Network Resource Allocation Under Fairness Constraints

Shyam S Chandramouli

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ABSTRACT

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This work considers the basic problem of allocating resources among a group of agents in a network, when the agents are equipped with single-peaked preferences over their assignments. This generalizes the classical *claims* problem, which concerns the division of an estate's liquidation value when the total claim on it exceeds this value. The claims problem also models the problem of rationing a single commodity, or the problem of dividing the cost of a public project among the people it serves, or the problem of apportioning taxes. A key consideration in the claims problem is *equity*: the good (or the "bad," in the case of apportioning taxes or costs) should be distributed as fairly as possible. The main contribution of this dissertation is a comprehensive treatment of a generalization of this classical rationing problem to a network setting.

Bochet et al. recently introduced a generalization of the classical rationing problem to the network setting and designed an allocation mechanism—the *egalitarian* mechanism—that is *Pareto optimal*, *envy free* and *strategyproof*. In chapter 2, it is shown that the egalitarian mechanism is in fact *group strategyproof*, implying that no coalition of agents can collectively misreport their information to obtain a (weakly) better allocation for themselves. Further, a complete characterization of the set of all group strategyproof mechanisms is obtained.

The egalitarian mechanism satisfies many attractive properties, but fails *consistency*, an important property in the literature on rationing problems. It is shown in chapter 3 that no Pareto optimal mechanism can be envy-free and consistent. Chapter 3 is devoted to the *edge-fair* mechanism that is Pareto optimal, group strategyproof, and consistent. In a related model where the agents are located on the *edges* of the graph rather than the nodes, the edge-fair rule is shown to be envy-free, group strategyproof, and consistent.

Chapter 4 extends the egalitarian mechanism to the problem of finding an optimal exchange in non-bipartite networks. The results vary depending on whether the commodity being exchanged is divisible or indivisible. For the latter case, it is shown that no efficient mechanism can be

strategyproof, and that the egalitarian mechanism is Pareto optimal and envy-free. Chapter 5 generalizes recent work on finding stable and balanced allocations in graphs with unit capacities and unit weights to more general networks. The existence of a stable and balanced allocation is established by a transformation to an equivalent unit capacity network.

Table of Contents

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Chapter 1

Introduction

The problem of dividing an amount of a resource when the total claim on it exceeds its supply is thousands of years old. Many allocation problems are solved by a pricing resources: high energy prices induce consumers to conserve energy, high salaries attract workers to particular occupations etc. A theme common to all the models discussed in this dissertation is that prices are not allowed for legal or ethical reasons. This is true in many public decision making problems such as allocating students to public schools, exchanging organs among patients, etc. Furthermore, there are many markets where there is lack of perfect competition either due to the connectivity constraints or the indivisiblity of the good and the markets become thin. How these thin markets allocate resources depends primarily on the institutions that govern these transactions. The goal of the study here is to identify mechanisms with attractive efficiency, fairness and incentive properties in a variety of problems.

The “claims problem” is the most thoroughly studied formal model of distributive justice in a moneyless market. It is equivalent to the problems of rationing a commodity among its consumers or dividing a tax or the cost of a public project among a group of citizens. It was formalized by O’Neill [46] and Aumann and Maschler [5] who propose many allocation rules. A feature unifying its classical solutions is that they can all be obtained by maximizing an additive measure of welfare over the possible divisions of the resource [63]. We find that this insight is more general. It extends to a network-constrained resource allocation problem encompassing division with single-peaked preferences [56], random matching under dichotomous preference [13] and a version of the kidney exchange problem [49].

Concretely, we consider the problem of matching the supplies of a resource with competing demands when side payments or price adjustments are not possible. The key constraint is that the supplies of the resource can only flow from suppliers to demanders through connections in a network which is a modeling tool used to encode diverse operational constraints:

- In a networked market for a commodity under fixed prices, a supplier is connected to demanders she has signed supply contracts with, or to whose specifications she is tailoring the commodity, etc.
- In the kidney exchange model, a supplier is connected to a demander if there are no blood type or immunological incompatibilities between them.
- In matching problems, the network encodes preferences deeming agents acceptable or unacceptable: a supplier is connected to a demander if they find each other mutually acceptable.

The claims problem can be thought of as a special case in which a single supplier is connected to each demander and the total demand exceed supply. In the claims or matching problems, preferences are assumed to be increasing over the amounts received or the probability of being matched respectively. In contrast, we consider the possibility of agents having satiated preferences over these amounts: each supplier and demander has a unique preferred transfer or peak; more is preferred to less up to that point, and less to more beyond it. Such preferences are known as "single peaked". They arise from the convexity of preferences over an underlying consumption or production space.[1].

Recently, ideas from combinatorial optimization have played an important role in policy making. In the healthcare sector, the work of Roth et al. [47, 49] has had an enormous impact. Their work gained popularity among transplant surgeons led to the amendment of the National Organ Transplant Act (NOTA) of 1984 to allow for kidney exchange or kidney paired donations, thereby saving a lot of lives. The follow up work of Roth et al. [48, 50] made further progress in exchanging kidneys among patients by allowing for multi way exchanges. In the context of school choice, Sonmez et al. [1] study a student optimal stable mechanism (SOSM) which is a variant of the

[1]For instance, if suppliers have strictly convex production sets and prices are fixed, their profits are single-peaked in their output. Alternatively, if an employee is paid an hourly wage and her disutility of labor is a convex function of labor supplied, her preferences over time worked are single-peaked

deferred -acceptance algorithm of Gale and Shapley [29]. Major school districts including Boston, Denver and New York City have already adopted versions of SOSM advocated in their work. In the context of cadet branching, the low retention rates of junior officers has been a major issue for the U.S. army since the late 1980's. Sonmez and Switzer [55] study a matching with contracts model that could potentially improve the retention among junior officers.

As mentioned earlier, the problems discussed in this chapter have a network structure and feasible flows in this network determine the allocations to the agents in the problem. Efficiency and/or design constraints forces us to focus on allocation for agents that is induced by a maximum flow in the underlying network. The existing work in operations research and computer science regarding the study of maximum flows is quite rich: we know algorithms with good running times for computing a maximum flow, we understand the structure of flow polytopes, connections with linear programming etc. However, this literature generally does not distinguish between different maximum flows.

From an economic perspective, some of these solutions can be unacceptable based on fairness considerations. When two nodes are connected identically but treated differently in regard to their final allocation, it might imply an unfair treatment by the central planner. Also, indifference between different solutions can lead to strategic manipulation by agents. Thus these considerations force us to choose particular subsets of solutions (mostly a unique maximum flow). As we see shall see later, in many networks an efficient allocation is one that is induced by a maximum flow.

The mechanisms and structural properties that we study in this work are particularly interesting when multiple units of the good are available. In the applications discussed earlier agents are typically endowed with a unit quantity of a good, so the strategic behavior of the agents are limited to their connectivity. Whereas in our problems, the agents may report their ideal demand/supply, called their "peak" to the central planner. So they may have an incentive to misreport this value as well as to improve their allocation.

The rest of the dissertation is stuctured as follows: We start by summarizing the results of Sprumont [56] for the case of a unit supplier/demander. The uniform rule of Sprumont obtains an allocation that is *Pareto optimal*, *consistent*, *envy free* and *strategyproof* with respect to the peaks of the agents in the network. An immediate complication arises when the network structure is bipartite with agents on either side of the network and arbitrary connectivity. The notion of envy freeness is no longer compatible with that of consistency in the set of Pareto optimal solutions. This creates a dichotomy in the study of allocation mechanisms when the network structures are complex. Hence, a mechanism planner has to choose between envy freeness and consistentcy.

In Chapter 2, the focus is on obtaining envy free allocation for agents in a supply/demand bipartite network (The agents control the nodes in this chapter)[2]. Bochet et al. [11, 12] introduce two different models when the network structure is bipartite. In the two sided model of Bochet et al. [12], the suppliers and demanders are in either side of the bipartite network. In the one sided model of Bochet et al. [11] only one side of the network has agents and the other side of the network has goods to be rationed to these agents. Each good can be allocated to any agent that has a connection to it. The agents in these models do not care as to whom they supply or receive the goods from. They derive their utility from the total net allocation. Agents have single peaked preferences over their allocation. They have an ideal quantity that they would like to receive. Bochet et al. introduce the egalitarian mechanism as a generalization of uniform rule for both these problems. The egalitarian mechanism is *Pareto optimal*, *envy free*, *strategyproof* with respect to the peaks in both these models. Our main contribution to this literature is a proof that the egalitarian mechanism is in fact peak group strategyproof in both these models i.e. it is robust against coordinated misreporting by a groups of agents. We identify the structural properties that makes a mechanism peak groupstrategyproof. We show that any mechanism that is Pareto optimal and *strongly invariant* is peak group strategyproof and vice-versa. This not only helps us understand the structure of peak groupstrategyproof mechanisms but also makes it easier to verify if a mechanism is in fact robust against coordinated misreports. Moreover, our technique simplifies the existing proofs of strategyproofness.

In the model of Sprumont [56] an agent does not have any incentive to misreport his link. A misreporting agent gets disconnected from the network thereby receiving zero utility. But when the network structure is bipartite it is possible for agents have the possibility to misreport their connectivity to improve their allocation on other connected edges. We show that in the two sided model, the egalitarian mechanism is link groupstrategyproof if the coalition is restricted to agents

[2]In chapters 2-4, the utility of an agent i is the total amount of flow that the agent shares or sends to his neighbors in the network

on one side of the network only. Finally, we extend these results to the case of a capacitated network as weall as to the case of indivisible goods.

In Chapter 3, we shift the focus to studying rules that are consistent. In recent work, Moulin and Sethuraman [43] study consistent rules and their extensions to bipartite networks, establishing that the uniform gains and uniform losses methods have infinitely many consistent extensions whereas the propotional method has only one. In their follow up work, Moulin and Sethuraman [44] study loss calibrated rationing methods that are consistently extendable to bipartite networks. They show that most standard parametric methods do not admit such consistent extensions. They do not model the strategic behavior of the agents and assume the peak of the agents to be known or observable. We ask if then an efficient mechanism that are both consistent and strategyproof.

In the first part of the chapter, we still assume that the agents are on the nodes. We introduce the edge fair mechanism and show that its outcome can be found by solving to a sequence of linear programming problems. The edge fair mechanism is Pareto optimal, consistent and peak groupstrategyproof. In essence, it retains many of the attractive properties of the egalitarian mechanism and is a sound alternative when consistency is important. In the second part of the chapter, we assume that the agents are on the edges, and that the nodes are simply transshipment points. Such a model is know in the literature as a flow game. We continue to study the edge fair algorithm when the agents are on the edges. We show that the allocation is Pareto optimal, envy free, consistent and group strategyproof [3]. Moreover, the allocation induced by the edge fair rule is still a *core* allocation.

In Chapter 4, we extend many of the familiar rules to general non-bipartite networks. In these problems, agents are on the nodes and they own a specified quantity of a homogeneous good. Each agent derives utility when he/she exchanges or shares the good with the neighbors, the utility increasing in the amount shared. We find fair allocation rules on these general non-bipartite networks by suitably transforming it to bipartite networks, both for divisible and indivisible goods. Note that when the goods are indivisible and agents own exactly one unit of a good, it boils down to the well-known pairwise kidney exchange problem of Roth et al. [49] for which we know that the egalitarian lottery mechanism has very attractive properties. When the goods are indivisible,

[3] Note that in this model, agents report the capacity of their edge

we obtain a similar egalitarian lottery mechanism for the agents on the nodes that is Lorenz dominant and is also envy free. This egalitarian mechanism can be seen as an extension of the probabilistic egalitarian rule discussed earlier for bipartite networks. The egalitarian mechanism is weakly link group strategyproof for the agents and it is impossible for any mechanism to be peak groupstrategyproof in this model.

Finally, in Chapter 5, we study stable and balanced allocations for flows in networks. In Chapters 2-4, the amount of commodity that an agent sends/receives directly contributes to his utility for the good. In contrast, in the current model the flow f_{ij} is the surplus created when i and j are involved in a partnership, and this surplus f_{ij} has to be shared between these two agents. The central planner decides the share of the surplus that each agent receives. The planner wishes to find solutions that are stable and balanced. The study of stable solutions dates back to the work of Gale and Shapley [29] where they study the problem of matching medical students to residency programs. They do so by a deferred-acceptance algorithm and prove that the outcome of their algorithm is a stable solution[4]. Later, Shapley and Shubik [53] study assignment games in which nodes are still unit capacitated and agents are preference homogeneous. They obtain stable solutions and establish the equivalence between core solutions and stable solutions.

The recent literature on network bargaining by Kleinberg and Tardos [38], Bateni et al. [9] and Koenmann et al. [27] all study extensions of the Shapley and Shubik model to more general networks with arbitrary capacities. They also study the notion of balanced outcomes: in every pairwise contract, the allocation of an agent with respect to his best outside option is the same. In some sense, this treats every agent in a fair way i.e. an agent with relatively a better allocation inherently has better bargaining power in the network. All the aforementioned literature restrict their attention to strictly integral contracts. Since the focus of the dissertation has been on flows in networks, we relax the model to allow for fractional exchanges. We show that when such exchanges are allowed, we can always find a stable outcome in contrast to the integral case where stable solutions may not exist [38]. We also try to find balanced outcomes in these fractional exchange case by reducing to simpler networks. Again, it is impossible to find a strategyproof mechanism that selects a stable and balanced outcomes.

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[4]Agents have strict preferences in the Gale and Shapley student assignment model

Chapter 2

The Egalitarian Mechanism

2.1 Introduction

Motivated by applications in diverse settings, Bochet et al. [11, 12] study a model in which a single commodity is reallocated between a given set of agents with single-peaked preferences. In this environment, each agent is endowed with a certain quantity of the commodity and has an ideal consumption level (his *peak*) of that commodity. An agent who is endowed with more than his ideal consumption level can thus be thought of as a *supplier*, and an agent who is endowed with less than his ideal consumption level can be thought of as a *demander*. Furthermore, transfers are possible only between certain pairs of agents, represented by a graph. The goal is to reallocate the commodity to balance supply and demand to the extent possible. The key difference from conventional economic models on this topic is the inability to use money: motivating applications include assigning (or reassigning) patients to hospitals, assigning students to schools, and allocating emergency aid supplies. On the other hand, it is easy to see that the resulting problem is essentially a transportation problem in a (bipartite) network. The distinguishing feature here is that the preferences of the agents (such as their peaks) and the other agents they are linked to is typically private information, so the agents must be given an incentive to report this information truthfully.

Bochet et al. [12] propose a clearinghouse mechanism (a centralized organization of the market) that prescribes an allocation that is efficient with respect to (reported) preferences and (reported) feasible links between agents. They identify a unique *egalitarian* allocation—so named because of the intimate connection with the egalitarian solution of an associated supermodular game—that Lorenz dominates and “envy free” among all Pareto efficient allocations for this problem.

Furthermore, they show that the egalitarian mechanism is strategyproof with respect to both links and peaks: no *individual* agent can strictly benefit by misreporting his peak or the set of agents he is linked to. In a companion paper, Bochet et al. [11] consider a “one-sided” model where the demanders are not strategic, and their demands have to be met exactly. For this model, they propose an egalitarian mechanism that is strategyproof with respect to peaks, but not with respect to links.

Our main result is that the egalitarian mechanism is *group* strategyproof with respect to peaks in both the one-sided and two-sided models of Bochet et al. Furthermore, we show that under the egalitarian mechanism it is a weakly dominant strategy for *any* coalition of suppliers (or any coalition of demanders) to truthfully report their links. These results thus properly generalize the corresponding (individual) strategyproofness results of Bochet et al. Our proofs result in an improved understanding of the two models and simpify some of the earlier proofs of strategyproofness.

The models of Bochet et al. [12, 11] generalize many well-known and well-understood models in the literature; If there is a single demander (or a single supplier), the problem reduces to a classical rationing problem of the sort considered by Sprumont [56]. The egalitarian rule then reduces to the “uniform” rule, and admits many characterizations [56, 18, 54]. If the peaks are all identically 1, the problem reduces to a matching problem with dichotomous preferences, discussed in Bogomolnaia and Moulin [13]: in this case, the flow between a supplier-demander pair can be thought of as the probability that this pair is matched. Some of the negative results related to link strategyproofness discussed later are true even in this restricted setting as has already been observed there; we mention these results in the appropriate sections for the sake of completeness. Finally, Megiddo [41, 42] considered the problem of finding an “optimal” flow in a multiple-source, multiple-sink network, and proposed an algorithm to find a lexicographically optimal flow. The egalitarian algorithm described in Bochet et al. [12, 11] is essentially Megiddo’s algorithm to compute a lexicographically optimal flow. An implication of our result is that Megiddo’s algorithm is group strategyproof with respect to the source and sink capacities, that is, if the agents are located on the edges incident to sources and sinks, and all other edge-capacities are common knowledge, then no coalition of agents have an incentive to misreport their capacities. This observation is useful in settings in which equitably sharing resources is important, such as the *sharing* problem of Brown [14].

The rest of the chapter is organized as follows: In section 2.2 we describe the uniform rule and summarize other results that are most relevant to the rest of the chapter, in section 2.3 we describe the two sided model of Bochet et al. [12] and survey their main results about the Egalitarian Mechanism. We conclude with our contribution on the strategic issues related to the Egalitarian Mechanism. In section 2.5 we discuss the one sided extension of the Sprumont model, discuss the egalitarian mechanism and related strategic issues. Finally, in section 2.4.1 and section 2.4.2, we generalize the Egalitarian Mechanism when the goods are indvisible and when the connections are capacity constrained.

2.2 Uniform Rule

2.2.1 Model

The resource allocation problem discussed in this section originated from the claims problem of O'Neill [46]. The claims problem concerns the division of an estate's liquidation value when claims on it exceed this value. It is equivalent to the problem of rationing a commodity among its consumers or dividing a tax or the cost of a public project among a group of citizens.

More formally, an amount $K \in R_+$ has to be divided among a set N of agents with claims adding up to more than K. For each $i \in N$, let $c_i \in R$ denote agent i's claim, and $c = (c_i), i \in N$ denote the vector of claims ($\sum_{i \in N} c_i \geq K$). In the bankruptcy application, K is the liquidation value of a bankrupt firm, the members of N are creditors, and c_i is the claim of creditor i against the firm. A closely related application is to estate division: a man dies and the debts he leaves behind, written as the coordinates of c, are found to add up to more than the worth of his estate, K. How should the estate be divided? Alternatively, each c could simply be an upper bound on agent i's consumption. When a pair (c, K) is interpreted as a tax assessment problem, the members of N are taxpayers, the coordinates of c are their incomes, and they must cover the cost K of a project among themselves. The inequality $c_i \geq K$ indicates that they can jointly afford the project. In this context, c_i could also be seen as the benet that consumer i derives from the project. See

Thomson [60] for a brief survey of research on the claims problem.

O'Neill [46] considers strategic manipulations where agents can merge with other agents to form a bigger agent or split themselves into duplicate copies each with lesser capacity. He shows that proportional rule is the only mechanism that is merge or split proof. The seminal paper by Sprumont [56] considers a similar model in which agents can even misreport their claims. In this model, there is an infinitely divisible good that must be divided (no-free disposal) among a set of agents with single-peaked preferences. Agents report their claim profiles (preferences) to the mechanism designer and an allocation vector x is determined by the mechanism. The *uniform rule* of Sprumont allocates to each agent either his peak or a common amount, in such a way that the total quantity is fully distributed to agents whether they collectively over-demand or under-demand shares of the quantity. The Uniform rule strives to be as egalitarian as possible, under the restriction that the division of the quantity must be Pareto efficient.

Typical everyday applications include: a manager wanting to allocate an amount of overtime hours among a given set of employees, and there is a fixed hourly wage; government wanting to allocate a public good to the demanding participants, etc. If agents have a concave utility function, they then have single-peaked preferences over shares of the good.

The uniform rule is uniquely characterized by Pareto efficiency, envy freeness and peak strategyproofness. We describe the model and the rule below.

We follow the description and notation as in the original work of Sprumont [56]. There is a one (normalized) unit of a divisible good with a supplier that has to allocated among a set of $N = \{1, 2, ...N\}$ agents. The preference relations are assumed to be *single peaked*, denoted by R_i: i.e. there exists a $s_i \in [0, 1]$ (the peak of R_i) such that for any two possible allocations x_i, x'_i:

$$x'_i < x_i \leq s_i \implies x_i P_i x'_i \tag{2.1}$$

$$s_i \leq x_i < x'_i \implies x_i P_i x'_i \tag{2.2}$$

where P_i denotes the strict preference relation over R_i. The set of single peaked preferences will be denoted by $\mathcal{R}$.

A *division problem* is the report $(R_i)_{i \in N}$ of the preference profile and the number of units K that is to be allocated (In our current description of the model, $K = 1$; generalization to arbitrary

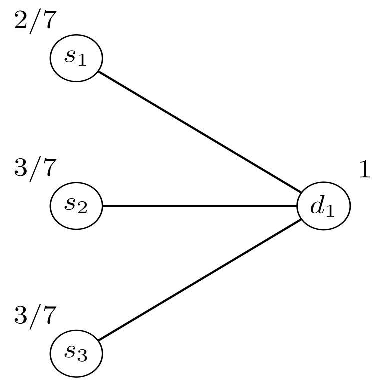

Figure 2.1: Sprumont Model

value of K follows directly)

A *feasible allocation* $x = (x_i)_{i \in N} \in R_+^N$ such that $\sum_{i \in N} x_i = 1$. Denote the set of feasible allocation profiles by $\mathcal{F}$.

A *Mechanism* or *Rule* is a function $\varphi : \mathcal{R}^N \to \mathcal{F}$ such that it maps each input preference profile R to a feasible allocation profile in $\mathcal{F}$. The allocation of agent i under profile $\mathcal{R}$ by a mechanism φ is given by $x_i = \varphi_i(R)$.

We are interested in finding a unique feasible allocation which is efficient, fair and strategyproof allocation for all the agents. We define these economic constraints more mathematically below and also discuss their importance in fair allocation literature.

Efficiency: A mechanism or rule is φ is efficient if for all $R \in \mathcal{R}^N$,

$$[\textstyle\sum_{i \in N} s_i(R_i) \leq 1] \implies [\phi_i(R) \geq s_i(R_i) \text{ for all } i \in N], \text{ and} \tag{2.3}$$

$$[\textstyle\sum_{i \in N} s_i(R_i) \geq 1] \implies [\phi_i(R) \leq s_i(R_i) \text{ for all } i \in N] \tag{2.4}$$

If an allocation is *efficient*, then there does not exist another allocation which is weakly better for all the agents and strictly better for at least one agent. Hence, it is a Pareto optimal allocation. In the current context, efficiency simply requires that if the preferred shares add up to more (less) than the amount required, then no agent should get more (less) than his preferred share.

Envy Freeness: For all $R \in \mathcal{R}^N$ and $i, j \in N$, $\varphi_i(R) R_i \varphi_j(R)$. In an envy free mechanism, any agent $i \in N$ prefers his allocation over all other agents for all preference profiles in $\mathcal{R}^N$.

<u>*Strategy-proofness*</u>: A mechanism φ is strategyproof if for all $R \in \mathcal{R}^N$, all $i \in N$ and all $R'_i \in \mathcal{R}, \varphi_i(R_i, R_{-i}) R_i \varphi_i(R'_i, R_i)$. That is, in a strategy proof mechanism it is a dominant strategy for the agents to reveal their preferences truthfully.

Sprumont [56] showed that the properties of strategy proofness, efficiency and envy freeness characterize an allocation rule that he called the uniform rule.

2.2.2 Sprumont's Uniform Rule

Definition 1 *(Sprumont[[56]]) The Uniform Rule ϕ^* is defined as follows:*

$$\phi_i^*(R) = \begin{cases} min\{s_i(R_i), \lambda(R)\}, & \sum_{i \in N} s_i(R_i) \geq 1 \\ max\{s_i(R_i), \mu(R)\}, & \sum_{i \in N} s_i(R_i) \leq 1 \end{cases}$$

for all $i \in N$, where $\lambda(R)$ solves the equation $\sum_{i \in N} min\{s_i(R_i, \lambda(R)\} = 1$ and $\mu(R)$ solves the equation $\sum_{i \in N} min\{s_i(R_i, \mu(R)\} = 1$

The uniform rule gives to each agent his most preferred share, as long as it falls within certain bounds which are the same for everyone and chosen so as to satisfy the feasibility condition.

The uniform rule applied to the network 2.1 above splits the unit good in the following way: $(s_1, s_2, s_3) = (2/7, 5/14, 5/14)$. Agent s_1 receives his peak and does not envy other agents; Agents s_2, s_3 are symmetric and receive the same fraction. Any increase in the allocation of agents s_2, s_3 violates feasibility or envy freeness. We will define these properties in later sections. The rest of this chapter is related to generalizing the Sprumont's model and uniform rule to a bipartite network.

2.3 Two Sided Model (Divisible Goods)

2.3.1 Transfers with bilateral constraints

We have a set S of suppliers with generic element i, and a set D of demanders with generic element j. A set of transfers of the single commodity from suppliers to demanders results in a vector $(x,y) \in \mathbb{R}_+^S \times \mathbb{R}_+^D$ where x_i (resp. y_j) is supplier i's (resp. demander j's) *net transfer,* with $\sum_S x_i = \sum_D y_j$.

The commodity can only be transferred between certain pairs of supplier i, demander j. The bipartite graph G, a subset of $S \times D$, represents these constraints: $ij \in G$ means that a transfer is possible between $i \in S$ and $j \in D$. We assume throughout that the graph G is connected, else we can treat each connected component of G as a separate problem.

We use the following notation. For any subsets $T \subseteq S$, $C \subseteq D$ the restriction of G is $G(T,C) = G \cap \{T \times C\}$ (not necessarily connected). The set of demanders compatible with the suppliers in T is $f(T) = \{j \in D | G(T,\{j\}) \neq \varnothing\}$. The set of suppliers compatible with the demanders in C is $g(C) = \{i \in S | G(\{i\},C) \neq \varnothing\}$. For any subsets $T \subseteq S$, $C \subseteq D$, $x_T := \sum_{i \in T} x_i$ and $y_C := \sum_{j \in C} y_j$.

A transfer of goods from S to D is realized by a G-flow φ, i.e., a vector $\varphi \in \mathbb{R}_+^G$. We write $x(\varphi), y(\varphi)$ for the transfers implemented by φ, namely:

$$\text{for all } i \in S : x_i(\varphi) = \sum_{j \in f(i)} \varphi_{ij}; \text{ for all } j \in D : y_j(\varphi) = \sum_{i \in g(j)} \varphi_{ij} \tag{2.5}$$

We say that the net transfers (x,y) are *feasible* if they are implemented by some G-flow. We write $\Phi(G)$ for the set of feasible flows, and $\mathcal{A}(G)$ for the set of feasible net transfers. We define similarly $\mathcal{A}(G(S',D'))$ for any $S' \subseteq S$, $D' \subseteq D$. These sets are described as follows.

Lemma 1: *For any $S' \subseteq S$, $D' \subseteq D$ the three following statements are equivalent:*

i) $(x,y) \in \mathcal{A}(G(S',D'))$

ii) for all $T \subseteq S'$, $x_T \leq y_{f(T)}$ and $x_{S'} = y_{D'}$

iii) for all $C \subseteq D'$, $y_C \leq x_{g(C)}$ and $y_{D'} = x_{S'}$

Proof: This is a standard application of the Marriage Lemma, see, e.g., [2].

The two sided model arise in many practical scenarios. The applications include matching call center employees to customers, hospitals sharing/diverting patients, service providers sharing customers like in airline or hotel industry, matching organ or blood donors with recipients, etc.

2.3.2 Maximal flow under capacity constraints

Assume, *in this section only*, that each supplier $i \in S$ has a (hard) capacity constraint s_i, i.e., cannot send more than s_i units of the commodity. Similarly each demander $j \in D$ cannot receive more than d_j units.

We write $\Phi(G, s, d)$ for the set of feasible flows φ such that $x(\varphi) \leq s$, and $y(\varphi) \leq d$, and $\mathcal{A}(G, s, d)$ for the corresponding set of feasible constrained transfers.

The problem of finding the maximal feasible flows between suppliers and demanders thus constrained, is well understood. We can apply the celebrated max-flow/min-cut theorem to the oriented capacity graph $\Gamma(G, s, d)$ obtained from G by adding a source σ connected to all suppliers, and a sink τ connected to all demanders; by orienting the edges from source to sink; by setting the capacity of an edge in G to infinity, that of an edge $\sigma i, i \in S$, to s_i, and that of $j\tau, \tau \in D$, to d_j. A σ-τ cut (or simply a cut) in this graph is a subset X of nodes that contains σ but not τ. The capacity of a cut X is the total capacity of the edges that are oriented from a node in X to a node outside of X (such edges are said to be “in the cut”).

We illustrate next this construction.

Example 3: *Canonical flow representation*

Figure 2.3 shows the canonical flow representation of Example 2.2. The maximum flow from σ to τ is bounded by the capacity of any σ-τ cut, in particular the minimum capacity σ-τ cut. The max-flow/min-cut theorem says that the maximum σ-τ flow has value equal to the capacity of the minimum σ-τ cut. Agents in the minimum cut are in the market segment with long supply; agents outside the minimum cut belong to the segment with long demand. In Figure 2.3 , the minimum capacity cut contains supplier 1 and demander 1 only (and σ) and has a capacity of 24 which is the value of a maximum flow. Note that in the subset of efficient allocations where the long side

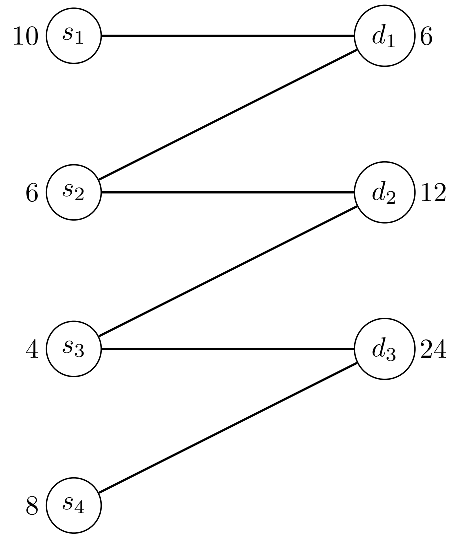

Figure 2.2: A two sided network with suppliers (on the left) and demanders (on the right)

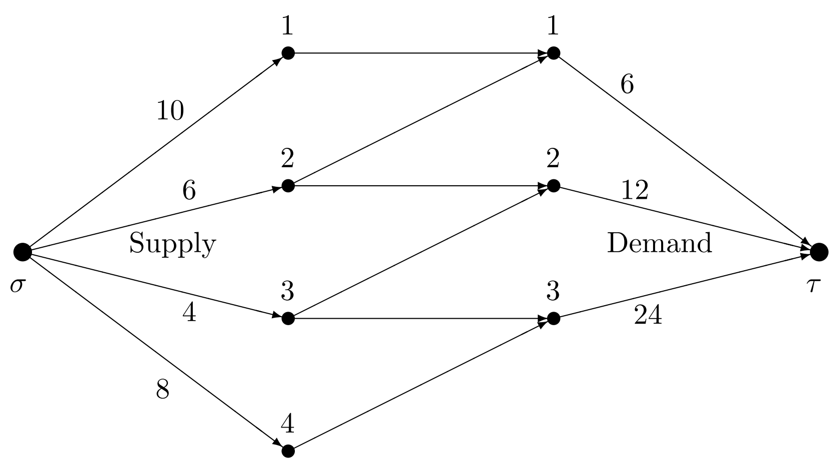

Figure 2.3: The max-flow problem for the above network

gets always rationed, any allocation will involve a net transfer of 24. This implies that supplier 1 will send only 6 units on demander .

These observations are summarized as follows: if we fix a maximum flow from σ to τ and a minimum-capacity σ-τ cut, then every edge in the cut must carry a flow equal to its capacity; moreover every edge that is oriented from a node outside of the cut to a node in the cut should carry zero flow. This leads to a key decomposition result.

Lemma 1 *i) There exists a partition S_+, S_- of S, and a partition D_+, D_- of D, where at most one of $S_+ = D_- = \varnothing$, or $S_- = D_+ = \varnothing$ is possible, with the following properties:*

$$G(S_-, D_-) = \varnothing,\ D_+ = f(S_-),\ S_+ = g(D_-)$$

$$s_{S'} \leq d_{f(S')\cap D_-} \text{ for all } S' \subseteq S_+;\ d_{D'} \leq s_{g(D')\cap S_-} \text{ for all } D' \subseteq D_+ \tag{2.6}$$

ii) The maximal flow is $s_{S_+} + d_{D_+}$. The flow $\varphi \in \Phi(G, s, d)$, with net transfers x, y is maximal if and only if

$$\varphi = 0 \text{ on } G(S_+, D_+),\ x = s \text{ on } S_+,\ y = d \text{ on } D_+$$

iii) The profile of transfers $(x, y) \in \mathcal{A}(G, s, d)$ is achieved by a maximal flow if and only if

$$x_S = y_D = s_{S_+} + d_{D_+} \tag{2.7}$$

Proof. Refer to the Appendix.

The inequalities (2.6) express that the supply from S_+ is short with respect to the demanders in D_-, whereas the demand in D_+ is short with respect to the supply in S_-.

Example 4: *Several possible decompositions*

In general, the decomposition is not unique as there are several minimum cuts, all with identical capacities. If there is a unique min-cut, for instance as in Figure 3, the decomposition of the market in two segments is unique too (this holds true for an open and dense set of vectors (s, d)). If it is not unique, there is a partition S_+, S_- (resp. D_+, D_-) where S_- (resp. D_-) is the largest possible,

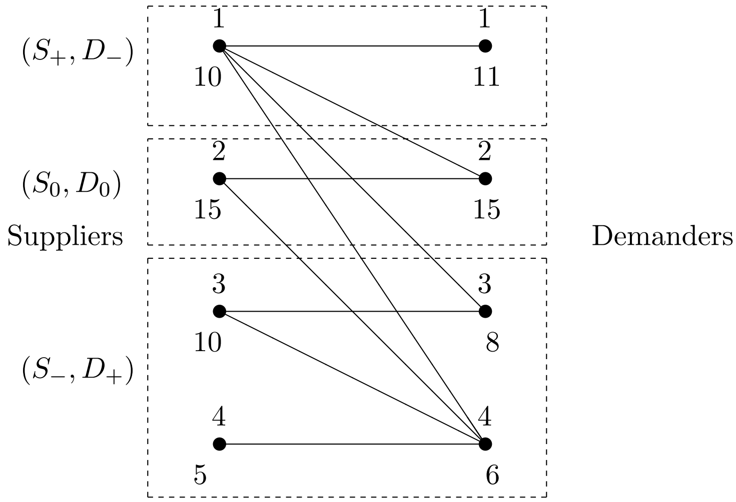

Figure 2.4: Decomposition with a balanced subgraph

and one where it is smallest. In Figure 2.4, there are two ways to decompose the demand and the supply sides. One possible decomposition is $D_- = \{1, 2\}$, $D_+ = \{3, 4\}$, $S_+ = \{1, 2\}$, $S_- = \{3, 4\}$. The other is $D'_- = \{1\}$, $D'_+ = \{2, 3, 4\}$, $S'_+ = \{1\}$, $S'_- = \{2, 3, 4\}$.

In contrast, a familiar graph-theoretical result, the Gallai-Edmonds decomposition (see Lovasz and Plummer [40]), determines a unique partition of the market but in up to *three* segments. In one segment supply is overdemanded and the corresponding demanders must be rationed; in the second segment supply is underdemanded, and these suppliers transfer less than their ideal share; and in the third segment supply exactly balances demand. In Figure 4 the three segments of this decomposition are depicted as (S_+, D_-), (S_-, D_+) and (S_0, D_0) respectively.

2.3.3 The Egalitarian mechanism

Definition: *Given the agents* (S, D)*, a rule* ψ *selects for every economy* $(G, R) \in 2^{S \times D} \times \mathcal{R}^{S \cup D}$ *a feasible allocation* $\psi(G, R) \in \mathcal{A}(G)$.

We give two definitions of our egalitarian solution. The first one is a constructive algorithm. The second one is based on the fact that, within the subset of Pareto optimal allocations, this allocation equalizes individual shares in the strong sense of Lorenz dominance defined later.

We fix a problem (G, s, d) such that $s_i, d_j > 0$ for all i, j (clearly if $s_i = 0$ or $d_j = 0$ we can ignore supplier i or demander j altogether). We define independently our solution for the suppliers and for the demanders.

The definition for suppliers is by induction on the number of agents $|S| + |D|$. Consider the parameterized capacity graph $\Gamma(\lambda), \lambda \geq 0$: the only difference between this graph and $\Gamma(G, s, d)$ is that the capacity of the edge $\sigma i, i \in S_-$ is $\min\{\lambda, s_i\}$, which we denote by $\lambda \wedge s_i$. (In particular, the edge from j to τ still has capacity d_j). We set $\alpha(\lambda)$ to be the maximal flow in $\Gamma(\lambda)$. Clearly α is a piecewise linear, weakly increasing, strictly increasing at 0, and concave function of λ, reaching its maximum when the total σ-τ flow is d_{D_+}. Moreover, each breakpoint is one of the s_i (type 1), and/or is associated with a subset of suppliers X such that

$$\sum_{i \in X} \lambda \wedge s_i = \sum_{j \in f(X)} d_j \tag{2.8}$$

Then we say it is of type 2. In the former case, the associated supplier reaches his peak and so cannot send any more flow. In the latter case, the group of suppliers in X is a *bottleneck*, in the sense that they are sending enough flow to satisfy the collective demand of the demanders in $f(X)$ and these are the only demanders they are connected to; any further increase in flow from any supplier in X would cause some demander in $f(X)$ to accept more than his peak demand.

If the given problem does not have any type-2 breakpoint, then the egalitarian solution is obtained by setting each supplier's allocation to his peak value. Otherwise, let λ^* be the first type-2 breakpoint of the max-flow function; by the max-flow min-cut theorem, for every subset X satisfying (2.25) at λ^* the cut $C^1 = \{\sigma\} \cup X \cup f(X)$ is a minimal cut in $\Gamma(\lambda^*)$ providing a certificate of optimality for the maximum-flow in $\Gamma(\lambda^*)$. If there are several such cuts, we pick the one with the largest X^* (its existence is guaranteed by the usual supermodularity argument). The egalitarian solution is obtained by setting

$$x_i = \min\{\lambda^*, s_i\}, \text{ for } i \in X^*, \ y_j = d_j, \text{ for } j \in f(X^*),$$

and assigning to other agents their egalitarian share in the reduced problem $(G(S \diagdown X^*, D \diagdown f(X^*)), s, d)$. That is, we construct $\Gamma^{S \diagdown X^*, D \diagdown f(X^*)}(\lambda)$ for $\lambda \geq 0$ by changing in $\Gamma(G(S \diagdown X^*, D \diagdown f(X^*)), s, d)$ the capacity of the edge σi to $\lambda \wedge s_i$, and look for the first type-2 breakpoint λ^{**} of the corresponding max-flow function. An important fact is that $\lambda^{**} > \lambda^*$. Indeed there exists a subset X^{**} of $S \diagdown X^*$

such that

$$\sum_{i \in X^{**}} \lambda^{**} \wedge s_i = \sum_{j \in f(X^{**}) \setminus f(X^*)} d_j$$

If $\lambda^{**} \leq \lambda^*$ we can combine this with equation (2.25) at X^* as follows

$$\sum_{i \in X^* \cup X^{**}} \lambda^* \wedge s_i \geq \sum_{i \in X^*} \lambda^* \wedge s_i + \sum_{i \in X^{**}} \lambda^{**} \wedge s_i = \sum_{j \in f(X^* \cup X^{**})} d_j$$

contradicting our choice of X^* as the largest subset of S_- satisfying (2.25) at λ^*.

The solution thus obtained recursively is the egalitarian allocation for the suppliers. A similar construction works for demanders: We consider the parameterized capacity graph $\Delta(\mu), \mu \geq 0$, with the capacity of the edge $\tau j, j \in D$ set to $\mu \wedge d_j$. We look for the first type-2 breakpoint μ^* of the maximal flow $\beta(\mu)$ of $\Delta(\mu)$, and for the largest subset of demanders Y such that

$$\sum_{j \in Y} \mu \wedge d_j = \sum_{i \in g(Y)} s_i$$

etc.. Combining these two egalitarian allocations yields the egalitarian allocation $(x^e, y^e) \in \mathbb{R}_+^{S \cup D}$ for the overall problem.

Two simple examples

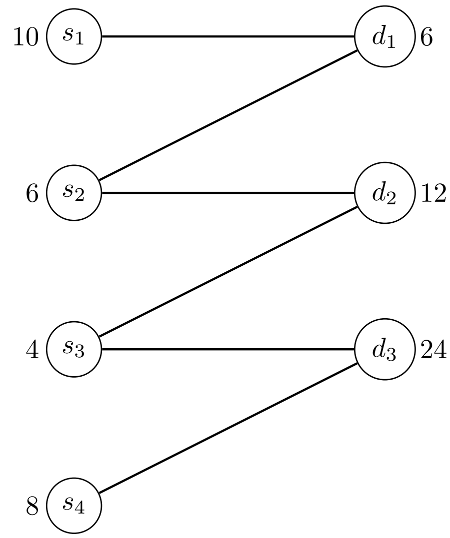

Figure 2.5: Short supply and short demand co-exist

Example 1: *Short supply and short demand coexist*

Short supply and short demand typically coexist in two independent segments of the market. This is illustrated in Figure 2.5. Supplier 1 can only transfer to demander 1, whose demand is short against 1's long supply. The two demanders $2, 3$ are similarly captive of suppliers $2, 3, 4$, whose supply is short against their long demand. Note that decentralized trade may fall short of efficiency. Indeed demander 1 and supplier 2 achieve their ideal consumption by a bilateral transfer of 6 units. However after this transfer supplier 1 is unable to trade, and demanders $2, 3$ have to share a short supply of 12 against their long demand of 36. It is more efficient to transfer 6 units from supplier 1 to demander 1 and let suppliers $2, 3, 4$ send their 18 units to demanders $2, 3$.

The first market segment contains the *long* supplier 1 and the *short* demander 1. On the other hand, demanders 2,3 compete for transfers from suppliers 2,3,4. These agents form the short supply/long demand segment. Our egalitarian solution rations the long side of the market in each of the two segments. Consider the efficient profile of net transfers $(x, y) = ((6, 6, 4, 8), (6, 8, 10))$ (x for suppliers, y for demanders). Here demanders 2,3 split equally the transfer from supplier 3, their only common link. However the profile $((6, 6, 4, 8), (6, 9, 9))$ is feasible and Lorenz dominates (x, y), it is our egalitarian solution.

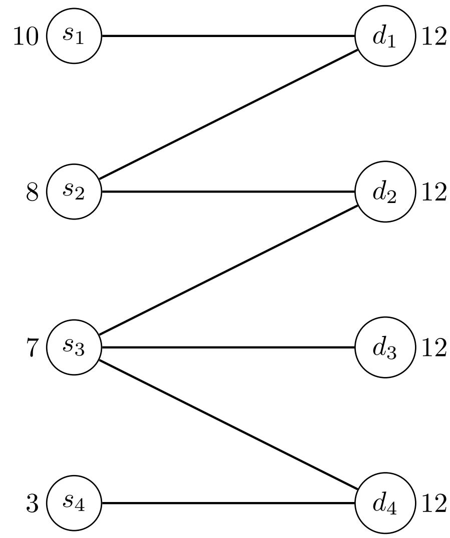

Figure 2.6: Agents on the short side are not treated identically

There is a unique min cut given by $C^1 = \{\sigma\} \cup \{X\} \cup \{f(X)\}$ where $X = \{\text{supplier 1}\}$. Agents in the minimum cut form the partition (S_-, D_+) whereas $S_+ = \{\text{suppliers 2,3,4}\}$ and $D_- = \{\text{demanders 2,3}\}$. We start with (S_-, D_+). The algorithm looks for λ_1 such that $\min\{s_1, \lambda_1\} = 6$, giving $\lambda_1 = 6$. For the other segment, the egalitarian algorithm stops at $\lambda_2 = 9$. Indeed $\min\{d_2, \lambda_2\} + \min\{d_3, \lambda_2\} = s_2 + s_3 + s_4$.

Another implication of the bilateral constraints is that agents with identical preferences cannot always be treated equally.

Example 2: *Identical preferences, different transfers*

This is illustrated in Figure 2.6. There is a single market segment with a long demand, so the suppliers unload their peak transfer. The bilateral constraints, restrict the (non negative) transfers y_i to the four demanders as follows:

$$10 \leq y_1 \leq 12;\ 6 \leq y_2 \leq 12;\ y_3 \leq 7$$

$$\sum_1^4 y_i = 28;\ y_1 + y_2 \geq 18 \Leftrightarrow y_3 + y_4 \leq 10$$

Without the bilateral constraints, we can achieve $y_i = 7, i = 1, 2, 3, 4$. Under these constraints, the most egalitarian profile is $y_1 = 10, y_2 = 8, y_3 = y_4 = 5$. We now illustrate the algorithm by revisiting the examples of Section 2.

Recall that there is a single segment in which the demand is long. The algorithm first stops at $\lambda_1 = 10$. Indeed $\min\{d_1, \lambda_1\} = s_1$. The algorithm next stops at $\lambda_2 = 8$ since $\min\{d_2, \lambda_2\} = s_2$. Finally, the algorithm stops at $\lambda_3 = 5$ since $\min\{d_3, \lambda_3\} + \min\{d_4, \lambda_3\} = s_3 + s_4$.

In the next few subsections, we briefly summarize the results of Bochet et al. [12]. Refer to the work of Bochet et al. for a more detailed exposition.

2.3.4 Pareto optimality and the Core

We now have a bipartite graph G between S and D as before, but we replace the hard capacity constraint of the previous section by a soft ideal consumption. Each supplier i has *single-peaked preferences*[1] R_i (with corresponding indifference relation I_i) over her *net transfer* x_i, with peak s_i, and each demander j has single-peaked preferences R_j (I_j) over her net transfer y_j, with peak d_j. We write $\mathcal{R}$ for the set of single peaked preferences over $\mathbb{R}_+$, and $\mathcal{R}^{S \cup D}$ for the set of preference profiles.

The feasible net transfer $(x, y) \in \mathcal{A}(G)$ is Pareto optimal if for any other $(x', y') \in \mathcal{A}(G)$ we have

$$\{\text{for all } i, j\text{: } x'_i R_i x_i \text{ and } y'_j R_j y_j\} \Rightarrow \{\text{for all } i, j\text{: } x'_i I_i x_i \text{ and } y'_j I_j y_j\}$$

We write $\mathcal{PO}(G, R)$ for the set of Pareto optimal net transfers.

Proposition 1:

Fix the economy (G, R), and two partitions S_+, S_- and D_+, D_- corresponding to the profile of peaks (s, d) at R (as in Lemma 1).

i) if the G-flow φ implements Pareto optimal net transfers (x, y), then transfers occur only between S_+ and D_-, and between S_- and D_+:

$$\varphi_{ij} > 0 \Rightarrow ij \in G(S_+, D_-) \cup G(S_-, D_+)$$

[1] Writing P_i for agent i's strict preference, we have for every x_i, x'_i: $x_i < x'_i \leq s_i \Rightarrow x'_i P_i x_i$, and $s_i \leq x_i < x'_i \Rightarrow x_i P_i x'_i$.

ii) $(x, y) \in \mathcal{PO}(G, R)$ *if and only if* $(x, y) \in \mathcal{A}(G)$ *and*

$$x \geq s \text{ on } S_+,\ y \leq d \text{ on } D_-, \text{ and } x_{S_+} = y_{D_-}$$

$$x \leq s \text{ on } S_-,\ y \geq d \text{ on } D_+ \text{ and } x_{S_-} = y_{D_+}$$

An important feature of the Pareto set is that it only depends upon the profile of peaks s, d, and not upon the full preference profile R. The same is true of our egalitarian solution. To emphasize this important simplification, we speak of a *transfer problem* (S, D, G, s, d) or simply (G, s, d), keeping in mind the underlying single-peaked preferences.

The following subset of $\mathcal{PO}(G, R)$ will play an important role:

$$\mathcal{PO}^*(G, s, d) = \mathcal{PO}(G, R) \cap \{(x, y) | x \leq s; y \leq d\}$$

By Proposition 1, this is the set of efficient allocations where the short side gets its optimal transfer:

$$x = s \text{ on } S_+,\ y \leq d \text{ on } D_-, \text{ and } y_{D_-} = s_{S_+}$$

$$x \leq s \text{ on } S_-,\ y = d \text{ on } D_+ \text{ and } x_{S_-} = y_{D_+}$$

Moreover by Lemma 2, the net transfers in $\mathcal{PO}^*(G, s, d)$ are precisely those implemented by all the maximal flows of the capacity graph $\Gamma(G, s, d)$.

We focus on allocations in $\mathcal{PO}^*(G, s, d)$, because under the Voluntary Trade (requiring $x_i R_i 0, y_j R_j 0$ for all i, j; see Section 8) property, they are the only allocations Pareto optimal for any choice of preferences in $\mathcal{R}$ with peaks (s, d).

We first give an alternative characterization of the Pareto* set, critical to the analysis of the egalitarian solution. Define two cooperative games, (S, v) and (D, w), of which the players are respectively the suppliers and the demanders:

$$v(T) = \min_{T' \subseteq T} \{s_{T'} + d_{f(T \setminus T')}\} \text{ for all } T \subseteq S \tag{2.9}$$

$$w(E) = \min_{E' \subseteq E} \{d_{E'} + s_{g(E \setminus E')}\} \text{ for all } E \subseteq D \tag{2.10}$$

Lemma 2 *The games* (S, v) *and* (D, w) *are submodular. Moreover*

$$v(S) = w(D) = s_{S_+} + d_{D_+};\ v(S_-) = d_{D_+};\ w(D_-) = s_{S_+} \tag{2.11}$$

The core of the game (S, v), denoted $Core(S, v)$, is the set of allocations $x \in \mathbb{R}^S_+$ such that $x_T \leq v(T)$ for all $T \subset S$, and $x_S = v(S)$; similarly the core of the game (D, w) is the set of allocations $y \in \mathbb{R}^D_+$ such that $y_E \leq w(E)$ for all $E \subset D$, and $y_D = w(D)$. Notice that $v(T) \leq s_T$ for all $T \subset S$, therefore $x \in Core(S, v)$ implies $x \leq s$; similarly $y \in Core(D, w) \Rightarrow y \leq d$.

Lemma 3 *Fix the problem (G, s, d), and two partitions S_+, S_- and D_+, D_- as in Lemma 1. Then the allocation (x, y) is in $\mathcal{PO}^*(G, s, d)$ if and only if it satisfies one of the two equivalent properties*

i) $x \in Core(S, v)$ and $y \in Core(D, w)$

ii) $\{x = s$ on S_+, and on S_-, $x \in Core(S_-, v)\}$ and $\{y = d$ on D_+, and on D_-, $y \in Core(D_-, w)\}$

For any problem (G, s, d), the allocation x^e (resp. y^e) is the egalitarian selection in $Core(S, v)$ (resp. $Core(D, w)$).

We turn now to the Lorenz dominant position of our solution inside $\mathcal{PO}^*(G, s, d)$. For any $z \in \mathbb{R}^N$, write z^* for the *order statistics* of z, obtained by rearranging the coordinates of z in increasing order. For $z, w \in \mathbb{R}^N$, we say that z *Lorenz dominates* w, written $z >_{LD} w$, if for all $k, 1 \leq k \leq n$

$$\sum_{a=1}^{k} z^{*a} \geq \sum_{a=1}^{k} w^{*a}$$

Lorenz dominance is a partial ordering, so not every set, even convex and compact, admits a Lorenz dominant element. On the other hand, in a convex set A there can be at most one Lorenz dominant element. The appeal of a Lorenz dominant element in A is that it maximizes over A, *any* symmetric and concave collective utility function.

Theorem 1 *The allocation (x^e, y^e) is the Lorenz dominant element in $PO^*(G, s, d)$* [2].

We introduce the incentives and equity properties which form the basis of the characterization result in the next section. Those properties bear on the profile of individual preferences R, therefore instead of a transfer *problem* (G, s, d), we consider now a transfer *economy* (G, R). We use the notation $s[R_i], d[R_j]$ for the peak transfer of supplier i and demander j.

[2]Note that this solution is not Lorenz dominant in the entire Pareto set.

We now turn to equity properties. The familiar equity test of no envy must be adapted to our model because of the feasibility constraints. If supplier 1 envies the net transfer x_2 of supplier 2, it might not be possible to give him x_2 anyway because the demanders connected to agent 1 have insufficient demands. Even if we can exchange the net transfers of 1 and 2, this may require us to construct a new flow and alter some of the other agents' allocations. In either case we submit that supplier 1 has no legitimate claim against the allocation x. An envy argument by agent 1 against agent 2 is legitimate only if it is feasible to improve upon agent 1's allocation without altering the allocation of anyone other than agent 2.

No Envy: *For any preference profile $R \in \mathcal{R}^{S \cup D}$ and any $i_1, i_2 \in S$ such that $\psi_{i_2}(R) P_{i_1} \psi_{i_1}(R)$, there exists no $(x,y) \in \mathcal{A}(G)$ such that*

$$\psi_i(R) = x_i \text{ for all } i \in S \setminus \{i_1, i_2\};\ \psi_j(R) = y_j \text{ for all } j \in D$$

$$\text{and } x_{i_1} P_{i_1} \psi_{i_1}(R) \tag{2.12}$$

and a similar statement where we exchange the role of demanders and suppliers.

Note that if i_1, i_2 have identical connections, $i_1 j \in G \Leftrightarrow i_2 j \in G$, then we can exchange their allocations without altering any other net transfer, therefore No Envy implies $\psi_{i_1}(R) I_{i_1} \psi_{i_2}(R)$.

The familiar horizontal equity property must be similarly adapted to account for the bilateral constraints on transfers.

Equal Treatment of Equals (ETE): *For any preference profile $R \in \mathcal{R}^{S \cup D}$ and any $i_1, i_2 \in S$* such that $R_{i_1} = R_{i_2}$, *there exists no $(x,y) \in \mathcal{A}(G)$ such that*

$$\psi_i(R) = x_i \text{ for all } i \in S \setminus \{i_1, i_2\}\ \psi_j(R) = y_j \text{ for all } j \in D$$

$$|x_{i_1} - x_{i_2}| < |\psi_{i_1}(R) - \psi_{i_2}(R)| \tag{2.13}$$

and a similar statement where we exchange the role of demanders and suppliers.

Again, if i_1, i_2 have identical connections ETE implies $\psi_{i_1}(R) = \psi_{i_2}(R)$. In general ETE requires the rule to equalize as much as possible the allocations of two agents with identical preferences.

Voluntary trade: *For all $R \in \mathcal{R}^{S \cup D}$, $i \in S \cup D$, we have $\psi_i(R) R_i 0$.*

Lemma 4 *i) Any mechanism that allocates a No Envy solution in the set of Pareto optimality solutions is also an Equal Treatment of Equals allocation; ii)The egalitarian transfer rule ψ^e satisfies No Envy*

We turn now to strategic issues related to the Egalitarian Mechanisms. Bochet et al. [12] show that the egalitarian mechanism is both link strategyproof and peak strategyproof. Here we show that the egalitarian mechanism is in fact peak groupstrategyproof and link groupstrategyproof when limited to coalition only among suppliers and demanders.

2.3.5 Strategic Issues

Firstly, we define certain notions that we use in the rest of the section.

Link monotonicity requires that an agent on either side of the market weakly benefits from the access to new links. This ensures that no agent has an incentive to close a feasible link; equivalently it is a dominant strategy to reveal all feasible links to the manager.

Link Monotonicity: *For any economy $(G, R) \in 2^{S\times D} \times \mathcal{R}^{S\cup D}$, and any $i \in S$, $j \in D$, we have* $\psi_k(G \cup \{ij\}, R)R_k\psi_k(G, R)$, for $k = i, j$. That is, having an additional link can only improve the allocation of an agent.

Link Strategyproof: *For any economy $(G, R) \in 2^{S\times D} \times \mathcal{R}^{S\cup D}$, and any $i \in S \cup D$,* let A_i be the agents compatible with agent i in network G. Suppose agent i misreports his compatible partners, say A'_i and hence network G' is revealed to the mechanism, then we have $\psi_i(G, R)R_i\psi_i(G', R)$. That is, in a link strategyproof mechanism it is a dominant strategy ro reveal your links truthfully.

Proposition 1 (Bochet et al. [12]) *The egalitarian transfer rule is link-monotonic and hence link strategyproof.*

Note that the addition of a link ij may well hurt agents other than i, j. In Figure 2.7, we show an example with short demand in which our rule picks the allocation $x_1 = 3$ and $x_2 = 1$. Adding the link between supplier 2 and demander 1 gives $x'_1 = x'_2 = 2$.

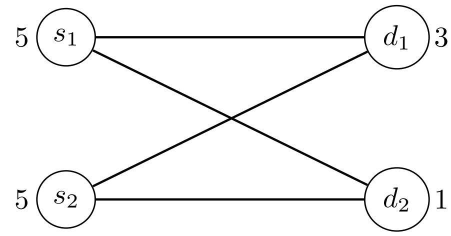

Figure 2.7: A new link may hurt non-involved agents

Now, we turn to strategic manipulation of links by groups of agents. In such a coalition group, each agent has two types of strategies: (i) Shrinking the set of agents to whom he/she is connected to; (ii) Reporting an agent feasible who is not feasible to begin - hence creating a spurious link.

Link Groupstrategyproof: *For any economy* $(G,R) \in 2^{S\times D}\times\mathcal{R}^{S\cup D}$*, and any subset of agents* $M \subseteq S\cup D$, let A_i be the agents compatible with agent i in network G, $i \in M$. Suppose agent i misreports his compatible partners, say A_i', for $i \in M$ and hence network G' is revealed to the mechanism, then we have $\psi_i(G,R)R_i\psi_i(G',R)$, for $i \in M$.

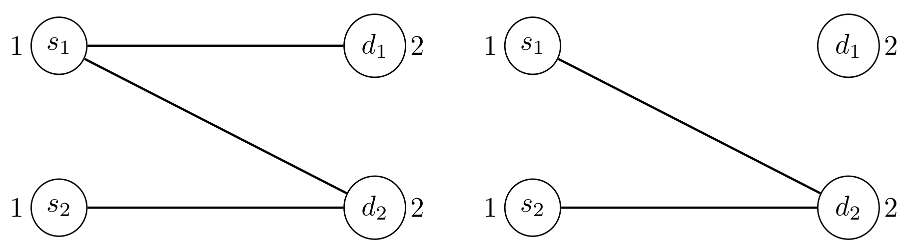

Figure 2.8: Egalitarian Mechanism is not link GSP w.r.t. to both suppliers and demanders

That the egalitarian mechanism is *not* link group strategyproof in the two-sided model is not difficult to see. Consider the network shown in Figure 2.8. The network (a) represents the true network, with the peaks shown next to the agent labels. The egalitarian allocation gives 1 unit to each supplier and to each demander on this example. Suppose however supplier 1 and demander 2 collude, and supplier 1 does not report his link to demander 1. In the resulting network, shown in (b), each supplier still receives his peak allocation; demander 2 now receives her peak, and demander 1 receives nothing. Note that both members of the coalition weakly improve, and demander 2 strictly improves, proving that the egalitarian mechanism is in general *not* link group strategyproof [3]. The following result, however, shows that the egalitarian mechanism satisfies a limited form of link group strategyproofness.

Theorem 1 *In the two sided model, the egalitarian mechanism is link group strategy proof when the coalition is restricted to the set of suppliers only (demanders only).*

Proof. We prove the result for an arbitrary coalition of suppliers; the result for the demanders follow by a similar argument. Let A_i be the set of demanders that supplier i is linked to, and let A'_i be supplier i's report. We may assume without loss of generality that any given demander finds all the suppliers acceptable: if demander j finds supplier i unacceptable, then supplier i cannot have a link to demander j regardless of his report, so clearly i's manipulation opportunities are more restricted. Let ϕ and $\phi^{'}$ be (any) egalitarian flows when the suppliers report A and A' respectively, and let x and x' be the corresponding allocation to the suppliers. We show that no coalition of suppliers can weakly benefit by misreporting their links unless each supplier in the coalition gets exactly their egalitarian allocation.

The proof is by induction on the number of type 2 breakpoints in the algorithm to compute the egalitarian allocation. Suppose the given instance has n type 2 breakpoints, and suppose $X_1, X_2, \ldots, X_n$ are the corresponding bottleneck sets of suppliers. If $n = 0$, every supplier is at his peak value in the egalitarian allocation, and clearly this allocation cannot be improved. Suppose $n \geq 1$. Define

$$\tilde{X}_\ell = \{i \in X_\ell \mid \sum_{j \in A_i} \phi'_{ij} \ \geq \ \sum_{j \in A_i} \phi_{ij}\},$$

and

$$\hat{X}_\ell = \{i \in X_\ell \mid \sum_{j \in A_i} \phi'_{ij} \ \leq \ \sum_{j \in A_i} \phi_{ij}\}.$$

We shall show, by induction on ℓ, that for each $\ell = 1, 2, \ldots, n$:

(a) $\phi^{'}_{ij} = 0$ for any $i \in \tilde{X}_{\ell'}$, $j \in \cup_{i' \in X_\ell} A_{i'}$, $\ell' > \ell$; and

[3] This example may suggest that if we require each member of the deviating coalition to strictly improve their allocation, then the egalitarian mechanism may be link group strategyproof. However, this is also false, as shown by Bogomolnaia and Moulin [13]. They construct an example involving 4 agents on each side with all peaks identically 1 in which a coalition of agents from both sides deviate and all *strictly* improve.

(b) $X_\ell \subseteq \hat{X}_\ell$.

The theorem follows from part (b) above.

Any supplier $k \in X_\ell \setminus \tilde{X}_\ell$ must have $A_k = A'_k$ as otherwise supplier k is part of the deviating coalition and does worse. Consider now a supplier $i \in \tilde{X}_\ell$ with $x_i < s_i$ and a supplier $k \in X_\ell \setminus \tilde{X}_\ell$. We have the following chain of inequalities:

$$\sum_{j \in A'_k} \phi'_{kj} \;=\; \sum_{j \in A_k} \phi'_{kj} \;<\; \sum_{j \in A_k} \phi_{kj} = x_k \;\le\; x_i \;=\; \sum_{j \in A_i} \phi_{ij} \;\le\; \sum_{j \in A_i} \phi'_{ij} \;\le\; \sum_{j \in A'_i} \phi'_{ij}.$$

To see why, note that as $k \in X_\ell \setminus \tilde{X}_\ell$, the second inequality is true by definition, and also $A_k = A'_k$ (justifying the first equality). Also $k, i \in X_\ell$ and $x_i < s_i$, implies $x_k < s_i$, as suppliers k and i both belong to the same bottleneck set and supplier i is below his peak; this justifies the third inequality. The fourth and fifth inequalities follow from the fact that $i \in \tilde{X}_\ell$ and the fact that ϕ'_{ij} must be zero for all $j \in A_i \setminus A'_i$. This chain of inequalities implies that $x'_k < x_k \le s_k$ and $x'_k < x'_i$. Therefore, when the suppliers report A', supplier k must be a member of an "earlier" bottleneck set than supplier i. An immediate consequence is that demanders in $A'_k = A_k$ do not receive any flow from supplier i when the report is A'.

By the induction hypothesis, supplier $i \in X_\ell$ does not send any flow to the demanders in $\cup_{1 \le i' \le \ell - 1} \cup_{k \in X_{i'}} A_{k'}$. Therefore

$$\{j \mid \phi'_{ij} > 0, j \in A_i\} \subseteq \{j \mid \phi_{ij} > 0, j \in A_i\}.$$

This observation, along with the fact that every $i \in \tilde{X}_\ell$ weakly improves, and the fact that X_ℓ is a type 2 breakpoint implies that $\sum_{j \in A_i} \phi'_{ij} = \sum_{j \in A_i} \phi_{ij}$, establishing (b). Furthermore, in such a solution, every demander $j \in A_i$ for $i \in \tilde{X}_\ell$ must receive all his flow from the suppliers in $\tilde{X}_\ell$. In particular, the demanders in X_ℓ cannot receive any flow from suppliers in $X_{\ell'}$ for $\ell' > \ell$, establishing (a). To complete the proof we need to establish the basis for the induction proof, i.e., the case of $\ell = 1$. This, however, follows easily: it is easy to verify that the set $X_1 \setminus \tilde{X}_1$ must be empty, so $X_1 = \tilde{X}_1$. As X_1 is a type 2 bottleneck set, it is not possible for *every* member of X_1 to do weakly better unless the allocation remains unchanged. Thus, both (a) and (b) follow. ∎

In the rest of the section we discuss properties for which the graph G is fixed, so we write a rule simply as $\psi(R)$ for $R \in \mathcal{R}^{S\cup D}$. The next incentive property is the familiar *strategyproofness*. It is useful to decompose it into a monotonicity and an invariance condition.

Peak Monotonicity: *An agent's net transfer is weakly increasing in her reported peak: for all* $R \in \mathcal{R}^{S\cup D}$, $i \in S, j \in D$ *and* $R'_i, R'_j \in \mathcal{R}$

$$s[R'_i] \le s[R_i] \Rightarrow \psi_i(R'_i, R_{-i}) \le \psi_i(R)$$

$$d[R'_j] \le d[R_j] \Rightarrow \psi_j(R'_j, R_{-j}) \le \psi_j(R)$$

Invariance: *For all* $R \in \mathcal{R}^{S\cup D}$, $i \in S$ *and* $R'_i \in \mathcal{R}$

$$\{s[R_i] < \psi_i(R) \text{ and } s[R'_i] \le \psi_i(R)\} \text{ or } \{s[R_i] > \psi_i(R) \text{ and } s[R'_i] \ge \psi_i(R)\} \tag{2.14}$$

$$\Rightarrow \psi_i(R'_i, R_{-i}) = \psi_i(R)$$

and similarly $\psi_j(R'_j, R_{-j}) = \psi_j(R)$ *when agent* $j \in D$ *such that* $\psi_j(R) \neq d[R_j]$ *reports* $R'_j \in \mathcal{R}$ *with peak* $d[R'_j]$ *on the same side of* $\psi_j(R)$ *as* $d[R_j]$.

Peak Strategyproofness: *For all* $R \in \mathcal{R}^{S\cup D}$, $i \in S, j \in D$ *and* $R'_i, R'_j \in \mathcal{R}$

$$\psi_i(R) R_i \psi_i(R'_i, R_{-i}) \text{ and } \psi_j(R) R_j \psi_j(R'_j, R_{-j})$$

Each one of Peak Monotonicity or Invariance implies *own-peak-only:* my net transfer only depends upon the peak of my preferences, and not on the way I compare allocations across my peak.

Lemma 5 *For any rule that allocates* $\psi(R) \in PO^*$, $\forall\, R \in \mathcal{R}^{S\cup D}$, *strategyproofness and invariance are equivalent. (Note:* $\psi = (x, y)$*)*

Proof: First we show that, under PO^*, strategyproofness implies invariance: As the allocation is in PO^* we have $x_i \le s_i$. Thus, to prove invariance we need to show that when $x_i < s_i$, and $s'_i \ge x_i$ we have $x'_i = x_i$. Suppose not and we have $x'_i < x_i$. Then agent i benefits by misreporting his peak as s_i when his true peak is s'_i, which violates strategyproofness. Similarly, if $x'_i > x_i$, we can construct a profile R^* such that $x'_i P_{i^*} x_i$. As a PO^* + Strategyproof rule is peak-monotonic and as a consequence own peak only (Bochet et al. [12]), $x_i(R_i^*, R_{-i}) = x_i(R)$. Hence, i benefits by reporting s_i' when his true peak is s_i, which violates strategyproofness again.

We now show the converse. Suppose the rule is invariant but not strategyproof. Under a PO^* rule, $x_i = s_i$ for every agent $i \in S_+$, hence those agents never misreport. Every agent in $i \in S_-$ is such that $x_i \leq s_i$. So, any agent who deviates and improves his allocation is such that $s_i' \geq x_i < s_i$ and $x_i' P_i x_i$. But this is not possible under an invariant rule. Hence, the rule is indeed strategyproof. ∎

We prove the following structural lemma before giving a simpler proof of strategy proofness of the Egalitarian Mechanism.

Lemma 6 *For a problem (G, s, d), suppose the decomposition is S_+ and S_- (with D_+, D_- defined as before), and the egalitarian allocation is x. Consider the problem (G, s', d) with $s_j' = s_j$ for all $j \neq i$, with the decomposition being S_+' and S_-'.*

(a) If $i \in S_-$ and $s_i' \geq s_i$, $S_+' = S_+$ and $S_-' = S_-$.

(b) If $i \in S_+$ and $s_i' \leq s_i$, $S_+' = S_+$ and $S_-' = S_-$.

Proof. By definition, S_- is the smallest (both in terms of cardinality and inclusion) min-cut in the graph $G(s)$ (see §2.5.1 for the definition). For $i \in S_-$, the arc (s, i) does not contribute to the cut-capacity. If $s_i' \geq s_i$, the capacity of any cut is weakly greater in (G, s', d) than in (G, s, d), whereas the capacity of the cut S_- stays the same, so part (a) follows by the minimality of S_-. Similarly, for $i \in S_+$, the arc (s, i) contributes to cut-capacity, the capacity of the cut S_- is smaller in (G, s', d) than in (G, s, d) by *exactly* $s_i - s_i'$, whereas the capacity of any cut is weakly smaller in (G, s', d) than in (G, s, d) by *at most* $s_i - s_i'$. Again, part (b) follows by the minimality of S_-. ∎

The egalitarian transfer rule ψ^e of Bochet et al. [12] *is characterized by Pareto optimality, Strategyproofness, Voluntary Trade, and Equal Treatment of Equals.* Bochet et al. also conjecture that the egalitarian transfer rule is group strategyproof, i.e., robust against coordinated misreport of preferences by subgroups of agents. We settle this conjecture below.

Peak Groupstrategyproof: *For all* $R \in \mathcal{R}^{S\cup D}$, $M \subseteq S \cup D$ *and* each agent $i \in M$ misreport to $R_i' \in \mathcal{R}$ [4]

$$\psi_i(R)_i \psi_i(R_M', R_{-M}), \;\; \forall \;\; i \in M$$

i.e. it is dominant strategy for agents to reveal their true peaks even when they can coordinate with other agents and jointly misreport.

Theorem 2 *In the two-sided model, the egalitarian mechanism is peak group strategyproof.*

Proof. Suppose not. Focus on a counterexample G with the *smallest* number of nodes. Suppose the true peaks of the suppliers and demanders are s and d respectively, and suppose their respective misreports are s' and d'. We can assume that $d_j > 0$ for every demander j, as otherwise deleting j would result in a smaller counterexample. Fix a coalition A of suppliers and a coalition B of demanders : note that A contains all the suppliers k with $s_k' \neq s_k$, and B includes all demanders ℓ with $d_\ell' \neq d_\ell$.

Let (x, y) and (x', y') be the respective allocations to the suppliers and demanders when they report (s, d) and (s', d') respectively. Let S_+, S_-, D_+, D_- be the decompsition when the agents report (s, d), and let S_+', S_-', D_+', D_-' be the decomposition when the agents report (s', d'). We shall show that when the agents report (s', d') rather than (s, d), the only allocation in which each agent in $A \cup B$ is (weakly) better off, then $x_k' = x_k$ for all $k \in A$ and $y_\ell' = y_\ell$ for all $\ell \in B$. This establishes the required contradiction.

Let $Y' := D_+ \cap D_-'$. Note that $g(Y') \subseteq S_+'$, for, otherwise, there will be a supplier in S_-' with a link to a demander in D_-'. We now make two simple observations about the suppliers in $S_- \cap g(Y')$:

- For any such supplier k, $s_k' = x_k'$ and $x_k \leq s_k$. Also, $d_\ell = y_\ell$ and $y_\ell' \leq d_\ell'$ for any $\ell \in Y'$.
- When the report is s', every such supplier can send flow only to the demanders in Y': this is because $f(S_-) \subseteq D_+$, and each supplier in $g(Y')$ can send flow only to the agents in D_-'. Therefore $\sum_{k \in S_- \cap g(Y')} x_k' \;\leq\; \sum_{\ell \in Y'} y_\ell'$.

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[4] We allow agents who receive their peak allocation to also misreport, as such a misreport can improve the allocation of other agents without altering the allocation of these agents; On the contrary, Barbera et al. [7] allows only misreports of agents who do not receive their peak allocation

- When the report is s, the demanders in Y' can receive flow only from such suppliers: the demanders in Y' can receive flow only from the suppliers in S_- and they are connected only to the suppliers in $g(Y')$. Therefore $\sum_{k\in S_-\cap g(Y')} x_k \geq \sum_{\ell\in Y'} y_\ell$.

Finally, note that $s'_k = s_k$ for all $k \notin A$, and $d'_\ell = d_\ell$ for all $\ell \notin B$. These observations first lead to

$$\sum_{\substack{k\in S_-\cap g(Y')\\ k\notin A}} s_k + \sum_{\substack{k\in S_-\cap g(Y')\\ k\in A}} x'_k \;=\; \sum_{\substack{k\in S_-\cap g(Y')\\ k\notin A}} s'_k + \sum_{\substack{k\in S_-\cap g(Y')\\ k\in A}} x'_k \;=\; \sum_{k\in S_-\cap g(Y')} x'_k \;\leq\; \sum_{\ell\in Y'} y'_\ell. \tag{2.15}$$

Note that every demander ℓ in $Y'\cap B$ receives *exactly* his peak allocation d_ℓ for a truthful report, so for the coalition B of demanders to do weakly better in the (G, s', d') problem, $y'_\ell = d_\ell$ for each such ℓ. Therefore,

$$\sum_{\ell\in Y'} y'_\ell \;=\; \sum_{\ell\in Y'\setminus B} y'_\ell + \sum_{\ell\in Y'\cap B} y'_\ell \;\leq\; \sum_{\ell\in Y'\setminus B} d'_\ell + \sum_{\ell\in Y'\cap B} d_\ell \;=\; \sum_{\ell\in Y'} d_\ell. \tag{2.16}$$

Finally,

$$\sum_{\ell\in Y'} d_\ell \;=\; \sum_{\ell\in Y'} y_\ell \;\leq\; \sum_{k\in S_-\cap g(Y')} x_k \;\leq\; \sum_{\substack{k\in S_-\cap g(Y')\\ k\notin A}} s_k + \sum_{\substack{k\in S_-\cap g(Y')\\ k\in A}} x_k. \tag{2.17}$$

For every supplier in A to be (weakly) better off when reporting s', we must have $x'_k \geq x_k$ for each $k \in S_-\cap g(Y')$. Combining this with inequalities (A.1) and (A.3), we conclude that all the inequalities in (A.1)-(A.3) hold as equations. In particular, $x'_k = x_k$ for all $k \in S_-\cap g(D')$, and $y'_\ell = y_\ell$ for $\ell \in Y'$. Therefore, whether the report is s or is s', the suppliers in $S_-\cap g(Y')$ send all of their flow only to the demanders in Y'; and that these demanders receive all of their flow only from the suppliers in $S_-\cap g(Y')$. Therefore, removing the suppliers in $S_-\cap g(Y')$ and the demanders in Y' does not affect the egalitarian solution for either problem. As we picked a smallest counterexample, Y' must be empty.

We now turn to the other case. Let $\tilde{X} := S_+\cap S'_-$. Note that $f(\tilde{X})\cap D_- \subseteq D'_+$, for otherwise there will be a supplier in S'_- linked to a demander in D'_-. Consider the demanders in $f(\tilde{X})\cap D_-$:

- For any such demander ℓ, $d'_\ell = y'_\ell$ and $y_\ell \leq d_\ell$. Also, $s_k = x_k$ and $x'_k \leq s'_k$ for any $k \in \tilde{X}$.
- When the report is s', every such demander can receive flow only from the suppliers in $\tilde{X}$: such demanders are linked only to the suppliers in S_+ and can receive flow only from the suppliers in S'_-. Therefore $\sum_{k\in\tilde{X}} x'_k \;\geq\; \sum_{\ell\in f(\tilde{X})\cap D_-} y'_\ell$.

- When the report is s, the suppliers in $\tilde{X}$ send flow only to the demanders in D_-, and they can send flow only to the demanders they are connected to, so the suppliers in $\tilde{X}$ can send flow only to the demanders in $f(\tilde{X}) \cap D_-$. Therefore $\sum_{k\in\tilde{X}} x_k \leq \sum_{\ell\in f(\tilde{X})\cap D_-} y_\ell$.

Finally, note that $s'_k = s_k$ for all $k \notin A$, and $d'_\ell = d_\ell$ for all $\ell \notin B$. Putting all this together, we have:

$$\sum_{\substack{\ell\in f(\tilde{X})\cap D_-) \\ \ell\notin B}} d_\ell + \sum_{\substack{\ell\in f(\tilde{X})\cap D_-) \\ \ell\in B}} d'_\ell = \sum_{\ell\in f(\tilde{X})\cap D_-} d'_\ell = \sum_{\ell\in f(\tilde{X})\cap D_-} y'_\ell, \tag{2.18}$$

and

$$\sum_{\ell\in f(\tilde{X})\cap D_-} y'_\ell \leq \sum_{k\in\tilde{X}} x'_k \leq \sum_{k\in\tilde{X}\setminus A} s'_k + \sum_{k\in\tilde{X}\cap A} x'_k = \sum_{k\in\tilde{X}\setminus A} s_k + \sum_{k\in\tilde{X}\cap A} x'_k. \tag{2.19}$$

Note that every supplier k in $\tilde{X} \cap A$ receives *exactly* his peak allocation s_k for a truthful report, so for the coalition A of suppliers to do weakly better in the (G, s', d') problem, $x'_k = s_k$ for each such k. Thus,

$$\sum_{k\in\tilde{X}\setminus A} s_k + \sum_{k\in\tilde{X}\cap A} x'_k = \sum_{k\in\tilde{X}} s_k = \sum_{k\in\tilde{X}} x_k \leq \sum_{\ell\in f(\tilde{X})\cap D_-} y_\ell \leq \sum_{\substack{\ell\in f(\tilde{X})\cap D_- \\ \ell\notin B}} d_\ell + \sum_{\substack{\ell\in f(\tilde{X})\cap D_- \\ \ell\in B}} y_\ell \tag{2.20}$$

For every demander in B to be (weakly) better off, we must have $y'_\ell \geq y_\ell$ for each $\ell \in f(\tilde{X}) \cap D_-$. Combining this with inequalities (A.4)-(A.6), we conclude that all the inequalities in (A.4)-(A.6) hold as equations. In particular, $x'_k = x_k$ for all $k \in \tilde{X}$, and $y'_\ell = y_\ell$ for $\ell \in f(\tilde{X}) \cap D_-$. Therefore, whether the report is s or is s', the suppliers in $\tilde{X}$ send all of their flow only to the demanders in $f(\tilde{X}) \cap D_-$; and that these demanders receive all of their flow only from the suppliers in $\tilde{X}$. Therefore, removing the suppliers in $\tilde{X}$ and the demanders in $f(\tilde{X}) \cap D_-$ does not affect the egalitarian solution for either problem. As we picked a smallest counterexample, $\tilde{X}$ must be empty.

We now establish that the decomposition does not change in a smallest counterexample. We already know that $Y' = \emptyset$, which implies $D'_- \subseteq D_-$. Suppose this containment is strict so that there is a demander $j \in D_- \setminus D'_-$. Then, $g(j) \subseteq S_+$. As $\tilde{X} = \emptyset$, $g(j) \subseteq S'_+$, which implies demander j cannot receive any flow when the report is s' (i.e. $x'_j = 0$). This is a contradiction since, $d'(j) > 0$, then the egalitarian solution allocates the Pareto value $x'_j = d'_j$ for all $j \in D'_+$. (w.l.o.g we can skip the case $d'_j = 0$ as we can delete such a j to obtain the new decomposition or just place it in D_-). Therefore $D'_- = D_-$, which implies $D'_+ = D_+$, $S'_+ = S_+$, and $S'_- = S_-$.

To complete the argument, let A be as defined earlier. Let $A_+ = A \cap S_+$ and $A_- = A \cap S_-$, $B_+ = A \cap D_+$ and $B_- = A \cap D_-$. Now, for any $j \in B_+$, $d'_j \neq d_j$ implies $y'_j = d'_j \neq d_j$ causing j to do worse by reporting d'_j. Hence, it follows, $\forall j \in B_+$, $d'_j = d_j$. By a similar argument, we could establish $s'_j = s_j \; \forall j \in A_+$.

For any $i \in A_-$, $s'_i < x_i$ implies $x'_i \leq s'_i < x_i$, causing i to do worse by reporting s'_i. Likewise, any $i \in B_-$, $d'_i < y_i$ implies $y'_i \leq d'_i < y_i$, causing i to do worse by reporting d'_i. So any improving coalition A must be such that $s'_i \geq x_i$ for all $i \in A_-$ and $d'_i \geq y_i$ for all $i \in B_-$. But in this case the egalitarian solution does not change for either problem. ∎

An easy implication is the following result, whose proof is an immediate consequence of the results we have already established.

Theorem 3 *In the two sided model, the egalitarian mechanism is group strategyproof w.r.t. to both links and peaks when the coalition is restricted to the set of suppliers only (demanders only).*

A natural question is if every strategyproof rule in our problem is also group strategyproof[5]. As it turns out, the answer is "no" as shown by the following example. Consider the following mechanism, if the report of $d_0 >= 5$, then apply the egalitarian mechanism and if the report of $d_0 < 5$, follow the edge fair mechanism (Increase the flow on all the edges till a point that no edge can carry more flow in any maximum flow, for a detailed description of this mechanism, refer to the next chapter).

This rule is clearly strategyproof. But agent d_0 and s_1 can collude such that agent d_0 misreports his peak as 4 (when his/her true peak is 6). This improves the allocation of agent s_1 by 1 unit, keeping the allocation of d_0 to be the same.

We know from Bochet et al. [12] that any rule that is peak monotonic and invariant is strategyproof. From the above discussion, strategyproofness is characterized by PO^* and invariance. So,

[5] Barbera et al. [7] study environments where this is indeed the case.

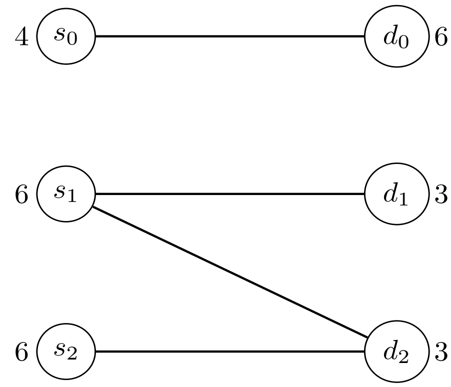

Figure 2.9: Invariance and GSP are not equivalent

the natural question is what other additional property is needed to make a mechanism groupstrategyproof. Next, we show that any groupstrategyproof mechanism can be characterized by PO^* and the following *stronger* invariance property:

Strong Invariance: For all $R \in \mathcal{R}^{S \cup D}$, $i \in S$ and $R'_i \in \mathcal{R}$

$$\{s[R_i] < x_i(R) \text{ and } s[R'_i] \leq x_i(R)\} \text{ or } \{s[R_i] > x_i(R) \text{ and } s[R'_i] \geq x_i(R)\} \tag{2.21}$$

$$\implies x_j(R'_i, R_{-i}) = x_j(R) \;\; \forall j \in S \text{ and} \tag{2.22}$$

$$y_l(R'_i, R_{-i}) = y_l(R) \;\; \forall l \in D \tag{2.23}$$

and a similar strong invariance property can be defined with respect to the demanders.

In other words, while invariance implies that the allocation of a supplier is unchanged whenever his peak misreport is above his allocation, strong invariance implies that the allocation of *every* agent is unchanged when a particular agent misreports his peak over his current allocation.

Theorem 4 *Any mechanism $\psi = (x, y)$, that always selects an allocation from PO^* satisfies strong invariance if and only if it is group strategy-proof.*

Proof: To prove, PO^* + strong invariance $\implies$ Peak group strategyproof, it is enough to follow the ideas in the proof of Theorem 2. Suppose the mechanism always chooses an allocation from PO^* and is Strongly Invariant. Now suppose the mechanism is also *not* group strategyproof, then there is a smallest counterexample in which a set of agents misreport and improve their allocation. Let the smallest such network be (G, s, d) and the agents misreport to form the network (G, s', d'). From Theorem 2, it follows that the Gallai-Edmonds decomoposition doesn't change. Within the unchanged decompositions, from the proof idea in theorem 2, strong invariance implies the bottleneck subsets are the same in both the problems (G, s, d) and (G, s', d') and hence the allocation remain the same in either problem. Thus, we have a contradiction and the mechanism is indeed group strategyproof.

Now, we turn to prove the other direction of the result i.e. any rule that is PO^* and peak GSP is strongly invariant. We discuss the result only for the suppliers, by symmetry a similar reasoning follows for the demanders. Suppose such a rule is not strongly invariant. Since agents in S_+ receive their peak, strong invariance property needs to be discussed only for the agents in S_- where $x_i \leq s_i$. Now, consider an agent $i \in S_-$ such that $x_i < s_i$. Consider a report by agent i such that $s'_i \geq x_i$. From Lemma 5 it follows that PO^* + strategyproof implies invariance. Hence, $x'_i = s_i$. Furthermore, it follows from the earlier discussion that the decomposition and maximum flow does not change in this new problem. Hence, $\sum_{k \in S_-} x_k = \sum_{k \in S_-} x'_k$. Suppose $x'_k = x_k$ $\forall$ $k \in S_-$ then we are done. Suppose, $x'_k \neq x_k$ for some agent $k \in S_-$, then there exists at least one agent j such that $s_j \geq x'_j > x_j$ (agent j improves the allocation). Thus, the pair of agents i and j represent a colluding group who can deviate and (weakly) improve the allocation which contradicts the peak GSP property of the rule. ∎

Szwagrzak [57] introduces the class of separably convex rules. Each such rule is parametrized by a profile of real-valued functions, one for each agent. These rules are closely related to the parametric rules characterized by Young [63] in the context of bankruptcy problems.

We follow the notation and description of Szwagrzak [57] below to describe his main results. Let $\mathcal{H}$ denote the class of strictly convex and differentiable functions $h_0 : R \rightarrow R$

Separably Convex Rule: Let $h \in \mathcal{H}^N$, for each preference profile $R \in \mathcal{R}^N$,

$$\psi(R) = \arg\min \ \{\sum_{i \in N} h_i(x_i) : x \in \mathcal{PO}(\mathcal{R})\}$$

where each h_i is a convex function. Szwagrzak also notes that the constraint set in the definition of separably convex rules is a compact and convex set. Thus, by the strict convexity of the objective, the minimizer is unique.

Minimizing an additively separable strictly convex function over a base polyhedron is an important and well studied problem in combinatorial optimization. A number of algorithms to solve this class of problems can be readily applied to compute the allocations recommended by the separably convex rules: see Nagano [19], Groenvelt [31], Fujishigee [28]. The egalitarian allocation of Bochet et al. [12] minimizes any symmetric additively separable convex function over $\mathcal{PO}(\mathcal{R})$.

Theorem 5 (Szwagrzak [57]) *The separably convex rules are group strategyproof*

Karol uses the same proof technique as in Theorem 2 to establish the result. As separably convex rules pick a solution from the Pareto set, misreports by agents does not change the Gallai-Edmonds decomposition in the smallest counterexample. Within the original decompositions, the allocation produced by these separably convex rules remains the same.

Corollary 1 *Theorem 4 implies that all the separably convex rules are strongly invariant.*

2.4 Related Extensions

2.4.1 Indivisible Goods

In section 2.3 of this chapter, our study was focused on a two sided model with divisible goods. The two sided model in section 2.3 was an extension of the model of Sprumont [56] to bipartite networks. In this section, we study the two sided model with suppliers and demanders on either side of the network but with indivisible goods. Klaus et al. [25] study a probabilistic version of the uniform rule 2.2 when the goods are indivisible in the Sprumont's model. We summarize the results of Klaus et al. and give a brief description of the probabilistic version of the egalitarian rule in section 2.3.3.

Probabilistic uniform rule: The main contribution of Klaus et al. [25] is that there is no "utility gap" when the goods are indivisible. The lorenz dominant uniform allocation of section 2.2 can still be obtained as an expected utility over all possible random allocations of the probabilistic version of the uniform rule.[6]. The mechanism in this model is a lottery which assigns probabilities over the set of feasible allocations $\mathcal{F}$. Formally, a lottery vector μ is such that $|\mu| = |\mathcal{F}|$, $\mu_f \in [0, 1]$, $\forall\ f \in \mathcal{F}$, $\sum_{f\in\mathcal{F}} \mu_f = 1$. Note each f is a feasible allocation vector for the agents in the network.

The expected utility of an agent $i \in N$ is defined as:

$$x_i = \sum_{f\in F} \mu_f x_i^f \tag{2.24}$$

where x_i^f is the allocation of agent i in a particular allocation f.

Let $R_i \in \mathcal{R}$ represent the preference profile for agent i, $p(R_i)$ represent the peak of this profile and x_λ be the allocation of agents at a bottleneck point of this allocation rule, k be the total amount of goods to be rationed. We define the rule for the case of excess demand when $\sum_{i\in N} p(R_i) > k$ below; The case of excess supply is similar.

Definition 2 *(Probabilistic uniform rule, Klaus et al. [25])*
Let $N' = \{i \in N | p(R_i) \geq x_\lambda + 1\} = \{1, 2, 3....\tilde{n}\}$ *and* $\tilde{N} = \{i \in N | p(R_i) \leq x_\lambda\} = \{\tilde{n} + 1, ..., n\}$. *At*

[6] More strongly, every feasible allocation for the agents in the Sprumont's model can be obtained as a feasible expected utility for the agents when the goods are indivisible

the bottleneck, each agent in N' receives his peak amount and each agent in $\tilde{N}$ receives either x_λ or $x_\lambda + 1$. Note that for each $i \in N', (x_\lambda + 1)\ P_i\ x_\lambda$ and that exactly $n'(\lambda - x_\lambda)$ agents in N' can receive $x_\lambda + 1$.

Lottery μ: We obtain the final allocation by placing equal probability on all allocations where all agents in $\tilde{N}$ receive their peak amounts, $n'(\lambda - x_\lambda)$ agents in N' receive $x_\lambda + 1$ and the remaining agents in N' receive x_λ. Hence, final allocation is obtained by placing equal probabilities on exactly $\binom{n'}{n'(\lambda - x_\lambda)}$ allocations.

Intuitively, at a bottleneck point, each agent who has not received his peak allocation prefers an additional unit to be allocated to them. The probabilistic uniform rule assigns each remaining unit with every agent having equal probability of being assigned. Klaus et al. establish that the Uniform probabilistic rule is characterized by Pareto efficiency, envy freeness[7] and strategyproofness. We generalize the ideas of Klaus et al. to obtain a probablisitc egalitarian mechanism for agents rationing indivisible goods on bipartite networks.

Probabilistic egalitarian mechanism: We follow the same model and notation as discussed earlier in section 2.3 for divisible goods. The only difference is that the goods are indivisible in the rest of this section.

Recall the discussion on how the breakpoints were obtained in section 2.3.3 for the suppliers. Each breakpoint is one of the s_i (type 1), and/or is associated with a subset of suppliers X such that

$$\sum_{i \in X} \lambda \wedge s_i = \sum_{j \in f(X)} d_j \tag{2.25}$$

Then we say it is of type 2. In the former case the associated supplier reaches his peak and so cannot send any more flow. In the latter case the group of suppliers in X is a *bottleneck*, in the sense that they are sending enough flow to satisfy the collective demand of the demanders in $f(X)$ and these are the only demanders they are connected to; any further increase in flow from any supplier in X would cause some demander in $f(X)$ to accept more than his peak demand.

Once a bottleneck point is obtained in the egalitarian rule we construct the lottery as discussed earlier. Without loss of generality, let $\bar{X} = \{i \in X^{**} | p(R_i) \geq x_\lambda + 1\} = \{1, 2...., \bar{n}\}$ and $\tilde{N} = \{i \in$

[7]Unlike earlier, envy freeness and equal treatment of equals are not equivalent when the goods are indivsible

$N|p(R_i) \leq x_\lambda\} = \{\bar{n}+1, ..., n\}$. Then in the final allocation each agent in $\tilde{N}$ receives his peak amount and each agent in $\bar{N}$ receives either x_λ or $x_\lambda+1$. Note that for each $i \in \bar{N}, (x_\lambda+1)P_i x_\lambda$ and exactly $\bar{n}(\lambda - x_\lambda)$ agents in $\bar{N}$ can receive $x_\lambda+1$. The randomized "lottery" μ places equal probability on all allocations where all agents in $\tilde{N}$ receive their peak amounts, $\bar{n}(\lambda - x_\lambda)$ agents in $\bar{N}$ receive $x_\lambda + 1$ and the remaining agents in $\bar{N}$ receive x_λ. Hence, the utility profile is obtained by placing equal probabilities on exactly $\binom{n'}{n'(\lambda - x_\lambda)}$ allocations. If $p(R_i \leq x_\lambda)$, then $U_i(R)(p(R_i)) = 1$ and if $p(R_i) \geq x_\lambda+1$, then $U_i(R)(x_\lambda+1) = \lambda - x_\lambda$ and $U_i(R)(x_\lambda) = 1-(\lambda - x_\lambda)$. We perform this lottery at each bottleneck point of the egalitarian mechanism. The solution thus obtained recursively is the probabilistic egalitarian allocation for the suppliers. A similar construction works for demanders. Combining these two egalitarian allocations yields the egalitarian allocation $(x^e, y^e) \in \mathbb{R}_+^{S \cup D}$ for the overall problem.

We conclude this chapter with a simple example of the probabilistic egalitarian rule. In the following network in figure 2.10, demanders d_1, d_2 receive their peak allocations in every outcome of the lottery. The expected outcome for suppliers $(s_1, s_2, s_3, s_4, s_5)$ is $(5/2, 5/2, 8/3, 8/3, 8/3)$. This expected utility is obtained through a lottery that assigns a equal probability of $\frac{1}{6}$ to the following allocations: $(3,2,3,3,2), (3,2,3,2,3), (3,2,2,3,3), (2,3,3,3,2), (2,3,3,2,3), (2,3,2,3,3)$.

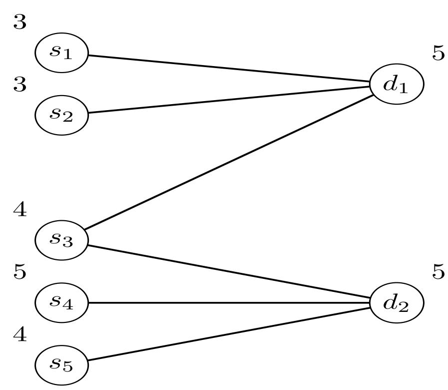

Figure 2.10: An example for the probabilistic egalitarian rule

Note that there is no "utility gap" even when the goods are indivisible if the network is bipartite. Hence it follows that the probabilistic egalitarian rule is a Pareto efficient, envy free and groupstrategyproof mechanism for the agents when the goods are indivisible.

2.4.2 Capacitated Edges

We study the two sided model in section 2.3 but with capacitated edges. We denote the capacity of an edge by $u_{ij} > 0$. If $u_{ij} = \infty$, we have the model of Bochet et al. [12]. Hence, the results in this section generalize the earlier known results on equity and strategic properties of the egalitarian mechanism to a network with capacity constraints.

Recall our notation from earlier section. For any subset $T \subseteq S$, the set of demanders compatible with the suppliers in T is $f(T) = \{j \in D | (i,j) \in E, \quad i \in T\}$. Similarly, the set of suppliers compatible with the demanders in $C \subseteq D$ is $g(C) = \{i \in S | (i,j) \in E, \; j \in C\}$. We abuse notation and say $f(i)$ and $g(j)$ instead of $f(\{i\})$ and $g(\{j\})$ respectively. For any subsets $T \subseteq S$, $C \subseteq D$, $x_T := \sum_{i \in T} x_i$ and $y_C := \sum_{j \in C} y_j$.

A transfer of the commodity from S to D is realized by a flow φ, which specifies the amount of the commodity transferred from supplier i to demander j using the edge $(i,j) \in E$. The flow φ induces an allocation vector for each supplier and each demander as follows:

$$\text{for all } i \in S : x_i(\varphi) = \sum_{j \in f(i)} \varphi_{ij}; \text{ for all } j \in D : y_j(\varphi) = \sum_{i \in g(j)} \varphi_{ij} \tag{2.26}$$

The flow φ is *feasible* if (i) $\varphi_{ij} \leq u_{ij}$ for all $(i,j) \in E$ and $\varphi_{ij} = 0$ for all $(i,j) \notin E$; (ii) $x_i(\varphi) \leq s_i$ for all $i \in S$; and (iii) $y_j(\varphi) \leq d_j$ for all $j \in D$. Let $\mathcal{F}(G,s,d,u)$ be the set of feasible flows for the problem (G,s,d,u). A feasible flow φ^* is a maximum flow if

$$\varphi^* \in \arg \max_{\varphi \in \mathcal{F}(G,s,d,u)} \sum_{i \in S} x_i(\varphi).$$

Let $\mathcal{F}^*(G,s,d,u)$ be the set of maximum flows for the problem (G,s,d,u). For reasons that will be clearer later, we shall focus mostly on finding a maximum flow for any given problem. As a result, it is important to understand the set $\mathcal{F}^*(G,s,d,u)$, which we turn to next.

The Gallai-Edmonds Decomposition. The problem under consideration is the well-known problem of finding a maximum flow in a capacitated bipartite network. The following result characterizes the structure of maximum flows and is essentially a version of the Gallai-Edmonds decomposition. It can proved by a straightforward application of the max-flow min-cut theorem.

Lemma 7 *There exists a partition S_+, S_- of S, and a partition D_+, D_- of D such that the flow φ with net transfers x, y is a maximum flow if and only if*

$$\varphi_{ij} = u_{ij} \;\; \forall ij \in G(S_-, D_-), \;\; x_i = s_i \;\; \forall i \in S_+, \;\; y_j = d_j \;\; \forall j \in D_+ \tag{2.27}$$

Proof: Refer to the appendix

The Egalitarian mechanism

From the structural Lemma 7 edges ij such that $i \in S_-, j \in D_-$ are saturated. Firstly, set flow on such edges to u_{ij} and remove them from the network and adjust the peaks of the suppliers and demanders connected to those edges i.e. for every edge ij such that $i \in S_-, j \in D_-$, do $s_i \leftarrow s_i - \varphi_{ij}, d_j \leftarrow d_j - \varphi_{ij}$. Since in any maximum flow there is no flow between the agents in S_+ to the agents in D_+, the network again is decomposed into two disjoint components. Hence, we define independently our solution for the suppliers and for the demanders.

The definition for suppliers is by induction on the number of agents $|S| + |D|$. Consider the parameterized capacity graph $\Gamma(\lambda), \lambda \geq 0$: the only difference between this graph and $\Gamma(G, s, d)$ is that the capacity of the edge $\sigma i, i \in S_-$ is $\min\{\lambda, s_i\}$, which we denote by $\lambda \wedge s_i$. (In particular, the edge from j to τ still has capacity d_j). We set $\alpha(\lambda)$ to be the maximal flow in $\Gamma(\lambda)$. Clearly α is a piecewise linear, weakly increasing, strictly increasing at 0, and concave function of λ, reaching its maximum when the total σ-τ flow is d_{D_+}.

Let b_i^1, b_i^2 denote the type 1 and type 2 bottleneck points respectively for an agent $i \in S$. At the start of the mechanism, set $b_i^1 = b_i^2 = \infty$, $\forall\, i \in S$. We say the breakpoint λ is of type I, when some agent i is constrained by his/her peak capacity (set $b_i^1 = s_i = \lambda$). A breakpoint λ is of type II, when for some agent i more flow cannot be supported by the edges incident to it in any other maximum flow; then set $b_i^2 = \lambda$. A breakpoint λ is of type III, if it is associated with a subset of suppliers X such that

$$\sum_{i \in X} \lambda \wedge b_i^1 \wedge b_i^2 = \sum_{j \in f(X)} d_j \tag{2.28}$$

In type III the group of suppliers in X is a *bottleneck*, in the sense that they are sending enough flow to satisfy the collective demand of the demanders in $f(X)$ and these are the only demanders they are connected to; any further increase in flow from any supplier in X would cause some demander in $f(X)$ to accept more than his peak demand.

If the given problem does not have any type-2 breakpoint, then the egalitarian solution obtains by setting each supplier's allocation to his peak value. Otherwise, let λ^* be the first type-2 breakpoint of the max-flow function; by the max-flow min-cut theorem, for every subset X satisfying (2.25) at λ^* the cut $C^1 = \{\sigma\} \cup X \cup f(X)$ is a minimal cut in $\Gamma(\lambda^*)$ providing a certificate of optimality for the maximum-flow in $\Gamma(\lambda^*)$. If there are several such cuts, we pick the one with the largest X^* (its existence is guaranteed by the usual supermodularity argument). The egalitarian solution obtains by setting

$$x_i = \min\{\lambda^*, b_i^1, b_i^2\}, \text{ for } i \in X^*, \; y_j = d_j, \text{ for } j \in f(X^*),$$

and assigning to other agents their egalitarian share in the reduced problem $(G(S \diagdown X^*, D \diagdown f(X^*)), s, d)$. That is, we construct $\Gamma^{S \diagdown X^*, D \diagdown f(X^*)}(\lambda)$ for $\lambda \geq 0$ by changing in $\Gamma(G(S \diagdown X^*, D \diagdown f(X^*)), s, d)$ the capacity of the edge σi to $\lambda \wedge s_i$, and look for the first type-2 breakpoint λ^{**} of the corresponding max-flow function. An important fact is that $\lambda^{**} > \lambda^*$.

The solution thus obtained recursively is the egalitarian allocation for the suppliers. A similar construction works for demanders: We consider the parameterized capacity graph $\Delta(\mu), \mu \geq 0$, with the capacity of the edge $\tau j, j \in D$ set to $\mu \wedge d_j$. We look for the first type-3 breakpoint μ^* of the maximal flow $\beta(\mu)$ of $\Delta(\mu)$, and for the largest subset of demanders Y such that

$$\sum_{j \in Y} \mu \wedge b_1^j \wedge b_2^j = \sum_{i \in g(Y)} s_i$$

etc.. Combining these two egalitarian allocations yields the egalitarian allocation $(x^e, y^e) \in \mathbb{R}_+^{S \cup D}$ for the overall problem.

We illustrate below the egalitarian mechanism in a simple capacitated network. In figure 2.11, the capacity of the edges $s_3d_3, s_2d_3, s_2d_2, s_1d_2, s_1d_1$ are 3,3,2,5 and 5 respectively. The first bottleneck is at $\lambda = 3$ when supplier s_3 reaches the peak value; The next bottleneck occurs when $\lambda = 5$ when we have a type II bottleneck at supplier node 2. Finally, at $\lambda = 8$, we have a type III bottleneck involving all the suppliers. The egalitarian allocation for the network is then: $(x_1, x_2, x_3) = (8, 4, 3)$ and $(d_1, d_2, d_3) = (7, 5, 3)$.

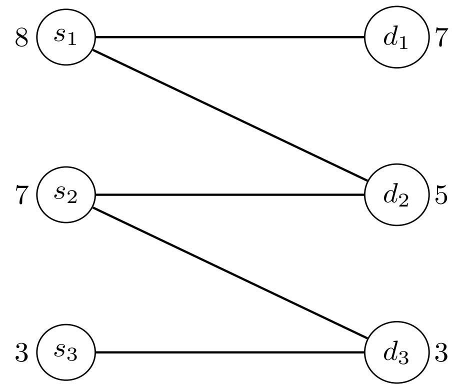

Figure 2.11: An example for the egalitarian rule in a network with capacities

Pareto optimality and the Core

Fix the economy (G, R). Let S_+, S_- and D_+, D_- be the Gallai-Edmonds decomposition applied to the network G with edge capacities given by u, supplies given by the peaks of the suppliers and the demands given by the peaks of the demanders. Then:

(a) If the flow φ implements Pareto optimal net transfers (x, y), then:

$$ij \in G(S_-, D_-) \implies \varphi_{ij} = u_{ij};\; ij \in G(S_+, D_+ \cup (f(S_-) \cap D_-)) \implies \varphi_{ij} = 0 \tag{2.29}$$

(b) The transfers (x, y) induced by a feasible flow φ are Pareto optimal if and only if

$$x \geq s \text{ on } S_+,\;\; y \leq d \text{ on } D_- \text{ and } x_{S_+} = y_{D_-} - \varphi(S_-, D_-) \tag{2.30}$$

$$x \leq s \text{ on } S_-,\;\; y \geq d \text{ on } D_+ \text{ and } x_{S_-} = y_{D_+} + \varphi(S_-, D_-) \tag{2.31}$$

where $\varphi(S_-, D_-)$ is the net flow from component S_- to D_-. From earlier discussions, $\varphi(S_-, D_-) = \sum_{i \in S_-, j \in D_-} u_{ij}$

The following subset of $\mathcal{PO}(G, R)$ will play an important role:

$$\mathcal{PO}^*(G, s, d) = \mathcal{PO}(G, R) \cap \{(x, y) | x \leq s; y \leq d\}$$

From the discussion above, this is the set of efficient allocations where the short side gets its optimal transfer:

$$x = s \text{ on } S_+,\ y \leq d \text{ on } D_-, \text{ and } y_{D_-} = s_{S_+}$$

$$x \leq s \text{ on } S_-,\ y = d \text{ on } D_+ \text{ and } x_{S_-} = y_{D_+}$$

Moreover by Lemma 2, the net transfers in $\mathcal{PO}^*(G, s, d)$ are precisely those implemented by all the maximal flows of the capacity graph $\Gamma(G, s, d)$.

We focus on allocations in $\mathcal{PO}^*(G, s, d)$, because under the Voluntary Trade (requiring $x_i R_i 0, y_j R_j 0$ for all i, j; see Section 8) property, they are the only allocations Pareto optimal for any choice of preferences in $\mathcal{R}$ with peaks (s, d).

We first give an alternative characterization of the Pareto* set, critical to the analysis of the egalitarian solution. Define two cooperative games, (S, v) and (D, w), of which the players are respectively the suppliers and the demanders:

$$v(T) = \min_{T' \subseteq T} \{s_{T'} + d_{f(T \setminus T') \cap D_+} + \sum_{i \notin T', j \in f(T') \cap D_-} u_{ij}\} \text{ for all } T \subseteq S \tag{2.32}$$

$$w(E) = \min_{E' \subseteq E} \{d_{E'} + s_{g(E \setminus E') \cap S_+ +} \sum_{j \notin E', i \in f(E') \cup S_-} u_{ji}\} \text{ for all } E \subseteq D \tag{2.33}$$

The games (S, v) and (D, w) are submodular. Moreover

$$v(S) = w(D) = s_{S_+} + d_{D_+} + \sum_{i \in S_-, j_{in D_+}} u_{ij}; \tag{2.34}$$

The core of the game (S, v), denoted $Core(S, v)$, is the set of allocations $x \in \mathbb{R}_+^S$ such that $x_T \leq v(T)$ for all $T \subset S$, and $x_S = v(S)$; similarly the core of the game (D, w) is the set of allocations $y \in \mathbb{R}_+^D$ such that $y_E \leq w(E)$ for all $E \subset D$, and $y_D = w(D)$. Notice that $v(T) \leq s_T$ for all $T \subset S$, therefore $x \in Core(S, v)$ implies $x \leq s$; similarly $y \in Core(D, w) \Rightarrow y \leq d$.

Fix the problem (G, s, d), and two partitions S_+, S_- and D_+, D_- as in Lemma 2. Then the allocation (x, y) is in $\mathcal{PO}^*(G, s, d)$ if and only if it satisfies one of the two equivalent properties

i) $x \in Core(S, v)$ and $y \in Core(D, w)$

ii) $\{x = s$ on S_+, and on S_-, $x \in Core(S_-, v)\}$ and $\{y = d$ on D_+, and on D_-, $y \in Core(D_-, w)\}$

For any problem (G, s, d), the allocation x^e (resp. y^e) is the egalitarian selection in $Core(S, v)$ (resp. $Core(D, w)$).

Properties of the egalitarian mechanism

Theorem 6 *The allocation (x^e, y^e) is the Lorenz dominant element in $PO^*(G, s, d)$* [8]*.*

Proof: For $z, w \in \mathbb{R}^N$, we say that z *lexicographically dominates* w if the first coordinate a in which z^* and w^* are not equal is such that $z^{*a} > w^{*a}$. We show that the egalitarian solution lexicographically dominates any other solution. Recall that in an arbitrary submodular cooperative game, the egalitarian core selection introduced in [**?**] *Lorenz dominates* every other core allocation. As the set $\mathcal{PO}^*(G, s, d)$ is the intersection of the cores of two submodular games (Lemma 5), it has a unique Lorenz dominant element, which must also be lexicographically optimal . As the lexicographically optimal element is always unique, it must also be Lorenz dominant.

We prove the result for the suppliers by induction on the number of agents. An analogous argument for the demanders, omitted as usual, completes the proof. The result is clearly true when there is a single supplier, and when the max-flow function (defined earlier) $\alpha(\lambda)$ does not have any type-3 breakpoints. In the latter case, every supplier will be allocated his peak, which clearly Lorenz dominates every other allocation. Let λ^* be the first type-3 breakpoint of the max-flow function $\alpha(\lambda)$, and let X^* be the corresponding largest bottleneck set of suppliers (2.25). The following facts about the egalitarian allocation are clear:

- Each supplier $i \in X^*$ will send b_i^1 or b_i^2 or λ^*, whichever is smaller.
- Each supplier $i \notin X^*$ with $b_i^1, b_i^2 \leq \lambda^*$ will send $\min(b_i^1, b_i^2)$.
- Each supplier $i \notin X^*$ with $\min(b_i^1, b_i^2) > \lambda^*$ will send a flow that is *strictly above* λ^*.

(the last statement because the next breakpoint $\lambda^{**} > \lambda^*$).

Let W be the set of suppliers (both in X^* and outside) with allocation at or below λ^*. The allocations of the suppliers in W with allocation such that $x_i = b_i^1$ cannot be improved because they are already receiving their peak allocation. From construction, the allocation of an agent with $x_i = b_i^2$ can only be improved if agents with allocation smaller than x_i reroute some of the flow

[8]Note that this solution is not Lorenz dominant in the entire Pareto set.

through agent i; In which case, the egalitarian mechanism lexicographically dominates the new allocation. Hence, the allocations of the suppliers in W cannot be improved.

It is also clear that in any other allocation at least one of the suppliers in $X^* \setminus W$ who is not constrained by capacities must send at most λ^*. This is because, in the egalitarian allocation, they split equally the $d_{f(X^*)} - s_{X^* \cap W}$ units of flow they collectively send. In any other allocation, they send *at most* these many units of flow, so the smallest allocation of a supplier in $X^* \setminus W$ is at most λ^*. And if this smallest allocation is exactly λ^*, the allocation coincides with the egalitarian allocation on $X^* \cup W$. Thus the egalitarian allocation lex-dominates any allocation that does not agree with it on the allocations of the suppliers in $W \cup X^*$. We can therefore fix the allocations of the suppliers in $W \cup X^*$ to their egalitarian allocation for the purposes of proving lex-dominance. Let $\mathcal{W}$ be the subset of Pareto optimal allocations that gives each supplier in $W \cup X^*$ their egalitarian allocation. Note that in every allocation in $\mathcal{W}$, each demander $j \in f(X^*)$ receives his peak demand, all of which flows from the suppliers in X^*. Thus, none of these demanders receives additional flow from the suppliers in $S \setminus X^*$ in any allocation in $\mathcal{W}$. By construction, no supplier in X^* has links to a demander in $D \setminus f(X^*)$. Thus proving lex-dominance of the egalitarian allocation for the original problem is equivalent to proving the following statement: when restricted to the suppliers in $S \setminus X^*$, the egalitarian allocation lexicographically dominates all the allocations in $\mathcal{W}$. The restriction of the egalitarian allocation to the suppliers in $S \setminus X^*$ is identical to the egalitarian allocation of the subproblem $(S \setminus X^*, D \setminus f(X^*))$. This, however, is a smaller problem, so, by the induction hypothesis, the egalitarian allocation of this subproblem lexicographically dominates any other Pareto optimal allocation, and, in particular, those in $\mathcal{W}$.$\blacksquare$. This proof also implies that the egalitarian allocation is in the core of the game described earlier.

Theorem 7 *The egalitarian transfer rule satisfies No Envy*

Proof: Let R be a profile at which supplier 1 envies supplier 2 via (x, y). We have $\psi_1^e(R) < s_1$, because 1 is not envious if $\psi_1^e(R) = s_1$. Single-peakedness of R_1, and the fact that 1 prefers both x_1 and $\psi_2^e(R)$ to $\psi_1^e(R)$, implies $\psi_1^e(R) < x_1, \psi_2^e(R)$. As above, conservation of flows implies $x_1 + x_2 = \psi_1^e(R) + \psi_2^e(R)$. Therefore $x_2 < \psi_2^e(R)$. We see that for ε small enough, the allocation $\varepsilon x + (1-\varepsilon)\psi^e(R)$ is in $\mathcal{PO}^*(G, s, d)$. It is a Pigou Dalton transfer from 2 to 1 in this set, contradicting the Lorenz dominance of $\psi^e(R)$.$\blacksquare$

Theorem 8 *A mechanism is peak group strategyproof if and only if it is strongly invariant and picks an allocation in* PO^**; The egalitarian mechanism is peak groupstrategyproof*

Proof: Refer to the appendix

2.5 The One-sided Model (Divisible Goods)

Related extension of Sprumont's [56] model to bipartite networks is one where agents on one side of the network have preferences whereas the other side of network consists of goods that must by fully allocated.

2.5.1 Model

Recall that in the one-sided model, we are given a bipartite graph G with suppliers S indexed by i and demanders D indexed by j. Demander j has a demand of d_j that must be satisfied exactly, whereas supplier i has single-peaked preferences with peak s_i; therefore, a supplier may be required to send more or less than his peak. In addition supplier i is required to send at least ℓ_i and at most u_i units of flow; we may assume without loss of generality that $\ell_i \leq s_i \leq u_i$. The peaks of the demanders, their preferences, and the ℓ_i and u_i are common knowledge; in contrast, for any supplier i, his peak s_i and the set $f(i)$ of demanders he is linked to may be private information held only by that supplier i and hence must be elicited by the mechanism.

Let $\lambda := (\lambda_i)_{i \in S}$ be non-negative. Construct the following network $G(\lambda)$: introduce a source s and a sink t; arcs of the form (s, i) for each supplier i with capacity λ_i, arcs of the form (j, t) for each demander j with capacity d_j; an infinite-capacity arc from supplier i to demander j if supplier i and demander j share a link. Let $\ell = (\ell_i)_{i \in S}$, $u = (u_i)_{i \in S}$, and $s = (s_i)_{i \in S}$. It is straightforward to verify that the given problem admits a feasible solution if and only if the maximum s-t flow in $G(\ell)$ and $G(u)$ are, respectively, $\sum_{i \in S} \ell_i$ and $\sum_{j \in D} d_j$. Consider now a maximum s-t flow in the network $G(s)$. By the max-flow min-cut theorem, there is a cut C (a cut is a subset of nodes that contains the source s but not the sink t) whose capacity is equal to that of the max-flow. If the set of suppliers in C is X and the set of demanders in C is Y, it is clear that $Y = f(X)$: if $Y \not\subseteq f(X)$, then C has infinite capacity, and if $Y \supset f(X)$ then C's capacity can be improved by deleting the demanders in $Y \setminus f(X)$. Bochet et al. [11] show that in any Pareto-optimal allocation x for the suppliers, $x_i \leq s_i$ for each $i \in X$ and $x_i \geq s_i$ for each $i \in S \setminus X$.

If the min-cut is not unique, it is again well-known (see [40]) that there is a min-cut with the largest X (largest in the sense of inclusion), and a min-cut with the smallest X (again in the sense of inclusion). Call these sets $\overline{X}$ and $\underline{X}$. It is easy to check that every supplier in $\overline{X} \setminus \underline{X}$ will be at his peak value in *all* Pareto optimal solutions. In the notation of Bochet et al. [11], $M_0 := \overline{X} \setminus \underline{X}$, $M_- := \underline{X}$, and $M_+ := S \setminus \overline{X}$. To keep things simple, however, we shall dispense with M_0 and use the partition $M_- = \underline{X}$, $M_+ = S \setminus \underline{X}$. In this case the partition of the demanders becomes $Q_+ = f(M_-)$ and $Q_- = D \setminus f(M_-)$. We note that our M_- is still uniquely determined for each problem. In what follows, often it will be important to talk about the set of suppliers involved in the cut, rather than the cut itself: we abuse notation and talk about the cut X when in fact the set of nodes in the cut is really $s \cup X \cup f(X)$.

2.5.2 Egalitarian Mechanism

Suppose $(x_i)_{i \in S}$ is a Pareto optimal allocation. From the earlier discussion it is clear that $x_i \in [s_i, u_i]$ for every supplier $i \in M_+$, and $x_i \in [\ell_i, s_i]$ for every supplier $i \in M_-$. Bochet et al. [11] prove that the egalitarian allocation, which is defined independently for the suppliers in M_- and M_+, Lorenz dominates all other Pareto optimal allocations.

For the suppliers in M_-, the egalitarian allocation is found by the following algorithm. Let λ be a parameter whose value is increased continuously from zero, and let $m_i(\lambda) = \text{median}(\ell_i, \lambda, s_i)$. Consider the graph $G(m(\lambda))$, where the capacity of the arc (s, i) is $m_i(\lambda)$. By the earlier discussion, we know that each supplier in M_- will send at least ℓ_i and at most s_i units of flow in a Pareto optimal solution, and that every demander j in $f(M_-)$ will receive exactly d_j units of flow. We now study the sequence of networks $G(m(\lambda))$—specifically the maximum s-t flow in such networks—as λ is increased from zero. It is not hard to see that the maximum s-t flow in $G(m(\lambda))$ is a weakly-increasing, piecewise linear function of λ with at most $2n$ breakpoints. Moreover, each breakpoint is one of the ℓ_i, or one of the s_i (type 1), or is associated with a subset of suppliers X such that

$$\sum_{i \in X} m_i(\lambda) = \sum_{j \in f(X)} d_j \tag{2.35}$$

This we call a type-2 breakpoint. At a type-1 breakpoint, the associated supplier is at his peak and so will not send any more flow (recall that every supplier $i \in M_-$ will send flow at most his peak s_i); at a type-2 breakpoint, however, the group of suppliers in X are sending enough flow to satisfy the collective demand of the demanders in $f(X)$, so any further increase in flow from *any* supplier in X would cause *some* demander in $f(X)$ to accept more than his peak demand.

If the given problem does not have any type-2 breakpoint, then the egalitarian solution obtains by setting each supplier's allocation to his peak value. Otherwise, let λ^* be the first type-2 breakpoint of the max-flow function; by the max-flow min-cut theorem, for every subset X satisfying (2.35) at λ^* the cut $C^1 = \{s\} \cup X \cup f(X)$ is a minimal cut in $G(m(\lambda^*))$ providing a certificate of optimality for the maximum-flow in $G(m(\lambda^*))$. If there are several such cuts, we pick the one with the largest X^* (its existence is guaranteed by the usual supermodularity argument). The egalitarian solution is obtained by setting

$$x_i = \text{median}(\ell_i, \lambda^*, s_i), \text{ for } i \in X^*, \quad y_j = d_j, \text{ for } j \in f(X^*),$$

and assigning to other agents their egalitarian share in the reduced problem involving the suppliers in $M_- \setminus X^*$ and the demanders in $Q_+ \setminus f(X^*)$. It is straightforward to verify that the first type-2 breakpoint λ^{**} of this reduced problem will satisfy $\lambda^{**} > \lambda^*$.

For the suppliers in M_+, a similar algorithm is used to determine the egalitarian allocation: here, each demander $j \in Q_-$ receives exactly d_i units of flow, whereas every supplier $i \in M_+ = g(Q_-)$ sends at least s_i and at most u_i units of flow in a Pareto optimal solution. As before, we consider the graph $G(m(\lambda))$, where the capacity of the arc (s, i) is $m_i(\lambda) := \text{median}(s_i, \lambda, u_i)$. We increase λ gradually and observe that the maximum s-t flow in $G(m(\lambda))$ is a weakly-increasing, piecewise linear function of λ with at most $2n$ breakpoints. Moreover, each breakpoint is one of the s_i, or one of the u_i (type 1), or is associated with a subset of suppliers X such that

$$\sum_{i \in X} m_i(\lambda) \;=\; \sum_{j \in f(X)} d_j$$

This we call a type-2 breakpoint. At a type-1 breakpoint, the associated supplier is at his upper bound and so cannot send any more flow; at a type-2 breakpoint, however, the group of suppliers in X are sending enough just enough flow to satisfy the collective demand of the demanders in $f(X)$, so any decrease in flow from *any* supplier in X would cause *some* demander in $f(X)$ to receive an amount strictly below his peak demand. As before, if the given problem does not have any type-2 breakpoint, then the egalitarian solution obtains by setting each supplier's allocation to his upper bound. Otherwise, let λ^* be the first type-2 breakpoint of the max-flow function, and let X^* be the (largest) associated bottleneck set of suppliers (as before). The egalitarian solution is obtained by setting

$$x_i = \text{median}(s_i, \lambda^*, u_i), \text{ for } i \in X^*, \quad y_j = d_j, \text{ for } j \in f(X^*),$$

and assigning to other agents their egalitarian share in the reduced problem involving the suppliers in $M_+ \setminus X^*$ and the demanders in $Q_- \setminus f(X^*)$. This completely defines the egalitarian solution.

2.5.3 Strategic Issues

We turn now to strategic aspects of the rationing problem with constraints. In the one-sided model, only the suppliers are modeled as "agents," who possess potentially two pieces of information that could be modeled as private: the set of demanders they are compatible with, and their own preference over allocations [9]. As the egalitarian mechanism is "peak-only" [11], it is sufficient for the suppliers to report only their peaks, rather than their entire preference ordering.

It is a simple matter to verify that the egalitarian mechanism is *not* link strategyproof. Consider a supplier with a peak of 1, connected to two demanders, each with a demand of 1, see Figure 2.12. If the supplier reveals both links, his egalitarian allocation is 2, whereas by suppressing one of the links, his egalitarian allocation improves to 1. Therefore in the rest of this section we focus only on peak strategyproofness.

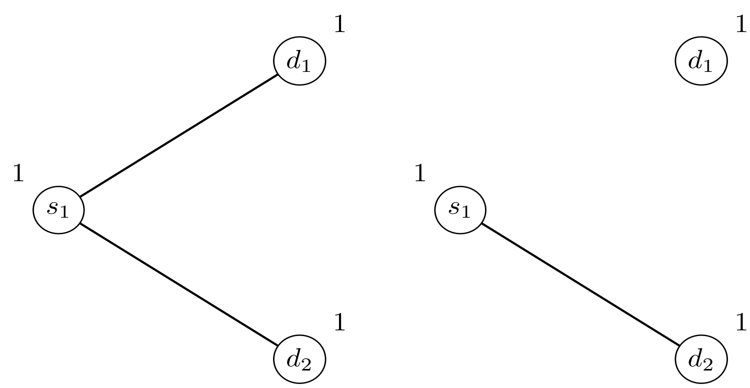

Figure 2.12: Counterexample for Link SP

Bochet et al. [11] show that the egalitarian mechanism, is peak strategyproof. Our main result in this section is that, in fact, the egalitarian mechanism is peak groupstrategyproof. To set the stage for this, we use Lemma 6 from the previous section to give an alternative proof that the

[9]The set of demanders, their individual demands, as well as the the lower and upper bounds on arc-flows are assumed to be common knowledge.

egalitarian mechanism is strategyproof. We note that this is a much simpler proof in comparison to the original proof that appears in Bochet et al. [11]

Theorem 9 *The egalitarian mechanism is peak strategyproof.*

Proof. For the problem (G, s, d) let x be the egalitarian allocation, and let M_+ and M_- be defined as before. Consider the problem (G, s', d) with $s_k = s'_k$ for all $k \neq i$. Suppose $i \in M_-$. If $s'_i \geq s_i$, Lemma 6 proves that the decomposition does not change; it is easy to see that the egalitarian allocation is unaffected as well, because the algorithm to compute operates identically in the problems (G, s, d) and (G, s', d). Similarly, if $i \in S_+$ and $s'_i \leq s_i$, the decomposition does not change (by Lemma 6), and the egalitarian allocation is unaffected as well. Suppose agent i reports s'_i as his peak and the allocation changes to x'_i. To prove strategyproofness, it suffices to show that any $i \in M_-$ (weakly) prefers x_i to x'_i for all $s'_i < s_i$, and that any $i \in M_+$ (weakly) prefers x_i to x'_i for all $s'_i > s_i$.

Fix an $i \in M_-$, and suppose that i reports a peak of $s'_i < s_i$. In this case the decomposition *may* change; let M'_- and M'_+ be the new decomposition. If $i \in M'_-$, an application of Lemma 6 to the problem (G, s', d) shows that the decomposition does not change, and that $x'_i = x_i$. Suppose $i \in M'_+$. Let $D' := Q_+ \cap Q'_-$, and $X' := M_- \cap M'_+$, and note that by our supposition $X' \ni i$. Note also that $g(D') \cap M_- \subseteq X'$, as no agent in M'_- has a link to any demander in Q'_-. Furthermore, if $i \notin g(D')$, $x'_i = 0$, and again the result follows: recall that $f(i) \subseteq Q_+$; and if $i \notin g(D')$, $f(i) \subseteq Q'_+$, and the links from M'_+ to Q'_+ do not carry any flow. So we may assume that $i \in g(D')$. We now make two simple observations about the agents in $X' \cap g(D')$ in the problem (G, s', d): first every such agent sends flow only to the demanders in D', and therefore $\sum_{k \in X' \cap g(D')} x'_k \leq \sum_{j \in D'} d_j$. Also, as every agent in $X' \cap g(D')$ is (weakly) above his reported peak, $x'_k \geq s_k$ for every $k \in X' \cap g(D')$, $k \neq i$, and $x'_i \geq s'_i$. This implies

$$\sum_{k \in X' \cap g(D'), k \neq i} s_k + x'_i \leq \sum_{j \in D'} d_j. \tag{2.36}$$

We next claim that in the problem (G, s, d), $\sum_{k \in X' \cap g(D')} x_k = \sum_{j \in D'} d_j$. To see why, observe that the demands of D' are covered in the problem (G, s, d) by the suppliers in $M_- \cap g(D')$; but every demander in D' moves from Q_+ to Q'_-, so every supplier in $M_- \cap g(D')$ must move to M'_+ (as there cannot be an edge between a supplier in M'_- and a demander in Q'_-). This implies that any supplier supplying a positive amount to a demander in D' in the problem (G, s, d) must be in $X' \cap g(D')$. Note also that for each $k \in X' \cap g(D')$, $x_k \leq s_k$. These, along with $X' \cap g(D') \subseteq M_-$, imply

$$\sum_{k \in X' \cap g(D'), k \neq i} s_k + x_i \geq \sum_{j \in D'} d_j. \tag{2.37}$$

Inequalities (2.36) and (2.37) imply $x'_i \leq x_i$, as required.

Now fix an $i \in M_+$, and suppose that i reports a peak of $s'_i < s_i$. In this case the decomposition *may* change; let M'_- and M'_+ be the new decomposition. If $i \in M'_+$, as before, an application of Lemma 6 to the problem (G, s', d) shows that the decomposition does not change, and that $x'_i = x_i$. Suppose $i \in M'_-$. Let $D' := Q_- \cap Q'_+$, and $X' := M_+ \cap M'_-$, and note that by our supposition $X' \ni i$. Note also that $f(X') \cap Q_- \subseteq D'$, as no agent in M'_- can have a link to any demander in Q'_+. We now make two simple observations about the demanders in $f(X') \cap D'$ in the problem (G, s', d): first every such demander can receive flow only from the agents in X', and therefore $\sum_{k \in X'} x'_k \geq \sum_{j \in f(X') \cap D'} d_j$. Also, as every agent in X' is (weakly) below his reported peak (in the new problem), $x'_k \leq s_k$ for every $k \in X'$, $k \neq i$, and $x'_i \leq s'_i$. This implies

$$\sum_{k \in X', k \neq i} s_k + x'_i \geq \sum_{j \in f(X') \cap D'} d_j. \tag{2.38}$$

We next claim that in the problem (G, s, d), $\sum_{k \in X'} x_k = \sum_{j \in f(X') \cap D'} d_j$: in (G, s, d) the suppliers in X' send flow only to the demanders in $f(X') \cap D'$, who receive flow only from these suppliers. Furthermore, $x_k \geq s_k$ for each $k \in X'$. In particular,

$$\sum_{k \in X', k \neq i} s_k + x_i \leq \sum_{j \in f(X') \cap D'} d_j. \tag{2.39}$$

Inequalities (2.38) and (2.39) imply $x'_i \geq x_i$, as required. ∎

In fact, the ideas in the proof of Theorem 9 can be used to prove the following result, which weakens the conditions under which the decomposition is guaranteed not to change.

Lemma 8 *For a problem (G, s, d), suppose the decomposition is M_+ and M_- (with Q_+, Q_- defined as before), and the egalitarian allocation is x. Consider the problem (G, s', d) with $s'_j = s_j$ for all $j \neq i$, with the decomposition being M'_+ and M'_-.*

(a) If $i \in M_-$ and $s'_i > x_i$, $M'_+ = M_+$ and $M'_- = M_-$.

(b) If $i \in M_+$ and $s'_i < x_i$, $M'_+ = M_+$ and $M'_- = M_-$.

Proof. By definition, M_- is the smallest (both in terms of cardinality and inclusion) min-cut in the graph $G(s)$ (see §2.5.1 for the definition). For $i \in M_-$, the arc (s, i) does not contribute to the cut-capacity. If $s'_i \geq s_i$, the capacity of any cut is weakly greater in (G, s', d) than in (G, s, d), whereas the capacity of the cut M_- stays the same, so the result follows. Suppose now that $x_i < s'_i < s_i$, the max s-t flow in $G(s')$ is weakly below that of $G(s)$, but the egalitarian allocation x is still feasible, so x continues to be a max-flow, so M_- continues to be a min-cut in $G(s')$. We need to show that it remains the minimal min-cut. First observe that $M'_- \subseteq M_-$, as M'_- is the minimal min-cut in $G(s')$ whereas M_- is *a* min-cut for $G(s')$. If $i \in M'_-$, then the capacity of the cut M'_- is the same in $G(s)$ and $G(s')$, so the minimality of M_- in the problem (G, s, d) implies $M'_- = M_-$. Suppose $i \notin M'_-$. Let $X = M_- \setminus M'_-$, and note that $i \in X$. Note also that $Q_+ = f(M_-)$ and $Q'_+ = f(M'_-)$, so that the net change in the cut capacity when the suppliers in X move from M_- to M'_+ is precisely $\sum_{k \in X} s'_k - \sum_{j \in Q_+ \setminus Q'_+} d_j$. In the problem (G, s, d), however, the demanders in $Q_+ \setminus Q'_+$ receive flow only from the suppliers in X, each of whom sends no more than his peak: thus, $\sum_{k \in X} x_k \geq \sum_{j \in Q_+ \setminus Q'_+} d_j$, and $s_k \geq x_k$ for each k. An easy implication is that $s'_k \geq x_k$ for each $k \in X$, $k \neq i$, and $s'_i > x_i$. Thus the net change in cut capacity in moving from M_- to M'_- is strictly positive, which implies M'_- cannot be a min-cut. A similar argument establishes part (b).

∎

We conclude this section with a proof that the egalitarian mechanism is, in fact, group strategyproof.

Theorem 10 *The egalitarian mechanism is peak groupstrategyproof.*

Proof. Suppose not. Focus on a counterexample G with the *smallest* number of nodes. Suppose the true peaks of the suppliers are s and suppose they misreport their peaks to be s'. Fix a coalition A of agents: note that this coalition includes all the agents k with $s'_k \neq s_k$. Let x and x' be the respective allocations to the agents when they report s and s' respectively. As with the earlier proof, let M_+, M_- be the decompsition when the agents report s, and let M'_+, M'_- be the decomposition when the agents report s'. We shall show that when the agents report s' rather than s the only allocation in which each agent in A is (weakly) better off is one in which $x'_k = x_k$ for all $k \in A$, establishing the required contradiction.

Let $D' := Q_+ \cap Q'_-$. Note that $g(D') \subseteq M'_+$, for otherwise there will be a supplier in M'_- with a link to a demander in Q'_-. We now make two simple observations about the agents in $M_- \cap g(D')$:

- When the report is s', every such agent can send flow only to the demanders in D': this is because $f(M_-) \subseteq Q_+$, and each agent in $g(D')$ can send flow only to the agents in Q'_-. Therefore $\sum_{k \in M_- \cap g(D')} x'_k \le \sum_{j \in D'} d_j$.
- When the report is s, the demanders in D' can receive flow only from such agents: the demanders in D' can receive flow only from the suppliers in M_- and they are connected only to the suppliers in $g(D')$. Therefore $\sum_{k \in M_- \cap g(D')} x_k \ge \sum_{j \in D'} d_j$.

Note also that $s'_k \le x'_k$ and $x_k \le s_k$ for any $k \in M_- \cap g(D')$, and that $s'_k = s_k$ for all $k \notin A$. These observations lead to

$$\sum_{\substack{k \in M_- \cap g(D') \\ k \notin A}} s_k + \sum_{\substack{k \in M_- \cap g(D') \\ k \in A}} x'_k = \sum_{\substack{k \in M_- \cap g(D') \\ k \notin A}} s'_k + \sum_{\substack{k \in M_- \cap g(D') \\ k \in A}} x'_k \le \sum_{k \in M_- \cap g(D')} x'_k \le \sum_{j \in D'} d_j, \tag{2.40}$$

and

$$\sum_{j \in D'} d_j \le \sum_{k \in M_- \cap g(D')} x_k \le \sum_{\substack{k \in M_- \cap g(D') \\ k \notin A}} s_k + \sum_{\substack{k \in M_- \cap g(D') \\ k \in A}} x_k. \tag{2.41}$$

For every agent in A to be (weakly) better off when reporting s', we must have $x'_k \ge x_k$ for each $k \in A$. Combining this with inequalities (2.40) and (2.41), we conclude that $x'_k = x_k$ for each $k \in M_- \cap g(D') \cap A$. Moreover, these inequalities also imply that $x'_k = x_k = s_k$ for each $k \in M_- \cap g(D')$, $k \notin A$. Thus, $x'_k = x_k$ for all $k \in M_- \cap g(D')$. Also, whether the report is s or is s', the suppliers in $M_- \cap g(D')$ send all of their flow only to the demanders in D'; and that these demanders receive all of their flow only from the suppliers in $M_- \cap g(D')$. Therefore, removing the suppliers in $M_- \cap g(D')$ and the demanders in D' does not affect the egalitarian solution for either problem. As we picked a smallest counterexample, $D' = \emptyset$.

We now turn to the other case. Let $\tilde{X} := M_+ \cap M'_-$. Note that $f(\tilde{X}) \cap Q_- \subseteq Q'_+$, for otherwise there will be a supplier in M'_- linked to a demander in Q'_-. Consider the demanders in $f(\tilde{X}) \cap Q_-$:

- When the report is s', every such demander can receive flow only from the suppliers in $\tilde{X}$: such demanders are linked only to the suppliers in M_+ and can receive flow only from the suppliers in M'_-. Therefore $\sum_{k \in \tilde{X}} x'_k \;\geq\; \sum_{j \in f(\tilde{X}) \cap Q_-} d_j$.
- When the report is s, the suppliers in $\tilde{X}$ send flow only to the demanders in Q_-, and they can send flow only to the demanders they are connected to, so the suppliers in $\tilde{X}$ can send flow only to the demanders in $f(\tilde{X}) \cap Q_-$. Therefore $\sum_{k \in \tilde{X}} x_k \leq \sum_{j \in f(\tilde{X}) \cap Q_-} d_j$.

Note also that $s'_k \geq x'_k$ and $x_k \geq s_k$ for any $k \in \tilde{X}$, and that $s'_k = s_k$ for all $k \notin A$. Putting all this together we have:

$$\sum_{k \in \tilde{X} \setminus A} s_k + \sum_{k \in \tilde{X} \cap A} x'_k \;=\; \sum_{k \in \tilde{X} \setminus A} s'_k + \sum_{k \in \tilde{X} \cap A} x'_k \;\geq\; \sum_{k \in \tilde{X}} x'_k \;\geq\; \sum_{j \in f(\tilde{X}) \cap Q_-} d_j, \tag{2.42}$$

and

$$\sum_{j \in f(\tilde{X}) \cap Q_-} d_j \;\geq\; \sum_{k \in \tilde{X}} x_k \;\geq\; \sum_{k \in \tilde{X} \setminus A} s_k + \sum_{i \in \tilde{X} \cap A} x_i. \tag{2.43}$$

For every agent in A to be (weakly) better off when reporting s', we must have $x_k \leq x'_k$ for each $k \in A$. Combining this with inequalities (2.42) and (2.43), we conclude that $x'_k = x_k$ for each $k \in \tilde{X} \cap A$. Moreover, these inequalities also imply that $x'_k = x_k = s_k$ for each $k \in \tilde{X} \setminus A$. Thus, $x'_k = x_k$ for all $k \in \tilde{X}$. Note that the suppliers in $\tilde{X}$ send all of their flow flow to the demanders in $f(\tilde{X}) \cap Q_-$, whether the report is s or s'; also the demanders in $f(\tilde{X}) \cap Q_-$ receive all of their flow from the suppliers in $\tilde{X}$, whether the report is s or s'. Therefore, removing the suppliers in $\tilde{X}$ and the demanders in $f(\tilde{X}) \cap Q_-$ does not affect the egalitarian solution for either problem. As we picked a smallest counterexample, $\tilde{X} = \emptyset$.

We now establish that the decomposition does not change in a smallest counterexample. We already know that $D' = \emptyset$, which implies $Q'_- \subseteq Q_-$. Suppose this containment is strict so that there is a demander $j \in Q_- \setminus Q'_-$. Then, $g(j) \subseteq M_+$. As $\tilde{X} = \emptyset$, $g(j) \subseteq M'_+$, which implies demander j cannot receive any flow when the report is s'. Therefore $Q'_- = Q_-$, which implies $Q'_+ = Q_+$, $M'_+ = M_+$, and $M'_- = M_-$.

To complete the argument, let A be as defined earlier. Let $A_+ = A \cap M_+$ and $A_- = A \cap M_-$. For any $i \in A_-$, $s'_i < x_i$ implies $x'_i \leq s'_i < x_i$, causing i to do worse by reporting s'_i. Likewise, any $i \in A_+$, $s'_i > x_i$ implies $x'_i \geq s'_i > x_i$, causing i to do worse by reporting s'_i. So any improving coalition A must be such that $s'_i \geq x_i$ for all $i \in A_-$ and $s'_i \leq x_i$ for all $i \in A_+$. But in this case the egalitarian solution does not change for either problem. ∎

2.6 Further Work

- As we have discussed before, the egalitarian mechanism is uniquely characterized by Pareto optimality, strategyproof and equal treatment of equals. It will be interesting to see if group strategyproofness and any other additional property can characterize the egalitarian mechanism.

- We prove in Theorem 4 that strong invariance and Pareto optimality characterizes the set of all peak groupstrategyproof mechanisms. Is there a similar notion to characterize the link groupstrategyproof mechanisms?

- A mechanism is said to be "bossy" if an agent can worsen the allocation of another agent without actually improving his own allocation (strictly utility). That, the egalitarian mechanism is "bossy" is clearly seen from the following example. The open question is, is it possible to construct a non bossy mechanism for these problems discussed here?

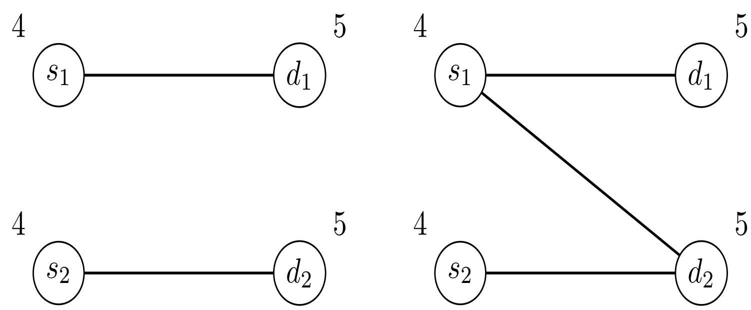

Figure 2.13: Bossiness of the egalitarian rule

- In section 2.4.2 we discussed the egalitarian mechanism for a network with capacities. When we allow for agents to report these finite capacities, agents can be strategic with respect to these capacity reports. It will be interesting to see if the egalitarian mechanism is robust against coordinated misreports of the capacities.

Chapter 3

The Edge Fair Mechanism

3.1 Introduction

We study the problem of fair division of a maximum flow in a capacitated bipartite network. This model generalizes a number of matching and allocation problems that have been studied extensively over the years, motivated by applications in school choice, kidney exchange, etc. The common feature in these application contexts is that the associated market is moneyless, so that fairness is achieved by equalizing the allocation *as much as possible.* This last caveat is to account for additional considerations, such as Pareto efficiency and strategyproofness, that may be part of the planner's objective.

Specifically, we are given a bipartite network $G = (S \cup D, E)$, and we think of S as the set of supply nodes and D as the set of demand nodes. Each arc $(i, j) \in E$ connects a supply node i to a demand node j, and has capacity $u_{ij} \geq 0$. There is a single commodity (the resource) that is available at the supply nodes and needs to be transferred to the demand nodes: we assume that supply node i has s_i units of the resource, and demand node j requires d_j units of it. The capacity of an arc (i, j) is interpreted as an upper bound on the direct transfer from supply node i to demand node j. The goal is to satisfy the demands "as much as possible" using the available supplies, while also respecting the capacity constraints on the arcs.

A well-studied special case of our problem is that of allocating a single resource (or allocating the resource available at a single location) amongst a set of agents with varying (objectively verifiable) claims on it. This is the special case when there is a single supply node that is connected to every one of the demand nodes in the network by an arc of large-enough capacity. If the sum of the claims of the agents exceeds the amount of the resource available, the problem is a standard rationing problem (studied in the literature as "bankruptcy" problems or "claims" problems). There is an extensive literature devoted to such problems that has resulted in a thorough understanding of many natural methods including the *proportional* method, the *uniform gains* method, and the *uniform losses* method. A different view of this special case is that of allocating a single resource amongst agents with single-peaked preferences over their net consumption. Under this view, studied by Sprumont [56], Thomson [60], and many others, the goal is to design a mechanism for allocating the resource that satisfies appealing efficiency and equity properties, while also eliciting the preferences of the agents truthfully. The *uniform rule*, which is essentially an adaptation of the uniform gains method applied to the reported peaks of the agents, occupies a central position in this literature: it is strategy-proof (in fact, group strategy-proof), and finds an envy-free allocation that Lorenz dominates every other efficient allocation; furthermore, this rule is also *consistent*. (We will define consistency, Lorenz dominance, etc. precisely in Section 3.2.) A natural two-sided version of Sprumont's model has agents initially endowed with some amount of the resource, so that agents now fall into two categories: someone endowed with less than her peak is a potential demander, whereas someone endowed with more than her peak is a potential supplier. The simultaneous presence of demanders and suppliers creates an opportunity to trade, and the obvious adaptation of the uniform rule gives their peak consumption to agents on the short side of the market, while those on the long side are uniformly rationed (see [37], [8]). This is again equivalent to a standard rationing problem because the nodes on the short side of the market can be collapsed to a single node. The model we consider generalizes this by assuming that the resource can only be transferred between certain pairs of agents. Such constraints are typically logistical (which supplier can reach which demander in an emergency situation, which worker can handle which job request), but could be subjective as well (as when a hospital chooses to refuse a new patient by declaring red status). This complicates the analysis of efficient (Pareto optimal) allocations, because short demand and short supply typically coexist in the same market.

The general model we consider in this paper has been the subject of much recent research and was first formulated by Bochet et al. [11, 12]. The authors work with a bipartite network in both papers and assume that each node is populated by an agent with single-peaked preferences over his consumption of the resource: thus, each supply node has an "ideal" supply (its peak) quantity, and each demand node has an ideal demand. These preferences are assumed to be private information, and Bochet et al. [11, 12] propose a clearinghouse mechanism that collects from each agent only their "peaks" and picks Pareto-optimal transfers with respect to the reported peaks. Further, they show that their mechanism is strategy-proof in the sense that it is a dominant strategy for each agent to report their peaks truthfully. While the models in the two papers are very similar, there is also a critical difference: in [12], the authors require that no agent be allowed to send or receive any more than their peaks, whereas in [11] the authors assume that the *demands* must be satisfied exactly (and so some supply nodes will have to send more than their peak amounts). The mechanism of Bochet et al.—the *egalitarian* mechanism—generalizes the uniform rule, and finds an allocation that Lorenz dominates all Pareto efficient allocations. Later, Chandramouli and Sethuraman [15] show that the egalitarian mechanism is in fact group strategyproof: it is a dominant strategy for any group of agents (suppliers or demanders) to report their peaks truthfully [1]. Szwagrzak [57, 59, 58] expands the study of allocation rules in these networked economies by introducing broader class of mechanisms with various fairness properties. His work also develops alternative characterizations of these mechanisms (in particular, the egalitarian mechanism) and provides a unified view of the allocation problem on networks. Szwagrzak [57] studies the property of *contraction invariance* of an allocation rule: when the set of feasible allocations contracts such that the optimal allocation is still in this smaller set, then the allocation rule should continue to select the same allocation. He shows that the egalitarian rule is contraction invariant. These results suggest that the egalitarian mechanism may be the correct generalization of the uniform rule to the network setting. However, it is fairly easy to show that the egalitarian mechanism is not consistent: if the link from a supply node i and demand node j is dropped, and s_i and d_j are adjusted accordingly, applying the egalitarian mechanism to the reduced problem will not necessarily give the same allocation to the agents. Motivated by this observation, Moulin and Sethuraman [43, 44] study rules for network rationing problems that extend a given rule for a standard rationing problem while preserving consistency and other natural axioms. In particular, they propose a family of rules that generalize the uniform

[1] Szwagrzak [57] genrealize the proof methodology of Chandramouli and Sethuraman [15] to establish that all separably convex rules are group strategyproof

rule to the bipartite network setting. While they are able to show that their extension satisfies consistency, it is not known if any of these rules is strategyproof.

Our main contribution in this paper is a new *group strategy-proof* mechanism (the "edge-fair" mechanism) that is a consistent extension of the uniform rule. Our proof shows that for any Pareto efficient mechanism, group strategyproofness is equivalent to a property that we call *strong invariance* that is often straightforward to verify. (In particular, the group strategy-proofness of the egalitarian mechanism also follows immediately, even if one works with a capacitated model.) Along the way we show that consistency imposes very severe restrictions: for instance, no consistent rule can find allocations that are envy-free, even in the limited sense introduced by Bochet et al. [12] for such problems. The mechanism we propose does not find the Lorenz optimal allocation, but we show that no consistent mechanism can.

In the second part of the chapter, we consider a model where the supplies and demands at the nodes are given, but that each edge is controlled by an independent agent with single-peaked preferences on the amount transferred along that edge. The planner still wishes to implement a maximum flow (it is now a design constraint), and the goal is to divide this reasonably among the edges of the network. For this model we show that a Lorenz optimal allocation need not exist, but that our mechanism can still be applied and finds a division of the max-flow that is envy free, consistent and group strategyproof.

The rest of the paper is organized as follows: in Section 3.2 we consider the standard model of maximizing the total flow in a capacitated bipartite network. We state the well-known Gallai-Edmonds decomposition, and describe the edge-fair algorithm that selects a particular max-flow for any given problem. An easy argument shows that the edge-fair algorithm makes a consistent selection of max-flows across related problems. Section 3.3 considers the model in which agents are located on the nodes of the network and have single-peaked preferences over their allocations—the equivalence of group strategy-proofness and strong invariance, and the fact that the edge-fair rule satisfies strong invariance are the key results in this section. In section 3.4 we turn to the problem in which agents are on the edges of the network, and study the implications of consistency.

3.2 Maximum Flows Review and Consistency

We consider the problem of transferring a single commodity from a set S of suppliers to a set D of demanders using a set E of edges. Supplier i has $s_i \geq 0$ units of the commodity, and demander j wishes to consume $d_j \geq 0$. Associated with each edge is a distinct supplier-demander pair: the edge $e = (i, j)$ links supplier i to demander j, and has a non-negative, possibly infinite, capacity u_{ij}. Transfer of the commodity is allowed between supplier i and demander j only if $(i, j) \in E$, in which case at most u_{ij} units of the commodity can be transferred along this edge[2]. The goal is to find an "optimal" transfer of the commodity from the suppliers to the demanders. We let $G = (S \cup D, E)$ be the natural bipartite graph and we speak of the *problem* (G, s, d, u).

We use the following notation. For any subset $T \subseteq S$, the set of demanders compatible with the suppliers in T is $f(T) = \{j \in D | (i, j) \in E, \ \ i \in T\ \}$. Similarly, the set of suppliers compatible with the demanders in $C \subseteq D$ is $g(C) = \{i \in S | (i, j) \in E, \ \ j \in C\}$. We abuse notation and say $f(i)$ and $g(j)$ instead of $f(\{i\})$ and $g(\{j\})$ respectively. For any subsets $T \subseteq S$, $C \subseteq D$, $x_T := \sum_{i \in T} x_i$ and $y_C := \sum_{j \in C} y_j$.

A transfer of the commodity from S to D is realized by a flow φ, which specifies the amount of the commodity transferred from supplier i to demander j using the edge $(i, j) \in E$. The flow φ induces an allocation vector for each supplier and each demander as follows:

$$\text{for all } i \in S : x_i(\varphi) = \sum_{j \in f(i)} \varphi_{ij}; \text{ for all } j \in D : y_j(\varphi) = \sum_{i \in g(j)} \varphi_{ij} \tag{3.1}$$

The flow φ is *feasible* if (i) $\varphi_{ij} \leq u_{ij}$ for all $(i, j) \in E$ and $\varphi_{ij} = 0$ for all $(i, j) \notin E$; (ii) $x_i(\varphi) \leq s_i$ for all $i \in S$; and (iii) $y_j(\varphi) \leq d_j$ for all $j \in D$. Let $\mathcal{F}(G, s, d, u)$ be the set of feasible flows for the problem (G, s, d, u). A feasible flow φ^* is a maximum flow if

$$\varphi^* \in \arg \max_{\varphi \in \mathcal{F}(G,s,d,u)} \sum_{i \in S} x_i(\varphi).$$

Let $\mathcal{F}^*(G, s, d, u)$ be the set of maximum flows for the problem (G, s, d, u). For reasons that will be clearer later, we shall focus mostly on finding a maximum flow for any given problem. As a result, it is important to understand the set $\mathcal{F}^*(G, s, d, u)$, which we turn to next.

[2] Equivalently, we could assume that an edge exists between every supplier i and every demander j, but that $u_{ij} = 0$ for all $(i, j) \notin E$.

The Gallai-Edmonds Decomposition. The problem under consideration is the well-known problem of finding a maximum flow in a capacitated bipartite network. The following result characterizes the structure of maximum flows and is essentially a version of the Gallai-Edmonds decomposition. It can proved by a straightforward application of the max-flow min-cut theorem.

Lemma 9 *There exists a partition S_+, S_- of S, and a partition D_+, D_- of D such that the flow φ with net transfers x, y is a maximum flow if and only if*

$$\varphi_{ij} = u_{ij} \;\; \forall ij \in G(S_-, D_-), \;\; x_i = s_i \;\; \forall i \in S_+, \;\; y_j = d_j \;\; \forall j \in D_+ \tag{3.2}$$

Proof: Refer to the appendix ∎

We briefly describe some key axioms that we want our rules to satisfy.

Edge consistency. The key axiom in our paper is *edge consistency* (or simply consistency, hereafter). Suppose we have a rule φ that picks a flow z for a given problem (G, s, d, u). Fix an edge $(i, j) \in G$ connecting supply node i and demand node j, and define the reduced problem as follows: the new graph is $G' = G \setminus \{e\}$; the supplies and demands of all the nodes other than i and j stay the same, $s'_i = s_i - z_{ij}$ and $d'_j = d_j - z_{ij}$; and the capacities on the edges that remain stay the same. Let z' be the flow picked by the rule φ for the reduced problem (G', s', d', u). The rule φ is edge-consistent if $z = z'$ for every reduced problem G' that can be obtained from G by omitting an edge. Observe that z restricted to the edges in G' is a max-flow for the reduced problem, and edge-consistency requires that the rule not "reallocate" the flow amongst the remaining edges if some edge is dropped from the problem and the corresponding supplies and demands are adjusted in the obvious way.

Continuity. The mapping $\varphi : (G, s, d, u) \to \mathbb{R}^{|E|}$ is jointly continuous in s, d, and u. Roughly speaking, this simply says that a rule is continuous only if small changes in supplies, demands or edge-capacities result in small changes on the edge-flows picked by the rule.

Symmetry. Consider any permutation π of the supply nodes and a permutation σ of the demand nodes. Define the graph G' as follows: $(i, j) \in G$ if and only if $(\pi(i), \sigma(j)) \in G'$. The supplies s'

and demands d' are defined analogously by permuting the original supplies and demands according to the respective permutations. A rule φ is symmetric if and only if for every π and every σ, $z_{ij} = z'_{\pi(j),\sigma(j)}$ where z and z' are the outcomes of the rule for the problems (G, s, d, u) and (G', s', d', u') respectively.

Individual rationality from equal division requires all agents to be at least as well off when the good is evenly distributed

One-sided resource-monotonicity requires that all agents gain upon an increase in the commodity if (i) initially there is not enough of the commodity and (ii) after the increase there is not enough of the commodity; and gain upon a decrease in the commodity if (i) initially there is too much of the commodity and (ii) after the decrease there is still too much of the commodity.

Connection to literature: The uniform rule of Sprumont [56] as described in section 2.2 is a consistent rule. Consider the example in figure 2.1, suppose we allocate 5/14 units to agent s_3 and remove him from the network and adjust the peak of d_1 to $9/14(1-5/14)$. The uniform rule on the reduced network allocates for agents $s_1, 2/7$ units and for $s_2, 5/14$ units which coincides with the uniform rule allocation in the original network. Sonmez [54] show that consistency along with one sided resouce monotonicity and individual rationality from equal division characterize the uniform rule.

More recently, Moulin and Sethuraman [43] study consistent rules and their extensions to bipartite networks. Their main contribution is that (i) uniform gains and uniform losses method have infinitely many consistent extensions whereas the propotional method has only one. In their follow up work, Moulin and Sethuraman [44] study loss calibrated rationing methods that are consistently extendable to bipartite networks. They also show that these are the only methods that meet this property whereas most standard parametric methods do not admit such consistent extensions.

3.2.1 The Edge Fair Rule

Given two max-flows φ and φ' sorted in increasing order we say that φ *lexicographically dominates* φ' if the first coordinate k in which φ and φ' are not equal is such that $\varphi_k > \varphi'_k$. (Note that the k-th smallest entry in the flows φ and φ' may be on different edges.) The max-flow φ is lex-optimal if it lexicographically dominates all other max-flows $\mathcal{F}^*(G, s, d, u)$. It is clear that a lex-optimal flow exists and is unique.[3] The edge-fair algorithm, formally described next, finds this lex-optimal flow by solving a sequence of linear programming problems.

We fix a problem (G, s, d) such that $s_i, d_j > 0$ for all i, j (clearly, if $s_i = 0$ or $d_j = 0$ we can ignore supplier i or demander j altogether). Let $E_0 := E$ and $\mathcal{F}_0 := \mathcal{F}^*(G, s, d, u)$, the set of all max-flows for the given problem. The edge-fair algorithm (or rule) proceeds iteratively, solving a linear programming problem in each step. In any iteration t, it starts with a candidate set of max-flows $\mathcal{F}_t$, and a set of active edges E_t, and solves the following linear programming problem:

$$\max_{\varphi \in \mathcal{F}_t} \left\{ \lambda_{t+1} \;\middle|\; \varphi_e \geq \lambda_{t+1}, \;\; \forall e \in E_t \right\}.$$

Suppose λ^*_{t+1} is the optimal value of this linear programming problem. Then,

$$\mathcal{F}_{t+1} \; = \; \left\{ \varphi \in \mathcal{F}_t \mid \varphi_e \geq \lambda^*_{t+1} \;\; \forall e \in E_t \right\},$$

and

$$E_{t+1} \; = \; \left\{ e \in E_t \mid \varphi_e > \lambda^*_{t+1} \;\text{ for some }\; \varphi \in \mathcal{F}_{t+1} \right\}.$$

The edges in $E_t \backslash E_{t+1}$ are declared *inactive*, and the algorithm proceeds to the next value of t if any active edges remain. As at least one edge becomes inactive in each step, the algorithm terminates in $O(|E|)$ steps.

It is often useful to think about this algorithm in a different, but equivalent way. First, observe that any edge whose flow is *fixed* in every max-flow will carry exactly this amount in the outcome of the edge-fair algorithm as well. Thus, we focus only on those edges (i, j) with the property that $0 < \varphi_{ij} < u_{ij}$ for *some* flow $\varphi \in \mathcal{F}^*(G, s, d, u)$. In particular, from the observations in Proposition 2 on the set of Pareto Optimal solutions, we could fix $z_{ij} = u_{ij}$ for $ij \in G(S_-, D_+)$ and $z_{ij} = 0$ for $ij \in G(S_+, D_-)$ and remove these edges from the network. The reduced problem now decomposes into 2 disjoint components: one in which the suppliers are rationed (and every demander gets what they ask for), and the other in which the demanders are rationed, but each supplier sends his entire supply. As the algorithm is completely symmetric, we simply describe it for the case of rationed demanders. In this case each supplier will be allocated his peak in every max-flow; and

[3]The term lex-optimal flow is also used to mean a flow whose induced allocation for the suppliers (or demanders) lex-dominates the induced allocation for the suppliers in any other flow [41, 42].

any flow that respects edge-capacities while allocating each supplier his peak, while allocating each demander no more than his peak is a max-flow. Thus, the linear programming problem that must be solved in each step can be explicitly described: the only edges that need to be considered are those between S_+ and D_-.

$$\text{Maximize} \quad \lambda_{t+1}$$

$$\text{subject to}$$

$$\textstyle\sum_j z'_{ij} = s_i, \ \ \forall \{i \in S_+, ij \in E_t\}$$

$$\textstyle\sum_i z'_{ij} \leq d_j, \ \ \forall \{j \in D_-, ij \in E_t\}$$

$$\lambda_{t+1} \leq z'_{ij}, \ \ \forall \{ij \in E_t\}$$

$$u_{ij} \geq z'_{ij} \geq 0$$

Initially, every such edge is active, and the algorithm tries to maximize the minimum amount carried by an active edge in any max-flow.

Theorem 11 *The edge fair rule is symmetric, continuous, and consistent.*

Proof: Symmetry follows because the rule is invariant, by definition, to permutations of supply nodes, demand nodes, and edge-capacities. Continuity is equally clear. Consistency is also immediate by the definition of the algorithm: we may assume that the edge (i, j) that is dropped to obtain the reduced problem is still present but carries a constant flow z_{ij}, where z is the outcome chosen by the rule for the original problem. Thus, the set of feasible solutions to the reduced problem is a subset of the set of feasible solutions to the original problem at every stage of the algorithm: As the outcome z for the original problem is a member of both sets, it will be chosen in both cases. ∎

Example. We illustrate the algorithm on the problem shown in Figure 3.1 with 8 supply nodes and 8 demand nodes. All edges have infinite capacity except for the edges $(s_7 d_3)$ and $(s_8 d_4)$, which have capacity 0.5 each. It is clear that these two capacitated edges must carry 0.5 unit of flow

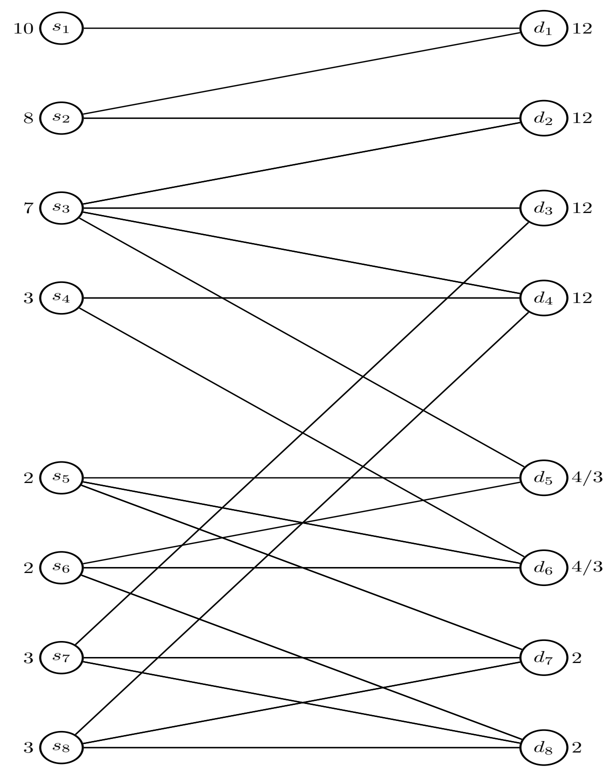

Figure 3.1: Gallai-Edmonds Decomposition and the Edge Fair Allocation

each in *every* max-flow, so their flow can be fixed; by consistency, we could omit these edges from further consideration, and adjust the supplies at s_7 and at s_8 and the demands at d_3 and at d_4 down by 0.5 unit each. Similarly, the edges $(s_3 d_5)$ and $(s_4 d_6)$ carry no flow in any max-flow, and so can be omitted as well. The problem now decomposes into two components: one involving the first 4 supply and demand nodes, where the demand nodes are rationed in any max-flow; and the other involving the last 4 supply and demand nodes, where the supply nodes are rationed in any max-flow.

First consider the problem involving the first four supply and demand nodes. Each supply node sends all its supply, whereas each demand node receives at most what it wants. The edge-fair algorithm applied to this problem gives the following flow: first $z_{21} = 2$; then $z_{32} = z_{33} = z_{34} = 7/3$; then $z_{44} = 3$, after which $z_{22} = 6$, and finally $z_{12} = 10$. The resulting allocation for the demanders in this problem is $(12, 25/3, 7/3, 16/3)$; recall that demand nodes 3 and 4 also get 0.5 units of flow from suppliers $s7$ and $s8$ respectively, so the final allocation for the demand nodes is $(12, 25/3, 17/6, 35/6)$.

Now consider the edge-fair algorithm applied to the last 4 supply and demand nodes. Here the supply nodes are rationed whereas the demand is exactly met. It is easy to check that the the edge-fair rule sends a flow of 2/3 on each edge in this component so that the resulting allocation for the supply nodes is $(2, 2, 4/3, 4/3)$; as the last 2 supply nodes also send 0.5 units of flow to the other component, the final allocation for these supply nodes is $(2, 2, 11/6, 11/6)$.

To summarize, the edge-fair algorithm applied to this example results in an allocation of $(10, 8, 7, 3, 2, 2, 11/6, 11/6)$ for the supply nodes and $(12, 25/3, 17/6, 35/6, 4/3, 4/3, 2, 2)$ for the demand nodes. In contrast, it can be verified that the egalitarian allocation results in an allocation of $(10, 8, 7, 3, 23/12, 23/12, 23/12, 23/12)$ for the supply nodes, and $(10, 8, 11/2, 11/2, 4/3, 4/3, 2, 2)$ for the demand nodes. This also highlights an important distinction between the edge-fair allocation and the egalitarian one: in our example, demand nodes 3 and 4 have identical demands, and it is *possible* to give them the same allocation, as shown by the Egalitarian allocation; the edge-fair rule, however, treats these demand nodes differently. In particular, demand node 4 is better off under the edge-fair rule because of its improved connectivity.

3.3 Model 1: Agents on nodes

In this section we consider the version of the problem where the nodes of the network are populated by agents. Specifically, each supply node i is occupied by a *supplier* i and each demand node j is occipied by a *demander* j. Thus, we our problem becomes one of transferring a single commodity from a set S of suppliers to a set D of demanders using the set E of edges. The edge e has a capacity u_e that is known to all the agents. A transfer of the commodity from S to D is realized by a flow φ, which specifies the amount of the commodity transferred from supplier i to demander j using the edge $(i, j) \in E$. The flow φ induces an allocation vector for each supplier and each demander as follows:

$$\text{for all } i \in S : x_i(\varphi) = \sum_{j \in f(i)} \varphi_{ij}; \text{ for all } j \in D : y_j(\varphi) = \sum_{i \in g(j)} \varphi_{ij} \tag{3.3}$$

As we shall see in a moment, suppliers and demanders only care about their *net* transfers, and not on how these transfers are distributed across the agents on the other side.

Each supplier i has *single-peaked preferences*[4] R_i (with corresponding indifference relation I_i) over her *net transfer* x_i, with peak s_i, and each demander j has single-peaked preferences R_j (I_j) over her net transfer y_j, with peak d_j. We write $\mathcal{R}$ for the set of single peaked preferences over $\mathbb{R}_+$, and $\mathcal{R}^{S \cup D}$ for the set of preference profiles.

We think of the graph G as fixed, and focus our attention on mechanisms that elicit preferences from the agents and maps the reported preference profile to a flow. For reasons that will be clear later, we focus on allocation rules that are *peak* only: the flow (and hence the induced allocation vector for the suppliers and demanders) depends on the reported preference profile of the agents only through their peaks. Thus it makes sense to talk of the *problem* (s, d): this emphasizes the fact that the *peaks* of the agents are private information whereas the other part of the problem (specifically, the graph G and the edge capacities u) are known. To summarize: suppliers and demanders report their peaks; the allocation rule is applied to the graph G with edge-capacities u, and the data (s, d) where s and d are the reported peaks of the suppliers, and demanders. Our focus will be on mechanisms in which no agent has an incentive to misreport his peak.

A mechanism is said to be *strategyproof* if for any graph G it is a weakly dominant strategy for an agent to truthfully report their peak. It is *group strategyproof* if for any graph G it is a weakly dominant strategy for any coalition of agents to truthfully report their peaks.

3.3.1 Efficiency and Equity

Pareto Optimality: A feasible net transfer (x, y) as defined in the previous section is Pareto Optimal if there is no other allocation (x', y') such that every agent is weakly better off and atleast one agent is strictly better off in it. In mathematical terms, if R_i and I_i denote the preference and indifference relations respectively for agent i, then

$$\{\forall\ i, j:\ \ x'_i R_i x_i \ \text{ and } \ y'_j R_j y_j\} \implies \{\forall\ i, j:\ \ x'_i I_i x_i \ \text{ and } \ y'_j I_j y_j\} \tag{3.4}$$

The following result shows that set of Pareto optimal transfers for peak-only rules is intimately related to the set of max-flows.

[4] Writing P_i for agent i's strict preference, we have for every x_i, x'_i: $x_i < x'_i \le s_i \Rightarrow x'_i P_i x_i$, and $s_i \le x_i < x'_i \Rightarrow x_i P_i x'_i$.

Proposition 2 *Fix the economy* (G, R). *Let* S_+, S_- *and* D_+, D_- *be the Gallai-Edmonds decomposition applied to the network* G *with edge capacities given by* u*, supplies given by the peaks of the suppliers and the demands given by the peaks of the demanders. Then:*

(a) If the flow φ *implements Pareto optimal net transfers* (x, y)*, then:*

$$ij \in G(S_-, D_-) \implies \varphi_{ij} = u_{ij}; \;\; ij \in G(S_+, D_+ \cup (f(S_-) \cap D_-)) \implies \varphi_{ij} = 0 \tag{3.5}$$

(b) The transfers (x, y) *induced by a feasible flow* φ *are Pareto optimal if and only if*

$$x \geq s \;\; on \;\; S_+, \;\; y \leq d \;\; on \;\; D_- \;\; and \;\; x_{S_+} = y_{D_-} - \varphi(S_-, D_-) \tag{3.6}$$

$$x \leq s \;\; on \;\; S_-, \;\; y \geq d \;\; on \;\; D_+ \;\; and \;\; x_{S_-} = y_{D_+} + \varphi(S_-, D_-) \tag{3.7}$$

where $\varphi(S_-, D_-)$ *is the net flow from component* S_- *to* D_-*. From earlier discussions,* $\varphi(S_-, D_-) = \sum_{i \in S_i, j \in D_-} u_{ij}$

We are particularly interested in Pareto optimal flows and transfers in which no supplier or demander is allocated more than their peak: such solutions are Pareto optimal for *any* single-peaked preferences of the agents as long as the peaks are s and d respectively. Following Bochet et al., we call this set PO^* and note that $(x, y) \in PO^*$ if and only if (x, y) is Pareto optimal, $x \leq s$, and $y \leq d$. In particular, $(x, y) \in PO^*$ if and only if

$$x = s \;\text{ on }\; S_+, \;\; y \leq d \;\text{ on }\; D_- \;\text{ and }\; x_{S_+} = y_{D_-} - \varphi(S_-, D_-) \tag{3.8}$$

$$x \leq s \;\text{ on }\; S_-, \;\; y = d \;\text{ on }\; D_+ \;\text{ and }\; x_{S_-} = y_{D_+} + \varphi(S_-, D_-) \tag{3.9}$$

In the rest of the section, by a Pareto optimal solution we mean a flow inducing net transfers $(x, y) \in PO^*$. We proceed now to discussions related to fairness.

Ranking: One notion of fairness is that suppose two agents with different peaks have identical connections, then the agent with higher peak should have higher net allocation. This is true for the uniform rule where there is only 1 type of divisible good. This can be formalized in the following way for a general bipartite graph discussed here: (A similar statement can be made about the demanders)

1. **Ranking** (RK) : $s_i \leq s_j \implies x_i \leq x_j \forall\ i, j$ such that $f(i) = f(j)$

2. **Ranking*** (RK*): $s_i \leq s_j \implies s_i - x_i \leq s_j - x_j\ \ \forall i, j$ such that $f(i) = f(j)$

We start with a proof of statement (i). Suppose $x_i > x_j$, we show a transfer from agent i to agent j is possible and contradicts the lexicographic solution on the edges. Construct a new solution x' such that $z'_{kl} = z_{kl}\ \ \forall\ \ k \in S\backslash\{i, j\},\ \ l \in f(k),\ \ z'_{il} = z'_{jl} = \frac{z_{il}+z_{jl}}{2}\ \ \forall\ \ l \in f(i)$. The allocation x' is clearly feasible and x does not lexicographic dominate x'. Hence, we arrive at a contradiction. Using the similar idea of routing the flows from agent i to agent j and by contradiction we can prove statement (ii).

Connectivity Fairness: As discussed earlier in section 3.2, by construction edge fair algorithm increases flow on every edge until there is an edge which cannot send more flow. This intuitively is giving more preference to nodes with better connectivity. Review the example in figure 3.1 and the discussion following it, inspite of the peaks of agents d_3 and d_4 being identical, the allocation of d_3 is better because of having better connectivity.

No Envy: A rule $(x, y) \in \mathcal{F}(G, s, d, u)$ satisfies *No Envy* if for any preference profile $R \in \mathcal{R}^{S\cup D}$ and $i, j \in S$ such that $x_j P_i x_i$, there exists no (x', y') such that

$$x_k = x'_k \text{ for all } k \in S\backslash\{i, j\};\ y_l = y'_l \text{ for all } l \in D \text{ and} \tag{3.10}$$

$$x'_i P_i x_i \tag{3.11}$$

and a similar statement when we interchange the role of suppliers and demanders.

Equal Treatment of Equals: A rule $(x, y) \in \mathcal{F}(G, s, d, u)$ satisfies *Equal Treatment of Equals* if for any preference profile $R \in \mathcal{R}^{S\cup D}$ and $i, j \in S$ such that $s_i = s_j$, if $x_j \neq x_i$ then there exists no (x', y') such that

$$x_k = x'_k \text{ for all } k \in S\backslash\{i, j\};\ y_l = y'_l \text{ for all } l \in D \text{ and} \tag{3.12}$$

$$|x'_i - x'_j| < |x_j - x_i| \tag{3.13}$$

and a similar statement when we interchange the role of suppliers and demanders.

If an allocation rule always results in a Pareto optimal allocation and satisfies No Envy, then it also satisfies Equal Treatment of Equals (Refer to Proposition 5 in Bochet et al. [12]).

The egalitarian rule of Bochet et al. [12] is a selection from the Pareto set PO^* as is the edge-fair allocation rule. They also show that the egalitarian rule is envy free but not consistent as show in the following example: In figure 3.2 where we remove the node d_2 from the network on the left. The egalitarian allocation on the reduced network improves the allocation of s_2 by sending 1 unit of flow on the edge $s_2 - d_2$

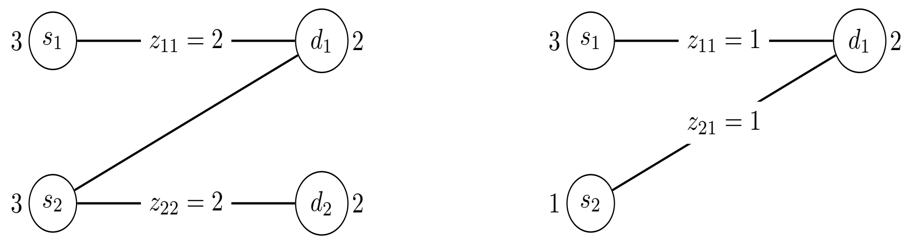

Figure 3.2: Inconsistency of the egalitarian rule and envy of edge fair rule

We have already seen that the edge-fair rule is also consistent. But here is an example where the edge-fair rule has envy. But one can show that no consistent rule is envy-free (under PO^*) using the same example.

Lemma 10 *There is no Pareto optimal (PO^*) mechanism which is simultaneously envy free for agents and edge consistent*

Consider the network in figure 3.2. Suppose the mechanism is envy free: Any envy free solution should allocate 2 units to each supplier 1 and 2. This establishes a unique edge flow: $(z_{11}, z_{21}, z_{22}) = (2, 0, 2)$. Lets remove the edge $s_2 - d_2$ with $z_{22} = 2$ units allocation. If this mechanism was also consistent, then on this reduced network the mechanism should have an allocation $(z_{11}, z_{21}) = (2, 0)$ on the edges. But the no-envy solution on this reduced graph would allocate $(z_{11}, z_{21}) = (1, 1)$

Now, suppose the given mechanism is edge consistent and also $(z_{11}, z_{21}, z_{22}) = (2, 0, 2)$ is an allocation given by edge consistent rule. Removing the edge $s_2 - d_2$, in the reduced graph the allocation from the edge consistent rule should be $(z_{11}, z_{21}) = (2, 0)$ but this is not an envy free allocation in the reduced graph. As a consequence, if the mechanism is edge consistent it cannot allocate $(z_{11}, z_{21}, z_{22}) = (2, 0, 2)$ in the original network but this is the only envy free solution on that network. The same example also shows that any edge consistent mechanism violates the equal treatment of equals property. ∎

These results imply that no rule can be Pareto efficient, Envy-free and Consistent. Both the egalitarian and edge-fair rules find Pareto efficient allocations; where they differ is that the egalitarian rule relaxes consistency but is envy-free, but the edge-fair rule relaxes envy-freeness but is consistent. The egalitarian rule is peak group strategyproof. A natural question is if the edge-fair rule satisfies this property as well. We answer this question in the affirmative in the next section.

3.3.2 Strategic Issues

Peak Groupstrategyproof: *For all* $R \in \mathcal{R}^{S \cup D}$, $M \subseteq S \cup D$ *and* each agent $i \in M$ misreport to $R'_i \in \mathcal{R}$

$$\psi_i(R)_i \psi_i(R'_M, R_{-M}), \;\; \forall \;\; i \in M$$

i.e. it is dominant strategy for agents to reveal their true peaks even when they can coordinate with other agents and jointly misreport.

Theorem 12 *The edge fair rule is strongly invariant and hence peak group strategyproof*

Proof. When the edges are uncapacitated ($u_{ij} = \infty, \forall \, ij \in E(G)$): From the discussion in section 3.3 it is clear that the edge fair rule always picks a Pareto optimal allocation from PO^*. In chapter 2, Theorem 4, we proved that any peak group strategyproof rule is characterized by PO^* and *strong invariance*. Hence, it is enough to prove the strong invariance property of the edge fair rule to establish its peak group strategyproofness. Consider an agent i with $x_i < s[R_i]$ and suppose the new report of agent i is such that $s[R'_i] \geq x_i(R)$. In the original network, the edge fair rule identifies bottleneck points $(\lambda_1, \lambda_2 ...., \lambda_n)$ and allocates flow on them progressively till we identify a maximum flow. As described in section 3.2, the edge flow induced by the edge fair rule identifies a lexicographic optimal flow i.e. it lexicographically dominates every other edge flow which is a maximum flow. Thus, the bottleneck points obtained in either problem coincide, otherwise it contradicts this lex-optimal property of the edge fair rule. Consequently, the corresponding node allocations remains the same in either problem implying the edge fair rule is strongly invariant.

When the edges $ij, i \in S, j \in D$ have finite capacities: the discussion in chapter 2, Theorem 8 any peak GSP mechanism is still characterized by PO^* and strong invariance. Hence, following the same discussion as the uncapacitated case, we conclude that the edge fair rule is strongly invariant.■

3.4 Model 2: Agents on edges

As in Section 3.3, we consider the problem of transferring a single commodity from the set S of suppliers to the set D of demanders using a set E of edges: each edge $e = (i, j)$ links a distinct supplier-demanded pair. However, here we think of the supplier and demander nodes as passive, whereas each edge e is controlled by a distinct agent who has single-peaked preferences R_e over the amount of flow on edge e. We think of the "peak" u_e of his preference relation as the capacity of the associated edge. We write $\mathcal{R}$ for the set of single peaked preferences over $\mathbb{R}_+$, and $\mathcal{R}^E$ for the set of preference profiles. Transfer of the commodity is allowed between supplier i and demander j only if $(i, j) \in E$. We let $G = (S \cup D, E)$ be the natural bipartite graph.

As before we focus our attention on *peak* only mechanisms: in a such a mechanism, the flow depends on the preferences of the agents only through their peaks, so we could simply ask each agent e to report their peak u_e. We assume that the supplies s_i and demands d_j are fixed, and the only varying quantity are the reported peaks (equivalently, edge-capacities).

Pareto flows. The set of Pareto efficient allocations can be complicated because of the peaks of the edge-agents. For example, consider the network in figure 3.3 with two suppliers $\{s_1, s_2\}$, two demanders $\{d_1, d_2\}$, and edges $\{(s_1, d_1), (s_2, d_2), (s_1, d_2), (s_2, d_1)\}$. Suppose all the peaks are 1. Then the flow given by sending $z_{12} = 1$ unit of flow along the edge (s_1, d_2) is Pareto optimal; as is the flow given by sending a unit along each of the edges (s_1, d_1) and (s_2, d_2). In the latter flow 2 units are sent from the supply to demand nodes whereas only 1 unit is transferred in the former.

In contrast to model 1, therefore, it is possible that a Pareto optimal flow does not result in a maximum-flow from supply to demand nodes. For that reason, we assume that the planner implements a max-flow in the given problem (G, s, d, u), and we consider the question of how this max-flow is distributed across the edge-agents. In other words, we focus on the fair division of a max-flow, interpreting max-flow as a design constraint. Let $\mathcal{F}$ be the set of max-flows.

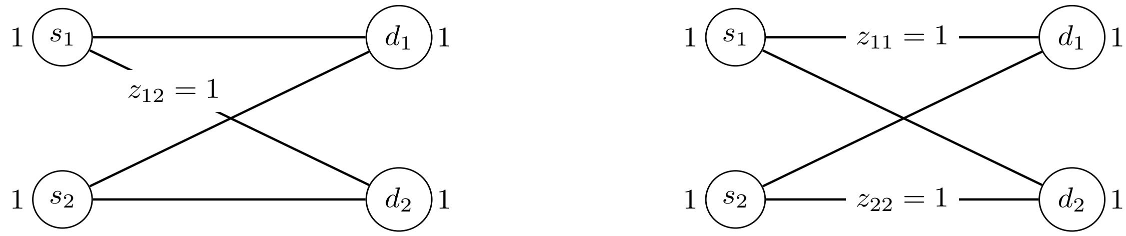

Figure 3.3: Every Pareto flow does not allocate a maximum flow

Restricting ourselves only to max-flows, it is easy to see that the Pareto set is convex: the average of any two max-flows is itself a max-flow. In contrast to model 1, any change in flow along an edge affects the agent's utility directly; in model 1, because the agents were located at the nodes, it is possible for different edge-flows to give the same allocation to the set of agents.

It is natural to try to formulate this "edge"-flow problem as a bipartite rationing problem on an auxiliary graph. For example, consider the Gallai-Edmonds decomposition for the given network (G, s, d, u), and suppose the partitions are S_+, S_- for the suppliers, and D_+, D_- for the demanders. From the GE decomposition, every edge between S_- and D_- carries flow equal to capacity, so their allocation if fixed in all solutions in $\mathcal{F}$; likewise for all edges between S_+ and D_+. This suggests the following idea: create a bipartite graph with one node on the left for each edge, and one node on the right for each element of $S_+ \cup D_+$; each edge that still remains is incident to either S_+ or D_+, but not both; moreover, the given problem is a *rationing* problem in the sense that the nodes on the right must be fully allocated. Thus it appears that we have rewritten the flow problem as a bipartite rationing problem of the sort considered in Section 3.3. That this analogy must be wrong is implied by the following result.

Proposition 3 *There is no Lorenz Dominant allocation among the edge flows in the set* $\mathcal{F}$

Proof. Consider the network of Figure 3.4. The actual network is shown in Figure (a) and the lexicographic solution is shown in (b). However, the solution $\phi_c := \{z_{11} = 1.4, z_{12} = 1.6, z_{21} = 3, z_{22} = 3.1\}$ is also a maximum-flow; the lex-solution does not dominate this flow, nor is it dominated by this one. ∎

Remark. If we draw the bipartite graph suggested in the discussion before the statement of

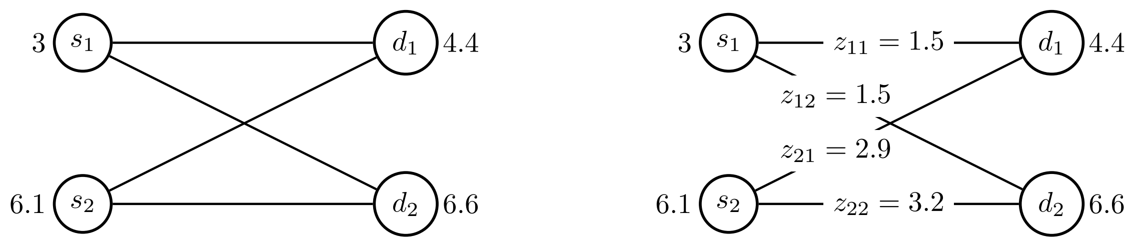

Figure 3.4: Absence of Lorenz Dominance element in Model 2

the proposition, and treat it as a bipartite rationing problem, we find that edges $(1, 1)$ and $(2, 1)$ will carry a flow of 1.5 and 3.05 units each, and this exceeds the total demand at $d1$. These implied "side-constraints" are not accounted for in translating the given problem to a bipartite rationing problem.

Allocation Rules. We can apply the edge-fair rule discussed earlier on this model as well. The edge-fair rule finds a lex-optimal max-flow. It is clear that the rule is also edge consistent. Our next result shows that every edge-consistent rule is also group strategyproof.

Theorem 13 *Fix a graph G with the supply vector s and demand vector d. Suppose we have an allocation rule that maps reports of edge-capacities to a flow. Every edge-consistent allocation rule is group strategyproof.*

Proof. Consider a coalition of agents $A = \{e \in E | u'_e \neq u_e\}$, i.e., they misreport their true peaks. Let the misreported profile be denoted by $R' \in \mathcal{R}^{|E|}$ and the resulting network by G'. Note that the edge-fair rule always results in an allocation $z \leq u$, hence any agent $e \in A$ should report $u'_e \geq z_e$; otherwise, $z'_e \leq u'_e < z_e$ and the agent e is worse off in profile R'. Let $B := \{e \in A | z_e = u_e\}$. The agents in B should have the allocation $z'_e = u_e$ when the reports are R' as every such agent received their peak allocation in profile R. Consider the graph $G \setminus B$ by removing the agents in B to form the reduced graph $(G \setminus B, s_{G \setminus B}, d_{G \setminus B}, u')$, where $s_{G \setminus B}, d_{G \setminus B}$ are the adjusted peaks of supply and demand nodes respectively after fixing the flow on the agents in B.

By edge-consistency of the rule, the allocation $z'_e = z'_e(-B)$ for all $e \in G \setminus B$. From the discussion above, the report of every agent $e \in G \setminus B$ is such that $u'_e \geq z_e$. Also, note that $z_e < u_e \;\; \forall \;\; e \in G \setminus B$.

By increasing the capacity of an unsaturated edge, the total value of the maximum flow does not change and the bottleneck points remain the same when the edge fair rule is applied to components $(G\backslash B, s_{G\backslash B}, d_{G\backslash B}, c')$ and $(G\backslash B, s_{G\backslash B}, d_{G\backslash B}, c)$. Hence, every agent $e \in A$ receives the allocation $z'_e = z_e$. ∎

We next turn to equity properties of allocations and allocation rules. Given that different edges may connect possibly different suppliers and demanders who may have supply or demand different amounts of the commodity, one has to be careful in formulating these notions. Following Bochet et al. [12], we formulate these properties for a pair of agents (equivalently, edges). In general these properties take the following form: Fix a problem (G, s, d, u), and consider the allocation z given by a rule φ. For every pair of edges e and e', fix the flows on all edges other than e and e' and ask if there is a "better" feasible flow in $\mathcal{F}$.

An allocation is envy-free if whenever e prefers $z_{e'}$ to z_e (for some agents e and e'), there is no other allocation $\hat{z} \in \{z' \in \mathcal{F} \mid z'_f = z_f, \;\; \forall f \neq e, e' \;\}$ such that e prefers $\hat{z}$ to z. An allocation z satisfies equal treatment of equals (ETE) if for each e and e' with $u_e = u_{e'}$, there is no other allocation $\hat{z} \in \{z' \in \mathcal{F} \mid z'_f = z_f, \;\; \forall f \neq e, e' \;\}$ with $|\hat{z}_e - \hat{z}_{e'}| < |z_e - z_{e'}|$.

The following result shows the relationship between these two properties.

Proposition 4 *Consider the problem (G, s, d, u) and an allocation rule z that makes a selection from the Pareto set $\mathcal{F}$. If z is envy-free it satisfies ETE.*

Proof. Suppose the rule z violates ETE, we would like to show it violates No Envy or the flow is $\notin$ PO*. Fix a profile R^E and two edge agents e and e' such that $u_e[R_e] = u_{e'}[R_{e'}] = c^*$ and suppose there exists z' satisfying the definition above . Now, we have that, $z'_e + z'_{e'} = z_e + z_{e'}$ because z and z' coincide on $E\backslash\{e, e'\}$. Assume without loss $z_e(R) < z_{e'}(R)$, then only two cases are possible: $z_e(R) < z'_e \leq z'_{e'} < z_{e'}(R)$ or $z_e(R) < z'_{e'} \leq z_e < z_{e'}(R)$. Assume first case: $c^* \geq z_{e'}(R)$ implies a violation of No Envy. Now in case (ii), the allocation $z''_e = \frac{z_e + z'_e}{2} \;\; \forall ij \in E$ is such that $z''_e \in PO^*$ and we are in case (i) again. ∎

By construction, the edge-fair rule selects a maximum flow allocation from the Pareto set. The edge-fair rule also finds an envy-free allocation. Define the set of agents $A := \{e|e \in E, z_e > 0\}$,

$B := \{e | e \in E, z_e = 0\}$, $E = A \cup B$. If $z_e > 0$ under the edge fair rule, then the agent e carries a *positive* flow in some maximum flow solution. Similarly, $e \in B$ do not carry a positive flow in any maximum flow solution. So even if $z_e R_{e'} z_{e'}$ for some $e' \in B, e \notin B$, there is no maximum flow solution in which $z_{e'} > 0$ to possibly redistribute and improve the allocation of agent e'. On the other hand, $e', e \in A$ implies e is in a higher bottleneck set than e' since the allocation rule is monotone. Suppose, there is envy through the solution z', consider the solution $\frac{z'+z}{2}$, which is still feasible because the set $\mathcal{F}$ is convex. This is a contradiction to the earlier obtained solution of the LP at the step when e' was a bottleneck. Hence, the edge fair rule satisfies no envy in this model and thus treats equals equally.

3.5 Further Work

- **Extending the Uniform Rule:** In the language of Moulin and Sethuraman [43], an allocation rule φ on a bipartite network (G, V, E) is said to be an extension of the Sprumont's Uniform rule if φ coincides with the allocation of uniform rule if G is a network with unit demander (supplier) connected to multiple suppliers (demanders).

  The notion of "extending a rule" is such that we are not compromising on properties of existing fair allocation mechanisms. Instead, we are actually generalizing them in a suitable way to more general network structures. Moulin and Sethuraman [43] study a broad class of extensions of well known basic rules but they do not consider strategic issues.

  Both Egalitarian and Edge fair are extensions of the uniform rule whereas edge fair is a consistent extension of the uniform rule. The edge fair mechanism is also peak group strategyproof. It will be an interesting to understand the structure of mechanisms which are peak group strategyproof, consistent and also extends the uniform rule.

- The egalitarian mechanism is link groupstrategyproof w.r.t. to coordinated misreport among agents on the same side. We conjecture here that the edge fair mechanism might also be attractive as a link groupstrategyproof mechanism.

- Does the properties discussed in this work about the edge fair mechanism characterize it? If not, what are the other interesting properties of this mechanism that makes it unique?

Chapter 4

General Networks

4.1 Introduction

The study of fair allocation on economic networks has gained a lot of importance in recent years with growing applications in many public policy domains like fair exchange of kidneys among patients [49], matching students to public schools [1], matching cadets in ROTC [55] etc. The common theme in all these problems is that participating agents supply/demand a single unit of an indivisible good and the set of agents with whom they can exchange or transact this good is modeled by a link. The focus is to identify fair and strategyproof allocation mechanisms for the agents in the network and thus is in very similar spirit of the classical marriage problem of Gale and Shapley [29].

On the other hand, Bochet et al. [11, 12] study the problem of fair division of a maximum flow in a capacitated bipartite network. Their model generalizes and studies the exchange of divisible goods on a economic networks when agents have arbitrary supply/demand constraints. The aforementioned problems were typically unit supply/demand model. The common feature in all these models is that the associated market is moneyless, so that fairness is achieved by equalizing the allocation *as much as possible*. This last caveat is to account for additional considerations, such as Pareto efficiency and strategyproofness, that may be part of the planner's objective.

A special case of the problem studied by Bochet et al. [11, 12] problem is that of allocating a single resource (or allocating the resource available at a single location) amongst a set of agents with varying (objectively verifiable) claims on it. This is the special case when there is a single supply node that is connected to every one of the demand nodes in the network by an arc of large-enough capacity. If the sum of the claims of the agents exceeds the amount of the resource available, the problem is a standard rationing problem (studied in the literature as the "bankruptcy" problems or the "claims" problems). There is an extensive literature devoted to such problems that has resulted in a thorough understanding of many natural methods including the *proportional* method, the *uniform gains* method, and the *uniform losses* method. A different view of this special case is that of allocating a single resource amongst agents with single-peaked preferences over their net consumption. Under this view, studied by Sprumont [56], Thomson [60] and many others, the goal is to design a mechanism for allocating the resource that satisfies appealing efficiency and equity properties, while also eliciting the preferences of the agents truthfully. The *uniform rule*, which is essentially an adaptation of the uniform gains method applied to the reported peaks of the agents, occupies a central position in this literature: it is strategy-proof (in fact, group strategy-proof), and finds an envy-free allocation that Lorenz dominates every other efficient allocation; furthermore, this rule is also *consistent*. A natural two-sided version of Sprumont's model has agents initially endowed with some amount of the resource, so that agents now fall into two categories: someone endowed with less than her peak is a potential demander, whereas someone endowed with more than her peak is a potential supplier. The simultaneous presence of demanders and suppliers creates an opportunity to trade, and the obvious adaptation of the uniform rule gives their peak consumption to agents on the short side of the market, while those on the long side are uniformly rationed (see [37], [8]). This is again equivalent to a standard rationing problem because the nodes on the short side of the market can be collapsed to a single node. The model we consider generalizes this by assuming that the resource can only be transferred between certain pairs of agents. Such constraints are typically logistical (which supplier can reach which demander in an emergency situation, which worker can handle which job request), but could be subjective as well (as when a hospital chooses to refuse a new patient by declaring red status). This complicates the analysis of efficient (Pareto optimal) allocations, because short demand and short supply typically coexist in the same market.

As mentioned earlier, Bochet et al. [11, 12]. work with a bipartite network in both papers and assume that each node is populated by an agent with single-peaked preferences over his consumption of the resource: thus, each supply node has an "ideal" supply (its peak) quantity, and each demand node has an ideal demand. These preferences are assumed to be private information, and Bochet et al. [11, 12] propose a clearinghouse mechanism that collects from each agent only their "peaks" and picks Pareto-optimal transfers with respect to the reported peaks. Further, they show that their mechanism is strategy-proof in the sense that it is a dominant strategy for each agent to report their peaks truthfully. While the models in the two papers are very similar, there is also a critical difference: in [12], the authors require that no agent be allowed to send or receive any more than their peaks, whereas in [11] the authors assume that the *demands* must be satisfied exactly (and so some supply nodes will have to send more than their peak amounts). The mechanism of Bochet et al.—the *egalitarian* mechanism—generalizes the uniform rule, and finds an allocation that Lorenz dominates all Pareto efficient allocations. Later, Chandramouli and Sethuraman [15, 16] show that the egalitarian mechanism is in fact strongly invariant, peak and link group strategyproof: it is a dominant strategy for any group of agents (suppliers or demanders) to report their peaks truthfully. Szwagrzak [57] studies the property of contraction invariance of an allocation rule: when the set of feasible allocations contracts such that the optimal allocation is still in this smaller set, then the allocation rule should continue to select the same allocation. Szwagrzak shows that the egalitarian rule is contraction invariant. These results suggest that the egalitarian mechanism may be the *correct* generalization of the uniform rule to the network setting.

The egalitarian rule of Bochet et al. [11, 12] constructs a fair allocation for the agents in a bipartite network with divisible goods. This rule is a generalization of the well known Sprumont's [56] uniform rule. On the other hand, when the goods are, Klaus et al. [25] proposed the probablistic uniform rule in the single agent model. Their main contribution is the fact that there is no net utility loss for agents in both divisible/indivisible models. The fractional part of an agents allocation (expected utility from this mechanism) is the probability with which he has a claim on an extra unit of good. The mechanism of Klaus et al. [25] mechanism is based on a simple idea that at each bottleneck point of the Sprumont's model, we randomize over all possible feasible allocations. Similarly, we define an egalitarian mechanism for indivisible goods. The idea is to randomize over all possible feasible flows at each bottleneck. The expected utility from the indivisible goods case is exactly the same as in the divisible goods case. Our research is motivated by these results. From the results above, the generalized model of allocation of divisible goods on bipartite networks is well understood. This result can be viewed as a generalization of identifying a fair maximum matching on general networks to that of identifying a fair maximum b-matching problem on non-bipartite networks. Our contributions parallel the contribution of Bochet et al. [11, 12] in generalizing unit capacity models.

Specifically, we are given a non-bipartite network $G = (N, E)$, and we think of N as the set of agents involved in this network. Each arc $(i, j) \in E$ connects two participating agents and has capacity $u_{ij} \geq 0$. There is a single commodity (the resource) that is available at each node and needs to be exchanged with the neighbors on the network: we assume that node i has b_i units of the resource. The capacity of an arc (i, j) is interpreted as an upper bound on the direct transfer from agent i to agent j. The agents derive utility whenever they exchange a good with their neighbors. The goal is to find a maximum "fair" exchange among participating agents, while also respecting the capacity constraints on the arcs.

The rest of the paper is organized as follows: in Section 3.3 we consider the divisible goods case and in Section 3.4 we consider the indivisible goods case.

4.2 Egalitarian Mechanism (Divisible goods)

In this section we consider the version of the problem where the nodes of the network are populated by agents. Each node has a specific number of units of a particular good that it can exchange with the agents it is connected to. Thus, our problem becomes one of exchanging a single commodity among the set of agents V using the set E of edges.

An exchange of the commodity among the agents is realized by a flow f, which specifies the amount of the commodity exchanged among the agents i and j using the edge $(i, j) \in E$. The flow f induces an allocation vector for each agent as follows:

$$\text{for all } i \in V : x_i(f) = \sum_{j \in N(i)} f_{ij}; \tag{4.1}$$

where $N(i)$ is the neighbors of agent i in network G. As we shall see in a moment, agents only care about their *net* transfers, and not on how these transfers are distributed across the other agents.

In the following sections, we assume that the peaks of the agents are fixed, and focus our attention on mechanisms that asked each agent to report the agents they are connected to and maps the reports to a profile to a unique b-matching. Our focus will be on mechanisms in which no agent has an incentive to misreport his compatible neighbors and also efficient.

Allocation Rules

We define an allocation on the edges as f_{ab} $, ab \in E$ as feasible, if $f_{ab} = f_{ba}$ $\forall ab \in E$ and the flow induced on the nodes by f (as x defined earlier) and for feasibility, $x_a \leq b_a$ $\forall a \in V$. Any rule which picks an allocation from this set is called a feasible allocation rule. In the rest of the section, we discuss the egalitarian rule for this model.

Given a network (G, V, E), we make the following transformation to construct the bipartite network (G_b, V_b, E_b). We represent $V_b = A \cup B$ where $A = V$ and $B = V$. are the two sides of the bipartite network. There is an edge between agent $i \in A$ and $j \in B$ only if there is an edge ij in the given network G. Connect the agents in A to a supply node s and the agents in B to a sink node t. We refer to as a flow, any feasible shipment of goods from the source node to the sink node.

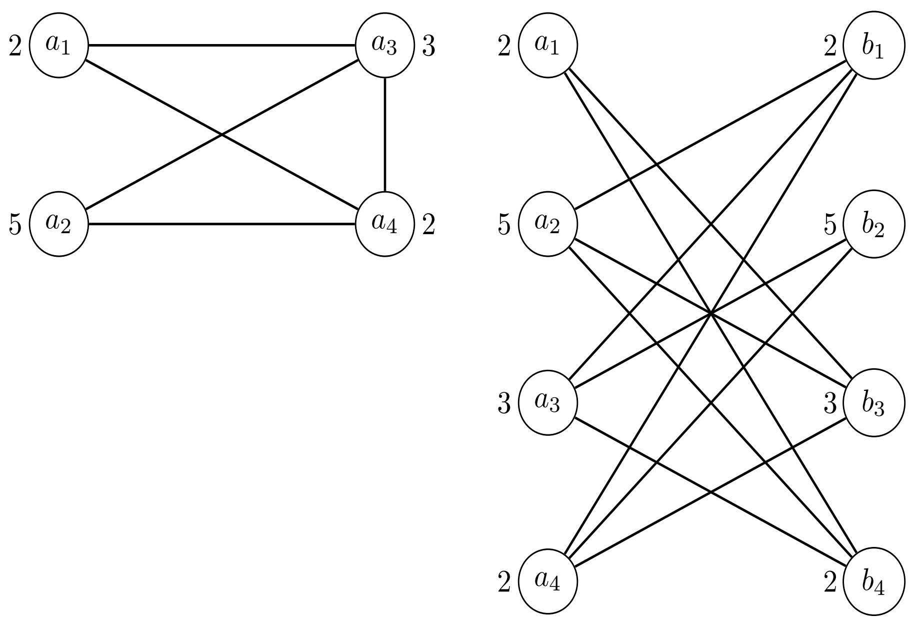

Figure 4.1: Transformation to bipartite network

Lemma 11 *Every feasible flow in the modified network G_b corresponds to a feasible allocation (exchange) in the original network G. BIMS's Egalitarian rule on the modified network G_b results* *in a Pareto optimal, Lorenz dominant, envy free and peak strategy proof allocation for the agents in the original network G.*

Proof : Consider a solution (y_a, y_b, g_{ab}) for the agents in the modified bipartite network where y_a, y_b refers to an allocation for agent $a \in A$ and $b \in B$ respectively and g_{ab} is the flow on the edge ab. We shall show that every feasible solution in the modified bipartite network maps to a solution in the original network and vice-versa. The result then follows (As Bochet et al. [11, 12] have established the fairness properties of egalitarian mechanism in bipartite networks).

Firstly, consider a feasible exchange in the original problem. Let x be the node allocation induced by the edge f in the original network. In particular, denote the solution by (x_a, f_{ab}) for agent a, b in the network. In the modified network, set $g_{a_i b_j} = g_{a_j b_i} = f_{a_i a_j} \ \ \forall i, j \in V$. Clearly, this solution is feasible in the modified network and $y_{a_i} = y_{b_i} = x_{a_i} \ \ \forall i \in V$.

Now, consider a solution (y_a, y_b, g_{ab}) in the modified bipartite network. Construct the following solution, $f_{a_i a_j} = (g_{a_j b_i} + g_{a_i b_j})/2 \ \ \forall i, j \in V$. This also defines a node allocation in the original network with $x_{a_i} = (y_{a_i} + y_{b_i})/2 \ \ \forall i \in V$. This is a feasible allocation since, $f_{ij} = f_{ji} \ \ \forall ij \in E$ and $x_i \leq b_i \ \ \forall i \in V$. ∎

The mechanism is also peak strategyproof if the preference profiles are private information of the agents. In the next section, we discuss the case of exchanging indivisible goods among agents in a network. We show that there is no mechanism that is peak strategyproof but we design a mechanism which is egalitarian in nature and retains many other attractive properties.

4.3 Egalitarian Mechanism (Indivisible goods)

In this section, we focus on allocation rules that are required to use only integral amounts on all the edges in the network i.e. we aim to identify a integer flow in the network. If the network is bipartite then there always exists an integer maximum flow. But when the network is non-bipartite there is a gap between an integer and fractional maximum flow. For an example, consider the triangle network with nodes (a, b, c) each with peak $= 1$; In the divisible goods case, a Pareto allocation would assign 1 unit to each node (0.5 units on each edge). The net social utility is 3 units in this case. If the goods are indivisible, only 1 of the 3 edges can be picked in any maximal allocation (Pareto). The net social utility is 2 units. In a fair allocation, we would pick each edge with equal probability, giving an allocation of 2/3 for each agent. Thus in non-bipartite networks there is a gap in the achievable utility between divisible and indivisible goods.

In this section we identify an egalitarian mechanism when the agents can exchange only *integral* quantities of the goods between themselves. We reduce it to an equivalent bipartite network and apply the well-known fair allocation rules on this modified network.

Given a network G and a positive integer b_i for each node $i \in V$ and a integral capacity u_{ij} for each edge $ij \in E$, a *u-capacitated b-matching* or (b, u) matching is a vector $x \in Z^V$ such that

$$x_i = \sum_{j \in N(i)} f_{ij} \text{ for all } i \in V \tag{4.2}$$

$$x_i \leq b_i \text{ for all } i \in V \tag{4.3}$$

$$0 \leq f_{ij} \leq u_{ij} \text{ for all } ij \in E \tag{4.4}$$

The case when $b_i = 1$, $\forall\, i \in V$, is the usual matching problem. Let X denote the set of all x satisfying the equations (4.2-4.4) (this is the set of all feasible b-matchings). Clearly, a given graph G can have more than one b-matching and our interest is in finding a maximum cardinality b-matching that satisfies additional properties. A maximum weight u-capacitated b-matching problem can be found in strongly polynomial time by a reduction to the maximum weight b-matching problem. Following is a linear programming formulation of the maximum weight u-capacitated b-matching problem:

$$\text{Max } \sum_i x_i \tag{4.5}$$

$$\text{subject to} \tag{4.6}$$

$$x_i \leq b_i, \ \forall i \in V(G) \tag{4.7}$$

$$x_i = \sum_{j \in N(i)} f_{ij} \ \forall i \in V(G) \tag{4.8}$$

$$f_{ij} \leq u_{ij}, \ \forall ij \in E(G) \tag{4.9}$$

$$f_{ij} \in Z \tag{4.10}$$

We restrict our attention to the case $u_{ij} = \infty$ in the rest of the thesis but the results can be easily extended to general capacity case. Given a network, a b-matching with maximum weight can be identified in polynomial time. Algorithms for finding such maximum weight b-matchings are discussed in Cook and Cunningham [22]. They also describe a matroid formulation of the b-matching problem.

We briefly summarize some of the main ideas below that are useful to our work. The pair, $(V, X) := M$ is a matroid. We will refer to it as the b-matching matroid. A base in M is a vector x that is generated by some maximum b-matching of G.

The "*rank*" function : $2^V \to Z^+ \cup \{0\}$ of the matroid M is defined for every $S \subseteq V$ as

$$rank(S) := \max_{I \subset S, I \in \mathcal{I}} |I|$$

The rank function is submodular and in our context rank(S) can be interpreted as the size of a maximum b-matching restricted to the set S. P_{rank} refers to the polymatroid associated with this particular rank function and is given by

$$P_{rank} = \{x \in R^V | x \geq 0, x(S) \leq f(S), \ \forall \ S \subseteq V\} \tag{4.11}$$

where $x(S) = \sum_{i \in S} x_i$

Our focus is on selecting a maximum b-matching such that the utility of the agents satisfy some fairness criterion.

- **Connection to the kidney exchange problem**: The pairwise kidney exchange problem of [49] and the subsequent related literature study a unit exchange problem between patients who are connected to their compatible donors through links in the network. Each agent needs exactly one unit of good (in this case, kidney). Since, not every patient is compatible with everyone else, we may not able to match every patient with a compatible donor, hence the goal is to identify maximize the total number of matches.

  This problem can be viewed as one of finding a maximum matching on a non-bipartite network. When there are multiple maximum matchings, the mechanism designer has to identify a particular maximum matching satisfying certain efficiency, fairness and incentive objectives.

Roth et al. [49] construct the “Egalitarian Mechanism” - which is a lottery mechanism over the set of maximum matchings. They show that the mechanism is strategyproof and finds a fair allocation.

In our problem each agent i in the network has b_i units and derives utility with every exchange of that good with his/her neighbors in the network.

- **Connection to the ordinal transportation problem**: Balinski and Yu [6], studied the ordinal transportation problem where the goal is to identify a stable b-matching on bipartite networks. We differ from their study in many ways. Our model is on general non-bipartite networks, our preference structure is dichotomous (agents have single peaked preferences over the net amount exchanged) and moreover, we are interested in fairness notions like Pareto optimality and envy freeness rather than stability.

The utility of an agent i is the number of times an agent is matched in a given b-matching i.e. x_i. A utility profile $U \in R^V$ is said to be a feasible utility profile if the profile is an outcome of a feasible b-matching in the network.

Lemma 12 *The set of Pareto optimal allocations coincides with the set of maximum b-matchings*

Proof: Let x denote the allocation (utility) vector of agents in a network. Then the size of the b-matching that produced this utility profile is $\frac{\sum_{i \in V} x_i}{2}$. As each exchange between 2 agents on a network adds a utility of one to each of those agents.

Suppose, a flow f associated with a maximum b-matching results in an allocation profile x. As we have a maximum b-matching, there is no augmenting path (of odd length) with respect to f. Then, x must be Pareto optimal. For otherwise, there is an allocation y that Pareto dominates x, $(\frac{\sum_{i \in V} y_i}{2} > \frac{\sum_{i \in V} x_i}{2})$ contradicting the assumption that x is a maximum cardinality b-matching.

Now, suppose x is a Pareto optimal allocation for the agents in the network induced by a flow f, Then there exists no other allocation y in the network such that every agent is weakly better off. This implies, there is no augmenting path and the given solution is a maximum b-matching. ∎

The following generalization of Gallai and Edmonds theorem further helps us understand the structure of Pareto optimal solutions.

Partition V as $\{V^U, V^O, V^P\}$ such that

$$V^U = \{i \in V : \exists x \in X \;\; s.t. \;\; x_i < b_i\} \tag{4.12}$$

$$V^O = \{i \in V \backslash V^U : \exists \text{set of agents } \tilde{j} \in V^U s.t. \sum_{j \in \tilde{j}} f_{ij} = b_i\}, \tag{4.13}$$

$$V^P = V \backslash (V^U \cup V^O) \tag{4.14}$$

where $f_{ij} = 1$ if i, j are matched in a feasible solution. V^U is the set of agents unmatched ($x_i < b_i$) in atleast one maximum b-matching. V^O is the set of agents perfectly matched ($x_i = b_i$) in every maximum b-matching and have atleast 1 neighbor in V^U. V^P is the set of agents who are again perfectly matched but do not have a link with any agent in V^U. In lemma, 13 below, we discuss precisely how to obtain these components.

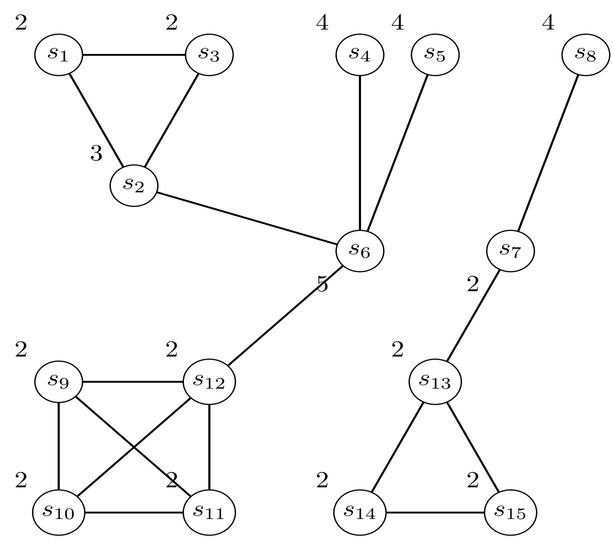

Figure 4.2: GED of a non-bipartite network with arbitrary peaks; In this figure, $\{s_6, s_7\} \in V^O, \{s_1, s_2, s_3, s_4, s_5, s_8\} \in V^U, \{s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_{15}\} \in V^P$

Let $I \subseteq V$ and $N(I) = \{j | \; ij \in E, i \in I\}$. Then $(I, N(I))$ is a reduced sub problem of the original problem. Define $i^j := \{j | f_{ij} > 0\}$

Lemma 13 *Let $I = V \backslash V^0$ and let f be a Pareto efficient flow for the original problem ($\mu(i)$ is the set of neighbors that is matched to an agent i), Then*

- *For any agent* $i \in V^0$, $i^j \subseteq V^U$

- *For any even component*[1] $(J, N(J))$ *such that* $J \subseteq V^P$ *and for any agent* $i \in J, i^j \subseteq J \backslash i$

- *For any unsaturated component* $J(|J| \geq 2)$, *the maximum size of a b-matching within every unsaturated component is* $\Sigma_{j=1}^{|J|} b_j - 1$. *Moreover, for any unsaturated component* (J, R_J), *either*

  - *exactly one agent* $k \in J$ *sends flow to an agent in* V^O *under the Pareto efficient flow* f *whereas all remaining agents in* J *exchange within themselves: for any other agent* $i \in J \backslash \{k\}$, $i^j \subseteq J$ *or*

  - *exactly one agent* $k \in J$ *remains unsaturated under the Pareto-efficient flow* f *whereas all the remaining agents in* J *are saturated so that* $i^j \subseteq J$ *for any agent* $i \in J \backslash k$

Proof: Refer to the appendix

A mechanism is *deterministic* if for a given network, it picks a maximum b-matching as an outcome from the set of maximum b-matchings. Roth et al. [49] study priority mechanisms in the context of deterministic mechanisms. They also show establish that the randomized egalitarian mechanism is superior in the sense that it is not only efficient and strategyproof but it also produces an outcome that is envy free and strongly efficient in the sense of Lorenz dominance. The mechanism of Roth et al. is a "lottery" mechanism, that randomly picks a maximum b-matching from the Pareto optimal set. The distribution or lottery is chosen in such a way that the aforementioned economic properties are exhibited by the mechanism. We define this more precisely below.

Lottery Mechanism: Let $\mathcal{F}$ be the set of feasible flows (recall each flow induces a b-matching)[2] in G. A matching lottery $l : P_f, f \in \mathcal{F}$ is a probability distribution over $\mathcal{F}$. For every flow $f \in \mathcal{F}$, P_f is the probability of choosing a flow f in lottery l, and $\sum_{f \in \mathcal{F}} P_f = 1$. A matching lottery l can be also viewed as a fractional b-matching which is defined to be a convex combination of several integral b-matchings. Let $\mathcal{L}$ be the set of matching lotteries. Given $l \in \mathcal{L}$, define the utility x_i^l of vertex i to be the expected total exchanges that is involving i , i.e., $x_i^l = \sum_{f \in \mathcal{U}} l_f x_i^f$. The expected

[1] The definition of odd and even components will become more clear in the proof

[2] Hence depending on the context we will use f to denote either a flow or the corresponding induced b-matching

utility induced by lottery l to be the vector $x_l = (x_i^l), i \in V$. Let $P = x_l, l \in \mathcal{L}$ be the set of all feasible utility profiles.

It is now clear from our definition that a feasible utility profile can be understood as a fractional b-matching (a convex combination of integral b-matchings). The P_{rank} polymatroid we defined earlier is exactly the set of feasible utility profiles.

Roth et al. [49] described the egalitarian Mechanism for the pairwise kidney exchange problem. Li et al. [39] develop a water filling algorithm which gives a simpler and intuitive proof of the Roth's mechanism. Here, we generalize the ideas of Li et al. to our model. The main idea is that the allocation problem on a general network can be reformulated as an equivalent problem on a *bipartite* network. Once this is established, we could use the familiar allocation rules for bipartite networks to find an allocation for the original non-bipartite network. We describe this procedure below in more detail.

Step 1: *Transformation to an equivalent bipartite network:*

Construction of Nodes: Define $\mathcal{C} = \{C_1, C_2....C_k\}$ as the set of unsaturated components in the underdemanded component V^U where $k = |C|$. Construct the following bipartite graph $G_B = (A, B, E)$ where each node in A corresponds to a node in V. We use $A^{C_1}, ..., A^{C_k}$ to denote k disjoint sets corresponding to $C_1, C_2, ..C_k$ respectively. We use a_i as a label for node $i \in A$. In particular, let $A^{PO} = \{a_i | a_i \in V^P \cup N^O\}$; $A^U = \{a_i | a_i \in V^U\}$.

The construction of B is as follows. B Can be partitioned into 3 parts: $B^{PO} \cup B^O \cup B^C$. Each node in B^{PO} corresponds to a node in $B^P \cup B^O$. We use $c_i \in B^{PO}$ to denote the node corresponding to $i \in B^P \cup B^O$. Each node in B^P corresponds to a node V^P. We use $c'_i \in B^O$ to denote the node corresponding to $i \in V^P$. B^C consists of k disjoint sets $B^{C_1}, B^{C_2}....B^{C_k}$, where B^{C_i} contains a node with value $\Sigma_{j \in C_i} b_j - 1$ if and only if $|B^{C_i}| \geq 2$; B^{C_i} is empty otherwise[3]

Construction of Edges: (1) For each $i \in V^P \cup V^O$, we have edge $(a_i, c_i) \in E$ where $a_i \in A^{PO}$ and $c_i \in B^{PO}$; (2) For a vertex $a_i \in A^C$ and another vertex $c'_j \in B^O$ we have $(a_i, c'_j) \in E$ if $ij \in E$; (3) For each vertex, $a \in A^{C_i}$ add an edge to the node in B^{C_i}.

---

[3]This is the size of a maximum b-matching within a particular odd component

Step 2: On the transformed network, run the egalitarian mechanism for indivisible goods in chapter 2 [section 2.4.1] to find the final allocation.

Step 3: *Construct a randomized mechanism: lottery over b-matchings:* The outcome of the egalitarian mechanism is a fractional utility profile for the agents. We show later that such a utility profile can be obtained as a lottery over integral b-matchings. We generate a maximum b-matching from this probability profile and conduct exchanges on the network.

Example: Consider the original network as given in figure 4.2, The corresponding bipartite transformation outlined in step 1 is shown in figure 4.3. When the egalitarian mechanism is applied to this bipartite network, we get the following utility profile for the agents: Every agent $i \in V^P \cup V^O$ receive their peak allocation. The agents s_1, s_2, s_3 receive 2 units of utility each. The utility of agents s_2, s_4, s_5 is $\frac{7}{3}$ each. This fractional utility is obtained as a lottery over integral b-matchings. In this case, 2 units of agents s_4, s_5 are always exchanged with agent s_6. Agent s_2 has to exchange 2 units within its odd component in every maximum b-matching. Agents s_2, s_4, s_5 compete for the extra unit of exchange that s_6 can do. The egalitarian mechanism picks any of the possibility with a probability of $\frac{1}{3}$.

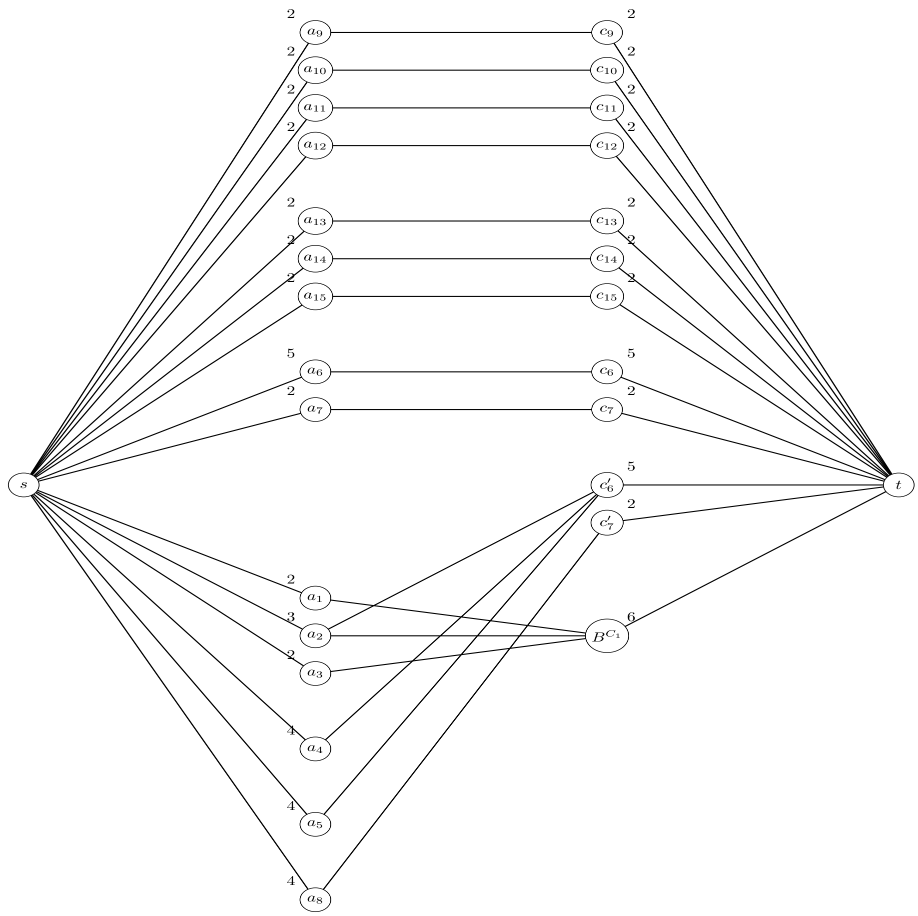

Figure 4.3: Transformation into bipartite network of the original graph in Figure 4.2; a_i refers to s_i

Next, we describe a more mathematical programming approach of obtaining the Egalitarian allocation. We do this by representing the model through a sequence of linear programming problems and solving them iteratively. Such an approach not only builds an alternate framework of modeling the problem but also sheds insights on many economic properties of the mechanism as we shall see later.

Egalitarian Mechanism: A Linear Programming Approach

(1) Step 1. Solve the linear program LP_1

$$\text{Maximize } \lambda_1$$

$$\text{subject to}$$

$$x_v = \lambda_1, \forall v \in V,$$

$$x \in P_{rank}$$

We call an element $v \in V$ a tight element if x_v participates in some tight constraint in P_{rank}. Let D_1 be the set of tight elements. In other words, increasing x_v would violate some constraint in P_{rank} when other x_u is fixed for $u \neq v$ are fixed. In many linear programming algorithms, we can easily detect such tight elements. Another way to test the tightness of an element is to solve a closely related linear program in which we fix other x_u and maximize x_v.

(2) In general, at step k, we solve the linear program LP_k

$$\text{Maximize } \lambda_k$$

$$\text{subject to}$$

$$x_v = \lambda_j, \ \ \forall\, v \in D_j, \ \ \forall\, j < k,$$

$$x_v = \lambda_k, \forall v \in V \setminus \cup_{j<k} D_j,$$

$$x \in P_{rank}.$$

Let D_k be the set of elements that become tight in this step.

(3) The algorithm stops with $x = x_v = \lambda_j$ for $v \in D_j$ if $\cup_{j=1}^{k} D_j = V$

Lemma 14 *Every feasible maximum flow on the transformed bipartite network is equivalent to a feasible utility profile in the original network*

Proof: A utility profile in the original network is induced by a maximum b-matching f or a lottery over $\mathcal{F}_z$. It is sufficient to prove the statement only for integral feasible flows as any other maximum flow can be obtained as a linear combination of these integral flows.

Given a utility profile x induced by a maximum b-matching, we will construct an equivalent maximum flow, f' in the transformed bipartite network. Since each agent $i \in N^P \cup N^O$ is saturated, set $f'_{ij} = b_i$ on all edges $ij, i \in A^{PO}, j \in B^{PO}$. Now, focus on each component C_k: if C_k has only one node i with utility x_i, then this utility is derived only by exchanging with the agents in B^O. Let $f'_{ij} = f_{ij}\ \forall j \in N(i) \cap B^O$. We know in any C_k, $|C_k| - 1$ exchanges are made within the same component, so a flow of $|C_k| - 1$ can be obtained by sending f'_{ij} units of flow to the agents in B^{C_k}. From the GED, at most one agent in a particular odd component is matched with an agent in V^O. Also, If $f_{ij} > 0$ for an agent $i \in V^C$ and $j \in V^O$, then the node i is saturated in that particular b-matching: for such $ij \in E(G)$ send a unit flow $f'_{ij} = 1$ in the bipartite network. Agent i is also saturated in the transformed bipartite network.

We now prove that every integral maximum flow in the bipartite network corresponds to a feasible utility profile in the original network. Consider a maximum flow of $|F|$ in the modified bipartite network. This is also the size of the maximum b-matching in the original network. For a given set A^{C_i} with $|A^{C_i}| \geq 2$, there is at most one vertex $a_j \in A^{C_i}$ such that there exists a vertex $c'_h \in B^O$ with $f'_{a_j c'_h} = 1$. In this case, we include it in the b-matching $f_{ij} = 1$. In any maximum flow, for each C_i, agents in A^{C_i} send $|C_i| - 1$ units of flow to B^{C_i}. Consequently from the GED lemma, $|C_i| - 1$ vertices can be matched among themselves. There are exactly $\sum_{j \in V^O} b_j$ units exchanged by agents in A^C with agents in V^O. Each such exchange constitutes one unit flow to B^O. So all vertices in V^O are saturated. From GED Lemma, we can match all vertices in V^P among themselves in f. It is easy to see that x is exactly the utility profile corresponding to f'.

Corollary 1 *The outcome of an egalitarian mechanism is a maximum flow allocation f (not necessarily integral). Such a maximum flow can be obtained as a convex combination of integral maximum flows ($f = \sum_{f' \in \mathcal{F}_z} \lambda_{f'} f'$).*[4] *Setting $P_{f'} = \lambda_{f'}$ for all $f' \in \mathcal{F}_z$ we obtain a probability distribution over the set of integral b-matchings and a lottery based on λ induces a feasible utility profile in the original network. This is the familiar Birkhoff - Von Neumann theorem in combinatorial optimization.*[5]

[4] $\mathcal{F}_z$ is the set of integral maximum flows

[5] For more details on the Birkhoff - Von Neumann theorem, refer to [2]

Lorenz Dominance & No Envy: As discussed in Klaus [25], No envy and ETE are not equivalent here. It follows from the above discussion that the egalitarian mechanism applied to this bipartite network yields a feasible utility profile. Bochet et al. [12] have established the egalitarian mechanism is Lorenz dominant among all feasible flows. No Envy also follows from the allocation rule of Bochet et al. [11, 12]

Consistency and Extension of rules: As discussed in Chandramouli and Sethuraman [16], the egalitarian rule is an extension of the uniform rule that is not consistent. They propose the edge fair rule which is a consistent extension of the uniform rule to bipartite networks. In the previous section, if after the transformation into bipartite network, if we apply the edge fair mechanism to that network, we would have a consistent extension of the uniform rule to the non-bipartite indivisible goods network.

In the language of Moulin and Sethuraman [43], an allocation rule φ on a bipartite network (G, V, E) is said to be an extension of the Sprumont's Uniform rule if φ coincides with the allocation of uniform rule if G is a network with one demander (supplier) connected to multiple suppliers (demanders). i.e.

$$\varphi_i = U_i, \ \ \forall \ \ i \in V \tag{4.15}$$

where U_i is the utility of agent i under uniform rule.

In this spirit, Bochet et al. [11, 12], Chandramouli and Sethuraman [15, 16] develop mechanisms that are extensions of the uniform rule. Moulin and Sethuraman [43] study a more general class of extensions of some basic well known rules. We extend this definition further to general non-bipartite networks. An allocation rule x on a general network (G, V, E) is said to be an extension of the egalitarian rule φ if x coincides with ϕ if the network G is bipartite i.e. $x_i = \varphi_i, \ \ \forall \ \ i \in V$

Lemma 15 *The egalitarian rule described for non-bipartite networks is an extension of the probabilistic egalitarian rule for bipartite networks*

Proof: If the network is bipartite, the odd components set, V^U forms an independent set. So, V^U is a collection of nodes (with peaks b_i) each of which is not saturated in at least one maximum b-matching. Hence, the set B^U is empty and each agent $i \in V^U$ is only connected to agents in V^O who are completely saturated. In the bipartite context, if i were a supplier then, any agent

$j \in N(i)$ has $x_j = b_j$ in all Pareto allocations. It is clear from the analysis in section 2.4.1 that if the given network G is bipartite, then the Step 1 of our algorithm which transforms the given network into a equivalent bipartite network, outputs G. As we apply the probabilistic egalitarian rule in step 2 of our algorithm, the result follows. ∎

Strategic Issues

Peak Strategyproofness: In the discussion so far, we have ignored the possibility of agents having control over the values b_i. Suppose agents report b_i to the mechanism designer who then decides the final allocation, then the agents could misreport the peaks to improve their allocation which is always $\leq$ to the reported b. If the agents report $b_i' > b_i$, then there is a possibility of agents having an allocation more than his true b_i. In that case, we need to know the full preference relation of an agent to compare his utilities when his allocations are on either side of his true peak b_i.

Single peaked preferences: In the spirit of Bochet et al. [11, 12], we would like to continue our assumption of single peaked preferences for the agent allocations. Mathematically, given a preference profile R_i for agent i and two possible allocations x_i, x_i' then:

$$x_i' < x_i \leq p[R_i] \implies x_i P_i x_i' \tag{4.16}$$

$$p[R_i] \leq x_i < x_i' \implies x_i P_i x_i' \tag{4.17}$$

A mechanism is *peak strategyproof* if it is a dominant strategy for the agents to reveal their peaks truthfully. Given this preference structure, the following example shows that there is no peak strategyproof mechanisms in the set of Pareto optimal allocations. Consider agents (a, b, c) connected to each other and peak $b_i = 1$ for all the agents. If the agents report their true peaks, then any allocation mechanism is such that $\{(x_a, x_b, x_c) | x_a + x_b + x_c \leq 2\}$. Suppose lets say $x_a < 1$. Suppose agent a misreports his peak $b_a = 2$, then we have a unique maximum b-matching and the allocation is $(2, 1, 1)$. If $2 P_a x_a$ for agent a, he improves his allocation.

Link Strategyproofness: A mechanism is *link strategyproof* if it is a dominant strategy for the agents to reveal all of his/her neighbors. Roth et al. [49] established the link strategyproofness of the egalitarian mechanism in the pairwise kidney exchange problem. The egalitarian mechanism is not link group strategyproof even in the kidney exchange problem. To see that, consider the network in Figure 4.4 where each node has 1 unit of good to exchange. Agent s_4 is matched in every maximum matching, but s_7 is missed in some maximum matching and receives a utility strictly less than 1 in the egalitarian allocation. Now, s_4 and s_7 can coordinate and not report about the existence of the link between s_4 and s_3. If that link is not reported, then, (s_4, s_5, s_6, s_7) form a separate group and all the agents are matched and receive peak utility but agent s_7 improves his allocation. In this case, however, s_4 is indifferent.

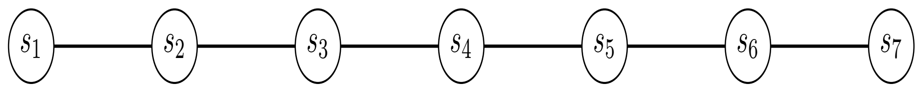

Figure 4.4: Egalitarian mechanism is not link gsp in the strong sense

Weakly link groupstrategyproof: A mechanism is weakly link group strategyproof if in a deviating coalition group, every agent strictly improves his/her allocation.

The agents in over demanded and perfectly demanded components can be ignored because they already receive their peak allocation ($x_i = b_i$). The only deviating subsets are those agents in the odd components. Hence, for the rest of the proof we restrict our attention to coalition groups which consists only of agents in the odd components. We may assume without loss of generality that any given agent in V^{PO} finds all the agents in odd components acceptable: if a agent $j \in V^{PO}$ finds an odd component agent $i \in V^U$ unacceptable, then agent i cannot have a link to demander j regardless of his report, so clearly i's manipulation opportunities are more restricted. Recall that in a bipartite network [chapter 2], we established that the egalitarian mechanism is link group strategyproof when manipulations are restricted to agents on one side only. The situation here is similar where only the agents in V^U misreport their connectivity to the agents in V^{PO}. Using the same proof technique as in chapter 2 [refer to theorem 1], it follows that the egalitarian mechanism is weakly link groupstrategyproof.

4.4 Further Work

- The main contribution of this work is a generalization of the Roth et al. [49] framework to that of a network with arbitrary node capacities. It will be interesting to see how much of the rich literature that followed the work of Roth et al. [49] on matchings would also generalize. Some possible avenues include the dynamic exchange of multiple objects between agents, ordinal preferences on the agents/objects, multi-way exchanges etc.

- In the current work, the agents are on the nodes. In contrast, in the model of Kalai and Zemel [34] the agents are on the edges of $s-t$ flow model and the total surplus if the value of the maximum s-t flow. The agents wish to share this surplus in a fair way. Kalai and Zemel construct a surplus sharing rule that is in the core. But in this core, only the agents in a minimum cut have a positive surplus. Is there a more "fairer" away of sharing the surplus created by the agents in such a network?

- In Section 4.3 we discussed how there is no peak strategyproof mechanism when the preferences are single peaked. Are there other preference domains under which one could construct peak strategyproof mechanisms?

Chapter 5

Stable and Balanced Maximum Flows

5.1 Introduction

Network exchange theory has long been an important subject of study in economics, operations research and sociology. The goal is to determine how an agents location in a network influences his ability to negotiate for resources [21]. Social and economic networks are even more popular than before with the emergence of online media like Facebook, Twitter etc. Someone who has a better connectivity has access to more and wider information and enjoys an important presence in the network. When each connection generates a surplus for the two agents involved, a natural question is how this surplus is divided between the agents. If the agents are strategic then certain surplus sharing rules might give some agents an incentive to form their own network. Our goal is to understand and design surplus sharing rules in which this does not happen. The framework of study that we follow in this note is that of a bargaining solution. It was initially introduced by Nash [45] in a multi agent model trying to share a particular resource. In the assignment games model of Shapley and Shubik [53], the network is bipartite and the agents on the nodes and these agents can form at most one contract with any of its neighbors. Shapley and Shubik show that (i) Any stable solution divides a maximum weighted matching in the underlying network; (ii) Stable solutions can be computed in polynomial time as a set of inequalities; Kleinberg and Tardos(KT) [38], extended the work to general networks (bipartite or non-bipartite). The capacity of the nodes are all identically 1 and agents cannot have fractional exchanges. The main contributions of Kleinberg and Tardos [38] is that within the set of stable outcomes, a *balanced* outcome exists and the set of all balanced solutions can be enumerated in polynomial time.

The model consists of an undirected graph $G = (V, E)$ with edge capacities $u_{ij}, ij \in E(G)$ and positive, integral vertex capacities $b_i, i \in V(G)$. The vertices represent the agents, and the edges represent the pairwise contracts that the corresponding agents can be involved in. Kleinberg and Tardos [38] and later Koenemann et al. [27] allow each edge to be picked only once but allow for each edge to have a weight which represents the total value of the corresponding contract. These weights might account for the relative importance of a particular connection over an other. In our model, we allow each agent to share any amount of a divisible homogeneous good on each edge. If a contract is formed between two vertices, this share (flow) is divided between them, whereas if the contract is not formed neither vertex receives any profit from that edge. The capacity of each agent (b_i) along with the edge capacity limits the total amount of flow that is shared between the agents on a network. It is clear from this description, that the amount of flow that an agent can share or send in the network is limited by his connectivity, his/her vertex capacity and also the capacity of the edges connected to him. An agent with a better connectivity has a better bargaining power in the network.

A solution in our model is defined in the following way: First, identify a feasible flow (or exchange), f, in the given network. A feasible flow is an exchange among the agents in the network that respects the node and edge capacities. Next, the solution to divide the flow amongst the agents who contributed to the flow i.e. the flow f_{ij} on an edge ij is shared among agents i and j. Since an agent incident to many edges, his node allocation is obtained by adding his share on every edge.

In the network bargaining literature, fairness is mainly studied by finding a stable and balanced outcome [38]. A solution is *stable* if the share an agent earns from a flow on a particular edge is at least as much as his/her *outside option*. An agents outside option, in this context, refers to the maximum profit that the agent can receive by rerouting a fraction of the flow to one of his neighbors, under the condition that the new flow would benefit both agents involved. The notion of a balanced outcome as studied by Kleinberg and Tardos [38] generalizes the notion of a Nash bargaining solution to the network setting. In a balanced solution the value of each contract is split according to the following rule: both endpoints must earn their outside options, and any remaining surplus is to be divided equally among them. This notion of *balanced* is fair in the sense that whenever we divide the flow of a particular contract, we make sure both agents receive at least their outside option. Moreover, the division of surplus is not dictatorial, i.e. it is divided equally among them. The division of the flow between the agents i and j is such that the quantity[1] that i gets over its outside option is the same as that for agent j.

This model is similar to the ones we considered in the last three chapters of the dissertation. The agents are again on the vertices of a bipartite network, the edges represent their compatible partners to whom they can send flow. The main difference is the way we model how the agents derive utility from the flow they send to their neighbors. Suppose S and D represent either sides of the bipartite network, and say an edge ij carries a flow f_{ij} i.e. agent $i \in S$ send f_{ij} units of flow to an agent $j \in D$. In the models of earlier chapters, both the agent i, j derive a utility of f_{ij} from this particular contract (or exchange) between them through edge ij. Such a model is applicable in those settings where the side S takes the role of a supplier of a particular commodity and the side D takes the role of a demander of the same commodity. The amount of commodity that an agent sends/receives directly contributes to his utility for the good. In contrast, in the current model the flow f_{ij} is the surplus created when i and j decide to be involved in a partnership, and this surplus f_{ij} has to be shared between these two agents. The mechanism designer or central planner decide the share of the flow that each agent receives from a contract. The planner wishes to find a solution that is stable and balanced (defined more formally later). In turn, the allocation of a particular agent is governed by his relative position in the network and his bargaining power. Since we insist on stable and balanced outcomes, we later show that there exists no strategy proof mechanism that always finds a balanced outcome. Thus, we look for allocations in the core that are stable and balanced.

5.2 Bipartite Network Model

In this section we consider the version of the problem where the nodes of the network are populated by agents and edges represent the connectivity between agents. The two agents on the either side of

[1] When we refer to quantity, we refer to how much an agent receives from a unit value on that edge

an edge can exchange a positive flow with each other. Thus, our problem becomes one of exchanging a single commodity among the set of agents V using the set E of edges. For an agent $i \in V$, $N(i)$ represents the neighbors of agent i in network G. An exchange of the commodity among the agents is realized by a feasible flow f: f_{ij} is the amount of the commodity exchanged by the agents i and j using the edge $(i, j) \in E$. The net surplus created by their exchange is shared between them. Each agent $i \in V$ has a peak capacity b_i which is the maximum amount of flow that agent i can share with its neighbors. We assume the capacity of an edge ij, $u_{ij} \leq \min(b_i, b_j)$. Otherwise, redefine $u_{ij} := \min\{b_i, b_j\}$. Whenever an agent transfers flow to his neighboring agents, the flow on an edge ij should be less than its edge capacity u_{ij}.

In the following sections, we assume that the peaks of the agents and connectivity are fixed, and focus our attention on proving the existence of a stable and balanced solution. We do this by transforming to a unit flow network. We define a solution as follows: for a feasible flow f in the network G and for every edge $ij \in E(G)$; also for every edge $ij \in E(G)$ with $f_{ij} > 0$, the solution specifies the fraction of the flow allocated to an agent i denoted by $\gamma_{i \leftarrow j}$. Clearly, $\gamma_{i \leftarrow j} + \gamma_{j \leftarrow i} = 1$. Define for each $i \in V$, $\gamma_i := \min_{j \in N(i)} \gamma_{i \leftarrow j}$, the minimum utility per unit flow that an agent i receives from all his contracts. The utility for an agent $i \in V$ is his/her total share of the flow f and is given by $U_i = \sum_{j \in N(i), f_{ij} > 0} \gamma_{i \leftarrow j} f_{ij}$.

Kleinberg and Tardos [38] discuss stable and balanced outcomes in networks with unit node capacity. In the assignment games model of Shapley and Shubik [53], the network is bipartite and the agents on the nodes and these agents can form at most one contract with any of its neighbors. Shapley and Shubik show that (i) Any stable solution divides a maximum weighted matching in the underlying network; (ii) Stable solutions can be computed in polynomial time as a set of inequalities; Kleinberg and Tardos(KT) [38], extended the work to general networks (bipartite or non-bipartite). Even in their model, each agent can form at most one contract with his/her neighbors. In other words, the capacity of the nodes are all identically 1 and agents cannot have fractional exchanges. The main contributions of Kleinberg and Tardos [38] is that a balanced outcome exists and the set of all balanced solutions can be enumerated in polynomial time.

Continuing the study of bargaining solutions on networks, Bateni et al. [9] take a mathematical programming view to obtaining stable and balanced outcomes. For the KT model they show that the (i) set of stable and balanced outcomes coincides with the core and prekernel respectively of the corresponding cooperative game; (ii) the nucleolus is a "fair" outcome and can be computed in polynomial time by adapting the algorithm of Faigle [26]. Finally, they show that in the constrained bipartite bargaining game (i.e. one side of the bipartite network has arbitrary capacities, whereas the other side has all agents with unit node capacity) there are solutions in the core that are not stable. This does not happen in the standard case where all the node capacities are identically 1.

The more recent wok of koenemann et al. [27] is closer to our model. They study the general bipartite network bargaining game in which both sides of the network have arbitrary node capacities. They find very restrictive conditions under which a balanced outcome of this bargaining game coincide with the prekernel. In this case, we can again use the algorithm of Faigle [26] to compute the entire prekernel in polynomial time. The model of koenemann et al. assumes that each edge can be picked at most once, and fractional exchanges are disallowed. They prove the existence of stable and balanced outcomes by transforming any given instance into an equivalent matching problem of the sort considered in KT.

Our model differs from the aforementioned work in the following way: (i) We allow fractional exchanges, i.e. we allow agents to share fractional flows in the network; (ii) We allow each edge to be used more than once as long as the node and edge capacities are respected. In the rest of the chapter, we describe our model in more detail and adapt some of the proof techniques from the literature to our model. We start by giving a more precise definition of stable and outcomes in general networks.

Stable Outcome: An outcome, γ, associated with a feasible flow, f, is set to be stable if:
(a) $\gamma_{i \leftarrow j} + \gamma_{j \leftarrow i} = 1,\ \forall\ ij \in E(G)\ \ s.t. f_{ij} > 0$,
(b) $\gamma_i + \gamma_j \geq 1,\ \forall\ ij \in E(G)\ \ s.t.\ f_{ij} < u_{ij}$

Intuitively, if there is an edge ij that is unsaturated and can carry more flow, then the nodes connected to it should be sharing fractions in their partnerships such that the minimum fraction that they get from their partners should sum up to to at least 1. Otherwise, i and j could do better by sending more flow between each other. For an unsaturated node i, $\gamma_i = 0$ by definition.

Let α_{ij} denote the **best outside option** for an agent i for the edge ij. Then, $\alpha_{ij} = \max_k\{1 - \gamma_k | k \in N(i) \backslash j, f_{ik} < u_{ik}\}$. Intuitively, α_{ij} is the maximum utility rate agent i can get by reallocating an ϵ units of flow from j to other agents.

Next we discuss the best outside options when the solution γ is a stable solution; There are 2 cases: (a) Agent i is unsaturated ($\sum_{j\in N(i)} f_{ij} < b_i$): Then $\alpha_{ij} = 0\ \forall j \in N(i)$. This is because, for an unsaturated node i, $\gamma_i = 0$ and stable division implies $\gamma_k = 1\ \forall k\ \ s.t. f_{ik} < u_{ik}$. But these are precisely the set of agents that we could reroute our flow to.
(b) Agent i is saturated ($\sum_{j\in N(i)} f_{ij} = b_i$): Then $\alpha_{ij} = \max_k\{1 - \gamma_k | k \in N(i)\backslash j, f_{ik} < u_{ik}\}$.
In other words, α_{ij} is the maximum fractional share on a flow that an agent can receive in the current stable solution without j.

Note the difference in the definition of the best outside option α here compared to that of Kleinberg and Tardos [38] paper. In Kleinberg and Tardos [KT] paper, there is only one value of α defined for every vertex. Here, we define the outside option α for every edge with a strictly positive quantity of flow on it.

Balanced Outcome: An outcome γ, is said to be balanced if on any edge ij such that $f_{ij} > 0$, the surplus f_{ij} generated by agents i and j is shared by them in a fair way. A balanced outcome is one in which the extra utility that each agent (i or j) gets with respect to to his best outside option remains the same. Mathematically speaking, a balanced outcome is one that satisfies the following set of equations:

$$\gamma_{i\leftarrow j} - \alpha_{ij} = \gamma_{j\leftarrow i} - \alpha_{ji}\ \ \forall\, ij \in E(G)\ \ s.t.\ f_{ij} > 0 \tag{5.1}$$

Example:

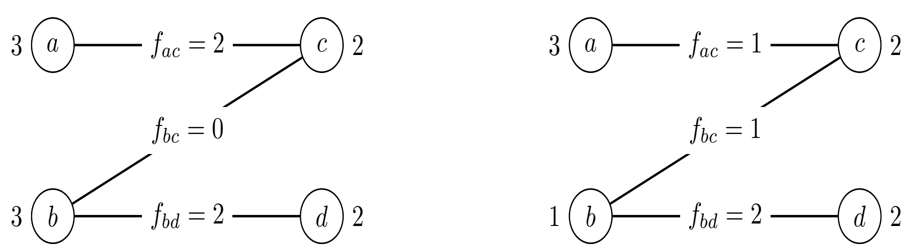

Figure 5.1: Example of a balanced outcome

Example: Consider the bipartite network in Figure 5.1 with 4 nodes {a,b,c,d} with node capacities {2,3,2,2} respectively; The edges {ac,bc,bd}. The maximum flow in which $f_{ac} = 2, f_{bc} =$

$0, f_{bd} = 2$, the net split given by $(x_a, x_b, x_c, x_d) = (0, 1, 2, 1)$ is a balanced outcome. This outcome is obtained by setting $\gamma_a = \gamma_b = 0$ and $\gamma_{c\leftarrow a} = 1, \gamma_{d\leftarrow b} = \gamma_{b\leftarrow d} = 0.5$. The best outside option for agents are: $\alpha_{ac} = 0, \alpha_{ca} = 1, \alpha_{db} = 0, \alpha_{bd} = 0$.

If the maximum flow is rather chosen as $f_{ac} = 1, f_{bc} = 1, f_{bd} = 2$, the solution still by the definitions above gives $(x_a, x_b, x_c, x_d) = (0, 1, 2, 1)$ which is a balanced outcome. The split in this case is such that $\gamma_a = \gamma_b = 0$ and $\gamma_{c\leftarrow a} = 1, \gamma_{d\leftarrow b} = \gamma_{b\leftarrow d} = 0.5, \gamma_{c\leftarrow b} = 1$. On edge ac, the best outside option for agents are $\alpha_{ac} = 0, \alpha_{ca} = 1$. On edge bc, $\alpha_{bc} = 0, \alpha_{cb} = 1$, on edge bd, $\alpha_{db} = \alpha_{bd} = 0$.

Linear Programming Formulation

Consider the linear programming formulation of the maximum flow (primal) and minimum cut (dual) problems. These formulations along with complementary slackness help us understand the connection between a stable outcome and a maximum flow.

Let f_{ij} be the flow on edge $ij \in E(G)$ and y_i, v_{ij} be the dual variables associated with the node and edge capacity constraints respectively. Specifically given a solution (f, γ) to the network bargaining game (G, b, u) we can define a corresponding utility vector x, where x_i is the net allocation of agent $i \in V$ from all his/her exchanges.

$$x_i = \sum_{j \in N(i)} \gamma_{i\leftarrow j} f_{ij} \tag{5.2}$$

$$\begin{aligned} \text{maximize} \quad & \sum_{ij \in E} f_{ij} \\ \text{subject to} \quad & \\ & \sum_{j \in N(i)} f_{ij} \leq b_i, \quad i \in V(G) \\ & f_{ij} \leq u_{ij}, \quad ij \in E(G) \\ & f_{ij} \geq 0, \quad ij \in E(G) \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & \sum_{i \in V(G)} y_i b_i + \sum_{ij \in E(G)} v_{ij} u_{ij} \\ \text{subject to} \quad & \\ & v_{ij} + y_i + y_j \geq 1, \quad ij \in E(G) \\ & v_{ij} \geq 0, \quad ij \in E(G) \\ & y_i \geq 0, \quad i \in V(G) \end{aligned}$$

The Complementary slackness (CS) conditions imply that a pair of feasible solutions (f^*, y^*, v^*) are optimal if and only if

- For each $ij \in E(G)$, $(f^*_{ij} - u_{ij})v^*_{ij} = 0$
- For each $i \in V(G)$, $(\sum_{j \in N(i)} f^*_{ij} - b_i)y^*_i = 0$
- For each $ij \in E(G)$, $(v^*_{ij} + y_i + y_j - 1)f^*_{ij} = 0$

Theorem 14 *There is a stable outcome, γ, of a flow f if and only if f is a maximum flow*

Proof: Let A and B be either sides of the bipartite network such that $A \cup B = V$. We first prove that an outcome γ of a flow f, is stable iff f is a maximum flow. Suppose the flow is not maximum and let γ be a stable solution associated with this flow. Then there is an augmenting path $i_1 - i_2 - i_3 - \cdots i_n$ where $i_1 \in A$ and $i_n \in B$. This means, i_1, i_n are unsaturated and hence $\gamma_{i_1} = \gamma_{i_n} = 0$. Since γ is stable and $i_1 - i_2$ is unsaturated, from the definition of stability we have $\gamma_{i_1} + \gamma_{i_2} \geq 1$ which implies $\gamma_{i_2} = 1$. This also means that i_2 is necessarily a saturated node. Moreover, all the edges connected to $i_2 j$ and carry positive flow should split the surplus on the edge such that $\gamma_{i_2 \leftarrow j} = 1, \gamma_{j \leftarrow i_2} = 0$. Hence, $\gamma_{i_3} = 0$. Repeating the argument iteratively, the nodes on the augmenting path have γ value that alternate between 0 and 1 with the first node $i_1 \in A$ ($\gamma_{i_1} = 0$) and the last node $i_n \in B$ has $\gamma_{i_n} = 1$, contradicting $\gamma_{i_n} = 0$. Thus, if f is not a maximum flow then the outcome γ cannot be stable.

Now, we prove that given a maximum flow f^*, we construct a stable outcome γ. Let (y^*, v^*) be an optimal dual solution. Set $\gamma_i = 0$ for all unsaturated nodes. For all edges $ij \in E$ such that $f^*_{ij} > 0$, assign the following split: $(\gamma_{i \leftarrow j}, \gamma_{j \leftarrow i}) = (y^*_i + rv^*_{ij}, y^*_j + (1 - r)v^*_{ij})$ for any r such that $0 \leq r \leq 1$. Note that for all edges such that $f^*_{ij} > 0$, we have $(\gamma_{i \leftarrow j} + \gamma_{j \leftarrow i}) = 1$ (from CS (iii) statement). In a maximum flow it is not possible for both the nodes connected to an edge to be unsaturated. Hence, $v^*_{ij} = 0$ for all edges $ij \in E(G)$ such that, $f^*_{ij} < u_{ij}$ (from CS (i) statement). Consequently, $\gamma_{i \leftarrow j} = y_i, \gamma_{j \leftarrow i} = y_j$. If a node is unsaturated, then $\gamma_j = y_j = 0$ from CS (ii) statement. Since $v^*_{ij} \geq 0$ we have, $\gamma_i + \gamma_j = \min_{k \in N(i) | f^*_{ik} > 0} \gamma_{i \leftarrow k} + \min_{l : N(j) | f^*_{jl} > 0} \gamma_{j \leftarrow l} \geq y^*_i + y^*_j \geq= 1$. In particular, when a node i is unsaturated, we have $\gamma_i = y^*_i = v^*_{ij} = 0$ which implies $y^*_j = 1$ and $\gamma_j = 1$. ∎

Corollary: If γ is stable with respect to some maximum flow, then it is also stable w.r.t. to any other maximum flow.

Polytope of Stable Solutions

We turn now to characterize the set of all stable solutions (in the spirit of KT). Once we fix a flow f. the set of all stable solutions for this flow can be written as:

$$\begin{aligned}
\gamma_{i\leftarrow k} + \gamma_{j\leftarrow l} &\geq 1, \ \forall \ ij \in E(G) \ \ s.t. \ f_{ij} < u_{ij} \\
\gamma_{i\leftarrow j} + \gamma_{j\leftarrow i} &= 1, \ \forall \ ij \ \ s.t. \ f_{ij} > 0 \\
\gamma_{i\leftarrow j} \geq 0, \gamma_{j\leftarrow i} &\geq 0, \ \forall \ ij \ \ s.t. \ f_{ij} > 0 \\
\gamma_{i\leftarrow j} = 0, \gamma_{j\leftarrow i} &= 0, \ \forall \ ij \ \ s.t. \ f_{ij} = 0
\end{aligned}$$

As in the KT paper, we have two variables per inequality in the above polytope which describes the set of all stable solutions. Hence, a feasible solution to the above can be obtained in polynomial time by the Apsvall-Shiloach [4] procedure.

Transformation to a unit matching network

In this section we show that there exists a balanced solution in any given bipartite network. Instead of showing that there exists a balanced outcome in the set of all maximum flows, we show that there is a balanced outcome within the smaller set of all integral maximum flows.[2]

Identify a particular integral maximum flow f in $G(V, E)$. Denote the set of edges with strictly positive flow by $f^0 := \{ij \in E(G) | f_{ij} > 0\}$. Construct another unit capacity network $G'(V', E')$ and flow f' on G' such that:

- For each $i \in V(G)$, create b_i copies of node i in $V(G')$. Label them $i_1, i_2, ..i_{b_i}$

[2]Integral maximum flows in bipartite networks are just the maximum b-matchings

- For each $ij \in E(G) \cap f^0$, pick an unmatched node i_k and j_l and construct an edge between them [For convenience, choose k, l as the smallest unmatched copy of nodes i, j respectively]. Assign unit flow on these edges and split them in the same fraction $\gamma_{i_k \leftarrow j_l} = \gamma_{i \leftarrow j}, \gamma_{j_l \leftarrow i_k} = \gamma_{j \leftarrow i}$.

- For each $ij \in E(G) \backslash f^0$, add edges between $\{i_k j_l | 1 \leq k \leq b_i, 1 \leq l \leq b_j\}$ in $E(G')$

- For an unsaturated node $i \in G$ $[x_i < b_i]$: define a set i^j in G' such that, $i^j := \{j_l \in V(G') \mid ij \in E(G), f_{ij} < u_{ij},\ i_k j_l \notin E(G')$ for any $1 \leq k \leq b_i\}$. In the graph G', for every node $k \in i^j$ match all the unmatched copies of i to node k.

- The edges such that $f_{ij} < u_{ij}$ can be used for redirecting the flow even if the node is saturated. For every edge $ij \in E(G)$ such that $f_{ij} < u_{ij}$, define $i^j := \{j_l | i_k j_l \notin E(G')$ for any $1 \leq k \leq b_i\}$ and $j^i := \{i_k | j_l i_k \notin E(G')$ for any $1 \leq l \leq b_j\}$. Construct an edge between every node i^j and j^i.

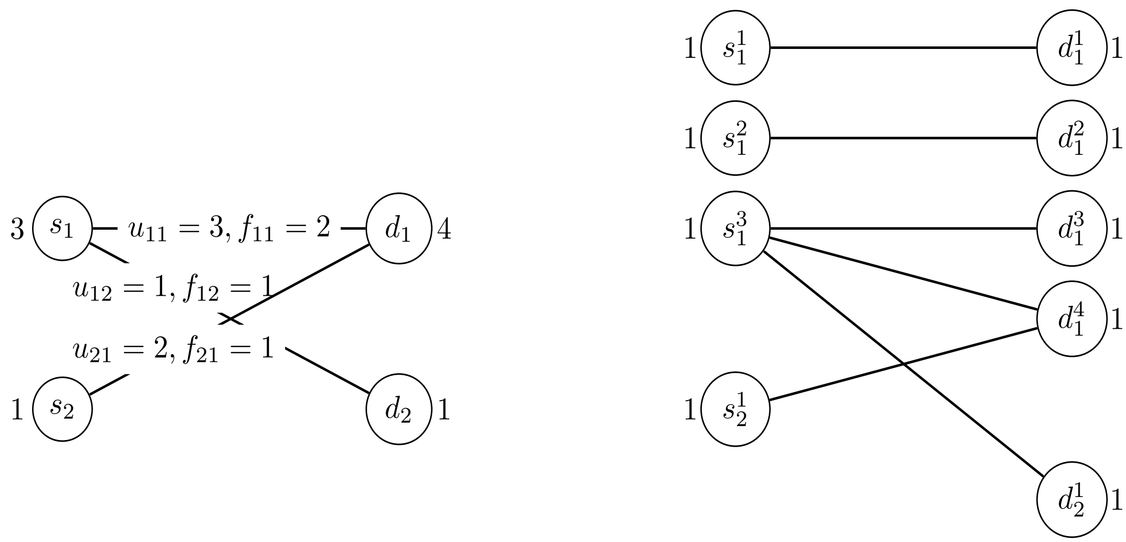

Figure 5.2: Bipartite transformation: Example 1

As defined earlier, α_{ij} is the best outside option for node i w.r.t. to a matched edge j $(f_{ij} > 0)$ in network G. The best outside option $\alpha_{i_k j_l}(G')$ in the modified network is defined as the best outside fraction that node i_k can receive from other connected neighbors other than the copies of node j $(j_l, 1 \leq l \leq b_j)$ in G'.

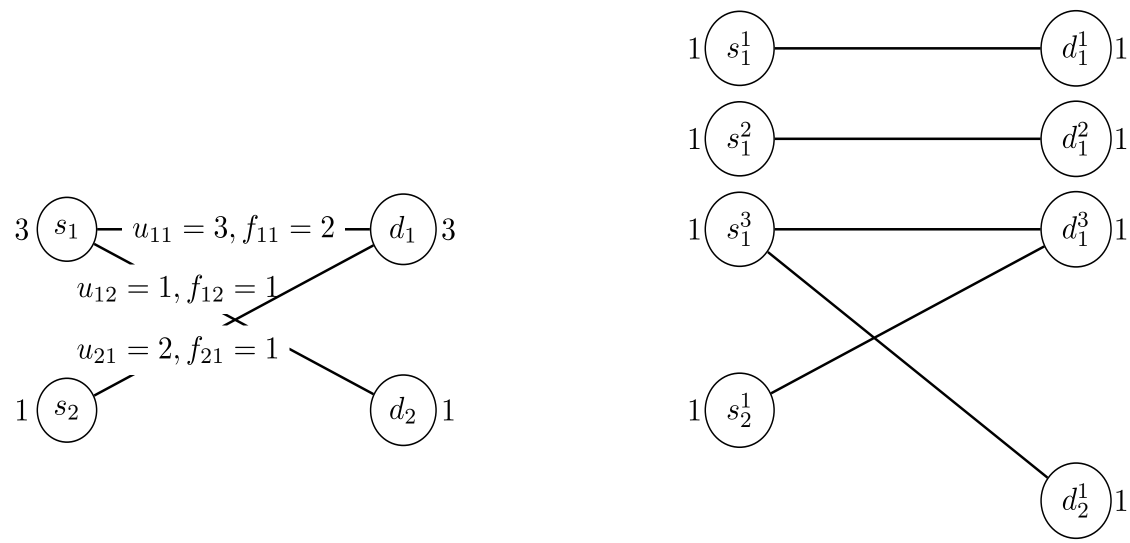

Figure 5.3: Bipartite transformation: Example 2

Lemma 16 *For every* $ij \in E(G)$ *with* $f_{ij} > 0$, $\alpha_{ij}(G) = \alpha_{i_k j_l}(G')$ $\forall\, k, l$ *such that* $i_k, j_l \in V(G')$ *and* $f_{i_k j_l} > 0$

Proof: From the defintion of best outside option we have, $\alpha_{ij} = \max_v\{1 - \gamma_v | v \in N(i) \backslash j, f_{iv} < u_{iv}\}$. Define $N_{ij} := \{v \in V(G) | v \in N(i) \backslash j, f_{iv} < u_{iv}\}$. We will prove the result for a particular i_k, j_l and by symmetry of construction, the result follows for every copy of i that sends flow to a copy of node j. Consider this particular node i_k that is connected to j_l. Hence, this node i_k is not connected to any other copy of node j. From steps (3-5) of the construction of the transformation network, any node $v \in N_{ij}$ such that $f_{iv} < u_{iv}$ will contribute to an edge $i_k v_r$ in the network G' for some r. And also it follows from construction, $\gamma_{i \leftarrow v}$ for some $v \in N_{ij}$ is the same as in $\gamma_{i_k \leftarrow v_r}$ for some $v_r \in V(G')$. ∎

Theorem 15 *The graph* G *with a flow* f *has a stable and balanced outcome if and only if* $G'(V', E')$ *with a flow* f' *has a stable and balanced outcome*

Proof: From Lemma 16 the best outside option of agent i for the edge with j in network G, $\alpha_{ij} = \alpha_{i_k j_l}$, the best outside option of agent i_k for the edge with j_l in network G'. Moreover, we know that balanced outcome can be obtained as a solution to a set of equations of the form,

$\gamma_{i\leftarrow j} - \alpha_{ij} = \gamma_{j\leftarrow i} - \alpha_{ji}\ \forall\, ij \in E(G)\ s.t.\ f_{ij} > 0$. But every edge ij such that $f_{ij} > 0$ has a edge $i_k j_l$ in G' carrying a unit flow. For a solution in G' to be balanced, we would have for those edges:$\gamma_{i_k\leftarrow j_l} - \alpha_{i_k j_l} = \gamma_{j_l\leftarrow i_k} - \alpha_{j_l i_k}$. Note that the equations in both networks are identical and hence a balanced division $\gamma_{i_k\leftarrow j_l}$ in network G' maps directly to a balanced division $\gamma_{i\leftarrow j}$ in network G.

Generating Balanced Outcomes: Kleinberg and Tardos [38] show how to generate all balanced solutions in a unit matching network. Moreover, Bateni et al. [9] show that the set of all balanced solutions coincides with the pre kernel of the game. Faigle et al. [26] design an algorithm that generates a point in the pre kernel in polynomial time. Since Theorem 15 has established an equivalence between the original and transformed network, we can use the algorithm of Faigle et al. [26] to obtain a balanced outcome in the transformed network and map it to a solution in the original network.

Strategyproof Mechanisms: Any mechanism that picks a balanced outcome is not strategyproof. Consider a path network $(b - a - c)$ with peaks $(2, 4, 2)$ respectively (only edges in the network are (ab, ac)). The unique balanced outcome is $(2, 1, 1)$. Suppose the node capacities are instead $(3, 2, 2)$. Then, given a flow say $(f_{ab} = 1, f_{ac} = 2)$, the only balanced outcome is $(2.5, 0.5, 0)$. Hence, if the mechanism always identifies a balanced outcome in a given network and agent a has the option to report his peak, then in the first network, agent a can misreport to 3 and improve his allocation.

Core and Prekernel

We discuss a connection between network bargaining games and cooperative game theory. The vector x can be seen as a solution to a corresponding cooperative game (V, ν) defined as follows: for every subset $S \subseteq N$ of players, we define its value $\nu(S)$ as the maximum flow surplus created by agents in S alone. This definition is the generalization of the notion used in assignment games [53]. The subsets $S \subseteq N$ can deviate and form coalitions and the value $\nu(S)$ of each coalition is interpreted as the net surplus that the players in S would receive if they decide to deviate. We haven't discussed the exact utility structure for an agent i.

Any mechanism that attempts to split the surplus among agents (i.e. mechanism identifies the fraction $\gamma_{i \leftarrow j}$ for each $ij \in E$ such that $f_{ij} > 0$) should split in such a way that no subset of agents have any incentive to deviate. Hence, the grand coalition is formed with each agent $i \in V$ better off in the grand coalition than any other smaller subset containing i.

An outcome $\{x_i\}$ is in the *core* if for all subsets $S \subseteq N$, $\sum_{i \in S} x_i \geq \nu(S)$, and for the grand coalition N, $\sum_{i \in N} x_i = \nu(N)$. Given an allocation x, the excess of a coalition S is defined as $\nu(S) - x(S)$. Intuitively, from the discussion above it means, the excess of every coalition should be non-positive for the grand coalition to form.

The *power* of agent i with respect to agent j in the outcome x is $s_{ij} := \max\{\nu(S) - \sum_{k \in S} x_k : S \subseteq N, i \in S, j \notin S\}$. The *pre kernel* is the set of outcomes x that satisfy for every $i, j \in V$, $s_{ij}(x) = s_{ji}(x)$.

The core of a game may be empty but a prekernel exists for every game. In our game, both core and the pre kernel are non-empty.

Lemma 17 *Every stable split of a feasible flow f results in a node utility x which is in the core of the bargaining game*

Consider a dual feasible solution $y_i, i \in V$, v_{ij}, $ij \in E$ and lets say $v_{ij} = v'_{ij} + v'_{ji}$ where i gets the fraction v'_{ij} of the flow between i and j and j gets v'_{ji} fraction of the flow.

Firstly, we show that the core condition holds for the grand coalition V i.e. $\sum_i x_i = \nu(V)$.

$$\sum_{i \in V} x_i = \sum_{i \in V} (\sum_{k \in N(i)} f_{ik} y_i + \sum_{j \in V} v'_{ij} f_{ij}) = \sum_{i \in V} b_i y_i + \sum_{i,j \in V} v_{ij} u_{ij} \tag{5.3}$$

The equality follows as agent i receives y_i for unit of every flow it shares and an addition fraction of v'_{ij} for every unit of flow it shares with an agent $j \in N(i)$. The second equation follows from complementary slackness conditions. If $y_i > 0$ then $\sum_{j \in N(i)} f_{ij} = b_i$ and if $v_{ij} > 0$ then the edge ij is saturated i.e. $f_{ij} = u_{ij}$.

which is the value of a dual optimal solution. Hence, by strong duality, it equals the primal solution which is the value of a maximum flow or in other words, the net surplus created when all the V agents participate in the game $\nu(V)$.

Now consider, $S \subset N$. Construct the linear program 5.2 restricted to set S,

$$\sum_{i \in S} x_i = \sum_{i \in S} (\sum_{k \in N(i) \cup S} f_{ik} y_i + \sum_{j \in V} v'_{ij} f_{ij}) \geq \sum_{i \in S} b_i y_i + \sum_{i,j \in S} v_{ij} u_{ij} \tag{5.4}$$

which is the value of a dual feasible solution. Hence, by weak duality this is greater than any primal feasible solution. $\nu(S)$ is primal feasible solution obtained by setting $f_{ij} = 0$ if either i or j is not in S. The flow on the edges ij such that both $i, j \in S$ is obtained through complementary slackness conditions. Hence, $\sum_{i \in S} x_i \geq \nu(S)$ ∎

The converse is not true i.e. there maybe allocations in the core that are not stable. See Bateni et al. [9] for an example.

Pareto Optimality

A feasible stable division γ (inducing a utility profile x) as defined in the previous section is Pareto optimal if there is no other division (γ', x') such that every agent is weakly better off and atleast one agent is strictly better off in it. In mathematical terms, if R_i and I_i denote the preference and indifference relations respectively for agent i, then

$$\{\forall \ i \ \ x'_i R_i x_i\} \text{ and } \implies \{\forall \ i : \ \ x'_i I_i x_i \ \ \} \tag{5.5}$$

A Pareto solution need not be stable for the same reason that a core solution can be unstable.

Theorem 16 *A stable solution is also a Pareto optimal allocation for the agents*

Proof: Let F be the value of the maximum flow in the network. From Theorem 14 we know that every stable solution has to divide a flow of value F among the agents. Hence the corresponding node allocations from a stable split should be such that $\sum_{i \in V} x_i = F$. If the allocation x is not Pareto optimal, then there exists an allocation y in which there is at least one agent j such that $y_j > x_j$ and $y_k \geq x_k \ \forall \ k \in V \backslash j$. But the net surplus that is divided among the agents can never exceed F. Hence, such a allocation y does not exist.

Generating a Unique Outcome: Bateni et al. [9] choose the nucleolus as a unique allocation for agents in the network. The nucleolus is a"fair" solution as it lexicographically minimizes the excess allocations among a set of agents. As we discussed earlier, a core allocation need not be stable for the general bipartite bargaining game. So, it is still an open question if the nucleolus of the general bipartite game is stable and balanced. Meanwhile, we propose a random priority type mechanism:

Choose an integral maximum flow in the original network. Make the transformation to a unit matching network. Randomly arrange the agents in V in an order. Choose the balanced outcome which gives the highest allocation (utility) to the first agent in the list. Keeping the allocation of first agent fixed, choose the balanced outcome which gives the highest to the second agent in the list. Repeat the procedure till all the agents are processed. This mechanism is "fair" in the sense that the agent who is given priority is done through a lottery. Hence, every agent has the possibility of getting a balanced outcome of his choice in the final solution. Also, the mechanism produces a stable, balanced solution in the core and is also Pareto optimal.

5.3 Extension to general networks

In bipartite networks, stable and balanced outcomes exist when the exchanges allowed are either strictly integral or fractional. In general non-bipartite networks this is not the case. There may be networks where stable outcomes do not exist [38]. Consider a simple network with 3 nodes, (a, b, c) each with node capacity 1 and every node connected with the other two. The maximum matching in this network is 1, and any of the three edges can be chosen. Lets say we pick the matching ab and we share the surplus as f_a and $f_b = 1 - f_a$ between agents a and b. Then, $\gamma_a = f_a, \gamma_b = f_b, \gamma_c = 0$. This implies either $\gamma_a + \gamma_c < 1$ or $\gamma_b + \gamma_c < 1$ or both causing instability. Whereas when we allow for fractional exchanges, $f_{ab} = f_{bc} = f_{ca} = 0.5$ with the surplus on each edge being shared equally is a stable outcome. Our first result is that when we allow for fractional exchanges (non integral flows) between nodes, we can always find a stable outcome.

We then use the ideas of Bateni et al. [9] to prove the existence of a balanced outcome when fractional exchanges are allowed in a unit capacitated non-bipartite network. Then, we transform a given network into a unit capacity network and solve the fractional matching problem and establish equivalence between stable and balanced outcomes between both networks.

A network G is given with node capacities b_i for every node $i \in V(G)$ and edge capacities $u_{ij}, ij \in E(G)$. A fractional matching is one that assigns a flow $0 \leq f_{ij} \leq u_{ij}$ for every edge $ij \in E(G)$ and $x_i = \sum_{j \in N(i)} f_{ij} \leq b_i, i \in V(G)$. A maximum fractional matching problem is one that maximizes $\sum_{i \in V} x_i$ among all fractional matchings. The agents who create the surplus share the surplus on that edge. The definitions of stable,balanced, core and pre kernel are the same as in previous sections. We prove the following theorems based on the ideas of Bateni et al. [9]. We discuss below the connection of stability and core in such problems using a linear programming formulation. In our discussions, we will restrict our focus to $u_{ij} = 1$. Then we transform a model with non-unit peaks to a unit peak model to find the stable and balanced outcome in the original network.

Theorem 17 *(i) An outcome (f, x) is stable if and only if the utility vector x is in the core; (ii) A stable outcome always exists when fractional exchanges are allowed; Such a stable outcome always splits the surplus created by a maximum fractional matching.*

Proof: The proof of the above statement follows easily using the same proof technique as in the previous section. Observe the linear programming problem in section 5.2 is also the formulation to find a maximum fractional matching on any network and the rest of the proof follows identically.

Let F be the value of a maximum fractional matching. Consider the following linear programs where the primal represents the definition of a core and the dual problem represents the maximum fractional matching:

$$\begin{aligned} \text{minimize} \quad & \sum_{i \in V} x_i \\ \text{subject to} \quad & \\ x_i + x_j \geq 1, \quad & \forall\ ij \in E(G) \\ x_i \geq 0, \quad & i \in V(G) \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & \sum_{ij \in E(G)} f_{ij} \\ \text{subject to} \quad & \\ \sum_{k \in N(i)} f_{ik} \leq 1, \quad & \forall\ i \in V(G) \\ f_{ij} \geq 0, \quad & ij \in E(G) \end{aligned}$$

Since by definition $\sum_{i \in V} x_i = F'$, where $F' = \sum_{ik \in E} f'_{ik}$, we need to be prove $F' = F$ if (f', x) is a stable outcome. The primal and dual objective function values are same for every fractional matching but only for maximum fractional matchings, the corresponding primal vector x is feasible. Hence, by strong duality, stable outcomes correspond to maximum fractional matchings.

Stable $\implies$ Core: Consider an edge $ij \in E(G)$. For a saturated edge ij $(f_{ij} = 1)$ we have $x_i + x_j = 1$ by the definition of a stable outcome. For an edge ij such that $f_{ij} < 1$, by the definition of an outside option, $\alpha_i \geq 1 - x_j$ and stability implies $x_i \geq \alpha_i$ which results in $x_i + x_j \geq 1$.

Core $\implies$ Stability: Suppose x is a solution in the core, then $x_i + x_j \geq 1$ for any pair i,j $\in V(G)$. We just need to show $x_i + x_j = 1$ for nodes such that $f_{ij} = 1$. Consider any maximum weight fractional solution f, it is clearly feasible to the dual problem. Since, $\nu(V) = \sum_{k \in V} x_k = F$. By strong duality and complementary slackness conditions we have that, $f_{ik}(x_i + x_k - 1) = 0$ for each $ik \in E(G)$. It means, if $f_{ik} > 0$, then $x_i + x_j = 1$, we have indeed proved a stronger result.

Theorem 18 *An outcome (f, x) is stable and balanced if and only if the utility vector x is in the intersection of core and prekernel*

Proof: Following Theorem 17 it only remains to prove that the notions of a balanced outcome and a pre kernel coincide with each other. Note that unlike the earlier section, here the notion of balancedness has to be satisfied by every pair of agents (i, j) such that $f_{ij} > 0$. Recall that, in defining α_{ij} we just wanted to route ϵ fraction of the flow f_{ij} to a connected agent other than j, and compute the best utility that agent i can receive. Following the steps in Theorem 4.3 of Bateni et al. [9] we have the result. Use the algorithm of Faigle et al. [26] to construct a balanced outcome in polynomial time.

Generating a Unique Outcome: Given a general network and a maximum fractional matching, make the same transformation as in section 5.2. The only difference here is instead of unit flows on all matched edges, assign unit flow on every edge except the last edge on which assign the fractional part of the flow. Alternately, define $i^j := \{j | f_{i_k j_l < 1}\}$ for every node and construct the edges in the transformed network accordingly. In the transformation, only construct balanced outcome equations for the edges with a unit flow. Establishing equivalence between the two networks follows from the Theorems 17 and **??** proved earlier. To generate a unique outcome, one could follow the random priority mechanism discussed earlier. But the overall process involved in generating a unique outcome is not polynomial in time. Identifying such polynomial time algorithms is still an open question.

5.4 Further Work

- For the networks discussed in this section, does there exist a polynomial time algorithm for enumerating the set of all balanced outcomes?

- Bateni et al. [9] argue that the nucleolus is a unique fair outcome and prove that the nucleolus is stable and balanced. Is the nucleolus a stable and balanced outcome for the general bipartite network bargaining game as well?

- In our current work, we split the entire flow between two agents by the same fraction. In general, we could allow for the fraction to depend on the total flow that is divided between them (among other factors). This would generalize the existing models and identify sharing mechanisms that result in stable and balanced outcomes

- The best outside option of an agent i with respect to an agent j is defined as the maximum utility agent i receives by routing an ϵ fraction of the flow on edge ij to another neighbor of his. In such a situation, we are not considering the complete bargaining power of agent j. Consider the following example: if $f_{ij} = 2$ and say i can find another neighbor to reroute ϵ unit flow but does not have enough connectivity to reroute all the 2 units of flow that he shares with j. Then j should enjoy more bargaining power in the network as his presence is important for i to send more flow. One way of modeling it could be by defining the best outside option of an agent as the utility an agent receives when he reroutes the entire amount of flow on a particular edge.

Chapter 6

Conclusions

We consider the problem of fair allocation of resources in a network environment. The current work addresses a wide range of efficiency, fairness and strategic issues that arise in economic networks and leads to other interesting questions.

In chapter 2, our contributions generalize many of the results in the standard rationing literature to bipartite networks. We make a stronger case for the egalitarian mechanism by showing that it is peak group strategyproof, thus proving a conjecture of Bochet et al. [11, 12]. We also characterize peak group strategyproof mechanisms using a property that we call *strong invariance*. As for link group strategyproofness, we show that the egalitarian mechanism is strategyproof for one-sided coalitions (i.e., a coalition of suppliers or a coalition of demanders), but not for two-sided ones. Several open questions remain: In a bipartite network with capacities, is the egalitarian mechanism strategyproof with respect to the capacity reports of the agents? A proof of this would generalize link strategyproofness of the egalitarian mechanism to a more general model. Another challenging open question is to characterize link group strategyproof mechanisms.

Although the egalitarian mechanism appears to be *the* correct generalization of the uniform rule to bipartite networks, it fails *consistency*. For the standard rationing model, the uniform rule is in fact consistent, but a similar result is impossible for the network model: we show that no envy-free rule can be consistent. This motivates the need for alternative mechanisms for applications where consistency is important. Moulin and Sethuraman [43, 44] study consistent extensions of several rules to bipartite networks, but they work with a model in which the peaks of the agents are observable; in particular, they do not prove that any of their rules are strategyproof, if the agents have single-peaked preferences over their net allocations. In chapter 3, we introduce the edge fair mechanism as a compelling alternative to the egalitarian mechanism: the edge-fair mechanism is Pareto efficient, consistent, and strategyproof with respect to peaks. We have been unable to characterize the edge-fair rule using these properties, but believe that such a characterization (with perhaps some additional properties) will be insightful. Furthermore, we conjecture that the edge-fair rule is link group strategyproof. Finally, characterizing the mechanisms that are Pareto efficient, strategyproof, and consistent in this setting is an interesting open question.

In chapter 4, we extended the study of the egalitarian mechanism to more general non-bipartite networks by transforming them to equivalent bipartite networks. In these general networks there is a need to distinguish the case of divisible goods from that of indivisible goods for the simple reason that the size of a max-cardinality fractional matching in a non bipartite network can be more than the size of a max-cardinality integer matching. Many (but not all) of the results of Chapter 2 generalize. Several related questions are still open: For example, in the model of Kalai and Zemel [34] the agents are on the edges of an s-t flow network; assuming that the agents collectively generate utility equal to the value of a maximum-flow, how should the total utility be divided? Kalai and Zemel identify an allocation in the core, but in their solution the only agents with a positive utility are the agents on the edges of a minimum cut. Is there a fairer way of sharing the surplus?

Finally, in chapter 5, we made a connection with the literature on stable allocations. We show that when fractional exchanges are allowed, stable and balanced outcomes always exist. It is still an open question to identify a good rule that always yields a balanced outcome. Bateni et al. [9] propose the nucleolus as a fair outcome in constrained bargaining games. In general, does the nucleolus result in a balanced outcome in more general bargaining games? If not, what additional constraints on the network structure would make the nucleolus a balanced outcome? Also, in our current work, we split the entire flow between two agents by the same fraction. In general, we could allow for the fraction to depend on the total flow that is divided between them (among other factors). This would generalize the existing models and choose “good” functions under which the networks can have stable and balanced outcomes.

There are some other general directions for future research that are loosely related to the research described in this dissertation. For example, there is a growing literature on dynamic kidney exchange (see Unver [61]) and strategic issues that arise in such settings (see Ashlagi et al. [3]). When agents have multiple units of a good and also arrive sequentially over time, the problem becomes more complex. Mechanism design in such complex environments remains a challenging problem that is well worth addressing. Also, social and economic networks are even more popular than before with the emergence of online media like Facebook, Twitter etc. Agents with better connectivity have access to more information and enjoy an important presence in the network. An empirical study of Cook and Yamagashi [21] determines how an agent's location in the network influences his ability to negotiate for resources. It would be interesting to study how our mechanisms perform in practical settings.

Part I

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Part II

Appendices

Appendix A

Related Proofs

Gallai-Edmonds decomposition (GED) for a bipartite network without edge capacities:

Proof of Lemma 1: The max-flow from σ to τ is clearly finite, and so must be the capacity of a minimum σ-τ cut. Fix a min-cut, and let X and Y be the set of suppliers and demanders respectively in that min-cut. Then we claim that $Y = f(X)$. If there exists demander $j \in Y$ such that $j \notin f(X)$, then the cut's capacity can be reduced by deleting the demander j; if, however, there exists demander $j \notin Y$ such that $j \in f(X)$, then the cut has infinite capacity.

Set $S_- = X$, $D_+ = Y$, $S_+ = S \setminus X$, and $D_- = D \setminus Y$. By construction, $G(S_-, D_-) = \emptyset$, $D_+ = f(S_-)$, and $S_+ = g(D_-)$. The capacity of the cut $\sigma \cup X \cup Y$ is, by definition, $s_{S \setminus X} + d_Y$, which equals $s_{S_+} + d_{D_+}$. Moreover, in any maximum-flow, the edges oriented from S_+ to D_+ are backward edges in the cut, so they must carry zero flow. The edges from σ to S_+ and the edges from D_+ to τ are the edges in the cut, so these edges carry flow equal to their respective capacities. This establishes (ii) of the lemma. Parts (i) and (iii) follow from Lemma 1.■

Gallai-Edmonds decomposition (GED) for a bipartite network with finite edge capacities:

Proof of Lemmas 7, 9: Let $\lambda := (\lambda_i)_{i \in S}$ be non-negative. Construct the following network $G(\lambda)$: introduce a source s and a sink t; arcs of the form (s, i) for each supplier i with capacity λ_i, arcs of the form (j, t) for each demander j with capacity d_j; an arc of capacity u_{ij} from supplier i to demander j if supplier i and demander j share a link. Consider now a maximum s-t flow φ in the network $G(s)$. By the max-flow min-cut theorem, there is a cut C (a cut is a subset of nodes that contains the source s but not the sink t) whose capacity is equal to that of the max-flow. Let X be the set of suppliers in C and Y be the set of demanders in C. If the min-cut is not unique, it is again well-known (see [40]) that there is a min-cut with the largest X (largest in the sense of inclusion), and a min-cut with the smallest X (again in the sense of inclusion). Call these sets $\overline{X}$ and $\underline{\mathrm{X}}$. Define $S_- := \underline{\mathrm{X}}$ and $S_+ := S \setminus S_-$; and define $D_+ = f(S_-) \cap \{j | y_j(\varphi) = d_j, \ \ \forall \varphi \in \mathcal{F}^*\}$ and $D_- = D \setminus D_+$. We note that the partition is uniquely determined for each problem.[1]

Let C be the cut that has precisely $\underline{\mathrm{X}}$ as its set of suppliers and Y as its set of demanders. We claim that $Y \subseteq f(\underline{\mathrm{X}})$. For otherwise, there is a supplier $j \in Y \setminus f(X)$ who contributes d_j to the capacity of the cut C, and omitting j from C would reduce this capacity by $d_j > 0$, resulting in a smaller capacity cut. Moreover, by the max-flow min-cut theorem, every edge (i, j) with $i \in C$ and $j \notin C$ must carry flow equal to its capacity, and that the value of the max-flow is precisely the sum of the capacites of such edges. Thus, $\varphi_{ij} = u_{ij}$ for every edge (i, j) with $i \in S_-$ and $j \in D_-$; the edge (s, i) carries a flow of s_i for each supplier $i \in S_+$; and the edge (j, t) carries a flow of d_j for each demander $j \in D_+$.■

Gallai-Edmonds decomposition for non-bipartite networks without edge capacities:

Proof of Lemma 13: Let us construct an instance of the given b-matching problem. From the given graph (G, V, E) construct the following graph (G', V', E'): Create b_i duplicate copies of node i with unit peak each. Label these nodes as $(i^1, i^2....i^{b_i})$. In the graph G', nodes $i^k, (k \in 1, 2, ..b_i)$ and $j^l, (l \in 1, 2, ..b_j)$ are connected by an edge if i and j are connected in the original graph G. There is no edge between i^k and $i^r, (k, r \in (1, 2, ...b_i))$ in G'. Now, we have a graph with unit peaks. Applying the GED on this unit graph and merging the duplicate copies back establishes the decomposition.

The first two statements of the lemma follows from the GED of a unit peak non-bipartite network. For a proof of the unit peak case, refer to Roth et al. [49]. When $(|J| = 1)$ the odd component has only 1 element, there is no b-matching within that component. In a unit peak network, when $|J| \geq 2$, the size of a maximum matching within every odd component is one less than the number

[1] It is easy to check that every supplier in $\overline{X} \setminus \underline{\mathrm{X}}$ will transfer his entire supply in *all* maximum flows.

of nodes in the component. By symmetry, all the duplicate (identical) copies of a node will be in the same odd component. Hence, when we merge these duplicate nodes into a parent node with original peak in the decomposition above, it follows that the sum of the peaks of the agents in an odd component is $\Sigma_{j=1}^{|J|} b_j$. Again it follows from the structure of the GED in the unit peak network that exactly one agent in every odd component sends flow to an agent in V^O or remains unsaturated. ■

Peak GSP of egalitarian mechanism in a network with capacitated edges:

Proof of Theorem 8 We follow the proof technique introduced in Chandramouli & Sethuraman [15] for the first part of the theorem: PO^*, strong invariance $\implies$ Peak GSP.

Suppose such a rule is not peak group strategyprof then lets focus on a network G with the *smallest* number of nodes. Suppose the true peaks of the suppliers and demanders are s and d respectively, and suppose their respective misreports are s' and d'. We can assume that $d_j > 0$ for every demander j, as otherwise deleting j would result in a smaller counterexample. Fix a coalition A of suppliers and a coalition B of demanders : note that A contains all the suppliers k with $s'_k \neq s_k$, and B includes all demanders ℓ with $d'_\ell \neq d_\ell$.

Let (x, y) and (x', y') be the respective allocations to the suppliers and demanders when they report (s, d) and (s', d') respectively. Let S_+, S_-, D_+, D_- be the decompsition when the agents report (s, d), and let S'_+, S'_-, D'_+, D'_- be the decomposition when the agents report (s', d'). We shall show that when the agents report (s', d') rather than (s, d), the only allocation in which each agent in $A \cup B$ is (weakly) better off, then $x'_k = x_k$ for all $k \in A$ and $y'_\ell = y_\ell$ for all $\ell \in B$. This establishes the required contradiction.

Let $Y' := D_+ \cap D'_-$. If $Y' = \{\emptyset\}$, then consider the set of suppliers $S_- \cap S'_+$. Every supplier $i \in S_- \cap S'_+$ do not send flow to any demander j in D'_+. Hence, these suppliers can send flow to only demanders in $f(S_- \cap S'_+) \cap D'_-$. Now observe, $z_{ij} = u_{ij}, z'_{ij} \leq u_{ij}$ when the reports are s and s' respectively for every agent $i \in S_- \cap S'_+, j \in f(S_- \cap S'_+) \cap D'_-$. Hence, every supplier $i \in S_- \cap S'_+$ sends weakly less flow to every agent connected to him. Hence, $s'_i = x'_i \leq x_i \leq s_i$. So, we can conclude $A = \{\emptyset\}$ when $Y' = \{\emptyset\}$.

We now consider the case $Y' \neq \{\emptyset\}$ and make observations about the suppliers $X' := g(Y') \cap S_- \cap S'_+$. Let $Y'' := f(X') \cap D'_- \cap D_-$

- For any such supplier k, $s'_k = x'_k$ and $x_k \leq s_k$. Also, $d_\ell = y_\ell$ and $y'_\ell \leq d'_\ell$ for any $\ell \in Y'$.
- When the report is s', every such supplier can send flow only to the demanders in $Y' \cup Y''$: this is because no link exists between agents in X' and demanders in $D'_- \backslash \{Y' \cup Y''\}$ and $z_{ij} = 0 \; \forall \; ij \in G(S'_+, D'_+)$ in a pareto optimal allocation. Also, observe that $z_{ij} \leq u_{ij} \; \forall \; ij \in G(X', Y'')$ and $z_{ij} = u_{ij} \; \forall \; ij \in G(S'_-, Y')$. Therefore $\sum_{k \in X'} x'_k \leq \sum_{\ell \in Y'} y'_\ell - \sum_{ij \in G(S'_-, Y')} u_{ij} + \sum_{ij \in G(X', Y'')} u_{ij}$
- When the report is s, $z_{ij} = u_{ij} \; \forall \; ij \in G(X', Y'')$. The agents in Y' can receive flow only from agents in X' and $g(Y') \cap S'_- \cap S_-$. The agents in Y' can receive at most $\sum_{ij \in G(S'_-, Y')} u_{ij}$ units of flow from the suppliers $g(Y') \cap S'_- \cap S_-$. Hence, the remaining allocation has to be supplied from X'. Also, note that $f(X') \supseteq Y'$. Therefore $\sum_{k \in X'} x_k \geq \sum_{\ell \in Y'} y_\ell - \sum_{ij \in G(S'_-, Y')} u_{ij} + \sum_{ij \in G(X', Y'')} u_{ij}$.

Let $f(S'_-, Y') := -\sum_{ij \in G(S'_-, Y')} u_{ij} + \sum_{ij \in G(X', Y'')} u_{ij}$. Finally, note that $s'_k = s_k$ for all $k \notin A$, and $d'_\ell = d_\ell$ for all $\ell \notin B$. These observations first lead to

$$\sum_{\substack{k \in X' \\ k \notin A}} s_k + \sum_{\substack{k \in X' \\ k \in A}} x'_k \; = \; \sum_{\substack{k \in X' \\ k \notin A}} s'_k + \sum_{\substack{k \in X' \\ k \in A}} x'_k \; = \; \sum_{k \in X'} x'_k \; \leq \; \sum_{\ell \in Y'} y'_\ell + f(S'_-, Y') \tag{A.1}$$

Note that every demander ℓ in $Y' \cap B$ receives *exactly* his peak allocation d_ℓ for a truthful report, so for the coalition B of demanders to do weakly better in the (G, s', d') problem, $y'_\ell = d_\ell$ for each such ℓ. Therefore,

$$\sum_{\ell \in Y'} y'_\ell \; = \; \sum_{\ell \in Y' \backslash B} y'_\ell + \sum_{\ell \in Y' \cap B} y'_\ell \; \leq \; \sum_{\ell \in Y' \backslash B} d'_\ell + \sum_{\ell \in Y' \cap B} d_\ell \; = \; \sum_{\ell \in Y'} d_\ell. \tag{A.2}$$

Finally,

$$\sum_{\ell \in Y'} d_\ell + f(S'_-, Y') \; = \; \sum_{\ell \in Y'} y_\ell \; + f(S'_-, Y') \; \leq \; \sum_{k \in X'} x_k \; \leq \; \sum_{\substack{k \in X' \\ k \notin A}} s_k + \sum_{\substack{k \in X' \\ k \in A}} x_k \tag{A.3}$$

For every supplier in A to be (weakly) better off when reporting s', we must have $x'_k \geq x_k$ for each $k \in X'$. Combining this with inequalities (A.1) and (A.3), we conclude that all the inequalities in (A.1)-(A.3) hold as equations. In particular, $x'_k = x_k$ for all $k \in X'$, and $y'_\ell = y_\ell$ for $\ell \in Y'$. Therefore, whether the report is s or is s', the suppliers in X' send all of their flow only to the demanders in Y' and Y''; Moreover, the edges from X' to Y'' and S'_- to Y' are saturated and that the demanders in Y' receive all of their flow only from the suppliers in X' and from the saturated edges from S'_- Therefore, removing the suppliers in X' and the demanders in Y' and the saturated edges from X' to Y'' and S'_- to Y' does not affect the allocation rule for either problem. As we picked a smallest counterexample, Y' must be empty.

We now turn to the other case. Let $\tilde{X} := S_+ \cap S'_-$. Define $\tilde{\tilde{Y}} := f(\tilde{X}) \cap D_- \cap D'_-$ and Consider the demanders in $\tilde{Y} := f(\tilde{X}) \cap D_- \cap D'_+$

- For any such demander ℓ, $d'_\ell = y'_\ell$ and $y_\ell \leq d_\ell$. Also, $s_k = x_k$ and $x'_k \leq s'_k$ for any $k \in \tilde{X}$.
- When the report is s', every such demander can receive flow from the suppliers in $\tilde{X}$ and suppliers in $g(\tilde{Y}) \cap S_- \cap S'_-$. The supplier $i \in \tilde{X}$ send flow $z_{ij} = u_{ij}$ to every demander $j \in \tilde{Y}$ in the graph $G(\tilde{X}, \tilde{\tilde{Y}})$. Suppliers in S_- send at most $\sum_{ij \in G(S_-, \tilde{Y})} u_{ij}$ units of flow to $\tilde{Y}$. But note that $f(\tilde{X}) \supseteq \tilde{Y}$ and hence $\tilde{X}$ can send flow to agents in $D'_+ \backslash \tilde{Y}$. Therefore $\sum_{k \in \tilde{X}} x'_k \;\geq\; \sum_{\ell \in \tilde{Y}} y'_\ell - \sum_{ij \in G(S_-, \tilde{Y})} u_{ij} + \sum_{ij \in G(\tilde{X}, \tilde{\tilde{Y}})} u_{ij}$.
- When the report is s, the suppliers in $\tilde{X}$ send flow only to the demanders in D_-, and they can send flow only to the demanders they are connected to. so the suppliers in $\tilde{X}$ can send flow only to the demanders in $\tilde{Y} \cup \tilde{\tilde{Y}}$. The agents in $\tilde{X}$ can send at most $\sum_{ij \in G(\tilde{X}, \tilde{\tilde{Y}})} u_{ij}$ units of flow to the agents in $\tilde{\tilde{Y}}$. Also, the agents in $\tilde{Y}$ receive flow $\sum_{ij \in G(S_-, \tilde{Y})} u_{ij}$ from S_- Therefore $\sum_{k \in \tilde{X}} x_k \;\leq\; \sum_{\ell \in \tilde{Y}} y_\ell - \sum_{ij \in G(S_-, \tilde{Y})} u_{ij} + \sum_{ij \in G(\tilde{X}, \tilde{\tilde{Y}})} u_{ij}$.

Lets denote $\tilde{f}(S_-, \tilde{Y}) := -\sum_{ij \in G(S_-, \tilde{Y})} u_{ij} + \sum_{ij \in G(\tilde{X}, \tilde{\tilde{Y}})} u_{ij}$.

Finally, note that $s'_k = s_k$ for all $k \notin A$, and $d'_\ell = d_\ell$ for all $\ell \notin B$. Putting all this together, we have:

$$\sum_{\substack{\ell \in \tilde{Y} \\ \ell \notin B}} d_\ell + \sum_{\substack{\ell \in \tilde{Y} \\ \ell \in B}} d'_\ell + \tilde{f}(S_-, \tilde{Y}) \;=\; \sum_{\ell \in \tilde{Y}} d'_\ell + \tilde{f}(S_-, \tilde{Y}) \;=\; \sum_{\ell \in \tilde{Y}} y'_\ell + \tilde{f}(S_-, \tilde{Y}) \tag{A.4}$$

and

$$\sum_{\ell \in \tilde{Y}} y'_\ell + \tilde{f}(S_-, \tilde{Y}) \;\leq\; \sum_{k \in \tilde{X}} x'_k \;\leq\; \sum_{k \in \tilde{X} \backslash A} s'_k + \sum_{k \in \tilde{X} \cap A} x'_k \;=\; \sum_{k \in \tilde{X} \backslash A} s_k + \sum_{k \in \tilde{X} \cap A} x'_k. \tag{A.5}$$

Note that every supplier k in $\tilde{X} \cap A$ receives *exactly* his peak allocation s_k for a truthful report, so for the coalition A of suppliers to do weakly better in the (G, s', d') problem, $x'_k = s_k$ for each such k. Thus,

$$\sum_{k \in \tilde{X} \setminus A} s_k + \sum_{k \in \tilde{X} \cap A} x'_k \;=\; \sum_{k \in \tilde{X}} s_k \;=\; \sum_{k \in \tilde{X}} x_k \;\leq\; \sum_{\ell \in \tilde{Y}} y_\ell + \tilde{f}(S_-, \tilde{Y}) \;\leq\; \sum_{\substack{\ell \in \tilde{Y} \\ \ell \notin B}} d_\ell + \sum_{\substack{\ell \in \tilde{Y} \\ \ell \in B}} y_\ell + \tilde{f}(S_-, \tilde{Y}) \quad \text{(A.6)}$$

For every demander in B to be (weakly) better off, we must have $y'_\ell \geq y_\ell$ for each $\ell \in \tilde{Y}$. Combining this with inequalities (A.4)-(A.6), we conclude that all the inequalities in (A.4)-(A.6) hold as equations. In particular, $x'_k = x_k$ for all $k \in \tilde{X}$, and $y'_\ell = y_\ell$ for $\ell \in \tilde{Y}$. Therefore, whether the report is s or is s', the suppliers in $\tilde{X}$ send all of their flow only to the demanders in $\tilde{Y}$ and to the demanders in $\tilde{\tilde{Y}}$; Moreover, the edges from $\tilde{X}$ to $\tilde{\tilde{Y}}$ are saturated in both problems; So are the edges S_- to $\tilde{Y}$. and that the demanders in $\tilde{Y}$ receive all of their flow only from the suppliers in $\tilde{X}$ and through the saturated edges from S_- in both the problems. Therefore, removing the suppliers in $\tilde{X}$ and the demanders in $\tilde{Y}$ and the saturated edges from $\tilde{X}$ to $\tilde{\tilde{Y}}$ and S_- to $\tilde{Y}$, we do not affect the allocation rule for either problem. As we picked a smallest counterexample, $\tilde{X}$ must be empty.

We now establish that the decomposition does not change in a smallest counterexample. We already know that $Y' = \emptyset$, which implies $D'_- \subseteq D_-$. Suppose this containment is strict so that there is a demander $j \in D_- \setminus D'_-$. The links from S_- to j are completely saturated. As $\tilde{X} = \emptyset$, j receives flow only from the suppliers in $S_- \cap S'_-$. Also, the flow on the edges from a supplier $i \in S_- \cap S'_-$ to j is such that $z'_{ij} \leq u_{ij} = z_{ij}$. Hence, the allocation for agent j is such that, $y'_j = d'_j \leq y_j$. But now note that, if $j \in B$ then, $d'_j \geq y_j$ or if $j \notin B$ then $y'_j = d_j \leq y_j \leq d_j$. In both the cases, we have the equality $y'_j = d'_j = y_j$. This implies, $g(j) \cap S_- \cap S'_+ = \{\emptyset\}$; The links from S_- to j is saturated in both the problems (Follows from the fact that the given rule allocates the pareto value to the agents in both the networks, in particular $y'_k = d'_k$ when the reports are d'). Hence, we can remove those saturated edges and adjust the peaks of suppliers and demanders. The adjusted demand of agent j now is $d'_j = 0$. w.l.o.g we can skip the case $d'_j = 0$ as we can delete such a j to obtain the new decomposition or just place it in D_-. Therefore $D'_- = D_-$, which implies $D'_+ = D_+$, $S'_+ = S_+$, and $S'_- = S_-$.

To complete the argument, let A be as defined earlier. Let $A_+ = A \cap S_+$ and $A_- = A \cap S_-$, $B_+ = A \cap D_+$ and $B_- = A \cap D_-$. Now, for any $j \in B_+$, $d'_j \neq d_j$ implies $y'_j = d'_j \neq d_j$ causing j

to do worse by reporting d'_j. Hence, it follows, $\forall j \in B_+$, $d'_j = d_j$. By a similar argument, we could establish $s'_j = s_j \forall j \in A_+$.

For any $i \in A_-$, $s'_i < x_i$ implies $x'_i \leq s'_i < x_i$, causing i to do worse by reporting s'_i. Likewise, any $i \in B_-$, $d'_i < y_i$ implies $y'_i \leq d'_i < y_i$, causing i to do worse by reporting d'_i. So any improving coalition A must be such that $s'_i \geq x_i$ for all $i \in A_-$ and $d'_i \geq y_i$ for all $i \in B_-$.

Now, we use the strong invariance property of the rule to conclude the result. Partition the agents in $A_- = A_s \cup A_x$ where $A_s := \{x_i = s_i | i \in A_-\}$ and $A_x := \{x_i < s_i | i \in A_-\}$. Lets start with an agent $i \in A_s$, such an agent reports $s'_i > x_i = s_i$ and receives $x'_i = s_i$. Now, consider the alternate set of reports such that $s''_j = s'_j$ for all agents $j \neq i$ and $s''_i = s_i$ and denote the corresponding network by $G(S'', D'')$. Strong invariance property implies that when the peak report $s'_{ii} \geq x'_i = s_i$ then the allocation profile of the agents remains the same in the networks $G(S', D')$ and $G(S'', D'')$. Hence, we can find a *smaller* counterexample by removing i from A_-. Hence, we can remove all the agents from A_s and still find a smaller counterexample. Hence, we can assume the smallest counterexample $A_s = \{\emptyset\}$.

On similar lines, strong invariance property also implies that when an agent i with $x_i < s_i$ misreports such that $s'_i > x_i$ then $x'_i = x_i \ \forall \ i \in S$. Hence, applying this argument for each agent iteratively, we can conclude that when the set of agents in A_x inflate their peaks, the allocation does not change i.e. $x'_i = x_i \ \forall \ i \in S$. Hence, no agent improves his allocation under this rule, concluding the result.

Now, we turn to prove the other direction of the result i.e. any rule that is PO^* and peak GSP is strongly invariant. We discuss the result only for the suppliers, by symmetry a similar reasoning follows for the demanders. Suppose such a rule is not strongly invariant. Since agents in S_+ receive their peak, strong invariance property needs to be discussed only in the context of the agents in S_- where $x_i \leq s_i$. Now, consider an agent $i \in S_-$ such that $x_i < s_i$. Consider a report by agent i such that $s'_i \geq x_i$. From Lemma 5 it follows that PO^* + strategyproof implies invariance. Hence, $x'_i = s_i$. Furthermore, it follows from the earlier discussion that the decomposition and maximum flow does not change in this new problem. Hence, $\sum_{k \in S_-} x_k = \sum_{k \in S_-} x'_k$. Suppose $x'_k = x_k \ \forall \ k \in S_-$ then we are done. Suppose, $x'_k \neq x_k$ for some agent $k \in S_-$, then there exists at least one agent j such that $s_j \geq x'_j > x_j$ (agent j improves the allocation). Thus, the pair of agents i and j

represent a colluding group who can deviate and (weakly) improve the allocation which contradicts the peak GSP property of the rule. ∎