Columbia University

Department of Economics
Discussion Paper Series

Gambling Reputation: Repeated Bargaining with Outside Options

Jihong Lee, Qingmin Liu

Discussion Paper No.: 1213-29

Department of Economics
Columbia University
New York, NY 10027

June 2013

Gambling Reputation: Repeated Bargaining with Outside Options*

Jihong Lee† Seoul National University

Qingmin Liu‡ Columbia University

March 2013

Abstract

We study the role of incomplete information and outside options in determining bargaining postures and surplus division in repeated bargaining between a long-run player and a sequence of short-run players. The outside option is not only a disagreement point but reveals information privately held by the long-run player. In equilibrium, the uninformed short-run players' offers do not always respond to changes in reputation and the informed long-run player's payoffs are discontinuous. The long-run player invokes inefficient random outside options repeatedly in order to build reputation to a level where the subsequent short-run players succumb to his extraction of a larger payoff, but he also runs the risk of losing reputation and relinquishing bargaining power. We investigate equilibrium properties when the discount factor goes to 1 and when the informativeness of outside option diffuses. In both cases, bargaining outcomes become more inefficient and the limit reputation building probabilities are interior.

Keywords: bargaining, incomplete information, outside option, reputation, repeated game

JEL codes: C61, C73, C78

*The authors are grateful to a co-editor and the anonymous referees for their insightful suggestions. We also thank Heski Bar-Isaac, Yeon-Koo Che, Drew Fudenberg, Johannes Hörner, Navin Kartik, Mark Machina, George Mailath, Paul Niehaus, Nicola Persico, Hamid Sabourian, Larry Samuelson, Andy Skrzypacz and Bob Wilson, as well as seminar participants at a number of institutions and conferences. Klaus Schmidt kindly sent us his doctoral dissertation, and Euncheol Shin and Jangsu Yoon provided valuable research assistance. Jihong Lee's research was supported by a National Research Foundation Grant funded by the Korean government (NRF-2011-330-B00063).

†Department of Economics, Seoul National University, Seoul 151-746, Korea; jihonglee@snu.ac.kr

‡Department of Economics, Columbia University, 420 West 118th Street, New York, NY 10027, U.S.A.; qingmin.liu@columbia.edu

1 Introduction

Many real world negotiations take place repeatedly in the shadow of outside options, such as those provided by experts, arbitrators or even courts. Consider, for example, a firm that is in disputes with its employees regarding wage increases or with its customers regarding compensation for damages.[1] These disputes often involve interaction between a single privately informed long-run player and a sequence of short-run players. The recent high-profile litigations surrounding Merck, a pharmaceutical firm, offer an interesting case in point. Merck refused to settle and contested every case in court. After losing the first case with a compensation verdict of $253 million in 2005, it continued to fight in court over the following two years. After winning most of the cases, the firm ended up settling further 27,000 cases out of court for $4.85 billion in total, an amount far smaller than experts predicted at the beginning.[2]

In these examples, the bargainers obtain random outside options when they fail to reach an agreement. Moreover, the outside option represents not merely a "disagreement point" in a repeated setup; it can partially reveal the informed party's private information. The decision on whether to take the outside option must take into account not only the amount of information that this decision will disclose *per se* but also learning from the subsequent realization of uncertain payoffs. While the bargaining literature has long recognized the fundamental roles played by *outside options* and *incomplete information* (e.g., Nash (1950, 1953), Harsanyi and Selten (1972)), this linkage between the two essential ingredients of bargaining is yet to be explored. Our goal is to investigate how the additional source of learning from random outside options determines bargaining strategies and outcomes and to provide an analytical tool to study other related repeated interactions.

We consider a discrete-time repeated bargaining model in which a long-run player (e.g., a firm) bargains with a sequence of short-run players (e.g., customers or employees). In each period, a new short-run player enters the game and the two parties bargain (e.g., over damage compensation or wage increase). If they reach an agreement, the corresponding transfer is made from the long-run player to the short-run player who subsequently leaves the game. If they disagree, the players invoke an uncertain outside option (e.g., through a court verdict), which is inefficient due to a deadweight cost. For each pair of long-run and short-run players, the disagreement payoffs are drawn independently from a finite set according to a distribution that takes one of two types, "good" or "bad", and is privately known by the long-run player. The long-run player has an incentive to build a reputation for having a good distribution of outside

[1] The sheer existence of collective governance arrangements such as courts is a demonstration of the prominence of these applications.

[2] Source: New York Times, http://www.nytimes.com/2007/11/09/business/09merck.html

options. We analyze the reputation equilibria in which the players' strategies are functions of reputation (i.e., posterior belief on the good type) and reputation is valuable.

Our first main results (Theorem 1 and Corollary 1) establish the existence of a reputation equilibrium and its behavioral and payoff properties. Even though the short-run players are unrestricted in the set of offers, their equilibrium offers respond discontinuously to the long-run player's reputation. We show that every reputation equilibrium features two threshold levels of reputation, $0 < p^* < p^{**} < 1$. When reputation falls between the two thresholds, the long-run player always turns down the equilibrium demand and the inefficient random outside option is invoked, and as a result, the belief updating process is driven solely by the realizations of random signals. When reputation is above p^{**}, the long-run player accepts the short-run players' low equilibrium demand. There is no further learning and bargaining outcome is efficient, but the long-run player extracts the full benefits of reputation. When reputation is below p^*, the long-run player randomizes, the outside option is invoked only occasionally, and the negative reputational effect of an adverse signal is reduced, and may even be overturned, by the act of rejection. The long-run player's payoffs are discontinuous in his reputation.

The short-run player trades off the deadweight cost of disagreement and the high expected outside option against the bad type. This trade-off endogenously determines p^{**}. By revealing his private information, the long-run player relinquishes his bargaining power and the short-run players extract all the surplus in the ensuing bargaining. Therefore, the long-run player's benefit from revealing his type is merely the payoff that the current short-run player is willing to give up in order to induce acceptance: this amount equals the one-off deadweight cost that the short-run player would eschew in the case of an agreement. The long-run player weighs this one-off deadweight cost against the probability of building reputation to p^{**} beyond which he obtains a larger share of the surplus from each subsequent short-run player. Since the reputation building probability is increasing in the prior belief, this trade-off endogenously determines p^*.

The precise calibration of the rate of information revelation plays a role in the determination of equilibrium incentives in our model, as in Cramton (1984), Chatterjee and Samuelson (1987) and Abreu and Gul (2000), among others. In our repeated setup, the possibility of learning from random outside options gives rise to an additional incentive to reject a myopically attractive offer: not only does the long-run player not reveal himself to be a bad type but he can get lucky and increase his reputation. Thus, he can "gamble" his reputation at the expense of efficiency. Indeed, this incentive dominates in the limit as we next show.

We establish the limit properties of reputation equilibria as the long-run player becomes increasingly patient (Theorem 2). We first show that, as the discount factor δ goes to 1, the limiting equilibrium outcome is unique: the lower reputation threshold p^* converges to 0, while the upper threshold p^{**} remains unchanged. Thus, as the long-run player becomes more patient, the players adopt incompatible bargaining postures, and inefficiency arises, over a wider range of beliefs. Explicit bounds are derived for the equilibrium payoffs as well as for the reputation building probability, i.e., the probability with which the posterior reaches the upper threshold starting from prior $p \in (0, p^{**})$. As $\delta \to 1$, the accumulated signals become increasingly informative and, consequently, reputation building is fast and incurs no discounting cost, but the reputation building probability is interior: reputation can be built, though not always. Nonetheless, the reputation gain is small at a low prior. These results on the long-run distributions of payoff and belief contrast with the high payoff bounds established by Fudenberg and Levine (1989, 1992) and the impermanence of reputation shown in Cripps, Mailath and Samuelson (2004).[3]

We also explore the limiting equilibrium properties in a parametrized model where, as the real time interval Δ goes to 0, the value and informativeness of outside options shrink correspondingly at the rate that keeps the discounted sum of payoffs and aggregate informativeness over a unit of real time asymptotically constant. The reputation equilibrium is shown to be unique for generic Δ (Theorem 3). We obtain in closed-form the limit schedules of the reputation building probability and discounted payoffs. The reputation building probability is always interior despite the fact that reputation building takes real time. As $\Delta \to 0$, the one-off incentive offered by the short-run player to induce agreement vanishes and the long-run player's reputation concerns prevail. Consequently, the lower threshold p^* again converges to 0, and the equilibrium in the limit features only one threshold p^{**}. Diffusive outside options shut down the signaling channel of information revelation and amplify the incentives to gamble reputation. This contrasts with the equilibrium dynamics obtained by Gul and Pesendorfer (2012) and Daley and Green (2012) in dynamic games where exogenous information arrives in a Brownian motion.

Although we have chosen to present our analysis in a bargaining setup in which one party pays the other and the offer space is unbounded, it can be readily adapted to a standard surplus-splitting bargaining model. The presence of informative and random payoff realizations is a salient feature of many repeated interactions beyond the bargaining setup that we consider, from repeated sales between a long-lived seller and a sequence of short-lived buyers to entry deterrence by an incumbent facing a series of potential entrants. With appropriate interpretations of the disagreement points, the tools developed in this paper can be adapted to analyze such applications, which we elaborate on in Section 6.

The rest of the paper is organized as follows. The next section describes the model of repeated bargaining with random outside options. Section 3 presents our main results on

[3]See also Benabou and Laroque (1992) and Bar-Issac (2003) for related applications in which private information is fully revealed in the long run.

reputation equilibrium and its limit properties as δ goes to 1. Section 4 investigates the limiting equilibrium when information diffuses. We discuss several extensions of our analysis as well as the related literature in Section 5. Some concluding remarks are offered in Section 6. Formal proofs are relegated to the Appendix, and the Supplemental Material (Lee and Liu (2013)) contains additional results and proofs that are left out for expositional reasons.

2 The Model

2.1 Repeated Bargaining with Outside Options

We consider a repeated bargaining model in discrete time. Periods are indexed by $t = 0, 1, 2, \ldots$. A single long-run player 1 faces an infinite sequence of short-run players 2 with a new player 2 entering in every period.

The game within each period t is as follows. Player 2 makes a demand $s \in \mathbb{R}$, which player 1 can accept or reject. If the demand is accepted, player 1 pays s to player 2 who then leaves the game. If the demand is rejected, a transfer $v \in \mathbb{R}$ from player 1 to player 2 is drawn with probability $f^{\theta}(v)$, where $\theta \in \{G, B\}$ is privately known by player 1. Let $E^{\theta}[v]$ denote the expectation of v under f^{θ}. This outside option is inefficient: it incurs a cost $c > 0$ to player 2 with the cost to player 1 normalized to 0.

Let $p_t \in [0, 1]$ denote player 1's *reputation*, i.e., player 2's belief on $\theta = G$, at the beginning of period t, with $p_0 \in (0, 1)$ being the commonly known prior. Players observe the realized transfer and whether it results from voluntary agreement or outside option; rejected demand is not publicly observable. Player 1 minimizes his repeated game expected transfers/payments to the short-run players with a discount factor $\delta \in (0, 1)$.[4] Each player 2 maximizes his stage-game expected payoff.

We make the following assumptions.

Assumption 1 *(Full Support) f^G and f^B share a common support $V \subset \mathbb{R}$, which is a finite set with at least two elements.*

Assumption 2 *(Strict Monotone Likelihood Ratio Property) $\frac{f^B(v)}{f^G(v)}$ is strictly increasing in v.*

Assumption 3 $E^B[v] - E^G[v] > c$.

Assumption 1 ensures that no single realization of v can reveal player 1's type. Assumption

[4] Whenever we refer to player 1's "payoff" below, we mean the negative of his transfer.

2 implies that higher realizations of v are more likely to arise from type B.[5] Assumption 3 says that the difference between the expected values of the outside option to player 2 from the two player 1 types outweighs the cost; this provides an incentive for player 2 to induce the costly outside option.

2.2 Strategies and Equilibrium

A Markov (behavioral) strategy of the short-run player, d, maps his belief to a probability distribution over all possible demands, i.e., $d : [0, 1] \to \triangle(\mathbb{R})$. A Markov strategy of the long-run player of type $\theta \in \{G, B\}$ is a function r^θ that specifies a probability of rejection of each demand at each belief, i.e., $r^\theta : [0, 1] \times \mathbb{R} \to [0, 1]$. We write player 1's discounted average expected transfer to player 2 at belief p as $S^\theta(p)$. We suppress the dependence of S^θ on the strategy profile and the discount factor to save on notation.

We say that a strategy profile $\left(d, r^G, r^B\right)$ together with beliefs $\{p_t\}$ is a *reputation equilibrium* if (i) it is a perfect Bayesian equilibrium, (ii) $S^\theta(p)$ is non-increasing in p over $[0, 1]$, and (iii) once player 1's type is revealed, belief no longer changes.

Perfect Bayesian equilibrium is defined in Fudenberg and Tirole (1991). Even though they only consider finite games, its extension to an infinite game is straightforward (see Hörner and Vieille (2009), for instance). Monotone equilibrium payoffs capture reputation as a valuable asset. Similar monotonicity conditions are also invoked by Benabou and Laroque (1992) and Mathis, McAndrews and Rochet (2009) in reputation setups and by Fudenberg, Levine and Tirole (1987) in a single-sale bargaining setup. Assuming that belief does not change from 0 or 1 ensures that, once player 1's type is revealed, the continuation game is played as if it has complete information. This assumption is standard in dynamic Bayesian games, including bargaining literature with payoff uncertainties. See Nöldeke and van Damme (1990) for a pathological example that can arise without the restriction.

In equilibrium, the short-run player can make a demand that will be rejected for sure; let us refer to such a demand as a *losing* demand; a demand that is offered and accepted with positive probability by either type of long-run player in equilibrium will be referred to as a *serious* demand.

[5] We can weaken this assumption to a version of a statistical identifiability condition. Assumption 2 simplifies the exposition.

3 Gambling Reputation

3.1 Equilibrium

In this section, we investigate the dynamics of the reputation equilibrium of our bargaining game. In a reputation equilibrium, once the long-run player has revealed his type $\theta \in \{G, B\}$, each short-run player demands $E^{\theta}[v]$ and type θ accepts it for sure. Our results below are concerned with behavior and payoffs at interior reputation levels.

Our first main result establishes that, with a sufficiently patient long-run player, a reputation equilibrium exists, and every reputation equilibrium features two serious demands and two threshold levels of reputation.

Theorem 1 *There exists $\bar{\delta} \in (0,1)$ such that, for any $\delta > \bar{\delta}$, a reputation equilibrium exists, and every reputation equilibrium admits only two serious demands, $s^* = E^B[v] - c > s^{**} = E^G[v]$, and two reputation thresholds, $0 < p^* < p^{**} = 1 - \frac{c}{E^B[v] - E^G[v]}$, such that the following hold:*

(a) If $p \in (p^{**}, 1)$, s^{**} is demanded and accepted by both types of player 1 for sure.

(b) If $p \in (p^*, p^{**})$, losing demands are made and outside options are invoked for sure.

(c) If $p \in (0, p^*)$, s^* is the only serious demand and is made with positive probability; type G rejects every equilibrium demand; type B is indifferent between rejecting and accepting s^* and rejection by type B occurs with positive probability.[6]

(d) At p^*, only losing demands are made; at p^{**}, player 2 is indifferent between losing demands and s^{**}, which is the only serious demand and is accepted for sure by both types if demanded.

Proof. See Appendices B and C. ■

Since acceptance of the higher serious demand leads to revelation of type B, we immediately obtain from Theorem 1 the payoff implications below.

[6] There is, however, payoff-equivalent multiplicity regarding type B's exact randomizing behavior in this region of beliefs. For instance, there could be $\hat{p} < p^*$ such that, in $(0, \hat{p})$, rejection occurs with an interior probability while, in $[\hat{p}, p^*)$, rejection occurs with probability 1. See Appendix B.3 for more details and the Supplemental Material (Section 2) for an example. Besides, for each equilibrium in which player 1 rejects a demand with an interior probability, there are other outcome-equivalent equilibria involving player 2's randomization instead; see the Supplemental Material (Section 1.1).

Corollary 1 *For any reputation equilibrium with two thresholds, p^* and p^{**}, established in Theorem 1, we have the following:*

(a) For any $p \in (p^{**}, 1)$, $S^B(p) = \underline{S} = E^G[v]$.

(b) For any $p \in (0, p^*)$, $S^B(p) = \overline{S} = E^B[v] - (1-\delta)c$.

(c) For any p, $S^G(p) = \underline{S}$.

The reputation equilibrium in our model has the following noteworthy features. First, even though the short-run players are not restricted in the set of demands, they do not adjust their offers in order to induce acceptance and avoid the deadweight cost c of outside options. Their equilibrium offers are inflexible and respond discontinuously to the long-run player's reputation, and this results in inefficiency of bargaining outcomes as well as discontinuity in the payoffs. There is empirical support for this prediction of our model. For instance, in her study of repeated shareholder litigations involving long-run underwriters, Alexander (1991) finds that, beyond very few exceptions, the estimated strength of the case does not matter for the settlement amount.

Second, the long-run player's payoffs admit two flat boundaries: one at low reputation levels and the other at high reputation levels. When reputation falls in the intermediate region, this player's bargaining posture resembles a gambling process: he refuses every equilibrium demand and resorts to outside options, and his reputation evolves according to the realizations of random signals until it settles at one of the two boundaries. These reputation dynamics provide one possible explanation of the bargaining postures illustrated by our motivating example in the Introduction. Facing a series of product liability litigations, the firm may have suffered damage to its reputation by losing the first few cases in court but repeated gambles eventually proved successful as later court victories propelled reputation to a level where all further cases settled at a low amount.

Third, type G's payment is equal to $E^G[v]$, his expected value of each outside option, at all reputation levels. However, this results from voluntary agreement only when reputation is above the upper threshold p^{**}. At lower reputation levels, type G rejects all equilibrium demands and his transfers to the short-run players are determined by random outside options drawn with distribution f^G. Indeed, type G's response to the cutoff level of demand $E^G[v]$ depends on the reputation level and is determined endogenously in equilibrium.[7]

[7] This indeterminacy of type G's response to the cutoff contrasts with the reputational bargaining literature that assumes a behavioral type who follows a commitment cutoff strategy (e.g., Myerson (1991) and Abreu and Gul (2000)). Indeed, one can construct equilibria in which $E^G[v]$ is demanded and rejected for sure at some reputation levels, while it is demanded and accepted for sure at others. See Section 1.2 of the Supplemental Material.

The intuition behind the dynamics of the reputation equilibrium is as follows. First, when reputation is high, the short-run player's expected payoff from the outside option is low. Thus, there exists a threshold level of reputation beyond which the incentive to avoid the deadweight cost dominates and player 2 proposes a demand low enough to induce acceptance by both types of player 1. This demand corresponds to s^{**} and the threshold is p^{**}. Second, consider the incentives when reputation is below p^{**}. For player 2, the expected value of the outside option is high relative to the deadweight cost; this induces an aggressive demand that type G rejects. Type B faces the following trade-off: the randomness of the outside option offers a non-negligible chance of settling forever at a belief above p^{**} with low future payments, but by accommodating player 2's demand and revealing his type, he could claim his contribution to the total surplus, amounting to the one-off saving of the deadweight cost c. When reputation is close to p^{**}, the benefit from gambling with random outside options dominates. The lower threshold p^* is then determined by the balance between the one-off saving of the deadweight cost and the benefit from gambling reputation. Finally, when reputation is below p^*, equilibrium requires that player 2's demand leaves type B indifferent. Since acceptance of any demand higher than s^{**} leads to revelation of type B, the corresponding serious demand is s^* which does not respond to reputation.

We next provide an outline of our constructive proof. The equilibrium value function is discontinuous and admits two fixed payoff boundaries: one for $p < p^*$ (equal to $\overline{S}$) and the other for $p > p^{**}$ (equal to $\underline{S}$). The task boils down to identifying the lower reputation threshold p^* and the short-run player's randomization at p^{**}. Formally, we define, for each $\alpha \in [0,1]$, a contraction mapping T_α on type B's value function S as follows:

$$[T_\alpha(S)](p_t) = \begin{cases} E^B[v] & \text{if } p_t = 0 \\ \min\left\{\overline{S}, (1-\delta)E^B[v] + \delta E^B[S(p_{t+1})\,|p_t]\right\} & \text{if } p_t \in (0, p^{**}) \\ \alpha E^G[v] + (1-\alpha)\left[(1-\delta)E^B[v] + \delta E^B[S(p_{t+1})\,|p_t]\right] & \text{if } p_t = p^{**} \\ \underline{S} & \text{if } p_t > p^{**} \end{cases}$$

The equilibrium properties stated in Theorem 1 motivate the definition of T_α above, in which α corresponds to the randomization at p^{**}, and the second line is set up to determine p^*. We show that each T_α admits a unique fixed point S_α, which is a candidate equilibrium value function, and $p^*(\alpha) := \sup\left\{p : S_\alpha(p) = \overline{S}\right\}$ is a candidate for the lower reputation threshold. (For more details, see Appendix C.1: From Equilibrium Contraction Mapping.)

Then, we verify that there indeed exists an $\alpha \in [0,1]$ such that the fixed point S_α together with $p^*(\alpha)$ correspond to a reputation equilibrium of the following kind: at $p \in (0, p^*(\alpha))$, type B plays a mixed strategy such that the posterior right after the rejection but before the outside option is exactly $p^*(\alpha)$, and $S^B(p^*(\alpha)) = \overline{S}$. (See Appendix C.2: From Contraction Mapping to Equilibrium.)

3.2 Limit Properties: $\delta \to 1$

Our next main result concerns the properties of reputation equilibrium as player 1 becomes increasingly patient. We examine both equilibrium strategies and payoffs.

Starting from belief $p \in (0, p^{**})$, let $R(p)$ denote the probability with which reputation reaches (i.e., hits exactly or moves above) p^{**} in equilibrium. We suppress the dependence of R on δ to save on notation. Note that, when it falls below p^* in equilibrium, reputation can still bounce back to p^* via player 1's own randomization after which gambling can occur again; $R(p)$ takes this into account.

Denote by ρ the solution of

$$E^B\left[\rho^{\log\left(\frac{f^G(v)}{f^B(v)}\right)}\right] = 1. \tag{1}$$

We show in Appendix D that $\rho > 1$. Let $\lambda(p) = \log\left(\frac{p}{1-p}\right)$, $\lambda^{**} = \log\left(\frac{p^{**}}{1-p^{**}}\right)$, and $\underline{\lambda} = \log\left(\frac{f^G(\underline{v})}{f^B(\underline{v})}\right)$, where $\underline{v} = \min V$.

Theorem 2 *For each δ, fix a reputation equilibrium with two reputation thresholds p^* and p^{**}. We have the following:*

(a) "Limit Uniqueness": $\lim_{\delta\to 1} p^* = 0$.

(b) "Reputation Building Probability": For any $p \in (0, p^{**})$, $\lim_{\delta\to 1} R(p)$ exists and

$$\lim_{\delta\to 1} R(p) \in \left[\rho^{\lambda(p)-\lambda^{**}-\underline{\lambda}}, \rho^{\lambda(p)-\lambda^{**}}\right] \subset (0,1).$$

(c) "Payoffs": For all but countably many $p \in (0, p^{**})$,

$$\lim_{\delta\to 1} S^B(p) = E^G[v]\left(\lim_{\delta\to 1} R(p)\right) + E^B[v]\left(1 - \lim_{\delta\to 1} R(p)\right).$$

Proof. See Appendix D. ■

Part (a) shows that in any limiting reputation equilibrium the lower threshold p^* converges to 0, and p^{**} is independent of δ by Theorem 1. Therefore, in the limit, disagreement occurs and inefficient outside options are invoked for all $p \in (0, p^{**})$, while agreement is achieved and the bargaining outcome is efficient for $p > p^{**}$. Except at p^{**}, the equilibrium outcome is uniquely determined.

Part (b) obtains tight bounds on the reputation building probability. As $\delta \to 1$, the accumulated signals become increasingly informative, but the reputation building probability is interior starting from any prior $p \in (0, p^{**})$: reputation can be built, though not always. As $p \to 0$, $\lambda(p) \to -\infty$ and hence $\rho^{\lambda(p)-\lambda^{**}} \to 0$, i.e., the reputation building probability becomes arbitrarily small.

Part (c) further shows that type B's payment in the limit is the weighted average between $E^G[v]$ and $E^B[v]$ with the reputation building probability as the coefficient on the former.[8] Combined with part (b), this implies that even though the reputation building probability is strictly interior, conditional on the event that reputation is built, the speed of reputation building is fast relative to $\delta \to 1$ and there is no discounting cost in the limit. Furthermore, as $p \to 0$, the limit payment converges to $E^B[v]$: at a very low prior, reputation building is essentially futile in terms of payoff gain.

Example 1 *Let* $V = \{0, 2, 4\}$, $c = \frac{1}{10}$, $f^G(0) = f^B(4) = \frac{10}{27}$, $f^G(2) = f^B(2) = \frac{1}{3}$, $f^G(4) = f^B(0) = \frac{8}{27}$. *Then,* $p^{**} = 0.6625$ *and* $\rho \simeq 2.7287$. *Figure 1 plots the bounds on the limit reputation building probability established in Theorem 2 above.*

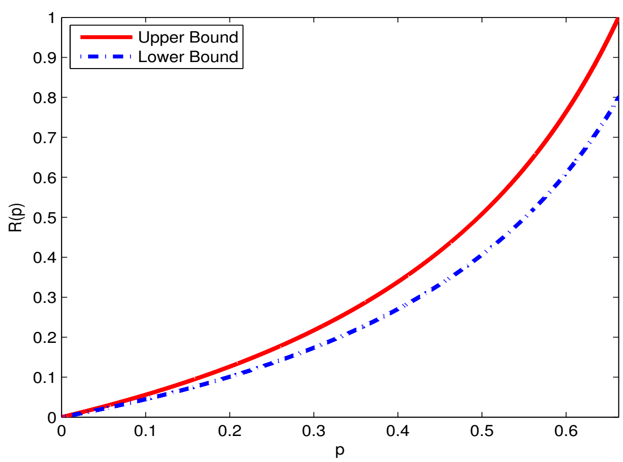


Figure 1: Bounds on reputation building probability

[8]There are countably many $p < p^{**}$ that can be reached from p^{**} after a finite path of signal realizations. For these beliefs, $S^B(p)$ depends on $S^B(p^{**})$ that is in general not the same as $E^G[v]$ due to player 2's randomization.

4 Informativeness of Outside Options

In the previous section, we took $\delta \to 1$ but kept the signal structure fixed as in the traditional repeated game and reputation literature (e.g., Fudenberg and Levine (1992)). An equivalent way of interpreting δ is to fix the interest rate $r > 0$ and treat the real time interval $\Delta > 0$ between fixed actions as a parameter: $\delta = e^{-r\Delta}$. However, in our model, as $\delta \to 1$ or $\Delta \to 0$, outside options arrive more frequently and the aggregate precision of signals within unit time explodes. In this section, we investigate the limit properties of our model while keeping asymptotically constant the aggregate informativeness of signals within a unit of real time.

We focus on a symmetric binary case of our model. Let the repeated bargaining game be played in discrete periods in real time $t = 0, \Delta, 2\Delta, \ldots$. The set of outside options is $V = \{-\sqrt{\Delta}, \sqrt{\Delta}\}$ with $f^G(-\sqrt{\Delta}) = f^B(\sqrt{\Delta}) = \frac{1+\mu\sqrt{\Delta}}{2} \in \left(\frac{1}{2}, 1\right)$. Hence, $E^G[v] = -\mu\Delta$ and $E^B[v] = \mu\Delta$. Both Assumptions 1 and 2 are satisfied; we take $c = \frac{2\mu\Delta}{\kappa}$ for some constant $\kappa > 1$, in accordance with Assumption 3 (i.e., $E^B[v] - E^G[v] = 2\mu\Delta > \frac{2\mu\Delta}{\kappa} = c$). We suppress the dependence on Δ to simplify notation.

This parametrization corresponds to the binary approximation of a Brownian motion: the sum of signals per unit of real time is approximately normal with a type-dependent mean ($-\mu$ for type G and μ for type B) and a type-independent variance (normalized to 1); see, for example, Cox and Miller (1965). The discounted sum of expected outside option payments for the two types in the limit are $\lim_{\Delta\to 0} \frac{E^G[v]}{1-e^{-r\Delta}} = -\frac{\mu}{r}$ and $\lim_{\Delta\to 0} \frac{E^B[v]}{1-e^{-r\Delta}} = \frac{\mu}{r}$, respectively.[9]

Recall that Theorem 1 obtains properties of all reputation equilibria. However, multiple equilibria could arise due to the long-run player's randomization below the lower threshold and the short-run player's randomization exactly at the upper threshold. In the parametrized model, we can strengthen Theorem 1 into the following.

Theorem 3 *Consider the symmetric binary model parametrized by Δ. There exists $\underline{\Delta} > 0$ such that the reputation equilibrium outcome is unique for all but at most countably many $\Delta < \underline{\Delta}$. In the generically unique equilibrium with two reputation thresholds p^* and p^{**}, we have the following:*

[9]Our parametrization differs from the treatment in several other papers that investigate discrete-time approximations of continuous-time repeated games. Fudenberg and Levine (2007, 2009), for instance, fix the stage game payoffs and let signals diffuse by letting $\Delta \to 0$. See also Abreu, Milgrom and Pearce (1991) for the case of Poisson signals. In these models, signals are informative about action choices but do not directly affect stage-game payoffs. In our game, v serves as both signal and payoff. Indeed, if we fix V to be independent of Δ, the discounted average payoffs for both types are finite but collapse into the same level while the discounted sum of payoffs explodes. However, the limiting properties are well-defined. See Lee and Liu (2012) for a formal analysis.

(a) At any $p \in (0, p^*]$, type B rejects the equilibrium demand with probability $\frac{p}{1-p}\frac{1-p^*}{p^*} \leq 1$.

(b) At $p^{**} = \frac{\kappa-1}{\kappa}$, which is independent of Δ, player 2's mixing probability is uniquely determined.

Proof. See Appendix E. ■

With symmetric binary signals, the equilibrium value function over the intermediate levels of reputation (p^*, p^{**}) is recursively written as the following second-order difference equation

$$S^B(p^n) = (1 - e^{-r\Delta})E^B[v] + e^{-r\Delta}\left(\frac{1+\mu\sqrt{\Delta}}{2}S^B(p^{n-1}) + \frac{1-\mu\sqrt{\Delta}}{2}S^B(p^{n+1})\right), \qquad (2)$$

where, from p^n, the posterior after a favorable (unfavorable) signal is p^{n+1} (p^{n-1}). This can be solved explicitly with the two payoff boundaries $\overline{S}$ and $\underline{S}$, as illustrated in Figure 2 below. The explicit derivation of the equilibrium value function can be further exploited to pin down the generic uniqueness of the equilibrium.[10]

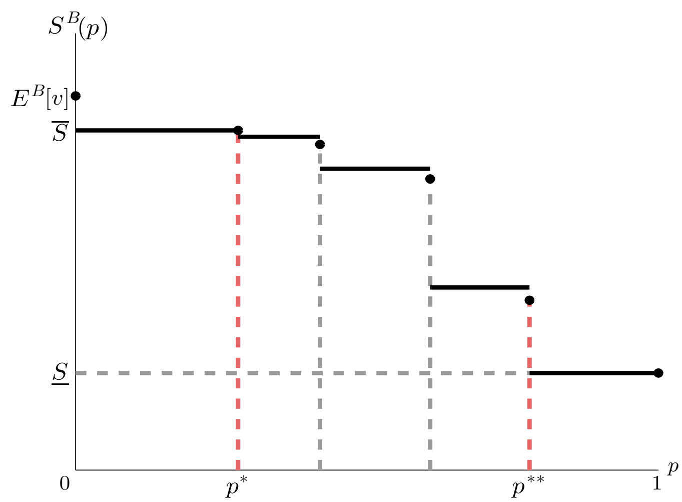


Figure 2: Equilibrium value function with symmetric binary signals

As before, we denote by $R(p)$ the probability that, starting from $p \in (0, p^{**})$, reputation hits or goes above p^{**} in equilibrium. Note that $R(p)$ is a function of Δ.

Theorem 4 *Consider the generically unique reputation equilibrium of the symmetric binary model parametrized by Δ. We have the following:*

[10]There are alternative ways to parametrize the symmetric binary case of our model, for which generic uniqueness can be obtained similarly.

(a) $\lim_{\Delta\to 0} p^* = 0$.

(b) For any $p \in (0, p^{**})$, $\lim_{\Delta\to 0} R(p) = \frac{1}{\kappa-1}\frac{p}{1-p} \in (0,1)$.

(c) For any $p \in (0, p^{**})$,

$$\lim_{\Delta\to 0} \frac{rS^B(p)}{1-e^{-r\Delta}} = \mu\left[1 - \left(\frac{1}{\kappa-1}\frac{p}{1-p}\right)^{\gamma}\right] - \mu\left(\frac{1}{\kappa-1}\frac{p}{1-p}\right)^{\gamma},$$

where $\gamma = \frac{1}{2} + \frac{\sqrt{\mu^2+2r}}{2\mu} > 1$.

(d) At any $p \in [0,1]$, $\lim_{\Delta\to 0} \frac{rS^B(p)}{1-e^{-r\Delta}}$ *is continuous. In particular,*

$$\lim_{\Delta\to 0} \frac{rS^B(0)}{1-e^{-r\Delta}} = \mu \text{ and } \lim_{\Delta\to 0} \frac{rS^B(p^{**})}{1-e^{-r\Delta}} = -\mu.$$

Proof. See Appendix F. ■

Theorem 4 is the counterpart of Theorem 2. The new results simultaneously take limits on signal precision as well as the long-run player's patience, while Theorem 2 considers the limit only on patience. We explain below the role of signal diffusion by comparing the driving forces behind the two theorems.

In Theorem 2 above, increased patience puts a greater weight on the low future payments above p^{**}, and since the aggregate informativeness of signals in a unit time explodes as $\Delta \to 0$, reputation building, if it occurs, is fast in real time; in the limit, there is no discounting cost. If signal precision decreases with Δ, however, reputation building takes real time. Nonetheless, since the sum of signals per unit of time is informative even as $\Delta \to 0$, reputation building must happen with positive probability for a finite amount of real time. For the same reason, the difference between the two types' discounted sum of payments does not disappear as $\Delta \to 0$. Thus, despite the fact that the discounting cost is positive for any fixed interest rate r, the benefit of reputation building does not vanish.

Regarding the cost of reputation building, note that, in our infinitely repeated game, choosing not to gamble leads to the revelation of type B and a constant flow of high payment thereafter. Since the high serious demand is less than the expected value of the outside option by c, the benefit of opting out of gambling is just the one-off saving of c, which vanishes with Δ. It follows that, just as in Theorem 2, the incentive to gamble reputation dominates for any $p < p^{**}$ as Δ becomes small.

We derive the limit reputation building probability in closed form. In contrast to Theorem 2, reputation building takes real time here, and hence, to obtain limit payoffs, we compute the discounted reputation building probability, which amounts to

$$\hat{R}(p) = \left(\frac{1}{\kappa - 1}\frac{p}{1-p}\right)^{\gamma}.$$

Comparing the discounted reputation building probability with its undiscounted counterpart derived in part (b) reveals exactly how much discounting takes place: since $\gamma > 1$, $\hat{R}(p) < \frac{1}{\kappa-1}\frac{p}{1-p}$, and in addition, $\hat{R}(p) \to \frac{1}{\kappa-1}\frac{p}{1-p}$ as $r \to 0$. It is also shown that the value function in the limit is continuous everywhere even though it is a step function for any fixed $\Delta > 0$.

Example 2 *Let* $\mu = 0.2$ *and* $\kappa = 5$. *Then,* $p^{**} = 0.8$. *Figure 3 plots the undiscounted reputation building probability* ($r = 0$) *against the discounted reputation building probability at* $r = 0.05$*; Figure 4 simulates type B's limit payments at* $r = 0$ *and at* $r = 0.05$.

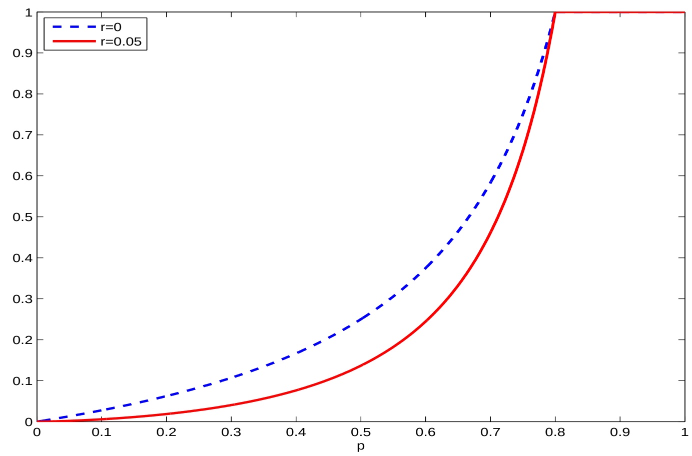


Figure 3: Discounted reputation building probability, $\left(\frac{1}{\kappa-1}\frac{p}{1-p}\right)^{\gamma}$

5 Discussion

In this section, we discuss several extensions of our analysis above and relate our contributions to the existing literature in more detail.

5.1 Extensions

Non-Markov Equilibria. In our analysis, we have restricted attention to strategies that condition actions only on the reputation level of the long-run player at each history. This enables us to highlight the role of reputation in shaping outcomes of the repeated interactions that we

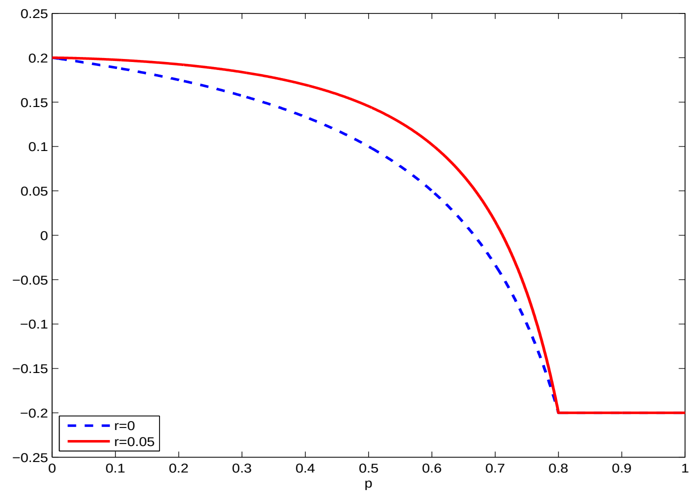


Figure 4: Payment, $\lim_{\Delta \to 0} \frac{rS^B(p)}{1-e^{-r\Delta}}$

consider. If we allow for non-Markov strategies, many new equilibrium possibilities arise in our repeated game. To see this, note that, when player 1's type is known, our model admits a folk theorem: when $p = 0$, any payment in the closed interval $\left[E^B[v] - c, E^B[v]\right]$ can be supported by a subgame perfect equilibrium. Then, by simply allowing for non-Markov behavior after the bad type reveals himself, our equilibrium construction can be extended to deliver a wider range of equilibrium payoffs. Formal details of these non-Markov equilibria appear in the Supplemental Material (Section 3.1).

Non-Monotone Equilibria. In a reputation equilibrium, the long-run player's payoffs (payments) are monotone increasing (decreasing) in reputation. Since the good type's equilibrium expected payment at $p = 1$ is equal to $E^G[v]$, i.e., the expected value of his outside option, the monotonicity property then implies that $S^G(p) = E^G[v]$ for all $p \in [0, 1]$, and this endogenizes the stationary cutoff demand equal to $E^G[v]$. It turns out that the precise details of our equilibrium dynamics change if the restriction is relaxed. In the Supplemental Material (Section 3.2), we construct examples of non-reputation equilibria with non-monotone payoffs. By allowing type G to adopt non-stationary cutoffs, it is shown that both long-run types' equilibrium payments could oscillate.

(Un)observability of Demands. We have assumed that the details of bargaining are observable if and only if there is an agreement. This assumption is consistent with many applications. For instance, in the cases of shareholder-auditor bargaining documented by Alexander (1991) and Palmrose (1991), the details of disagreement are private information and the terms of agreement are publicly observable. We can also extend our model by considering voluntary

disclosure of an accepted demand and/or voluntary concealment of a rejected demand. Our equilibrium is robust under the following natural specification of beliefs upon observing a confidential agreement or open disagreement: player 2 assigns probability 1 to the bad type. This equilibrium survives refinements such as the intuitive criterion. This argument eliminates any benefit of confidentiality and suggests that other factors not captured in the current model are responsible for confidential agreements observed in the real world. For example, a confidential agreement may reduce the arrival of new disputes. On the other hand, allowing for observability of rejected demands brings about a fresh signaling issue.

5.2 Related Literature

Bargaining. The bargaining literature has long recognized the fundamental roles of outside options and incomplete information in determining bargaining strategies and outcomes. The study of outside option in bargaining dates at least back to Nash (1950, 1953); the previous bargaining literature on incomplete information has focused on the role of private information about *valuations* (e.g., Cramton (1984), Gul, Sonnenschein and Wilson (1986) and Chatterjee and Samuelson (1987)), *patience* (e.g., Chatterjee and Samuelson (1987) and Abreu, Pearce and Stacchetti (2012)) or *bargaining postures* (e.g., Myerson (1991), Abreu and Gul (2000) and Abreu and Pearce (2007)). These models consider negotiations over a single sale.

In this paper, we explore an interplay between outside options and incomplete information in a *repeated* bargaining model: outside options provide informative signals and determine the players' immediate disagreement payoffs. Part of the mechanics of incomplete information in our paper is not new. When players have private information about their outside options, their decisions on whether or not to take the outside option must take into account the amount of information that this decision will disclose. Cramton (1984) and Chatterjee and Samuelson (1987), who consider private information about valuations instead, also analyze bargaining models in which a precise calibration of the rate of information revelation plays a role in the determination of equilibrium incentives. The distinct aspect of our model is the learning from random outside options and strategic responses in a repeated setup, which give rise to the gambling phenomenon. Indeed, as shown in Theorems 2 and 4, this phenomenon prevails in the limit.

A different kind of linkage between outside options and incomplete information in single-sale bargaining is considered by Compte and Jehiel (2002) who show that introducing outside options into the Myerson-Abreu-Gul setup of single-sale bargaining with commitment types may cancel out the delay and inefficiency that such informational asymmetry otherwise generates. Atakan and Ekmekci (2012) consider a search market as a way of endogenizing outside options and explore the role of reputation.

Reputation. Two aspects of our model differentiate our analysis from the canonical reputation approach of Fudenberg and Levine (1989, 1992). First, the long-run player in our model has private information about his payoffs rather than bargaining posture. Thus, this player builds a reputation for having a strong outside option rather than being insistent.[11] Second, we have informative outside options, and this makes the reputation building for the bad type essentially futile with a very small prior: Corollary 1 shows that the reputation gain amounts only to the one-off cost of the outside option, i.e., $(1-\delta)c$.

To bring our analysis closer to Fudenberg and Levine (1989, 1992), we could assume an insistent type who accepts a demand if and only if it is no larger than some cutoff C, and make outside options uninformative such that $f^B = f^G$ with expectation $E[v]$.[12] This is a Fudenberg-Levine-style model but its stage game has the following features. First, it is an extensive form game. Second, not all of the long-run player's strategies are identifiable since only actual transfers are observed. Third, the Stackelberg strategy is not well-defined since the most aggressive insistent strategy (i.e., cutoff equal to $E[v]-c$) makes player 2 indifferent between offering a compatible demand and a losing demand; hence, one should consider $C > E[v]-c$. In the Supplemental Material (Section 4), we obtain a payoff bound similar to that of Fudenberg and Levine (1989, 1992) under Markov assumption.[13] This direct comparison between our model and the alternative model confirms that informative outside options are indeed the source of the low reputation benefit.

Exogenous Information in Dynamic Games. Our model is closely related to Gul and Pesendorfer (2012) who study a dynamic model of political campaigns where the exogenous signal is a Brownian motion. With asymmetric information, the equilibrium in their model is characterized by two cutoff levels of the median voter's belief: one cutoff delineates the region of hard information provision from that of signaling through mixed actions and the other cutoff is the belief at which the voter is convinced of the candidate's policy. While the equilibrium of our discrete-time model resembles this equilibrium, our continuous-time limit, with signals parametrized to approximate a Brownian motion, is different: Theorem 4 shows that there is only one threshold p^{**} in the limit. Indeed, this distinction reveals a different force behind our bargaining game. In Gul and Pesendorfer (2012), stopping the arrival of external information

[11] In our model, a specific cutoff strategy is derived for type G but his response to the cutoff itself is flexible. See Appendix A.2.

[12] Uninformative outside options violate Assumptions 2 and 3. Hence, the alternative model is not a limit of our model as outside options become less informative. This is also confirmed in Section 4.

[13] Schmidt (1991) considers reputation in a finite horizon repeated bargaining model without informative outside options.

saves the accumulated cost of future information provision. In contrast, in our repeated bargaining setup, the decision to stop gambling reveals the type and confers the bargaining power to the short-run players who extract all gains from trade in the continuation game. Hence, the cost saving is only the one-off amount equal to c, which vanishes in the continuous-time limit.[14]

The dynamic signaling model of Daley and Green (2012) contains a similar equilibrium structure in which exogenous signals drive the belief process between two thresholds. This paper further differs from ours because the two thresholds in their equilibrium are simultaneously determined by the informed seller's trade-off between delayed trade of a single asset and high competitive prices at high beliefs. See Kremer and Skrzypacz (2007) for another dynamic signaling model with exogenous information. Repeated models with exogenous signals in discrete time are analyzed by Benabou and Laroque (1992), Bar-Isaac (2003) and Mathis, McAndrews and Rochet (2009). These studies derive reputation dynamics in which the informed long-run player draws random signals above an endogenously determined threshold but any reputation built in this process must eventually disappear.

In all aforementioned works, with the exception of Gul and Pesendorfer (2012), the informed player faces a competitive price environment and hence his payoff responds continuously to the belief. In contrast, the uninformed players in our bargaining model are fully strategic, and the signals are themselves disagreement payoffs; beliefs are updated from the realized transfers. These features are not only relevant for applications, but they are of conceptual importance because studying strategic price formation is a prime motivation of bargaining models. Indeed, we obtain the following important implication from strategic uninformed players: their equilibrium offers change only discretely to reputation, even though there is *a priori* no constraint on the offer space.[15] Moreover, in the discrete-time model, this results in the long-run player's value function being discontinuous. All these issues pose analytical challenges in our model. The incentives of the short-run players determine the two fixed boundaries of the long-run player's equilibrium value, as well as the upper threshold p^{**}, and the precise details of the short-run player's mixing at p^{**} are critical for matching the gambling process with the equilibrium value function.

[14] Note that we have normlized the long-run player's cost of the outside option to be zero. This is without any loss since otherwise the short-run players would also extract this cost saving once the long-run player has revealed his type.

[15] See Alexander (1991) for empirical support of this prediction of our model.

6 Conclusion

In this paper we demonstrate the role of informative outside options in determining reputation dynamics in a repeated bargaining model. The possibility of learning from the informative outside options gives the long-run player with weak outside options an additional incentive to reject a myopically attractive offer: not only does he not reveal himself to be the bad type, he can also get lucky and improve his reputation if the signal happens to be favorable. Thus, he can gamble his reputation. Nonetheless, at a very low prior, reputation building is essentially futile in terms of payoff gain.

A direction to enrich our analysis would be to explore the interaction between the reputation dynamics and detailed institutional features of the application. For instance, our bargaining setup could be extended to address other potentially relevant features of negotiation, from coalition formation (e.g., class action) to other more complex bargaining protocols and outside option processes (e.g., strategic third party). Another interesting direction for future research is to consider a long-lived uninformed player, which would generate a tension between incentives for experimentation versus reputation building with informative signals.

The tools developed in this paper can be applied to analyze other repeated interactions where informative and random payoff realizations give rise to incentives for the gambling reputation phenomenon with two fixed payoff boundaries. We wrap up the paper by selecting and discussing some examples below.

Repeated Sales. A seller serves a sequence of identical buyers. The seller privately knows his unit production cost, which is either high or low. Each buyer only consumes one unit of the product and his valuation is commonly known to be higher than the high cost. Each buyer makes an offer. A disagreement invokes a random and fair but imperfect third party arbitration that results in an informative signal about the seller's private cost.[16] Applying our analysis to this model, we will obtain gambling reputation dynamics: transactions are conducted with direct involvement of third parties when the belief on the high cost seller lies between two thresholds, while the low cost seller bets his reputation until it reaches one of the boundaries.

Entry Deterrence. An incumbent faces a sequence of potential entrants over spatially separated markets. The incumbent has private information about technology or consumer brand loyalty, and this stochastically affects the parties' profits. Each entrant decides whether to enter and the incumbent decides whether to start a price war. We can interpret entry as "disagreement" and the profits after entry as "informative outside options." Applying our analysis to this model, we will again derive gambling dynamics: entry is deterred only when the incumbent's reputation is high, and the incumbent will fight for sure when the reputation is between

[16]Gambetta (1993) and Dixit (2009) report an intriguing example of the Sicilian Mafia's role as an arbitrator.

two thresholds, betting on the random payoffs to improve his reputation. We emphasize the difference between this model and the standard chain-store model *à la* Kreps and Wilson (1982) and Milgrom and Roberts (1982). In the above model, the incumbent is not building a reputation for being tough *per se*. Such an incumbent will not scare the entrant away; rather, the incumbent is building a reputation of having a superior technology or high consumer loyalty, convincing the potential entrants that entry will not be profitable.

Appendix

A Preliminary Results

In this section, we offer several preliminary results of our analysis that will be utilized later. Also, throughout the rest of the paper, we introduce the following notation. Let $\overline{v}$ and $\underline{v}$ denote the largest and smallest elements in V, respectively. Also, let $v^* \in V$ be such that $\frac{f^B(v)}{f^G(v)} \geq 1$ if $v > v^*$ and $\frac{f^B(v)}{f^G(v)} < 1$ if $v \leq v^*$. The existence of v^* is ensured by Assumption 2.

A.1 Stopped Martingale

Consider an auxiliary belief updating process $\{p_t\}_{t=0}^{\infty}$ starting from a prior p_0 that is driven by the realizations of outside options according to the true distribution f^B. Then, by Bayes' rule, the posterior on type G upon a realization of $v \in V$ at p_t is

$$p_{t+1} = \frac{p_t f^G(v)}{p_t f^G(v) + (1 - p_t) f^B(v)}. \tag{3}$$

Fix $p^{**} \in (0, 1)$. Let the stopping time τ designate the first time such that $p_t \geq p^{**}$. Let $M(p_0)$ be the probability with which $\tau < \infty$, i.e., p_t reaches p^{**} in finite time.

Lemma 1 $\lim_{p_0 \to 0} M(p_0) = 0$.

Proof. From (3),

$$\frac{p_{t+1}}{1 - p_{t+1}} = \frac{p_t}{1 - p_t} \frac{f^G(v)}{f^B(v)}. \tag{4}$$

Hence,

$$\begin{aligned} E^B\left[\left.\frac{p_{t+1}}{1 - p_{t+1}}\right| p_t\right] &= \sum\nolimits_{v \in V} \left(\frac{p_t}{1 - p_t} \frac{f^G(v)}{f^B(v)}\right) f^B(v) \\ &= \sum\nolimits_{v \in V} \frac{p_t}{1 - p_t} f^G(v) \\ &= \frac{p_t}{1 - p_t}. \end{aligned}$$

That is, $\frac{p_t}{1-p_t}$ is a martingale and $\frac{p_{t\wedge\tau}}{1-p_{t\wedge\tau}}$ is a stopped martingale, where $t \wedge \tau := \min\{t, \tau\}$.

By the definition of stopping time τ, (4), and the (strict) monotone likelihood ratio property (MLRP), $\frac{p_{t\wedge\tau}}{1-p_{t\wedge\tau}}$ is bounded above by $\frac{p^{**}}{1-p^{**}}\frac{f^G(\underline{v})}{f^B(\underline{v})}$. Therefore, by the martingale stopping theorem (e.g., Theorem 6.2.2, Ross (1996)),

$$E^B\left[\lim_{t\to\infty}\frac{p_{t\wedge\tau}}{1-p_{t\wedge\tau}}\right] = E^B\left[\frac{p_{t\wedge\tau}}{1-p_{t\wedge\tau}}\right] = \frac{p_0}{1-p_0}.$$

By the definition of the stopped martingale,

$$E^B\left[\lim_{t\to\infty}\frac{p_{t\wedge\tau}}{1-p_{t\wedge\tau}}\right] \geq M(p_0)\frac{p^{**}}{1-p^{**}}.$$

Hence,

$$M(p_0)\frac{p^{**}}{1-p^{**}} \leq \frac{p_0}{1-p_0} \overset{\text{as } p_0\to 0}{\to} 0.$$

It follows that $\lim_{p_0\to 0} M(p_0) = 0$. ■

A.2 Serious Demands

Here, we present some useful properties of the players' behavior in a reputation equilibrium. Let us begin with type G's equilibrium strategy: type G rejects any demand strictly above $E^G[v]$, i.e., his outside option value, while accepting any demand strictly below it.[17]

Lemma 2 *In any reputation equilibrium, for any $p \in (0,1)$, $r^G(p,s) = 0$ if $s < E^G[v]$ and $r^G(p,s) = 1$ if $s > E^G[v]$.*

Proof. When player 1 is known to be type G, i.e., when $p = 1$, the unique reputation equilibrium is such that player 2 demands $E^G[v]$ and player 1 accepts a demand if and only if it is less than or equal to $E^G[v]$. Hence, $S^G(1) = E^G[v]$. By monotonicity of $S^G(p)$, therefore, every reputation equilibrium is such that $S^G(p) \geq E^G[v]$ for all $p \in [0,1]$. By always rejecting player 2's demands, G can guarantee $E^G[v]$ as the (discounted average) expected transfer. It therefore follows that $S^G(p) = E^G[v]$ for all $p \in [0,1]$.

Now, suppose that player 2 demands $s < E^G[v]$ at some history. Accepting s yields an expected transfer equal to $(1-\delta)s + \delta E^G[v] < E^G[v]$, while rejection yields $(1-\delta)E^G[v] + \delta E^G[v] = E^G[v]$. Thus, G must accept s for sure. A similar argument shows that G must reject $s > E^G[v]$ for sure. ■

We next derive the following property of type B's equilibrium strategy.

[17] Recall that the long-run player's cost of the outside option is normalized to 0.

Lemma 3 *Fix any δ and any reputation equilibrium. Also, fix any p, and consider any equilibrium demand $s > E^G[v]$ that could be offered at this history. If B's equilibrium strategy accepts s with a positive probability, then it accepts any $s' < s$ for sure.*

Proof. Note that rejected demands are not observable. Let X denote B's expected transfer from rejecting any demand at this history. By Lemma 2, accepting s reveals that player 1 is B and hence yields an expected transfer equal to $(1-\delta)s + \delta E^B[v]$, which is at most X since B weakly prefers to accept s.

Suppose that another demand $s' < s$ is offered on or off the equilibrium path. Since, by monotonicity, $S^B(p) \leq S^B(0) = E^B[v]$ for all p, B's expected transfer from accepting s' is at most $(1-\delta)s' + \delta E^B[v] < X$. Thus, B must strictly prefer to accept s'. ■

Our next lemma concerns the short-run player's equilibrium demand. If player 1 is patient enough, there are only two serious demands despite the fact that *a priori* player 2 has the option to demand anything in the real line. Any other demands must be either off the equilibrium path, or offered and rejected for sure in equilibrium.

Lemma 4 (Serious Demands) *Fix any $\delta > \frac{c}{E^B[v]-E^G[v]+c}$, and consider any reputation equilibrium. A serious demand at any $p \in (0,1)$ is either $E^G[v]$ or $E^B[v] - c$.*

Proof. We prove Lemma 4 by way of contradiction. Fix any p. Let s be a serious demand at p. We consider the following cases.

<u>Case 1</u>: $s < E^G[v]$.

But then, by Lemma 2, G accepts s and, hence, $S^G(p) = (1-\delta)s + E^G[v] < E^G[v]$, which contradicts that $S^G(p) = E^G[v]$ for all p.

<u>Case 2</u>: $s > E^B[v]$.

By Lemma 2, for s to be a serious demand, B must accept s. Since accepting $s > E^B[v]$ reveals B, B's subsequent expected transfer as of the next period is $E^B[v]$. If B rejects s, his current period expected transfer is $E^B[v] < s$, while future transfers are bounded above by $E^B[v]$. Therefore, B must strictly prefer to reject s, a contradiction.

<u>Case 3</u>: $s \in \left(E^G[v], E^B[v] - c\right)$.

But then, consider player 2 demanding $E^B[v] - c$ instead of s. Player 2's expected payoff from the deviation is $p\left(E^G[v] - c\right) + (1-p)\left(E^B[v] - c\right)$ since, by Lemma 2, G rejects $E^B[v] - c$ for sure and B's rejection also yields $E^B[v] - c$ in expectation. Note that G also rejects s for sure and hence B accepts $s < E^B[v] - c$ with a strictly positive probability by assumption. Thus, the deviation is profitable, a contradiction.

<u>Case 4</u>: $s \in \left(E^B[v] - c, E^B[v]\right]$ and B rejects s with probability $r^B \in (0,1)$.

But then, consider player 2 demanding $s-\varepsilon > E^B[v]-c$ for some $\varepsilon \in \left(0, r^B\left(s - E^B[v] + c\right)\right)$. By Lemma 2, G rejects this for sure while, by Lemma 3, B accepts this for sure. Hence, player 2's expected payoff from this deviation is $p\left(E^G[v] - c\right) + (1-p)(s-\varepsilon)$, while the payoff from s is $p\left(E^G[v] - c\right) + (1-p)(1-r^B)s + (1-p)r^B\left(E^B[v] - c\right)$. Since $\varepsilon < r^B\left(s - E^B[v] + c\right)$, such a deviation is profitable, a contradiction.

Case 5: $s \in \left(E^B[v] - c, E^B[v]\right]$ and B accepts s for sure.

We proceed in the following steps.

Step 1: If there is another equilibrium demand $s' \neq s$, then $s' = E^G[v]$.

Proof of Step 1. Suppose not; so, $s' \neq E^G[v]$ is offered in equilibrium. There are several cases to consider here.

(i) $s' < E^G[v]$ or $s' \in \left(E^G[v], E^B[v] - c\right)$

In this case, by Lemma 3, B accepts s' for sure. But we have already shown in Cases 1 and 3 above that this cannot be possible.

(ii) $s' \in \left(E^B[v] - c, s\right)$

We know from Lemmas 2 and 3 that G rejects s' for sure, while B accepts it for sure. Thus, player 2 strictly prefers to demand s over s', a contradiction.

(iii) $s' > s$

In this case, s' must be accepted by B since, otherwise, player 2 obtains

$$p\left(E^G[v] - c\right) + (1-p)\left(E^B[v] - c\right),$$

which, since $s > E^B[v] - c$, is strictly less than what he obtains from demanding s, amounting to

$$p\left(E^G[v] - c\right) + (1-p)s.$$

But then, by Lemma 3, any $s'-\varepsilon \in (s, s')$ is accepted for sure by B and we can invoke arguments similar to those for Case 4 above to show the existence of a profitable deviation for player 2, a contradiction.

Step 2: Rejection reveals G.

Proof of Step 2. This follows immediately from Step 1 and Lemmas 2 and 3.

Now, by Step 2, the expected transfer from rejection equals

$$(1-\delta)E^B[v] + \delta E^G[v], \tag{5}$$

while that from accepting s, since this reveals B, is

$$(1-\delta)s + \delta E^B[v]. \tag{6}$$

But since $s > E^B[v] - c$ and $\delta > \frac{c}{E^B[v] - E^G[v] + c}$, (6) is strictly larger than (5) and, hence, B could profitably deviate by rejecting s. This is a contradiction. ■

B Proof of Theorem 1: Equilibrium Properties

Before presenting our contraction mapping arguments for existence, we first establish properties (a)-(d) of a reputation equilibrium via a series of lemmas.

B.1 Part (a): $p > p^{**}$

Lemma 5 *Fix any δ, and consider any reputation equilibrium. For any p, player 2's expected payoff is at least $E^G[v]$.*

Proof. Suppose not; so, for some p, player 2's expected payoff is less than $E^G[v] - \varepsilon$ for some $\varepsilon > 0$. Now, consider player 2 demanding $E^G[v] - \frac{\varepsilon}{2}$. By Lemma 2, G accepts this for sure and B's rejection yields $E^B[v] - c > E^G[v]$ by Assumption 3. Thus, player 2's corresponding expected payoff is at least $E^G[v] - \frac{\varepsilon}{2}$. This is a contradiction. ■

Let $\overline{S} = E^B[v] - (1-\delta)c$, and define $\bar{\delta}$ implicitly such that

$$\overline{S} = (1-\bar{\delta})E^B[v] + \bar{\delta} f^B(\underline{v})E^G[v] + \bar{\delta}(1 - f^B(\underline{v}))\overline{S}, \tag{7}$$

where $\underline{v}$ is the smallest element in V. Given Assumption 3, it is straightforward to see that such $\bar{\delta} < 1$ exists. Also, note that if $\delta = \frac{c}{E^B[v]-E^G[v]+c}$ we have

$$\overline{S} = (1-\delta)E^B[v] + \delta E^G[v]. \tag{8}$$

Comparing (8) with (7), we see that $\bar{\delta} > \frac{c}{E^B[v]-E^G[v]+c}$.

Throughout the analysis below, assume that $\delta > \bar{\delta}$, and consider any reputation equilibrium. Since $\delta > \bar{\delta}$, Lemma 4 holds.

Lemma 6 *For any $p \in (p^{**}, 1)$, where $p^{**} = \frac{E^B[v]-E^G[v]-c}{E^B[v]-E^G[v]}$, $E^G[v]$ is demanded and accepted for sure by both types and, hence, $S^G(p) = S^B(p) = E^G[v]$.*

Proof. Fix any $p > p^{**}$. Let us proceed in the following steps.

Step 1: $E^G[v]$ is the unique equilibrium demand.

Proof of Step 1. Suppose otherwise; so, there exists another demand $s \neq E^G[v]$ offered in equilibrium. There are two cases to consider.

<u>Case 1</u>: $s < E^G[v]$.

But then, by Lemma 2, G accepts $s < E^G[v]$ and, hence, $S^G(p) < E^G[v]$, which contradicts that $S^G(p) = E^G[v]$ for all p.

<u>Case 2</u>: $s > E^G[v]$.

In this case, by Lemma 2, G rejects s for sure and, by Lemma 4, s can be accepted by B only if $s = E^B[v] - c$. Note that player 2's expected payoff from rejection conditional on player 1 being B is also $E^B[v] - c$. Thus, by demanding s, player 2's expected payoff is

$$p\left(E^G[v] - c\right) + (1-p)\left(E^B[v] - c\right),$$

which, since $p > p^{**}$, is strictly less than $E^G[v]$. This contradicts Lemma 5.

Step 2: Acceptance of $E^G[v]$ will not reduce the posterior.

Proof of Step 2: Let r^G and r^B denote the equilibrium rejection probability by G and B, respectively. We need to establish that $r^B \geq r^G$ and $r^G < 1$.

First, suppose that $r^B < r^G$. Player 2's expected payoff then is

$$\begin{aligned}
& p\left[r^G\left(E^G[v] - c\right) + (1-r^G)E^G[v]\right] + (1-p)\left[r^B\left(E^B[v] - c\right) + (1-r^B)E^G[v]\right] \\
< \; & p\left[r^B\left(E^G[v] - c\right) + (1-r^B)E^G[v]\right] + (1-p)\left[r^B\left(E^B[v] - c\right) + (1-r^B)E^G[v]\right] \\
= \; & pr^B\left(E^G[v] - c\right) + (1-p)r^B\left(E^B[v] - c\right) + (1-r^B)E^G[v] \\
= \; & pr^B(-c) + (1-p)r^B\left(E^B[v] - E^G[v] - c\right) + E^G[v] \\
\leq \; & p^{**}r^B(-c) + (1-p^{**})r^B\left(E^B[v] - E^G[v] - c\right) + E^G[v] \;\; \text{(because } p > p^{**}\text{)} \\
= \; & \frac{E^B[v] - E^G[v] - c}{E^B[v] - E^G[v]}r^B(-c) + \frac{c}{E^B[v] - E^G[v]}r^B\left(E^B[v] - E^G[v] - c\right) + E^G[v] \\
= \; & E^G[v].
\end{aligned}$$

But this contradicts Lemma 5.

Next, suppose that $r^G = 1$; so, from above, $r^B = 1$. But then, since $p > p^{**}$, player 2's expected payoff is strictly less than $E^G[v]$. This contradicts Lemma 5.

Step 3: $E^G[v]$ is accepted for sure by both types.

Proof of Step 3. It follows from Steps 1 and 2 that, for any $p > p^{**}$, $S^B(p) \leq E^G[v]$; otherwise, B can simply accept the equilibrium demand at every $p > p^{**}$. Since rejecting $E^G[v]$ yields at best $(1-\delta)E^B[v] + \delta E^G[v] > E^G[v]$, B must accept $E^G[v]$ for sure.

Finally, G must also accept $E^G[v]$ for sure. Otherwise, since B accepts this demand for sure, player 2's expected payoff is strictly less than $E^G[v]$. This contradicts Lemma 5. ■

B.2 Payoffs and Strategies at $p < p^{**}$

Lemma 7 *For any $p \in (0, p^{**}]$, $S^B(p) \leq \overline{S}$.*

Proof. Suppose not; so, for some $p \in (0, p^{**}]$, $S^B(p) > \overline{S}$. There are two cases to consider.

<u>Case 1</u>: There is no serious demand.

Note that rejected demands are not observable. Let X be B's expected transfer from rejection. By assumption, there exists some $\varepsilon > 0$ such that $X > \overline{S} + \varepsilon$. Since every demand is rejected, player 2's expected payoff is

$$p\left(E^G[v] - c\right) + (1-p)\left(E^B[v] - c\right). \tag{9}$$

Next, consider player 2 demanding $E^B[v] - c + \varepsilon$. G rejects this for sure and, by accepting, B's expected transfer is at most $(1-\delta)\left(E^B[v] - c + \varepsilon\right) + \delta E^B[v]$, but this is strictly smaller than X and hence B would accept it for sure. Thus, player 2's expected payoff from demanding $E^B[v] - c + \varepsilon$ is $p\left(E^G[v] - c\right) + (1-p)\left(E^B[v] - c + \varepsilon\right)$, which is strictly larger than (9). This is a contradiction.

Case 2: There is a serious demand.

By Lemma 4, the serious demand is either $E^G[v]$ or $E^B[v] - c$. Thus, B's expected transfer from accepting a demand is at most $(1-\delta)\left(E^B[v] - c\right) + \delta E^B[v] = \overline{S}$. Since rejected demands are not observable, it then follows that $S^B(p) \leq \overline{S}$. ■

Lemma 8 *For any $p \in (0, p^{**})$, one of the following holds:*

(i) $S^B(p) \leq \overline{S}$, and there are only losing demands.

(ii) $S^B(p) = \overline{S}$, and $E^B[v] - c$ is the only serious demand, which B is indifferent between accepting and rejecting. Furthermore, rejection by B must occur with positive probability and it strictly increases reputation.

Proof. Fix any $p \in (0, p^{**})$. There are several cases to consider.

Case 1: $E^G[v]$ is the only demand.

Let r^G and r^B denote the equilibrium rejection probability by G and B, respectively. Player 2's expected payoff is

$$p\left[r^G\left(E^G[v] - c\right) + (1-r^G)E^G[v]\right] + (1-p)\left[r^B\left(E^B[v] - c\right) + (1-r^B)E^G[v]\right]. \tag{10}$$

Also, if player 2 offers a demand larger than $\frac{E^B[v]}{1-\delta}$, it must be rejected for sure and, hence, he can guarantee

$$p\left(E^G[v] - c\right) + (1-p)\left(E^B[v] - c\right). \tag{11}$$

Note that, since $p < p^{**}$, (11) is strictly larger than $E^G[v]$.

We now go through each of the following possible sub-cases:

(1.1) $r^B \leq r^G < 1$. Then, since $E^B[v] - c > E^G[v]$ by Assumption 3, (10) is less than or equal to

$$\begin{aligned} & p\left[r^G\left(E^G[v] - c\right) + (1-r^G)E^G[v]\right] + (1-p)\left[r^G\left(E^B[v] - c\right) + (1-r^G)E^G[v]\right] \\ = \; & r^G\left[p\left(E^G[v] - c\right) + (1-p)\left(E^B[v] - c\right)\right] + \left(1 - r^G\right)E^G[v], \end{aligned}$$

which is less than (11) since $r^G < 1$ and $p < p^{**}$. This implies that player 2 would not demand $E^G[v]$, a contradiction.

(1.2) $r^B < r^G = 1$. Then, (10) becomes

$$p\left(E^G[v] - c\right) + (1-p)\left[r^B\left(E^B[v] - c\right) + (1 - r^B)E^G[v]\right],$$

which is less than (11) since $r^B < 1$. Thus, player 2 would not demand $E^G[v]$, a contradiction.

(1.3) $r^B > r^G \geq 0$. In this case, $S^B(p)$ is given by rejection and, since B's future transfers are bounded below by $E^G[v]$, we have

$$S^B(p) \geq (1-\delta)E^B[v] + \delta E^G[v] > E^G[v]. \tag{12}$$

Also, since $r^B > r^G$, accepting $E^G[v]$ must improve reputation and, hence, monotonicity implies that $S^B(p) \leq (1-\delta)E^G[v] + \delta S^B(p)$, or $S^B(p) \leq E^G[v]$. This contradicts (12).

(1.4) $r^B = r^G = 1$. Then, given Lemma 7, part (i) of the claim holds.

Case 2: $\left\{E^G[v], s\right\}$ for some $s \neq E^G[v]$ is in the support of player 2's equilibrium strategy.

In this case, clearly, it must be that $s > E^G[v]$ and hence, by Lemma 2, G rejects it for sure. We proceed by considering each possible sub-case:

(2.1) B accepts s with positive probability. Then, by Lemma 4, $s = E^B[v] - c$ and, hence, by Lemma 3, B accepts $E^G[v]$ for sure. Player 2's expected payoff from demanding $E^G[v]$ is, therefore, at most $E^G[v]$, which is less than (11) since $p < p^{**}$. This implies that $E^G[v]$ cannot be demanded, a contradiction.

(2.2) B rejects s for sure. In this case, we can apply the same arguments as for Case 1 above to consider each possible response to $E^G[v]$.

Case 3: $E^G[v]$ is not demanded.

If there is no serious demand, by Lemma 7, (i) holds. It then remains to show that, otherwise, part (ii) of the claim holds. In this case, by Lemma 4, $E^B[v] - c$ is the only serious demand and, by Lemma 2, only B accepts it. Since accepting this demand reveals B, the corresponding expected transfer amounts to $\overline{S}$. Let X denote B's expected transfer from rejection. Clearly, $X \geq \overline{S}$. We first show that $S^B(p) = X = \overline{S}$.

Suppose not; so, there exists some $\varepsilon > 0$ such that $X > \overline{S} + \varepsilon$. Then, consider player 2 demanding $E^B[v] - c + \varepsilon$. By accepting this demand, B's expected transfer is at most $(1-\delta)(E^B[v] - c + \varepsilon) + \delta E^B[v] = \overline{S} + (1-\delta)\,\varepsilon < X$ and, hence, B must accept it for sure. This implies that there exists a profitable deviation for player 2 from demanding $E^B[v] - c$, a contradiction.

Next, we show that rejection by B must occur in equilibrium. Otherwise, by Lemma 2, rejection reveals G and, hence, yields the expected transfer $(1-\delta)E^B[v] + \delta E^G[v] < \overline{S}$, where the inequality holds since $\delta > \frac{c}{E^B[v] - E^G[v] + c}$. This contradicts that $S^B(p) = \overline{S}$.

Finally, since G rejects all equilibrium demands and B accepts $E^B[v] - c$, rejection strictly increases reputation. ■

B.3 Parts (b) and (c): $p \in (p^*, p^{**})$ and $p \in (0, p^*)$

Lemma 9 *There exists some $p^* \in (0, p^{**})$ such that $S^B(p) = \overline{S}$ for all $p \in (0, p^*)$ and $S^B(p) < \overline{S}$ for all $p > p^*$.*

Proof. Suppose not. Then, by Lemma 8 and monotonicity, there are two cases to consider.

Case 1: $S^B(p) = \overline{S}$ for all $p \in (0, p^{**})$.

Consider $p = p^{**} - \varepsilon$ for some small $\varepsilon > 0$. By Lemma 8, rejection weakly improves reputation and, therefore, for sufficiently small ε, the posterior after the smallest realization of outside option, $\underline{v}$, must be above p^{**}. Thus, by Lemmas 6 and 7, we have

$$S^B(p) \leq (1 - \delta)E^B[v] + \delta f^B(\underline{v})E^G[v] + \delta(1 - f^B(\underline{v}))\overline{S}. \tag{13}$$

But since $\delta > \bar{\delta}$, the right-hand side of (13) is strictly less than $\overline{S}$, a contradiction.

Case 2: $S^B(p) < \overline{S}$ for all $p \in (0, p^{**})$.

By Lemma 8, in this case, there are only losing demands at every $p \in (0, p^{**})$. Then, reputation is updated purely by the realizations of random variable v; i.e., for any $p_t \in (0, p^{**})$, the posterior p_{t+1} after v is given by Bayes' formula (3) . Consider a stochastic process $\{p_t\}_{t=0}^{\infty}$ defined by the prior $p_0 \in (0, p^{**})$ and Bayes' formula (3). Let $M(p_0)$ be the probability with which p_t first reaches p^{**} in finite time. It follows from Lemma 1 in Appendix A.1 above that $\lim_{p_0 \to 0} M(p_0) = 0$.

Next, since belief is updated purely by the realizations of random variable v from any $p \in (0, p^{**})$, the (discounted average) expected payment $S^B(p_0)$ is obtained by a sequence of constant flow transfers with an expectation of $E^B[v]$ until the posterior reaches or exceeds p^{**}. However, we have just shown that $\lim_{p_0 \to 0} M(p_0) = 0$. Then, $\lim_{p_0 \to 0} S^B(p_0) = E^B[v] > \overline{S}$, a contradiction. ■

Parts (b) and (c) of Theorem 1 follow from combining Lemma 9 with Lemma 8. In addition, we obtain the following.

Lemma 10 *Fix p^* as defined in Lemma 9. There exists $\hat{p} \leq p^*$ such that part (ii) of Lemma 8 holds for any $p \in (0, \hat{p})$: $E^B[v] - c$ is the only serious demand, which B is indifferent between accepting and rejecting, and rejection by B must occur with a positive probability and it strictly increases reputation.*

Proof. Given an equilibrium value function $S^B(p)$, observe that $p^* := \sup\left\{p : S^B(p) = \overline{S}\right\}$. Define $\hat{p}$ implicitly such that

$$p^* = \frac{\hat{p} f^G(\underline{v})}{\hat{p} f^G(\underline{v}) + (1-\hat{p}) f^B(\underline{v})},$$

i.e., p^* is the updated posterior from $\hat{p}$ if the realized signal is $\underline{v}$.

Now, suppose the contrary of the claim; so, for all $p \in (0, p^*)$, part (i) of Lemma 8 holds. Fix any $p \in (0, \hat{p})$. By the definitions of p^* and $\hat{p}$, and since only losing demands are made, the posterior at the next period is bounded above by p^*. Thus,

$$S^B(p) = (1-\delta)E^B[v] + \delta\overline{S} > \overline{S}.$$

But this contradicts that $S^B(p) = \overline{S}$. ■

B.4 Part (d): p^* and p^{**}

Lemma 11 *At p^*, rejection occurs for sure.*

Proof. Suppose not; then, by part (ii) of Lemma 8, the serious demand must be $E^B[v] - c$ and acceptance of this demand leads to the continuation payment equal to $\overline{S}$. Also, rejection strictly increases reputation, say, to p'. Putting together these facts, we obtain

$$S^B(p^*) = \overline{S} = (1-\delta)\, E^B[v] + \delta E^B\left[S_\alpha(p_{t+1})\,|p_t = p'\right] = S^B(p') < \overline{S},$$

where the last inequality follows from the definition of p^* and monotonicity of $S^B(p)$. This is a contradiction ■

Lemma 12 *At p^{**}, we have the following:*
(i) $E^G[v]$ is the only serious demand.
(ii) If $E^G[v]$ is offered, it must be accepted for sure by both types.

Proof. (i) Suppose not; so, there is another serious demand, s. By Lemma 4, $s = E^B[v]-c$. Then, B accepts $E^G[v]$ for sure by Lemma 3. We also know that G rejects s for sure by Lemma 2. Therefore, rejection must strictly improve reputation. Thus, by Lemmas 6 and 7, B's expected transfer from rejection here is at most

$$(1-\delta)E^B[v] + \delta F(v^*)E^G[v] + \delta(1-F(v^*))\overline{S} < \overline{S},$$

where the inequality follows from $\delta > \bar{\delta}$. This contradicts that s is accepted in equilibrium.

(ii) Suppose not; consider the following two cases.

Case 1: G accepts $E^G[v]$ for sure.

Then, B must reject $E^G[v]$ and, hence, given that this is the only serious demand, acceptance must strictly increase reputation. Thus, by Lemma 6, the corresponding expected transfer for B is $(1-\delta)E^G[v] + \delta E^G[v] = E^G[v]$, which is clearly less than that from rejection. This is a contradiction.

Case 2: G rejects $E^G[v]$ with probability $r^G > 0$.

Let r^B denote B's corresponding rejection probability. We know that $p^{**}(E^G[v]-c) + (1-p^{**})(E^B[v]-c) = E^G[v]$. This implies that, if $r^B < r^G$, player 2's expected payoff is less than $E^G[v]$, which contradicts Lemma 5. Thus, $r^B \geq r^G$ and, hence, accepting $E^G[v]$ weakly improves reputation and the corresponding payment to type B is at most $(1-\delta)E^G[v] + \delta S^B(p^{**}) < S^B(p^{**})$, where $S^B(p^{**})$ must also be the payment from rejection at p^{**} (which happens with a positive probability) because $E^G[v]$ is the only serious demand and rejected offers are not observable. This contradicts that $r^B \geq r^G > 0$. ■

C Proof of Theorem 1: Construction

C.1 From Equilibrium to Contraction Mapping

Consider the reputation process $\{p_t\}$ governed by (3) in Appendix A above. Let $F^B(v) = \sum_{v \leq v'} f^B(v')$; as defined before, $v^* \in V$ is such that $\frac{f^B(v)}{f^G(v)} \geq 1$ if and only if $v > v^*$.

Fix $p^{**} = \frac{E^B[v]-E^G[v]-c}{E^B[v]-E^G[v]}$. Let $\mathbf{S}$ be the set of bounded non-increasing real-valued functions $S : [0,1] \to \left[E^G[v], E^B[v]\right]$. We know that $\mathbf{S}$ is a Banach space under the supremum norm. For each $\alpha \in [0,1]$, define an operator T_α on $\mathbf{S}$ as follows:

$$[T_\alpha(S)](p_t) = \begin{cases} E^B[v] & \text{if } p_t = 0 \\ \min\left\{\overline{S}, (1-\delta)E^B[v] + \delta E^B\left[S(p_{t+1})\,|p_t\right]\right\} & \text{if } p_t \in (0, p^{**}) \\ \alpha E^G[v] + (1-\alpha)\left[\begin{array}{c} (1-\delta)E^B[v] + \delta F^B(v^*)E^G[v] \\ +\delta\left(1-F^B(v^*)\right)E^B\left[S(p_{t+1})\,|p_t, v > v^*\right] \end{array}\right] & \text{if } p_t = p^{**} \\ \underline{S} & \text{if } p_t > p^{**}. \end{cases}$$

The definition of T_α is motivated by the equilibrium properties derived in Appendix B above. We explain the above definition of the operator in more detail below:

- If $p_t > p^{**}$, by Lemma 6 above, the value function is tied down by the payoff boundary, $\underline{S} = E^G[v]$.

- If $p_t = p^{**}$, player 2 must randomize. Upon a losing demand, belief updating depends on the realization of v and, hence, the corresponding continuation payment is given by

$$(1-\delta) E^B[v] + \delta F^B(v^*) E^G[v] + \delta \left(1 - F^B(v^*)\right) E^B\left[S(p_{t+1}) | p_t, v > v^*\right]. \tag{14}$$

  Accepting the serious demand $E^G[v]$ leads to an immediate payment $E^G[v]$ and the posterior p_{t+1} is unchanged at p^{**}. The latter implies that, at the next period, player 1 again entertains random demands between $E^G[v]$ and losing demands, and therefore, only $E^G[v]$ and (14) appear in computing the continuation payment. This in turn implies that the equilibrium value at p^{**} must itself be a convex combination of $E^G[v]$ and (14). We denote by $\alpha \in [0,1]$ the coefficient on the former. Thus, α is one-to-one to the probability with which player 2 demands $E^G[v]$ at p^{**}. Note that if $\delta > \frac{c}{F^B(v^*)(E^B[v]-E^G[v])+c}$, (14) is bounded above by $\overline{S} = E^B[v] - (1-\delta)c$. Therefore $[T_\alpha(S)](p^{**})$ is just the following expression appearing in the corresponding definition in Section 3.1 above:

$$\alpha E^G[v] + (1-\alpha)\left[(1-\delta) E^B[v] + \delta E^B\left[S(p_{t+1}) | p_t\right]\right].$$

- If $p_t \in (0, p^{**})$, by Lemma 8 above, either the first payoff boundary $\overline{S}$ binds, or the equilibrium features rejection (which happens when $p_t \in (p^*, p^{**})$) and hence the equilibrium value is given by $(1-\delta) E^B[v] + \delta E^B[S(p_{t+1}) | p_t]$. Therefore, by monotonicity of the value function, for $p_t \in (0, p^{**})$, the value must be

$$\min\left\{\overline{S}, (1-\delta) E^B[v] + \delta E^B\left[S(p_{t+1}) | p_t\right]\right\}.$$

- Note that T_α does not involve p^*, which is unknown and to be endogenized below. The other threshold, p^{**}, is the unique point that makes player 2 indifferent and therefore we take it as given in the definition of contraction mapping.

We establish the following properties of T_α for any $\alpha \in [0,1]$.

Lemma 13 *For each $\alpha \in [0,1]$, T_α is a contraction mapping with a Lipschitz constant $\delta < 1$. Hence, T_α admits a unique fixed point S_α. Furthermore,*

(a) S_α is non-increasing in α, i.e., $S_\alpha(p) \leq S_\beta(p)$ for all p whenever $\alpha \geq \beta$; and

(b) S_α is continuous in α in supremum norm.

Proof. We first check Blackwell's two sufficient conditions for contraction mapping.

(i) Monotonicity: Suppose $S \leq S'$. Then,

$$
\begin{aligned}
\left[T_{\alpha}(S)\right](p_t) &= \begin{cases} E^B[v], & \text{if } p_t = 0 \\ \min\left\{\overline{S}, (1-\delta)E^B[v] + \delta E^B\left[S(p_{t+1})|p_t\right]\right\} & \text{if } p_t \in (0, p^{**}) \\ \alpha E^G[v] + (1-\alpha)\left[\begin{array}{c} (1-\delta)E^B[v] + \delta F^B(v^*)E^G[v] \\ +\delta\left(1-F^B(v^*)\right)E^B\left[S(p_{t+1})|p_t, v > v^*\right] \end{array}\right] & \text{if } p_t = p^{**} \\ E^G[v] & \text{if } p_t > p^{**} \end{cases} \\
&\leq \begin{cases} E^B[v], & \text{if } p_t = 0 \\ \min\left\{\overline{S}, (1-\delta)E^B[v] + \delta E^B\left[S'(p_{t+1})|p_t\right]\right\} & \text{if } p_t \in (0, p^{**}) \\ \alpha E^G[v] + (1-\alpha)\left[\begin{array}{c} (1-\delta)E^B[v] + \delta F^B(v^*)E^G[v] \\ +\delta\left(1-F^B(v^*)\right)E^B\left[S'(p_{t+1})|p_t, v > v^*\right] \end{array}\right] & \text{if } p_t = p^{**} \\ E^G[v] & \text{if } p_t > p^{**} \end{cases} \\
&= \left[T_{\alpha}(S')\right](p_t).
\end{aligned}
$$

(ii) Discounting:

$$
\begin{aligned}
\left[T_{\alpha}(S+a)\right](p_t) &= \begin{cases} E^B[v] & \text{if } p_t = 0 \\ \min\left\{\overline{S}, (1-\delta)E^B[v] + \delta E^B\left[S(p_{t+1}) + a|p_t\right]\right\} & \text{if } p_t \in (0, p^{**}) \\ \alpha E^G[v] + (1-\alpha)\left[\begin{array}{c} (1-\delta)E^B[v] + \delta F^B(v^*)E^G[v] + \\ \delta\left(1-F^B(v^*)\right) \cdot \\ E^B\left[S(p_{t+1}) + a|p_t, v > v^*\right] \end{array}\right] & \text{if } p_t = p^{**} \\ E^G[v] & \text{if } p_t > p^{**} \end{cases} \\
&\leq \delta a + \begin{cases} E^B[v] & \text{if } p_t = 0 \\ \min\left\{\overline{S}, (1-\delta)E^B[v] + \delta E^B\left[S(p_{t+1})|p_t\right]\right\} & \text{if } p_t \in (0, p^{**}) \\ \alpha E^G[v] + (1-\alpha)\left[\begin{array}{c} (1-\delta)E^B[v] + \delta F^B(v^*)E^G[v] + \\ \delta\left(1-F^B(v^*)\right) \cdot \\ E^B\left[S(p_{t+1})|p_t, v > v^*\right] \end{array}\right] & \text{if } p_t = p^{**} \\ E^G[v] & \text{if } p_t > p^{**} \end{cases} \\
&= \delta a + \left[T_{\alpha}(S)\right](p_t).
\end{aligned}
$$

By the definition of T_α, its unique fixed point S_α satisfies:

$$S_\alpha(p_t) = \begin{cases} E^B[v] & \text{if } p_t = 0 \\ \min\left\{\overline{S}, (1-\delta)\, E^B[v] + \delta E^B\left[S_\alpha(p_{t+1})\,|p_t\right]\right\} & \text{if } p_t \in (0, p^{**}) \\ \alpha E^G[v] + (1-\alpha)\begin{bmatrix} (1-\delta)\, E^B[v] + \delta F^B(v^*)\, E^G[v] \\ +\delta\left(1 - F^B(v^*)\right) E^B\left[S_\alpha(p_{t+1})\,|p_t, v > v^*\right] \end{bmatrix} & \text{if } p_t = p^{**} \\ \underline{S} & \text{if } p_t > p^{**}. \end{cases} \tag{15}$$

Next, we derive the two stated properties of the unique fixed point S_α:

(a) Monotonicity of S_α in α.

For any $S \in \mathbf{S}$, $S_\alpha = \lim_{n\to\infty} (T_\alpha)^n(S)$. Note that by definition, if $\alpha \geq \beta$, then

$$T_\alpha(S) \leq T_\beta(S).$$

Hence, by monotonicity of T_α (in S; the first of Blackwell's conditions above), and by the above inequality, we have

$$T_\alpha(T_\alpha(S)) \leq T_\alpha(T_\beta(S)) \leq T_\beta(T_\beta(S)).$$

Iterating the same argument, we obtain, for any n, $(T_\alpha)^n(S) \leq (T_\beta)^n(S)$. Hence, $S_\alpha \leq S_\beta$.

(b) Continuity of S_α in α.

Consider a sequence $\alpha_n \to \alpha$. We want to show that $S_{\alpha_n} \to S_\alpha$ in sup-norm $\|\cdot\|$. We can write

$$T_{\alpha_n}(S)(p_t) = \begin{cases} E^B[v] & \text{if } p_t = 0 \\ \min\left\{\overline{S}, (1-\delta)\, E^B[v] + \delta E^B\left[S_\alpha(p_{t+1})\,|p_t\right]\right\} & \text{if } p_t \in (0, p^{**}) \\ \alpha_n E^G[v] + (1-\alpha_n)\begin{bmatrix} (1-\delta)\, E^B[v] + \delta F^B(v^*)\, E^G[v] \\ +\delta\left(1 - F^B(v^*)\right) E^B\left[S_\alpha(p_{t+1})\,|p_t, v > v^*\right] \end{bmatrix} & \text{if } p_t = p^{**} \\ E^G[v] & \text{if } p_t > p^{**}. \end{cases}$$

Note that this differs from (15) only at p^{**}.

Then, by definition,

$$\begin{aligned} & \|T_{\alpha_n}(S) - T_\alpha(S)\| \\ = \ & |\alpha_n - \alpha| \cdot \left|(1-\delta)\, E^B[v] + \delta F^B(v^*)\, E^G[v] + \delta\left(1 - F^B(v^*)\right) E^B_{v^*}\left[S(p_{t+1})\,|p_t\right] - E^G[v]\right| \\ \leq \ & |\alpha_n - \alpha| \left(E^B[v] + E^G[v]\right). \end{aligned}$$

Therefore, for any $\varepsilon > 0$, there exists N such that, if $n > N$, $\|T_{\alpha_n}(S) - T_\alpha(S)\| < \varepsilon$ for *any* $S \in \mathbf{S}$.

Since δ is a Lipschitz constant of the contraction mapping T_α, we have, for $n > N$,

$$\begin{aligned}
\left\|(T_{\alpha_n})^2(S) - (T_\alpha)^2(S)\right\| &= \left\|(T_{\alpha_n})^2(S) - T_\alpha(T_{\alpha_n}(S)) + T_\alpha(T_{\alpha_n}(S)) - (T_\alpha)^2(S)\right\| \\
&\leq \left\|(T_{\alpha_n})^2(S) - T_\alpha(T_{\alpha_n}(S))\right\| + \left\|T_\alpha(T_{\alpha_n}(S)) - (T_\alpha)^2(S)\right\| \\
&\leq \varepsilon + \delta\left\|T_{\alpha_n}(S) - T_\alpha(S)\right\| \\
&\leq \varepsilon + \delta\varepsilon \\
&= (1+\delta)\,\varepsilon.
\end{aligned}$$

Thus, if $n > N$, $\left\|(T_{\alpha_n})^2(S) - (T_\alpha)^2(S)\right\| < (1+\delta)\,\varepsilon$ for *any* $S \in \mathbf{S}$.

Now, fix $n > N$, and assume for the purpose of induction that, for any integer $m > 0$,

$$\left\|(T_{\alpha_n})^m(S) - (T_\alpha)^m(S)\right\| < \left(1 + \delta + \cdots + \delta^{m-1}\right)\varepsilon \text{ for any } S \in \mathbf{S}.$$

Then, we obtain

$$\begin{aligned}
&\left\|(T_{\alpha_n})^{m+1}(S) - (T_\alpha)^{m+1}(S)\right\| \\
= \; &\left\|(T_{\alpha_n})^{m+1}(S) - T_\alpha(T_{\alpha_n})^m(S) + T_\alpha(T_{\alpha_n})^m(S) - (T_\alpha)^{m+1}(S)\right\| \\
\leq \; &\left\|T_{\alpha_n}(T_{\alpha_n})^m(S) - T_\alpha(T_{\alpha_n})^m(S)\right\| + \left\|T_\alpha(T_{\alpha_n})^m(S) - T_\alpha(T_\alpha)^m(S)\right\| \\
\leq \; &\varepsilon + \delta\left\|(T_{\alpha_n})^m(S) - (T_\alpha)^m(S)\right\| \\
\leq \; &\varepsilon + \varepsilon\delta\left(1 + \delta + \cdots + \delta^{m-1}\right) \\
= \; &\left(1 + \delta + \cdots + \delta^m\right)\varepsilon.
\end{aligned}$$

That is, for any m, and for any $S \in \mathbf{S}$,

$$\left\|(T_{\alpha_n})^m(S) - (T_\alpha)^m(S)\right\| < \frac{\varepsilon}{1-\delta}.$$

Thus, when $n > N$, $\|S_{\alpha_n} - S_\alpha\| < \frac{\varepsilon}{1-\delta}$ as $m \to \infty$. This proves the continuity of S_α in α. ■

C.2 From Contraction Mapping to Equilibrium

The family of fixed points $\{S_\alpha : \alpha \in [0,1]\}$ obtained in the previous section offers potential candidates for the equilibrium payoffs $S^B(p)$. To go from the contraction mapping to an equilibrium, we need to identify the exact randomization at p^{**}. We proceed as follows.

C.2.1 Defining $p^*(\alpha)$

For each $\alpha \in [0,1]$, we define

$$p^*(\alpha) := \sup\left\{p : S_\alpha(p) = \overline{S}\right\},$$

where $S_\alpha(p)$ is the fixed point of T_α. That is, $p^*(\alpha)$ is the supremum of p such that the upper payment boundary is binding.

The next result guarantees that $p^*(\alpha)$ is well-defined.

Lemma 14 *For any $\alpha \in [0,1]$, there exists $p \in (0, p^{**})$ such that $S_\alpha(p) = \overline{S}$.*

Proof. Suppose to the contrary that there does not exist such a p. Then by the definition of S_α in (15) above, $S_\alpha(p) < \overline{S}$ for all $p > 0$. Therefore, from the definition of the fixed point S_α, the value of $S_\alpha(p_0)$ for $0 < p_0 < p^{**}$ is obtained by aggregating a sequence of constant flow payoff $E^B[v]$ until the posterior belief reaches or exceeds p^{**}. However, from the martingale convergence property established in Lemma 1, the probability of the latter event converges to 0 as $p_0 \to 0$. Then, $\lim_{p_0 \to 0} S_\alpha(p_0) = E^B[v] > \overline{S}$, a contradiction. ■

Then, by the definition of the fixed point and Lemma 14, we can immediately obtain that, with sufficiently large δ (as required in Theorem 1), $p^*(\alpha) \in (0, p^{**})$. By monotonicity, $S_\alpha(p) = \overline{S}$ for any $p \in (0, p^*(\alpha))$. However, we do not know whether $S_\alpha(p^*(\alpha)) = \overline{S}$; as we see below, this becomes relevant for our arguments.

C.2.2 Candidate Equilibrium Σ_α

For each $\alpha \in [0,1]$, let $p^* = p^*(\alpha) = \sup\left\{p : S_\alpha(p) = \overline{S}\right\}$. Also, let $p^{**} = \frac{E^B[v]-E^G[v]-c}{E^B[v]-E^G[v]}$. Then, consider the strategy profile Σ_α and associated belief system as follows:

1. Player 2's strategy:
   (a) At $p = 0$, it demands $E^B[v]$ for sure.
   (b) At any $p \in (0, p^{**})$, it demands $E^B[v] - c$ for sure.
   (c) At $p = p^{**}$, it demands $E^G[v]$ with probability $x = \frac{\alpha}{1-\delta+\alpha\delta} \in [0,1]$ and $E^B[v] - c$ with probability $1 - x$.
   (d) At any $p \in (p^{**}, 1]$, it demands $E^G[v]$ for sure.
2. Type G's strategy: for all p, it accepts s if and only if $s \leq E^G[v]$.[18]
3. Type B's strategy:
   (a) At $p = 0$, it accepts s if and only if $s \leq E^B[v]$.

[18] We could construct another equilibrium in which this type accepts $E^G[v]$ only at $p \geq p^{**}$.

(b) At any $p \in (0, p^*]$,

- it rejects s for sure if $s > E^B[v] - c$ and accepts s for sure if $s < E^B[v] - c$;
- it rejects $E^B[v] - c$ with probability $r(p) = \frac{p}{p^*} \frac{1-p^*}{1-p} \in [0, 1]$.

(c) At any $p \in (p^*, p^{**})$, it accepts s if and only if $s \leq \max\left\{\xi(p), E^G[v]\right\}$, where $\xi(p) = \frac{S_\alpha(p) - \delta E^B[v]}{1-\delta}$.

(d) At $p = p^{**}$, it accepts s if and only if $s \leq \max\left\{\xi(p^{**}), E^G[v]\right\}$, where $\xi(p^{**}) = \frac{X - \delta E^B[v]}{1-\delta}$ and

$$X = (1-\delta)E^B[v] + \delta F^B(v^*) E^G[v] + \delta\left(1 - F^B(v^*)\right) E^B\left[S_\alpha(p_{t+1}) | p_t = p^{**}, v > v^*\right].$$

(e) At any $p \in (p^{**}, 1]$, it accepts s if and only if $s \leq E^G[v]$.

4. Beliefs:

(a) The belief is updated by Bayes' rule whenever possible.

(b) At any $p \in (0, 1)$, the posterior belief assigns probability 1 to type B after acceptance of a demand strictly higher than $E^G[v]$; there is no change of belief after acceptance of a demand lower than or equal to $E^G[v]$.

(c) At any $p \in (p^{**}, 1)$, the posterior belief assigns probability 1 to type G after rejection (which is off-path).

C.2.3 Verification

We next show that Σ_α is an equilibrium for some α in two lemmas. By the definition of the fixed point, we have $S_\alpha(p^*(\alpha)) = \min\left\{\overline{S}, L(\alpha)\right\}$, where

$$L(\alpha) := (1-\delta) E^B[v] + \delta E^B\left[S_\alpha(p_{t+1}) | p_t = p^*(\alpha)\right].$$

Lemma 15 *The proposed strategy profile Σ_α and the associated beliefs form a reputation equilibrium if and only if $S_\alpha(p^*(\alpha)) = L(\alpha) = \overline{S}$.*

Proof. *The "only if" part:* Suppose that Σ_α is a reputation equilibrium. Fix any $p \in (0, p^*(\alpha))$. At this belief, Σ_α requires that type B be indifferent between accepting and rejecting $E^B[v] - c$; furthermore, right after rejection but before the outside option, the posterior jumps exactly to $p^*(\alpha)$. This implies that

$$S^B(p) = \overline{S} = (1-\delta) E^B[v] + \delta E^B\left[S_\alpha(p_{t+1}) | p_t = p^*(\alpha)\right]. \quad (16)$$

Moreover, Σ_α says that, at $p^*(\alpha)$, rejection must occur for sure. This means that

$$S_\alpha\left(p^*\left(\alpha\right)\right) = \left(1-\delta\right)E^B\left[v\right] + \delta E^B\left[S_\alpha\left(p_{t+1}\right)|p_t = p^*\left(\alpha\right)\right]. \tag{17}$$

Putting (16) and (17) together, we obtain that $S_\alpha\left(p^*\left(\alpha\right)\right) = L(\alpha) = \overline{S}$.

The "if" part: Suppose that $S_\alpha\left(p^*\left(\alpha\right)\right) = L(\alpha) = \overline{S}$. We claim that the fixed point S_α gives the value function associated with Σ_α. This follows from the definition; only the value at p^{**} needs some explanation.

At p^{**}, the definition of S_α implies that

$$S_\alpha\left(p^{**}\right) = \alpha E^G[v] + \left(1-\alpha\right)\left[\begin{array}{c}\left(1-\delta\right)E^B[v] + \delta F^B\left(v^*\right)E^G[v] \\ +\delta\left(1-F^B\left(v^*\right)\right)E^B\left[S_\alpha\left(p_{t+1}\right)|p_t = p^{**}, v > v^*\right]\end{array}\right].$$

Now, by the definition of Σ_α, player 2 demands $E^G[v]$ with probability $x = \frac{\alpha}{1-\delta+\alpha\delta}$. Then, we can re-arrange the above expression to obtain

$$S_\alpha\left(p^{**}\right) = x\left(1-\delta\right)E^G\left[v\right] + x\delta S_\alpha\left(p^{**}\right) + \left(1-x\right)\left[\begin{array}{c}\left(1-\delta\right)E^B[v] + \delta F^B\left(v^*\right)E^G[v] \\ +\delta\left(1-F^B\left(v^*\right)\right)E^B\left[S_\alpha\left(p_{t+1}\right)|p_t = p^{**}, v > v^*\right]\end{array}\right].$$

This is precisely the Bellman equation for type B's expected payment at p^{**} under Σ_α.

It remains to verify that Σ_α forms a reputation equilibrium when $S_\alpha\left(p^*\left(\alpha\right)\right) = \overline{S}$. First, consider player 2's strategy. Recall that, at p^{**}, we have

$$E^G[v] = p^{**}E^G[v] + (1-p^{**})E^B[v] - c.$$

Thus, at this belief, player 2 is indifferent between offering $E^G[v]$, which is accepted for sure, and a losing demand. Also, $E^B[v]$ is the payoff that player 2 can guarantee from type B via the outside option. It is then clear that the short-run player's offer is optimal against player 1's strategies at $p > p^{**}$ and at $p \leq p^*\left(\alpha\right) < p^{**}$. For $p \in \left(p^*\left(\alpha\right), p^{**}\right)$, note that $\xi(p) < E^B[v] - c$ (since $S_\alpha(p) < \overline{S}$) and, therefore, it is optimal for player 2 to make a losing demand as prescribed.

Second, consider type G. Fix any (on- or off-path) history at which this long-run player has to respond to offer s. Note that he expects the transfer $E^G[v]$ from the outside option; according to the equilibrium, the continuation payment at the next period is also equal to $E^G[v]$. Thus, it is optimal to accept s if and only if $s \leq E^G[v]$.

Finally, consider type B. We know from the "if" part of the proof of Lemma 15 in the main text that the equilibrium payoff $S^B(p)$ is indeed given by $S_\alpha(p)$ if $S_\alpha\left(p^*\left(\alpha\right)\right) = \overline{S}$. To show that deviation is not possible, fix any belief p and any (on- or off-path) demand s. His strategies at $p = 0$ and $p = 1$ are clearly best responses. Consider the following remaining cases.

Case 1: $p \in (0, p^*(\alpha)]$

In this case, since rejected offers are not observable, right after type B's rejection in the candidate equilibrium the posterior is $p^*(\alpha)$, and hence his expected payment is

$$(1-\delta) E^B[v] + \delta E^B[S_\alpha(p_{t+1})|p_t = p^*(\alpha)] = S_\alpha(p^*(\alpha)) = \overline{S}$$

where the first equality follows from the fact that rejection occurs with probability 1 at $p^*(\alpha)$ in the candidate equilibrium and the second inequality follows by the condition of Lemma 15. Note that $\overline{S}$ is also the payment from accepting $E^B[v] - c$ and revealing type B. Hence, at $p \in (0, p^*]$, it is optimal for type B to accept s if and only if $s \leq E^B[v] - c$.

Case 2: $p \in (p^*(\alpha), p^{**})$

By definition, $\xi(p)$ is the demand such that the continuation payment from accepting such a demand and revealing type B is $S_\alpha(p)$, where S_α corresponds to the value function computed from the candidate equilibrium strategy of rejecting the on-path demand. If $\xi(p) > E^G[v]$, accepting $s \leq \xi(p)$ is a best response. If $\xi(p) \leq E^G[v]$, then the candidate equilibrium calls type B to accept $s \leq E^G[v]$, which is also type G's strategy. Hence, the posterior will not change after the acceptance of $s \leq E^G[v]$. Hence, acceptance leads to a payoff of $(1-\delta)s + \delta S_\alpha(p) \leq S_\alpha(p)$. Hence, the candidate equilibrium's prescription of accepting s is indeed a best response.

Case 3: $p = p^{**}$

Note that $\xi(p^{**})$ is the demand such that the continuation payment from accepting such a demand and revealing type B is exactly X, i.e., the payment given by rejection according to the candidate equilibrium. Hence, the same argument for Case 2 applies here.

Case 4: $p \in (p^{**}, 1)$

Clearly, it is optimal to accept s if $s \leq E^G[v]$. Suppose that $s > E^G[v]$. By parts (b) and (c) of the proposed beliefs above, accepting this offer leads to payment $(1-\delta)s + \delta E^B[v]$ while the continuation payment from rejection is at most $(1-\delta)E^B[v] + \delta E^G[v]$. Thus, rejection is optimal if $\delta > \frac{1}{2}$. Recall that $\bar{\delta} > \frac{c}{E^B[v]-E^G[v]+c} > \frac{1}{2}$, where the last inequality follows from the assumption that $E^B[v] - E^G[v] > c$. ■

Given Lemma 15, the construction from contraction mapping to equilibrium is concluded by the next lemma. Recall from the definition of the fixed point that $S_\alpha(p^*(\alpha)) = \min\left\{\overline{S}, L(\alpha)\right\}$, where $L(\alpha) := (1-\delta) E^B[v] + \delta E^B[S_\alpha(p_{t+1})|p_t = p^*(\alpha)]$.[19] The proof turns out to be non-trivial. Note that although S_α is continuous and monotone in α (Lemma 13), as we alter the parameter α the entire fixed point S_α shifts, including the value of $p^*(\alpha)$.

Lemma 16 *There exists $\alpha \in [0,1]$ such that $S_\alpha(p^*(\alpha)) = L(\alpha) = \overline{S}$.*

[19] Note that $L(\alpha)$ may not be monotone in α even though S_α is.

Proof. We shall prove the lemma by way of contradiction. Suppose that the lemma is false; then, there are several cases to consider.

Case 1: For all $\alpha \in [0,1]$, $L(\alpha) > \overline{S}$.

Then, since S_α is decreasing in α (Lemma 13), we have

$$L(1) = (1-\delta)\, E^B[v] + \delta E^B \left[S_1\left(p_{t+1}\right) | p_t = p^*\left(1\right)\right] > \overline{S}.$$

Note also that $S_1(p^*(1)) = \min\{\overline{S}, L(1)\} = \overline{S}$.

Let $h = (v_1, v_2, \ldots)$ denote a sequence of realized signals v and $\phi(p,h)$ the posterior updated from p after h via Bayes' formula (3). Then, let H be the at most countable set of finite sequences of signals such that either $\phi\left(p^*\left(1\right), h\right) \geq p^{**}$ or $\phi\left(p^*\left(1\right), h\right) < p^*\left(1\right)$ but neither of the two inequalities will hold for any sub-history of h. Thus, by the definitions of the fixed point and $p^*(1)$, for any $h \in H$, $S_1\left(\phi\left(p^*\left(1\right), h\right)\right)$ is either $E^G[v]$ or $\overline{S}$.[20]

Let $\Pr(\cdot)$ be the probability measure over H induced by the signals. For any small $\eta > 0$, there exists a finite subset of H, say, $\hat{H}$, such that $\Pr\left(\hat{H}|H\right) > 1-\eta$. Since $\phi\left(p,h\right)$ is continuous and monotone in p, for any finite sequence of signals $h = (v_1, v_2, ..., v_n) \in \hat{H}$, we can find $\varepsilon_h > 0$ such that the following condition holds: for any $p \in \left[p^*\left(1\right), p^*\left(1\right) + \varepsilon_h\right]$, $\phi\left(p,h\right) > p^{**}$ or $\phi\left(p,h\right) < p^*\left(1\right)$ for the first time along h. Let $\varepsilon = \min_{h\in\hat{H}}\left\{\varepsilon_h\right\}$. Hence, $\left[p^*\left(1\right), p^*\left(1\right)+\varepsilon\right] \subset \left[p^*\left(1\right), p^*\left(1\right)+\varepsilon_h\right]$ for any $h \in \hat{H}$. That is, the interval $\left[p^*\left(1\right), p^*\left(1\right)+\varepsilon\right]$ reaches the same stopping regions $[p^{**}, 1)$ or $\left(0, p^*\left(1\right)\right)$ at the same time along any h in $\hat{H}$.

By the definition of $p^*(1)$, for any $p \in (p^*(1), p^{**})$,

$$S_1(p) = (1-\delta)E^B[v] + \delta E^B\left[S_1(p_{t+1})|p_t = p\right].$$

Therefore, for any $p \in \left[p^*\left(1\right), p^*\left(1\right)+\varepsilon\right]$, we have

$$\begin{aligned} \left|S_1(p) - L(1)\right| &= \delta\left|E^B\left[S_1(p_{t+1})|p_t = p\right] - E^B\left[S_1(p_{t+1})|p_t = p^*(1)\right]\right| \\ &\leq \delta\Pr\left(H - \hat{H}|H\right) E^B[v] \leq \delta\eta E^B[v]. \end{aligned} \tag{18}$$

Since $L(1) > \overline{S}$, for η is very close to 0, (18) implies that $S_1(p) > \overline{S}$. But this contradicts the definition of $p^*\left(1\right)$.

Case 2: For some $\beta \in [0,1]$, $L(\beta) < \overline{S}$.

In this case, $S_\beta\left(p^*\left(\beta\right)\right) = \min\left\{\overline{S}, L\left(\beta\right)\right\} = L\left(\beta\right) < \overline{S}$. Define $\alpha^* := \inf\left\{\alpha : L\left(\alpha\right) < \overline{S}\right\}$. By Lemma 13, $\alpha^* \leq \beta$. If $L(\alpha^*) = \overline{S}$, then the claim holds for α^*; so, suppose otherwise.

[20]When $\alpha = 1$, player 2 demands $E^G[v]$ for sure, and hence, $S_1\left(p^{**}\right) = E^G[v]$ by the definition of the fixed point.

Case 2.1: $L(\alpha^*) < \overline{S}$.

(i) $\alpha^* = 0$

In this case, the argument is almost symmetrical to that of Case 1 above. A contradiction can be derived by showing that $S_0(p^*(0) - \varepsilon) < \overline{S}$ for some $\varepsilon > 0$. Let H be the at most countable set of finite sequences of signals such that starting from $p^*(0)$, the posterior after any $h \in H$, which we write as $\phi(p^*(0), h)$, is such that either $\phi(p^*(0), h) > p^{**}$ or $\phi(p^*(0), h) < p^*(0)$, but neither of the two inequalities will hold for any sub-history of h. It will be made clear later that the strict inequalities in this statement are critical, as compared to Case 1.

In terms of payoffs, for any $h \in H$, $S_0(\phi(p^*(0), h))$ is either $\underline{S}$ or $\overline{S}$. Note that when $\alpha = 0$, $S_0(p^{**}) > \underline{S}$ by definition of the fixed point. Again, let Pr be the probability measure over H induced by the signals. For any small $\eta > 0$, there exists $\hat{H}$, a finite subset of H, such that $\Pr\left(\hat{H}|H\right) > 1 - \eta$. For any finite sequence of signals $h = (v_1, v_2, ..., v_n) \in \hat{H}$, we can find $\varepsilon_h > 0$ such that the following condition holds: $\phi(p^*(0) - \varepsilon_h, h) > p^{**}$ or $\phi(p^*(0) - \varepsilon_h, h) < p^*(0)$ for the first time along h. The existence of ε_h is guaranteed by the continuity of $\phi(p, h)$ in p from Bayes' formula. The monotonicity of $\phi(p, h)$ in p moreover implies that the entire interval $[p^*(0) - \varepsilon_h, p^*(0)]$ reaches the same stopping regions $(p^{**}, 1)$ or $(0, p^*(0))$ at the same time along history h.

Let $\varepsilon = \min_{h \in \hat{H}} \{\varepsilon_h\}$. Hence $[p^*(0) - \varepsilon, p^*(0)] \subset [p^*(0) - \varepsilon_h, p^*(0)]$ for any $h \in \hat{H}$. That is, the interval $[p^*(0) - \varepsilon, p^*(0)]$ reaches the same stopping regions $(p^{**}, 1)$ or $(0, p^*(0))$ at the same time along any h in $\hat{H}$. Therefore, for any $p \in [p^*(0) - \varepsilon, p^*(0)]$, we have

$$\begin{aligned} & \left|(1-\delta) E^B[v] + \delta E^B[S_0(\phi(p, v))] - (1-\delta) E^B[v] + \delta E^B[S_0(\phi(p^*(0), v))]\right| \\ = \ & \delta \left|E^B[S_0(\phi(p, v))] - E^B[S_0(\phi(p^*(0), v))]\right| \\ \leq \ & \delta \Pr\left(H - \hat{H}|H\right) E^B[v] \leq \delta\eta E^B[v]. \end{aligned} \tag{19}$$

If $L(0) = (1-\delta) E^B[v] + \delta E^B[S_0(\phi(p^*(0), v))] < \overline{S}$, then when η is very close to 0, (19) implies that

$$(1-\delta) E^B[v] + \delta E^B[S_0(\phi(p, v))] < \overline{S}$$

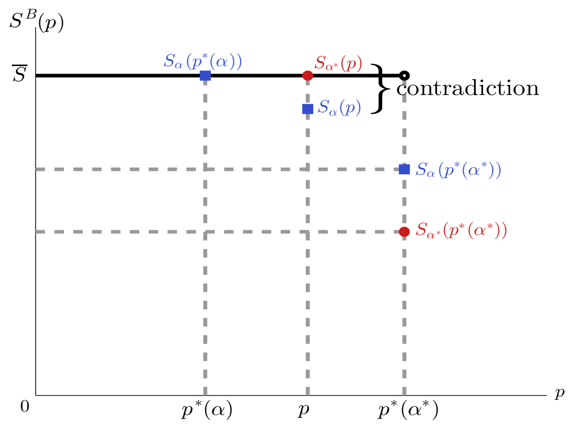
for any $p \in [p^*(0) - \varepsilon, p^*(0)]$. But then by the definition of the fixed point, $S_0(p) = \overline{S}$ for any $p < p^*(0)$. This is a contradiction.

(ii) $\alpha^* > 0$

Then, by the definition of the fixed point, $S_{\alpha^*}(p^*(\alpha^*)) = \min\left\{\overline{S}, L(\alpha^*)\right\} < \overline{S}$. Consider $\alpha \in (\alpha^* - \varepsilon, \alpha^*)$ for some small $\varepsilon > 0$. Let us proceed in the following steps as illustrated by Figure 5.

- $S_{\alpha^*}(p^*(\alpha^*)) \leq S_\alpha(p^*(\alpha^*)) < \overline{S}$. This follows from the continuity and monotonicity of S_α in α (Lemma 13).

Figure 5: $L(\alpha^*) < \overline{S}$ and $\alpha^* > 0$



- $S_\alpha(p^*(\alpha)) = \min\{\overline{S}, L(\alpha)\} = \overline{S}$. This follows from the fact that $L(\alpha) \geq \overline{S}$ by the definition of α^*.

- $p^*(\alpha) < p^*(\alpha^*)$. This follows from the two steps above because $S_\alpha(\cdot)$ is a decreasing function.

- For any $p \in (p^*(\alpha), p^*(\alpha^*))$, $S_\alpha(p) < \overline{S} = S_{\alpha^*}(p)$. This follows from the previous step and the definition of $p^*(\cdot)$.

But since $\alpha < \alpha^*$, the last step above contradicts that S_α is decreasing in α (Lemma 13).

Case 2.2: $L(\alpha^*) > \overline{S}$.

Then, by the definition of fixed point, $S_{\alpha^*}(p^*(\alpha^*)) = \min\{\overline{S}, L(\alpha^*)\} = \overline{S}$. Note that $\beta > \alpha^*$ such that $L(\beta) < \overline{S}$. Then, by the definition of α^*, there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ such that $L(\alpha^* + \varepsilon_n) < \overline{S}$ and $\beta > \alpha^* + \varepsilon_n$. Let us proceed in the steps below as illustrated by Figure 6.

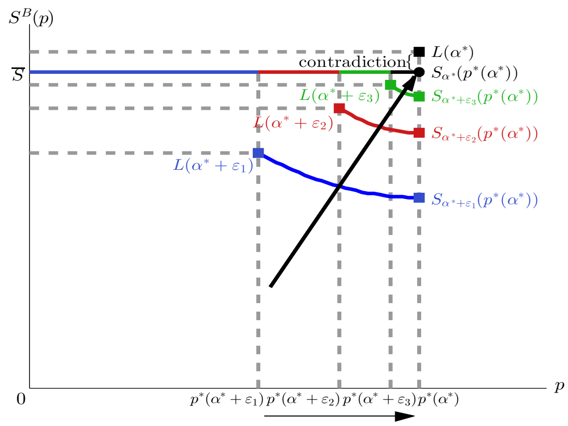
- $p^*(\alpha^* + \varepsilon_n) \leq p^*(\alpha^*)$. This follows from the fact that S_α is decreasing in α.

- $\overline{S} > L(\alpha^* + \varepsilon_n) = S_{\alpha^*+\varepsilon_n}(p^*(\alpha^* + \varepsilon_n)) \geq S_{\alpha^*+\varepsilon_n}(p^*(\alpha^*))$. This follows from the previous step because $S_\alpha(\cdot)$ is a decreasing function.

Now, by the continuity of S_α in α (Lemma 13), and since $\varepsilon_n \to 0$, $\|S_{\alpha^*+\varepsilon_n} - S_{\alpha^*}\| \to 0$. Note that

$$S_{\alpha^*+\varepsilon_n}(p^*(\alpha^*)) = (1-\delta) E^B[v] + \delta E^B[S_{\alpha^*+\varepsilon_n}(p_{t+1}) | p_t = p^*(\alpha^*)].$$

Figure 6: $L(\alpha^*) > \overline{S}$



Thus, the previous steps imply that

$$\begin{aligned}
\overline{S} &> (1-\delta) E^B[v] + \delta E^B \left[S_{\alpha^*+\varepsilon_n}(p_{t+1}) \,|p_t = p^*(\alpha^*) \right] \\
&\to (1-\delta) E^B[v] + \delta E^B \left[S_{\alpha^*}(p_{t+1}) \,|p_t = p^*(\alpha^*) \right] = L(\alpha^*) > \overline{S}.
\end{aligned}$$

This is a contradiction. ■

D Proof of Theorem 2

D.1 Part (a): Limit Uniqueness

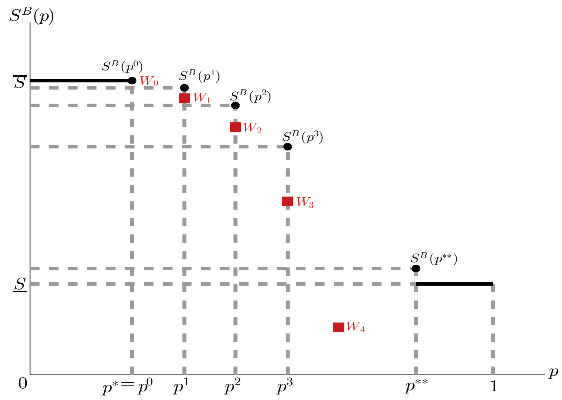
Let us begin with an outline of the proof. Fix the equilibrium payment of type B, S^B. For each fixed discount factor δ, we define an auxiliary *decreasing*, *divergent*, sequence of payment levels, W_n, $n = 0, 1, ...$, such that $W_0 = S^B(p^*)$ and $W_n \leq S^B(p^n)$ for each $n = 1, 2, ...$ ... and $W_n \leq S^B(p^n)$ for some sequence of "sparse" belief levels, p^n, $n = 0, 1, ...$ starting from $p^0 = p^*$. We shall show that $\min\{n : W_n \leq E^G[v]\} \to \infty$ as $\delta \to 1$. That is, for any finite n, W_n is always above $E^G[v]$ as δ approaches 1. Since $S^B(p^n)$ is above W_n, we know that for any n, $p^n < p^{**}$ as $\delta \to 1$. Since the sequence, p^n, $n = 0, 1, 2, ...$ is "sparse" by definition, this is possible only when p^* is close to 0. This intuition is illustrated in Figure 7.

D.1.1 Auxiliary Process W_n

The auxiliary sequence of payments is defined via the following first-order recursive equation:

$$W_n = (1-\delta)E^B[v] + \delta(1 - f^B(\underline{v}))E^B[v] + \delta f^B(\underline{v}) W_{n+1}, \tag{20}$$

Figure 7: W_n and $S^B(p)$



where $\underline{v} \in V$ is the smallest (best) signal. Let $W_0 = S^B(p^*)$, that is, type B's equilibrium payoff at the lower threshold belief p^*. It is clear that W_n is strictly decreasing and divergent.

Write $p^0 = p^*$. Let p^n be the posterior obtained from p^* after n consecutive realizations of $\underline{v}$. That is,

$$p^{n+1} = \frac{p^n f^G(\underline{v})}{p^n f^G(\underline{v}) + (1-p^n) f^B(\underline{v})} > p^n.$$

We first obtain the following two lemmas.

Lemma 17 *For any $n > 0$, $W_n < S^B(p^n)$ whenever $S_n > E^G[v]$.*

Proof. Given $W_0 = S^B(p^0)$, we prove the claim by induction. Suppose that $W_n \leq S^B(p^n)$ and $S^B(p^n) > E^G[v]$. By Theorem 1, the latter assumption implies that $p^n \in (p^*, p^{**})$ where only rejection occurs in equilibrium and, hence, we have

$$S^B(p^n) = (1-\delta)E^B[v] + \delta\left[\sum_{v \neq \underline{v}} f^B(v)\, S^B\left(\frac{p^n f^G(v)}{p^n f^G(v) + (1-p^n) f^B(v)}\right)\right] + \delta\left(1 - f^B(\underline{v})\right) S^B(p^{n+1}) \tag{21}$$

Now we compare (21) with (20). By induction, $W_n \leq S^B(p^n)$. Moreover, $S^B(p) < E^B[v]$ for all $p > 0$. Hence, $W_{n+1} < S^B(p^{n+1})$. ■

Lemma 18 $\lim_{\delta \to 1} S^B(p^*) = E^B[v]$ *(the limit exists and is equal to $E^B[v]$).*

Proof. Consider $p^* - \varepsilon$ for some small $\varepsilon > 0$. We know from Lemma 10 that, in equilibrium, rejection occurs at $p^* - \varepsilon$ such that the belief weakly improves immediately after rejection, say, to p'. Thus, for small enough ε, there exists some $v' \in V$ such that, for any $v < v'$,

$$\frac{p' f^G(v)}{p' f^G(v) + (1-p') f^B(v)} > p^*. \tag{22}$$

We know that

$$S^B(p^* - \varepsilon) = (1-\delta)E^B[v] + \delta \sum_{v \in V} f^B(v)\, S^B\left(\frac{p' f^G(v)}{p' f^G(v) + (1-p') f^B(v)}\right) = \overline{S}. \tag{23}$$

We also know that $S^B(p) \leq \overline{S} = E^B[v] - (1-\delta)\, c$ for all $p > 0$. Thus, (22) and (23) imply that there exists some $\varepsilon' > 0$ such that, for any $p \in (p^*, p^* + \varepsilon')$, $S^B(p) \to E^B[v]$, as $\delta \to 1$. By monotonicity of $S^B(\cdot)$, the claim then follows. ∎

D.1.2 Limit of p^* via W_n

Now, suppose that it takes $N(\delta)$ consecutive best signals to hit or exceed p^{**} from p^* in equilibrium. Then, Lemma 17 implies that

$$N(\delta) \geq \widehat{N}(\delta) = \min\left\{n : W_n \leq E^G[v]\right\}.$$

By standard formula, the solution to the first-order difference equation (20) is given by $W_n = \frac{b(1-a^n)}{1-a} + a^n W_0$, where $a = \frac{1}{\delta f^B(\underline{v})}$ and $b = -\frac{1-\delta f^B(\underline{v})}{\delta f^B(\underline{v})} E^B[v]$. Thus, $W_n \leq E^G[v]$ is equivalent to

$$n \geq \frac{\log\left(\frac{E^G[v] - \frac{b}{1-a}}{W_0 - \frac{b}{1-a}}\right)}{\log a} = \frac{\log\left(\frac{E^B[v] - E^G[v]}{E^B[v] - S^B(p^*)}\right)}{\log\left(\frac{1}{\delta f^B(\underline{v})}\right)}.$$

Note that $S^B(p^*) < E^B[v]$ for any $\delta < 1$ and by Lemma 21, $\lim_{\delta\to 1} S^B(p^*) = E^B[v]$. Hence

$$\liminf_{\delta \to 1} \widehat{N}(\delta) \geq \liminf_{\delta \to 1} \frac{\log\left(\frac{E^B[v] - E^G[v]}{E^B[v] - S^B(p^*)}\right)}{\log\left(\frac{1}{\delta f^B(\underline{v})}\right)} = \infty,$$

It then follows that $\liminf_{\delta \to 1} N(\delta) = \infty$ and, hence, $\lim_{\delta\to 1} p^*$ exists and the limit is 0.

D.2 Part (b): Reputation Building Probability $R(p)$

Our idea is to use the success probability $Q(p)$ of reaching p^{**} before dropping to p^* computed from the generalized gambler's ruin process to approximate the reputation building probability $R(p)$. Observe that to compute $R(p)$, the overall reputation building probability, we need to consider the randomization at the low region $(0,p^*]$ because even when reputation drops below p^*, it could bounce back with positive probability. We shall show as $p^* \to 0$ the gap between $Q(p)$ and $R(p)$ vanishes. Hence, we first need to derive some relevant properties of the equilibrium at low beliefs, i.e., at $p \in (0,p^*]$, where we know from Theorem 1 that type B may sometimes accept an equilibrium demand and, hence, reveal himself. We first show that the posterior belief upon a rejection at any belief p less than p^* can be bounded above by p^* and below by a constant that is independent of p. Using this, we then find a constant lower bound of the total probability with which player 1 voluntarily reveals himself at any $p < p^*$.

D.2.1 Reputation Building via Randomization at $p < p^*$

For any $p \in (0,1)$ and any $v \in V$, define

$$\phi_v^1(p) = \frac{pf^G(v)}{pf^G(v) + (1-p)\, f^B(v)},$$

that is, $\phi_v^1(p)$ is the posterior obtained from Bayesian updating upon signal v. Define recursively, for $k \geq 1$, $\phi_v^{k+1}(p) = \phi_v^1\left(\phi_v^k(p)\right)$. Also, let $\phi_v^{-k}(p)$ represent the inverse of $\phi_v^k(p)$, that is, starting from $\phi_v^{-k}(p)$, k consecutive realizations of signal v take the posterior belief exactly to p. Note that, by the MLRP, there exists some $v^* \in V$ such that $\phi_v^1(p) > p$ if and only if $v \leq v^*$.

Lemma 19 *Fix any $p \in (0,p^*]$, and suppose that rejection occurs at p. Let p' be the posterior immediately after the rejection but before the signal. Then, $p' \in \left[\phi_{\underline{v}}^{-1}(p^*), p^*\right]$, where $\underline{v} = \min V$.*

Proof. Suppose not. There are two cases to consider.

Case 1: $p' < \phi_{\underline{v}}^{-1}(p^*)$.
Then, since $S^B(p) = \overline{S}$ for all $p \in (0,p^*)$, and by the definition of $\phi_{\underline{v}}^{-1}(p^*)$, we have

$$\begin{aligned} S^B(p) &= (1-\delta)E^B[v] + \delta \sum\nolimits_{v\in V} f^B(v)\, S^B\left(\frac{p'f^G(v)}{p'f^G(v) + (1-p')\, f^B(v)}\right) \\ &= (1-\delta)E^B[v] + \delta\overline{S} > \overline{S}. \end{aligned}$$

But this contradicts that $S^B(p) = \overline{S}$.

Case 2: $p' > p^*$.

But then, since both $S^B(p)$ and $S^B(p')$ are determined by the continuation payoff from rejection, we have

$$\begin{aligned} S^B(p) &= (1-\delta)E^B[v] + \delta \sum\nolimits_{v\in V} f^B(v)\, S^B\left(\frac{p'f^G(v)}{p'f^G(v) + (1-p')\, f^B(v)}\right) \\ &= S^B(p') < \overline{S}, \end{aligned}$$

where the last inequality follows from the definition of p^*. But this contradicts $S^B(p) = \overline{S}$. ■

With slight abuse of notation, for any p, let $r^B(p)$ denote the total equilibrium probability of rejection by type B. We know from Theorem 1 that, if $p < p^{**}$, type G rejects all equilibrium demands for sure.

Fix any $p \in \left(0, \phi_{\underline{v}}^{-1}(p^*)\right)$. It must be that

$$r^B(p) \leq \frac{p}{1-p} \frac{1-\phi_{\underline{v}}^{-1}(p^*)}{\phi_{\underline{v}}^{-1}(p^*)}.$$

Otherwise, the posterior after rejection will not reach $\left[\phi_{\underline{v}}^{-1}(p^*), p^*\right]$ as required by Lemma 19. Let

$$y(p) := 1 - \frac{p}{1-p} \frac{1-\phi_{\underline{v}}^{-1}(p^*)}{\phi_{\underline{v}}^{-1}(p^*)} = \frac{\phi_{\underline{v}}^{-1}(p^*) - p}{(1-p)\phi_{\underline{v}}^{-1}(p^*)} \in (0,1).$$

Thus, $y(p)$ gives a lower bound on the probability of acceptance (and revelation) at p.

Note that $\phi_{\underline{v}}^{-1}(p^*) = \frac{\phi_{\underline{v}}^{-2}(p^*) f^G(\underline{v})}{\phi_{\underline{v}}^{-2}(p^*) f^G(\underline{v}) + \left(1-\phi_{\underline{v}}^{-2}(p^*)\right) f^B(\underline{v})}$. Hence, by simple algebra, we obtain

$$y\left(\phi_{\underline{v}}^{-2}(p^*)\right) = \frac{\phi_{\underline{v}}^{-1}(p^*) - \phi_{\underline{v}}^{-2}(p^*)}{(1-\phi_{\underline{v}}^{-2}(p^*))\phi_{\underline{v}}^{-1}(p^*)} = \frac{f^G(\underline{v}) - f^B(\underline{v})}{f^G(\underline{v})}.$$

Then, Lemma 19 implies that, for any $p \leq p^*$ (and hence $\phi_{\underline{v}}^{-2}(p) \leq \phi_{\underline{v}}^{-2}(p^*)$),

$$y\left(\phi_{\underline{v}}^{-2}(p)\right) \geq y\left(\phi_{\underline{v}}^{-2}(p^*)\right) = \frac{f^G(\underline{v}) - f^B(\underline{v})}{f^G(\underline{v})}. \tag{24}$$

D.2.2 Bounding the Probability of Revelation

Define $\varrho(p)$ as the aggregate probability with which, starting from p, type B *reveals* his type in equilibrium. Also, for any $p < p^{**}$, let $P(p, n, v)$ denote the posterior belief obtained from p after a sample equilibrium path over n periods in which player 1 rejects the demand followed by the realization of signal v in each period.

Lemma 20 *There exists some $\eta \in (0,1)$, independent of p, such that, for any $p \in (0, p^*]$, $\varrho(p) \geq \eta$.*

Proof. We proceed in following steps.

Step 1: There exists a finite integer k, independent of p, such that $\phi_{\overline{v}}^{k}(p) < \phi_{\underline{v}}^{-2}(p)$, where $\overline{v} = \max V$ and $\underline{v} = \min V$. (That is, k consecutive worst signals reduce reputation to a level from which two best signals will not bring it back.)

Proof of Step 1. Bayes' rule implies that

$$\log\left(\frac{\phi_{\overline{v}}^{k}(p)}{1-\phi_{\overline{v}}^{k}(p)}\right) = \log\left(\frac{p}{1-p}\right)+k\log\left(\frac{f^{G}(\overline{v})}{f^{B}(\overline{v})}\right) \quad \text{and}$$
$$\log\left(\frac{p}{1-p}\right) = \log\left(\frac{\phi_{\underline{v}}^{-2}(p)}{1-\phi_{\underline{v}}^{-2}(p)}\right)+2\log\left(\frac{f^{G}(\underline{v})}{f^{B}(\underline{v})}\right).$$

Hence, $\phi_{\overline{v}}^{k}(p)<\phi_{\underline{v}}^{-2}(p)$ is equivalent to

$$\log\left(\frac{p}{1-p}\right)+k\log\left(\frac{f^{G}(\overline{v})}{f^{B}(\overline{v})}\right)<\log\left(\frac{p}{1-p}\right)-2\log\left(\frac{f^{G}(\underline{v})}{f^{B}(\underline{v})}\right),$$

which is in turn equivalent to

$$k>-\frac{2\log\left(\frac{f^{G}(\underline{v})}{f^{B}(\underline{v})}\right)}{\log\left(\frac{f^{G}(\overline{v})}{f^{B}(\overline{v})}\right)}>0. \tag{25}$$

Step 2: Fix any $p\in(0,p^{*}]$ and any integer k satisfying (25). We have

$$\varrho(p)\geq\left(f^{B}(\overline{v})\right)^{2k}\min\left\{\frac{f^{G}(\underline{v})-f^{B}(\underline{v})}{f^{G}(\underline{v})},1-\left(\frac{f^{G}(\overline{v})}{f^{B}(\overline{v})}\right)^{k}\right\}=:\eta\in(0,1). \tag{26}$$

Proof of Step 2. Consider $P(p,2k,\overline{v})$, where k is given by (25) above and $\overline{v}$ is the worst signal; that is, starting at p, consider the posterior belief after a continuation history of $2k$ periods in which rejection followed by signal $\overline{v}$ happens in each period. There are two cases to consider.

<u>Case 1</u>: $P(p,2k,\overline{v})\leq\phi_{\underline{v}}^{-2}(p)$.

In this case, (24) above implies immediately that

$$\varrho(p)\geq\left(f^{B}(\overline{v})\right)^{2k}y\left(\phi_{\underline{v}}^{-2}(p^{*})\right)=\left(f^{B}(\overline{v})\right)^{2k}\frac{f^{G}(\underline{v})-f^{B}(\underline{v})}{f^{G}(\underline{v})}. \tag{27}$$

<u>Case 2</u>: $P(p,2k,\overline{v})>\phi_{\underline{v}}^{-2}(p)$.

Note that

$$\log\left(\frac{P(p,2k,\overline{v})}{1-P(p,2k,\overline{v})}\right) = \log\left(\frac{p}{1-p}\right)+2k\log\left(\frac{f^{G}(\overline{v})}{f^{B}(\overline{v})}\right)+\sum_{n=1}^{2k}\log\left(\frac{1}{r^{B}(P(p,n,\overline{v}))}\right)$$
$$= \log\left(\frac{\phi_{\overline{v}}^{2k}(p)}{1-\phi_{\overline{v}}^{2k}(p)}\right)+\sum_{n=1}^{2k}\log\left(\frac{1}{r^{B}(P(p,n,\overline{v}))}\right).$$

Hence, if $P\left(p,2k,\overline{v}\right)>\phi_{\underline{v}}^{-2}(p)$, then we have

$$\log\left(\frac{p}{1-p}\right)+2k\log\left(\frac{f^{G}\left(\overline{v}\right)}{f^{B}\left(\overline{v}\right)}\right)+\sum_{n=1}^{2k}\log\left(\frac{1}{r^{B}\left(P\left(p,n,\overline{v}\right)\right)}\right)>\log\left(\frac{\phi_{\underline{v}}^{-2}(p)}{1-\phi_{\underline{v}}^{-2}(p)}\right). \tag{28}$$

However, by definition of k, $\phi_{\overline{v}}^{k}\left(p\right)<\phi_{\underline{v}}^{-2}(p)$, that is,

$$\log\left(\frac{p}{1-p}\right)+k\log\left(\frac{f^{G}\left(\overline{v}\right)}{f^{B}\left(\overline{v}\right)}\right)<\log\left(\frac{\phi_{\underline{v}}^{-2}(p)}{1-\phi_{\underline{v}}^{-2}(p)}\right). \tag{29}$$

Putting (28) and (29) together, we obtain

$$\log\left(\frac{p}{1-p}\right)+2k\log\left(\frac{f^{G}\left(\overline{v}\right)}{f^{B}\left(\overline{v}\right)}\right)+\sum_{n=1}^{2k}\log\left(\frac{1}{r^{B}\left(P\left(p^{*},n,\overline{v}\right)\right)}\right)>\log\left(\frac{p}{1-p}\right)+k\log\left(\frac{f^{G}\left(\overline{v}\right)}{f^{B}\left(\overline{v}\right)}\right),$$

which yields

$$\log\left(\prod_{n=1}^{2k}r^{B}\left(P\left(p,n,\overline{v}\right)\right)\right)<k\log\left(\frac{f^{G}\left(\overline{v}\right)}{f^{B}\left(\overline{v}\right)}\right),$$

and hence,

$$\prod_{n=1}^{2k}r^{B}\left(P\left(p,n,\overline{v}\right)\right)<\left(\frac{f^{G}\left(\overline{v}\right)}{f^{B}\left(\overline{v}\right)}\right)^{k}. \tag{30}$$

Therefore,

$$\begin{aligned}
\varrho(p) &\geq \sum_{n=1}^{2k}\left[\left(1-r^{B}\left(P\left(p,n,\overline{v}\right)\right)\right)\left(f^{B}\left(\overline{v}\right)\right)^{n}\prod_{\ell=0}^{n-1}r^{B}\left(P\left(p,\ell,\overline{v}\right)\right)\right]\\
&> \left(f^{B}\left(\overline{v}\right)\right)^{2k}\sum_{n=1}^{2k}\left[\left(1-r^{B}\left(P\left(p,n,\overline{v}\right)\right)\right)\prod_{\ell=0}^{n-1}r^{B}\left(P\left(p,\ell,\overline{v}\right)\right)\right]\\
&= \left(f^{B}\left(\overline{v}\right)\right)^{2k}\sum_{n=1}^{2k}\left[\prod_{\ell=0}^{n-1}r^{B}\left(P\left(p,\ell,\overline{v}\right)\right)-\prod_{\ell=0}^{n}r^{B}\left(P\left(p,\ell,\overline{v}\right)\right)\right]\\
&= \left(f^{B}\left(\overline{v}\right)\right)^{2k}\left[1-\prod_{\ell=1}^{2k}r^{B}\left(P\left(p,\ell,\overline{v}\right)\right)\right]\\
&> \left(f^{B}\left(\overline{v}\right)\right)^{2k}\left[1-\left(\frac{f^{G}\left(\overline{v}\right)}{f^{B}\left(\overline{v}\right)}\right)^{k}\right],
\end{aligned} \tag{31}$$

where the last inequality follows from (30).

The statement of Step 2 then follows from (27) and (31). ■

D.2.3 Success Probability $Q(p)$ from Generalized Gambler's Ruin

Recall that ρ is defined by $E^B\left[\rho^{\log\left(\frac{f^G(v)}{f^B(v)}\right)}\right] = 1$. By Jensen's inequality, $E^B\left[\log\left(\frac{f^G(v)}{f^B(v)}\right)\right] < \log\left(E^B\left[\frac{f^G(v)}{f^B(v)}\right]\right) = 0$, where the last equality follows from the fact that the expectation is taken over $f^B(v)$. Hence, by Lemma 7.3.1 in Ethier (2010), $\rho > 1$.

We know that, in equilibrium, whenever $p_t \in (p^*, p^{**})$, the realization of signal $v \in V$ updates the belief such that

$$\log\left(\frac{p_{t+1}}{1-p_{t+1}}\right) = \log\left(\frac{p_t}{1-p_t}\right) + \log\left(\frac{f^G(v)}{f^B(v)}\right).$$

Let $Q(p_0)$ denote the "success probability" with which, starting from $p_0 \in (p^*, p^{**})$, the posterior belief p_t hits or exceeds p^{**} before hitting or falling below p^*. Then, by Theorem 7.3.2 in Ethier (2010), we have

$$L(p_0) \equiv \frac{\rho^{\lambda(p_0)-\lambda^*} - 1}{\rho^{\lambda^{**}-\lambda^*+\log\left(\frac{f^G(\underline{v})}{f^B(\underline{v})}\right)} - 1} \leq Q(p_0) \leq \frac{\rho^{\lambda(p_0)-\lambda^*-\log\left(\frac{f^G(\overline{v})}{f^B(\overline{v})}\right)} - 1}{\rho^{\lambda^{**}-\lambda^*-\log\left(\frac{f^G(\overline{v})}{f^B(\overline{v})}\right)} - 1} \equiv U(p_0), \tag{32}$$

where $\lambda(p) = \log\left(\frac{p}{1-p}\right)$, $\lambda^* = \log\left(\frac{p^*}{1-p^*}\right)$ and $\lambda^{**} = \log\left(\frac{p^{**}}{1-p^{**}}\right)$.

Since $p^* \to 0$ as $\delta \to 1$ by part (a) of Theorem 2, we have $\lambda^* \to -\infty$ as $\delta \to 1$. Since $\rho > 1$, applying l'Hôpital's rule, we obtain

$$\lim_{\delta\to 1} L(p_0) = \rho^{\lambda(p_0)-\lambda^{**}-\log\left(\frac{f^G(\underline{v})}{f^B(\underline{v})}\right)} \in (0,1) \quad \text{and} \quad \lim_{\delta\to 1} U(p_0) = \rho^{\lambda(p_0)-\lambda^{**}} \in (0,1). \tag{33}$$

D.2.4 Connecting $Q(p)$ and $R(p)$ in the Limit

Now, in order to work out the overall reputation building probability $R(p_0)$, i.e., the probability with which, starting from $p_0 \in (p^*, p^{**})$, p_t goes above p^{**}, we also have to consider the fact that, once the belief goes down to the region $(0, p^*]$, it may still bounce back. However, Lemma 20 shows that, at any such low belief, revelation occurs with probability at least $\eta \in (0,1)$. Hence, the interval $(0, p^*)$ becomes absorbing with a probability of at least η. Using this constant, we connect the success probability from the generalized gambler's ruin with reputation building probability in our game.

Lemma 21 *For any $p_0 \in (0, p^{**})$, $\lim_{\delta\to 1} R(p_0)$ exists and $\lim_{\delta\to 1} R(p_0) = \lim_{\delta\to 1} Q(p_0)$.*

Proof. Define $\Pi = \sup\left\{R(p) : p \in \left[p^*, \phi_{\underline{v}}^{2k}(p^*)\right]\right\}$. This supremum may not be achieved for any p, but, by definition, there exists a monotone sequence $p'_n \to p'$ such that $R(p'_n) \to \Pi$. (We first take a sequence of R's and then since p comes from a compact set, we take a further sequence of p'_n). Then, we have

$$R(p'_n) \leq Q(p'_n) + (1 - Q(p'_n))(1 - \eta)\Pi. \tag{34}$$

To see this, first recall that $Q(p_0)$ represents the probability with which the belief reaches p^{**} *before* falling to the region $(0, p^*]$. Therefore, with probability $1 - Q(p_0)$, the belief falls to some level in $(0, p^*]$. At such a belief, consider a sample equilibrium continuation history of $2k$ periods, where k is given by (25) in the proof of Lemma 20 above. We know from Lemma 20 that the aggregate revealing probability over such a sample history is at least η. With the remaining probability $1 - \eta$, the reputation building probability in the continuation game is bounded above by Π for the following reason: (i) by Lemma 19, the posterior at the end of $2k$ periods can be at most $\phi_{\underline{v}}^{2k}(p^*)$; (ii) if the posterior at the end of $2k$ periods falls short of p^*, the reputation building probability in the continuation game must be less than Π because the posterior must first bounce to at least p^* but this can only happen in equilibrium if player 1 sometimes accepts an equilibrium demand.

Since $L(p'_n) \leq Q(p'_n) \leq U(p'_n)$, (34) can be written as

$$R(p'_n) \leq U(p'_n) + (1 - L(p'_n))(1 - \eta)\Pi.$$

But since both L and U are continuous functions, taking limits of the above inequality, we obtain $\Pi \leq U(p') + (1 - L(p'))(1 - \eta)\Pi$, or

$$\Pi \leq \frac{U(p')}{1 - (1 - L(p'))(1 - \eta)}. \tag{35}$$

Note that $p' \in \left[p^*, \phi_{\underline{v}}^{2k}(p^*)\right]$ and $L(p') \leq U(p') \leq U\left(\phi_{\underline{v}}^{2k}(p^*)\right) \to 0$ as $p^* \to 0$. Thus, as $p^* \to 0$, $\Pi \to 0$.

Thus, since $\lim_{\delta\to 1} p^* = 0$, applying the same logic for any $p_0 \in (0, p^{**})$ yields

$$\limsup_{\delta\to 1} R(p_0) \leq \limsup_{\delta\to 1} \left[Q(p_0) + (1 - Q(p_0))(1 - \eta)\Pi\right] = \lim_{\delta\to 1} Q(p_0).$$

Note that $R(p_0) \geq Q(p_0)$ by definition. Hence,

$$\liminf_{\delta\to 1} R(p_0) \geq \lim_{\delta\to 1} Q(p_0) \geq \limsup_{\delta\to 1} R(p_0).$$

Therefore, $\lim_{\delta\to 1} Q(p_0) = \lim_{\delta\to 1} R(p_0)$. ■

Lemma 21, together with equations (32)and (33) , proves part (b) of Theorem 2.

D.3 Part (c): Payoffs

Let $\delta = e^{-r\Delta}$ for some $r \to 0$. Consider the equilibrium belief process p_t conditional on type B. Fix some small $\varepsilon > 0$ (as $\Delta \to 0$, the number of signals that could be observed in ε amount of real time explodes) and $p \in (\varepsilon, p^{**})$. Denote by $\tau\Delta$ the "real time" that it takes p_t to move out of (ε, p^{**}) in equilibrium.

Lemma 22 *Fix any $\varepsilon > 0$. There exists some $\Delta' > 0$ such that, if $\Delta < \Delta'$, we have $\tau\Delta < \varepsilon$ with probability at least $1 - \varepsilon$.*

Proof. From part (a) above, we know that $p^* \to 0$ as $\Delta \to 0$. Hence, there exists Δ'' such that $p^* < \varepsilon$ if $\Delta < \Delta''$. Hence, whenever $p_t \in (\varepsilon, p^{**}) \subset (p^*, p^{**})$, Theorem 1 says that only rejection occurs in equilibrium and, hence, belief is updated purely by Bayes' rule. Note that $\frac{p_t}{1-p_t}$ is a martingale conditional on rejection occurring, and therefore, by the martingale convergence theorem, p_t converges almost surely. Clearly, it cannot converge to some $p' \in (\varepsilon, p^{**})$ since both ε and p^{**} are fixed. Hence, since τ is finite almost surely, there exists N such that $\tau < N$ with probability at least $1 - \varepsilon$. Take $\Delta' = \min\left\{\frac{\varepsilon}{N}, \Delta''\right\}$, and the claim follows. ■

By part (b) above, we know that, for any $p_0 \in (0, p^{**})$, $R(p_0) \to Q(p_0)$ as $\delta \to 1$ (where $Q(\cdot)$ denotes the probability of the belief first reaching p^{**}). With slight abuse of notation, let $R(p_0, \varepsilon, \Delta)$ be the probability that, starting from $p_0 \in (\varepsilon, p^{**})$, the belief reaches p^{**} at the end of time ε. Then, $\lim_{\varepsilon\to 0} \lim_{\Delta\to 0} R(p_0, \varepsilon, \Delta) = \lim_{\delta\to 1} Q(p_0)$. By Lemma 22, it follows that $\lim_{\delta\to 1} S^B(p_0) = \lim_{\delta\to 1} \left[Q(p_0)E^G[v] + (1 - Q(p_0))E^B[v]\right]$.

E Proof of Theorem 3

To simplify exposition, in what follows, we let $\delta = e^{-r\Delta}$, $q = \frac{1+\mu\sqrt{\Delta}}{2}$, and $c = \frac{2\mu\Delta}{\kappa}$. As before, define $p^{**} = \frac{E^B[v]-E^G[v]-c}{E^B[v]-E^G[v]} = \frac{\kappa-1}{\kappa}$. Theorem 1 implies that there exists $\bar{\delta}_\Delta$ such that a reputation equilibrium exists if $\delta > \bar{\delta}_\Delta$. In particular, $\bar{\delta}_\Delta$ is determined implicitly by (7) in Section B above. The following guarantees that we can apply Theorem 1 in our parametrized model with $\delta = e^{-r\Delta}$.

Lemma 23 *There exists $\underline{\Delta} > 0$ such that $e^{-r\Delta} > \bar{\delta}_\Delta$ for any $\Delta < \underline{\Delta}$ and any $r > 0$.*

Proof. Plugging the relevant parameters into (7), we obtain

$$(1 + \mu\sqrt{\Delta})\bar{\delta}_\Delta^2 - (3 + \kappa + (1 - \kappa)\mu\sqrt{\Delta})\bar{\delta}_\Delta + 2 = 0.$$

The only solution in $(0,1)$ for this quadratic equation for sufficiently small Δ is

$$\bar{\delta}_{\Delta}=\frac{(3+\kappa+(1-\kappa)\mu\sqrt{\Delta})-\sqrt{(3+\kappa+(1-\kappa)\mu\sqrt{\Delta})^2-8(1+\mu\sqrt{\Delta})}}{2(1+\mu\sqrt{\Delta})}.$$

Taking $\lim_{\Delta\to 0}$ on both sides, we finally obtain

$$\lim_{\Delta\to 0}\bar{\delta}_{\Delta}=\frac{3+\kappa-\sqrt{\kappa^2+6\kappa+1}}{2}\in(0,1)\,.$$

Notice that $e^{-r\Delta}$ monotonically converges to 1 as $\Delta\to 1$. The lemma follows immediately. ■

Let $\phi^1(p)=\frac{pq}{pq+(1-p)(1-q)}$ and, for integer $k\geq 1$, define $\phi^{k+1}(p)=\phi^1(\phi^k(p))$ recursively. Let ϕ^{-k} be the inverse of ϕ^k. By the symmetry of signals, for any k and $k+1$, we have $\phi^k(p)=\phi^{-1}(\phi^{k+1}(p))$.

Consider the following second-order difference equation (SODE) for integer n:

$$S_n=(1-\delta)E^B[v]+\delta q S_{n-1}+\delta(1-q)S_{n+1} \tag{36}$$

with initial conditions $S_{-1}=S_0=\overline{S}=E^B[v]-(1-\delta)c$.

The explicit solution, for $n\geq 0$, is given by

$$S_n=E^B[v]+K_1\left[\frac{1+\left(1-4\delta^2q(1-q)\right)^{\frac{1}{2}}}{2\delta(1-q)}\right]^{n+1}+K_2\left[\frac{1-\left[1-4\delta^2q(1-q)\right]^{\frac{1}{2}}}{2\delta(1-q)}\right]^{n+1}, \tag{37}$$

where

$$\begin{aligned}K_1 &= -\frac{1}{2}\left[1-\frac{1-2\delta(1-q)}{\left(1-4\delta^2q(1-q)\right)^{\frac{1}{2}}}\right]\left(E^B[v]-S_0\right)\\ K_2 &= -\frac{1}{2}\left[1+\frac{1-2\delta(1-q)}{\left(1-4\delta^2q(1-q)\right)^{\frac{1}{2}}}\right]\left(E^B[v]-S_0\right).\end{aligned}$$

Note here that $\frac{1-2\delta(1-q)}{\left[1-4\delta^2q(1-q)\right]^{\frac{1}{2}}}<1$ and $E^B[v]-S_0=(1-\delta)\,c>0$ and hence $K_2<K_1<0$; $\{S_n\}$ is a decreasing and divergent sequence.

Simple algebra shows that, given $\delta>\bar{\delta}$ where $\bar{\delta}$ is given in (7) above for Theorem 1, $S_1>E^G[v]$. Define N as the following integer, which is finite because $\{S_n\}$ is divergent:

$$N=\sup\{n:S_n>E^G[v]\}. \tag{38}$$

Note that N is a function of δ.

Now, we consider a reputation equilibrium.

Lemma 24 *Consider the symmetric binary model. Fix any $\Delta < \underline{\Delta}$, where $\underline{\Delta}$ is as in Lemma 23, and assume that $S_{N+1} < E^G[v]$, where S_{N+1} is the $(N+1)$-th value of the solution to* (36) *and N is as defined by* (38)*. Then, consider any reputation equilibrium with two reputation thresholds $p^* = \sup\{p : S^B(p) = \overline{S}\}$ and $p^{**} = \frac{E^B[v]-E^G[v]-c}{E^B[v]-E^G[v]}$. We obtain the following:*

1. *For any $p \in (0, p^*)$, rejection occurs such that player 1's reputation immediately after rejection is p^*; thus, $r^B(p) = \frac{p}{p^*}\frac{1-p^*}{1-p} \in (0,1)$. Moreover, $S^B(p^*) = \overline{S}$, and p^* is uniquely determined by $p^* = \phi^{-N}(p^{**})$.*

2. *At p^{**}, there exists a unique probability $x \in (0,1)$ with which player 2 demands $E^G[v]$ (which is accepted for sure).*

Proof. 1. Let us first prove the first part of this lemma. Fix any $p \in (0, p^*)$. We proceed in the following steps.

Step 1: Player 1's reputation immediately after rejection, say, p^0, is such that $p^0 \le p^*$.

Proof of Step 1. Suppose not; so, $p^0 > p^*$. There are two cases to consider.

First, suppose that $p^0 \ge p^{**}$. Then, since $S^B(p) = E^G[v]$ for any $p \in (p^{**}, 1)$, we have

$$\begin{aligned} S^B(p) = \overline{S} &= (1-\delta)E^B[v] + \delta(1-q)S^B\left(\phi^1(p^0)\right) + \delta q S^B\left(\phi^{-1}(p^0)\right) \\ &= (1-\delta)E^B[v] + \delta(1-q)E^G[v] + \delta q S^B\left(\phi^{-1}(p^0)\right). \end{aligned}$$

But since $S^B\left(\phi^{-1}(p^0)\right) \le \overline{S}$ by monotonicity of the payoffs and $\delta > \bar{\delta}$, we have a contradiction.

Second, suppose that $p^0 \in (p^*, p^{**})$. By Theorem 1 of the main text, every equilibrium demand is rejected for sure at p^0 and, hence, given the definition of p^* and monotonicity of $S^B(p)$,

$$S^B(p^0) = (1-\delta)E^B[v] + \delta(1-q)S^B\left(\phi^1(p^0)\right) + \delta q S^B\left(\phi^{-1}(p^0)\right) < \overline{S}. \tag{39}$$

But (39) contradicts that

$$S^B(p^* - \varepsilon) = (1-\delta)E^B[v] + \delta(1-q)S^B\left(\phi^1(p^0)\right) + \delta q S^B\left(\phi^{-1}(p^0)\right) = \overline{S}.$$

Step 2: $S^B(p^0) = \overline{S}$.

Proof of Step 2. Suppose not; so, $S^B(p^0) < \overline{S}$. Given the definition of p^* and Step 1, it must then be that $p^0 = p^*$. We have

$$S^B(p^0) = (1-\delta)E^B[v] + \delta(1-q)S^B\left(\phi^1(p^0)\right) + \delta q\overline{S} < \overline{S}, \tag{40}$$

while, for any $p \in (0, p^*)$,

$$S^B(p) = (1-\delta)E^B[v] + \delta(1-q)S^B\left(\phi^1(p^0)\right) + \delta q\overline{S} = \overline{S}. \tag{41}$$

Comparing (41) with (40), we derive a contradiction.

Step 3: $p^0 = \phi^{-N}(p^{**})$. That is, p^0 is independent of p.

Proof of Step 3. For expositional ease, let $p^n = \phi^n(p^0)$. First, we show that $p^1 > p^*$. To see this, from Steps 1-2, we have

$$S^B(p^0) = (1-\delta)E^B[v] + \delta(1-q)S^B\left(p^1\right) + \delta q\overline{S} = \overline{S},$$

which exactly pins down $S^B(p^1) < \overline{S}$. Also, since $\delta > \bar{\delta}$, $S^B(p^1) > E^G[v]$. Thus, $p^1 \in (p^*, p^{**})$.

Then, given Step 2, and using the symmetry of signals, we obtain

$$\begin{aligned} S^B(p^1) &= (1-\delta)E^B[v] + \delta(1-q)S^B\left(p^2\right) + \delta q S^B(p^0) \\ &= (1-\delta)E^B[v] + \delta(1-q)S^B\left(p^2\right) + \delta q\overline{S}, \end{aligned}$$

which pins down $S^B(p^2)$, and so forth.

But by Theorem 1 of the main text, we know that $S^B(p) = E^G[v]$ for all $p > p^{**}$ and $S^B(p) > E^G[v]$ for all $p < p^{**}$. Thus, it must be that $S^B(p^n) = S_n$ only for positive integer $n \leq N$, where S_n solves (36) and N is given by (38).

Now, we want to show that $p^N = p^{**}$. Suppose not; so, $p^N < p^{**}$. Hence, at p^N, rejection occurs for sure and the corresponding equilibrium payoff must be such that

$$S^B(p^N) \geq (1-\delta)E^B[v] + \delta q S^B\left(p^{N-1}\right) + \delta(1-q)E^G[v]. \tag{42}$$

On the other hand, from the recursive equation (36), we have

$$S^B(p^N) = S_N = (1-\delta)E^B[v] + \delta q S^B\left(p^{N-1}\right) + \delta(1-q)S_{N+1}. \tag{43}$$

Thus, (43) contradicts (42) since we assume that $S_{N+1} < E^G[v]$.

Step 4: $p^* = p^0 = \phi^{-N}(p^{**})$. That is, the posterior after a rejection at $p < p^*$ is exactly p^*.

Proof of Step 4. Suppose not; so, by Step 1 above, $p^0 < p^*$. Step 3 shows that $p^0 = \phi^{-N}(p^{**})$ regardless of p. So, if we take $p \in (p^0, p^*)$, the posterior after a rejection is lower than p. But this contradicts Lemma 8: rejection cannot reduce player 1's reputation.

It follows immediately from Step 4 and Bayes' rule that $r^B(p) = \frac{p}{p^*}\frac{1-p^*}{1-p}$.

2. We know from Theorem 1 that, at p^{**}, $E^G[v]$ is the only possible serious demand and it will be accepted for sure if offered, leaving the belief unchanged at p^{**}. Letting x denote player 2's mixing probability on the demand $E^G[v]$, we can then write

$$S^B(p^{**}) = S_N = x\left[(1-\delta)E^G[v] + \delta S_N\right] + (1-x)X, \tag{44}$$

where

$$X \equiv (1-\delta)E^B[v] + \delta q S_{N-1} + \delta(1-q)E^G[v]. \tag{45}$$

Simple computation yields

$$x = \frac{X - S_N}{X - (1-\delta)E^G[v] - \delta S_N}.$$

Note first that $S_N < X$. This follows from comparing (45) above to the recursive equation

$$S_N = (1-\delta)E^B[v] + \delta q S_{N-1} + \delta(1-q)S_{N+1},$$

where, by assumption, $S_{N+1} < E^G[v]$. Also, we have $S_N > (1-\delta)E^G[v] + \delta S_N$ since $S_N > E^G[v]$ by definition. Thus, $x \in (0,1)$. ■

The next lemma pins down the generic uniqueness of the reputation equilibrium.

Lemma 25 *$S_{N+1} < E^G[v]$ for all but at most countably many $\Delta < \underline{\Delta}$.*

Proof. By the definition of N, $S_{N+1} \leq E^G[v] = -\mu\Delta$. The lemma is equivalent to saying that the set $\{\Delta \in (0, \underline{\Delta}) : S_{N+1} + \mu\Delta = 0\}$ is at most countable. Note that N is a function of Δ. For each integer $n = 1, 2, ...$, consider $D_n = \{\Delta \in (0, \underline{\Delta}) : S_n + \mu\Delta = 0\}$. It suffices to show that $\cup_{n=1}^{\infty} D_n$ is at most countable. To this end, notice that after substituting $\delta = e^{-r\Delta}$, $q = \frac{1+\mu\sqrt{\Delta}}{2}$ and $c = \frac{2\mu\Delta}{\kappa}$, it follows from (37) that

$$\begin{aligned} S_n = &\mu\Delta - \mu\Delta\frac{1-e^{-r\Delta}}{\kappa}\left[1 - \frac{1-e^{-r\Delta}(1-\mu\sqrt{\Delta})}{[1-e^{-2r\Delta}(1-\mu^2\Delta)]^{\frac{1}{2}}}\right]\left[\frac{1+[1-e^{-2r\Delta}(1-\mu^2\Delta)]^{\frac{1}{2}}}{e^{-r\Delta}(1-\mu\sqrt{\Delta})}\right]^{n+1} \\ &- \mu\Delta\frac{1-e^{-r\Delta}}{\kappa}\left[1 + \frac{1-e^{-r\Delta}(1-\mu\sqrt{\Delta})}{[1-e^{-2r\Delta}(1-\mu^2\Delta)]^{\frac{1}{2}}}\right]\left[\frac{1-[1-e^{-2r\Delta}(1-\mu^2\Delta)]^{\frac{1}{2}}}{e^{-r\Delta}(1-\mu\sqrt{\Delta})}\right]^{n+1}. \end{aligned} \tag{46}$$

Suppose $1 - \mu\sqrt{\Delta} \neq 0$. Then, S_n is an analytic function of Δ; thus, D_n, the set of zeros for $S_n + \mu\Delta = 0$, is at most countable (see, for instance, Theorem 3.7 in Conway (1978)). It follows that $\cup_{n=1}^{\infty} D_n$ is at most countable. ■

F Proof of Theorem 4

F.1 Part (a)

In this parametrized model, $p^{**} = \frac{\kappa-1}{\kappa}$ and $p^* = \phi^{-N(\Delta)}(p^{**})$, where $N(\Delta)$ is defined in Appendix E above. We know that $N(\Delta) \to \infty$ as $\Delta \to 0$, but since the precision q is a function of Δ, ϕ also changes with Δ. Lemma 26 below derives p^* as a function of N and Δ explicitly. Lemma 27 provides a sufficient condition for $p^* \to 0$ in terms of the speed of $N(\Delta) \to \infty$. Finally, we show that indeed this sufficient condition is satisfied.

Lemma 26 *With N as defined in* (38) *above, we have*

$$
\begin{aligned}
p^* &= \phi^{-N(\Delta)}(p^{**}) \\
&= \frac{(\kappa-1)(1-\mu\sqrt{\Delta})^{N(\Delta)}}{(\kappa-1)(1-\mu\sqrt{\Delta})^{N(\Delta)}+(1+\mu\sqrt{\Delta})^{N(\Delta)}}.
\end{aligned}
$$

Proof. The arguments in Appendix E above imply that $p^{**} = \phi^{N(\Delta)}(p^*)$. To pin this down further, we have $\phi^{-1}(p) = \frac{(1-\mu\sqrt{\Delta})p}{1+\mu\sqrt{\Delta}-2\mu\sqrt{\Delta}p}$, and hence, $\phi^{-1}(p^{**}) = \frac{(\kappa-1)(1-\mu\sqrt{\Delta})}{\kappa(1-\mu\sqrt{\Delta})+2\mu\sqrt{\Delta}}$. Thus, for $N(\Delta)$ given by (38) above, we obtain

$$
\begin{aligned}
\frac{1}{\phi^{-N(\Delta)}(p^{**})} &= \frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}} \cdot \frac{1}{\phi^{-N(\Delta)-1}(p^{**})} - \frac{2\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}} \\
&= \frac{1}{\phi^{-1}(p^{**})}\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{N(\Delta)-1} - \frac{2\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}} \cdot \frac{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{N(\Delta)-1}-1}{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)-1} \\
&= \frac{1}{\kappa-1}\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{N(\Delta)} + 1.
\end{aligned}
$$

Since $\phi^{-N(\Delta)}(p^{**}) = p^*$, the claim follows. ■

In order to compute the limit of p^* as $\Delta \to 0$, we need to conduct a rate comparison.

Lemma 27 *If* $N(\Delta)\sqrt{\Delta} \geq O\left(\log\left(\frac{1}{\Delta}\right)\right)$, *then* $\lim_{\Delta\to 0}\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{N(\Delta)} = \infty$ *and* $\lim_{\Delta\to 0} p^* = 0$.

Proof. Since $p^* = \frac{1}{1+\frac{1}{\kappa-1}\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{N(\Delta)}}$ by Lemma 26, it suffices to show that $\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{N(\Delta)} \to \infty$ for $p^* \to 0$. Note that, as $\Delta \to 0$,

$$
\left(1+\frac{2\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{\frac{1}{2}\left(\frac{1}{\mu\sqrt{\Delta}}-1\right)} \to e.
$$

Thus, $\lim_{\Delta\to 0}\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{N(\Delta)} = \infty$ if $N(\Delta)$ grows faster than $\frac{1}{\sqrt{\Delta}}$ as $\Delta \to 0$. This is sufficient if $N(\Delta)\sqrt{\Delta} = O\left(\log\left(\frac{1}{\Delta}\right)\right)$. ■

Now, to the first-order approximation, we obtain from (46) above:

$$
S_n > \mu\Delta - \frac{\mu r \Delta^2}{\kappa} \cdot \left[\frac{1+\sqrt{(2r+\mu^2)\Delta - 2r\mu^2\Delta^2}}{(1-r\Delta)(1-\mu\sqrt{\Delta})}\right]^{n+1} =: \widehat{S}_n. \tag{47}
$$

Define $\widehat{N}(\Delta) = \sup\{n : \widehat{S}_n > E^G[v] = -\mu\Delta\}$. Since S_n and $\widehat{S}_n$ are both decreasing, and by (47), it then follows that $\widehat{N} \leq N$ for any Δ. Since $\widehat{S}_{\widehat{N}(\Delta)+1} \leq -\mu\Delta$, we obtain

$$\widehat{N}(\Delta) + 2 \geq \frac{\log\left(\frac{2\kappa}{r\Delta}\right)}{\log\left(\frac{1+\sqrt{(2r+\mu^2)\Delta - 2r\mu^2\Delta^2}}{(1-r\Delta)(1-\mu\sqrt{\Delta})}\right)},$$

which implies that $\widehat{N}(\Delta)$, and hence $N(\Delta)$, grow faster than $\frac{1}{\sqrt{\Delta}}$ if

$$\frac{1}{\sqrt{\Delta}} \log\left(\frac{1+\sqrt{(2r+\mu^2)\Delta - 2r\mu^2\Delta^2}}{(1-r\Delta)(1-\mu\sqrt{\Delta})}\right) = O(1). \tag{48}$$

In particular, if this is true, we know that $\widehat{N}(\Delta)\sqrt{\Delta} = O\left(\log\left(\frac{2\kappa}{r\Delta}\right)\right)$. Since $N(\Delta) \geq \widehat{N}(\Delta)$, it follows from Lemma 27 that $\lim_{\Delta\to 0} p^* = 0$. We show condition (48) below.

As $\Delta \to 0$, we obtain

$$\begin{aligned}
&\frac{1}{\sqrt{\Delta}} \log\left(1+\sqrt{(2r+\mu^2)\Delta - 2r\mu^2\Delta^2}\right) = \log\left(1+\sqrt{(2r+\mu^2)\Delta - 2r\mu^2\Delta^2}\right)^{\frac{1}{\sqrt{\Delta}}} \\
&= \log\left(1+\sqrt{(2r+\mu^2)\Delta - 2r\mu^2\Delta^2}\right)^{\frac{1}{\sqrt{(2r+\mu^2)\Delta-2r\mu^2\Delta^2}}\cdot\frac{\sqrt{(2r+\mu^2)\Delta-2r\mu^2\Delta^2}}{\sqrt{\Delta}}} \\
&\to \log\exp(\sqrt{2r+\mu^2}) = \sqrt{2r+\mu^2} > 0,
\end{aligned}$$

since $\frac{\sqrt{(2r+\mu^2)\Delta-2r\mu^2\Delta^2}}{\sqrt{\Delta}} \to \sqrt{2r+\mu^2} > 0$; similarly, since $\frac{\mu\sqrt{\Delta}+r\Delta-r\mu\Delta\sqrt{\Delta}}{\sqrt{\Delta}} \to \mu$, we obtain

$$-\frac{1}{\sqrt{\Delta}} \log\left((1-r\Delta)(1-\mu\sqrt{\Delta})\right) \to \log\exp(\mu) = \mu.$$

Thus, as $\Delta \to 0$,

$$\frac{1}{\sqrt{\Delta}} \log\left(\frac{1+\sqrt{(2r+\mu^2)\Delta - 2r\mu^2\Delta^2}}{(1-r\Delta)(1-\mu\sqrt{\Delta})}\right) \to \sqrt{2r+\mu^2} + \mu > 0.$$

F.2 Part (b)

As in Appendix D.2, we use the success probability $Q(p)$ for the gambler's ruin process to approximate the reputation building probability $R(p)$. The bound on the error terms for a fixed discount factor obtained there continues to be valid. To compute the limit as $\Delta \to 0$, we apply the estimate on $N(\Delta)$ obtained in Appendix F.1.

F.2.1 Success Probability

Fix a prior $p_0 \in (p^*, p^{**})$. As before, let $Q(p_0)$ denote the probability that, conditional on type B, p_t exceeds p^{**} before dropping below p^* in equilibrium. Let $\lceil x \rceil$ denote the smallest integer larger than or equal to $x \in \mathbb{R}$. By the gambler's ruin result with symmetric binary signals (e.g., Theorem 7.1.1 in Ethier (2010)), and taking account of the integer problem, we obtain

$$Q(p_0) = \frac{\left(\frac{q}{1-q}\right)^{\left\lceil \frac{\lambda(p_0)-\lambda^*}{\lambda} \right\rceil} - 1}{\left(\frac{q}{1-q}\right)^{\left\lceil \frac{\lambda^{**}-\lambda(p_0)}{\lambda} \right\rceil + \left\lceil \frac{\lambda(p_0)-\lambda^*}{\lambda} \right\rceil} - 1},$$

where $\lambda = \log\left(\frac{q}{1-q}\right)$.

Now, we set $q = \frac{1+\mu\sqrt{\Delta}}{2}$ and take $\Delta \to 0$.

Lemma 28 $\lim_{\Delta\to 0} Q(p_0) = \frac{p_0}{(\kappa-1)(1-p_0)}$.

Proof. To simplify the calculation, let us re-write

$$Q(p_0) = \frac{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{\lceil z^* \rceil} - 1}{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{\lceil z^* \rceil + \lceil z^{**} \rceil} - 1} = \frac{1 - \frac{1}{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{\lceil z^* \rceil}}}{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{\lceil z^{**} \rceil} - \frac{1}{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{\lceil z^* \rceil}}}, \tag{49}$$

where

$$z^* := \frac{\lambda(p_0) - \lambda^*}{\lambda} = \frac{\log\left(\frac{p_0}{1-p_0}\frac{1-p^*}{p^*}\right)}{\log\left(\frac{q}{1-q}\right)} = \frac{\log\left(\frac{p_0}{1-p_0}\frac{1-p^*}{p^*}\right)}{\log\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)}$$

$$z^{**} := \frac{\lambda^{**} - \lambda(p_0)}{\lambda} = \frac{\log\left(\frac{p^{**}}{1-p^{**}}\frac{1-p_0}{p_0}\right)}{\log\left(\frac{q}{1-q}\right)} = \frac{\log\left(\frac{\kappa-1}{\frac{p_0}{1-p_0}}\right)}{\log\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)}.$$

Step 1: $\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{z^*} \to \infty$ as $\Delta \to 0$.

Proof of Step 1. Since $\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right) = 1 + o(1)$, $\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{\frac{1}{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)-1}} \to e$ as $\Delta \to 0$. Therefore,

$$\begin{aligned}
\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{\frac{\log\left(\frac{p_0}{1-p_0}\right)}{\log\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)}} &= \left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{\frac{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)-1}{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)-1}\frac{\log\left(\frac{p_0}{1-p_0}\right)}{\log\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)}} \\
&= \left[\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{\frac{1}{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)-1}}\right]^{\frac{\log\left(\frac{p_0}{1-p_0}\right)}{\left[\frac{1}{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)-1}\right]\log\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)}} \\
&\to \exp\left(\frac{\log\left(\frac{p_0}{1-p_0}\right)}{\log e}\right) = \frac{p_0}{1-p_0} > 0. \qquad (50)
\end{aligned}$$

We have already shown in Lemma 27 in Appendix F.1 above that $\lim_{\Delta\to 0}\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{N(\Delta)} = \infty$. Thus, by (50), we obtain

$$\begin{aligned}
\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{z^*} &= \left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{\frac{\log\left(\frac{p_0}{1-p_0}\right)+\frac{1}{\kappa-1}\log\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{N(\Delta)}}{\log\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)}} \\
&= \left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{\frac{N(\Delta)}{\kappa-1}}\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{\frac{\log\left(\frac{p_0}{1-p_0}\right)}{\log\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)}} \to \infty.
\end{aligned}$$

Step 2: $\lim_{\Delta\to 0} Q(p_0) = \frac{1}{\lim_{\Delta\to 0}\left[\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{\lceil z^{**}\rceil}\right]}$.

Proof of Step 2. Since $\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right) > 1$ and $z^* \le \lceil z^*\rceil < z^*+1$, and by Step 1, $\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{\lceil z^*\rceil} \to \infty$ as $\Delta \to 0$. The claim then follows from (49).

Now, by (50), we obtain

$$\lim_{\Delta\to 0}\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{z^{**}} = \lim_{\Delta\to 0}\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{\frac{\log\left(\frac{\kappa-1}{\frac{p_0}{1-p_0}}\right)}{\log\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)}} = \frac{(\kappa-1)(1-p_0)}{p_0} = \lim_{\Delta\to 0}\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{z^{**}+1},$$

which, by the squeeze theorem, implies $\lim_{\Delta\to 0}\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{\lceil z^{**}\rceil} = \frac{(\kappa-1)(1-p_0)}{p_0}$. Thus, by Step 2,

$$\lim_{\Delta\to 0} Q(p_0) = \frac{1}{\lim_{\Delta\to 0}\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{\lceil z^{**}\rceil}} = \frac{p_0}{(\kappa-1)(1-p_0)},$$

which belongs to $(0,1)$ for $p_0 < p^{**} = \frac{\kappa-1}{\kappa}$. ■

F.2.2 Reputation Building Probability

Next, recall from the proof of part (b) of Theorem 2 in Section D.2 that

$$R(p_0) \leq \left[Q\left(p_0\right) + \left(1 - Q\left(p_0\right)\right)(1-\eta)\,\Pi\right], \tag{51}$$

where, from (26) and (35),

$$\begin{aligned}
\Pi &\leq \frac{Q\left(p'\right)}{1-\left(1-Q\left(p'\right)\right)(1-\eta)}, \quad p' \in \left[p^*, \phi_{\underline{v}}^{2k}\left(p^*\right)\right] \\
\eta &= \left(f^B\left(\overline{v}\right)\right)^{2k} \min\left\{\frac{f^G\left(\underline{v}\right)-f^B\left(\underline{v}\right)}{f^G\left(\underline{v}\right)}, 1-\left(\frac{f^G\left(\overline{v}\right)}{f^B\left(\overline{v}\right)}\right)^k\right\}, \text{ where } k > -\frac{2\log\left(\frac{f^G(\underline{v})}{f^B(\underline{v})}\right)}{\log\left(\frac{f^G(\overline{v})}{f^B(\overline{v})}\right)}.
\end{aligned}$$

We know that $Q(p_0) \leq R(p_0)$, and have already solved for $\lim_{\Delta\to 0} Q(p_0)$ in Lemma 28. Thus, to show that indeed $\lim_{\Delta\to 0} R(p_0) = \lim_{\Delta\to 0} Q(p_0)$ for any $p_0 \in (0, p^{**})$, (51) implies that it suffices to show that $\lim_{\Delta\to 0}(1-\eta)\,\Pi = 0$.

With symmetric binary signals such that $f^G(\underline{v}) = f^B(\overline{v}) = q$, we have $k > 2$; without loss of generality, let us set $k = 3$. It is easy then to check that $\eta = q^5(2q-1)$. Also, Π is increasing in $Q(\cdot)$, which is itself increasing in p. Since we are considering an upper bound for $R(\cdot)$, let us take $p' = \phi_{\underline{v}}^{2k}(p^*)$. Since $(1-\eta)\Pi \leq \frac{(1-\eta)}{\frac{\eta}{Q(p')}+1-\eta} = \left(\frac{\eta}{1-\eta}\frac{1}{L(p')}+1\right)^{-1}$, we show that $\frac{\eta}{1-\eta}\frac{1}{Q(p')} \to \infty$ as $\Delta \to 0$.

First, substituting for $\eta = q^5(2q-1) = \left(\frac{1+\mu\sqrt{\Delta}}{2}\right)^5 \cdot \mu\sqrt{\Delta}$, we obtain

$$\frac{\eta}{1-\eta} = \frac{(1+\mu\sqrt{\Delta})^5\mu\sqrt{\Delta}}{32-(1+\mu\sqrt{\Delta})^5\mu\sqrt{\Delta}} = \frac{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^5\mu\sqrt{\Delta}}{\left(\frac{2}{1-\mu\sqrt{\Delta}}\right)^5-\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^5\mu\sqrt{\Delta}}. \tag{52}$$

It is straightforward to check that this goes to 0 as $\Delta \to 0$.

Second, to find $Q(p')$, note that $p' = \phi^6(p^*) = \phi^{-N(\Delta)-6}(p^{**}) = \frac{1}{1+\frac{1}{\kappa-1}\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{N(\Delta)-6}}$. By straightforward calculation, we obtain

$$Q(p') = \frac{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^6-1}{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{N(\Delta)}-1}. \tag{53}$$

Now, putting together (52) and (53), we obtain

$$\frac{\eta}{1-\eta}\frac{1}{Q(p')} = \frac{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^5 \mu\sqrt{\Delta}}{\left(\frac{2}{1-\mu\sqrt{\Delta}}\right)^5 - \left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^5 \mu\sqrt{\Delta}} \cdot \frac{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{N(\Delta)} - 1}{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^6 - 1}$$

$$= \frac{\mu\sqrt{\Delta}}{\left(\frac{2}{1+\mu\sqrt{\Delta}}\right)^5 - \mu\sqrt{\Delta}} \cdot \frac{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{N(\Delta)} - 1}{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^6 - 1}$$

$$= \frac{1}{\left(\frac{2}{1+\mu\sqrt{\Delta}}\right)^5 - \mu\sqrt{\Delta}} \cdot \frac{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{N(\Delta)} - 1}{\frac{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)-1}{\mu\sqrt{\Delta}} \cdot \sum_{i=0}^{5} \left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^i}.$$

By the results of Appendix F.1, we have

$$\lim_{\Delta\to 0}\left[\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{N(\Delta)} - 1\right] = \infty.$$

Also, $\lim_{\Delta\to 0}\left[\left(\frac{2}{1+\mu\sqrt{\Delta}}\right)^5 - \mu\sqrt{\Delta}\right) = 32$, $\lim_{\Delta\to 0}\frac{\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)-1}{\mu\sqrt{\Delta}} = \lim_{\Delta\to 0}\frac{2}{1-\mu\sqrt{\Delta}} = 2$, and

$$\lim_{\Delta\to 0}\left[\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^5 + \left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^4 + \left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^3 + \left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^2 + \left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right) + 1\right] = 6.$$

Thus, we obtain

$$\lim_{\Delta\to 0}\frac{\eta}{1-\eta}\frac{1}{Q(p')} = \frac{\lim_{\Delta\to 0}\left[\left(\frac{1+\mu\sqrt{\Delta}}{1-\mu\sqrt{\Delta}}\right)^{N(\Delta)} - 1\right]}{32\cdot 2\cdot 6} = \infty,$$

as required for $\lim_{\Delta\to 0}(1-\eta)\Pi = 0$.

Therefore, given part (a) of Theorem 4 and Lemma 28, we establish that, for any $p \in (0, p^{**}) = \left(0, \frac{\kappa-1}{\kappa}\right)$, $\lim_{\Delta\to 0} R(p) = \lim_{\Delta\to 0} Q(p) = \frac{p}{(\kappa-1)(1-p)}$ (and the limit exists).

F.3 Parts (c) and (d)

To compute the value function in the limit, we deal with two issues. First, in the diffusion limit, reputation building takes real time, and hence, a version of discounted reputation building probability is needed. This is obtained through Lemmas 29 and 30. Second, the payoff at exactly p^{**} becomes critical. For generic $\Delta > 0$, $S^B(p^{**})$ is not the same as $E^G[v]$; moreover, for any $p \in (0, p^{**})$, there will be Δ such that p is non-generic in the sense that it is reachable from p^{**} after a finite number of signals (see part (c) of Theorem 2). We consider a convergent sequence $\{\Delta_n\}$ and corresponding sets of generic beliefs $\{P_n\}$ (the computation of limiting payoffs at these beliefs does not involve p^{**}). This is the content of Lemma 32. Finally, in Lemmas 33 and 34 we show that the value function must be continuous in the limit, thereby deriving the payoff limits for all p.

F.3.1 Discounted Success Probability

Fix $p_0 \in (p^*, p^{**})$, and consider the random belief process $\{p_t\}$. Define

$$\tau = \inf \{t \geq 0 : p_t \geq p^{**} \text{ or } p_t \leq p^*\},$$

and let

$$z^* = \left\lceil \frac{\log\left(\frac{p_0}{1-p_0}\right) - \log\left(\frac{p^*}{1-p^*}\right)}{\log\left(\frac{q}{1-q}\right)} \right\rceil \text{ and } z^{**} = \left\lceil \frac{\log\left(\frac{p^{**}}{1-p^{**}}\right) - \log\left(\frac{p_0}{1-p_0}\right)}{\log\left(\frac{q}{1-q}\right)} \right\rceil.$$

We consider the *discounted* success probability, $\hat{Q}(p_0) := E^B\left[\delta^{\tau+1} 1_{\{p_\tau \geq p^{**}\}}\right]$.

Lemma 29 *For any $\delta \in (0,1)$, we have*

$$E^B\left[\delta^{\tau+1} 1_{\{p_\tau \geq p^{**}\}}\right] = \frac{\overline{\rho}^{z^*} - \underline{\rho}^{z^*}}{\overline{\rho}^{z^*+z^{**}} - \underline{\rho}^{z^*+z^{**}}},$$

where

$$\overline{\rho} = \frac{1 + \sqrt{1 - 4q(1-q)\delta^2}}{2(1-q)\delta} \text{ and } \underline{\rho} = \frac{1 - \sqrt{1 - 4q(1-q)\delta^2}}{2(1-q)\delta}.$$

Moreover, $\overline{\rho} > 1 > \underline{\rho} > 0$.

Proof. The formula follows immediately from Theorem 7.1.7 in Ethier (2010), which obtains the probability generating function for stopping times in the gambler's ruin process. Note here that the gambler's ruin process in our model starts at $t = 0$, and hence, at τ, $\tau + 1$ random variables are observed. Since $\delta \in (0,1)$ and $q \in \left(\frac{1}{2}, 1\right)$, it is straightforward to verify that $\overline{\rho} > 1 > \underline{\rho} > 0$. ■

In order to compute $\lim_{\Delta \to 0} \hat{Q}(p_0)$, notice that q is a function of Δ as is p^*, which is determined in equilibrium; thus, z^* and z^{**} change in Δ. We have shown in part (a) of Theorem 4 that $p^* \to 0$. The next lemma therefore holds for any $p_0 \in (0, p^{**})$.

Lemma 30 *For any $p_0 \in (0, p^{**})$, we have*

$$\lim_{\Delta \to 0} E^B \left[\delta^{\tau+1} 1_{\{p_\tau \geq p^{**}\}}\right] = \left(\frac{1}{\kappa - 1} \frac{p_0}{1 - p_0}\right)^{\frac{1}{2} + \frac{\sqrt{\mu^2 + 2r}}{2\mu}}.$$

Proof. By Lemma 29, $E\left[\delta^{\tau+1} 1_{\{p_\tau \geq p^{**}\}}\right] = \frac{\overline{\rho}^{z^*} - \underline{\rho}^{z^*}}{\overline{\rho}^{z^*+z^{**}} - \underline{\rho}^{z^*+z^{**}}}$. The computation is done in two steps.

Step 1: $\lim_{\Delta \to 0} \frac{\overline{\rho}^{z^*} - \underline{\rho}^{z^*}}{\overline{\rho}^{z^*+z^{**}} - \underline{\rho}^{z^*+z^{**}}} = \lim_{\Delta \to 0} \frac{1}{\overline{\rho}^{z^{**}}}$.

Proof of Step 1: We first show $\lim_{\Delta \to 0} \overline{\rho}^{z^*} = +\infty$. Plugging $q = \frac{1+\mu\sqrt{\Delta}}{2}$ and $\delta = e^{-r\Delta}$ into $\overline{\rho}$, we get

$$\overline{\rho} = \frac{1 + \sqrt{1 - e^{-2r\Delta}(1 - \mu^2\Delta)}}{e^{-r\Delta}(1 - \mu\sqrt{\Delta})}.$$

To the first-order approximation, we obtain

$$\begin{aligned}
\overline{\rho}^{z^*} &= \left[\frac{1 + \sqrt{1 - e^{-2r\Delta}(1 - \mu^2\Delta)}}{e^{-r\Delta}(1 - \mu\sqrt{\Delta})}\right]^{z^*} \\
&> \left[\frac{1 + \sqrt{1 - (1 - \mu^2\Delta)}}{1 - \mu\sqrt{\Delta}}\right]^{z^*} \\
&= \left(\frac{1 + \mu\sqrt{\Delta}}{1 - \mu\sqrt{\Delta}}\right)^{z^*} \to +\infty,
\end{aligned}$$

where the last limit follows from Step 1 in the proof of Lemma 28, which utilizes the fact that, as $\Delta \to 0$, $p^* = 0$ and hence $z^* \to +\infty$.

Moreover, $0 < \underline{\rho} < 1 < \overline{\rho}$ by Lemma 29. Therefore, $\overline{\rho}^{z^*+z^{**}} > \overline{\rho}^{z^*} \to +\infty$, and hence,

$$\lim_{\Delta \to 0} \frac{\overline{\rho}^{z^*} - \underline{\rho}^{z^*}}{\overline{\rho}^{z^*+z^{**}} - \underline{\rho}^{z^*+z^{**}}} = \lim_{\Delta \to 0} \frac{\overline{\rho}^{z^*}}{\overline{\rho}^{z^*+z^{**}}} = \lim_{\Delta \to 0} \frac{1}{\overline{\rho}^{z^{**}}}.$$

Step 2: $\lim_{\Delta \to 0} \frac{1}{\overline{\rho}^{z^{**}}} = \left(\frac{1}{\kappa-1} \frac{p_0}{1-p_0}\right)^{\frac{1}{2} + \frac{\sqrt{\mu^2+2r}}{2\mu}}$.

Proof of Step 2: Since $\log\left(\frac{q}{1-q}\right) = \log(1 + \mu\sqrt{\Delta}) - \log(1 - \mu\sqrt{\Delta})$, and letting $C = \log\left(\frac{p^{**}}{1-p^{**}}\right) - \log\left(\frac{p_0}{1-p_0}\right)$, we can rewrite

$$z^{**} = \frac{C}{\log(1 + \mu\sqrt{\Delta}) - \log(1 - \mu\sqrt{\Delta})}.$$

Note that

$$\lim_{\Delta\to 0}\log\overline{\rho}^{z^{**}} = \lim_{\Delta\to 0}\frac{C\cdot\left(\log\left(1+\sqrt{1-e^{-2r\Delta}(1-\mu^2\Delta)}\right)+r\Delta-\log(1-\mu\sqrt{\Delta})\right)}{\log(1+\mu\sqrt{\Delta})-\log(1-\mu\sqrt{\Delta})}.$$

Then, by l'Hôpital's rule, we obtain

$$\lim_{\Delta\to 0}\log\overline{\rho}^{z^{**}} = C\cdot\lim_{\Delta\to 0}\left[\frac{1}{\mu}\Big(\frac{1-\mu^2\Delta}{1+\sqrt{K(\Delta)}}\cdot\frac{2re^{-2r\Delta}(1-\mu^2\Delta)+e^{-2r\Delta}\mu^2}{2\sqrt{K(\Delta)/\Delta}}\right.$$
$$\left.+r\sqrt{\Delta}(1-\mu^2\Delta)+\frac{\mu}{2}(1-\mu\sqrt{\Delta})\Big)\right],$$

where $K(\Delta)=1-e^{-2r\Delta}(1-\mu^2\Delta)$.

Since $\frac{K(\Delta)}{\Delta}\to\mu^2+2r$ as $\Delta\to 0$, we obtain

$$\lim_{\Delta\to 0}\log\overline{\rho}^{z^{**}} = C\cdot\left(\frac{1}{2}+\frac{\sqrt{\mu^2+2r}}{2\mu}\right),$$

which, after substituting for C, implies

$$\lim_{\Delta\to 0}\overline{\rho}^{z^{**}} = \left(\frac{p^{**}}{1-p^{**}}\frac{1-p_0}{p^0}\right)^{\frac{1}{2}+\frac{\sqrt{\mu^2+2r}}{2\mu}}.$$

Substituting $p^{**}=\frac{\kappa-1}{\kappa}$ into this and taking the inverse, we establish the lemma. ∎

F.3.2 Discounted Payoffs

Lemma 31 *For any $p_0\in(p^*,p^{**})$, we have*

$$E^B\left[S^B(p_0)\right]=E^B[v]\left(1-E^B\left[\delta^{\tau+1}\right]\right)+\overline{S}E^B\left[\delta^{\tau+1}1_{\{p_\tau\le p^*\}}\right]+E^B\left[\delta^{\tau+1}S^B(p_\tau)1_{\{p_\tau\ge p^{**}\}}\right].$$

Proof. Our argument below adopts the idea behind the proof for Wald's equation. Define a random variable I_t as follows:

$$I_t=\begin{cases}1 & \text{if } \tau\ge t\\ 0 & \text{if } \tau<t\end{cases}.$$

Note that p_t is determined by $v_0, v_1, ..., v_{t-1}$, the realized outside option payments up to t;

therefore, $\{\tau \geq t\}$, and hence I_t, are independent of v_t. Then,

$$\begin{aligned}
E^B\left[\sum_{t=0}^{\tau}\delta^t v_t\right] &= E^B\left[\sum_{t=0}^{\infty}\delta^t v_t I_t\right] \\
&= E^B\left[\sum_{t=0}^{\infty}\delta^t E^B\left[v_t\right] I_t\right] \\
&= E^B\left[v\right] E^B\left[\sum_{t=0}^{\infty}\delta^t I_t\right] \\
&= E^B\left[v\right] E^B\left[\sum_{t=0}^{\tau}\delta^t\right] \\
&= \frac{E^B\left[v\right]}{1-\delta}\left(1-E^B\left[\delta^{\tau+1}\right]\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
E^B\left[S^B\left(p_0\right)\right] &= \left(1-\delta\right) E^B\left[\sum_{t=0}^{\tau}\delta^t v_t + \delta^{\tau+1}\frac{S^B\left(p_\tau\right)}{1-\delta}\right] \\
&= E^B\left[v\right]\left(1-E^B\left[\delta^{\tau+1}\right]\right) + E^B\left[\delta^{\tau+1}S^B\left(p_\tau\right)\right] \\
&= E^B\left[v\right]\left(1-E^B\left[\delta^{\tau+1}\right]\right) + E^B\left[\delta^{\tau+1}S^B\left(p_\tau\right)1_{\{p_\tau\leq p^*\}}\right] + E^B\left[\delta^{\tau+1}S^B\left(p_\tau\right)1_{\{p_\tau\geq p^*\}}\right] \\
&= E^B\left[v\right]\left(1-E^B\left[\delta^{\tau+1}\right]\right) + \overline{S}E^B\left[\delta^{\tau+1}1_{\{p_\tau\leq p^*\}}\right] + E^B\left[\delta^{\tau+1}S^B\left(p_\tau\right)1_{\{p_\tau\geq p^*\}}\right],
\end{aligned}$$

which completes the proof. ■

Now take a sequence $\{\Delta_n\} \to 0$. For each Δ_n, there exists a set of beliefs $P_n \subset (0, p^{**})$ such that $(0, p^{**})\setminus P_n$ is countable, and each $p \in P_n$ is not reachable from p^{**} (and hence from $p^*_{\Delta_n}$) via any path of signals. Let $P = \cup_{n=1}^{\infty} P_n$. By definition, $(0, p^{**})\setminus P$ is countable.

Lemma 32 *For any $p_0 \in P$, we have*

$$\lim_{\Delta_n\to 0} E^B\left[\frac{S^B\left(p_0\right)}{1-e^{-r\Delta}}\right] = \frac{\mu}{r}\left(1-\lim_{\Delta_n\to 0}\hat{Q}\left(p_0\right)\right) - \frac{\mu}{r}\lim_{\Delta_n\to 0}\hat{Q}\left(p_0\right),$$

where $\lim_{\Delta\to 0}\hat{Q}\left(p_0\right) = \left(\frac{1}{\kappa-1}\frac{p_0}{1-p_0}\right)^{\frac{1}{2}+\frac{\sqrt{\mu^2+2r}}{2\mu}}$.

Proof. Since $p_0 \in P$, if the posterior belief enters the region $[p^{**}, 1]$ starting from p_0, it cannot hit exactly p^{**}. Hence, $S^B\left(p_\tau\right)1_{\{p_\tau\geq p^{**}\}} = \underline{S}1_{\{p_\tau > p^{**}\}}$. Recall that $\lim_{\Delta_n\to 0} c\left(\Delta_n\right) = 0$, and therefore, we have

$$\lim_{\Delta_n \to 0} \frac{\overline{S}}{1-\delta} = \lim_{\Delta_n \to 0} \frac{E^B[v]}{1-\delta} = \frac{\mu}{r}$$
$$\lim_{\Delta_n \to 0} \frac{\underline{S}}{1-\delta} = \lim_{\Delta_n \to 0} \frac{E^G[v]}{1-\delta} = -\frac{\mu}{r}.$$

By Lemma 31,

$$\begin{aligned}
&\lim_{\Delta_n \to 0} E^B\left[\frac{S^B(p_0)}{1-\delta}\right] \\
&= \lim_{\Delta_n \to 0}\left[\frac{E^B[v]}{1-\delta}\left(1-E^B\left[\delta^{\tau+1}\right]\right)+\frac{\overline{S}}{1-\delta}E^B\left[\delta^{\tau+1}1_{\{p_\tau \le p^*\}}\right]+\frac{\underline{S}}{1-\delta}E^B\left[\delta^{\tau+1}1_{\{p_\tau \ge p^{**}\}}\right]\right] \\
&= \frac{\mu}{r}\lim_{\Delta_n \to 0}\left(1-E^B\left[\delta^{\tau+1}\right]+E^B\left[\delta^{\tau+1}1_{\{p_\tau \le p^*\}}\right]\right)-\frac{\mu}{r}\left(\lim_{\Delta_n \to 0}E^B\left[\delta^{\tau+1}1_{\{p_\tau \ge p^{**}\}}\right]\right) \\
&= \frac{\mu}{r}\left(1-\lim_{\Delta_n \to 0}E^B\left[\delta^{\tau+1}1_{\{p_\tau \ge p^{**}\}}\right]\right)-\frac{\mu}{r}\left(\lim_{\Delta_n \to 0}E^B\left[\delta^{\tau+1}1_{\{p_\tau \ge p^{**}\}}\right]\right). \qquad (54)
\end{aligned}$$

The proof is then completed by plugging in $\lim_{\Delta_n \to 0} E^B\left[\delta^{\tau+1}1_{\{p_\tau \ge p^{**}\}}\right]$ from Lemma 30. ■

F.3.3 Continuity

We know that $E^B\left[\frac{S^B(p)}{1-\delta}\right]$ defined on $p \in [0,1]$ is a discontinuous step function for any Δ. In particular, $S^B(p^{**})$ is determined by the short-run player's randomization at p^{**}. We want to show that nevertheless $\lim_{\Delta \to 0} E^B\left[\frac{S^B(p)}{1-\delta}\right]$ is continuous *everywhere* in p.

Lemma 33 $\lim_{\Delta \to 0} \frac{S^B(p^{**})}{1-\delta} = -\frac{\mu}{r}$.

Proof. As we take multiple limits and the order of quantifiers matters, let us write $\frac{S^B_\Delta(p^{**})}{1-e^{-r\Delta}}$ explicitly as a function of Δ (recall that p^{**} is independent of Δ). Suppose to the contrary that the claim is not true. Since $\frac{S^B_\Delta(p^{**})}{1-e^{-r\Delta}} \in \left[-\frac{\mu\Delta}{1-e^{-r\Delta}}, \frac{\mu\Delta}{1-e^{-r\Delta}}\right] \subset \left[-\frac{2\mu}{r}, \frac{2\mu}{r}\right]$ for sufficiently small Δ, there exists a sequence $\{\Delta_n\}$ such that $\frac{S^B_{\Delta_n}(p^{**})}{1-e^{-r\Delta_n}}$ converges to some constant $\eta > -\frac{\mu}{r}$ as $\Delta_n \to 0$. For this sequence Δ_n, we construct P_n and $P = \cup_{n=1}^\infty P_n$, previously defined just above Lemma 32. Since $[0, p^{**}] \setminus P$ is countable, there exists a sequence $\{p_m\} \subset P$ such that $\lim_{m \to \infty} p_m = p^{**}$.

Then, by Lemma 32,

$$\lim_{\Delta_n \to 0} \frac{S^B_{\Delta_n}(p_m)}{1-e^{-r\Delta_n}} = \frac{\mu}{r}\left(1-\lim_{\Delta_n \to 0}\hat{Q}(p_m)\right)-\frac{\mu}{r}\lim_{\Delta_n \to 0}\hat{Q}(p_m),$$

where $\lim_{\Delta_n \to 0} \hat{Q}(p_m) = \left(\frac{1}{\kappa-1}\frac{p_m}{1-p_m}\right)^{\frac{1}{2}+\frac{\sqrt{\mu^2+2r}}{2\mu}}$ and the latter goes to 1 as $p_m \to p^{**}$. Thus, as $p_m \to p^{**}$,

$$\lim_{\Delta_n \to 0} \frac{S^B_{\Delta_n}(p_m)}{1-e^{-r\Delta_n}} \to -\frac{\mu}{r}.$$

However, by monotonicity of $S^B_{\Delta_n}(p)$, $\frac{S^B_{\Delta_n}(p_m)}{1-e^{-r\Delta_n}} \geq E^B_{\Delta_n}\left[\frac{S^B_{\Delta_n}(p^{**})}{1-e^{-r\Delta_n}}\right]$. This contradicts the assumption that $\frac{S^B_{\Delta_n}(p^{**})}{1-e^{-r\Delta_n}} \to \eta > -\frac{\mu}{r}$. ■

Finally, we are ready to strengthen Lemma 32 to hold for *any* $p \in [0, p^{**}]$.

Lemma 34 *For any $p_0 \in [0, p^{**}]$, we have*

$$\lim_{\Delta_n \to 0} E^B\left[\frac{S^B(p_0)}{1-e^{-r\Delta}}\right] = \frac{\mu}{r}\left(1 - \lim_{\Delta_n \to 0} \hat{Q}(p_0)\right) - \frac{\mu}{r}\lim_{\Delta_n \to 0} \hat{Q}(p_0),$$

where $\lim_{\Delta \to 0} \hat{Q}(p_0) = \left(\frac{1}{\kappa-1}\frac{p_0}{1-p_0}\right)^{\frac{1}{2}+\frac{\sqrt{\mu^2+2r}}{2\mu}}$.

Proof. Since $\underline{S}1_{\{p_\tau \geq p^{**}\}} \leq S^B(p_\tau)1_{\{p_\tau \geq p^{**}\}} \leq S(p^{**})1_{\{p_\tau \geq p^{**}\}}$, it follows from Lemma 33 that $\lim_{\Delta \to 0} \frac{S^B(p_\tau)}{1-\delta} 1_{\{p_\tau \geq p^{**}\}} = -\frac{\mu}{r} 1_{\{p_\tau \geq p^{**}\}}$. Applying the dominated convergence theorem,

$$\begin{aligned} \lim_{\Delta \to 0} \frac{1}{1-\delta} E^B\left[\delta^{\tau+1} S^B(p_\tau) 1_{\{p_\tau \geq p^{**}\}}\right] &= \lim_{\Delta \to 0} \frac{\underline{S}}{1-\delta} E^B\left[\delta^{\tau+1} 1_{\{p_\tau \geq p^{**}\}}\right] \\ &= -\frac{\mu}{r} \lim_{\Delta \to 0} E^B\left[\delta^{\tau+1} 1_{\{p_\tau \geq p^{**}\}}\right]. \end{aligned}$$

By Lemma 31, we have

$$\lim_{\Delta \to 0} \frac{E^B\left[S^B(p_0)\right]}{1-\delta} = \frac{\mu}{r}\left(1 - \lim_{\Delta \to 0} E^B\left[\delta^{\tau+1} 1_{\{p_\tau \geq p^{**}\}}\right]\right) - \frac{\mu}{r}\left(\lim_{\Delta \to 0} E^B\left[\delta^{\tau+1} 1_{\{p_\tau \geq p^{**}\}}\right]\right).$$

The result follows from plugging in $\lim_{\Delta \to 0} E^B\left[\delta^{\tau+1} 1_{\{p_\tau \geq p^{**}\}}\right]$ from Lemma 30. ■

References

Abreu, D. and F. Gul (2000): "Bargaining and Reputation," *Econometrica*, 68, 85-117.

Abreu, D., P. Milgrom and D. Pearce (1991): "Information and Timing in Repeated Partnerships," *Econometrica*, 59, 1713-1733.

Abreu, D. and D. Pearce (2007): "Bargaining, Reputation, and Equilibrium Selection in Repeated Games with Contracts," *Econometrica*, 75, 653-710.

Abreu, D., D. Pearce and E. Stacchetti (2012): “One-Sided Uncertainty and Delay in Reputational Bargaining,” Working Paper, New York University.

Alexander, J. C. (1991): “Do the Merits Matter? A Study of Settlements in Securities Class Actions,” *Stanford Law Review*, 43, 497-598.

Atakan, A. E. and M. Ekmekci (2012): “Bargaining and Reputation in Search Markets,” Working Paper, Northwestern University.

Bar-Isaac, H. (2003): “Reputation and Survival: Learning in a Dynamic Signalling Model,” *Review of Economic Studies*, 70, 231-251.

Benabou, R. J. M. and G. Laroque (1992): “Using Privileged Information to Manipulate Markets: Insiders, Gurus and Credibility,” *Quarterly Journal of Economics*, 107, 921-958.

Chatterjee, K. and L. Samuelson (1987): “Bargaining with Two-Sided Incomplete Information: An Infinite Horizon Model with Alternating Offers,” *Review of Economic Studies*, 54, 175-192.

Compte, O. and P. Jehiel (2002): “On the Role of Outside Options in Bargaining with Obstinate Parties,” *Econometrica*, 70, 1477-1517.

Conway, J. B. (1978): *Functions of One Complex Variable I.* New York: Springer.

Cox, D. R. and H. D. Miller (1965): *The Theory of Stochastic Processes.* London: Chapman and Hall.

Cramton, P. (1984): “Bargaining with Incomplete Information: An Infinite-Horizon Model with Continuous Uncertainty,” *Review of Economic Studies*, 51, 579-593.

Cripps, M. W., G. J. Mailath and L. Samuelson (2004): “Imperfect Monitoring and Impermanent Reputations,” *Econometrica*, 72, 407-432.

Daley, B. and B. Green (2012): “Waiting for News in the Market for Lemons,” *Econometrica*, 80, 1433-1504.

Dixit, A. (2009): “Governance Institutions and Economic Activity,” *American Economic Review*, 99, 5-24.

Ethier, S. N. (2010): *The Doctrine of Chances: Probabilistic Aspects of Gambling.* Berlin: Springer.

FUDENBERG, D. AND D. K. LEVINE (1989): "Reputation and Equilibrium Selection in Games with a Patient Player," *Econometrica*, 57, 759-778.

FUDENBERG, D. AND D. K. LEVINE (1992): "Maintaining a Reputation When Strategies Are Imperfectly Observed," *Review of Economic Studies*, 59, 561-579.

FUDENBERG, D. AND D. K. LEVINE (2007): "Continuous Time Limits of Repeated Games with Imperfect Public Monitoring," *Review of Economic Dynamics*, 10, 173-192.

FUDENBERG, D. AND D. K. LEVINE (2009): "Repeated Games with Frequent Signals," *Quarterly Journal of Economics*, 124, 149-195.

FUDENBERG, D, D. K. LEVINE AND J. TIROLE (1987): "Incomplete Information Bargaining with Outside Opportunities," *Quarterly Journal of Economics*, 102, 37-50.

FUDENBERG, D. AND J. TIROLE (1991): *Game Theory*. Cambridge, MA: MIT Press.

GAMBETTA, D. (1993): *The Sicilian Mafia*. Cambridge, MA: Harvard University Press.

GUL. F. AND W. PESENDORFER (2012): "The War of Information," *Review of Economic Studies*, 79, 707-734.

GUL, F., H. SONNENSCHEIN AND R. B. WILSON (1986): "Foundations of Dynamic Monopoly and the Coase Conjecture," *Journal of Economic Theory*, 36, 155-190.

HARSANYI, J. C. AND R. SELTEN (1972): "A Generalized Nash Solution for Two-Person Bargaining Games with Incomplete Information," *Management Science*, 18, 80-106.

HÖRNER, J. AND N. VIEILLE (2009): "Public vs. Private Offers in the Market for Lemons," *Econometrica*, 77, 29-69.

KREMER, I. AND A. SKRZYPACZ (2007): "Dynamic Signaling and Market Breakdown," *Journal of Economic Theory*, 133, 58-82.

KREPS, D. AND R. WILSON (1982): "Reputation and Imperfect Information," *Journal of Economic Theory*, 27, 253-279.

LEE, J. AND Q. LIU (2012): "Gambling Reputation: Repeated Bargaining with Outside Options," Working Paper, Seoul National University and Columbia University.

LEE, J. AND Q. LIU (2013): "Supplement to 'Gambling Reputation: Repeated Bargaining with Outside Options'," Working Paper, Seoul National University and Columbia University.

Mathis, J., J. McAndrews and J-C. Rochet (2009): “Rating the Raters: Are Reputation Concerns Powerful Enough to Discipline Rating Agencies?” *Journal of Monetary Economics*, 56, 657-674.

Milgrom, P. and J. Roberts (1982): “Predation, Reputation and Entry Deterrence,” *Journal of Economic Theory*, 27, 280-312.

Myerson, R. B. (1991): *Game Theory: Analysis of Conflict.* Cambridge, MA: Harvard University Press.

Nash, J. F. (1950): “The Bargaining Problem,” *Econometrica*, 18, 155-162.

Nash, J. F. (1953): “Two-Person Cooperative Games,” *Econometrica*, 21, 128-140.

Nöldeke, G. and E. van Damme (1990): “Switching Away from Probability One Beliefs,” Working Paper, University of Bonn.

Palmrose, Z-V. (1991): “Trials of Legal Disputes Involving Independent Auditors: Some Empirical Evidence,” *Journal of Accounting Research*, 29, 149-185.

Ross, S. M. (1996): *Stochastic Processes*. New York: Wiley.

Schmidt, K. M. (1991): “Commitment in Games with Asymmetric Information,” *Ph.D. dissertation*, University of Bonn.