Study of Linear Information for Classes of Polynomial Equations

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by

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Abstract

Linear adaptive information for approximating a zero of f is studied where f belongs to the class of polynomials of unbounded degree. A theorem on constrained approximation of smooth functions by polynomials is established.

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1. Summary.

For a given positive ϵ we seek a point x^* such that $|x^*-\alpha_p| < \epsilon$, where α_p is a zero of a real polynomial p in the interval $[a,b]$. We assume that p belongs to the class F_1 of polynomials having a root in $[a,b]$ or to the class F_2 of polynomials which are nonpositive at a, nonnegative at b and have exactly one simple zero in $[a,b]$. The information on p consists of n values of arbitrary linear functionals which are computed adaptively. The point x^* is constructed by means of an algorithm which is an arbitrary mapping depending on the information on p.

We show that if $\epsilon \leq (b-a)/2$ then there exists no information and no algorithm for finding x^* for every p from F_1, no matter how large the value of n. This is a stronger result than that obtained for smooth functions in [7].

For the class F_2 we can find a point x^* for arbitrary p and ϵ. An optimal algorithm, i.e., an algorithm with the smallest error, is the bisection of the smallest known interval containing a root of p. We also exhibit optimal information operators, i.e., the linear functionals for which the error of an optimal algorithm that uses them is

minimal. It turns out that in the class of nonadaptive information, i.e., when functionals are given simultaneously, optimal information consists of the evaluations of a polynomial at n-equidistant points in [a,b]. This is a stronger result than that obtained for continuous functions in [9, p. 166]. In the class of adaptive continuous information, i.e., when the next continuous functional depends on the values of all previously computed functionals, optimal information consists of evaluations of a polynomial at n points generated by the bisection method. This is a stronger result than that obtained for C^{∞} functions in [6]. To prove this result we establish a theorem on constrained approximation of smooth functions by polynomials. More precisely we prove that a smooth function can be arbitrarily well uniformly approximated by a polynomial which satisfies constraints given by n arbitrary continuous linear functionals.

Our results indicate that the problem of finding an ϵ-approximation to a real zero of a real polynomial is essentially of the same difficulty as the problem of finding an ϵ-approximation to a zero of infinitely differentiable function, see [6,7]. This makes the results of [6] and [7] stronger. We stress that we did not assume the knowledge of the degree of a polynomial. The problem of finding an ϵ-approximation to a zero of a polynomial of known degree has been studied in many recent papers, e.g., [1,2,3,4,8].

2. Basic definitions and results.

Let $P = P[a,b]$ be the set of all real polynomials on the interval $[a,b]$ in $\mathbb{R}$, let $S(p)$ be the set of all zeros of p in $[a,b]$ for $p \in P$, and let $C^\infty = C^\infty[a,b]$ be the space of infinitely differentiable functions in $[a,b]$.

Define two subclasses F_1 and F_2 of P by

$$(2.1) \qquad F_1 = \{p \in P: S(p) \neq \emptyset, \ \|p\| \leq 1\},$$

where $\|\cdot\|$ is an arbitrary seminorm in C^∞ and

$$(2.2) \qquad F_2 = \left\{ \begin{array}{l} p \in P: p(a) \leq 0, \ p(b) \geq 0, \\ S(p) \text{ is a singleton and} \\ \quad f'(S(p)) \neq 0 \end{array} \right\}.$$

For a given ϵ, $\epsilon > 0$, define the set

$$(2.3) \qquad S(p,\epsilon) = \{x \in [a,b]: \mathrm{dist}(x,S(p)) < \epsilon\}, \quad \forall p \in P.$$

The set $S(p,\epsilon)$ is of course not empty for $p \in F_1 \cup F_2$. The problem is to find an ϵ-approximation to a zero of a polynomial p from F_1 or F_2, i.e., a point x^* such that

$$(2.4) \qquad x^* \in S(p,\epsilon).$$

To find x^* satisfying (2.4) we use an <u>information operator</u> N_n and an <u>algorithm</u> φ using N_n. These are defined as in [9].

Let $f \in C^\infty$ and

(2.5) $$N_n(f) = [L_1(f), L_2(f;y_1), \dots, L_n(f;y_1,\dots,y_{n-1})]$$

where

$$y_i = L_i(f;y_1,\dots,y_{i-1})$$

and

(2.6) $$L_{i,f}(\cdot) \stackrel{df}{=} L_i(\cdot;y_1,\dots,y_{i-1}): C^\infty \to \mathbb{R}$$

is a linear functional, $i = 1,2,\dots,n$. If $L_{i,f}(\cdot) = L_i(\cdot)$, $\forall i$, i.e., $L_{i,f}$ does not depend on the previously computed values $y_1,\dots,y_{i-1}$ the information operator is called <u>nonadaptive</u>; otherwise it is called <u>adaptive</u>. The total number of functional evaluations n is called the <u>cardinality</u> of N_n.

Knowing $N_n(p)$ we approximate x^* by an algorithm φ_i which is a mapping

(2.7) $$\varphi_i: N_n(F_i) \to [a,b], \quad i = 1,2.$$

The <u>error of the algorithm</u> φ_i in the class F_i is defined by

(2.8) $$e(\varphi_i) = \sup_{p \in F_i} \operatorname{dist}(S(p), \varphi_i(N_n(p))).$$

Thus $x^* = \varphi_i(N_n(p))$ satisfies (2.4) for every p in F_i iff $e(\varphi_i) < \varepsilon$. Note that (2.8) can be restated as

$$(2.9)\qquad e(\varphi_i) = \sup_{p \in F_i} e(\varphi_i, p)$$

where the local error $e(\varphi_i, p)$ is given by

$$(2.10)\qquad e(\varphi_i, p) = \sup\{\mathrm{dist}(S(\tilde{p}), \varphi(N_n(p))) : p, \tilde{p} \in F_i : N_n(p) = N_n(\tilde{p})\}.$$

Define the <u>radius</u> of <u>the information-operator</u> N_n (briefly <u>radius of information</u>) by

$$(2.11)\qquad r(N_n, F_i) = \sup_{p \in F_i} r_i(N_n, p),$$

where the <u>local radius</u> $r_i(N_n, p)$ is given by

$$(2.12)\qquad r_i(N_n, p) = \frac{1}{2} \sup\{\mathrm{dist}(S(\tilde{p}), S(\tilde{\tilde{p}})),\ \tilde{p}, \tilde{\tilde{p}} \in F_i : N_n(p) = N_n(\tilde{p}) = N_n(\tilde{\tilde{p}})\}.$$

Let $\Phi_i = \Phi_i(N_n)$ be the class of all algorithms of the form (2.7) using the information operator N_n. It is obvious that

$$(2.13)\qquad \inf_{\varphi_i \in \Phi_i} e(\varphi_i, p) = r_i(N_n, p), \qquad \forall p \in F_i$$

and

$$(2.14)\qquad \inf_{\varphi_i \in \Phi_i} e(\varphi_i) = r(N_n, F_i).$$

We are interested in algorithms for which the error $e(\varphi_i)$ is minimal. An algorithm φ_i^0 is optimal iff

$$(2.15)\qquad e(\varphi_i^0) = r(N_n, F_i).$$

The radius of information measures the strength of an information operator. We can solve the problem (2.4) iff $r(N_n, F_i) < \epsilon$. For a given n we want to find the functionals in (2.5) such that the radius of information is minimized. More precisely, let $\mathcal{N}_n$ be a class of information operators with cardinality at most n. Then the information operator N_n^0, $N_n^0 \in \mathcal{N}_n$ is <u>optimal</u> iff

(2.16) $$r(N_n^0, F_i) = \inf_{N \in \mathcal{N}_n} r(N, F_i).$$

In this paper, we solve the following problems:

(2.17) In Section 3 we prove that if $\epsilon < (b-a)/2$ then there exist no information and no algorithm for finding x^* for every p from F_1, no matter how large the number n of functional evaluations. This is a stronger result than that obtained for the class of infinitely differentiable functions in [6].

(2.18) In Section 4 we prove that the optimal nonadaptive information for solving (2.4) in the class F_2 consists of evaluations of a polynomial at n equidistant points in $[a,b]$. This is a stronger result than that obtained in [9, p. 166] for the class of continuous functions changing a sign at the endpoints of [a,b].

(2.19) In Section 5 we first prove Theorem 5.1 which is of intrinsic interest. Namely we assume that N_n of the form (2.5) is continuous, i.e., that $|L_{i,f}(g)| \le K_f\|g\|_\infty$ for $0 \le K_f < +\infty$, $\forall g,f \in C^\infty$, $i = 1,2,\dots,n$, and show that for an arbitrary function $f \in C^\infty$ and arbitrary N_n there exists a polynomial p having the same information as f, $N_n(p) = N_n(f)$, such that $\|p-f\|_\infty$ and $\|p'-f'\|_\infty$ are arbitrarily small. Using Theorem 5.1 we prove that the optimal adaptive continuous information for solving (2.4) in the class F_2 is the evaluation of a polynomial at n points generated by the bisection method (Theorem 5.2). This is a stronger result than that obtained in [6], assuming continuity of information. We also stress that using the same proof technique as in the proof of Theorem 5.2 one obtains Theorem 4.1 of [10] for the case of real polynomials and continuous information.

3. Class F_1.

In this section we show that there exists no information and no algorithm to solve (2.4) in the class F_1 with $\varepsilon < (b-a)/2$. A similar result was established in [7] for the class of infinitely differentiable functions. Here we present a sketch of the proof, since the idea is similar to that presented in [7]. Namely we prove

Theorem 3.1:

$$r(N_n, F_1) = (b-a)/2 \tag{3.1}$$

for arbitrary n and arbitrary adaptive information N_n of the form (2.5). □

Proof: Setting $\varphi(N(p)) = (a+b)/2$ we get $e(\varphi) \le (b-a)/2$. Thus $r(N_n, F_1) \le (b-a)/2$ due to (2.14). To prove the reverse inequality we construct for every γ, $0 < \gamma < (b-a)/2$, two polynomials $\tilde{p}$ and $\tilde{\tilde{p}}$ from F_1 such that $N_n(\tilde{p}) = N_n(\tilde{\tilde{p}})$ and $\text{dist}(S(\tilde{p}), S(\tilde{\tilde{p}})) \ge b-a-2\gamma$. Then (3.1) will follow from (2.11) with $\gamma \to 0$.

Construction of the polynomials $\tilde{p}$ and $\tilde{\tilde{p}}$ is similar to the construction of functions $\tilde{f}$ and $\tilde{\tilde{f}}$ from [7, section 2]. Define the functions h_i, $i = 1,2,\ldots,n+1$ as in [7], i.e.,

$$h_i(x) = \begin{cases} \exp(16((n+1)/\gamma)^4)\exp(-1/((x-x_{i-1})^2(x-x_i)^2)) \\ \qquad\qquad\qquad\qquad \text{if } x \in [x_{i-1},x_i] \\ 0 \qquad\qquad \text{otherwise,} \end{cases}$$

where $x_i = a + i\gamma/(n+1)$, $i = 0,1,\ldots,n+1$, and γ is an arbitrary number, $0 < \gamma < (b-a)/2$.

Let p_i be the polynomials such that

$$\max_{x\in[a,b]} |p_i(x)-h_i(x)| \le 10^{-2}/(n+1).$$

Let $d = \max(\|1\|, \max_{1\le i\le n+1} \|p_i\|)$, and take a positive δ such that

$$\delta < \begin{cases} 1/(4(n+1)d) & \text{if } d > 0, \\ +\infty & \text{if } d = 0. \end{cases}$$

Applying N_n to the constant polynomial $\delta(x) = \delta$ we get the information operator $N_{n,\delta}$, see (2.6)

$$N_{n,\delta}(p) = [L_{1,\delta}(p),\ldots,L_{n,\delta}(p)].$$

Let $c = [c_1,\ldots,c_{n+1}]$ be a nonzero solution of the homogeneous system

$$\Sigma_{i=1}^{n+1} c_i L_{j,\delta}(p_i) = 0, \qquad j = 1,2,\ldots,n.$$

Let $|c_k| = \max_{1\le i\le n+1} |c_i|$. Define the polynomial p^* by

$$p^* = \delta/|c_k|/(1+10^{-2}/(n+1))\Sigma_{i=1}^{n+1} c_i p_i .$$

Then for $c \in ((1+10^{-2}/(n+1))/(1-10^{-2}/(n+1)),3]$ let

$$p_c = \begin{cases} \delta + cp^* & \text{if} \quad c_k < 0, \\ \delta - cp^* & \text{if} \quad c_k > 0. \end{cases}$$

If $d = 0$ then $\|p_c\| = 0$. If $d > 0$ then

$$\|p_c\| \leq \delta\|1\| + c\|H\| \leq \|1\|/(4(n+1)d) + 3\delta(n+1)d \leq 1.$$

Observe that

$$p_c(x_i) \geq \delta - 3(n+1)10^{-2}/(n+1)\delta = \delta(1-3\cdot 10^{-2}) > 0$$

and

$$p_c((x_{k-1}+x_k)/2) \leq \delta - c\delta(1-10^{-2}/(n+1)) \leq \delta(1-(1+10^{-2}/(n+1)) < 0.$$

Thus p_c has a zero in $[a,b]$. The definition of p_c implies that $S(p_c) \subset [a,a+\gamma]$. The polynomial $\tilde{p}$ is defined as

$$\tilde{p} = p_c \text{ for some } c \text{ as above.}$$

To construct $\tilde{\tilde{p}}$ we proceed as above with x_i replaced by $x_i^* = b - i\gamma/(n+1)$, $i = 0,1,\ldots,n$, compare [7]. □

Theorem 3.1 states that the error of any algorithm is at least $(b-a)/2$. Thus if $\varepsilon \leq (b-a)/2$ then there exists no algorithm using linear information to solve the problem (2.4).

4. Class F_2- Optimal Nonadaptive Information.

In this section we prove that the optimal nonadaptive information for solving (2.4) in the class F_2 consists of evaluations of a polynomial at n equidistant points in [a,b]. This is a stronger result than that established in [9, p. 166] for the class of continuous functions.

Let $\mathcal{N}_n^{non}$ be the class of all nonadaptive information operators of the form (2.5) with cardinality at most n. Let

$$N_n(p) = [p(x_1),\dots,p(x_n)],$$

where $x_i = a + i(b-a)/(n+1)$, $i = 1,2,\dots,n$. Let p be an arbitrary polynomial from F_2 and $j = j(N_n(p))$ be the index such that $p(x_j) \le 0$ and $p(x_{j+1}) \ge 0$ where $x_0 = a$, $x_{n+1} = b$. Then it is clear that a zero of p lies in $[x_j, x_{j+1}]$ and zeros of all polynomials $\tilde{p}$ having the same information as p lie in $[x_j, x_{j+1}]$. Thus (2.11) and (2.12) imply that

$$(4.1)\qquad r(N_n^0, F_2) \le \frac{b-a}{2(n+1)}.$$

Then we prove

<u>Theorem 4.1</u>: The information N_n^0 is optimal in the class $\mathcal{N}_n^{non}$, i.e.,

$$\inf_{N_n \in \mathcal{N}_n^{non}} r(N_n, F_2) = r(N_n^0, F_2) = \frac{b-a}{2(n+1)}. \qquad \square$$

<u>Proof:</u> For arbitrarily small $\delta > 0$ and information $N_n(\cdot) = [L_1(\cdot), \ldots, L_n(\cdot)]$, $N_n \in \mathcal{N}_n^{non}$, we construct two polynomials p_1 and p_2 from F_2 such that $N_n(p_1) = N_n(p_2)$ and $\mathrm{dist}(S(p_1), S(p_2)) \geq (b-a)/(n+1) - \delta$. Then Theorem 4.1 will follow from (2.11), (2.12) and (4.1) with $\delta \to 0$.

Let $a = [a_0, \ldots, a_n]$ be a non-zero solution of the homogeneous system of n linear equations with $n + 1$ unknowns:

$$\Sigma_{i=0}^{n} a_i L_j(x^i) = 0, \qquad j = 1,2,\ldots,n.$$

Define the polynomial

$$p(x) = \Sigma_{i=0}^{n} a_i x^i.$$

Since p is of degree not larger than n there exists a subinterval $[c,d]$ of the interval $[a,b]$, $a < c$, $d < b$, such that $d - c \geq (b-a)/(n+1) - \delta$ and p is of a constant sign in $[c,d]$. Without loss of generality suppose that p is positive in $[c,d]$, see Fig. 4.1.

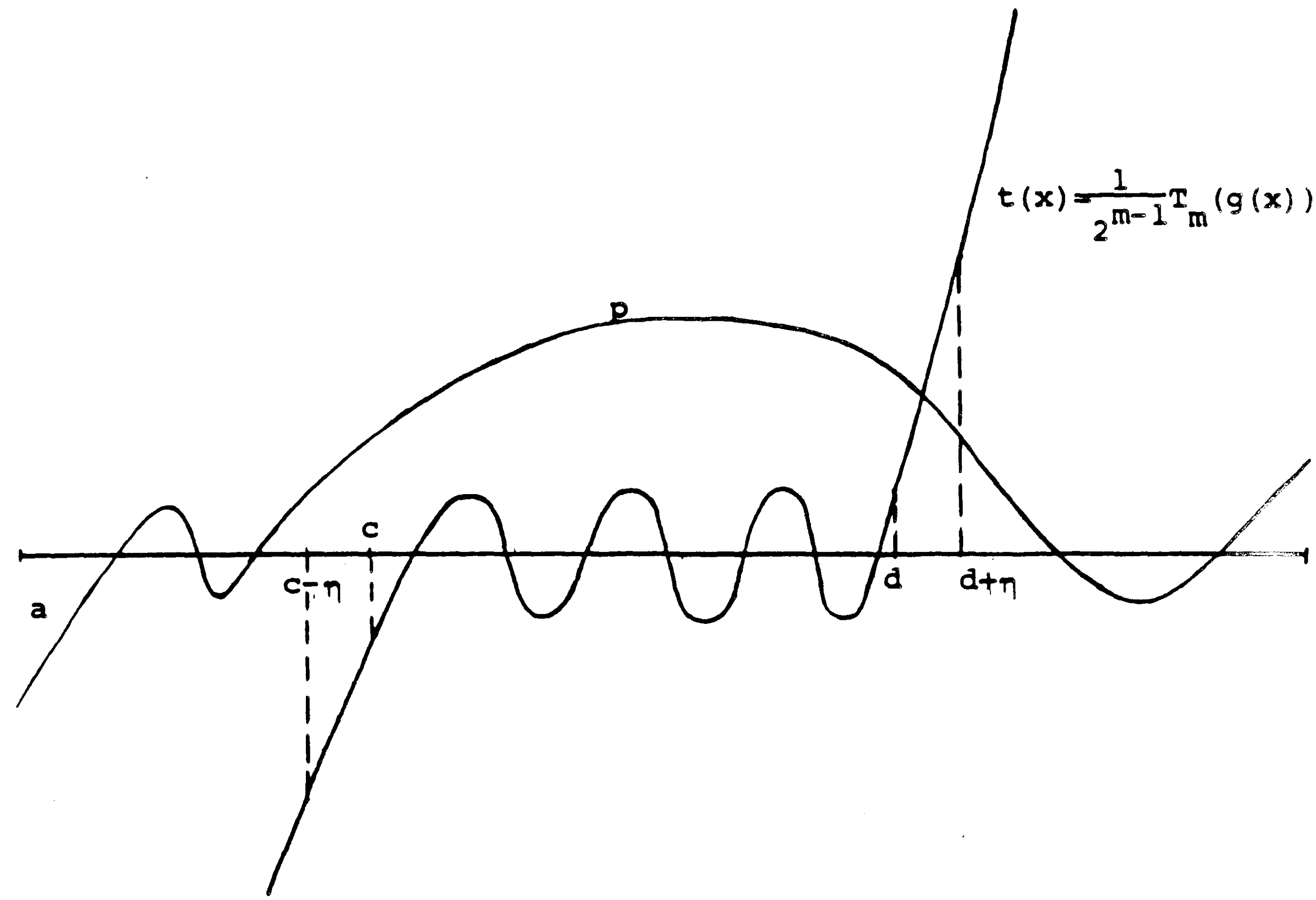

Fig. 4.1

Then take a normed Chebyschev polynomial $t(x) = 2^{-m+1}T_m(g(x))$, see Fig. 4.1, where g is a linear transformation of $[c,d]$ onto $[-1,1]$, i.e., $g(x) = \frac{2}{d-c}x - \frac{d+c}{d-c}$, m is sufficiently large odd integer and η is sufficiently small positive number, such that the following inequalities hold:

$$(4.2)\qquad \begin{cases} 2^{-m+1} < \min\limits_{x\in[c,d]} p(x) & \\ t(x) < -|p(x)| & x \in [a,c-\eta) \\ t(x) > |p(x)| & x \in (d+\eta,b] \\ t'(x) > |p'(x)| & x \in [c-\eta,c] \cup [d,d+\eta] \\ c-\eta > a \quad \text{and} \quad d+\eta < b. & \end{cases}$$

The numbers η and m exist due to well known properties of Chebyschev polynomials. Define

$$\text{(4.3)}\qquad \begin{aligned} p_1(x) &= t(x) + p(x), \\ p_2(x) &= t(x) - p(x). \end{aligned}$$

Then $N_n(p_1) = N_n(t) = N_n(p_2)$ and $p_i(a) < 0$, $p_i(b) > 0$, $i = 1,2$. Moreover each of p_1 and p_2 has a single and simple zero. $S(p_1) \subset [c-\eta,c]$, $S(p_2) \subset [d,d+\eta]$. Thus $p_i \in F_2$, $\forall i$. Since

$$\text{dist}(S(p_1),S(p_2)) \geq d-c \geq \frac{b-a}{n+1} - \delta$$

the proof is completed. □

5. Class F_2- Optimal Continuous Adaptive Information.

In this section we prove that the bisection information N_n^{bis}, defined as in [6], is optimal in the class of <u>all</u> adaptive continuous information η_c. This is a stronger result than that obtained in [6], assuming the class of continuous information operators.

We first prove

<u>Theorem 5.1</u>: For every function $f \in C^\infty[a,b]$, information $N_n \in \eta_c$ and $\delta > 0$, $\gamma > 0$, there exists a polynomial $w \in P[a,b]$ such that

$$(5.1) \qquad \|w-f\|_\infty \leq \delta, \quad \|w'-f'\|_\infty \leq \gamma,$$

and

$$(5.2) \qquad N_n(w) = N_n(f). \qquad \square$$

<u>Proof</u>: Recall that $N_{n,f}(\cdot) = [L_{1,f}(\cdot),\ldots,L_{n,f}(\cdot)]$, see (2.6). Consider the functionals $L_1^*,\ldots,L_{k_n}^*$ which form the maximal set of linearly independent functionals among $L_{1,f},\ldots,L_{n,f}$. Since $L_1^*,\ldots,L_{k_n}^*$ are linearly independent and continuous on $C^\infty[a,b]$, then they are linearly independent on $P[a,b]$. Therefore there exist polynomials p_i^*, $i = 1,\ldots,k_n$, $p_i^* \in P[a,b]$, such that

$$L_j^*(p_i^*) = \delta_{i,j}, \quad \forall\, i,j.$$

Consider a sequence of polynomials $\{w_m\}_{m=1}^{\infty}$ such that

$$\text{(5.3)} \qquad \begin{aligned} \|f - w_m\|_\infty &\to 0 \\ \|f' - w_m'\|_\infty &\to 0 \end{aligned} \qquad \text{as } m \to \infty.$$

Since L_j^* are continuous, then

$$\text{(5.4)} \qquad L_j^*(f-w_m) \to 0 \text{ as } m \to \infty,\ j = 1,\dots,k_n,$$

and also $L_{j,f}(f-w_m) \to 0$ as $m \to \infty$, $j = 1,\dots,n$. For each w_m define a polynomial p_m by

$$\text{(5.5)} \qquad p_m = \Sigma_{j=1}^{k_n} L_j^*(f-w_m)\cdot p_j^*.$$

Then $L_j^*(p_m) = L_j^*(f-w_m)$, $\forall j$,

$$\|p_m\|_\infty \le \Sigma_{j=1}^{k_n} |L_j^*(f-w_m)|\cdot\|p_j^*\|_\infty \le \max_{1\le j\le k_n} \|p_j^*\|_\infty \Sigma_{j=1}^{k_n} |L_j^*(f-w_m)|$$

and

$$\|p_m'\|_\infty \le \max_{1\le j\le k_n} \|p_j^{*\prime}\|_\infty \Sigma_{j=1}^{k_n} |L_j^*(f-w_m)|.$$

Conditions (5.3) and (5.4) imply that there exists an index m_0 such that for every $m \ge m_0$ the following inequalities hold:

$$\begin{cases} \max_{1\le j\le k_n} \|p_j^*\|_\infty \Sigma_{j=1}^{k_n} |L_j^*(f-w_m)| \le \frac{\delta}{2}, \\ \max_{1\le j\le k_n} \|p_j^{*'}\|_\infty \Sigma_{k=1}^{k_n} |L_j^*(f-w_m)| \le \frac{\gamma}{2}, \\ \|f - w_m\|_\infty \le \frac{\delta}{2}, \\ \|f' - w_m'\|_\infty \le \frac{\gamma}{2}. \end{cases}$$

Define the polynomial w^* by

(5.6) $$w^* = w_{m_0} + p_{m_0}.$$

Then $L_j^*(w^*) = L_j^*(f)$, $j = 1,\ldots,k_n$ and also $L_{j,f}(w^*) = L_{j,f}(f)$, $j = 1,\ldots,n$, which means that $N_n(w^*) = N_n(f)$. Moreover

$$\|w^*-f\|_\infty \le \|w^*-w_{m_0}\|_\infty + \|w_{m_0}-f\|_\infty \le \delta$$

and

$$\|w^{*'}-f'\|_\infty \le \|w^{*'}-w_{m_0}'\| + \|w_{m_0}'-f'\| \le \gamma$$

which means that w^* satisfies (5.1) and (5.2). □

In [6] the class of infinitely differentiable functions with simple zeros is studied. We use here the same notation as in [6] and assume that the reader is familiar with the proof technique presented there. Now we are ready to prove

<u>Theorem 5.2</u>: The bisection information N_n^{bis} is optimal in

the class $\mathcal{N}_c$, i.e.,

$$(5.7) \qquad \inf r(N_n, F_2) = r(N_n^{bis}, F_2) = \frac{b-a}{2^{n+1}}. \qquad \square$$

Proof: For every ε, $0 < \varepsilon < (b-a)/(2^n n)$, and every information $N_n \in \mathcal{N}_c$ we construct two polynomials w_1 and w_2 from F_2 such that $N_n(w_1) = N_n(w_2)$ and

$$\text{dist}(S(w_1), S(w_2)) \geq \frac{b-a}{2^n} - n\varepsilon.$$

Then the proof of Theorem 5.2 will follow from (2.11) and (2.12) with ε tending to zero.

Consider the function f_n constructed by induction in Lemma 2.2 of [6] with f_1 in the proof replaced by

$$(5.8) \qquad f_1(x) = \begin{cases} -\exp(-(x-a-\varepsilon/2)^{-2}) & x \in [a, a+\frac{\varepsilon}{2}], \\ 0 & x \in [a+\frac{\varepsilon}{2}, x_1-\frac{\varepsilon}{2}], \\ \exp(-(x-x_1+\varepsilon/2)^{-2}) & x \in [x_1-\frac{\varepsilon}{2}, b]. \end{cases}$$

Then as in the proof of Optimality Theorem of [6], construct f^* and f^{**}, $f^*, f^{**} \in C^\infty[a,b]$, such that $N_n(f^*) = N_n(f^{**})$, each of f^*, f^{**} has exactly one, simple zero $\alpha^* = S(f^*)$, $\alpha^{**} = S(f^{**})$ and $\alpha^{**} - \alpha^* \geq (b-a)/2^n - n\varepsilon$. The choice (5.8) of f_1 guarantees that $f_n(a) < 0$ and $f_n(b) > 0$, which yields that $f^*(a) < 0$, $f^*(b) > 0$ and $f^{**}(a) < 0$, $f^{**}(b) > 0$. Let

$$\tilde{\gamma} = \min\{f^{*\prime}(\alpha^*), f^{**\prime}(\alpha^{**})\}.$$

The number $\tilde{\gamma}$ is positive, since f^* and f^{**} are strictly increasing in neighborhoods

$$I^* = (x_n^* - \epsilon/2, x_n^*) \quad \text{and} \quad I^{**} = (x_n^{**}, x_n^{**} + \epsilon/2)$$

of their zeros.

Define $U^* = (u_1, u_2)$ and $U^{**} = (v_1, v_2)$, $U^* \subset I^*$, $U^{**} \subset I^{**}$ to be neighborhoods of α^* and α^{**} such that

$$f^{*\prime}(x) > \tilde{\gamma}/2 \quad \text{for} \quad x \in U^*$$

and

$$f^{**\prime}(x) > \tilde{\gamma}/2 \quad \text{for} \quad x \in U^{**}.$$

Let $\gamma = \tilde{\gamma}/4$ and

$$\delta = \frac{1}{2} \min\{\min_{x \in [a,u_1] \cup [u_2,b]} |f^*(x)|, \min_{x \in [a,v_1] \cup [v_2,b]} |f^{**}(x)|\}.$$

The definition of f^*, f^{**}, U^* and U^{**} implies that δ is positive. Applying Theorem 5.1 with the above δ and γ to the functions f^* and f^{**} we obtain two polynomials: w_1 and w_2, each of them having exactly one simple zero, distance between these zeros not less than $(b-a)/2^n - n\epsilon$ and $N_n(w_1) = N_n(f^*) = N_n(f^{**}) = N_n(w_2)$. This completes the proof of Theorem 5.2. □

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