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ON LAGRANGE-HERMITE INTERPOLATION*

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1. Introduction. Let the $p(n+1)$ numbers $y_i^{(m)}$, $0 \leqq i \leqq n$, $0 \leqq m \leqq p-1$, be given. It is well known that there exists a unique polynomial $P_{n,p}(t)$ of degree $p(n+1)-1$ such that

$$P_{n,p}^{(m)}(x_i) = y_i^{(m)}, \quad 0 \leqq i \leqq n, \quad 0 \leqq m \leqq p-1. \tag{1.1}$$

A classical problem is to find a formula for $P_{n,p}(t)$ in the form

$$P_{n,p}(t) = \sum_{i=0}^{n} \sum_{m=0}^{p-1} C_{m,i}^{n,p}(t) y_i^{(m)}.$$

The conditions on the $C_{m,i}^{n,p}(t)$ are that

$$D_t^j C_{m,i}^{n,p}(x_r) = \delta_{j,m}\delta_{r,i}, \quad 0 \leqq r \leqq n, \quad 0 \leqq j \leqq p-1, \tag{1.2}$$

where $D_t \equiv d/dt$ and $\delta_{j,m}$ is a Kronecker symbol. These conditions are used by Householder [5, pp. 193–195] to derive the formulas for $p = 1, 2$. The formula for $p = 3$ is given by Salzer [9]. The solution for $n = 0$ is given by Taylor's formula.

Many authors have reported on the case where p depends on i. General prescriptions for a solution in this more general case may be found in Fort [2, pp. 85–88], Greville [3], Hermite [4], Krylov [6, pp. 45–49], Kuntzmann [7, pp. 167–169], and Spitzbart [12]; but these prescriptions do not determine the structure of the interpolating polynomial. By restricting ourselves to the case where p is independent of i, which is the most important case in practice, we can determine the structure. Salzer [10] discovered some of the properties of $P_{n,p}(t)$ by semiempirical means.

We shall obtain, by a partial fraction expansion, a solution of surprising simplicity. [See (3.6), (3.7), or (3.8).] The solution depends upon the Bell polynomials which we now discuss.

2. Bell polynomials. Let $g = g(t)$ and define B_n by

$$e^{-\omega g} D_t^n e^{\omega g} = B_n(\omega) = B_n(\omega; g_1, \cdots, g_n), \qquad g_i \equiv g^{(i)}. \tag{2.1}$$

B_n is a polynomial in ω with coefficients which are polynomials in g_i. Define $U_{n,k}$ by

$$B_n(\omega; g_1, \cdots, g_n) = \sum_{k=0}^{n} U_{n,k}(g_1, \cdots, g_{n-k+1})\omega^k.$$

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Then

$$U_{n,k} = \frac{1}{k!} D_\omega^{\,k} B_n(0).$$

The $B_n(1; g_1, \cdots, g_n)$ were studied by Bell [1]. (See also Schlömilch [11, p. 4].) An explicit formula for B_n is

$$B_n = n! \sum \omega^j \prod_{i=1}^{n} \frac{1}{b_i!} \left(\frac{g_i}{i!}\right)^{b_i},$$

where $j = \sum_{i=1}^n b_i$ and where the sum is taken over all nonnegative integers b_i for which $\sum_{i=1}^n ib_i = n$.

Let $F(t) = f[g(t)]$. Then

$$F^{(n)} = \sum_{k=0}^{n} f^{(k)} U_{n,k}(g', \cdots, g^{(n-k+1)}), \tag{2.2}$$

or

$$F^{(n)} = B_n(f; g', \cdots, g^{(n)}), \qquad f^{(k)} \equiv f^k.$$

Generating functions and symbolic recurrence relations for the Bell polynomials may be found in Bell [1] and Riordan [8, pp. 35–38, 45–48]. The first five B_n are:

$$B_0 = 1,$$

$$B_1 = \omega g_1,$$

$$B_2 = \omega^2 g_1^{\,2} + \omega g_2,$$

$$B_3 = \omega^3 g_1^{\,3} + 3\omega^2 g_1 g_2 + \omega g_3,$$

$$B_4 = \omega^4 g_1^{\,4} + 6\omega^3 g_1^{\,2} g_2 + \omega^2(4g_1 g_3 + 3g_2^{\,2}) + \omega g_4.$$

3. The formula for the interpolatory polynomial. Let $P(t)/Q(t)$ be a proper rational function and let $Q(t)$ have a zero of multiplicity p at x_i. Let

$$\frac{1}{Q(t)} = \sum_{j=1}^{p} \frac{\alpha_{p,j}}{(t - x_i)^j} + \eta(t),$$

$$\frac{P(t)}{Q(t)} = \sum_{j=1}^{p} \frac{\beta_{p,j}}{(t - x_i)^j} + \lambda(t).$$

Then it is easy to show that

$$\beta_{p,p-j} = \sum_{k=0}^{j} \alpha_{p,p-k} \frac{P^{(j-k)}(x_i)}{(j-k)!}. \tag{3.1}$$

This result is the key to the solution of the Lagrange-Hermite interpolation problem. It permits us to write the interpolatory polynomial as a linear combination of the $y_i^{(m)}$.

Let

$$\pi(t) = \prod_{i=0}^{n} (t - x_i), \quad Q(t) = \pi^p(t), \tag{3.2}$$
$$R_i(t) = \frac{\pi(t)}{t - x_i}, \quad L_i(t) = \frac{R_i(t)}{R_i(x_i)}.$$

We calculate the contribution to $P_{n,p}(t)$ due to x_i and then sum on i. We have

$$P_{n,p}(t) \equiv Q(t)\frac{P_{n,p}(t)}{Q(t)} = Q(t)\sum_{j=1}^{p}\frac{\beta_{p,i,j}}{(t-x_i)^j} + \rho(t),$$
$$\beta_{p,i,p-j} = \sum_{k=0}^{j} \alpha_{p,i,p-k}\frac{P^{(j-k)}(x_i)}{(j-k)!}, \qquad \alpha_{p,i,p-k} = \frac{1}{k!} D_t^k R_i^{-p}(x_i).$$

Using

$$P^{(j-k)}(x_i) = y_i^{(j-k)},$$

we obtain, after some manipulation,

$$P_{n,p}(t) = L_i^p(t)\sum_{m=0}^{p-1} y_i^{(m)}\frac{(t-x_i)^m}{m!} \cdot \sum_{\nu=0}^{p-1-m}\frac{(t-x_i)^\nu}{\nu!} R_i^p(x_i) D_t^\nu R_i^{-p}(x_i) + \rho(t). \tag{3.3}$$

Let

$$S_\nu \equiv S_\nu(x_i) = (-1)^\nu(\nu-1)!\sum_{r=0;\, r\neq i}^{n}\frac{1}{(x_i-x_r)^\nu}. \tag{3.4}$$

It follows from (2.1), (3.2), and (3.4) that

$$R_i^p(x_i) D_t^\nu R_i^{-p}(x_i) = B_\nu(p; S_1, \cdots, S_\nu). \tag{3.5}$$

Using (3.5) and adding the contributions from all the x_i, we obtain as a solution to our problem

$$P_{n,p}(t) = \sum_{i=0}^{n} L_i^p(t)\sum_{m=0}^{p-1}\frac{(t-x_i)^m}{m!} y_i^{(m)} \sum_{\nu=0}^{p-1-m}\frac{(t-x_i)^\nu}{\nu!} \cdot B_\nu(p; S_1, \cdots, S_\nu). \tag{3.6}$$

Thus the essence of the pth order Lagrange-Hermite formula is contained in the $B_\nu(p; S_1, \cdots, S_\nu)$, $0 \leqq \nu \leqq p - 1$. Let

$$G_{p,i,m} = \sum_{\nu=0}^{p-1-m}\frac{(t-x_i)^\nu}{\nu!} B_\nu(p; S_1, \cdots, S_\nu).$$

Observe that $G_{p,i,m}$ may be obtained from the polynomial $G_{p,i,0}$ by truncating the highest m terms. Hence for each p, $P_{n,p}(t)$ may be easily obtained from $G_p \equiv G_{p,i,0}$. The first five G_p are:

$$G_1 = 1,$$
$$G_2 = 1 + (t - x_i)2S_1,$$
$$G_3 = 1 + (t - x_i)3S_1 + \tfrac{1}{2}(t - x_i)^2[3^2S_1^2 + 3S_2],$$
$$G_4 = 1 + (t - x_i)4S_1 + \tfrac{1}{2}(t - x_i)^2[4^2S_1^2 + 4S_2] + \tfrac{1}{6}(t - x_i)^3[4^3S_1^3 + 3\cdot4^2S_1S_2 + 4S_3],$$
$$G_5 = 1 + (t - x_i)5S_1 + \tfrac{1}{2}(t - x_i)^2[5^2S_1^2 + 5S_2] + \tfrac{1}{6}(t - x_i)^3[5^3S_1^3 + 3\cdot5^2S_1S_2 + 5S_3] + \tfrac{1}{24}(t - x_i)^4[5^4S_1^4 + 6\cdot5^3S_1^2S_2 + 5^2(4S_1S_3 + 3S_2^2) + 5S_4].$$

Equation (3.6) may be written in a number of other ways. Let

$$T_\nu \equiv T_\nu(x_i) = (\nu - 1)! \sum_{r=0;r\neq i}^{n} \left(\frac{x_i - t}{x_i - x_r}\right)^\nu.$$

Then

$$(3.7)\quad P_{n,p}(t) = \sum_{i=0}^{n} L_i^p(t) \sum_{m=0}^{p-1} \frac{(t - x_i)^m}{m!} y_i^{(m)} \sum_{\nu=0}^{p-1-m} \frac{1}{\nu!} B_\nu(p; T_1, \cdots, T_\nu).$$

Let

$$H_{p,i,m,k} = \sum_{\nu=k}^{p-1-m} \frac{U_{\nu,k}}{\nu!}(T_1, \cdots, T_{\nu-k+1}).$$

Then

$$P_{n,p}(t) = \sum_{i=0}^{n} L_i^p(t) \sum_{m=0}^{p-1} \frac{(t - x_i)^m}{m!} y_i^{(m)} \sum_{k=0}^{p-1-m} H_{p,i,m,k}\, p^k.$$

A formula for $P_{n,p}(t)$ in which the coefficients are polynomials in the $L_i^{(j)}(x_i)$ may be obtained as follows. Let

$$R_i^{-p}(t) = f[g(t)], \qquad f(u) = u^{-p}, \qquad g(t) = R_i(t).$$

Then using (2.2), and with $L^{(j)} \equiv L_i^{(j)}(x_i)$,

$$R^p(x_i)D_t^\nu R^{-p}(x_i) = \sum_{k=0}^{\nu} (-1)^k k! C(p + k - 1, k) U_{\nu,k}(L', \cdots, L^{(\nu-k+1)}).$$

Hence

$$P_{n,p}(t) = \sum_{i=0}^{n} L_i^p(t) \sum_{m=0}^{p-1} \frac{(t - x_i)^m}{m!} y_i^{(m)} E_{p,i,m},$$

$$(3.8)\quad E_{p,i,m} = \sum_{\nu=0}^{p-1-m} \frac{(t - x_i)^\nu}{\nu!} \sum_{k=0}^{\nu} (-1)^k k! C(p + k - 1, k) \cdot U_{\nu,k}(L', \cdots, L^{(\nu-k+1)}).$$

Observe that $E_{p,i,m}$ may be obtained from the polynomial $E_{p,i,0}$ by truncating the highest m terms. Hence for each p, $P_{n,p}(t)$ may be easily obtained from $E_p \equiv E_{p,i,0}$. The first five E_p are

$$E_1 = 1,$$

$$E_2 = 1 + (t - x_i)[-2L'],$$

$$E_3 = 1 + (t - x_i)[-3L'] + \tfrac{1}{2}(t - x_i)^2[-3L'' + 12(L')^2],$$

$$E_4 = 1 + (t - x_i)[-4L'] + \tfrac{1}{2}(t - x_i)^2[-4L'' + 20(L')^2] + \tfrac{1}{6}(t - x_i)^3[-4L''' + 60L'L'' - 120(L')^3],$$

$$E_5 = 1 + (t - x_i)[-5L'] + \tfrac{1}{2}(t - x_i)^2[-5L'' + 30(L')^2] + \tfrac{1}{6}(t - x_i)^3[-5L''' + 90L'L'' - 210(L')^3] + \tfrac{1}{24}(t - x_i)^4[-5L^{(4)} + 120L'L''' + 90(L'')^2 - 1260(L')^2L'' + 1680(L')^4].$$

As far as calculation with these formulas is concerned, observe that

$$L_i^{(j)}(x_i) = \frac{R_i^{(j)}(x_i)}{R_i(x_i)}.$$

The $R_i^{(j)}(x_i)$, $j \geqq 0$, may be obtained from $\pi(t)$ by repeated synthetic division.

4. Some applications. The interpolation formula may be used to generalize the Cauchy relations,

$$t^j \equiv \sum_{i=0}^{n} x_i^j L_i(t), \qquad j = 0, 1, \cdots, n.$$

Corresponding to the case $j = 0$, we have the following generalization.

$$(4.1) \qquad 1 \equiv \sum_{i=0}^{n} L_i^p(t) \sum_{\nu=0}^{p-1} \frac{(t - x_i)^\nu}{\nu!} B_\nu(p; S_1, \cdots, S_\nu).$$

Since the leading coefficient of t on the right side of (4.1) vanishes,

$$(4.2) \qquad \sum_{i=0}^{n} \frac{1}{[\pi'(x_i)]^p} B_{p-1}(p; S_1, \cdots, S_{p-1}) = 0.$$

This generalizes

$$\sum_{i=0}^{n} \frac{1}{\pi'(x_i)} = 0.$$

We can derive a formula for the confluent divided difference with the same number of repetitions of all arguments, $f[x_0, p; x_1, p; \cdots; x_n, p]$.

(This notation is introduced in Traub [13, pp. 241–242].) Since this divided difference is the coefficient of the highest degree term in (3.6), we obtain

$$f[x_0, p; x_1, p; \cdots; x_n, p] = \sum_{m=0}^{p-1} \frac{B_{p-1-m}(p; S_1, \cdots, S_{p-1-m})}{m!(p-1-m)!} \cdot \sum_{i=0}^{n} \frac{f^{(m)}(x_i)}{[\pi'(x_i)]^p}. \tag{4.2}$$

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