < G^{-1}\left(\frac{(1 + r_m)(1 - s)d}{(1 - \chi)(e + d)^\zeta}\right) \\ &\Longrightarrow \quad G^{-1}\left(\frac{(1 + r_b)ds}{\chi(e + d)^\zeta}\right) < I^{-1}\left(\frac{(1 + r_m)(1 - s)d}{(1 - \chi)(e + d)^\zeta}\right). \end{aligned}$$

Therefore, $(d, s) \in \tilde{\mathcal{S}}_R(e) \implies s \geq \underline{s}_R$.

**Step 2:** Next, I prove that when $(d, s) \in \tilde{\mathcal{S}}_R(e)$, $\underline{s}_R \leq s < s_{R,1}$, then only condition (frontier − R) is relevant to the definition of $\tilde{\mathcal{S}}_R(e)$; that is,

$$\left((d, s) \in \tilde{\mathcal{S}}_R(e) \text{ and } \underline{s}_R \leq s < s_{R,1}\right) \implies \left(\frac{(1 + r_m)(1 - s)d}{(1 - \chi)(e + d)^\zeta} < I(\phi_I) \text{ and } \frac{(1 + r_b)ds}{\chi(e + d)^\zeta} < \mathbb{E}(\phi)\right). \tag{B.10}$$

I establish this by proving the contraposition. First, if $\frac{(1+r_b)ds}{\chi(e+d)^\zeta} > \mathbb{E}(\phi)$, then $(d,s) \notin \tilde{\mathcal{S}}_R(e)$. If $\frac{(1+r_b)ds}{\chi(e+d)^\zeta} = \mathbb{E}(\phi)$, then:

$$\begin{aligned}
\mathbb{E}(\phi) = \frac{(1+r_b)ds}{\chi(e+d)^\zeta} &< \frac{(1+r_b)s_{R,1}d}{\chi(e+d)^\zeta} \\
&= \frac{(1+r_m)(1-s_{R,1})d}{(1-\chi)(e+d)^\zeta}\frac{G(\phi_I)}{I(\phi_I)} \\
&< \frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta}\frac{G(\phi_I)}{I(\phi_I)}
\end{aligned}$$

Therefore:

$$\frac{(1+r_m)(1-s_{R,1})d}{(1-\chi)(e+d)^\zeta} > \frac{\mathbb{E}(\phi)}{G(\phi_I)} I(\phi_I) > I(\phi_I) \implies (d,s) \notin \tilde{\mathcal{S}}_R(e).$$

Second, if $\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta} > I(\phi_I)$, then $(d,s) \notin \tilde{\mathcal{S}}_R(e)$. If $\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta} = I(\phi_I)$, then:

$$\begin{aligned}
I(\phi_I) = \frac{(1+r_b)(1-s)d}{(1-\chi)(e+d)^\zeta} &> \frac{(1+r_m)(1-s_{1,R})d}{(1-\chi)(e+d)^\zeta} \\
&= \frac{(1+r_b)s_{R,1}d}{\chi(e+d)^\zeta}\frac{I(\phi_I)}{G(\phi_I)} \\
&> \frac{(1+r_b)sd}{\chi(e+d)^\zeta}\frac{I(\phi_I)}{G(\phi_I)}
\end{aligned}$$

Therefore,

$$\frac{(1+r_b)ds}{\chi(e+d)^\zeta} < G(\phi_I) \implies G^{-1}\left(\frac{(1+r_b)ds}{\chi(e+d)^\zeta}\right) < \phi_I = I^{-1}\left(\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta}\right) \implies (d,s) \notin \tilde{\mathcal{S}}_R(e),$$

which finishes proving statement (B.10). It is then straightforward to show that condition (frontier − R) is equivalent to $0 \le d \le \overline{d}_{R,F}(s,e)$.

**Step 3:** Next, I prove that when $(d,s) \in \tilde{\mathcal{S}}_R(e)$, $s_{R,1} < s < s_{R,2}$, then only condition (c − market) is relevant to the definition of $\tilde{\mathcal{S}}_R(e)$; that is,

$$\begin{gathered}\left((d,s) \in \tilde{\mathcal{S}}_R(e) \text{ and } s_{R,1} < s < s_{R,2}\right) \implies \\ \left(\frac{(1+r_b)ds}{\chi(e+d)^\zeta} < \mathbb{E}(\phi) \text{ and } G^{-1}\left(\frac{(1+r_b)ds}{\chi(e+d)^\zeta}\right) > I^{-1}\left(\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta}\right)\right)\end{gathered} \tag{B.11}$$

Assume that $s_{R,1} < s < s_{R,2}$ and $(d,s) \in \tilde{\mathcal{S}}_R(e)$, then:

$$\begin{aligned}\frac{(1+r_b)sd}{\chi(e+d)^\zeta} &= \frac{1-\chi}{\chi}\frac{1+r_b}{1+r_m}\frac{s}{1-s}\frac{(1+r_m)(1-s)d}{(e+d)^\zeta} \\ &\leq \frac{1-\chi}{\chi}\frac{1+r_b}{1+r_m}\frac{s}{1-s}I(\phi_I) \\ &< \frac{1-\chi}{\chi}\frac{1+r_b}{1+r_m}\frac{s_{R,2}}{1-s_{R,2}}I(\phi_I) = \mathbb{E}(\phi)\end{aligned}$$

so that condition (c − bank) is satisfied with strict inequality. I next prove that when $(d,s) \in \tilde{\mathcal{S}}_R(e)$ and $s_{R,1} < s$, then condition (frontier − R) holds with strict inequality. As a first step, define $\Delta(x) = \frac{G(\phi_I)}{I(\phi_I)}I(x) - G(x)$. Note that $\Delta(0) = \Delta(\phi_I) = 0$. Moreover, $\Delta'(x) = \left(\frac{G(\phi_I)}{I(\phi_I)} - 1\right)(1-F(x)) - \frac{G(\phi_I)}{I(\phi_I)}xf(x)$, so that, following steps similar to the proof of lemma B.1, $\Delta(.)$ is increasing on $\left[0, \underline{\phi}\right]$ and decreasing on $\left[\underline{\phi}, \phi_I\right]$, where $\underline{\phi} < \phi_I$ is the unique solution to $\Delta'(x) = 0$. Thus,

$$0 \leq x \leq \phi_I \implies \frac{G(\phi_I)}{I(\phi_I)}I(x) \geq G(x),$$

so that:

$$0 \leq y \leq I(\phi_I) \implies G^{-1}\left(\frac{G(\phi_I)}{I(\phi_I)}y\right) \geq I^{-1}(y)$$

Moreover, using the definition of $s_{R,1}$, note that:

$$s > s_{R,1} \quad \Longrightarrow \quad \frac{(1+r_b)sd}{\chi(e+d)^\zeta} > \frac{G(\phi_I)}{I(\phi_I)} \frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta}$$

Thus,

$$s > s_{R,1} \implies G^{-1}\left(\frac{(1+r_b)sd}{\chi(e+d)^\zeta}\right) > G^{-1}\left(\frac{G(\phi_I)}{I(\phi_I)} \frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta}\right) \geq I^{-1}\left(\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta}\right),$$

This finishes establishing statement (B.11). It is then straightforward to show that condition (c − market) is equivalent to $0 \leq d \leq \overline{d}_{R,m}(s,e)$.

**Step 4:** Finally, I prove that when $(d,s) \in \tilde{\mathcal{S}}_R(e)$, $s_{R,2} < s \leq 1$, then only condition (c − bank) is relevant to the definition of $\tilde{\mathcal{S}}_R(e)$; that is,

$$\begin{gathered}\Big((d,s) \in \tilde{\mathcal{S}}_R(e) \text{ and } s_{R,2} < s < 1\Big) \Longrightarrow \\ \left(\frac{(1+r_m)d(1-s)}{(1-\chi)(e+d)^\zeta} < I(\phi_I) \text{ and } G^{-1}\left(\frac{(1+r_b)ds}{\chi(e+d)^\zeta}\right) > I^{-1}\left(\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta}\right)\right)\end{gathered} \tag{B.12}$$

This is straightforward: note that $s > s_{R,2}$ immediately implies the first part of statement (B.12), while the second part obtains because $s > s_{R,2} > s_{R,1}$. It is then straightforward to show that condition (c − bank) is equivalent to $0 \leq d \leq \overline{d}_{R,b}(s,e)$.

For the thresholds $s_{R,1}$ and $s_{R,2}$, it is simple to establish that when $s = s_{R,1}$, conditions (c − market) and (frontier − R) coincide, and both imply condition (c − bank); while, when $s = s_{R,2}$, conditions (c − market) and (c − bank) coincide, and imply condition (frontier − R). This concludes the proof of the proposition.

□

I next turn to the structure of the set $\tilde{\mathcal{S}}_K(e)$.

**Proposition B.6** (A parametrization of the set $\tilde{\mathcal{S}}_K(e)$)**.** *The set $\tilde{\mathcal{S}}_K(e)$ can be parametrized as:*

$$\tilde{\mathcal{S}}_K(e) = \left\{ (d,s) \in \mathbb{R}_+ \times [0, \overline{s}_K] \;\middle|\; \underline{d}_K(s,e) < d \leq \overline{d}_K(s,e) \right\},$$

*where:*

$$\underline{d}_K(s,e) = \begin{cases} 0 & \text{if} \quad 0 \leq s < \underline{s}_R \\ \underline{d}_{K,F}(s,e) & \text{if} \quad \underline{s}_R \leq s \leq \overline{s}_K \end{cases}$$

*The threshold $\overline{s}_K$ is given by:*

$$\overline{s}_K = \frac{1}{1 + \left(\frac{1}{\chi}\frac{M(\phi_M)}{G(\phi_M)} - 1\right)\frac{1+r_b}{1+r_m}}$$

*while the functions $\underline{d}_{K,F}(s,e)$ and $\overline{d}_K(s,e)$ are implicitly defined by:*

$$\begin{aligned} G^{-1}\left(\frac{(1+r_b)s\underline{d}_{K,F}(s,e)}{\chi(e+\underline{d}_{K,F}(s,e))^{\zeta}}\right) &= M^{-1}\left(\frac{(1+(1-s)r_m+sr_b)\,\underline{d}_{K,F}(s,e)}{(e+\underline{d}_{K,F}(s,e))^{\zeta}}\right) \\ \frac{(1+(1-s)r_m+sr_b)\,\overline{d}_K(s,e)}{(e+\overline{d}_K(s,e))^{\zeta}} &= M(\phi_M) \end{aligned}$$

*Proof.* For this proof, I proceed in three steps:

**Step 1:** If $(d,s) \in \tilde{S}_B(e)$, then $0 \leq s \leq \overline{s}_K$;

**Step 2:** Let $(d,s) \in \mathbb{R}_+ \times [0, \overline{s}_K]$; then if condition (c − joint) holds, condition (c − bank) also holds;

**Step 3:** Let $(d,s) \in \mathbb{R}_+ \times [0, \overline{s}_K]$; then conditions (frontier − R) and (c − joint) are equivalent to, respectively, lower and upper bounds on $d$.

**Step 1:** I prove the contraposition. Assume that $s > \overline{s}_K$. Note that in this case,

$$\frac{(1+r_b)s}{\chi}\frac{d}{(e+d)^\zeta} > \frac{G(\phi_M)}{M(\phi_M)}\left[(1+r_m)(1-s)+(1+r_b)s\right]\frac{d}{(e+d)^\zeta},$$

for any $d, e > 0$. Moreover, using steps analogous to the proof of proposition 2.17, one can establish that, $\forall 0 \leq y \leq M(\phi_M)$, $G^{-1}\left(\frac{G(\phi_M)}{M(\phi_M)}y\right) \geq M^{-1}(y)$. Thus,

$$\begin{aligned} G^{-1}\left(\frac{(1+r_b)sd}{\chi(e+d)^\zeta}\right) &> G^{-1}\left(\frac{G(\phi_M)}{M(\phi_M)}\left[(1+r_m)(1-s)+(1+r_b)s\right]\frac{d}{(e+d)^\zeta}\right) \\ &\geq M^{-1}\left(\left[(1+r_m)(1-s)+(1+r_b)s\right]\frac{d}{(e+d)^\zeta}\right), \end{aligned}$$

where the first line uses the second result above, and the second line uses the first result. This is a violation of (frontier − R), so $(d, s) \notin \tilde{\mathcal{S}}_K(e)$. This proves the result of announced.

**Step 2:** This result follows from using the fact that, when $s \leq \overline{s}_K$,

$$\frac{(1+r_b)s}{\chi}\frac{d}{(e+d)^\zeta} \leq \frac{G(\phi_M)}{M(\phi_M)}\left[(1+r_m)(1-s)+(1+r_b)s\right]\frac{d}{(e+d)^\zeta}.$$

When condition (c − joint) holds, we therefore have:

$$\frac{(1+r_b)s}{\chi}\frac{d}{(e+d)^\zeta} \leq G(\phi_M) < \mathbb{E}(\phi)$$

where the last inequality follows from the facts that $G(.)$ is strictly increasing. This shows that condition (c − bank) holds. Condition (c − bank) is therefore irrelevant to the definition of $\tilde{S}_B(e)$; only conditions (frontier − R) and (c − joint) matter.

**Step 3:** I first show that the condition (frontier − R) is tantamount to a lower bound on $d$. First, note that if $0 \leq \underline{s}_B$, then for any $d, e > 0$:

$$\frac{(1+r_b)sd}{\chi(e+d)^\zeta} \leq \frac{(1+r_m(1-s)+r_b s)d}{\chi(e+d)^\zeta}$$

This implies that, when $\frac{(1+r_m(1-s)+r_b s)d}{\chi(e+d)^\zeta} \leq M(\phi_M)$, ie condition (c − joint), then:

$$\begin{aligned} G^{-1}\left(\frac{(1+r_b)sd}{\chi(e+d)^\zeta}\right) &\leq G^{-1}\left(\frac{(1+r_m(1-s)+r_b s)d}{(e+d)^\zeta}\right) \\ &< M^{-1}\left(\frac{(1+r_m(1-s)+r_b s)d}{(e+d)^\zeta}\right) \end{aligned}$$

Thus, condition (frontier − R) is automatically verified. When $\underline{s}_B \leq s \leq \overline{s}_K$, it is straightforward to check that condition (frontier − R) is equivalent to $d \geq \underline{d}_{K,F}(s,e)$, where $\underline{d}_{K,F}(s,e)$ is defined in the statement of the proposition. Finally, it is also straightforward to show that condition (c − joint) corresponds to is equivalent to the upper bound $\overline{d}_K(s,e)$ given in the proposition.

□

The two previous propositions are useful to characterize the two sets that we are actually interested in, that is, the sets $\mathcal{S}_R(e)$ and $\mathcal{S}_K(e)$. Recall that $\mathcal{S}_R(e) = \tilde{\mathcal{S}}_R(e)$, so that the parametrization established in proposition 2.17 is also characterizes $\tilde{\mathcal{S}}_R(e)$. We are left with the task of describing the intersection $\tilde{\mathcal{S}}_R(e) \cap \tilde{\mathcal{S}}_K(e)$. This is the object of the following proposition.

**Proposition B.7** (A parametrization of the intersection $\tilde{\mathcal{S}}_R(e) \cap \tilde{\mathcal{S}}_K(e)$)**.** *The intersection $\tilde{\mathcal{S}}_R(e) \cap \tilde{\mathcal{S}}_K(e)$ can be parametrized as:*

$$\tilde{\mathcal{S}}_R(e) \cap \tilde{\mathcal{S}}_K(e) = \left\{ (d,s) \in \mathbb{R}_+ \times ]\underline{s}_R, \overline{s}_K[ \;\middle|\; \underline{d}_K(s,e) \leq d \leq \overline{d}_{R\cap K}(s,e) \right\},$$

*where:*

$$\overline{d}_{R\cap K}(s,e) = \begin{cases} \overline{d}_R(s,e) & \text{if} \quad \underline{s}_R \leq s < s_{R\cap K} \\ \overline{d}_K(s,e) & \text{if} \quad s_{R\cap K} \leq s \leq \overline{s}_K \end{cases}$$

*where the threshold* $s_{R\cup K}$ *is given by:*

$$s_{R\cup K} = \frac{1}{1 + \frac{1}{\frac{M(\phi_M)}{(1-\chi)I(\phi_I)} - 1}\frac{1+r_b}{1+r_m}},$$

*and satisfies* $\underline{s}_R < s_{R\cup K} < s_{R,1}$.

*Proof.* Note first that the intersection $\tilde{\mathcal{S}}_K(e) \cap \tilde{\mathcal{S}}_R(e)$ may only contain debt structures with $\underline{s}_R \leq s \leq \overline{s}_K$. Given this, the proof of this proposition proceeds in three steps:

**Step 1:** If $\underline{s}_R \leq s \leq s_{R,1}$, then $\overline{d}_R(s,e) = \underline{d}_K(s,e)$, so that $\tilde{\mathcal{S}}_K(e) \cap \tilde{\mathcal{S}}_R(e)$ contains no elements such that $\underline{s}_R \leq s \leq s_{R,1}$;

**Step 2:** If $s_{R,1} \leq s \leq \overline{s}_K$, then $\overline{d}_R(s,e) > \underline{d}_K(s,e)$, so that $\tilde{\mathcal{S}}_K(e) \cap \tilde{\mathcal{S}}_R(e)$ contains debt structures $(d,s)$ for each $s_{R,1} \leq s \leq \overline{s}_K$;

**Step 3:** There is a unique $s_{R,1} < s_{R\cup K} < \overline{s}_K$ such that $\overline{d}_K(s,e) \geq \overline{d}_R(s,e)$, if and only if, $s \leq s_{R\cup K}$.

**Step 1:** When $\underline{s}_R \leq s \leq s_{R,1}$, using proposition 2.17, $\overline{d}_R(s,e)$ solves:

$$I^{-1}\left(\frac{(1+r_m)(1-s)\overline{d}_R(s,e)}{(1-\chi)(e+\overline{d}_R(s,e))^\zeta}\right) = G^{-1}\left(\frac{(1+r_b)s\overline{d}_R(s,e)}{\chi(e+\overline{d}_R(s,e))^\zeta}\right).$$

This implies that:

$$x_1 \equiv \frac{(1+r_m)(1-s)\overline{d}_R(s,e)}{(1-\chi)(e+\overline{d}_R(s,e))^\zeta} \quad \leq \quad I(\phi_I)$$

$$x_2 \equiv \frac{(1+r_b)s\overline{d}_R(s,e)}{\chi(e+\overline{d}_R(s,e))^{\zeta}} \quad \leq \quad G(\phi_I)$$

Therefore,

$$\begin{aligned}
(1-\chi)x_1 + \chi x_3 = \frac{(1+r_m(1-s)+r_b s)\,\overline{d}_R(s,e)}{\left(e+\overline{d}_R(s,e)\right)^{\zeta}} &\leq (1-\chi)I(\phi_I) + \chi G(\phi_I) \\
&= M(\phi_I) \\
&< M(\phi_M)
\end{aligned}$$

Thus, $M^{-1}\left((1-\chi)+x_1+\chi x_2\right)$ is well-defined. Furthermore:

$$\begin{aligned}
I^{-1}(x_1) = G^{-1}(x_2) &\implies x_2 = G\left(I^{-1}(x_1)\right) \\
&\implies x_2 = x_1 + \int_0^{I^{-1}(x_1)} \phi dF(\phi) \\
&\implies \chi x_2 + (1-\chi)x_1 = M\left(I^{-1}(x_1)\right) \\
&\implies M^{-1}\left(\chi x_2 + (1-\chi)x_1\right) = I^{-1}(x_1) = G^{-1}(x_2)
\end{aligned}$$

Therefore, $\overline{d}_R(s,e)$ also solves:

$$M^{-1}\left(\frac{(1+r_m(1-s)+r_b s)\,\overline{d}_R(s,e)}{\left(e+\overline{d}_R(s,e)\right)^{\zeta}}\right) = G^{-1}\left(\frac{(1+r_b)s\overline{d}_R(s,e)}{\chi(e+\overline{d}_R(s,e))^{\zeta}}\right),$$

so that $\overline{d}_R(s,e) = \underline{d}_K(s,e)$. This concludes the proof of step 1.

**Step 2:** For $s_{R,1} \leq s \leq \overline{s}_K$, $\underline{d}_K(s,e)$ solves:

$$M^{-1}\left(\frac{(1+r_m(1-s)+r_b s)\,\underline{d}_K(s,e)}{(e+\underline{d}_K(s,e))^{\zeta}}\right) = G^{-1}\left(\frac{(1+r_b)s\underline{d}_K(s,e)}{\chi(e+\underline{d}_K(s,e))^{\zeta}}\right),$$

while $\overline{d}_R(s,e)$ solves one of the two conditions:

$$
\begin{aligned}
\frac{(1+r_m)(1-s)\overline{d}_R(s,e)}{(1-\chi)(e+\overline{d}_R(s,e))^{\zeta}} &= I(\phi_I) \\
&\text{or} \\
\frac{(1+r_b)s\overline{d}_R(s,e)}{\chi(e+\overline{d}_R(s,e))^{\zeta}} &= \mathbb{E}(\phi).
\end{aligned}
$$

In order to show that $\overline{d}_R(s,e) > \underline{d}_K(s,e)$, we need to prove that at $d = \underline{d}_K(s,e)$, we have:

$$
\begin{aligned}
\frac{(1+r_m)(1-s)\underline{d}_K(s,e)}{(1-\chi)(e+\underline{d}_K(s,e))^{\zeta}} &< I(\phi_I) \\
&\text{and} \\
\frac{(1+r_b)s\underline{d}_K(s,e)}{\chi(e+\underline{d}_K(s,e))^{\zeta}} &< \mathbb{E}(\phi).
\end{aligned}
$$

The second condition follows from the fact that because of the definition of $\underline{d}_K(s,e)$, it must be the case that:

$$x_2 \equiv \frac{(1+r_b)s\underline{d}_K(s,e)}{\chi(e+\underline{d}_K(s,e))^{\zeta}} \leq G(\phi_M) < \mathbb{E}(\phi).$$

Furthermore, let:

$$x_3 = \frac{(1+r_m(1-s)+r_b s)\underline{d}_K(s,e)}{(e+\underline{d}_K(s,e))^{\zeta}} = M\left(G^{-1}(x_2)\right)$$

Then:

$$
\begin{aligned}
x_1 \equiv \frac{(1+r_m)(1-s)\underline{d}_K(s,e)}{(1-\chi)(\underline{d}_K(s,e)+e)^{\zeta}} &= \frac{1}{1-\chi}\left(x_3 - \chi x_2\right) \\
&= \frac{1}{1-\chi}\left(M\left(G^{-1}(x_2)\right) - \chi x_2\right)
\end{aligned}
$$

Note that $\forall 0 \leq x \leq \phi_M$, $M(x) = \chi G(x) + (1-\chi)x(1-F(x))$. Therefore:

$$\begin{aligned} M\left(G^{-1}(x_2)\right) &= \chi x_2 + (1-\chi)G^{-1}(x_2)\left(1 - F\left(G^{-1}(x_2)\right)\right) \\ M\left(G^{-1}(x_2)\right) - \chi x_2 &= (1-\chi)G^{-1}(x_2)\left(1 - F\left(G^{-1}(x_2)\right)\right) \\ x_1 &= G^{-1}(x_2)\left(1 - F\left(G^{-1}(x_2)\right)\right) < I(\phi_I) \end{aligned}$$

where the last inequality follows because $I(x) = x(1-F(x))$ is maximized at $\phi_I$. This concludes the proof of step 2.

**Step 3:** Define $s_{R\cap K}$ as in the text of the proposition. I first prove that $\hat{s}_{R,1} < s_{R\cap K} < \hat{s}_{R,2}$. Note that $s_{R,1} < s_{R\cap K}$, if and only if:

$$M(\phi_M) > \chi G(\phi_I) + (1-\chi)I(\phi_I).$$

This statement is true because $M(\phi_M) > M(\phi_I) = \chi G(\phi_I) + (1-\chi)I(\phi_I)$. Next, $s_{R\cap K} < s_{R,2}$, if and only if,

$$M(\phi_M) > \chi \mathbb{E}(\phi) + (1-\chi)I(\phi_I).$$

To prove this inequality, define $\Delta(\chi) = M(\phi_M) - \chi\mathbb{E}(\phi) - (1-\chi)I(\phi_I)$ for $\chi \in [0,1]$. Note that:

$$\begin{aligned} \Delta(0) = \Delta(1) &= 0 \\ \frac{\partial \Delta}{\partial \chi} &= \int_0^{\phi_M} x dF(x) - \mathbb{E}(\phi) + I(\phi_I) \\ \frac{\partial \Delta}{\partial \chi}|_{\chi=0} &= G(\phi_I) - \mathbb{E}(\phi) < 0 \\ \frac{\partial \Delta}{\partial \chi}|_{\chi=1} &= I(\phi_I) > 0 \end{aligned}$$

$$\frac{\partial^2 \Delta}{\partial \chi^2} = \frac{\partial \phi_M}{\partial \chi} \phi_M f(\phi_M) > 0$$

Taken together, these properties imply that $\Delta(\chi)$ is strictly convex on $[0,1]$, and that $\forall \chi \in ]0,1]$, $\Delta(\chi) < 0$, so that the initial inequality holds, and $s_{R\cap K} < s_{R,2}$.

Next, I prove that $\overline{d}_K(s,e) \geq \overline{d}_R(s,e)$, if and only if, $s \leq s_{R\cup K}$. First note that by definition of $s_{R\cap K}$, $\overline{d}_B(s_{R\cap K}) = \overline{d}_R(s_{R\cap K})$. To prove the statement, it is therefore sufficient to show that $\overline{d}_K(s,e) - \overline{d}_R(s,e)$ is strictly decreasing in $s$. To show this, I establish that $\overline{d}_K(s,e)$ is strictly decreasing in $s$, and that $\overline{d}_R(s,e)$ is strictly increasing in $s$. The implicit definition of $\overline{d}_K(s,e)$ implies that:

$$\begin{aligned}
\frac{\partial \overline{d}_K(s,e)}{\partial s} &= \frac{-\overline{d}_K(s,e)(r_b - r_m)}{1 + (1-s)r_m + sr_b - \zeta M(\phi_M)(e + \overline{d}_K(s,e))^\zeta} \\
&= \frac{-\overline{d}_K(s,e)(e + \overline{d}_K(s,e))(r_b - r_m)}{(1 + (1-s)r_m + sr_b)e + (1-\zeta)M(\phi_M)(e + \overline{d}_K(s,e))^\zeta} < 0,
\end{aligned}$$

where the last inequality follows from the fact that $r_b > r_m$. Likewise, using the implicit definition of $\overline{d}_R(s,e)$, one obtains that:

$$\frac{\partial \overline{d}_R(s,e)}{\partial s} = \frac{\overline{d}_R(s,e)(e + \overline{d}_R(1 + r_m)}{\frac{(1+r_m)(1-s)}{1-\chi}e + (1-\zeta)I(\phi_I)(e + \overline{d}_R(s,e))^\zeta} > 0$$

This concludes the proof of step 3.

Step 3 shows that elements of $\tilde{\mathcal{S}}_K(e) \cap \tilde{\mathcal{S}}_R(e)$ with $s < s_{R\cup K}$ are therefore exactly those with $\underline{d}_K(s,e) \leq s \leq \overline{d}_R(s,e)$; while elements of $\tilde{\mathcal{S}}_K(e) \cap \tilde{\mathcal{S}}_R(e)$ with $s \geq s_{R\cup K}$ are *all elements* of $\tilde{S}_B(e)$, that is, those with $\underline{d}_K(s,e) \leq s \leq \overline{d}_K(s,e)$. This is the main statement of proposition B.6.

$\square$

## B.2 Appendix to section 2.3

*Proof of proposition 2.12.* Assume that $(b = sd, m = (1-s)d) \in \mathcal{S}_K(e)$ and that $((1 + r_m)(1-s) + (1+r_b)s)d \geq (1-\delta)(e+d)$; in that case, using the expression of the return function of the entrepreneur under a K-contract reported in appendix B.1:

$$\mathbb{E}\left[\tilde{\pi}(\phi; e, sd, (1-s)d)\right] = \left(\int_{\phi_K}^{+\infty} \phi dF(\phi)\right)(e+d)^{\zeta} - \phi_K(1-F(\phi_K))(e+d)^{\zeta}$$

where:

$$\phi_K = \frac{R_K(d,s,e) - (1-\delta)(e+d)}{(e+d)^{\zeta}}.$$

Using the results of lemmas B.2-B.4, we then have that:

$$\begin{aligned} & \mathbb{E}\left[\tilde{\pi}(\phi; e, sd, (1-s)d)\right] + \tilde{E}_{m,K}(R_b, R_m; e+d) + \tilde{E}_b(R_b; e+d) \\ = & \left(\textstyle\int_{\phi_K}^{+\infty} \phi dF(\phi)\right)(e+d)^{\zeta} - \phi_K(1-F(\phi_K))(e+d)^{\zeta} \\ & +\chi\left(\textstyle\int_0^{\phi_K} \phi dF(\phi)\right)(e+d)^{\zeta} - (1-\chi)F(\phi_K)(1-\delta)(e+d) \\ = & \mathbb{E}(\pi(\phi)) - (1-\chi)\textstyle\int_0^{\phi_K} \pi(\phi)dF(\phi). \end{aligned}$$

Since $\tilde{\mathbb{E}}_{m,K}(R_b, R_m; e+d) = (1+r_m)(1-s)d$ and $\tilde{\mathbb{E}}_b(R_b; e+d) = (1+r_b)sd$, this in turn implies that:

$$\mathbb{E}\left[\tilde{\pi}(\phi; e, sd, (1-s)d)\right] = \mathbb{E}(\pi(\phi)) - (1-\chi)\int_0^{\phi_K} \pi(\phi)dF(\phi) - ((1+r_m)(1-s) + (1+r_b)s)d.$$

When instead $((1+r_m)(1-s)+(1+r_b)s)d < (1-\delta)(e+d)$, the entrepreneur never defaults, and moreover $R_b = (1+r_b)sd$ and $R_m = (1+r_m)(1-s)d$. Thus:

$$\mathbb{E}\left[\tilde{\pi}(\phi; e, sd, (1-s)d)\right] = \mathbb{E}(\pi(\phi)) - (1+r_m)(1-s)d - (1+r_b)sd.$$

This proves the lemma's claim in the case $(b = sd, m = (1-s)d) \in S_K(e)$, with:

$$\overline{\phi}(e,d,s) = \frac{R_K(d,s,e) - (1-\delta)(e+d)}{(e+d)^\zeta},$$

where note that $\overline{\phi}(e,b,m) = 0$, if and only if, $(1-\delta)(e+b+m) \geq (1+r_b)b + (1+r_m)m$.

The proof is similar when $(b = sd, m = (1-s)d) \in \mathcal{S}_R(e)$, with in that case:

$$\overline{\phi}(e,d,s) = \frac{R_{m,l}(d,s,e) - (1-\chi)(1-\delta)(e+d)}{(1-\chi)(e+d)^\zeta}.$$

□

*Proof of proposition 2.14.* Consider, first, the sub-problem of a firm with internal finance $e$ restricted to use contracts in $\mathcal{S}_R(e)$. Abusing somewhat the notation of propositions 2.16 and 2.12, this problem can be written as:

$$\max_{(d,s)} \quad O_R(e,d,s) = \mathbb{E}(\pi(\phi)) - (1+r_b)sd - (1+r_m)(1-s)d - (1-\chi)\int_0^{\underline{\phi}(e,d,s)} \pi(\phi)dF(\phi)$$

$$\text{s.t. } \underline{s}_R \leq s \leq 1 \text{ and } 0 \leq d \leq \overline{d}_R(s,e)$$

The Lagrangian associated with this problem is:

$$\mathcal{L} = O_R(e,d,s) + \bar{\lambda}(\bar{d}(s,e) - d) + \underline{\lambda}d + \overline{\mu}(1-s) + \underline{\mu}(s - \underline{s}_R).$$

A first-order necessary conditions for optimality is:

$$\frac{\partial O_R}{\partial s} + \frac{\partial \bar{d}_R}{\partial s}\overline{\lambda} = (\overline{\mu} - \underline{\mu}),$$

where the derivative of the objective function with respect to $s$ is given by:

$$\frac{\partial O_R}{\partial s} = (r_m - r_b)d + \underline{\phi}(e,d,s)f(\underline{\phi}(e,d,s))(1+r_m)d(1-s)\frac{\partial I^{-1}}{\partial y}\left(\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^{\zeta}}\right).$$

Since $\frac{\partial I^{-1}}{\partial y} > 0$, we have that $\frac{\partial O_R}{\partial s} > 0$ if $r_b - r_m < 0$. Note, then, that if $\underline{s} \leq s \leq s_{R,M}$, then by theorem 2.16, $\frac{\partial \bar{d}_R}{\partial s} \geq 0$, so that $\overline{\mu} > 0$ and necessarily $s = 1$. This is a contradiction because $s_{R,M} < 1$. So, when $r_b - r_m < 0$, any solution to the restricted problem above must satisfy $s_{R,M} \leq s \leq 1$.

Then, there are two possibilities:

- Either the bank borrowing constraint is binding, that is, $\overline{\lambda} > 0$; in that case, it need not be that $\bar{\mu} > 0$, that is, the firm may still be borrowing from both sources although it is exhausting its bank borrowing capacity;

- The bank borrowing constraint is loose, that is $\overline{\lambda} = 0$. In that case, it must be that $\overline{\mu} > 0$, so that $s = 1$ and the firm is borrowing only from banks.

This establishes that, provided that the solution of the firm's problem is an element of $\mathcal{S}_R(e)$, then the firm is either entirely bank-financed, or its bank borrowing constraint is binding.

I next show that the solution to the firm's problem is necessarily an element of $\mathcal{S}_R(e)$. To establish this, let $\tilde{d}$ be a scale of borrowing that is feasible in $\mathcal{S}_K(e)$, that is, such that there exists $\tilde{s} \in [0, \bar{s}_K]$ such that $(\tilde{d}, \tilde{s}) \in \mathcal{S}_K(e)$. First, note that, for any $s$ such that $\tilde{d}$ is feasible in $\mathcal{S}_K(e)$:

$$\frac{\partial O_K}{\partial s}(e,\tilde{d},s) = (r_m - r_b)\tilde{d}\left(1 + (1-\chi)\underline{\phi}(\tilde{d},\tilde{s},e)f\left(\underline{\phi}(\tilde{d},\tilde{s},e)\right)\frac{\partial M^{-1}}{\partial y}\left(\frac{(1+r_m(1-s)+r_b s)\tilde{d}}{(e+\tilde{d})^{\zeta}}\right)\right) > 0, \tag{B.13}$$

where the latter inequality holds because $r_b < r_m$.

Furthermore, using the results of proposition 2.8, the joint borrowing constraint defining $\mathcal{S}_K(e)$ can be written as:

$$0 \leq (1 + r_b s + r_m(1-s))d \leq \hat{M}(e+d)^{\zeta}$$

When $r_b < r_m$, if $(\tilde{d}, \tilde{s})$ satisfy this constraint, then $(\tilde{d}, s)$ also satisfy it for any $s \geq \tilde{s}$. Therefore, all the points on the line defined, in $(d, s)$ space, by $d = \tilde{d}$ (that is, an anti-diagonal in $(b, m)$ where $b + m$ is a constant) are in the set $S_K(e)$ so long as they satisfy condition (frontier-K) from proposition 2.8. On this line, the objective function $O_K$ is increasing, as noted above. Finally, it is straightforward to show that, when the condition (frontier-K) is binding:

$$\underline{\phi}_K(e, \tilde{d}, s) = M^{-1}\left(\frac{(1 + r_b s + r_m(1-s))\tilde{d}}{(e+\tilde{d})^{\zeta}}\right) = I^{-1}\left(\frac{(1 + r_m(1-s))\tilde{d}}{(1-\chi)(e+\tilde{d})^{\zeta}}\right) = \underline{\phi}_R(e, \tilde{d}, s),$$

so that at the frontier between the two sets, $O_K(\tilde{d}, s, e) = O_R(\tilde{d}, s, e)$. Thus, the value of the firm at the point $(\tilde{d}, s) \in \mathcal{S}_R(e)$ at the intersection of the line $d = \tilde{d}$ and the frontier between the sets $S_K(e)$ and $S_R(e)$, is weakly larger than at any point on that line within the set $S_K(e)$. Since I established this for any $\tilde{d}$ feasible in $S_K(e)$, this shows that the global maximum to the firms' problem is in $S_R(e)$, and concludes the proof. □

*Proofs of propositions 2.16 and 2.17*

In order to establish the different claims of propositions 2.16 and 2.17, I first characterize the solution to the two subproblems:

$$\begin{aligned}
\max_{(d,s)} \quad O_K(e,d,s) \quad &= \quad \mathbb{E}(\pi(\phi)) - (1+r_b)sd - (1+r_m)(1-s)d - (1-\chi)\int_0^{\underline{\phi}_K(e,d,s)} \pi(\phi)dF(\phi) \\
\text{s.t.} \quad & 0 \leq (1+r_b)sd + (1+r_m)(1-s)d \leq \hat{M}(e+d)^{\zeta} \\
& G^{-1}\left(\frac{(1+r_b)sd}{\chi(e+d)^{\zeta}}\right) < M^{-1}\left(\frac{(1+r_b s + r_m(1-s))d}{(e+d)^{\zeta}}\right)
\end{aligned} \tag{B.14}$$

$$\begin{aligned}
\max_{(d,s)} \quad O_R(e,d,s) \quad &= \quad \mathbb{E}(\pi(\phi)) - (1+r_b)sd - (1+r_m)(1-s)d - (1-\chi)\int_0^{\underline{\phi}_R(e,d,s)} \pi(\phi)dF(\phi) \\
\text{s.t.} \quad & 0 \leq (1+r_b)sd \leq \chi\mathbb{E}(\phi)(e+d)^{\zeta} \\
& 0 \leq (1+r_m)(1-s)d \leq (1-\chi)\hat{I}(e+d)^{\zeta} \\
& I^{-1}\left(\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^{\zeta}}\right) < G^{-1}\left(\frac{(1+r_b)sd}{\chi(e+d)^{\zeta}}\right)
\end{aligned} \tag{B.15}$$

*Characterization of the solution to subproblem (B.14)* Let $(d,s)$ be a feasible debt structure in subproblem (B.14) with $s > 0$. Then, note that $G(0) < M^{-1}\left(\frac{(1+r_m)d}{(e+d)^{\zeta}}\right)$, so that $(d,0)$ is also feasible. Furthermore, $O_K(e,d,0) > O_K(e,d,s)$ because $\frac{\partial O_K}{\partial s} > 0$ when $r_b > r_m$, as indicated by equation (B.13). Thus, the maximum of subproblem (B.14) necessarily satisfies $\hat{s} = 0$, that is, is entirely market-financed. Moreover, the optimal amount of borrowing $\hat{d}$ is necessarily an interior solution, i.e. satisfies $(1+r_m)\hat{d} < \hat{M}(e+\hat{d})^{\zeta}$, so long as $e < \hat{k}_m$, where $\hat{k}_m$ is the unconstrained optimal size of the firm when liquidation is costless, as defined in lemma 2.13. To establish this, note that when $e < \hat{k}_m$:

$$\frac{\partial O_K}{\partial d}(e,0,0) = \zeta E(\phi)(e+d)^{\zeta-1} - (1+r_m) > 0.$$

Furthermore, the derivative of the objective function is give by:

$$
\begin{aligned}
\frac{\partial O_K}{\partial d}(e,d,0) &= \zeta E(\phi)(e+d)^{\zeta-1} - (1+r_m) - (1-\chi)\zeta(e+d)^{\zeta-1}\int_0^{\underline{\phi}_K(e,d,s)} \phi dF(\phi) \qquad \text{(B.16)}\\
&\quad -(1+r_m)(1-\chi)\underline{\phi}_K(e,d,s) f\left(\underline{\phi}_K(e,d,s)\right)\frac{\partial M^{-1}}{\partial y}\left(\frac{(1+r_m)d}{(e+d)^\zeta}\right)\left[1-\zeta\frac{d}{e+d}\right]
\end{aligned}
$$

In this last expression, all terms are bounded as $d \to \overline{d}_K(e)$, except the derivative $\frac{\partial M^{-1}}{\partial y}$, since $\lim_{y\to\hat{M}} \frac{\partial M^{-1}}{\partial y} = +\infty$. Thus, $\lim_{d\to\overline{d}_K(e)} \frac{\partial O_K}{\partial d}(e,d,0) = -\infty$. This establishes that the maximum in subproblem (B.14) is always such that $0 < \hat{d} < \overline{d}_K(e)$, and furthermore that it is attained when the derivative in equation (B.16) is equal to 0.

*Characterization of the solution to subproblem (B.15)* I start the analysis of subproblem (B.15) by characterizing its unconstrained (”interior”) solution $(d_{int}, s_{int})$. Manipulating the necessary first-order condition $\frac{\partial O_R}{\partial s} = 0$, the unconstrained solution must statisfy:

$$
I^{-1}\left(\frac{(1+r_m)s_{int}d_{int}}{(1-\chi)(e+d_{int})^\zeta}\right) = \phi_{int}
$$

where $\phi_{int}$ is the unique solution to:

$$
\frac{\phi_{int} f\left(\phi_{int}\right)}{1-F\left(\phi_{int}\right)} = \frac{r_b - r_m}{1+r_m}.
$$

Using this expression along with the necessary first-order condition $\frac{\partial O_R}{\partial d} = 0$, the interior solution is given by:

$$
\begin{aligned}
\hat{s}_{int}(e) &= 1 - \frac{\Gamma}{1+r_m}\frac{\hat{k}_{int}^\zeta}{\hat{k}_{int}-e}\\
\hat{d}_{int}(e) &= \hat{k}_{int} - e\\
\hat{O}_{R,int}(e) &= \Xi(1-\zeta)\hat{k}_{int}^\zeta + (1+r_b)e
\end{aligned}
$$

$$\begin{aligned}\hat{k}_{int} &= \left(\frac{\zeta\Xi}{1+r_b}\right)^{\frac{1}{1-\zeta}} \\ \Gamma &= (1-\chi)I\left(\phi_{int}\right) \\ \Xi &= \mathbb{E}(\phi) + (1-\chi)\left[\frac{r_b - r_m}{1+r_m} I\left(\phi_{int}\right) - \int_0^{\phi_{int}} \phi dF(\phi)\right]\end{aligned}$$

Provided that the interior solution does not violate any of the constraints in subproblem (B.15), it is the maximum of this problem. Note that, since $\phi_{int} < \hat{\phi}_I$, the second inequality in this problem (the market borrowing constraint) always holds strictly, so we need only study the two other inequalities, the bank borrowing constraint and the frontier with the set $S_K(e)$.[1]

By evaluating the bank borrowing constraint at the interior solution, one can establish that the bank borrowing constraint is binding at the interior solution, if and only if, $e \leq \tilde{e}$, where:

$$\tilde{e} = \hat{k}_{int}\left(1 - \frac{1}{\zeta}\frac{1}{1 + \frac{(1-\chi)(\mathbb{E}(\phi) - G(\phi_{int}))}{\chi\mathbb{E}(\phi) + (1-\chi)\frac{1+r_b}{1+r_m}I(\phi_{int})}}\right) \tag{B.17}$$

Along similar lines, the third constraint in subproblem (B.15), the frontier condition, is binding at the interior solution, if and only if $e > \tilde{e}^F$, where:

$$\tilde{e}^F = \hat{k}_{int}\left(1 - \frac{1}{\zeta}\frac{\frac{\chi G(\phi_{int}) + (1-\chi)\frac{1+r_b}{1+r_m}I(\phi_{int})}{\chi\mathbb{E}(\phi) + (1-\chi)\frac{1+r_b}{1+r_m}I(\phi_{int})}}{1 + \frac{(1-\chi)(\mathbb{E}(\phi) - G(\phi_{int}))}{\chi\mathbb{E}(\phi) + (1-\chi)\frac{1+r_b}{1+r_m}I(\phi_{int})}}\right) > \tilde{e} \tag{B.18}$$

Then, there are three possible cases:

- If $0 \leq e \leq \tilde{e}$, then, at the optimum of subproblem (B.15), the bank borrowing constraint is binding. (Indeed, note that, as long as a point satisfies the bank borrowing constraint,

[1] The inequality $\phi_{int} < \hat{\phi}_I$ is true so long as $r_b < 1 + 2r_m$, which it will under any reasonable calibration of the model.

it also satisfies the third constraint in subproblem (B.15), the frontier condition);

- If $\tilde{e} \leq e \leq \tilde{e}^F$, then the optimum of the subproblem is the interior solution described above;

- If $e > \tilde{e}^F$, then at the optimum of the subproblem (B.15), the third constraint in subproblem (B.15), the frontier condition, is binding.

Note that, as in the proof of proposition 2.12, it is straightforward to show that when $r_b > r_m$, $\frac{\partial O_r}{\partial s} < 0$ on the frontier defined by the third condition in subproblem (B.15). Thus, when $e > \tilde{e}^F$, the solution to subproblem (B.15) is dominated, locally, by a feasible point in subproblem (B.14), and therefore cannot be a the global solution to the firm's problem. This indicates that the threshold for switching from mixed to market-finance $e^*$ must satisfy $e^* < \tilde{e}^F$.

*The switching threshold $e^*$* In what follows, I prove that $\hat{O}_R(0) > \hat{O}_K(0)$. Since, as established before, $\hat{O}_R\left(\tilde{e}^F\right) < \hat{O}_K\left(\tilde{e}^F\right)$, and since both maximized functions are continuous in $e$, this establishes that there exists a switching threshold $e^*$ such that: $\hat{O}_R(e^*) = \hat{O}_K(e^*)$.

Note first that the objective function in the subproblem (B.14) can be written as:

$$O_K(d,0,0) = \left(E(\phi) - G\left(M^{-1}\left((1+r_m)d^{1-\zeta}\right)\right)\right)d^{\zeta}.$$

This objective function is defined for $0 \leq d \leq \overline{d}_K(0)$, or:

$$0 \leq d \leq \left(\frac{\hat{M}}{1+r_m}\right)^{\frac{1}{1-\zeta}}.$$

Alternatively, using the change of variable $\phi_K = M^{-1}\left((1+r_m)d^{1-\zeta}\right)$, this problem can reformulated as:

$$\max_{\phi_K \in [0,\phi_M]} \left[\int_{\phi_K}^{+\infty} (1 - F(\phi))\, d\phi\right] \left[\frac{1}{1+r_m} M(\phi_K)\right]^{\frac{\zeta}{1-\zeta}}.$$

The first order necessary condition for this problem is:

$$\frac{\zeta}{1-\zeta} \left[\int_{\phi_K}^{+\infty} (1 - F(\phi))\, d\phi\right] \left[1 - (1-\chi)\frac{\phi_K f(\phi_K)}{1 - F(\phi_K)}\right] = M(\phi_K). \tag{B.19}$$

In a similar manner, the objective function in the subproblem (B.15), when the bank borrowing constraint binds, can be written as:

$$O_R(d, 0, 0) = (1-\chi)\left(E(\phi) - G\left(I^{-1}\left(\frac{1+r_m}{1-\chi} d^{1-\zeta} - \frac{1+r_m}{1+r_b}\frac{\chi}{1-\chi}\mathbb{E}(\phi)\right)\right)\right) d^{\zeta}.$$

Given the results of proposition B.5, this objective function is defined for $1 \geq s \geq s_{R,2}$, or (given that the borrowing constraint binds):

$$\left(\frac{\chi \mathbb{E}(\phi)}{1+r_b}\right)^{\frac{1}{1-\zeta}} \leq d \leq \left(\frac{\chi \mathbb{E}(\phi)}{1+r_b} + \frac{(1-\chi) I(\phi_I)}{1+r_m}\right)^{\frac{1}{1-\zeta}}.$$

Using the change of variable $\phi_R = I^{-1}\left(\frac{1+r_m}{1-\chi} d^{1-\zeta} - \frac{1+r_m}{1+r_b}\frac{\chi}{1-\chi}\mathbb{E}(\phi)\right)$, this problem can reformulated as:

$$\max_{\phi_R \in [0,\phi_I]} (1-\chi) \left[\int_{\phi_R}^{+\infty} (1 - F(\phi))\, d\phi\right] \left[\frac{\chi \mathbb{E}(\phi)}{1+r_b} + \frac{(1-\chi) I(\phi_R)}{1+r_m}\right]^{\frac{\zeta}{1-\zeta}}.$$

The first order necessary condition for this problem is:

$$\frac{\zeta}{1-\zeta}(1-\chi) \left[\int_{\phi_R}^{+\infty} (1 - F(\phi))\, d\phi\right] \left[1 - \frac{\phi_R f(\phi_R)}{1 - F(\phi_R)}\right] = (1-\chi) I(\phi_R) + \frac{1+r_m}{1+r_b}\chi \mathbb{E}(\phi). \tag{B.20}$$

Given the assumption that $F(.)$ has a strictrly increasing hazard rate, the left hand side of B.20 is strictly decreasing, and moreover it always strictly smaller than the left hand side of B.19 (which is itself strictly decreasing). Since $M(\phi) = (1-\chi)I(\phi) + \chi G(\phi) \leq (1-\chi)I(\phi) + \chi\mathbb{E}(\phi)$, the right hand side of B.20 is always strictly larger than the right hand side of B.19; both are moreover strictly increasing.[2] This implies that: $\hat{\phi}_K > \hat{\phi}_R$ (ie, the optimal K-debt structure results in more frequent liquidation). Moreover, it also implies that: $(1-\chi)I(\hat{\phi}_R) + \frac{1+r_m}{1+r_b}\chi\mathbb{E}(\phi) > M(\hat{\phi}_K)$ (the optimal R-debt structure involves firms operating at larger scales). Combining the two facts and using the expressions of the objective functions then establishes that $\hat{O}_R(0) > \hat{O}_K(0)$.

[2]This is true provided that $r_b$ and $r_m$ are sufficiently close.

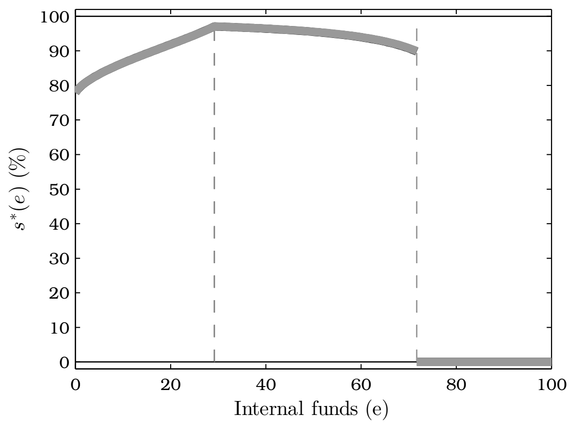
## B.3 Appendix to section 2.4

### B.3.1 Calibrations

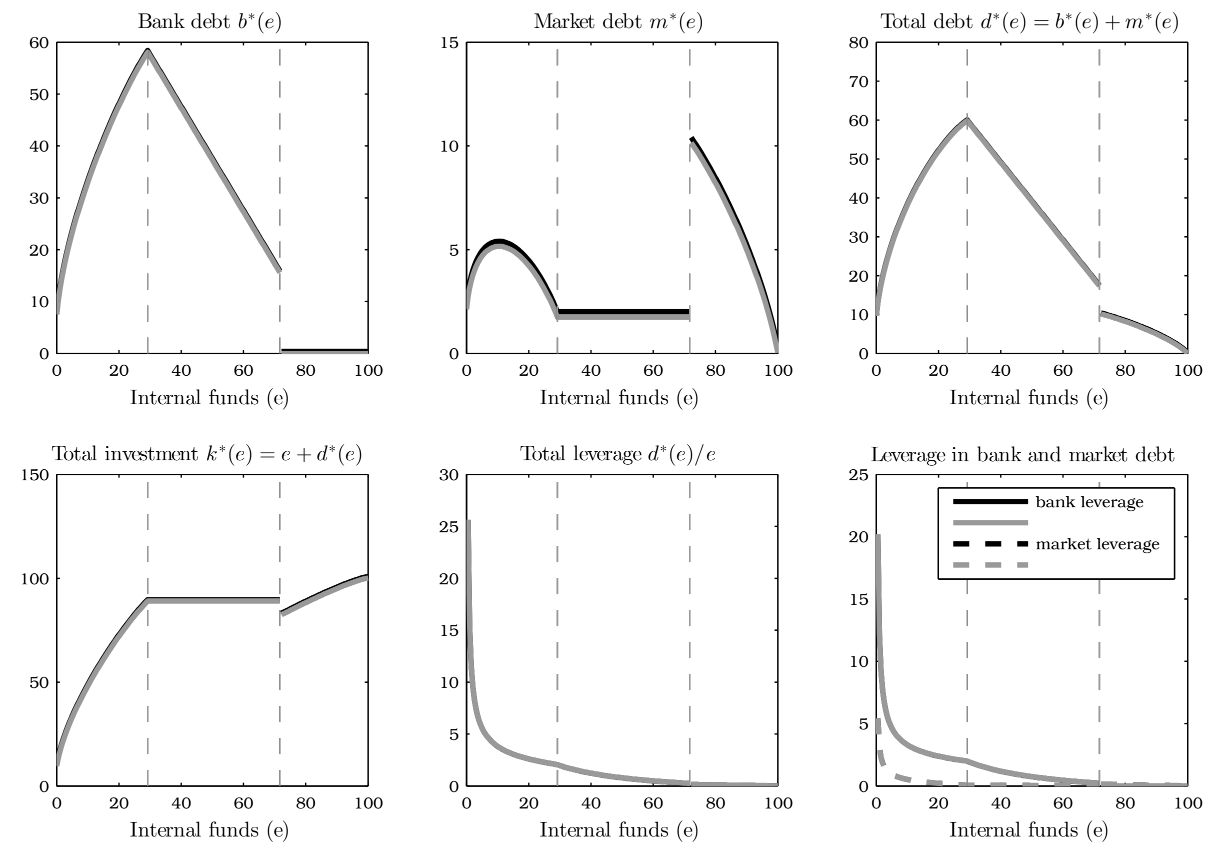
| | Baseline | High cost spread | High cost levels (1) | High cost levels (2) | Low average productivity | High productivity dispersion |
|---|---|---|---|---|---|---|
| Parameters: | | | | | | |
| $r_m$ | 0.05 | 0.05 | 0.06 | 0.06 | 0.05 | 0.05 |
| $r_b$ | 0.06 | 0.07 | 0.07 | 0.07 | 0.06 | 0.06 |
| $\chi$ | 0.60 | 0.60 | 0.60 | 0.60 | 0.60 | 0.60 |
| $\xi$ | 1.60 | 1.60 | 1.60 | 1.60 | 1.60 | 1.37 |
| $\lambda$ | 1.84 | 1.84 | 1.86 | 1.84 | 1.82 | 1.82 |
| $\zeta$ | 0.92 | 0.92 | 0.92 | 0.92 | 0.92 | 0.92 |
| Moments of $F(.)$: | | | | | | |
| $\mathbb{E}(\phi)$ | 1.65 | 1.65 | 1.67 | 1.65 | 1.63 | 1.65 |
| $\sigma(\phi)$ | 1.06 | 1.06 | 1.07 | 1.06 | 1.05 | 1.22 |
| $\frac{\sigma(\phi)}{\mathbb{E}(\phi))}$ | 0.64 | 0.64 | 0.64 | 0.64 | 0.64 | 0.74 |
| Shape of solution: | | | | | | |
| $s^*(0)$ | 0.78 | 0.78 | 0.78 | 0.78 | 0.78 | 0.79 |
| $\tilde{e}$ | 29.19 | 25.20 | 29.24 | 25.97 | 25.81 | 29.81 |
| $s^*(\tilde{e})$ | 0.97 | 0.96 | 0.97 | 0.97 | 0.97 | 0.98 |
| $e^*$ | 71.90 | 55.87 | 72.08 | 64.18 | 64.03 | 74.85 |
| $s^*(e^*)$ | 0.90 | 0.90 | 0.90 | 0.90 | 0.90 | 0.92 |
| $k^*$ | 88.94 | 79.31 | 89.04 | 79.28 | 79.09 | 88.90 |
| $\overline{e}$ | 100.00 | 100.00 | 100.00 | 88.83 | 88.40 | 100.00 |

*Table B.1: Calibrations of the model discussed in section 2.4. The parameters $\xi$ and $\lambda$ are the location and scale parameters of the Weibull distribution. $e^*$ refers to the level of equity such that the firm will choose not to borrow from either sources.*

## B.3.2 Comparative statics

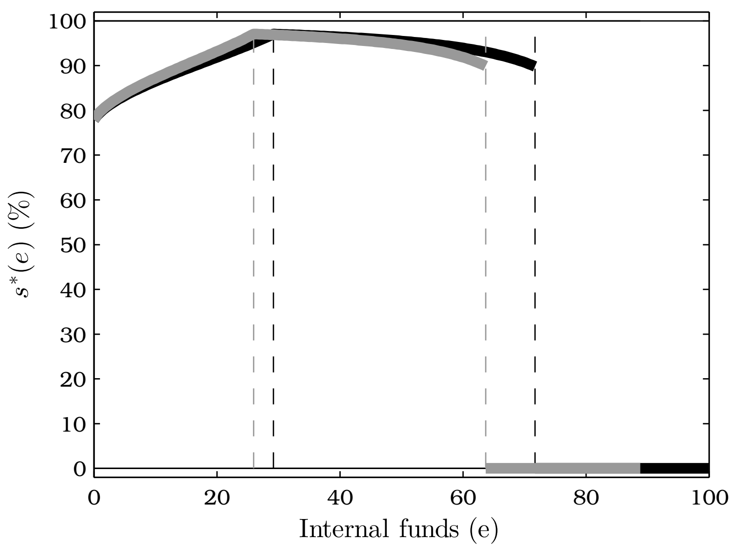



(a) Change in the bank share

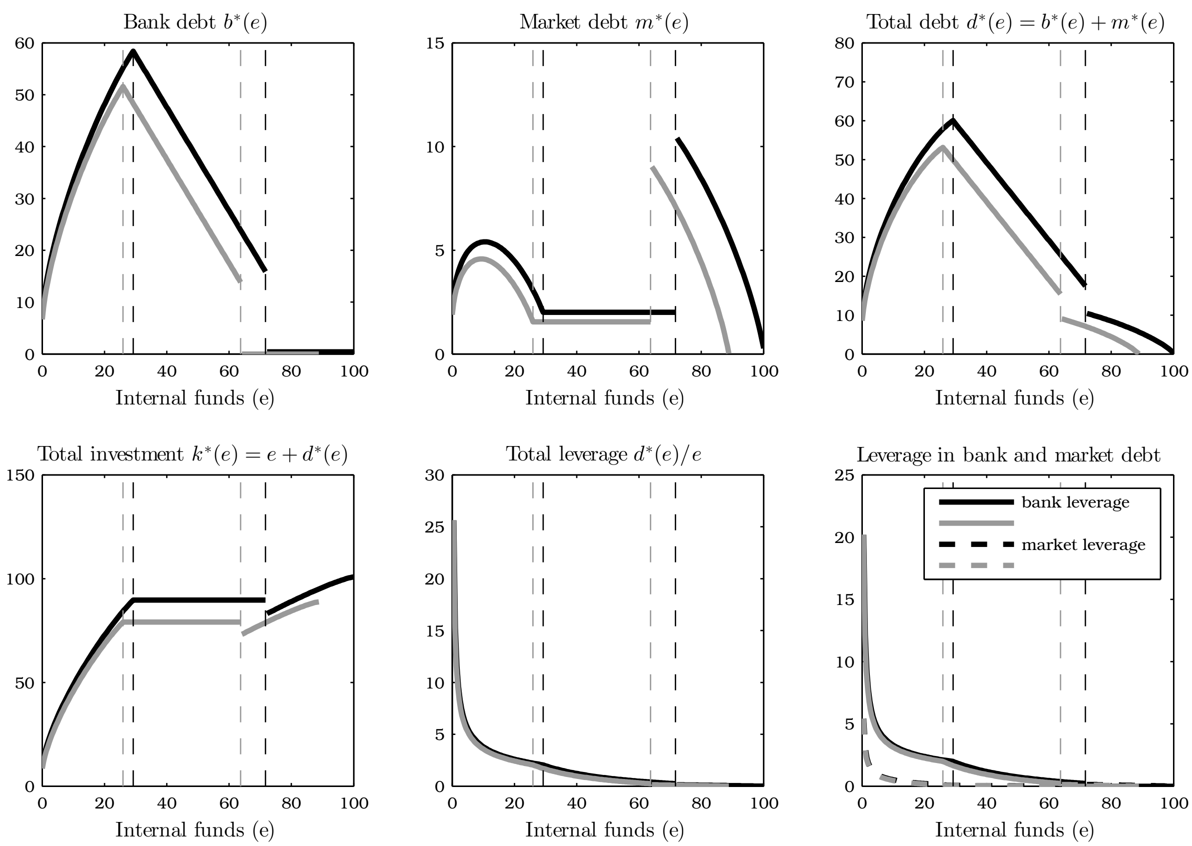



(b) Changes in borrowing

*Figure B.1: The effect of an increase in lending costs: first case. Graphs B.1(a) and B.1(b) report the characteristics of the debt structure in the "Baseline" calibration (black line), and in the "High cost levels (1)" calibration (grey line); see table B.1 for details.*

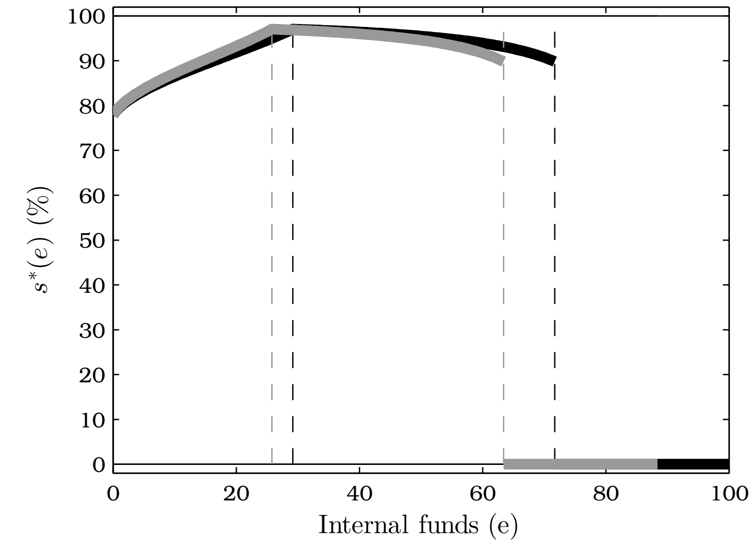



(a) Change in the bank share

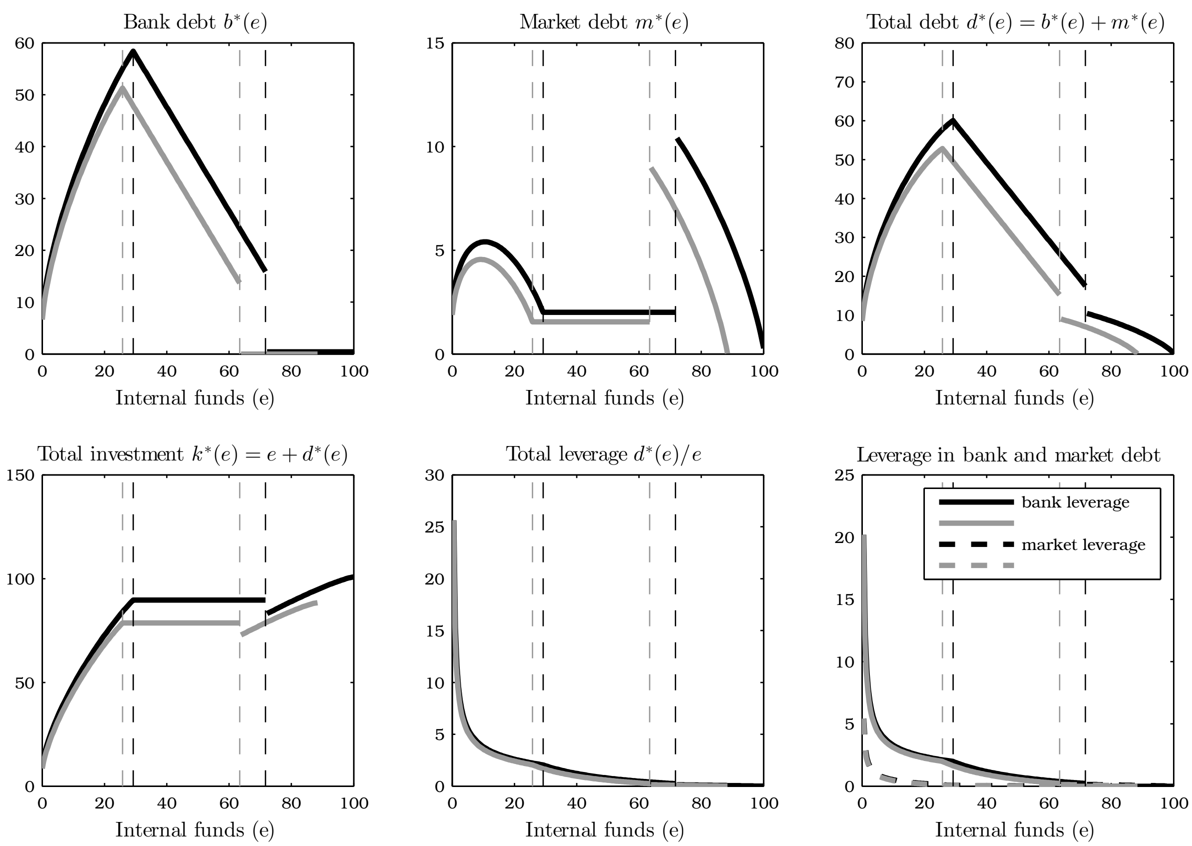



(b) Changes in borrowing

*Figure B.2: The effect of an increase in lending costs: second case. Graphs B.2(a) and B.2(b) report the characteristics of the debt structure in the "Baseline" calibration (black line), and in the "High cost levels (2)" calibration; see table B.1 for details.*

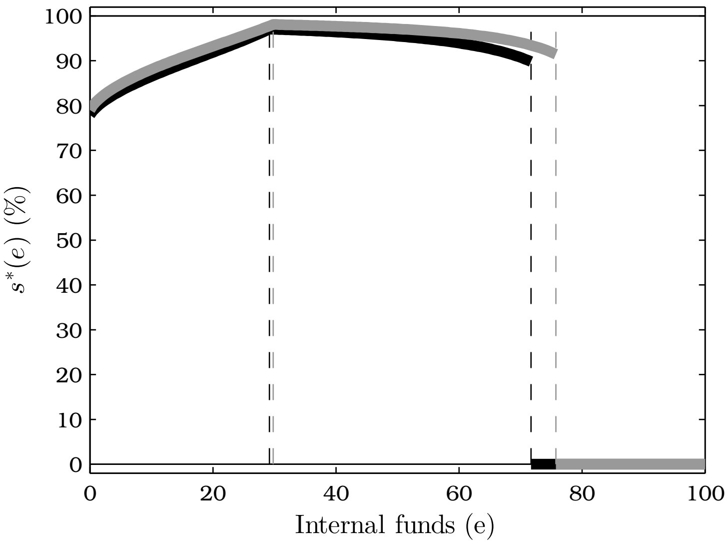



(a) Change in the bank share

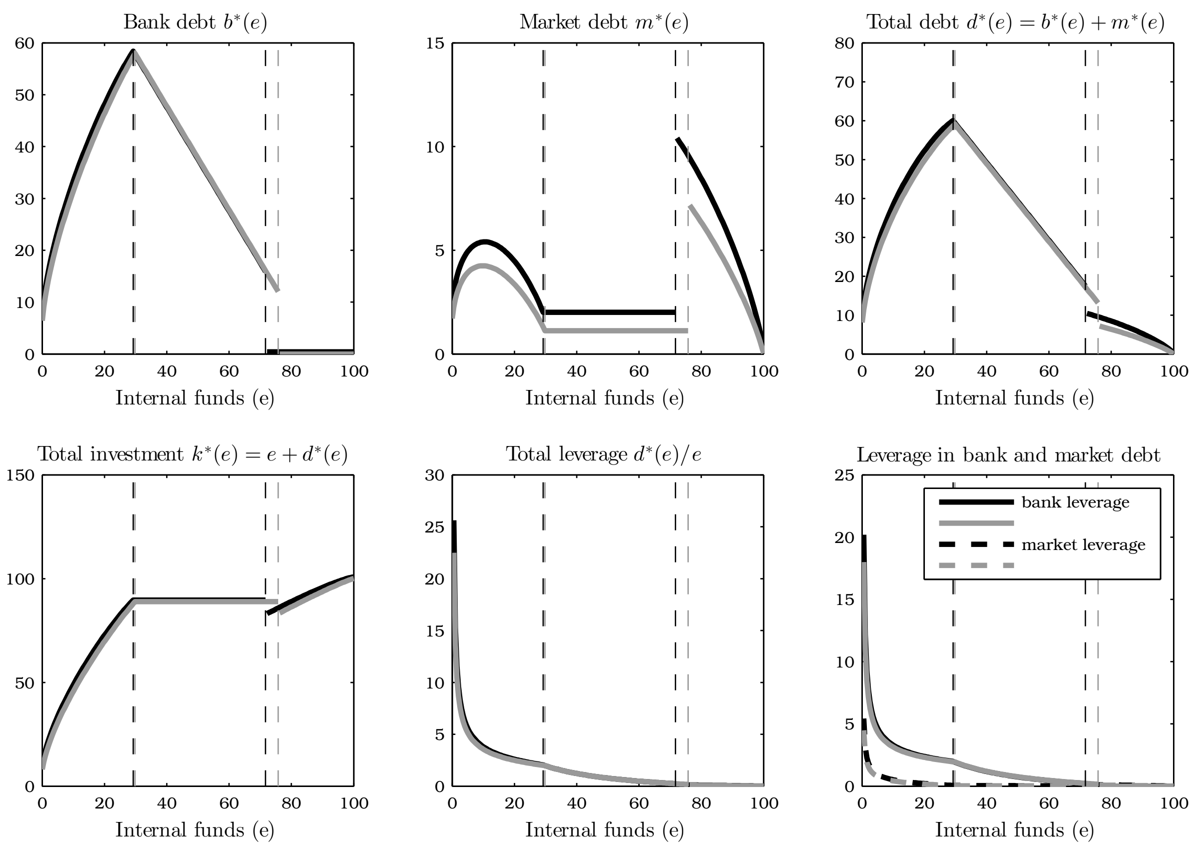



(b) Changes in borrowing

*Figure B.3: The effect of a fall in productivity. Graphs B.3(a) and B.3(b) report the characteristics of the debt structure in the "Baseline" calibration (black line), and in the "Low average productivity" calibration; see table B.1 for details.*




(a) Change in the bank share




(b) Changes in borrowing

*Figure B.4: The effect of an increase in productivity dispersion. Graphs B.4(a) and B.4(b) report the characteristics of the debt structure in the "Baseline" calibration (black line), and in the "High productivity dispersion" calibration; see table B.1 for details.*

## B.4 Appendix to section 2.5

### B.4.1 Measures of the debt structure in the QFR

**Breaking down the liability side of firms' balance sheets** Along with financial liabilities, the QFR balance sheets contain information on non-financial items, such as accounts payables and various forms of tax liabilities. I restrict my attention to financial liabilities, and more specifically to debt. This excludes, in particular, stockholders' equity and trade credit, an important component of liabilities, especially for smaller firms. Within the category of financial liabilities, some are current (due in more than one year) and some are non-current (due in one year or less). I construct measures of the composition of debt for both maturity categories, as well as measures combining short with long debt maturities. For large firms, the behavior of current debt during recessions differs substantially from that of non-current debt. The focus of this paper is on the behavior of total liabilities of firms; I do not adress changes in the maturity structure.

**Bank and non-bank liabilities** The QFR sample contains two subsamples of firms, one of small and medium-sized firms and one of large firms. Smaller firms report their liabilities with less detail than large firms (appendix B.4.2 contains more information on the differences between short and long form samples). In particular, small firms report as a group all their non-bank financial liabilities, whereas the large firm samples reports separately commercial paper and long-term debt. Table B.3 below provides a breakdown of current and non-current financial liabilities in the short and long sample. To maintain comparability across asset size classes, I focus on "non-bank" liabilities as whole. This includes, for largest firms, both commercial paper and long-term debt. As reported below, the facts described in section 2.5 are robust to the exclusion of commercial paper from measures of non-bank liabilities for large firms.

**Aggregates of financial liabilities** In the following discussion, I define two debt aggregates for each firm category: bank liabilities, denoted by $CB$ and $TB$; and non-bank financial liabilities, including commercial paper and bonds (denoted by $CNB$ and $TNB$). Variables beginning with a $C$ denote aggregates computed for current liabilities only, while variables beginning with a $T$ denote aggregates computed for total liabilities. I also contruct measures of total financial liabilities ($CFIN$ and $TFIN$), eliminating non-financial liabilities from the aggregate balance sheet measures. Table B.4 in appendix B.4.2 summarizes the construction of these variables.

### B.4.2 Variable definitions in the QFR

The QFR survey is constructed on a quarterly basis from two separate samples. The first sample (the "short-form" sample) is a representative sample from the universe of manufacturing firms with assets less than or equal to 25m\$. The second sample (the "long-form" sample) contains manufacturing firms with at least 25m\$. Firms with between 25m\$ and 250m\$ are sampled from the universe of manufacturing firms, while all existing manufacturing firms with more than 250m\$ in assets are included. For both samples, the QFR contains a variety of information on firms' real and financial variables. The samples differ in the detail with which firms report their balance sheets, as detailed below.

| Asset size class | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Assets ($m\$$) | $< 5$ | $[5, 10[$ | $[10, 25[$ | $[25, 50[$ | $[50, 100[$ | $[100, 250[$ | $[250, 1000[$ | $\geq 1000$ |

*Table B.2: Definition of asset size brackets in the QFR.*

| | Short form variables | | Long form variables | |
|---|---|---|---|---|
| | *Name* | *Description* | *Name* | *Description* |
| **Current liabilities** | STBANK | Short-term bank debt | STBANK | Short-term bank debt |
| | INSTBANK | Long-term bank debt with maturity $\leq 1$ year | INSTBANK | Long-term bank debt with maturity $\leq 1$ year |
| | STDEBTOTH | Other short-term debt (incl. commercial paper) | STDEBTOTH | Other short-term debt (excl. commercial paper) |
| | | | COMPAPER | Commercial paper |
| | INSTOTH | Other long-term debt with maturity $\leq 1$ year (incl. bonds) | INSTOTH | Other long-term debt with maturity $\leq 1$ year (excl. bonds) |
| | | | INSTBONDS | Bonds maturing in $\leq 1$ year |
| **Non-Current liabilities** | LTBNKDEBT | Long-term bank debt with maturity $> 1$ year | LTBNKDEBT | Long-term bank debt with maturity $> 1$ year |
| | LTOTHDEBT | Other debt with maturity $> 1$ year (incl. bonds) | LTOTHDEBT | Other debt with maturity $> 1$ year (excl. bonds) |
| | | | LTBNDDEBT | Bonds maturing in $> 1$ year |

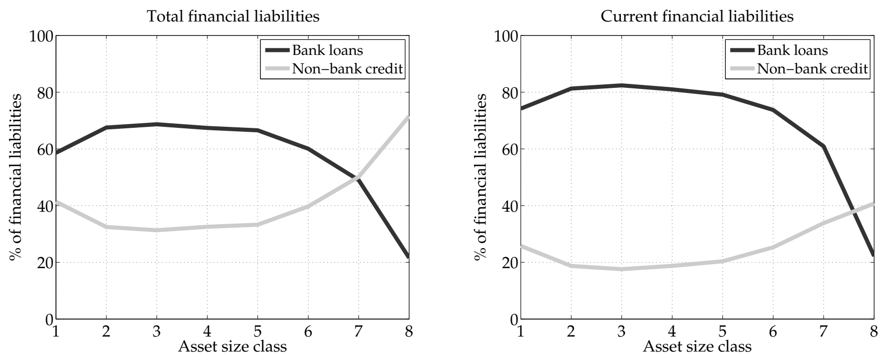
*Table B.3: Financial liabilities reported in the QFR, short and long form samples.*

| | | Definition for short form sample | Definition for long form sample |
|---|---|---|---|
| **Current liabilities** | CB | STBANK + INSTBANK | STBANK + INSTBANK |
| | CNB | STDEBTOTH + INSTOTH | COMPAPER + INSTBONDS+STDEBTOTH+INSOTH |
| | CNBX | n.a. | CNB - COMPAPER |
| | CFIN | CB + CNB | CB + CNB |
| **Total liabilities** | TB | CB + LTBNKDEBT | CB + LTBNKDEBT |
| | TNB | CNB + LTOTHDEBT | CNB + LTBNDDEBT + LTOTHDEBT |
| | TNBX | n.a. | CNBX + LTBNDDEBT + LTOTHDEBT |
| | TFIN | TB + TNB | TB + TNB |

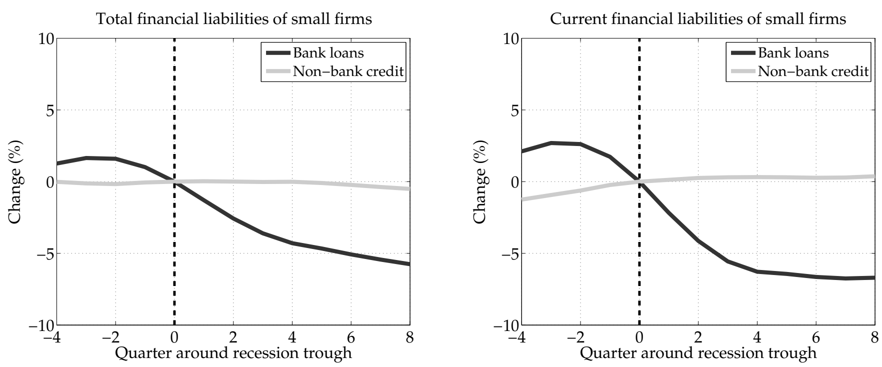
*Table B.4: Definitions of aggregates of financial liabilities.*

## B.4.3 Robustness

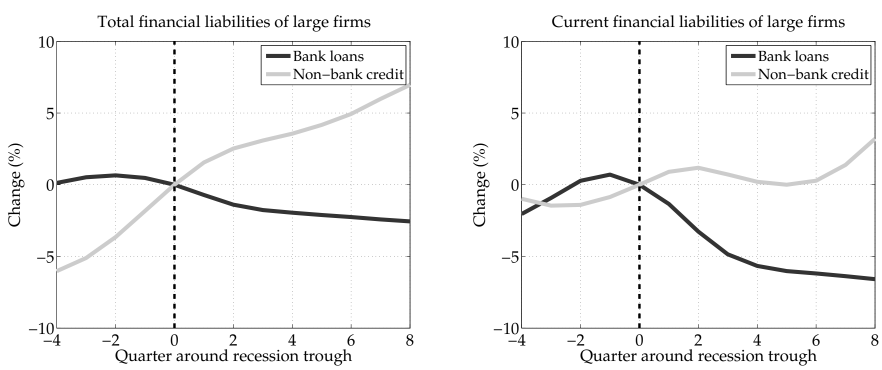
### Excluding commercial paper from non-bank liabilities




(a) Composition of debt in the cross-section




(b) Debt structure of small firms during recessions




(c) Debt structure of large firms during recessions

*Figure B.5: Robustness check 1: no commercial paper. Graphs B.5(a), B.5(b) and B.5(c) report the same series as graphs 2.12(a), 2.12(b) and 2.12(c), but excluding commercial paper from non-bank debt in the case of long-form sample firms.*

**An alternative definition of small and large firms**

To examine whether the facts discussed in section 2.5 are robust to other groupings of "small" and "large" firms, I use the definitions of Gertler and Gilchrist (1994). They use information on total sales of different asset size brackets in order to determine a cutoff between small and large firms (in terms of asset size). The (gross) growth rate of sales of small firms for a quarter is computed as a weighted average of the growth rate of total sales in the two categories that straddle the thirtieth percentiles of cumulative sales. A series for total sales in each size category is then constructed by taking the cumulative sum of the log of the quarterly gross growth rates obtained in this fashion.

I describe this method more formally. Fix $t$. Let $\left\{x^{(1)},...,x^{(n)}\right\}$ denote the asset size brackets for quarter $t$, and let $s_{i,t}$ and $x_{i,t}$ denote sales and total assets of firm $i$ at time $t$. First, let $S_t = \sum_i s_{i,t}$ and define a cutoff in terms of assets, $\underline{x}_t$, by:

$$\underline{x}_t = \max\left\{x \in \left\{x^{(1)},...,x^{(n)}\right\} \Big/ \frac{\sum_{x_{i,t}\leq x} s_{i,t}}{S_t} \leq 0.3\right\}.$$

Furthermore, let $\underline{x}_t^+$ be the cutoff immediately above $\underline{x}_t$ in the list $\left\{x^{(1)},...,x^{(n)}\right\}$. The cumulative sales of all firms with at most $\underline{x}_t^+$ are less than 30 % of the total of sales of manufacturing firms. Second, compute weights $w_t$ such that:

$$w_t\frac{\sum_{x_{i,t}\leq \underline{x}_t} s_{i,t}}{S_t} + (1-w_t)\frac{\sum_{x_{i,t}\leq \underline{x}_t^+} s_{i,t}}{S_t} = 0.3.$$

Third, for any series $y_{i,t}$, compute "weighted" growth rates according to:

$$G^y_{S,t-1,t} = w_t\frac{\sum_{\{i/x_{i,t}\leq \underline{x}_t\}} y_{i,t}}{\sum_{\{i/x_{i,t-1}\leq \underline{x}_t\}} y_{i,t-1}} + (1-w_t)\frac{\sum_{\left\{i/x_{i,t}\leq \underline{x}_t^+\right\}} y_{i,t}}{\sum_{\left\{i/x_{i,t-1}\leq \underline{x}_t^+\right\}} y_{i,t-1}}.$$

Roughly speaking, the goal of this approximation is to compute the growth rate of the variable $y$ for a synthetic size class which represents, on average, 30% of total sales. These weighted growth rates form the basis of a synthetic series for small firms for variable $y$, $y_{S,t}$, through:

$$y_{S,t} = \sum_{j=0}^{t} log(G^{Y}_{S,j-1,j}).$$

For large firms, Gertler and Gilchrist (1994) use[3]:

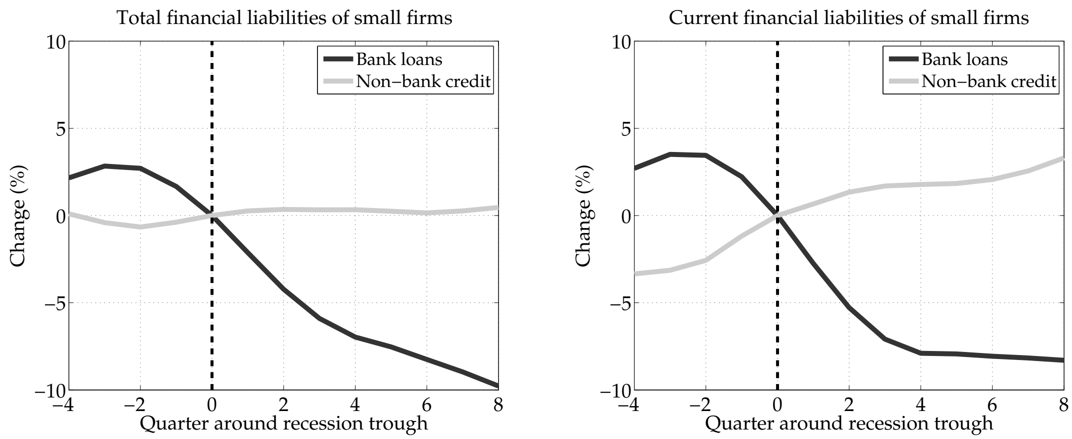
$$G_{L,t-1,t} = w_t \frac{\sum_{\{i/x_{i,t} \geq \underline{x}_t\}} y_{i,t}}{\sum_{\{i/x_{i,t-1} \geq \underline{x}_t\}} y_{i,t-1}} + (1-w_t) \frac{\sum_{\{i/x_{i,t} \geq \underline{x}_t^+\}} y_{i,t}}{\sum_{\{i/x_{i,t-1} \geq \underline{x}_t^+\}} y_{i,t-1}},$$

where $w_t$ is the same value as above, and likewise define a synthetic series for large firms, for the variable $y$, $y_{L,t}$, as:

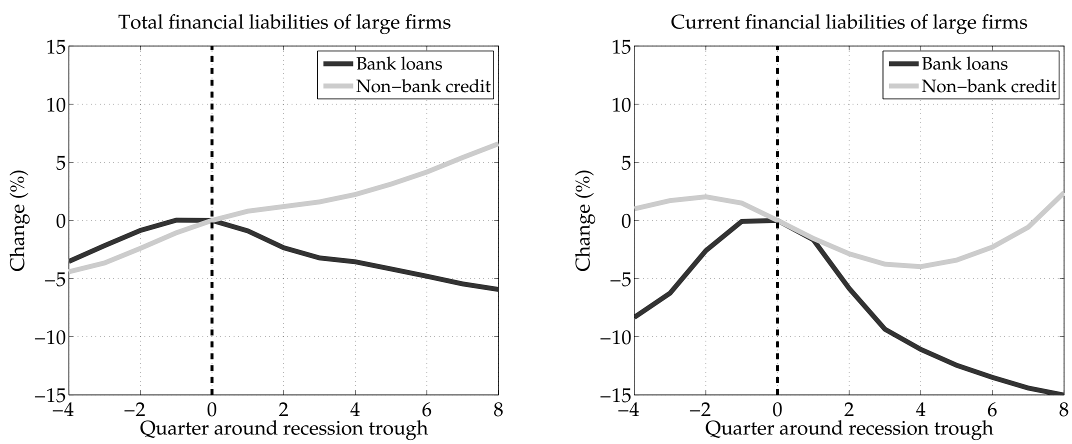
$$y_{L,t} = \sum_{j=0}^{t} log(G_{L,j-1,j}).$$

The results are reported in figures B.7(a) and B.7(b), when $y$ corresponds to the series $CB$, $CNB$, $TB$ and $TNB$ defined above. Consistently with the facts discussed in section 2.5, these figures suggest that, while the "synthetic" small firm experiences a contraction in her liabilities driven entirely by a reduction in bank loan, the large firm experiences a substitution of market credit for bank credit. Notably, the upward trend in non-bank liabilities is less noticeable in definition of small and large firms; in turn, over the first year after the recession, the substitution is less than one for one.

[3] In the later part of the sample, the cutoff $\underline{x}_t$ turns out to be $1b$\$, the largest asset class. For these years, I set the weight $w_t$ in $G_{L,t-1,t}$ to be equal to 1.




(a) The debt structure of small firms during recessions

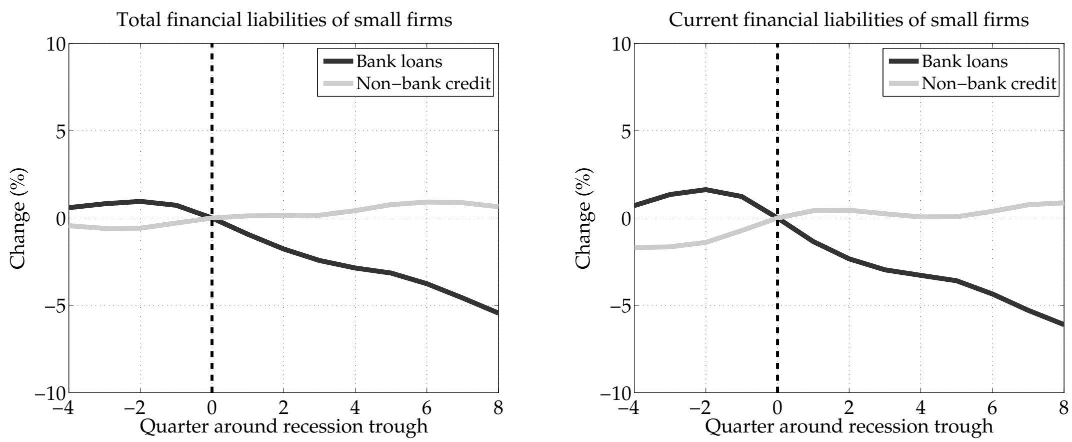



(b) The debt structure of large firms during recessions

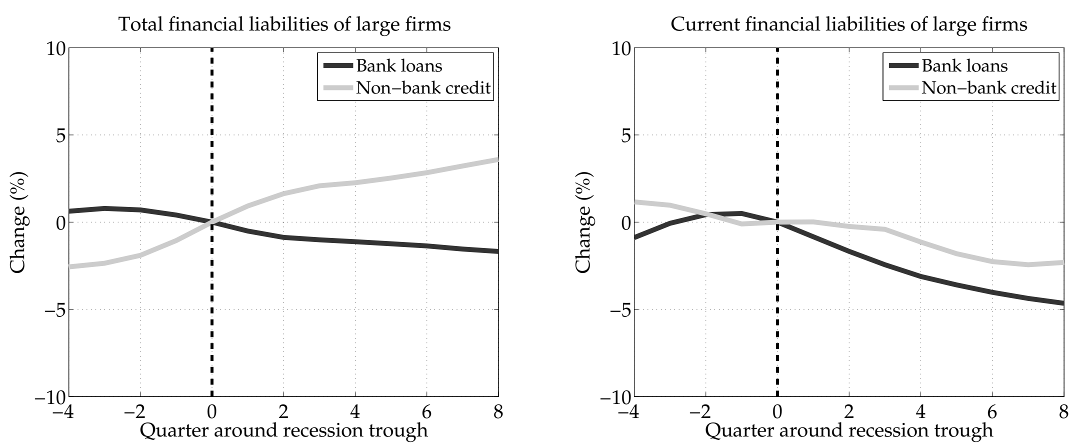
*Figure B.6: Robustness check 2: broader definition of large firms. Graphs B.7(a) and B.7(b) report the same series as graphs 2.12(b) and 2.12(c), but for the definition of "large" and "small" developed by Gertler and Gilchrist (1994); see text for details.*

### Excluding the Great Recession

The following graph reports changes in the composition of liabilities of small and large firms in recessions, excluding the 2007-2009 recession. The series are thus the average of the change in liabilities for the 1990-1991 and 2000 recession.



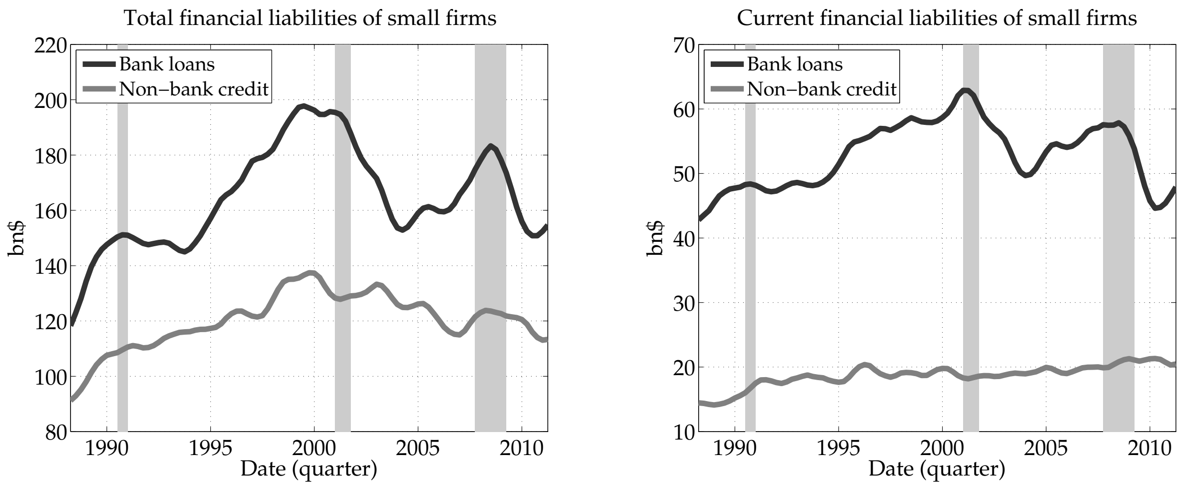

(a) Debt structure of small firms during recessions



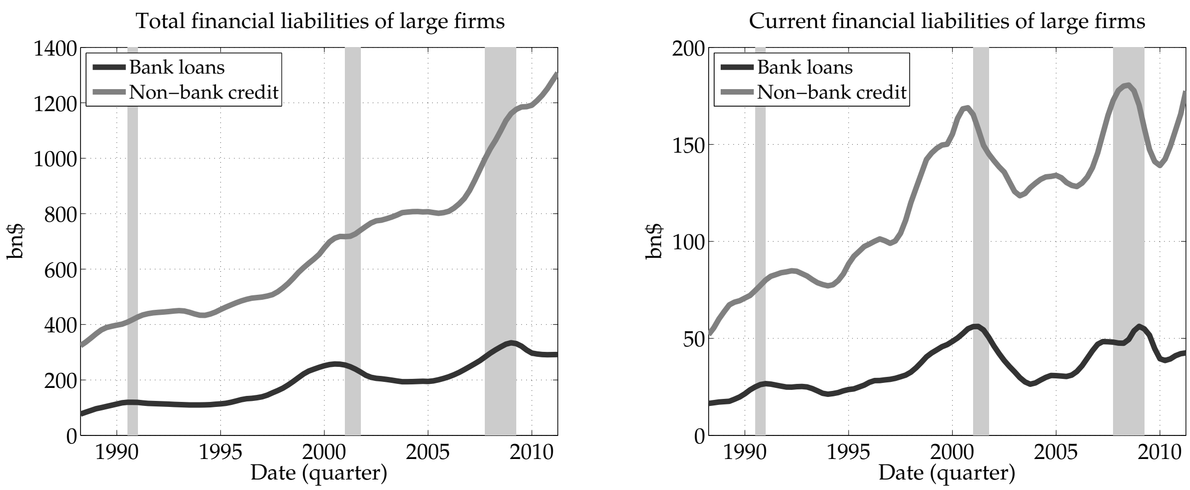

(b) Debt structure of large firms during recessions

*Figure B.7: Robustness check 3: excluding the 2007 recession. Graphs B.7(a) and B.7(b) report the same series as graphs 2.12(b) and 2.12(c), excluding the 2007-2009 recession.*

### B.4.4 Aggregate financial liabilities in levels

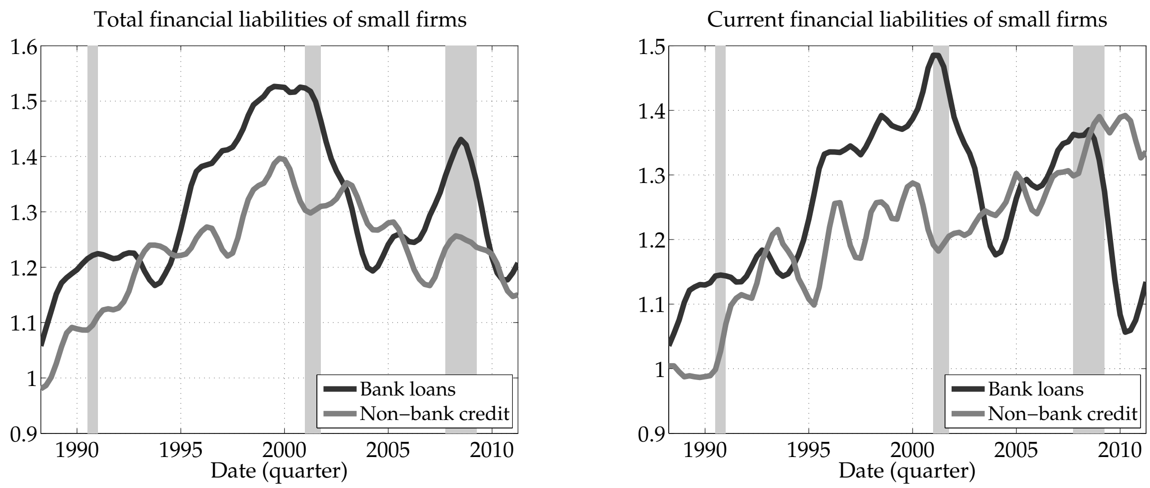



(a) **Small firms**

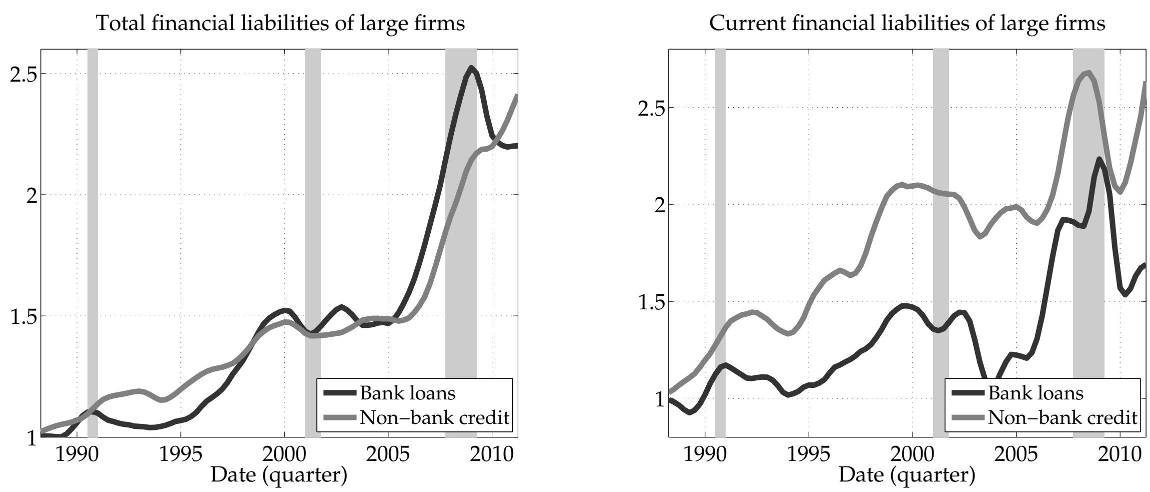



(b) **Large firms**

*Figure B.8: Financial liabilities in levels, baseline small/large classification. All series are smoothed with a 2 by 4 MA smoother to remove seasonal variation. Shaded areas indicate NBER recessions.*




(a) **Small firms**




(b) **Large firms**

*Figure B.9: Financial liabilities in levels, using the Gertler and Gilchrist (1994) definition of small and large firms. All series are smoothed with a 2 by 4 MA smoother to remove seasonal variation. Shaded areas indicate NBER recessions. Units on the y-axis are arbitrary, since the series are computed from synthetic growth rates; see text ofr details.*

# Appendix C

# Appendix for Chapter 3

## C.1 Data appendix

**Data on aggregate financial ratios** For the US, the data on the aggregate bank share is obtained from table L.102 of the Flow of Funds, the balance sheet of the the nonfinancial corporate business sector. The series in the left panel of figure 3.12 is the ratio of the sum of depository institution loans (line 27) and other loans and advances (line 28) to total credit market instruments oustanding (line 23). The ratio of debt to assets is measured as the ratio of total credit market instruments outstanding (line 23), to miscellaneous assets (line 16), a measure of assets excluding credit market instruments and deposits or money market fund shares. I exclude these financial assets from this ratio because the model's firms do not hold cash and do not lend to other firms.

For Italy, the data on the aggregate bank share is obtained from Bank of Italy (2008), table 5 (TDHET000). The aggregate bank share is computed as the ratio of total loans (short and long-term) to total debt. Total debt is measured as total liabilities minus shares and other equities issued by residents.[1] Similarly to the US, I construct the ratio of debt to assets as the ratio of total debt to a measure of total assets which excludes cash and cash-like securities, as well as credit market securities. I obtain this measure by substracting deposits, short-terms securities, bonds, derivatives, short-term loans and mutual fund shares from total liabilities of firms.

Figure 3.10 reports aggregate bank shares for a larger sample of countries, on average between 2000 and 2007. This graph is obtained using firm-level data from the OSIRIS database. For each year $t$ between 2000 and 2007, each country $c$, and each firm $j$ in that country's set of firms $F_{c,t}$, I construct measures of outstanding bank debt and outstanding total debt for each firm, $b^c_{j,t}$ and $d^c_{j,t}$. The construction of these measures is described below. For each country-year, the aggregate bank share is computed as $S_{c,t} = \frac{\sum_{j \in F_{c,t}} b_{j,t}}{\sum_{j \in F_{c,t}} d_{j,t}}$. The aggregate bank shares reported in the graph are the average of the $S_{c,t}$ for $t = 2000, ..., 2007$.[2]

**Data on business-cycle changes in debt composition in the US** The middle and right panels of figure 3.12 report changes in outstanding bank and non-bank credit for small and large manufacturing firms, in the US, from 2008 onwards. The data is from the Quarterly Financial Report of manufacturing firms. This dataset contains information on firms' balance sheets and income statements, and is reported in semi-aggregated form (by asset size categories). The QFR has two advantages over firm-level datasets: it includes small and private firms as well as large firms; and it has quarterly coverage. By contrast, the other

[1] The Bank of Italy distinguishes between loans from MFI's (monetary financial institutions, comprising the Central Bank, banks, money-market funds, electronic money institutions and the Cassa Depositi e Prestiti), and loans from other financial institutions. Excluding other financial institutions, the aggregate bank share for 2007Q3 is 50.3 %, closer to the aggregate bank share which can be matched by the model.

[2] In particular, the aggregate bank shares obtained in this way for Italy and the US are not the same as those obtained using the financial accounts of either country. I choose to use firm-level data in the construction of figure 3.10 for those countries, rather than the financial accounts data, to maintain comparability with other countries.

firm-level dataset I use for the US, created by Rauh and Sufi (2010), covers only public firms, is annual, and does not extend to 2012. It is thus less adapted to documenting facts on business-cycle changes in debt composition.

In chapter 2 and appendix B, I discuss in detail the QFR and the definitions of financial variables. The "small firm" category is defined as firms with less than $\$1bn$ in assets, and the "large firm" category as the remainder.[3] For both categories, total debt is defined as total liabilities excluding non-financial liabilities (such as trade credit) and stockholders' equity. It includes both short and long-term debt. Bank debt $b_t$ is reported as a specific item in the QFR; I define market credit $m_t$ as the remaining financial liabilities. The series called "bank debt" in figure 3.12 is given by: $\gamma_{b,small,t} = \frac{b_{small,t_0}}{b_{small,t_0}+m_{small,t_0}}\left(\frac{b_{small,t}}{b_{small,t_0}} - 1\right)$. The series called "market debt" is defined similarly.[4] I choose $t_0$ to be $2008Q3$. This is the quarter of the failure of Lehman Brothers, and it also marks the start in the decline of the aggregate bank share (see left panel). The series reported in this figure are smoothed by a 2 by 4 MA smoother to remove seasonal variation.

**Data on firm-level debt composition** Figure 3.8 is constructed using firm-level data from the OSIRIS database maintained by Bureau Van Djik. This dataset contains balance sheet information for publicly traded firms in emerging and advanced economies. I focus on the subsample of non-financial firms that are active between 2000 and 2010, and keep only firms that report consolidated financial statements.

Total debt $d^c_{j,t}$ is defined as total long-term interest bearing debt (data item number 14016), and bank debt $b^c_{j,t}$ as bank loans (data item number 21070). I focus on long-term

[3]Appendix B.4 shows that the pattern of debt substitution is robust to different definitions of size groups, in particular that adopted by Gertler and Gilchrist (1994).

[4]This definition preserves additivity; that is, the sum of the two lines in figure 3.12 corresponds to the percent change in total financial liabilities around $t_0$.

debt (inclusive of the currently due portions of it) because bank loans are a subset of this category in the OSIRIS database; currently due debt only features a "loans" category which include potentially credit instruments other than bank loans. Additionally, I define internal funds $e^c_{j,t}$ as shareholders funds (data item 14041). For each firm-year observation, the bank share is defined as: $s_{j,t,c} = \frac{b^c_{j,t}}{d^c_{j,t}}$. I keep only observations for which $e_{j,t,c} \geq 0$ and $s_{j,t,c} \in [0,1]$ (and for which both are non-missing). The resulting sample has 51921 firm-year observations, correspondind to 12931 distinct company names.

Observations are pooled by country-year $(t,c)$. For each country-year cell, let $\tilde{e}_{k,t,c}$ denote the $k-th$ quantile of the empirical distribution of firms across levels of internal funds $e_{j,t,c}$. I use $k \in \{5, 15, 25, ..., 95\}$. Define the *average* bank share within each quantile group as:

$$\tilde{s}_{k_i,t,c} = \frac{1}{N_{k_i,t,c}} \sum_{\tilde{e}_{k_i,t,c} \leq e_{j,t,c} < \tilde{e}_{k_{i+1},t,c}} s_{j,t,c},$$

where $N_{k_i,t,c}$ is the number of firms in $(t,c)$ with $\tilde{e}_{k_{i-1},t,c} \leq e_{j,t,c} < \tilde{e}_{k_i,t,c}$.[5] I then average out these shares over time: $\tilde{s}_{k_i,c} = \frac{1}{T}\sum_t \tilde{s}_{k_i,t,c}$. Figure 3.8 reports the pairs $(k_i, \tilde{s}_{k_i,c})$, for the subset of 8 countries that have the largest number of observations among advanced and emerging economies, respectively.

I do this for all countries, except for the US. For the US, my firm-level data is drawn from the dataset created by Rauh and Sufi (2010), arguably the highest-quality dataset on debt structure for publicly traded-firms. This dataset draws from Compustat (for balance-sheet data) and Dealscan (for bond issuances), and covers a larger number of firms than those available for the US wihtin the OSIRIS database, for a similar period. This database has direct measures of total debt $d_{j,t,US}$. Internal finance $e_{j,t,US}$ is defined as the difference between the "debt plus equity" variable and the "debt" variable. I keep only firm-year

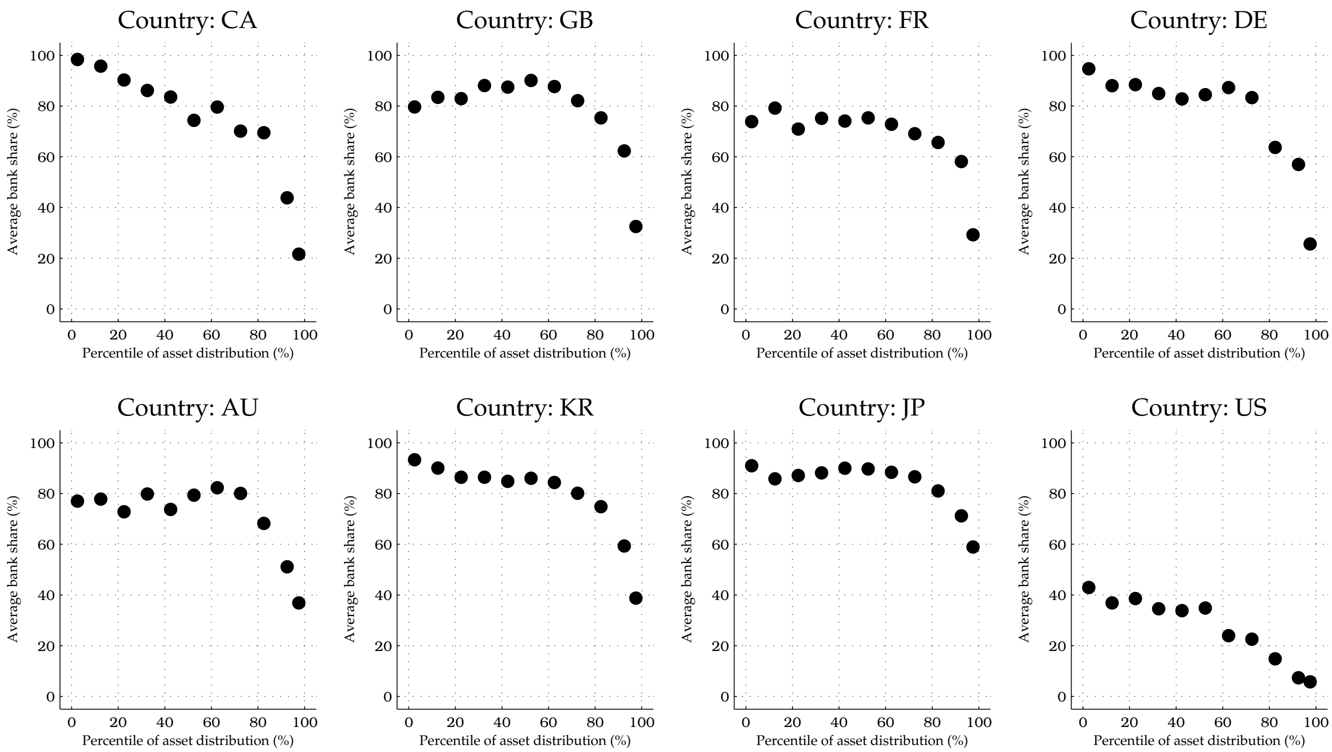
[5] Clearly, for $k_i = 5\%$, $k_{i-1} = 0$ and for $k_{i-1} = 95\%$, $k_i = 100\%$.

observations for which this measure of internal funds is positive. Finally, bank debt $b_{j,t,US}$ is defined as outstanding bank loans. All other variable definitions are identical.[6]

**The relationship between total assets and the bank share** A natural alternative measure of firm size are the value of its assets. In the cross-section, asset value is negatively related to the amount of bank in a firms' debt structure. I document this using the same cross-sectional statistics as for internal funds described above, that is, the average bank share for consecutive quantiles of the asset size distribution. For all countries in the OSIRIS database, asset value is the book value of fixed assets (data item 20085) which includes property, plant, equipment, intangibles, and other fixed assets. For the US, I use the measure of assets provided by Rauh and Sufi (2010). Fixed assets is the more directly relevant measure of assets since firms in the model only hold real assets; however, using total assets (inclusive of financial assets, a measure also available in the OSIRIS database) does not change substantially the results. The negative relationship between this measure of size and the bank share is reported in figure C.1.

[6] In the US sample of the OSIRIS database, the negative relationship between bank share $\tilde{s}_{k_i,US}$ and internal funds quantile $k_i$ also obtains, but the average level of the bank share is higher, and in fact larger than in other advanced countries.

(a) **Total assets and bank share in OECD-countries**




(b) **Total assets and bank share in non-OECD-countries**

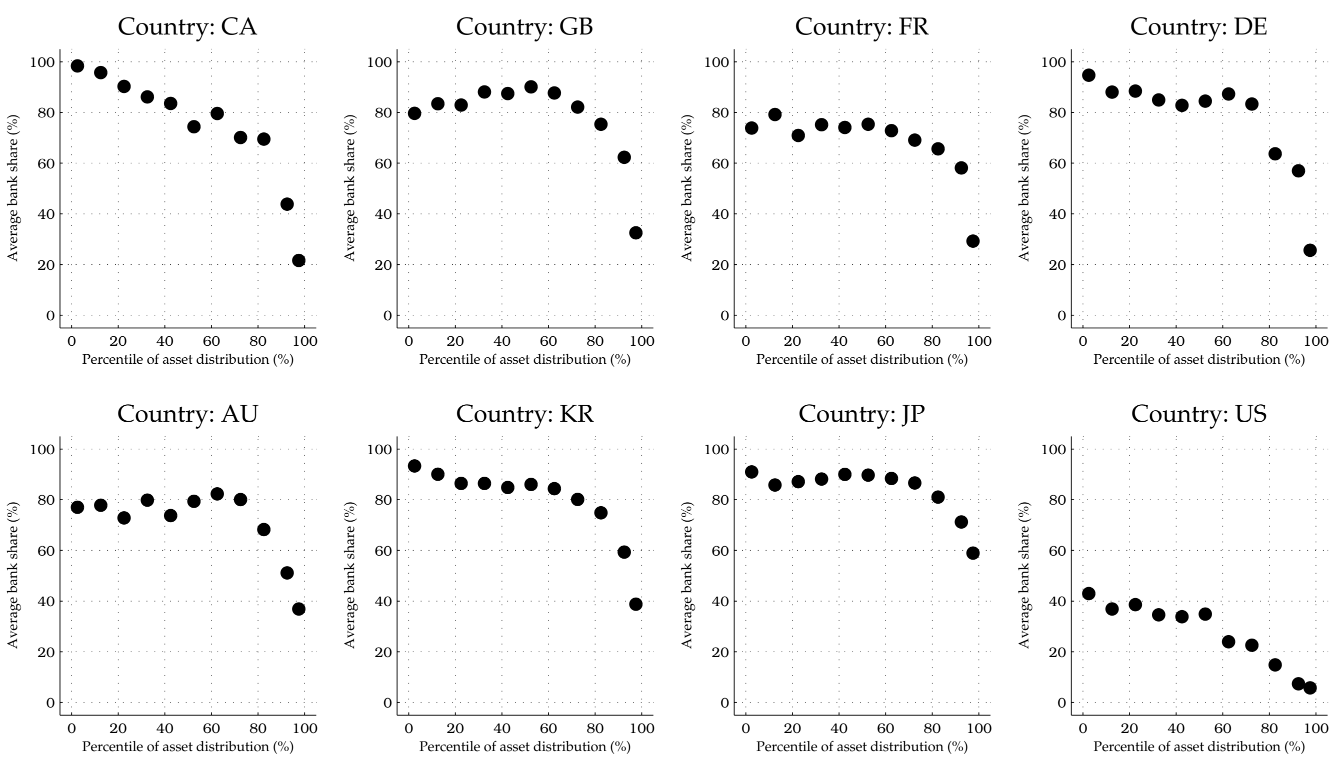



*Figure C.1: Bank share and fixed assets in the cross-section. Each graph reports, for a particualr country, the median ratio of bank loans to total firm liabilities, in each decile of the asset distribution. For the US, data from taken from Rauh and Sufi (2010); for other countries, data from Bureau Van Djik.*

## C.2 Analytical proofs

*Proof of lemma 3.1.* When $V$ is continuous, the objective function in problem (3.1) is continuous. The constraint correspondence in problem (3.1) is compact-valued and continuous. The theorem of the maximum then implies that $V^c$ is continuous. Let $(n_t^1, n_t^2) \in \mathbb{R}_+^2$ such that $n_t^1 > n_t^2$, and let $e_{t+1}^2$ be a value for next period net worth that solves problem (3.1), when $n_t = n_t^2$. We have $e_{t+1}^2 \leq n_t^2 < n_t^1$, so $e_{t+1}^2$ is also feasible when $n_t = n_t^1$. Therefore, $V^c(n_t^1) \geq n_t^1 - e_{t+1}^2 + (1-\eta)\beta V(e_{t+1}^2) > n_t^2 - e_{t+1}^2 + (1-\eta)\beta V(e_{t+1}^2) = V^c(n_t^2)$. This proves that $V^c$ is strictly increasing. Finally, when $n_t = 0$ the feasible set contains only $div_t = 0, e_{t+1} = 0$. So $V^c(0) = (1-\eta)\beta V(0)$. Therefore when $V(0) \geq 0$, $V^c(0) \geq 0$.

□

*Proof of proposition 3.4.* A useful result for the proof is that $V^c(n_t) \geq n_t$ when $V(0) \geq 0$. This is established by noting that the dividend policy $e_{t+1} = 0$ is always feasible at the dividend issuance stage, and that the value of this policy is $n_t + (1-\eta)\beta V(0) \geq n_t$.

Assume, first, that $\frac{R_{m,t}}{1-\chi} \leq \frac{R_{b,t}}{\chi}$. Then, $\frac{R_{m,t}}{1-\chi} \leq R_{b,t} + R_{m,t} \leq \frac{R_{b,t}}{\chi}$. The proof proceeds by comparing $V_t^L, V_t^R$ and $V_t^P$, the values of the firm under liquidation, restructuring or repayment, for each realization of $\pi_t$. There are five possible cases:

- when $\pi_t \geq \frac{R_{b,t}+R_{m,t}}{\chi}$, we have $V_t^L = \chi\pi_t - R_{b,t} - R_{m,t} < \pi_t - R_{b,t} - R_{m,t} \leq V^c(\pi_t - R_{b,t} - R_{m,t}) = V_t^P$. Moreover, since $\pi_t \geq \frac{R_{b,t}+R_{m,t}}{\chi} \geq \frac{R_{b,t}}{\chi}$, the reservation value of the bank is $R_{b,t}$, so the best restructuring offer for the firm is $l_t = R_{b,t}$. Therefore $V_t^P = V_t^R$. I will assume the firm chooses repayment.

- when $\frac{R_{b,t}+R_{m,t}}{\chi} > \pi_t \geq \frac{R_{b,t}}{\chi}$, we have $V_t^L = 0 \leq V^c(\pi_t - R_{b,t} - R_{m,t}) = V_t^P$, since $\pi_t \geq \frac{R_{b,t}}{\chi} \geq R_{b,t} + R_{m,t}$, $V^c(0) \geq 0$ and $V^c$ is strictly increasing. ($V_t^L < V_t^R$ so long as $\pi_t > R_{b,t} + R_{m,t}$). Moreover, $V_t^R = V_t^P$ for the same reason as above. Again, the firm chooses repayment.

- when $\frac{R_{b,t}}{\chi} > \pi_t \geq R_{b,t}+R_{m,t}$, the reservation value of the bank is $\chi\pi_t$. The restructuring offer at which the participation constraint of the bank binds, $l_t = \chi\pi_t$, is feasible because $\pi_t - l_t - R_{m,t} = (1-\chi)\pi_t - R_{m,t} \geq 0$. So $V_t^R \geq V^c\left((1-\chi)\pi_t - R_{m,t}\right)$. This implies $V_t^R > V^c\left(\pi_t - R_{b,t} - R_{m,t}\right) = V_t^P$, since $V^c$ is strictly increasing and $(1-\chi)\pi_t - R_{m,t} > \pi_t - R_{b,t} - R_{m,t}$. For the same reasons as above, $V_t^R \geq V_t^L$. So the firm chooses to restructure. Because $V^c$ is increasing, the optimal restructuring offer makes the participation constraint of the bank bind: $\hat{l}_t = \chi\pi_t$.

- when $R_{b,t} + R_{m,t} \geq \pi_t \geq \frac{R_{m,t}}{1-\chi}$, we have $V_t^L = 0 < V^c\left((1-\chi)\pi_t - R_{m,t}\right) = V_t^R$, where again the properties of $V^c$ were used. Moreover, $V_t^P = V_t^L$, since the firm does not have enough funds to repay both its creditors. So the firm chooses to restructure, again with $\hat{l}_t = \chi\pi_t$.

- when $\pi_t < \frac{R_{m,t}}{1-\chi}$, the firm is liquidated because any restructuring offer consistent with the participation constraint of the bank will leave the firm unable to repaye market creditors. Since in that case, $0 > \chi\pi_t - R_{b,t} - R_{m,t}$, the liquidation value for the firm is $V_t^L = 0$.

This shows that when $\frac{R_{m,t}}{1-\chi} \leq \frac{R_{b,t}}{\chi}$, the firm repays when $\pi_t \geq \frac{R_{b,t}}{\chi}$, restructures when $\frac{R_{b,t}}{\chi} \geq \pi_t \geq \frac{R_{m,t}}{1-\chi}$, and is liquidated otherwise. Moreover, this also establishes the two additional claims of the proposition, in the case $\frac{R_{m,t}}{1-\chi} \leq \frac{R_{b,t}}{\chi}$: $V_t^L = 0$ whenever liquidation is chosen, and the restructuring offer always makes the participation constraint of the bank bind: $\hat{l}_t = \chi\pi_t$. The claims of the proposition when $\frac{R_{m,t}}{1-\chi} > \frac{R_{b,t}}{\chi}$ can similarly be established, by focusing on the three sub-cases $\pi_t \geq \frac{R_{b,t}+R_{m,t}}{\chi}$, $\frac{R_{b,t}+R_{m,t}}{\chi} > \pi_t \geq R_{b,t}+R_{m,t}$ and $R_{b,t}+R_{m,t} > \pi_t$.

□

*Proof of proposition* . Given the results of proposition 3.4, the debt settlement outcomes yield the same conditional return functions for banks and market lenders, $\tilde{R}_{b,t}(\pi_t, R_{b,t}, R_{m,t})$ and $\tilde{R}_{m,t}(\pi_t, R_{b,t}, R_{m,t})$, as those reported in the appendix B.1 to chapter 2. Therefore, all the results of section 2.2.4 apply. Proposition C.2 is a subset of the results of proposition 2.8. $\square$

## Proof of proposition 3.8

The proof of the existence of a recursive competitive equilibrium of the economy of section 3.2 consists in two main steps. First, one needs to prove that the optimal debt structure problem of a single firm, problem (3.8), has a unique solution. Second, one must establish the existence and unicity of a steady-state distribution of firms across levels of internal finance. I start by introducing some notation.

**Preliminary notation** Throughout, I restate the firm's problem in terms of the variables $d_t = b_t + m_t$ and $s_t = \frac{b_t}{b_t+m_t}$. $d_t$ denotes total borrowing by a firm, and $s_t$ denotes the share of borrowing that is bank debt. Note that $(d_t, s_t) \in \mathbb{R}_+ \times [0,1]$. With some abuse of notation, I will keep denoting the set of feasible debt structures $(d_t, s_t)$ by $\mathcal{S}(e_t)$, and its partition established in proposition C.2 as $(\mathcal{S}_K(e_t), \mathcal{S}_R(e_t))$. Additionally, define the functions $G : \mathbb{R}_+ \to \mathbb{R}_+$, $I(.; e_t + d_t) : \mathbb{R}_+ \to \mathbb{R}_+$ and $M(; e_t + d_t) : \mathbb{R}_+ \to \mathbb{R}_+$ by:

$$\begin{aligned} G(x) &= x(1 - F(x)) + \int_0^x \phi dF(\phi) \\ I(x; e_t + d_t) &= x(1 - F(x)) - F(x)(1 - \delta)(e_t + d_t)^{1-\zeta} \\ M(x; e_t + d_t) &= (1 - \chi)I(x; e_t + d_t) + \chi G(x) \end{aligned}$$

Following lemmas B.2 to B.4 in appendix B.1, $G$ is strictly increasing on $\mathbb{R}^+$, while $I$ and $M$ have unique maxima. Moreover, the terms of debt contracts $(R_{b,t}, R_{m,t})$ for given $(e_t, d_t, s_t)$

can be expressed using the inverse mappings of these three functions, denoted by $G^{-1}$, $I^{-1}(,;e_t+d_t)$ and $M^{-1}(.;e_t+d_t)$. These inverse mappings are defined, respectively, on $[0,\mathbb{E}(\phi)]$, $\left[0,\hat{I}(e_t+d_t)\right]$ and $\left[0,\hat{M}(e_t+d_t)\right]$, where $\hat{I}(e_t+d_t)$ is the global maximum of $I$ and similarly $\hat{M}(e_t+d_t)$ is the global maximum of $M$. For example, the terms of bank contracts when $(d_t,s_t)\in\mathcal{S}_R(e_t)$ are given by:

$$R_{b,t} = R_b(d_t,s_t,e_t) = \begin{cases} (1+r_b)d_t s_t & \text{if} \quad 0 \leq \frac{(1+r_b)d_t s_t}{\chi(e_t+d_t)^\zeta} < (1-\delta)(e_t+d_t)^{1-\zeta} \\ \chi(1-\delta)(e_t+d_t) & \text{if} \quad (1-\delta)(e_t+d_t)^{1-\zeta} \leq \frac{(1+r_b)d_t s_t}{\chi(e_t+d_t)^\zeta} \\ +\chi(e_t+d_t)^\zeta G^{-1}\left(\frac{(1+r_b)d_t s_t-\chi(1-\delta)(e_t+d_t)}{\chi(e_t+d_t)^\zeta}\right) & \qquad \leq \mathbb{E}(\phi)+(1-\delta)(e_t+d_t)^{1-\zeta} \end{cases}$$

The expressions for $R_m(d_t,s_t,e_t)$ when $(d_t,s_t)\in\mathcal{S}_R(e_t)$ and for $R_b(d_t,s_t,e_t)$ and $R_m(d_t,s_t,e_t)$ when $(d_t,s_t)\in\mathcal{S}_K(e_t)$ are reported in section B.1. In what follows, I use these results to express the thresholds for restructuring and liquidation in terms of the productivity shock $\phi_t$, namely:

$$\begin{aligned} \underline{\phi}_R(e_t,d_t,s_t) &= \frac{R_m(d_t,s_t,e_t)-(1-\chi)(1-\delta)(e_t+d_t)}{(1-\chi)(e_t+d_t)^\zeta} && \text{(liquidation threshold when } (d_t,s_t)\in\mathcal{S}_R(e_t)) \\ \overline{\phi}_R(e_t,d_t,s_t) &= \frac{R_b(d_t,s_t,e_t)-\chi(1-\delta)(e_t+d_t)}{\chi(e_t+d_t)^\zeta} && \text{(restructuring threshold when } (d_t,s_t)\in\mathcal{S}_R(e_t)) \\ \underline{\phi}_K(e_t,d_t,s_t) &= \frac{R_b(d_t,s_t,e_t)+R_m(d_t,s_t,e_t)-(1-\delta)(e_t+d_t)}{(e_t+d_t)^\zeta} && \text{(liquidation threshold when } (d_t,s_t)\in\mathcal{S}_K(e_t)) \end{aligned}$$

*Proof of existence and unicity of a solution to problem 3.8.* The proof has three steps:

**Step 1:** Reformulate the optimization problem of the firm as the combination of a discrete choice and continuous choice problem.

**Step 2:** Show that the functional mapping $T$ associated with this new formulation maps the space $C(E)$ of real-valued, continuous functions on $[0,E]$, with the sup norm $\|.\|_s$, onto itself, where $E>0$ is an arbitrarily large upper bound for equity. Additionally, show that $T(C_0(E))\subseteq C_0(E)$, where $C_0(E)=\{V\in C(E) \text{ s.t. } V(0)\geq 0\}$. Since

$(C(E), \|.\|_s)$ is a complete metric space and $C_0(E)$ is a closed subset of $C(E)$ under $\|.\|_s$ which is additionally stable through $T$, if $T$ is a contraction mapping, then its fixed point must be in $C_0(E)$.

**Step 3:** Check that $T$ satisfies Blackwell's sufficiency conditions, so that it is indeed a contraction mapping.

Note that step 2 is crucial because lemma 3.1 requires the continuity of $V$ and the fact that $V(0) \geq 0$ for $V^c$ to be continuous, strictly increasing and satisfy $V^c(0) \geq 0$. In turn, these three conditions are necessary for characterizing of the set of feasible debt structures, that is, for proposition to hold.

**Step 1:** Define the mapping $T$ on $C(E)$ as:

$$\forall e_t \in [0, E], \quad TV(e_t) = \max_{R,K} \Big( T_R V(e_t), T_K V(e_t) \Big) \tag{A1}$$

where the mappings $T_R$ and $T_K$, also defined on $C(E)$, are given by:

$$\forall e_t \in [0, E], \quad T_R V(e_t) = \max_{(d_t, s_t) \in \mathcal{S}_R(e_t)} \int_{\phi_t \geq \underline{\phi}_R(e_t, d_t, s_t)} V^c \left( n_R \left( \phi_t; e_t, d_t, s_t \right) \right) dF(\phi_t) \tag{A1-R}$$

$$\text{s.t.} \quad V^c(n_t) = \max_{0 \leq e_{t+1} \leq n_t} n_t - e_{t+1} + (1-\eta)\beta V(e_{t+1})$$

$$n_R(\phi_t, e_t, d_t, s_t) = \begin{cases} \Big(\phi_t - \chi \overline{\phi}_R(e_t, d_t, s_t) - (1-\chi)\underline{\phi}_R(e_t, d_t, s_t)\Big)(e_t + d_t)^\zeta & \text{if} \quad \overline{\phi}_R(e_t, d_t, s_t) \leq \phi_t \\ (1-\chi)\Big(\phi_t - \underline{\phi}_R(e_t, d_t, s_t)\Big)(e_t + d_t)^\zeta & \text{if} \quad \underline{\phi}_R(e_t, d_t, s_t) \leq \phi_t \leq \overline{\phi}_R(e_t, d_t, s_t) \end{cases}$$

$$\underline{\phi}_R(e_t, d_t, s_t) = \begin{cases} 0 & \text{if} \quad 0 \leq (1+r_m)d_t(1-s_t) < (1-\chi)(1-\delta)(e_t + d_t) \\ I^{-1}\left(\frac{(1+r_m)d_t(1-s_t) - (1-\chi)(1-\delta)(e_t+d_t)}{(1-\chi)(e_t+d_t)^\zeta}; e_t + d_t\right) & \text{if} \quad (1-\chi)(1-\delta)(e_t + d_t) \leq (1+r_m)d_t(1-s_t) \\ & \qquad \leq (1-\chi)(1-\delta)(e_t+d_t) + (1-\chi)(e_t+d_t)^\zeta \hat{I}(e_t + d_t) \end{cases}$$

$$\overline{\phi}_R(e_t, d_t, s_t) = \begin{cases} 0 & \text{if} \quad 0 \leq (1+r_b)d_t s_t < \chi(1-\delta)(e_t + d_t) \\ G^{-1}\left(\frac{(1+r_b)d_t s_t - \chi(1-\delta)(e_t+d_t)}{\chi(e_t+d_t)^\zeta}\right) & \text{if} \quad \chi(1-\delta)(e_t + d_t) \leq (1+r_b)d_t s_t \\ & \qquad \leq \chi(1-\delta)(e_t+d_t) + \chi(e_t+d_t)^\zeta \mathbb{E}(\phi) \end{cases}$$

and:

$$\forall e_t \in [0,E], \quad T_K V(e_t) = \max_{(d_t,s_t)\in\mathcal{S}_K(e_t)} \int_{\phi_t \geq \underline{\phi}_K(e_t,d_t,s_t)} V^c\left(n_K\left(\phi_t; e_t, d_t, s_t\right)\right) dF(\phi_t) \qquad \text{(A1-K)}$$

$$\text{s.t.} \quad V^c(n_t) = \max_{0\leq e_{t+1}\leq n_t} n_t - e_{t+1} + (1-\eta)\beta V(e_{t+1})$$

$$n_K(\phi_t, e_t, d_t, s_t) = \left(\phi_t - \underline{\phi}_K(e_t, d_t, s_t)\right)(e_t+d_t)^\zeta \qquad \text{if} \quad \phi_t \geq \underline{\phi}_K(e_t,d_t,s_t)$$

$$\underline{\phi}_K(e_t,d_t,s_t) = \begin{cases} 0 & \text{if} \quad 0 \leq (1+r_m(1-s_t)+r_b s_t)d_t < (1-\delta)(e_t+d_t) \\ M^{-1}\left(\frac{(1+r_m(1-s_t)+r_b s_t)d_t-(1-\delta)(e_t+d_t)}{(e_t+d_t)^\zeta}; e_t+d_t\right) & \text{if} \quad (1-\delta)(e_t+d_t) \leq (1+r_m(1-s_t)+r_b s_t)d_t \\ & \quad \leq (1-\delta)(e_t+d_t)+(e_t+d_t)^\zeta \hat{M}(e_t+d_t) \end{cases}$$

Consider a solution $V$ to problem (3.8) and a particular value of $e_t \in [0,E]$. Since $(\mathcal{S}_K(e_t), \mathcal{S}_R(e_t))$ is a partition of $\mathcal{S}(e_t)$, the optimal policies $(\hat{d}_t, \hat{s}_t)$ (there may be several) must be in either $S_R(e_t)$ or $S_K(e_t)$. Assume that they are in $\mathcal{S}_R(e_t)$. Then the contracts $R_{b,t}, R_{m,t}$ associated with the optimal policies satisfy $\frac{R_{b,t}}{\chi} \geq \frac{R_{m,t}}{1-\chi}$. Given the results of proposition C.2, the constraints and objectives in problem (3.8) can be rewritten as in (A1-R). Since $V$ solves (3.8), this implies that $V(e_t) = T_R V(e_t)$. Moreover, in that case $T_R V(e_t) = V(e_t) \geq T_K V(e_t)$, by optimality of $(\hat{d}_t, \hat{s}_t)$. Thus, $TV(e_t) = V(e_t)$. The same equality obtains if $(\hat{d}_t, \hat{s}_t) \in \mathcal{S}_K(e_t)$. Any solution to $V$ to problem (3.8) must thus satisfy $TV = V$. The rest of the proof therefore focuses on the properties of the operators $T$, $T_K$ and $T_R$.

**Step 2:** Let $V \in C(E)$. By lemma 3.1, the associated continuation value $V^c$ is continuous on $\mathbb{R}^+$. Moreover, since $I^{-1}$ and $G^{-1}$ are continuous functions of $e_t, d_t$ and $s_t$, the functions $n_R$, $\underline{\phi}_R$ and $\overline{\phi}_R$ are continuous in their $(e_t, d_t, s_t)$ arguments. Define the mapping $O_R$ : $[0,E] \times \left[0, \bar{d}(E)\right] \times [0,1] \to \mathbb{R}_+$ by $O_R(e_t,d_t,s_t) = \int_{\phi_t \geq \underline{\phi}_R(e_t,d_t,s_t)} V^c\left(n_t^R\left(\phi_t; e_t, d_t, s_t\right)\right) dF(\phi_t)$. Here $\bar{d}(E)$ denotes the upper bound on borrowing for the maximum level of equity $E$.[7] By continuity of $V^c, n_t^R, \underline{\phi}_R$ and $\overline{\phi}_R$, the integrand in $O_R$ is continuous on the compact set

[7] See the results of proposition 2.16 for a proof that such an upper bound always exist.

$[0, E] \times \left[0, \bar{d}(E)\right] \times [0, 1]$, and therefore uniformly continuous. Hence, $O_R$ is continuous on $[0, E] \times \left[0, \bar{d}(E)\right] \times [0, 1]$. The constraint correspondence $\Gamma_R : e_t \to \mathcal{S}_R(e_t)$ maps $[0, E]$ into $\left[0, \bar{d}(E)\right] \times [0, 1]$. The characterization of the set $S_R(e_t)$ in proposition 2.16 moreover shows that the graph of the correspondence $\Gamma_R$ is closed and convex. Theorems 3.4 and 3.5 in Stokey, Lucas, and Prescott (1989) then indicate that $\Gamma_R$ is continuous. Given that $O_R$ is continuous and $\Gamma_R$ compact-valued and continuous, the theorem of the maximum applies, and guarantees that $T_R V \in C(E)$. In analogous steps, one can prove that $T_K V \in C(E)$. Therefore, $TV = \max(T_R V, T_K V) \in C(E)$. Moreover, let $V \in C_0(E)$. Then $V^c(0) \geq 0$ and $V^c$ is increasing, by lemma 1. Moreover $S_R(0) \neq \emptyset$, so one can evaluate $O_R$ at some $(d_{t,0}, s_{t,0}) \in S_R(0)$. Since $n_R \geq 0$, $V^c(0) \geq 0$ and $V^c$ is increasing, $O_R(0, d_{t,0}, s_{t,0}) \geq 0$. Therefore $T_R V(0) \geq 0$, so $TV(0) \geq 0$ and $TV \in C_0(E)$.

**Step 3:** Finally, I establish that the operator $T$ has the monotonicity and discounting properties. First, let $(V, W) \in C(E)$ such that $\forall e_t \in C(E)$, $V(e_t) \geq W(e_t)$. Pick a particular $e_t \in [0, E]$. By an argument similar to the proof of lemma 1, $\forall n_t \geq 0, V^c(n_t) \geq W^c(n_t)$, where $W^c$ denotes the solution to the dividend issuance problem when the continuation value is $W$ (and analogously for V). Since the functions $\underline{\phi}_R, \overline{\phi}_R$ and $n_R$ are independent of $V$, this inequality implies $O_R^V(e_t, d_t, s_t) \geq O_R^W(e_t d_t, s_t)$ for any $(d_t, s_t) \in \mathcal{S}_R(e_t)$, where the notation $O_R^W$ designates the objective function in problem (A1-R) when the continuation value function is $W$ (and analogously for $V$). Thus $T_R V(e_t) \geq T_R W(e_t)$. Similarly, one can show that $T_K V(e_t) \geq T_K W(e_t)$. Therefore, $TV(e_t) \geq TW(e_t)$, and $T$ has the monotonicity property. To establish the discounting property, it is sufficient to note that $(V + a)^c(n_t) = V^c(n_t) + \beta a$, so that for any $e_t \in [0, E]$ and $(d_t, s_t) \in \mathcal{S}_R(e_t)$, $O_R^{V+a}(e_t, d_t, s_t) = O_R^V(e_t, d_t, s_t) + \left(1 - F\left(\underline{\phi}_R(e_t, d_t, s_t)\right)\right) \beta a \leq O_R^V(e_t, d_t, s_t) + \beta a$. This shows that $T_R(V + a)(e_t) \leq T_R V(e_t) + \beta a$. A similar claim can be made for $T_K$. Therefore, the operator $T$ has the discounting property. The Blackwell sufficiency conditions hold, so that $T$ is a contraction mapping. $\square$

*Properties of the solution to problem (3.8).* Let $V$ denote the unique solution to problem (3.8).

**Monotonicity** First, it can be shown that $\forall (e_t^1, e_t^2) \in [0, E]$ s.t. $e_t^1 < e_t^2$, $T_R V(e_t^1) < T_R V(e_t^2)$. To show this, let $(d_t, s_t) \in S_R(e_t^1)$. I proceed in three steps:

**1 :** First, since $e_t^1 < e_t^2$, $S_R(e_t^1) \subset S_R(e_t^2)$ (a proof for this can be obtained using proposition 2.16; intuitively, this result indicates that increasing internal finance relaxes borrowing constraints). Therefore, $(d_t, s_t) \in S_R(e_t^2)$.

**2 :** Next, I show that $\overline{\phi}_R(e_t^1, d_t, s_t) > \overline{\phi}_R(e_t^2, d_t, s_t)$. (The functions at this point are well-defined because $(d_t, s_t) \in S_R(e_t^2)$). Given the expression for $\overline{\phi}_R$ in (A1-R), since $G^{-1}$ is strictly increasing on $\mathbb{R}_+$, it is sufficient to show that $e_t \to \frac{(1+r_b)d_t s_t - \chi(1-\delta)(e_t+d_t)}{\chi(e_t+d_t)^\zeta}$ is strictly decreasing in $e_t$. This is true because $\zeta < 1$. Next I prove that $\underline{\phi}_R(e_t^1, d_t, s_t) > \underline{\phi}_R(e_t^2, d_t, s_t)$. To see this, note that:

$$\frac{\partial \underline{\phi}_R(e_t, d_t, s_t)}{\partial e_t} = \frac{\frac{\partial y_I}{\partial e_t} - I_2\left(I^{-1}\left(y_I(e_t); e_t + d_t\right); e_t + d_t\right)}{I_1\left(I^{-1}\left(y_I(e_t); e_t + d_t\right); e_t + d_t\right)},$$

where $y_I(e_t) \equiv \frac{(1+r_m)d_t(1-s_t) - (1-\chi)(1-\delta)(e_t+d_t)}{(1-\chi)(e_t+d_t)^\zeta}$. Note that, letting $x = I^{-1}\left(y_I(e_t); e_t + d_t\right)$:

$$\frac{\partial y_I}{\partial e_t} - I_2(x; e_t + d_t) = -\zeta \frac{(1+r_m)d_t(1-s_t)}{(1-\chi)(e_t+d_t)^{\zeta+1}} - (1-\zeta)(1-\delta)(e_t+d_t)^{-\zeta}\left(1 - F(x)\right) < 0$$

Since $I_1 > 0$, this implies that $\underline{\phi_R}$ is strictly decreasing in $e_t$.

**3 :** Given the fact that $\underline{\phi}_R$ and $\overline{\phi}_R$ are strictly decreasing in $e_t$, the expression for $n_R(\phi_t; e_t, d_t, s_t)$ in (A1-R) then implies that $\forall \phi_t \geq \underline{\phi}_R(e_t^1, d_t, s_t)$, $n_R(\phi_t; e_t^1, d_t, s_t) < n_R(\phi_t; e_t^2, d_t, s_t)$. Thus, since $V^c$ is increasing,

$$\begin{aligned}
O_R(e_t^1, d_t, s_t) &= \int_{\phi_t \geq \underline{\phi}_R(e_t^1, d_t, s_t)} V^c\left(n_R\left(\phi_t; e_t^1, d_t, s_t\right)\right) dF(\phi_t) \\
&< \int_{\phi_t \geq \underline{\phi}_R(e_t^1, d_t, s_t)} V^c\left(n_R\left(\phi_t; e_t^2, d_t, s_t\right)\right) dF(\phi_t) \\
&\leq \int_{\phi_t \geq \underline{\phi}_R(e_t^2, d_t, s_t)} V^c\left(n_R\left(\phi_t; e_t^2, d_t, s_t\right)\right) dF(\phi_t) \\
&= O_R(e_t^2, d_t, s_t).
\end{aligned}$$

where the second line exploits the fact that $\underline{\phi}_R(e_t^1, d_t, s_t) > \underline{\phi}_R(e_t^2, d_t, s_t)$ and $V^c \geq 0$.

Since last inequality has been established for any $(d_t, s_t) \in S_R(e_t^1) \subset S_R(e_t^2)$, it shows that the objective function is uniformly increasing (strictly) in $e_t$, so that $T_R V(e_t^1) < T_R V(e_t^2)$. A similar but simpler proof using the expression for $\underline{\phi}_K$ in (A1-K) shows that $T_K V(e_t^1) < T_K V(e_t^2)$. Therefore, $TV(e_t^1) < TV(e_t^2)$, so that $V(e_t^1) < V(e_t^2)$. Therefore, the solution to problem 3.8 is strictly increasing in $e_t$. $\square$

*Existence and unicity of an invariant measure of firms across equity levels.* I next prove that, given a solution to problem (3.8), an invariant measure of firms across levels of $e_t$ exists. I start by introducing some preliminary notation.

**Preliminary notation** $\bar{e}$ denotes the level of net worth above which firms start issuing dividends. $\bar{E} = [0, \bar{e}]$ denotes the state-space of the firm problem (3.8). $\left(\bar{E}, \bar{\mathcal{E}}\right)$ is the measurable space composed of $\bar{E}$ and the family of Borel subsets of $\bar{E}$. For any value $e_t \in \bar{E}$, $\hat{d}(e_t)$ and $\hat{s}(e_t)$ denote the policy functions of the firms. The fact that these policy functions are such that $(\hat{d}(e_t), \hat{s}(e_t)) \in \mathcal{S}_R(e_t)$ will be denoted by $e_t \in \bar{E}_R$, and $e_t \in \bar{E}_K$ for the other case. $\hat{\underline{\phi}}_R(e_t)$ and $\hat{\bar{\phi}}_R(e_t)$ denote the liquidation threshold implied by the firm's policy functions when $e_t \in \bar{E}_R$, while $\underline{\hat{\phi}}_K(e_t)$ denotes the liquidation threshold when $e_t \in \bar{E}_K$. I use the notation: $\hat{r}(e_t) = r_m(1 - \hat{s}(e_t)) + r_b\hat{s}(e_t)$. Finally, recall that $F(.)$ denotes the CDF of $\phi_t$, the idiosyncratic productivity shock, and $\eta$ denotes the exogenous exit probability.

**Transition function** To define the transition function Q implied by firms' policy functions, one can proceed by constructing the probability of a firm having a level of internal finance smaller than or equal to $e_{t+1}$ next period, given that its current internal funds are $e_t$. Additionally, one must take into account the fact that the fraction $\eta$ of firms that exit exogenously, plus those that are liquidated endogenously, will be replaced by firms operating at the entry scale $e^e$. Insofar as the evolution of the measure of firms is concerned, this is equivalent to assuming that firms these firms transition to the level $e^e$.[8] The resulting probability of having an level of internal finance smaller than or equal to $e_{t+1}$, given that current internal finance is $e_t$, is denoted by $N(e_t, e_{t+1})$, and is given by the following expressions.

$\boxed{\bullet \textbf{ For } e_t \in \bar{E}_K:}$

- If $(1-\delta)(e_t + \hat{d}(e_t)) > (1 + \hat{r}(e_t))\hat{d}(e_t) + \bar{e}$, the firm never liquidates and always has internal funds of at least $\bar{e}$ after the debt settlement stage, so:

$$N(e_t, e_{t+1}) = \eta \mathbb{1}\{e^e \leq e_{t+1}\} + (1-\eta) \begin{cases} 0 & \text{if } e_{t+1} \leq \bar{e} \\ 1 & \text{if } \bar{e} \leq e_{t+1} \end{cases}$$

- If $(1+\hat{r}(e_t))\hat{d}(e_t) + \bar{e} \geq (1-\delta)(e_t + \hat{d}(e_t)) > (1+\hat{r}(e_t))\hat{d}(e_t)$, the firm is never liquidated

[8] However, one cannot assume this in the expression of the value function of the firm; this would indeed change firms' incentive to default, restructure and renegotiate.

but may have internal funds below $\bar{e}$ after the debt settlement stage, so:

$$N(e_t, e_{t+1}) = \eta \mathbb{1}\{e^e \leq e_{t+1}\} + (1-\eta) \begin{cases} 0 & \text{if } e_{t+1} < \underline{e}(e_t) \\ F\left(\frac{e_{t+1} - \underline{e}(e_t)}{(e_t + \hat{d}(e_t))^\varsigma}\right) & \text{if } \underline{e}(e_t) \leq e_{t+1} < \bar{e} \\ 1 & \text{if } \bar{e} \leq e_{t+1} \end{cases}$$

where $\underline{e}(e_t) = (1-\delta)(e_t + \hat{d}(e_t)) - (1 + \tilde{r}(e_t))\hat{d}(e_t)$ is the lower bound on these firms' internal funds (that is, their internal funds if $\phi_t = 0$).

- If $(1+\hat{r}(e_t))\hat{d}(e_t) \geq (1-\delta)(e_t + \hat{d}(e_t))$, the firm may be liquidated at the debt settlement stage (when $\phi_t \leq \underline{\hat{\phi}}_K(e_t)$), so:

$$\begin{aligned} N(e_t, e_{t+1}) \quad &= \quad \left(\eta + (1-\eta) F\left(\underline{\hat{\phi}}_K(e_t)\right)\right) \mathbb{1}\{e^e \leq e_{t+1}\} \\ &\quad + (1-\eta) \begin{cases} 0 & \text{if } e_{t+1} < 0 \\ F\left(\underline{\hat{\phi}}_K(e_t) + \frac{e_{t+1}}{(e_t + \hat{d}(e_t))^\varsigma}\right) - F\left(\underline{\hat{\phi}}_K(e_t)\right) & \text{if } 0 \leq e_{t+1} < \bar{e} \\ 1 - F\left(\underline{\hat{\phi}}_K(e_t)\right) & \text{if } \bar{e} \leq e_{t+1} \end{cases} \end{aligned}$$

**• For $e_t \in \bar{E}_R$:** There are two subcases, depending on whether $\underline{\hat{\phi}}_R(e_t) \gtrless \hat{\bar{\phi}}_R(e_t) - \frac{\bar{e}}{(1-\chi)(e_t + \hat{d}(e_t))^\varsigma}$.

**- If $\underline{\hat{\phi}}_R(e_t) > \hat{\bar{\phi}}_R(e_t) - \frac{\bar{e}}{(1-\chi)(e_t + \hat{d}(e_t))^\varsigma}$:**

- If $(1-\delta)(e_t + \hat{d}(e_t)) \geq (1 + \hat{r}(e_t))\hat{d}(e_t) + \bar{e}$, the firm never liquidates or restructures, and always has internal funds of at least $\bar{e}$ after the debt settlement stage, so:

$$N(e_t, e_{t+1}) = \eta \mathbb{1}\{e^e \leq e_{t+1}\} + (1-\eta) \begin{cases} 0 & \text{if } e_{t+1} \leq \bar{e} \\ 1 & \text{if } \bar{e} \leq e_{t+1} \end{cases}$$

- If $(1+\hat{r}(e_t))\hat{d}(e_t)+\bar{e} > (1-\delta)(e_t+\hat{d}(e_t)) \geq \frac{(1+r_b)\hat{s}(e_t)\hat{d}(e_t)}{\chi}$, the firm may sometimes have internal finance below $\bar{e}$ after the debt settlement stage, but still never liquidates or restructures, so:

$$N(e_t,e_{t+1}) = \eta\mathbb{1}\{e^e \leq e_{t+1}\} + (1-\eta)\begin{cases} 0 & \text{if } e_{t+1} < \underline{e}(e_t) \\ F\left(\frac{e_{t+1}-\underline{e}(e_t)}{(e_t+\hat{d}(e_t))^\zeta}\right) & \text{if } \underline{e}(e_t) \leq e_{t+1} < \bar{e} \\ 1 & \text{if } \bar{e} \leq e_{t+1} \end{cases}$$

where again, $\underline{e}(e_t) = (1-\delta)(e_t+\hat{d}(e_t)) - (1+\hat{r}(e_t))\hat{d}(e_t)$.

- If $\frac{(1+r_b)\hat{s}(e_t)\hat{d}(e_t)}{\chi} > (1-\delta)(e_t+\hat{d}(e_t)) \geq \frac{(1+r_m)(1-\hat{s}(e_t))}{1-\chi}$, then the firm will sometimes restructure at the debt settlement stage, but never liquidate, so:

$$\begin{aligned} N(e_t,e_{t+1}) &= \eta\mathbb{1}\{e^e \leq e_{t+1}\} \\ &\quad +(1-\eta)\begin{cases} 0 & \text{if } e_{t+1} < \underline{e}(e_t) \\ F\left(\frac{e_{t+1}-\underline{e}(e_t)}{(1-\chi)(e_t+\hat{d}(e_t))^\zeta}\right) & \text{if } \underline{e}(e_t) \leq e_{t+1} < \underline{e}(e_t) + (1-\chi)(e_t+\hat{d}(e_t))^\zeta \hat{\bar{\phi}}_R(e_t) \\ F\left(\frac{e_{t+1}-\underline{e}(e_t)}{(e_t+\hat{d}(e_t))^\zeta} + \chi\hat{\bar{\phi}}_R(e_t)\right) & \text{if } \underline{e}(e_t) + (1-\chi)(e_t+\hat{d}(e_t))^\zeta \hat{\bar{\phi}}_R \leq e_{t+1} < \bar{e} \\ 1 & \text{if } \bar{e} \leq e_{t+1} \end{cases} \end{aligned}$$

where now, $\underline{e}(e_t) = (1-\chi)(1-\delta)(e_t+\hat{d}(e_t)) - (1+r_m)(1-\hat{s}(e_t))\hat{d}(e_t)$.

- Finally, if $\frac{(1+r_m)(1-\hat{s}(e_t))\hat{d}(e_t)}{(1-\chi)} > (1-\delta)(e_t+\hat{d}(e_t))$, then the firm will sometimes be liquidated, sometimes restructure and sometimes repay, so:

$$\begin{aligned} N(e_t,e_{t+1}) &= \left(\eta + (1-\eta)F\left(\hat{\underline{\phi}}_R(e_t)\right)\right)\mathbb{1}\{e^e \leq e_{t+1}\} \\ &\quad +(1-\eta)\begin{cases} 0 & \text{if } e_{t+1} < 0 \\ F\left(\hat{\underline{\phi}}_R(e_t) + \frac{e_{t+1}}{(1-\chi)(e_t+\hat{d}(e_t))^\zeta}\right) - F\left(\hat{\underline{\phi}}_R(e_t)\right) & \text{if } 0 \leq e_{t+1} < (1-\chi)(e_t+\hat{d}(e_t))^\zeta\left(\hat{\bar{\phi}}_R(e_t) - \hat{\underline{\phi}}_R(e_t)\right) \\ F\left(\frac{e_{t+1}}{(e_t+\hat{d}(e_t))^\zeta} + \chi\hat{\bar{\phi}}_R(e_t) + (1-\chi)\hat{\underline{\phi}}_R(e_t)\right) - F\left(\hat{\underline{\phi}}_R(e_t)\right) & \text{if } (1-\chi)(e_t+\hat{d}(e_t))^\zeta\left(\hat{\bar{\phi}}_R(e_t) - \hat{\underline{\phi}}_R(e_t)\right) \leq e_{t+1} < \bar{e} \\ 1 - F\left(\hat{\underline{\phi}}_R(e_t)\right) & \text{if } \bar{e} \leq e_{t+1} \end{cases} \end{aligned}$$

**- If $\underline{\hat{\phi}}_R(e_t) \leq \hat{\bar{\phi}}_R(e_t) - \frac{\bar{e}}{(1-\chi)(e_t+\hat{d}(e_t))^\varsigma}$:**

- If $(1-\delta)(e_t+\hat{d}(e_t)) \geq \frac{(1+r_m)(1-\hat{s}(e_t))\hat{d}(e_t)+\bar{e}}{1-\chi}$, then even when the firm restructures, it will still have enough cash on hand to have internal funds $\bar{e}$ tomorrow, so that:

$$N(e_t, e_{t+1}) = \eta \mathbb{1}\{e^e \leq e_{t+1}\} + (1-\eta) \begin{cases} 0 & \text{if } e_{t+1} \leq \bar{e} \\ 1 & \text{if } \bar{e} \leq e_{t+1} \end{cases}$$

- If $\frac{(1+r_m)(1-\hat{s}(e_t))\hat{d}(e_t)+\bar{e}}{1-\chi} > (1-\delta)(e_t+\hat{d}(e_t)) \geq \frac{(1+r_m)(1-\hat{s}(e_t))\hat{d}(e_t)}{1-\chi}$, then the firm may restructure and not have sufficient cash to reach internal funds $\bar{e}$ tomorrow, so that:

$$N(e_t, e_{t+1}) = \eta \mathbb{1}\{e^e \leq e_{t+1}\} + (1-\eta) \begin{cases} 0 & \text{if } e_{t+1} < \underline{e}(e_t) \\ F\left(\frac{e_{t+1}-\underline{e}(e_t)}{(1-\chi)(e_t+\hat{d}(e_t))^\varsigma}\right) & \text{if } \underline{e}(e_t) \leq e_{t+1} < \bar{e} \\ 1 & \text{if } \bar{e} \leq e_{t+1} \end{cases}$$

  where again $\underline{e}(e_t) = (1-\chi)(1-\delta)(e_t+\hat{d}(e_t)) - (1+r_m)(1-\hat{s}(e_t))\hat{d}(e_t)$ is the lower bound on these firms' internal finance (that is, their internal finance when $\phi_t = 0$).

- Finally, if $\frac{(1+r_m)(1-\hat{s}(e_t))\hat{d}(e_t)}{1-\chi} \geq (1-\delta)(e_t+\hat{d}(e_t))$, then the firm will sometimes liquidate, so that:

$$\begin{aligned} N(e_t, e_{t+1}) &= \left(\eta + (1-\eta)F\left(\underline{\hat{\phi}}_R(e_t)\right)\right)\mathbb{1}\{e^e \leq e_{t+1}\} \\ &\quad + (1-\eta)\begin{cases} 0 & \text{if } e_{t+1} < 0 \\ F\left(\underline{\hat{\phi}}_R(e_t) + \frac{e_{t+1}}{(1-\chi)(e_t+\hat{d}(e_t))^\varsigma}\right) - F\left(\underline{\hat{\phi}}_R(e_t)\right) & \text{if } 0 \leq e_{t+1} < \bar{e} \\ 1 - F\left(\underline{\hat{\phi}}_R(e_t)\right) & \text{if } \bar{e} \leq e_{t+1} \end{cases} \end{aligned}$$

It is straightforward to check that given $e_t$, $N(e_t, .)$ is weakly increasing, has limits 0 and

1 at $-\infty$ and $+\infty$ and is everywhere continuous from above, using the expressions given above. Following theorem 12.7 of Stokey, Lucas, and Prescott (1989), there is therefore a unique probability measure $\hat{Q}(e_t, .)$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\hat{Q}(e_t, ]-\infty, e]) = N(e_t, e)\ \forall e \in \mathbb{R}$. This measure is 0 for $e < 0$ and 1 for $e > \bar{e}$, and so the restriction of $\hat{Q}$ to $\bar{\mathcal{E}}$ is also a probability measure on $\left(\bar{E}, \bar{\mathcal{E}}\right)$, which can be denoted $Q(e_t, .)$. Moreover, fixing $e \in \bar{E}$, the function $Q(., ]-\infty, e]) : \bar{E} \to [0,1]$ is measurable with respect to $\bar{\mathcal{E}}$. Indeed, given $z \in [0,1]$, the definition of $N(e_t, e)$ indicates that the set $H(z,e) = \left\{e_t \in \bar{E} \text{ s.t. } Q(e_t, [0,e]) \leq z\right\} = \left\{e_t \in \bar{E} \text{ s.t. } \hat{Q}(e_t, ]-\infty, e]) \leq z\right\} = \left\{e_t \in \bar{E} \text{ s.t. } N(e_t, e) \leq z\right\}$ is an element of $\bar{\mathcal{E}}$.[9] This in turn implies that the function $Q(., \mathbf{E})$ is $\bar{\mathcal{E}}$-measurable for any $\mathbf{E} \in \bar{\mathcal{E}}$. The function $Q : \bar{E} \times \bar{\mathcal{E}} \to [0,1]$ is such that $Q(e_t, .)$ is therefore a probability measure for any $e_t \in \bar{E}$, and $Q(., \mathbf{E})$ is $\bar{\mathcal{E}}$-measurable $\forall \mathbf{E} \in \bar{\mathcal{E}}$; hence, $Q$ is a transition function.

**Feller property** To establish that the transition function $Q$ has the Feller property, one must show that $\forall e_t \in \bar{E}$ and $\forall\ (e_{n,t})_n \in \bar{E}^{\mathbb{N}}$ such that $e_{n,t} \to e_t$, $Q(e_{n,t}, .) \Rightarrow Q(e_t, .)$, where $\Rightarrow$ denotes weak convergence. To establish this, given the definition of $Q$ it is sufficient to show that $N(e_{n,t}, e_{t+1}) \to N(e_t, e_{t+1})$ pointwise, at all values of $e_{t+1}$ and $e_t$ where $N(., e_{t+1})$ and $N(e_t, .)$ are continuous. This excludes, in particular, the cases $e_{t+1} = \bar{e}$ or $e_t = \bar{e}$. I give

[9] For example, if $z \leq \eta$ and $0 < e < \bar{e}$, the intersection of the set $H(z,e)$ with $\bar{E}_K$ is given by:

$$
\begin{aligned}
H(z,e) \cap \bar{E}_K &= \left\{e_t \in \bar{E}_K \text{ s.t. } (1-\delta)(e_t + \hat{d}(e_t)) \geq (1+\hat{r}(e_t))\hat{d}(e_t) + \bar{e}\right\} \cup \\
&\qquad \left\{e_t \text{ s.t. } (1+\hat{r}(e_t))\hat{d}(e_t) + \bar{e} > (1-\delta)(e_t + \hat{d}(e_t)) \geq (1+\hat{r}(e_t))\hat{d}(e_t) \text{ and } \underline{e}(e_t) > e\right\} \\
&= \left\{e_t \in \bar{E}_K \text{ s.t. } (1-\delta)(e_t + \hat{d}(e_t)) > (1+\tilde{r}(e_t))\hat{d}(e_t) + e\right\}
\end{aligned}
$$

This set is the inverse image of $]0, +\infty[$ by the function $g : \bar{E}_K \to \mathbb{R}, x \to (1-\delta)(x+\hat{d}(x)) - (1+\tilde{r}(x))\hat{d}(x) - e$. Since the policy functions are continuous, $g$ is continuous. Since the inverse image of an open set by a continuous function is an open set, $H(z,e) \cap \bar{E}_K$ is an open set, and hence a Borel set. The intersection $H(x,e) \cap \bar{E}_R$ has a more complicated expression, but also boils down to a finite union of sets that are open because of the continuity of policy functions. Given that $\bar{E}_K$ and $\bar{E}_R$ are intervals that form a partition of $\bar{E}$, this implies that $H(z,e) = (H(z,e) \cap \bar{E}_R) \cup (H(z,e) \cup \bar{E}_K)$ is a finite union of Borel sets, and therefore a Borel set. This line of reasoning applies for all $0 \leq z \leq 1$ and $0 \leq e \leq \bar{e}$.

the proof for the case $e_t \in \bar{E}_K$; the proof for the other case is similar.

First consider a simple case, when $e_t$ satisfies:

$$(1+\hat{r}(e_t))\hat{d}(e_t) > (1-\delta)(e_t+\hat{d}(e_t)) \tag{C.1}$$

Because $e_t \mapsto \eta + (1-\eta)F\left(\underline{\hat{\phi}}_K(e_t) + \frac{e_{t+1}}{(e_t+\hat{d}(e_t))^\varsigma}\right)$ is continuous on $\bar{E}_K$, $e \mapsto N(e, e_{t+1})$ is continuous in a neighborhood of $e_t$. Additionally, since $\hat{r}$ and $\hat{d}$ is continuous, the inequality (C.1) holds for $e_{n,t}$, when $n$ is sufficiently large. Combining these two observations implies that $N(e_{n,t}, e_{t+1}) \to N(e_t, e_{t+1})$. The case when the inequality above holds in the reverse direction is handled similarly.

Now consider a knife-edge case:

$$(1+\hat{r}(e_t))\hat{d}(e_t) = (1-\delta)(e_t+\hat{d}(e_t)) \tag{C.2}$$

The proglem is that the sequence $(e_{n,t})_n$ can have elements that satisfy either $(1+\hat{r}(e_{n,t}))\hat{d}(e_{n,t} \gtrless (1-\delta)(e_{n,t}+\hat{d}(e_{n,t}))$, which correspond to different expressions for $N(e_{n,t}, e_{t+1})$; one must check therefore check that $N(., e_{t+1})$ is continuous at $e_t$ such that (C.2) holds. At such a point, $\underline{e}(e_t) = 0$, so that:

$$\lim_{e \uparrow e_t} N(e_t, e_{t+1}) = \lim_{e \uparrow e_t} \eta + (1-\eta)F\left(\frac{e_{t+1} - \bar{e}(e_t)}{(e_t+\hat{d}(e_t))^\varsigma}\right) = \eta + (1-\eta)F\left(\frac{e_{t+1}}{(e_t+\hat{d}(e_t))^\varsigma}\right).$$

Moreover, $\underline{\hat{\phi}}_K(e_t) = 0$, so that:

$$\lim_{e \downarrow e_t} N(e_t, e_{t+1}) = \lim_{e \downarrow e_t} \eta + (1-\eta)F\left(\underline{\hat{\phi}}_K(e_t) + \frac{e_{t+1}}{(e_t+\hat{d}(e_t))^\varsigma}\right) = \eta + (1-\eta)F\left(\frac{e_{t+1}}{(e_t+\hat{d}(e_t))^\varsigma}\right).$$

This establishes the continuity of $N(., e_{t+1})$ at points $e_t$ such that equation (C.2) holds; thus,

$\forall e_{t+1} \in \bar{E}$, $\bar{e}_{t+1} \neq \bar{e}$, $N(., e_{t+1})$ is continuous on $\bar{E}_K$. $\square$

## C.3 Computational procedures

**Computation of a solution to the firm's problem (3.8)** The algorithm is a straightforward iteration in two fixed points, $V(.)$ and $\bar{e}$, except for the important insight that the feasible sets $\mathcal{S}_K(e_t)$ and $\mathcal{S}_R(e_t)$ can be computed *outside* of iterations used to compute fixed point of the mapping $T$ (value functions $V(.)$). This is again because the feasible sets are independent of the value function $V(.)$, but only depend on firms' internal funds $e_t$.[10]

1. Guess a value for $\bar{e}$ and choose a discrete grid with $N_e$ points on $[0, \bar{e}]$, $\{e_{i,t}\}_{i=1}^{N_e}$.[11]

2. For $i = 1, ..., N_e$, compute and store the frontiers of the sets $\mathcal{S}_K(e_{i,t})$ and $\mathcal{S}_R(e_{i,t})$ depicted in figure 3.4. This computation is done using the results from chapter 2. For example, the frontiers of the set $\mathcal{S}_R(e_{i,t})$ are given by:

$$\left\{ \left(s, \bar{d}_R(s)\right) \in [0,1] \times \mathbb{R}^+ \text{s.t.} s \in [s_{R,min}, 1] \right\},$$

   where $s_{R,min} = \frac{1}{1+\frac{1-\chi}{\chi}\frac{1+r_b}{1+r_m}}$ and $\bar{d}_R(.)$ is a function of $s$, the expression of which is reported in appendix B.1. The frontiers are computed by discretizing the interval $[s_{R,min}, 1]$ on a grid of size $N_s$, evaluating the function $\bar{d}_R(.)$ at each of these points, and storing the resulting points $\{s_j, \bar{d}(s_j)\}_{j=1}^{N_s}$.[12] The frontier of $\mathcal{S}_K(e_{i,t})$ is similarly approximated on a discrete grid covering $[0, s_{R,max}(e_{i,t})]$.

[10] The computation of the solution to the problem (3.8) would otherwise be considerably more expensive.

[11] The grid used is log-spaced and concentrates points close to 0.

[12] $N_s$ is taken to be sufficiently large to ensure that space between two gridpoints is at most 0.0005, which corresponds to 0.05% in terms of the share of bank credit as a fraction of total loans.

3. Choose an initial guess for the value function $V_0(.)$ at the points $\{e_{i,t}\}$. A useful initial guess is to first compute the value $V_S(.)$ of the firm in the static version of the model, and then use:

$$V_0(e_{i,t}) = V_S(e_{i,t}) + \frac{1}{1-(1-\eta)\beta}\left(\mathbb{E}(\phi)(\bar{e}+\hat{d}_S(\bar{e}))^{\zeta} + (1-\delta)(\bar{e}+\hat{d}_S(\bar{e})) - (1+r_m)\hat{d}_S(\bar{e}) - \bar{e}\right),$$

where $\hat{d}_S(\bar{e})$ denotes total borrowing when $e = \bar{e}$ in the solution to the static model. The term in parenthesis represents dividend issuance by an infinitely-lived firm with internal finance $\bar{e}$, market borrowing $\hat{d}_S(\bar{e})$ and no bank loans, assuming that this borrowing amount is riskless.

4. Apply the mapping $T$ defined in equation (A1) until convergence. This step takes into account the fact that the firm chooses market-only borrowing for some $e_t \geq e^*$. It proceeds as follows:

   (a) Guess a value for $e^*$, the switching threshold, and for $\left\{\hat{d}(e_{i,t}), \hat{s}(e_{i,t})\right\}$, the firm's policy functions, and an initial guess for the value function $V^{(0)}$.

   (b) Given a guess for $V^{(n)}(.)$, compute the implied values $TV_R^{(n)}(e_{i,t})$ for $e_{i,t} \leq e^*$ and $TV_K^{(n)}(e_{i,t})$ for $e_{i,t} \geq e^*$, assuming that the policy of the firm is given by $\left\{\hat{d}(e_{i,t}), \hat{s}(e_{i,t})\right\}$. Let $V^{(n+1)}(e_{i,t}) = TV_R^{(n)}(e_{i,t})$ for $e_{i,t} \geq e^*$ and $V^{(n+1)}(e_{i,t}) = TV_K^{(n)}(e_{i,t})$ for $e_{i,t} > e^*$. Repeat this Howard improvement step for $0 \leq n \leq N_H$.

   (c) Given $V^{(N_H)}(.)$, compute policy functions, value functions, and the implied value of the switching threshold $e^{*,(N_H)}$.

   Steps (a)-(c) are started with the guess $V^{(0)} = V_0$. They are repeated until $V^{(N_H)}$ and $V^{(0)}$ are sufficiently close, and $e^{*,(0)}$ and $e^{*,(N_H)}$ are sufficiently close. At each repetition of $(a)-(c)$ after the initial one, the initial guesses used in $(a)$ are the policy functions along with $TV^{(N_H)}$ and $e^{*,(N_H)}$ from the previous step. In steps (a)-(c), one needs to compute integrals of the form $\int_\phi V^c(n(\phi,d,s,e))dF(\phi)$, given the discrete approximation to the value function (which take into account the dividend issuance policy of the firm). I detail the method used below. The maximization over $(d,s)$ in step (c) is carried out using a numerical constrained maximization procedure, rather than a discretized grid; the constraints are the frontiers computed in step 1.

5. Given the final value function $V^{(N_H)}(.)$, approximate numerically the derivative $\frac{\partial V^{(N_H)}}{\partial e}(\bar{e})$ and check whether it is sufficiently close to $\frac{1}{(1-\eta)\beta}$. If not, adjust the guess for $\bar{e}$ (upwards if $(1-\eta)\beta\frac{\partial V^{(N_H)}}{\partial e}(\bar{e}) > 1$, downwards otherwise) and repeat steps (1)-(4). To accelerate convergence, interpolate the value function $V^{(N_H)}(.)$ on the new grid $[0,\bar{e}]$ and use the interpolated value as new starting guess in step 3.

**Computation of the value function** A problem in steps (b) and (c) is the computation of the iterate $TV$, given a guess for $V$ and values for the debt structure $(d_t,s_t)$. Here, I give an example of the computation of this value function when $(d_t,s_t)\in\mathcal{S}_R(e_t)$ and: $\frac{(1+r_m)(1-s_t)d_t}{1-\chi} > (1-\delta)(e_t+d_t)$. This is the most involved case, since the firm is sometimes liquidated, sometimes manages to restructure its bank loans, and sometimes repays its creditors in full.

Taking into account the dividend issuance policy of the firm, as well as the liquidation/restructuring choice, the objective function of the firm (A1-R) is given by:

$$\int_{\phi_t\geq\underline{\phi}_R(e_t,d_t,s_t)} V^c\left(n_R\left(\phi_t;e_t,d_t,s_t\right)\right)dF(\phi_t) = (1-\eta)\beta\int_{\underline{\phi}_R(e_t,d_t,s_t)}^{\overline{\phi}_R(e_t,d_t,s_t)} V\left((\phi-\underline{\phi}_R(e_t,d_t,s_t))(1-\chi)(e_t+d_t)^\zeta\right)dF(\phi) \tag{INT-1}$$

$$+(1-\eta)\beta\int_{\overline{\phi}_R(e_t,d_t,s_t)}^{\phi^M(e_t,d_t,s_t,\bar{e})} V\left((\phi-\chi\overline{\phi}_R(e_t,d_t,s_t))-(1-\chi)\underline{\phi}_R(e_t,d_t,s_t))(e_t+d_t)^\zeta\right)dF(\phi) \tag{INT-2}$$

$$+\left(1-F(\phi^M(e_t,d_t,s_t,\bar{e}))\right)\left((1-\eta)\beta V(\bar{e})-\phi^M(e_t,d_t,s_t,\bar{e})(e_t+d_t)^\zeta\right)$$
$$+\left(\int_{\phi^M(e_t,d_t,s_t,\bar{e})}^{+\infty}\phi dF(\phi)\right)(e_t+d_t)^\zeta$$

where $\phi^M(e_t,d_t,s_t,\bar{e})=\chi\overline{\phi}_R(e_t,d_t,s_t))+(1-\chi)\underline{\phi}_R(e_t,d_t,s_t)+\frac{\bar{e}}{(e_t+d_t)^\zeta}$ is the threshold for the idiosyncratic shock $\phi$ above which the firm starts issuing dividends. The intervals $(INT-1)$ and $(INT-2)$ are computed using the linear interpolation of $V(.)$ on two different sub-intervals. Namely, first find the index $1\leq k_0\leq n-1$ such that $(1-\chi)(e_t+d_t)^\zeta(\overline{\phi}_R(e_t,d_t,s_t)-\underline{\phi}_R(e_t,d_t,s_t))\in]e_{t,k_0};e_{t,k_0+1}]$. Define $\{\phi_k\}_{k=1}^{N_e}$, $\{A_k\}_{k=1}^{N_e}$ and $\{B_k\}_{k=1}^{N_e}$ by, for $1\leq k\leq k_0$:

$$\phi_k=(1-\chi)\underline{\phi}_R(e_t,d,s_t)+\frac{e_k}{(1-\chi)(e_t+d_t)^\zeta}$$
$$A_k=V(e_{k,t})-\left((1-\chi)\underline{\phi}_R(e_t,d_t,s_t)(e_t+d_t)^\zeta+e_{k,t}\right)\frac{V(e_{k+1,t})-V(e_{k,t})}{e_{k+1,t}-e_{k,t}}$$
$$B_k=\frac{V(e_{k+1,t})-V(e_{k,t})}{e_{k+1,t}-e_{k,t}}(1-\chi)\underline{\phi}_R(e_t,d,s_t)(e_t+d_t)^\zeta$$

and, for $k_0+1\leq k\leq N_e-1$:

$$\phi_k=\chi\overline{\phi}_R(e_t,d_t,s_t)+(1-\chi)\underline{\phi}_R(e_t,d,s_t)+\frac{e_k}{(e_t+d_t)^\zeta}$$
$$A_k=V(e_{k,t})-\left(\chi\overline{\phi}_R(e_t,d_t,s_t)+(1-\chi)\underline{\phi}_R(e_t,d,s_t)+e_{k,t}\right)\frac{V(e_{k+1,t})-V(e_{k,t})}{e_{k+1,t}-e_{k,t}}$$
$$B_k=\frac{V(e_{k+1,t})-V(e_{k,t})}{e_{k+1,t}-e_{k,t}}\left(\chi\overline{\phi}_R(e_t,d_t,s_t)+(1-\chi)\underline{\phi}_R(e_t,d,s_t)\right)(e_t+d_t)^\zeta$$

The two integrals are then approximated as:

$$(INT-1)+(INT-2)=\sum_{k=1}^{k_0}\int_{\phi_k}^{\phi_{k+1}}(A_k+B_k\phi)dF(\phi)$$
$$+\int_{\phi_{k_0}}^{\overline{\phi}_R(e_t,d_t,s_t)}(A_{k_0}+B_{k_0}\phi)dF(\phi)+\int_{\overline{\phi}_R(e_t,d_t,s_t)}^{\phi_{k_0+1}}(A_{k_0+1}+B_{k_0+1}\phi)dF(\phi)$$

$$+\sum_{k=k_0+1}^{N_e-1}\int_{\phi_k}^{\phi_{k+1}}(A_k+B_k\phi)dF(\phi).$$

This method works for $2 \leq k_0 \leq N_e - 2$; the extensions to the cases $k_0 = 1$ and $k_0 = N_e - 1$ are straightforward.

There are a number of other cases to be covered, depending on whether $(d_t, s_t) \in \mathcal{S}_R(e_t)$ or $(d_t, s_t) \in \mathcal{S}_K(e_t)$, whether the implied contracts are riskless or not, and on the threshold at which the firm has sufficient cash on hand to issue dividends after the debt settlement stage. However, in all these cases the value function can be approximated using linear interpolation, as described above. The exact expressions for the value function, in each case, are available on request.

**Computation of the invariant distribution** The invariant distribution of firms over the set $[0, \bar{e}]$ is approximated using a discrete set of weights $\{w_i\}_{i=1}^{N_e}$, which represent the mass of firms with internal finance in the interval $\left]\frac{1}{2}(e_{i-1}+e_i);\frac{1}{2}(e_i+e_{i+1})\right]$ for $2 \leq i \leq N_e - 1$, and the mass of firms in the interval $\left[0, \frac{1}{2}e_2\right]$ for $i = 1$ and $\left]\frac{1}{2}(e_{N_e-1}+e_{N_e}), e_{N_e}\right]$ for $i = N_e$. The invariant distribution is the solution to the matrix equation $wP = w$, where $w = (w_1, ..., w_{N_e})$ and $M$ is a $(N_e \times N_e)$ matrix representing the transition kernel $Q$. Specifically, the entry $m_{i,j}$ is given by $m_{i,1} = N(e_i, \frac{1}{2}e_2)$, $m_{i,j} = N(e_i, \frac{1}{2}(e_j+e_{j+1})) - N(e_i, \frac{1}{2}(e_j+e_{j-1}))$ for $2 \leq j \leq N_e-1$, and $m_{i,j} = 1 - N(e_i, \frac{1}{2}e_{N_e-1}))$ for $j = N_e$. Note that computing the invariant distribution requires the entry scale $e^e$, which can be obtained using the solution to the firms' problem (3.8) along with the entry condition (3.9).

**Computation of the perfect foresight response of the economy** Computing the perfect foresight response of the economy to aggregate shocks is straightforward, because there are no endogenous aggregate price paths to solve for, since the cost of funds of financial intermediaries, $r$, always equals $\frac{1}{\beta} - 1$. In a perfect foresight equilibrium, one must however ensure that firms' current decisions take into account future variation in the path of aggregate shocks; that is, the decision problem of each firm must be solved through backward iteration. I describe the algorithm used to compute the perfect foresight response when the aggregate shock affects the intermediation wedge $\gamma$. Let $\{\gamma^{(t)}\}_{t=0}^{+\infty}$ denote the path of the aggregate shock, and $\gamma^{(+\infty)} \equiv \lim_{t\to+\infty} \gamma^{(t)}$. $\gamma_{-1}$ denotes the value of the intermediation wedge before the shock occurs.

1. Compute the long-run steady-state of the economy, that is, the steady-state of the economy when $\gamma = \gamma^{(+\infty)}$. Denote by $\bar{e}^{(+\infty)}$, $V^{(+\infty)}$, $(\hat{d}^{(+\infty)}, \hat{s}^{(+\infty)})$ and $w^{(+\infty)} = \left\{ w_i^{(+\infty)} \right\}_{i=1}^{N_e}$ the upper bound on internal funds, the value function, policy functions and the discrete approximation to the steady-state distribution of firms in the long-run steady-state. Furthermore, let $\bar{e}^{(0)}$ and $w^{(0)}$ denote the upper bound oninternal funds and the discrete weights approximating the invariant measure of firms when $\gamma = \gamma_{-1}$.

2. Fix $T > 0$. Let $\bar{e}^{(T)} \equiv \bar{e}^{(+\infty)}$, $V^{(T)} \equiv V^{(+\infty)}$, $(\hat{d}^{(T)}, \hat{s}^{(T)}) \equiv (\hat{d}^{(+\infty)}, \hat{s}^{(+\infty)})$. For $t = T-1, ..., 1$:

    (a) Start with the guess $\bar{e}^{(t)} = \bar{e}^{(t+1)}$.

    (b) Compute the frontiers of the feasible sets $\mathcal{S}_K^{(t)}(.;\gamma^{(t)})$ and $\mathcal{S}_R^{(t)}(.;\gamma^{(t)})$ for values of $e_t$ on a discretized grid over $\left[0, \bar{e}^{(t)}\right]$.

    (c) Compute the value function $V^{(t)} = T^{(t)} V^{(t+1)}$ and the associated policy functions $(\hat{d}^{(t)}, \hat{s}^{(t)})$ on the discretized grid over $\left[0, \bar{e}^{(t)}\right]$. The index $(t)$ indicates that the aggregate shock takes the value $\gamma^{(t)}$ in the formulation of the firm problem.

    (d) Check whether $(1-\eta)\beta \frac{\partial V^{(t)}}{\partial e}(\bar{e}^{(t)}) = 1$ to a sufficient degree of precision. If not, restart $(a) - (c)$ with a new guess for $\bar{e}^{(t)}$.

(e) Compute the entry scale $e^{e,(t)}$ implied by $V^{(t)}$.

3. For $t = 0$, compute the policy functions $\left\{\hat{d}^{(t)}, \hat{s}^{(t)}\right\}_{t=0}^{T}$ on $\left[0, \bar{e}^{(0)}\right]$ that solve the maximization problem $V^{(0)} = T^{(0)}V^{(1)}$.

4. Next, given the sequence of upper bounds $\{\bar{e}^t\}_{t=1}^{T=0}$, policy functions $\left\{\hat{d}^{(t)}, \hat{s}^{(t)}\right\}_{t=0}^{T}$ and entry scales $\{e^{e,(t)}\}$ implied by the value functions, compute transition matrices $M_{t,t+1}$ for $0 \leq t \leq T-1$. As above, the transition matrix is defined using the transition kernel $N^{(t)}(e_t, e_{t+1})$. This transition kernel takes the same expression as described above, and only depends on policy functions of the firm at time $t$, the bounds $\bar{e}^{(t)}$ and $\bar{e}^{(t+1)}$ and the entry scale $e^{e,(t+1)}$, as well as the current value of the aggregate shock $\gamma^{(t)}$.

5. The transition matrices $M_{t,t+1}$ map discrete weights $w^{(t)}$ on $\left[0, \bar{e}^{(t)}\right]$ to discrete weights on $w^{(t+1)}$ on $\left[0, \bar{e}^{(t+1)}\right]$ through $w^{(t+1)} = w^{(t)}M_{t,t+1}$. Use this to compute the evolution of the approximate firm measure for $t = 0, ..., T$, starting with $w^{(0)}$.

6. Check whether the implied $w^{(T)}$ is sufficiently close to $w^{(+\infty)}$. If not, increase $T$ and repeat steps 1-5.

## C.4 Results omitted from main text

### C.4.1 A general equilibrium interpretation of the model

The following section extends the model of section 3.2 to include a representative household that interacts with financial intermediaries. In the absence of aggregate shocks, the stochastic steady-state equilibrium of this extended model has identical firm-level and aggregate outcomes as the recursive competitive equilibrium of definition 3.7. Moreover, when the household is risk-neutral, the real interest rate is fixed, and the perfect foresight responses of the extended model to aggregate shocks are identical to the responses described in section 3.5.

**Representative household** A representative household solves a standard consumption-savings problem. The household enters the period with assets $a_t$, which it has placed with financial intermediaries at the end of the previous period. It is also the final owner of firms, and funds entering firms.

After idiosyncratic firm risk has been realized and debt contracts have been settled, the household receives a payment of $(1+r_t)a_t$ from these intermediaries. It also receives dividends from existing firms, in the amount $d_t$. The cash on hand of the household, at the end of period $t$, is therefore givn by $(1+r_t)a_t + d_t$.

The household uses this cash for three purposes: consumption of final goods produced by firms or resold by financial intermediaires as a result of firm liquidations ($c_t$); investment in new firms ($\epsilon_t$); or deposits at financial institutions ($a_{t+1}$).

The household takes the amount of dividends, $d_t$, and the investment in new firms, $\epsilon_t$, as given. The determination of investment in entering firms is described below.

The recursive formulation of the household's optimization problem is:

$$V^H(a_t; s^t) \quad = \quad \max_{c_t, a_{t+1}} u(c_t) + \beta \mathbb{E}\left[V^H\left(a_{t+1}; s^{t+1}\right)|s^t\right]$$

$$\begin{aligned} \text{s.t.} \quad a_{t+1} \;&\leq\; (1+r_t)a_t + d_t - \epsilon_t - c_t \\ &\{r_t, d_t, \epsilon_t\} \subset s^t, \end{aligned}$$

where $s^t$ denotes a vector containing the history of aggregate state variables in the household's problem up to and including those of time $t$. The optimal consumption/savings plan of the household is characterized by:

$$1 = \mathbb{E}\left[(1+r_{t+1})\omega_{t,t+1}|s^t\right] \tag{C.3}$$

$$a_{t+1} = (1+r_t)a_t + d_t - c_t - \epsilon_t \tag{C.4}$$

where $\omega_{t,t+1} \equiv \beta\frac{u'(c_{t+1})}{u'(c_t)}$ is the household's stochastic discount factor.

**Financial intermediaries** There are two types of financial intermediaries: banks and market lenders. Market lenders receive $a_t^m$ units of deposits from the households, and banks receive $a_t^b$. Intermediaries lend out these funds to firms.

Perfect competition in the banking sector implies that total lending costs of banks must equal their total return on lending, which I denote by $B_t$. Bank lending costs involve two components: the compensation to the representative household; and intermediation costs. I assume that intermediation costs $\gamma_{b,t}$ per unit of funds lent are exogenously given. The zero profit condition of banks is then given by:

$$(1 + r_t + \gamma_{b,t})a_t^b = B_t.$$

With analogous notation, the zero profit conditions of market lenders is given by:

$$(1 + r_t + \gamma_{m,t})a_t^m = M_t,$$

where, by assumption, $\gamma_{m,t} \leq \gamma_{b,t}$.

Let $\mathbb{E}^B(e, m, b)$ denote the gross expected return to a bank from lending an amount $b$ to a firm with internal finance $e$ and market debt $m$, and let $\mathbb{E}^M(e, m, b)$ denote the similar quantity for a market lender.[13] I assume that lenders cannot be making strictly negative profits, in expectation, on each loan; that is:

$$\mathbb{E}^B\left[e,m,b\right] \geq (1+r_t+\gamma_{b,t})b \quad , \quad \mathbb{E}^M\left[e,m,b\right] \geq (1+r_t+\gamma_{m,t})m.$$

This assumption guarantees, in particular, that financial intermediaries will not cross-subsidize firms. Total lending returns are given by:

$$B_t = \int_{e_t} d\mu_t(e_t)\mathbb{E}^B\left[e_t, b_t^d(e_t;s^t), m_t^d(e_t;s^t)\right],$$

$$M_t = \int_{e_t} d\mu_t(e_t)\mathbb{E}^M\left[e_t, b_t^d(e_t;s^t), m_t^d(e_t;s^t)\right],$$

where $b_t^d(e_t;s^t)$ and $m_t^d(e_t;s^t)$ denote the demand for market and bank debt by firms with internal finance $e_t$, and $\mu_t(.)$ denotes the firm distribution.

**Firms** The optimization problem of individual firms is mostly unchanged with respect to the model of section 3.2. The main change involves the dividend issuance stage. Since the household is the ultimate owner of the firm, it uses the household's discount factor in order to value the dividend stream. The continuation value of a firm that reaches the dividend stage is thus given by:

$$V^c(n_t, e_t; s^t) = \max_{n_t \geq 0, \quad div_t + e_{t+1} \leq n_t} div_t + (1-\eta)\mathbb{E}_t\left[\omega_{t,t+1} V\left(e_{t+1}; s^{t+1}\right)\right],$$

where $n_t$ denotes the cash on hand of a firm that was not liquidated, and $\eta$ denotes the exogenous exit probability of the firm.

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[13]The expressions for these objects are derived in the context of the firm's optimization problem; expectations are taken with respect to $\phi$, the idiosyncratic productivity shock.

With the introduction of the household's discount factor, the dividend decision of the firm takes the following form. There is a unique solution $\bar{e}(s^t)$ to:

$$1 = (1-\eta)\mathbb{E}\left[\omega_{t,t+1}\frac{\partial V}{\partial e}\left(\bar{e}(s^t), s^{t+1}\right)|s^t\right]$$

As previously, a firm with $n_t < \bar{e}(s^t)$ accumulates all its internal finance towards equity next period, and a firm with $n_t \geq \bar{e}(s^t)$ accumulates exactly $\bar{e}(s^t)$ and issues dividends in the remaining amount. Given the firm distribution $\mu_t(e)$, total dividends rebated to the household are equal to:

$$d_t = (1-\eta)\int_{e_t,\phi}\left[\mathbb{1}_{\phi \geq \underline{\phi}(e_t)}\left(\hat{div}_t(n(e_t,\phi))\right)\right] d\mu_t(e_t)dF(\phi), \tag{C.5}$$

where $\underline{\phi}(e)$ denotes the endogenous exit threshold given the optimal debt structure with internal finance $e$, and $\hat{div}_t(n(e_t,\phi))$ denotes dividends distributed by a (non-exiting) firm with cash on hand $n(e_t,\phi)$.

**Entry** Entry involves the payment of a sunk cost $\kappa$ by the household, along with the financing of initial equity $e_t^e$. Free entry implies that the following zero-profit condition must hold:

$$\kappa + e_t^e = \mathbb{E}\left[\omega_{t,t+1}V(e_t^e; s^{t+1})|s^t\right]. \tag{C.6}$$

Given the policy function of the firm, this condition uniquely pins down the entry size $e_t^e$. Note that, because households fund the entry of new firms, the stream of future profits associated with this investment is discount by $\omega_{t,t+1}$. Additionally, the number of entering firms must equal the number of exiting firms in each period. Letting $\mu_t^e$ denote the fraction of the total mass of firms (normalized to 1) that exit at the end of period 1, investment in new firms for the household is therefore given by:

$$\epsilon_t = \mu_t^e(\kappa + e_t^e). \tag{C.7}$$

**Equilibrium** In the economy described above, there are two aggregate shocks: $\gamma_{b,t}$ and $\gamma_{m,t}$. The history of realizations of these shocks enters the vector of aggregate states $s^t$. Additionally, the vector of aggregate states contains the current firm size distribution, which the household uses to forecast future dividend payments and future investment in entering firms.

**Definition C.1** (Recursive competitive equilibrium)**.** *An competitive equilibrium of this economy is a set of:*

- *decision rules* $c_t(a_t; s^t)$ *and* $a_{t+1}(a_t; s^t)$ *and a value function* $V(a_t; s^t)$ *for the household;*
- *decision rules* $\hat{b}(e; s^t)$, $\hat{m}(e; s^t)$, $\hat{div}(n; s^t)$ *and* $\bar{e}(s^t)$ *and value functions* $V(e; s^t)$ *and* $V^c(e; s^t)$ *for the firm;*
- *deposits* $a_t^m$ *and* $a_t^b$ *at financial institutions and gross lending returns* $M_t$ *and* $B_t$*;*
- *an interest rate* $r_t$*;*
- *dividends payments to the households* $d_t$*;*
- *an entry scale* $e_t^e$*;*
- *an initial investment* $\epsilon_t$*;*
- *a transition mapping for the distribution of firms across equity sizes* $T(\mu_t; s^t)$*;*

*such that:*

1. *the decision rules and value functions of the household solve its optimization problem;*

2. *the decision rules and value functions of the firm solve its optimization problem;*

3. *the markets for bank loans, market loans, and deposits clear:*

$$a_t^m = \int_e d\mu_t(e_t)\hat{m}(e_t; s^t)$$

$$a_t^b = \int_e d\mu_t(e_t)\hat{b}(e_t; s^t)$$

$$a_t^b + a_t^m = a_t$$

4. *financial intermediaries make zero profits;*

5. *the market for final goods clears:*

$$c_t+\epsilon_t+\gamma_{b,t}a_t^b+\gamma_{m,t}a_t^m = \int_{e_t,\phi} \left(\mathbb{1}_{\phi \geq \underline{\hat{\phi}}(e_t)} + \chi \mathbb{1}_{\phi \leq \underline{\hat{\phi}}(e_t)}\right) y(e_t, \hat{b}(e_t; s^t), \hat{m}(e_t; s^t), \phi) d\mu_t(e_t) dF(\phi),$$

*where* $y(e_t, \hat{b}(e_t; s^t), \hat{m}(e_t; s^t), \phi) = \phi(e_t + \hat{b}(e_t; s^t) + \hat{m}(e_t, s^t))^\zeta + (1-\delta)(e_t + \hat{b}(e_t; s^t) + \hat{m}(e_t, s^t))$ *denotes firm-level output, and* $\underline{\hat{\phi}}(e_t) = \underline{\phi}\left(e_t, \hat{b}(e_t; s^t), \hat{m}(e_t, s^t)\right)$ *denotes the liquidation probabilities of firms given their optimal debt choices;*

6. *the entry scale of new firms satisfies the free-entry condition C.6;*

7. *dividends and investment in entering firms are given by the two conditions C.5 and C.7;*

8. *the transition mapping* $T(.; s^t)$ *is consistent with the decision rules of the firms.*

The only noteworthy thing in this defition are the market clearing conditions for financial intermediaries. Because of the assumption that financial intermediaries do not make individual loans that have a strictly negative expected value, the combination of the market clearing condition and the zero profit condition ensures that:

$$\mathbb{E}^B\left[e_t, \hat{b}(e_t; s^t), \hat{m}(e_t; s^t)\right] = (1 + r_t + \gamma_{b,t})\hat{b}(e_t; s^t),$$

$$\mathbb{E}^M\left[e_t, \hat{b}(e_t; s^t), \hat{m}(e_t; s^t)\right] = (1 + r_t + \gamma_{m,t})\hat{m}(e_t; s^t),$$

for $\mu_t$-almost any $e_t$. That is, in equilibrium, financial intermediaries make zero profits for each firm size $e_t$. This is guarantees that the pricing schedules described in section 3.2 are identical to those prevailing in the competitive equilibrium of this model.

**Equilibrium allocations in the extended model** The two particular cases of the extended model that correspond to those discussed in the main text are the following. First, when there are no aggregate shocks ($s^t = s$), the Euler equation of the household implies that $\beta(1+r) = 1$, and the characterization of the threshold for issuing dividends for the firm becomes:

$$1 = (1 - \eta)\beta\frac{\partial V}{\partial e}(\bar{e}).$$

These two conditions correspond, respectively, to equations 3.11 and 3.2. It is then clear that the equilibrium decisions of firms are identical to those described in section 3.3.

The second case is if the household is risk-neutral, so that $\omega_{t,t+1} = \beta$. The Euler equation of the household agains pins down the risk-free rate, regardless of other aggregate states. Since firms' individual decision problems only depend on the real interest rate, but not on the evolution of the firm size distribution, the perfect foresight response of such an economy to aggregate shocks is then identical to the one studied in section 3.5.

### C.4.2 Optimal debt structure in the static vs. dynamic model

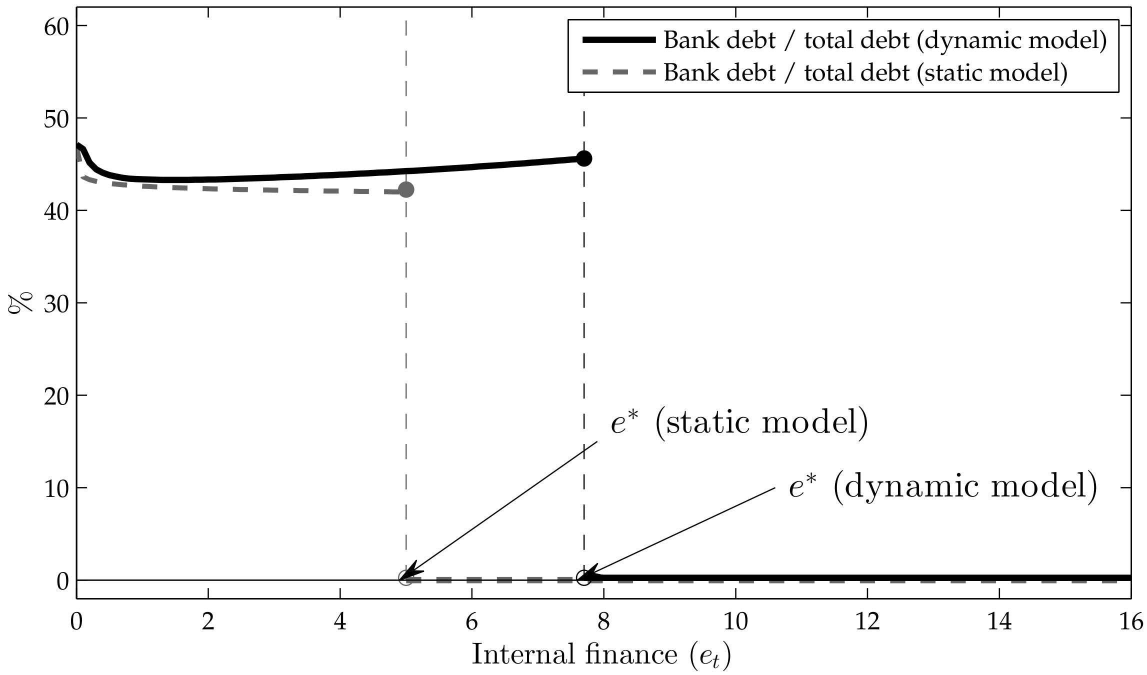


*Figure C.2: Optimal composition of debt in the static and dynamic models.*

The main text mentions the fact that one of the drivers of firms' debt choice, in the model, is the concavity of their value function $V(e_t)$. Figure C.2 illustrates this. It depicts the optimal debt structure of firms in the dynamic model (black line) and a static version of the model (grey line) with an identical calibration.[14] The static model can be thought of as the limiting case of the model studied in the main text, when $\beta = 0$ (provided that the cost of funds of financial intermediaries, $r$, is exogenously given). In that case, the firm has linear utility over final cash flows (after debt settlement). The resulting debt structure has two differences with respect to the dynamic case: firms wit330h mixed debt structures borrow less from banks; and the internal finance threshold at which firms switch to the market-only financing regime is smaller. Both of these differences indicate that the concavity of the

[14]This version of the model is identical to the static model I study in chapter 2.

valuation of cash flows, in the dynamic model, induces firms to increase their reliance on bank lending.

### C.4.3 The relationship between asset size and debt composition

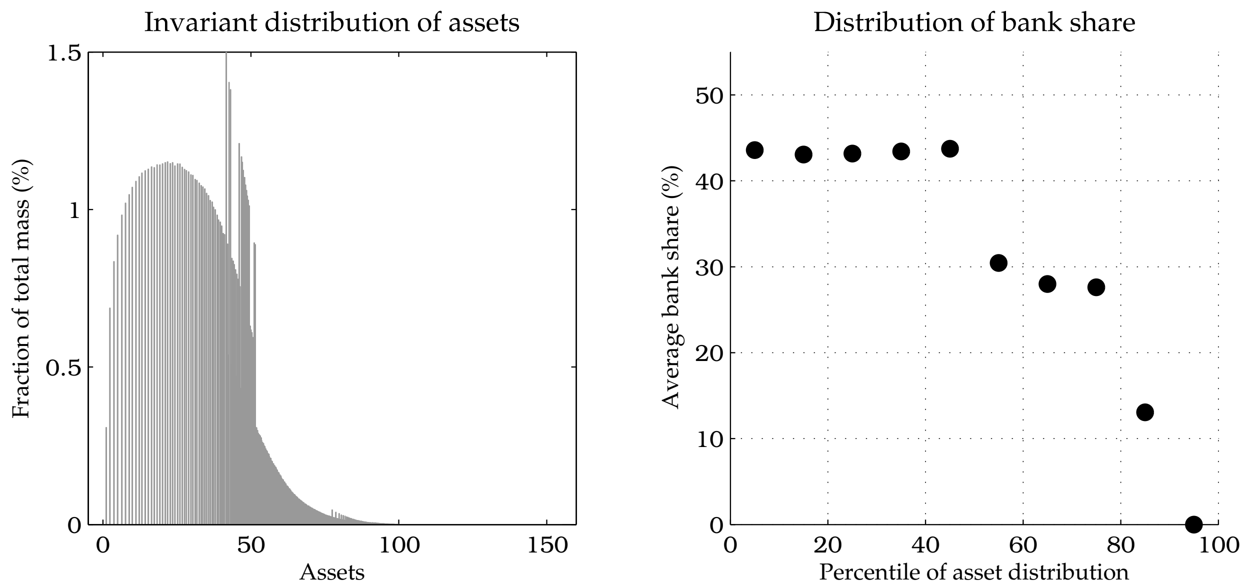


*Figure C.3: Asset size and debt structure in the steady-state of the model.*

The left panel of figure C.3 depicts the steady-state distribution of firms by asset size. Total assets of firms are given by $\hat{k}(e_t) = e_t + \hat{b}(e_t) + \hat{m}(e_t)$. The implied distribution depicted on the left panel is obtained by drawing a large sample from the approximate distribution of firms $\{w_i\}_{i=1}^{N_e}$, and computing the value of $\hat{k}(e_{d,t})$ for each draw indexed by $d$. This distribution is positively skewed, reflecting the fact that the invariant measure of firms across internal finance levels $e_t$ is also positively skewed, and additionally that total assets are a piecewise increasing function of internal finance. The "spikes" on the distribution occur because of the drop in total assets as firms switch from the mixed-finance to the market-finance regime. They coincide with asset levels that are optimal for both market-finance firms and mixed-financed firms. The right-hand side depicts the average bank share among firms within the successive quantiles of the asset distribution.[15] The bank share falls as one moves to the right of the invariant asset distribution. This is also the case in the data, as illustrated by figure C.1.

### C.4.4 Definition of firm dynamics moments reported in figure 3.9

Leverage ratios are defined as $\frac{\hat{b}(e_t)}{e_t}$ and $\frac{\hat{m}(e_t)}{e_t}$, respectively. The expected rate of profits is defined as:

$$\Pi^e(e_t) = \frac{1}{e_t} \int_{\phi \geq \underline{\hat{\phi}}(e_t)} \hat{n}\left(\phi; e_t\right) dF(\phi)$$

where the liquidation threshold $\underline{\hat{\phi}}(e_t) \equiv \underline{\phi}\left(e_t, \hat{b}(e_t), \hat{m}(e_t)\right)$ at the net worth function $\hat{n}(\phi; e_t) \equiv n\left(\phi; e_t, \hat{b}(e_t; s^t), \hat{m}(e_t; s^t)\right)$ both depend on the type of debt structure (mixed or market-financed) chosen by the firm, given its internal finance level $e_t$. Expected dividend issuance, the expected growth rate of internal finance, the expected growth rate of assets, and the expected growth rate of output are analogously defined as:

$$\begin{aligned}
D^e(e_t) &= \frac{1}{e_t} \int_{\phi : \hat{n}(\phi; e_t) > \bar{e}} \left(\hat{n}\left(\phi; e_t\right) - \bar{e}\right) dF(\phi) \\
E^e(e_t) &= \frac{1}{e_t} \int_{\phi} \left(\hat{e}(\phi, e_t) - e_t\right) dF(\phi) \\
Y^e(e_t) &= \int_{\phi} \left( \frac{\mathbb{E}(\phi)\left(\hat{e}(\phi, e_t) + \hat{b}\left(\hat{e}(\phi, e_t)\right) + \hat{m}\left(\hat{e}(\phi, e_t)\right)\right)^{\zeta}}{\phi\left(e_t + \hat{b}(e_t) + \hat{m}(e_t)\right)^{\zeta}} - 1 \right) dF(\phi) \\
K^e(e_t) &= \int_{\phi} \left( \frac{\left(\hat{e}(\phi, e_t) + \hat{b}\left(\hat{e}(\phi, e_t)\right) + \hat{m}\left(\hat{e}(\phi, e_t)\right)\right)}{\left(e_t + \hat{b}(e_t) + \hat{m}(e_t)\right)} - 1 \right) dF(\phi)
\end{aligned}$$

where $\hat{e}(\phi, e_t) \equiv \mathbb{1}_{\{\hat{n}(\phi; e_t) > \bar{e}\}} \bar{e} + \mathbb{1}_{\{\hat{n}(\phi; e_t) \leq \bar{e}\}} \hat{n}\left(\phi; e_t\right)$ denotes next period internal finance, given current productivity and current internal finance. The computation of the volatility of the growth rate of internal finance follows easily.

[15]The first dot, for example, is the average bank share of firms in the bottom 5% of the asset size distributions.

## C.4.5 Further details on comparative statics

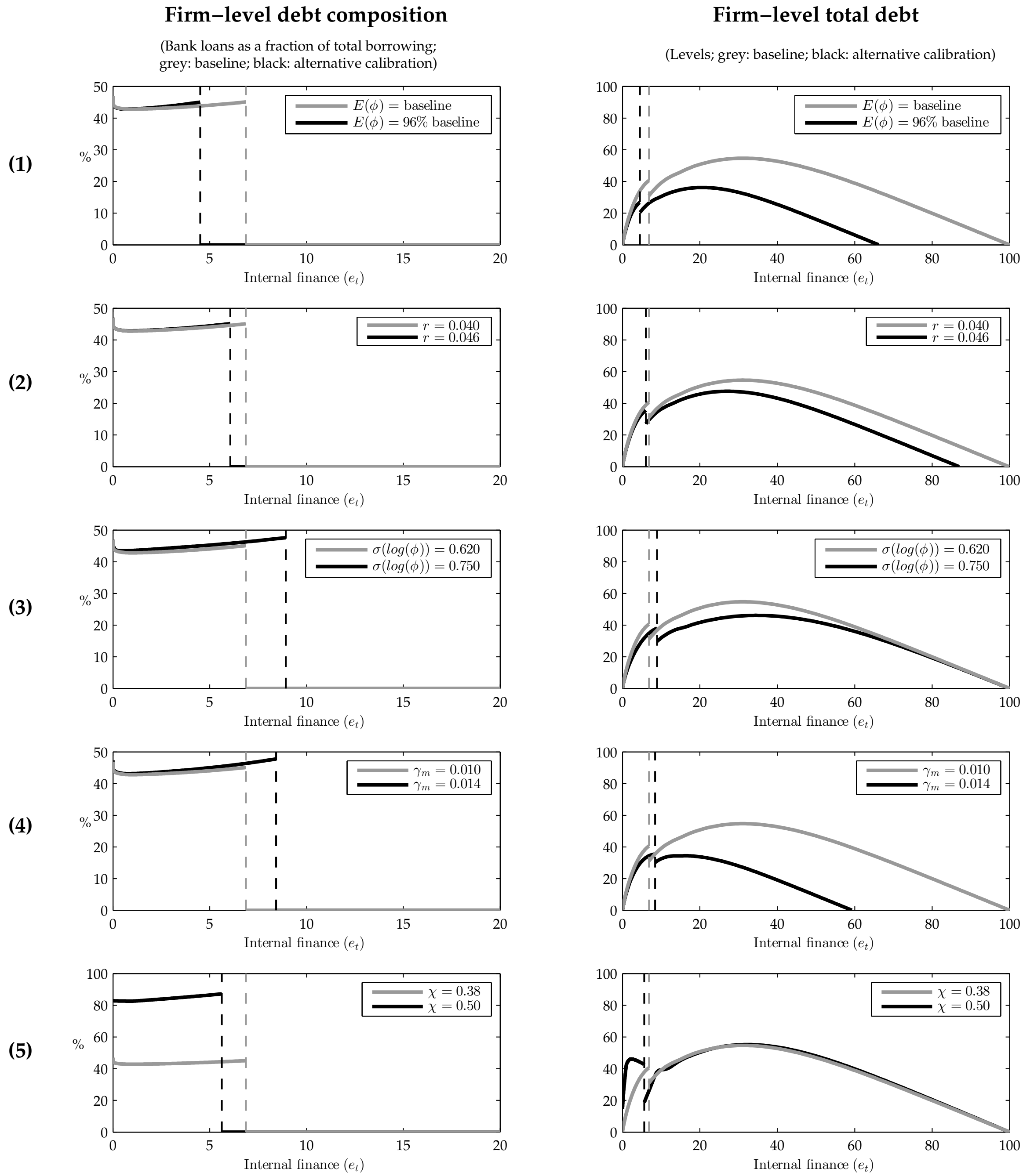


*Figure C.4: Differences in firm-level borrowing policies. The left column reports firms' optimal composition of debt, and the right column reports firms' optimal total borrowing. The grey lines correspond to the baseline calibration reported in table 3.1. The black lines correspond to one of the alternative calibrations reported in the comparative statics of figure 3.11.*

### C.4.6 The italian tax reform

In this appendix, I first briefly describe the introduction of tax shields into the model. I then illustrate the effects of the Italian reform described in section 3.6.2 on the borrowing and investment choices of firms.

#### Introducing tax shields

As mentioned in section 3.6.2, when gross income and debt payments are subject to differential taxation, the cash on hand of firm that repays its creditors can generally be written as:

$$n_t^R = (1-\tau)\pi_t - (1-\tau_b)R_{b,t} - (1-\tau_m)R_{m,t}.$$

One must specify how the firm and its creditor's income are taxed under payment, restructuring and liquidation. I make two key assumptions in this regard:

**Assumption C.2** (Tax treatment of restructuring and liquidation)**.**

- *Income tax liabilities are senior to bank and market debt payments in liquidation;*
- *There are no tax shields for debt payments that have been restructured.*

The first assumption is innocuous, and simply guarantees that firms will not find it beneficial to default in order to avoid the payment of tax liabilities. The second assumption guarantees that, when tax shields are identical ($\tau_b = \tau_m$), the restructuring choices of the firm are similar to the baseline model; it therefore helps to focus the discussion on the effects of *assymetric* tax treatment of debt.

With these assumptions, the payoffs to stakeholders in liquidation are given by:

$$\begin{aligned}
\tilde{R}_{b,t} &= \min\left(R_{b,t}, (1-\tau)\chi\pi_t\right) && \text{(bank lenders)} \\
\tilde{R}_{m,t} &= \min\left(\max(0, (1-\tau)\chi\pi_t - R_{b,t}), R_{m,t}\right) && \text{(market lenders)} \\
n_t^L &= \max\left(0, (1-\tau)\chi\pi_t - R_{b,t} - R_{m,t}\right) && \text{(firm)}
\end{aligned} \tag{C.8}$$

Moreover, in restructuring the firm will drive the bank down to its reservation value, $(1-\tau)\chi\pi_t$. In that case, given the second assumption, the cash on hand of the firm after restructuring will be given by:

$$n_t^R = (1-\tau)\pi_t - (1-\tau)\chi\pi_t - (1-\tau_m)R_{m,t}.$$

Given this, the following lemma is straightforward to establish.

**Lemma C.3** (Debt settlement outcomes)**.** *Assume that $V^c(.)$ is increasing, and $V^c(0) \geq 0$. Then, there are two types of debt settlement outcomes:*

- ***When*** $\frac{(1-\tau_b)R_{b,t}}{\chi} \geq \frac{(1-\tau_m)R_{m,t}}{1-\chi}$, *the firm chooses to repay its creditors in full, if and only if,* $\pi_t \geq \frac{(1-\tau_m)R_{b,t}}{\chi}$. *It successfully restructures its debt, if and only if,* $\frac{(1-\tau_m)R_{m,t}}{1-\chi} \leq \pi_t < \frac{(1-\tau_b)R_{b,t}}{\chi}$, *and it is liquidated when* $\pi_t < \frac{(1-\tau_m)R_{m,t}}{1-\chi}$.
- ***When*** $\frac{(1-\tau_b)R_{b,t}}{\chi} < \frac{(1-\tau_m)R_{m,t}}{1-\chi}$, *the firm repays its creditors in full if and only if* $\pi_t \geq \frac{(1-\tau_b)R_{b,t}+(1-\tau_m)R_{m,t}}{(1-\tau)}$, *and it is liquidated otherwise.*

*Moreover, in any successful restructuring offer, the bank obtains its reservation value* $(1-\tau)\chi\pi_t$, *and in all debt settlement outcomes resulting in liquidations,* $n_t^L = 0$.

Thus, under the two assumptions above, the structure of the debt settlment outcomes is similar to the baseline model: when the firm's bank liabilities are large enough, it will sometimes restructure debt contracts conditional on its productivity realizations; otherwise, it never uses the restructuring option. The difference is that, when $\tau_b \neq \tau_m$, the firm's decision to restructure debt contracts also depends on the relative values of the tax shield on bank and market debt, and not simply on liquidation losses (associated with the parameter $\chi$).

### Effects of the Italian tax reform

Lemma C.3 be used to fully characterize the set of feasible debt contracts, and therefore the general formulation of firms' optimal debt structure problem when there are differential tax treatments of debt. These derivations are available upon request, but follow the methods described in chapters 2 and 3.

I next turn to the effects of the policy experiment described in section 3.6.2. Namely, I compare firm-level borrowing and investment in an economy without and with the tax reform. In the baseline economy, all firms enjoy tax shields for bank debt, but only large firms enjoy a tax shield for market debt issuance:

$$\tau_b(e_t) = \tau \quad \forall e_t,$$

$$\tau_m(e_t) = \begin{cases} 0 & \text{if } e_t \leq e_{sm} \\ \tau & \text{if } e_t > e_{sm} \end{cases}$$

In the reformed economy, however, the tax shield applies to all debt issuances of all firms:

$$\tau_b(e_t) = \tau_m(e_t) = \tau \quad \forall e_t.$$

As in the case of the experiment of section 3.6.1, so as to clarify the exposition of the effects of the subsidy, I set the threshold for the reform at $e_{sm} = e^*$. Moreover, I focus on the borrowing policies of firms in the static version of the model. Indeed, the results of section

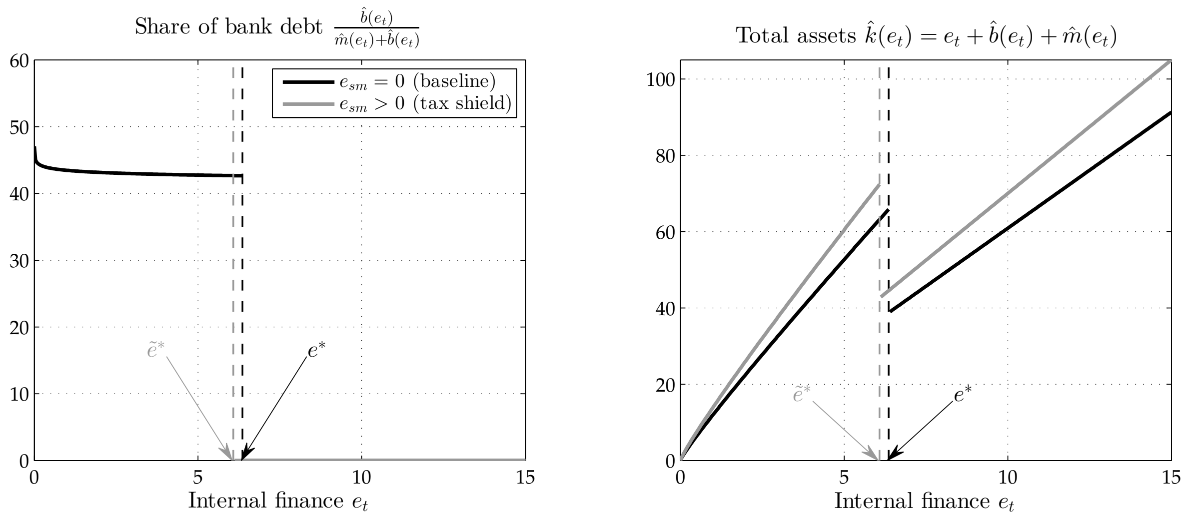


*Figure C.5: The effect of the tax reform on borrowing and investment.*

3.6.1 suggest that the bulk of the effects of this type of policy is mediated by firms' borrowing decisions, rather than by long-run changes in the firm size distribution.

Figure C.5 reports the results of the experiment, when the tax rate is $\tau = 2.5\%$. The reform has little incidence on the debt composition of firms, but induces a measure of firms to switch to pure market finance (left panel). In doing so, these firms operate at a lower scale than they otherwise would have (right panel). The investment of all other firms, however, gains from the introduction of the tax shield. These results are qualitatively analogous to those obtained in the experiment of section 3.6.1.