Which Approaches Do Students Prefer?
Analyzing the Mathematical Problem Solving Behavior of Mathematically Gifted Students

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ABSTRACT

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This study analyzed the mathematical problem solving behavior of mathematically gifted students. It focused on a specific fourth step of Polya's (1945) problem solving process, namely, looking back to find alternative approaches to solve the same problem. Specifically, this study explored problem solving using many different approaches. It examined the relationships between students' past mathematical experiences and the number of approaches and the kind of mathematics topics they used to solve three non-standard mathematics problems. It also analyzed the aesthetic of students' approaches from the perspective of expert mathematicians and the aesthetic of these experts' preferred approaches from the perspective of the students.

Fifty-four students from a specialized high school were selected to participate in this study that began with the analysis of their past mathematical experiences by means of a preliminary survey. Nine of the 54 students took a test requiring them to solve three non-standard mathematics problems using many different approaches. A panel of three

research mathematicians was consulted to evaluate the mathematical aesthetic of those approaches. Then, these nine students were interviewed. Also, all 54 students took a second survey to support inferences made while observing the problem solving behavior of the nine students. This study showed that students generally were not familiar with the practice of looking back. Indeed, students generally chose to supply only one workable, yet mechanistic approach as long as they obtained a correct answer to the problem.

The findings of this study suggested that, to some extent, students' past mathematical experiences were connected with the number of approaches they used when solving non-standard mathematics problems. In particular, the findings revealed that students' most recent exposure of their then-AP Calculus course played an important role in their decisions on selecting approaches for solution. In addition, the findings showed that students' problem solving approaches were considered to be the least "beautiful" by the panel of experts and were often associated with standard approaches taught by secondary school mathematics teachers. The findings confirmed the results of previous studies that there is no direct connection between the experts' and students' views of "beauty" in mathematics.

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Chapter I: INTRODUCTION

Need for the Study

Problem solving has been the focus of research in mathematics education for many years. The National Council of Teachers of Mathematics stated in the Principles and Standards for School Mathematics that:

> Problem solving means engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings. Solving problems is not only a goal of learning mathematics but also a major means of doing so… Problem solving is an integral part of all mathematics learning, and so it should not be an isolated part of the mathematics program. (NCTM, 2000, p. 52)

Problem solving has also been examined from many different points of view (Karp, 2007b; Kilpatrick, 1985; Lester, 1994; Schoenfeld, 1985; Schroeder & Lester, 1989; Silver, 1985; Stanic & Kilpatrick, 1988).

The seminal work of Polya (1945) identified four steps in the process of solving mathematics problems. These steps consist of understanding the problem, devising a plan, carrying out the plan, and looking back. Not all of the four steps, however, have received equal attention in problem solving research. For example, the second step, devising a plan, gained interest in the mathematics education community (Schoenfeld, 1985), but the fourth step, looking back, has attracted much less consideration so far (Lee, 2009).

In recent years, solving mathematics problems using many different approaches has drawn more attention than before. Some researchers, in fact, considered such practice to be beneficial for students' mathematics learning experience (Tabachneck, Koedinger, & Nathan, 1994). Certainly, this consideration appears warranted with evidence of students' learning outcomes, albeit conflicting evidence (Große & Renkl, 2006). In addition, problem solving can be analyzed in the preparation of mathematics teachers (Leikin & Levav-Waynberg, 2007; Silver et al., 2005).

Teaching and learning experiences are not the only focus of research in solving mathematics problems using many different approaches. Another focus involves investigating why some people solve one particular problem using different ways than others do. Some researchers have analyzed students' choice of approaches based on certain mathematics topics (Nesher et al., 2003). Others have explored the question of selecting a particular problem solving approach from an aesthetic point of view (Dreyfus & Eisenberg, 1986; Karp, 2008; Silver & Metzger, 1989; Sinclair, 2004). Typically, a problem solving approach is "beautiful" if it is particularly clear, simple, and unexpected.

Cognitive psychologists, in addition to mathematics educators, have also been interested in studying choices of problem solving approaches (Siegler, 1983). In particular, they examined how the order in which approaches are presented affects the whole process of thinking in problem solving. Their investigations, primarily on basic arithmetic skills, pointed towards an understanding of the development of approaches and the interactions among those approaches (Geary & Brown, 1991; Roberts et al., 1997).

It can be inferred thus far that research in mathematics education needs more analysis to explain the thinking processes involved in problem solving. Even existing research in cognitive psychology has typically concentrated on (limited) elementary school mathematics topics (Star, 1999). A joint effort from both fields in mathematics education and cognitive psychology is clearly needed. Interpreting the cognitive rationales for selecting certain approaches over many other approaches in solving advanced mathematics problems is an important area that still requires further study.

To access numerous different approaches to solving the same problem, one would need subjects capable of conceptualizing multiple approaches. Consideration for subjects, then, is crucial. One can start with a group of individuals who, as a whole, are more motivated to problem solve. One may alternatively consider individuals who are consistently more capable of producing a greater number of different approaches with less effort than others. To this end, problem solving for mathematically gifted students is one possible choice.

One attribute that allows gifted students to use many different approaches is creativity (Renzulli, 1986; Tannenbaum, 1983; Ervynck, 1991; Silver, 1997; Sriraman, 2005). The question then becomes how to locate and further delve into the presence of such creativity among mathematically gifted students. It is therefore important to reflect on which assessment is appropriate to elicit mathematical creativity from such students. Some researchers recommend non-standard mathematics problems as an effective means of exploring mathematical creativity in connection with mathematically gifted students (Leikin & Lev, 2007).

Past studies have led to a much needed, quite possibly neglected, path of investigation. Research in mathematics education demonstrates that problem solving using many different approaches improves students’ learning experiences. Explanations from within the mathematics education field about how students choose a particular approach have usually been limited to a specific branch in mathematics (e.g., algebra) or to the aesthetic considerations of subjects not at the high school level. Research in cognitive psychology reveals that approaches interact with each other in a successful problem solving process. Rationales for the choice of approaches from this field are limited to only elementary school mathematics topics (e.g., counting). Thus, it remains unclear whether cognitive processes do affect the unique preferences of gifted students for certain approaches in advanced mathematical problem solving. By synthesizing the literature from the fields in mathematics education and cognitive psychology, it may be possible to make a connection to teaching mathematically gifted students, specifically at the high school level, in order to fill this gap in the research. This is important to consider because gifted students likely provide a greater variety of problem solving approaches than other students (Krutetskii, 1976).

Purpose of the Study

This study is an effort to analyze the mathematical problem solving behavior of mathematically gifted students at the high school level. It attempts to examine: how the many different approaches become readily available to them, and whether any of these approaches are more favorable to them than others.

The current study is, in part, based on the study by Nesher et al. (2003) on preferred approaches. Nesher et al. investigated mathematical processes that affect students’ choices of many approaches to algebra problems. The current study examines students’ thinking processes in solving problems in several different branches of mathematics. Based on a study by Leikin and Lev (2007) on mathematical creativity, the current study builds upon their recommendation to use non-standard mathematics problems as productive measures of mathematical creativity. Criteria such as number of approaches also are not limited in the current study. The current study takes into account the mathematical aspect of aesthetic as well. Finally, it allows for a cognitive psychological examination of interactions among those approaches and the students' thinking processes.

Similar to Dreyfus and Eisenberg's (1986) study on aesthetic appreciation, the current study inquires into the aesthetic feelings leading to the formation of approaches in mathematical problem solving. Its focus on a group of high school students rather than on a group of college-level students is particularly appealing because of the limited research available on this group in the analysis of mathematical aesthetic. The current study also seeks to examine the interactions among approaches by looking at opportunities to replicate similar analyses, as conducted by Geary and Brown (1991). Instead of focusing on how young children use certain approaches for simple addition problems, the current study concentrates on how high school students select and use specific approaches to solve advanced mathematics problems.

In essence, the present research combines studies of students' mathematical problem solving experiences from mathematics education, gifted education, and cognitive psychology (see Figure 1). It analyzes mathematical creativity via many different problem solving approaches and specifically explores mathematical and cognitive reasons for preferred approaches. Finally, it examines the presence of aesthetic appreciation for particular approaches.

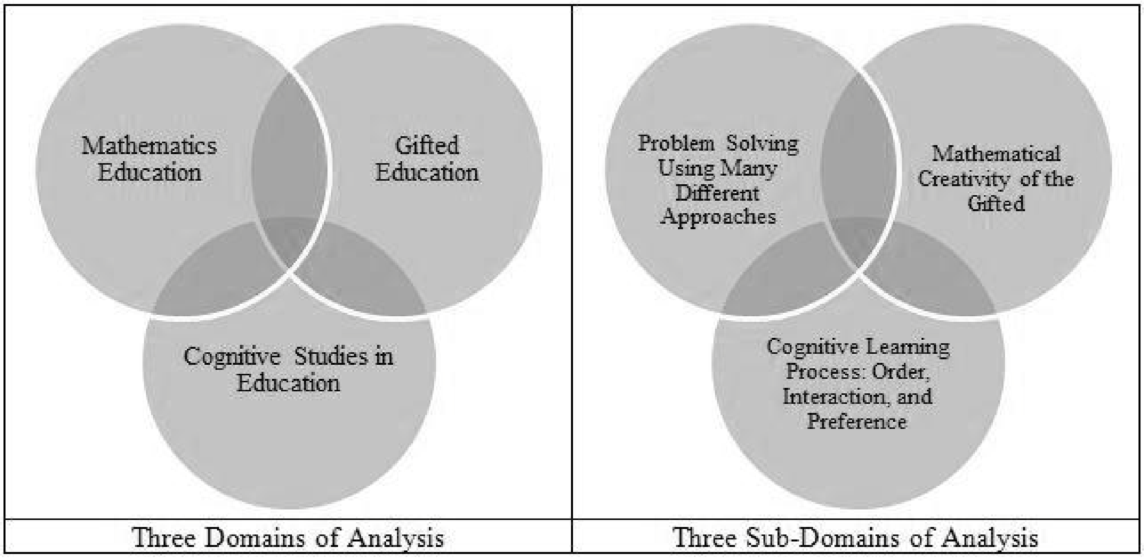

Figure 1. Domains and Sub-Domains of Analysis

Research Questions

This study aimed to explore influences on the preferences of problem solving approaches of mathematically gifted students at the high school level. It analyzed the development of such preferences from the perspectives of mathematics education and cognitive psychology. The following research questions guided the study:

1. How do gifted students' past mathematical experiences affect the number of approaches they used when solving non-standard mathematics problems?
2. How are gifted students' past mathematical experiences connected with the mathematics topics involved in their solutions?
3. To what extent are gifted students' approaches considered "beautiful" by experts?
4. To what extent are experts' preferred approaches considered "beautiful" by gifted students?

Procedures of the Study

Any study of mathematical problem solving, gifted education, and cognitive learning should make use of both qualitative and quantitative approaches and be solidly grounded in current literature. A review of this literature is presented in Chapter II.

This study involved 54 mathematically gifted students selected from a specialized high school in New York City. During the data collection, they were enrolled in an Advanced Placement (AP) Calculus course. In addition, three research mathematicians with no Columbia University affiliation participated as a panel of experts.

This study used five main instruments: 1) students' preliminary survey, 2) students' test, 3) experts' evaluation, 4) students' follow-up interview, and 5) students' validation survey. The preliminary survey was administered to provide details about the descriptions of students' past mathematical experiences. Of the 54 students, nine students, who were highly recommended by their AP Calculus teacher, took a test with three mathematics problems. They were specifically asked to provide as many different approaches as they could for each problem on the test. The three research mathematicians who served as a panel of experts were consulted to evaluate the aesthetic value of the students' approaches, in addition to other approaches that the researcher had prepared beforehand. Follow-up interviews were conducted with the nine students to examine their choice of presenting approaches and their reactions to the experts' preferred approaches. A validation survey was conducted with all 54 students to strengthen the inferences made from the problem solving behavior of the nine students.

To answer the first and second research questions, the researcher analyzed the past mathematical experiences of the nine students who took the test. In particular, a list of their mathematics courses, standardized tests, and other related mathematical experiences were compared with their test performances for: 1) the number of successful approaches which they supplied on the test (in connection with the first research question), and 2) the mathematics topics involved in their solutions (in connection with the second research question). To answer the third research question, the successful approaches supplied by the nine students on the test, as well as the approaches chosen by the 54 students in the validation survey as their first approach, were assessed according to the findings from the experts' evaluation of the aesthetic value. To answer the fourth research question, the findings from the interviews with the nine students who took the test, as well as those from the written responses provided by the 54 students in the validation survey, were analyzed for their aesthetic reactions towards the experts' preferred approaches.

Organization of the Study

This study is organized into six chapters. Chapter I, Introduction, presented an overview of the need for the study, the purpose of the study, the research questions, and the procedure of the study.

Chapter II, Literature Review, describes past studies from the fields of mathematics education, gifted education, and cognitive psychology. It begins with interpretations and definitions of mathematical problem solving using many different approaches. It continues with expositions of different perspectives and recommendations on this pedagogy, along with students' learning experiences. It also offers explanations from the point of view of cognitive psychologists for the development, interactions, and choices of problem solving approaches. Given the study's focus on students of gifted abilities, the chapter concludes with research on mathematical creativity in gifted education.

Chapter III, Methodology, details the research design of this study. It describes the criteria of selecting the 54 mathematically gifted students and the three research mathematicians who served as an expert panel. It also explains the procedure used to assess students' problem solving approaches through the experts' evaluation and the students' follow-up interview, and concludes with a discussion of the students' validation survey.

Chapter IV, Findings from Phases 1 and 3: Students' Preliminary Survey and Experts' Evaluation, describes the mathematical background of the subjects and the aesthetic evaluations of the panel of experts. Chapter V, Findings from Phases 2 and 4: Students' Test and Students' Follow-up Interview, describes the problem solving experiences of the nine students who took the test in terms of their test performance and interview responses. Chapter VI, Findings from Phase 5: Students' Validation Survey, describes the 54 students' choices of problem solving approaches and their attitudes towards problem solving using many different approaches.

Chapter VII, Summary, Conclusions, and Recommendations, recapitulates the study and highlights selected findings of significant consequences. It discusses the study's limitations and presents a discussion of recommendations for future research and for classroom practice.

Chapter II: LITERATURE REVIEW

Problem Solving: An Overview

Contemporary literature in mathematics education indicates that problem solving is a popular topic. Many issues have been discussed on problem solving in connection with many different topics in mathematics education. The important place of problem solving in school mathematics is understandable, given its strategic role in teaching and learning mathematics. A number of pedagogical approaches have been proposed to incorporate the problem solving experience in everyday mathematics classrooms. The topic has drawn considerable interest and attention from not only mathematics school teachers and mathematics educators, but also research mathematicians.

One research mathematician of Hungarian origin, George Polya (1945), was among several who made important contributions to the field of mathematics education. He analyzed how professional mathematicians solved mathematics problems and advocated that anyone could use problem solving in learning mathematics. He enumerated four distinct steps in the process of mathematical problem solving: 1) understanding the problem, 2) devising a plan, 3) carrying out the plan, and 4) looking back.

The first step, understanding the problem, begins with the identification of what the problem is asking for; that is, it is necessary to figure out the question being asked. For this purpose, it is important to recognize all available data in the problem. This also suggests being able to determine and differentiate necessary, sufficient, relevant, redundant, and contradictory conditions from the given information. Additional facts may be further derived from drawing appropriate figures or introducing suitable notations.

The second step is devising a plan. A well-devised plan makes the most of the straightforward connection between the data and the unknown. In addition, it also builds on comparably similar problem solving experience from the past. It is therefore important to think about analogous problems which may vary in appearance, from the structure of the data presented to the construction of the unknowns being requested. The use of similar techniques or established results in solving those related problems facilitates the way in which the problems are restated differently. Polya himself discussed many heuristic strategies to solve mathematics problems. They include: drawing a picture, solving an analogous simpler problem, considering a special case to find a general pattern, working backward, and adopting a different point of view.

The third step is carrying out the plan. Once the plan has been devised, carrying out the plan follows immediately. At this point, it is critical to carry out each step of the plan carefully. It is also no less important to be able to prove that each step is indeed logically correct.

The fourth step is looking back. A solved problem does not mean that the process of problem solving has ended. In retrospect, it is necessary to examine the obtained result by checking the argument along the way. Alternatively, it is valuable to derive the obtained result by using a different approach. Finally, this obtained result and many different approaches involved in deriving the result should be coordinated for future problem solving experience.

Polya devoted much time to supplying his model of problem solving with concrete exemplars. The model, as a result, gained many enthusiasts from a larger audience. He convinced them that problem solving processes were not only accessible for research mathematicians, but could also be utilized by and applied to the learning purposes of broader audiences. Many researchers in mathematics education have examined Polya's model of problem solving comprehensively and systematically. A review of prior literature reveals that much attention has focused specifically on the first three steps. In fact, many researchers were more attracted by the second step, devising a plan (Schoenfeld, 1985), and, understandably, this is what most classroom practitioners look for in their students to develop and implement in learning mathematics. This was, after all, the exigent reason why the model was constructed in the first place.

The truth is that Polya's model of problem solving does not end at the third step. The mathematics education community, however, seems to have done little to move forward and become aware of the fourth step, looking back. Only a limited number of studies in mathematics education have examined students' use of alternative approaches in problem solving. Some researchers in this field have been particularly successful in exploring the use of mathematical tasks requiring students to solve the same problem in many different approaches.

Problem Solving Using Many Different Approaches: Definitions and Interpretations

In order to describe a definite meaning of many different approaches, it is necessary to offer a common language. This becomes even more important because of the several distinct yet synonymous terminologies used in the current literature. Each will be explained along with the described problems and evaluated in relation to the purpose of the present study.

Leikin and Levav-Waynberg (2007) introduced the term "multiple-solution connecting task." In principle, they defined such a task as "one that may be attributed to different topics or to different concepts within a topic of the mathematics curriculum, and therefore may be solved in different ways" (p. 350). They especially considered the following three types of mathematical connections:

1. Connections based on similarities and differences between various representations of the same concept (see Appendix A).
2. Connections between different mathematical concepts and procedures (see Appendix B).
3. Connections between different branches of mathematics (see Appendix C).

In their study, Leikin and Levav-Waynberg (2007) also asked teachers for their interpretations of the meanings of problem solving in different ways. One of the teachers tackled three interpretations at once in a way that is more appropriate for the purpose of the current study:

> Maybe we'll sort different types of solutions into different groups. Say, for example now, it occurred to me that you could solve the same exercise differently: graphically or algebraically [different representation]…. By way of algebra you may solve [systems of equations] using the linear combination method or by substitution: these are two different ways [different tools within one topic]. A problem in space [geometry] can be solved by using vectors or trigonometry [tools associated with different topics]. (Leikin & Levav-Waynberg, 2007, p. 362)

Moreover, Silver, Leung, and Cai (1995) used the terms "multiple solutions," "solution methods," and "solution strategies" interchangeably. In addition, they also used the term "modes of explanation." They conducted a comparative study using a marble arrangement problem (see Appendix D), which was simple yet complex enough to provoke students' use of many different explanations in solving it. The researchers categorized "multiple solution strategies" into enumeration, grouping, and restructuring (see Appendix E). They also identified "modes of explanation" as either visual, verbal/symbolic, mixed, neither, or inconsistent (see Appendix F). On one hand, the categorization of "multiple solution strategies" is reasonable, but the choice of the problem allows limited analysis of different mathematics topics. On the other hand, the identification of "modes of explanation" is useful in assessing students' cognitive understanding of how they solve the problem.

As a whole, the descriptions of "multiple solution strategies" and "modes of explanation" are close to what was needed in the current study. The current study, however, more often uses the expressions "many different approaches" or "many different ways" to solve mathematics problems. An approach or a way of mathematical problem solving can be understood as an active process in arriving at an answer. It is dissimilar from a solution or a strategy, which can be interpreted as a well-furnished explanation of an answer. In the same manner, an answer or a result can be interpreted as a final complete product that the problem seeks to solve. The development of mathematical thinking in problem solving, then, is best viewed through the lens of many different approaches.

Problem Solving Using Many Different Approaches: Different Perspectives and Recommendations

Perspectives of Mathematics Educators

Despite lacking a certain systematic framework of analysis, mathematics educators have initiated preliminary dialogue on this topic in recent years. As with any potential development of a pristine research study, considerations should begin by reflecting on its objectives. Many earlier discussions were based on a variety of proposed benefits from the use of different problem solving approaches.

Silver et al. (2005) believed that students “can learn more from solving one problem in many different ways than [they] can from solving many different problems, each in only one way” (p. 288). They particularly advised students interested in mathematics to obtain more experience in problem solving with many different approaches. They regarded such experience as having “the potential advantage of providing students with access to a range of representations and solution strategies in a particular instance that can be useful in future problem-solving encounters” (p. 288). They also considered the use of many different approaches in order to “facilitate connection of a problem at hand to different elements of knowledge with which a student may be familiar, thereby strengthening networks of related ideas” (p. 288).

In order to compare their beliefs within common classroom practice, these mathematics educators interviewed several middle school mathematics teachers. Some of the teachers indeed shared similar views. First, these teachers valued a student-centered classroom environment. They welcomed an open classroom discourse on students’ use of different problem solving approaches. This warm gesture created a sense of acceptance among every student in the classroom. “[I]nstead of focusing on just one student,” one of the teachers explained that it was important to “help everyone feel comfortable to give their opinion, or share their strategy or their way of how they looked at it …” (Silver et al., 2005, p. 292). Another teacher expressed a way to facilitate discussions among students about the similarities, differences, and relationships between their own approaches. The teacher intended these discussions to foster a positive classroom environment, thereby creating more dynamics and flexibility in the students’ learning experience.

Second, teachers also indicated possible improvement in students’ conceptual understanding of the subject matter by looking at different perspectives. One teacher said, “It is important to consider several strategies when solving complex problems. Not only does it validate students’ different solutions, it offers them additional strategies for their mathematical ‘tool bag’” (Silver et al., 2005, p. 297). A different teacher hinted at the applications for future problem solving encounters, stating, “This is my philosophy about math: You can try it in one way, your favorite way, but you should always have a back-up. Because if your original way doesn’t work, then you have a backup” (p. 297). Another teacher made recommendations to demonstrate different problem solving approaches to students to “offer [them] a more useful strategy” (p. 297).

Third, teachers understood the significance of incorrect problem solving approaches as part of students’ learning experience. One teacher mentioned that it became “really important to take a look even at wrong answers” (Silver et al., 2005, p. 294). Such approaches should be made more noticeable at an earlier stage of students’ learning exposure on the subject matters. Although flawed, incorrect approaches were endorsed as an encouragement for students to improve their deeper understanding of related mathematics topics. A similar comment was raised by another teacher from a different study by Smith et al. (2005, as cited in Silver et al., 2005, p. 293). She cultivated in her students an open-minded way of thinking by becoming more courageous in communicating any conceptually erroneous approaches. She specifically considered revealing an incorrect approach as beneficial in “expos[ing] the fallacy of this approach as soon as possible and mov[ing] on to others” (Smith et al., 2005, as cited in Silver et al., 2005, p. 293). Teachers as a whole discussed many benefits of the students’ learning experience in solving mathematics problems with many different approaches.

Like Silver et al. (2005), Leikin and Levav-Waynberg (2007) were interested in surveying teachers for their thoughts about alternative approaches in problem solving. They interviewed several high school mathematics teachers in a comparable study on teachers’ beliefs. Their findings revealed positive attitudes towards the use of many different approaches in problem solving. Most teachers in fact considered it a benefit in connection with fostering students’ success in problem solving. They believed that working with many different approaches accommodated the learning experiences of students who had pronounced preferences in learning style. In return, they reasoned that struggling students could benefit from the presentation of various approaches, especially regarding difficulty level. Such presentation should be applied to problems with complex approaches that require sophisticated mathematics knowledge, yet are solvable using elementary approaches. As one teacher mentioned, when presented with different approaches, students would be able to choose the approach “that is easiest [for them] to understand” (Leikin & Levav-Waynberg, 2007, p. 363).

Some teachers saw another benefit in promoting students' aesthetic interest in mathematical problem solving. They asserted that working with and observing many different approaches might possibly cultivate students' appreciation of the beauty of mathematics. One teacher, for example, conveyed this message in order for the students to understand "how beautiful mathematics is" (Leikin & Levav-Waynberg, 2007, p. 363). This pedagogical aspect was also viewed as encouraging students to be more thoughtful in accepting and more critical in seeking alternative approaches. On this matter, a teacher noted that "[s]ome students dislike a certain method; maybe a different method can make them like the problem better" (Leikin & Levav-Waynberg, 2007, p. 363).

Other teachers valued the students' development of mathematical thinking and reasoning. As such, they assumed that students would then be more likely to establish a solid foundation for their future academic careers. A few other teachers acknowledged the significance of students' awareness of the connections among mathematics topics. Mathematics should be viewed "as a whole," that is, a collection of connected, as opposed to separate, mathematics topics (Leikin & Levav-Waynberg, 2007, p. 363). In general, Leikin and Levav-Waynberg (2007) concluded that these teachers indicated constructive opinions about the use of many different approaches.

Perspectives of Cognitive Psychologists

The promise of favorable students' learning outcomes has successfully fueled a number of recommendations for alternative approaches in problem solving. Yet these recommendations were apparently not issued by mathematics education researchers and mathematics teachers alone. Long before these groups expressed their opinions, analogous theories were in fact proposed by many cognitive psychologists who had a keen interest in educational psychology with applications in learning and cognition.

Collins, Brown, and Newman (1989) posed deliberations on using multiple perspectives in the instructional method by means of their cognitive apprenticeship approach. In their model, the students' learning processes progressed from five teaching methods: modeling, coaching, scaffolding, reflection, and articulation. These teaching methods took the form of either a recursive, a cyclical or a spiral pattern. The teachers' roles in supporting the students' learning experience gradually decreased as students themselves felt more confidence in communicating their understanding. The researchers argued that the more approaches and perspectives students explored, the more effective the implementation of this cognitive-based learning method. Some benefits of this method included improved apprenticeship in encouraging the value of real-world activities and assessments (Collins, Brown, & Newman, 1989). The method also enhanced students' motivation and engagement in overall learning (Collins, 1991), greater transfer and retention rates (Resnick, 1989), and higher order reasoning (Hogan & Tudge, 1999).

Spiro, Feltovich, Jacobson, and Coulson originated the cognitive flexibility theory (Spiro et al., 1991; Spiro & Jehng, 1990). Spiro et al. claimed that restructuring knowledge through changes in different approaches made learning new concepts possible. Such adaptations were based on the notion that the human mind could be trained to be flexible enough to accommodate different situations. New information and experience were processed via the transfer of knowledge and skills and further constructed to develop new meaning and understanding. In other words, they believed that learning through different perspectives associated with different situations deepened students' understanding and learning experience.

Tabachneck, Koedinger, and Nathan (1994) also recognized the purpose of adopting many different approaches in problem solving. They argued that on its own, each approach entailed disadvantages and weaknesses. In order to overcome these, they recommended students operate a combination of approaches, instead of counting on only one approach. More specifically, they maintained that students could benefit from employing this learning style in mathematical problem solving. In addition to teaching to solve one problem with many approaches, psychologists encouraged teaching a coherent interrelation among those approaches (Skemp, 1987; De Jong et al., 1998; Van Someren et al., 1998; Bodemer et al., 2004). Equally important, Reeves and Weisberg (1994) recommended showing students many analogical problems or examples concurrently. On the whole, cognitive psychologists took a positive stance on problem solving using many approaches, as did mathematics education researchers.

Challenges in Classroom Implementation

Despite the benefits of implementing this learning style, some discussions were not without uncertainties. A few teachers in the study by Silver et al. (2005) talked about issues in teaching problem solving with many approaches. The first issue was the limitation of instructional time. Teachers' concerns included: 1) "You don't have time to show all of these solutions," 2) how to "make it possible to explore and share several different solutions and to validate student thinking," and 3) how to "'fit everything in, and rush kids' to cover the content in the prescribed time" (p. 295). Schoenfeld (1991) also recognized this time factor. Teachers' common anxieties originated mostly from the nature of school administrations that put heavy emphasis on curricula and results. On one hand, teachers were responsible for covering many materials, which provided a time constraint on the academic calendar. On the other hand, they were accountable for ensuring the acceptability of their students' performance by the end of the year. Incorporating problem solving with many approaches thus offered greater challenges to the teachers. They would have to pack in more materials to teach within the same given timeframe. They would also have to convince students that these materials were worthy of learning, although less likely to be assessed.

The second issue was the limitation of students' perceived abilities. Teachers' concerns ranged from "Sometimes I am scared to put even two strategies up there because [the students] are barely able to get one" to "I would be afraid to have someone explain this [non-standard solution]. I have kids struggling to understand this stuff, and if a group comes up and starts explaining this, my kids would just shut down" (Silver et al., 2005, p. 295). Leikin et al. (2006) also acknowledged this issue. They observed reluctance among teachers to implement the teaching of problem solving with many different approaches. These teachers explained that exposing students to different approaches merely benefited them with a higher mathematics aptitude and were concerned that the presentation of different approaches might distract those with lower mathematics aptitude. In their opinion, such students would struggle to comprehend one approach, let alone all different approaches simultaneously. They also worried that these students might even lose interest in problem solving because of their growing frustration and confusion.

The third issue was the selection of approaches. Teachers' concerns ranged from "Explaining is important, but which solutions you focus on have to be tied to the goals of the lesson instead of always sharing everything" to "Do we need to 'share' strategies that are not brought up?" (Silver et al., 2005, p. 296). The fourth issue was presentation order. Teachers revealed their indecision over which approaches to present in which order and how much discourse they should use to follow up. Their concerns centered on the question of "Do I start with the most simplistic way and move up the ladder or is it random?" (Silver et al., 2005, p. 296). One teacher in the study by Smith et al. (2005, as cited in Silver et al., 2005) also pointed out that she was unsure of how to go about which approaches "to get out publicly and in what order" (Smith et al., 2005, p. 33).

The fifth issue was the presentation of incorrect approaches. Some teachers were worried that displaying erroneous approaches impaired students' orientations to their supposedly accurate conceptual understanding. They also anxiously anticipated possible passive learning behaviors from less motivated students. One teacher indicated that "students sometimes think, 'I will just sit here and wait' until…[the approaches offered by other students are] shown. They won't pay much attention until you get to theirs" (Silver et al., 2005, p. 297). The sixth issue was the reality of boredom. Teachers felt uneasy in supplying one single problem for an unusually longer period of time. Doing so, they believed, could especially prompt an unpleasant learning experience for higher-ability students. They asked, "How much time would you spend on this problem with a class? [How does a teacher tackle the] issue of boredom?" (Silver et al., 2005, p. 297).

A few other teachers in the study by Leikin and Levav-Waynberg (2007) expressed concern about teaching problem solving with many different approaches. One teacher specifically saw a practical conflict from the point of view of curriculum practice:

> This happens in plane geometry.… I told them: “I can show you how to solve the same problem using trigonometry.” The same problem in plane geometry can be solved using trigonometry. At the end of the lesson I thought I had made a mistake because they might conclude that they would be allowed to use trigonometry to solve a plane geometry problem in the matriculation exam [the Israeli matriculation exam requires solving plane geometry problems using geometrical theorems only]. (Leikin & Levav-Waynberg, 2007, p. 361)

Some teachers in the study by Leikin and Levav-Waynberg (2007) in fact showed genuine concern about students’ learning experience. They worried that students might confuse “whether the object of study is to solve the problem, the fact that there is more than one solution to the problem, or the principles behind the solutions and the connections between them” (p. 366).

In view of these constraints and concerns, mathematics education researchers and cognitive psychologists still felt firmly confident in their recommendations for problem solving using many different approaches. Silver et al. (2005) even went as far as pointing out the possibility of teachers’ weak mathematics content knowledge. They believed that this factor might contribute to a psychological threshold in integrating many different approaches into classroom practice.

<u>Students’ Learning Outcomes</u>

Proposals to advocate learning mathematics through problem solving in many different approaches demanded another step toward progress. These proposals might only become a pool of publicly agreed-upon hypotheses without any empirical findings on students’ learning outcomes. Große and Renkl (2006) examined the effects of learning problem solving with many different approaches through worked-out examples. They conducted two experiments involving university-level students. Each experiment was conducted with different sets of conditions and mathematics topics.

In the first experiment, Große and Renkl (2006) were interested in two factors: 1) number of approaches and 2) instructional support. Participating students were randomly assigned to six conditions: 1) two approaches with no support, 2) two approaches with self-explanation, 3) two approaches with instructional explanations, 4) one approach with no support, 5) one approach with self-explanation, and 6) one approach with instructional explanations. The mathematics topic used in the first experiment was combinatorics. In the second experiment, Große and Renkl (2006) were interested in the use of different mathematical representations. Participating students were randomly grouped into three conditions: 1) two approaches with two representations, 2) two approaches with one representation, and 3) one approach. The mathematics topic used in the second experiment was probability.

Their assessment on learning outcomes was based on procedural knowledge and conceptual knowledge. This assessment was conducted during the learning stage of the experiment. Procedural knowledge was evaluated using an elementary combinatorics problem to check the accuracy of students' answers. Conceptual knowledge was evaluated using an elementary combinatorics problem. It also involved careful discussions on the pros and cons of different approaches, the correctness of each approach, and their general applicability. The findings from the two experiments were inconsistent with each other. Exposing students to many different approaches did improve their procedural and conceptual understanding in the first experiment, but it did not make any significant difference in the second experiment. Große and Renkl (2006) reasoned that a different mathematics topic might have different effects on students' learning outcomes.

Moreover, Rittle-Johnson and Star (2007) analyzed the effect of comparing many different approaches on students' learning experience in problem solving. Their experiment involved seventh grade students who were randomly assigned to two groups. In this experiment, students were given algebra lessons on solving one linear equation with one variable. They were all exposed to many types of approaches, such as the conventional approach and the nonconventional approach. Although both groups were presented with similar problems along with similar approaches, the order in which the approaches were presented differed. In the first group, students solved these algebra problems by comparing and contrasting many different approaches. In the second group, students solved similar problems by reflecting on many different approaches, one at a time.

In order to evaluate their learning outcomes, students were assessed on their procedural knowledge, flexibility, and conceptual knowledge. Procedural knowledge was evaluated based on the students' accuracy in their answers and the type of approaches provided during the assessment. Flexibility was measured on three aspects: students' abilities to 1) generate and 2) recognize many different approaches, and 3) evaluate nonconventional approaches. Conceptual knowledge was evaluated based on students' understanding of algebraic symbolism and the effect of applying simultaneous operations to algebraic equations. The assessment brought about positive outcomes. Students in the first group performed better than those in the second group in procedural knowledge and flexibility. Students in both groups showed similar improvement in conceptual knowledge. This evidence supported the learning practice of solving algebraic equations by comparing many different approaches simultaneously rather than sequentially. Overall, in addition to the study by Große and Renkl (2006), Rittle-Johnson and Star's (2007) study demonstrated potentially favorable learning outcomes through problem solving with many different approaches in other branches of mathematics.

Problem Solving Using Many Different Approaches: Factors Affecting Choice of Approach

Perspectives of Mathematics Educators

In addition to the teaching and learning aspects of solving problem with many different approaches, mathematics education researchers have examined choices of approaches. Analysis of choice involves understanding why different people solve one particular problem using different ways than others. Given the many possible different approaches to solve the same problem, a student's decision to choose one approach may be less than arbitrary. Observations of students' problem solving experiences have prompted a search for specific explanations.

Silver et al. (1995) conducted a comparative study to investigate the problem solving experiences of American and Japanese students at the fourth grade level. Two problems assigned were simple but complex enough to be solved with many different approaches. One of these problems was the marble arrangement problem described earlier (see Appendix D). The students were asked to solve both problems using as many different approaches as they could think of. In addition, they were also asked to include explanations in their responses. The findings indicated that students in both countries employed essentially a similar type and frequency of particular approaches. On one hand, Japanese students were able to generate many different approaches more accurately than American students. On the other hand, Japanese students were able to explain such approaches with more rigorous mathematical concepts than American students. Japanese students appeared more skilled in multiplication and mathematical symbolism, whereas American students were more comfortable with addition and verbal statements.

Silver et al. (1995) discussed choices of approaches in the students' problem solving experience. They specifically observed a common pattern in the students' order of presentation of approaches. They noted that the first approach was "the only one purely directed toward the generation of a numerical answer" (p. 44). That is, students were more eager to figure out an accurate answer to a problem immediately. As soon as an accurate answer became accessible, the following approaches they used were aimed at validating their first approach. Despite this process, students unconsciously made use of their first approach as a stepping stone to generate additional approaches.

Like Silver et al. (1995), Star and Madnani (2004) were also interested in examining students' choices of approaches. Their study involved teaching sixth and seventh grade students basic algebraic operations to solve one linear equation with one variable. These simple equation transformations were demonstrated separately so that the students could work their way up to doing them jointly. As they worked, they were asked to explain their choice of approaches in combining several equation transformations. Student performance was measured by: 1) their abilities to solve isomorphic and transfer equation problems, 2) their flexibility, and 3) their conceptual understanding of equations. Students' abilities to solve isomorphic and transfer equation problems were assessed by means of solving problems that were, respectively, very similar and relatively similar to those demonstrated in the lessons. Students' flexibility was evaluated by their use of many different approaches, while students' conceptual understanding was assessed by concepts of equation, variable, and equivalency.

The findings revealed that students had various responses to their choices of problem solving approaches. Students' explanations were classified as either naïve or sophisticated. Naïve explanations included "most accurate way, with fewest errors, and arriving at right answer," "way that I'm more sure, confident, comfortable with, proud of, or happiest with," and "way that is most neatly written and organized" (Star & Madnani, 2004, p. 486). Sophisticated explanations included "shortest way, involving fewest steps," "quickest or fastest way," "easiest, least complicated, or least confusing way," and "it depends on various things, including the problem, how quickly a solver can execute steps, and the preferences or goals of a solver" (p. 486). Star and Madnani (2004) concluded that beginning problem solvers were capable on their own (with little instructional support) of identifying and describing choices of problem solving approaches. These were students who were also being exposed to solving one linear equation with one variable for the first time. Some students interpreted "best" approaches with more sophisticated explanations such as choosing the ones with the fewest steps and the fastest time required in problem solving. Not surprisingly, students with these interpretations performed better in transfer equation solving, flexibility, and conceptual knowledge. That is, better explanations for choosing particular approaches often led to better performance in mathematics literacy.

Like Star and Madnani (2004), Nesher, Hershkovitz, and Novotna (2003) also investigated students' choices of approaches to solving algebra problems. Specifically, they were interested in ninth grade students' use of independent variables when solving algebra word problems. These word problems involved a situation with three unknown quantities whose sum was known. In interviewing the students, the researchers found that the students' choices of independent variables were mainly influenced by the order in which the quantities were described in the word problems. At the same time, students favored independent variables with the smallest quantity in relation to the other two quantities discussed in the problems. By doing so, students unconsciously revealed their natural inclination to working with whole numbers as opposed to rational numbers.

Perspectives of Cognitive Psychologists

Unlike mathematics education researchers, cognitive psychologists analyze choice of approach in terms of interaction among approaches. Koedinger and Tabachneck (1994), for example, analyzed students' use of informal approaches in their problem solving process. Their study involved university students who were asked to solve two algebra word problems. Four approaches to solving the algebra word problems were identified and classified into two groups. The first group was labeled formal, schooled approaches, which included the use of either algebraic or diagrammatic representations. The second group was labeled informal, unschooled approaches, which included the use of either model-based reasoning or verbal explanation.

The findings revealed that students' high performance was not related to one choice of either a formal or an informal approach. Instead, students who attempted both formal and informal approaches were twice as likely to succeed as those who persisted on any one single approach. Each approach required a different familiarity with words and mathematical representations, computational efficiency, and demand on working memory capacity. Because of these differences in the strengths and weaknesses of the approaches, a problem became solvable more efficiently by complementing formal with informal approaches. The flexible and synergistic use of approaches resulted in a complementary effect by heightening the strengths and decreasing the weaknesses of those approaches.

In addition to interactions among approaches, cognitive psychologists analyze choice of approach in relation to mastery of certain attributes. Roberts et al. (1997) examined students' choices of problem solving approaches as related to spatial ability. Spatial ability is the mental skill to reason through manipulating geometric figures or to think in terms of visual representations. In their study, university students were grouped according to their performance on a spatial and verbal ability pre-test: 1) students with high spatial ability and 2) students with low spatial ability. Each student was tested individually on three direction tasks: the compass-point task, the one-person direction task, and the two-person direction task. These direction tasks required students to locate positions according to given directions. Roberts et al. (1997) identified beforehand two main approaches to solving these tasks. The first approach was the spatial approach, which made use of spatial representations to produce an accurate visualization of the directions. The second approach was the cancellation approach, which neutralized the effect of opposite directions to generate the estimated representations of the paths. The former appeared more mechanical, with naïve application of superficially necessary skills, whereas the latter required less cognitive demand in connection with spatial ability.

These three tasks were intentionally designed to cause inefficiency when students relied heavily on their spatial ability.

The findings demonstrated "an apparently counter-intuitive inverted aptitude-strategy relationship" (Roberts et al., 1997, p. 480). Students with high spatial ability ingeniously avoided the use of spatial ability when solving the direction tasks. They were aware that the use of spatial ability yielded an inefficient approach. Consequently, they demonstrated more flexibility in developing and adapting alternative approaches that increased accuracy and saved time in problem solving. Roberts et al. (1997) also noticed that the cancellation approach did not become immediately apparent to students with low spatial ability. They concluded that the level of spatial ability dictated their competency to acquire and assess more efficient choices of approaches.

Despite these results, it is still possible to interpret the findings of this study differently. That is, one might infer that strong evidence showing that students favored problem solving approach through visual reasoning was indeed lacking. Presmeg (1985, as cited in Presmeg, 1986), for example, found that almost all high achievers in mathematics at the senior high school level were identified as non-visualizers. An even more famous example was the case of Terence Tao, as described by Clements (1984): "While he has well developed spatial ability, when attempting to solve mathematical problems he has a distinct, though not conscious, preference for using verbal-logical, as opposed to visual thinking" (Clements, 1984, p. 235). For her part, Presmeg (1986) pointed out cognitive preference to explain why being identified as a visualizer did not compel one to solve mathematics problems visually all the time. While interviewing students doing problem solving, Presmeg (1986) noticed a cognitive progression from conceptual thinking to procedural thinking. The shift in cognitive preference was at first considered to be unnaturally habituated by school curriculum, but it later appeared to be naturally developed by the students themselves. In other words, Presmeg (1986) viewed the unconscious automation of the non-visual approach as a direct consequence of the repetitive practice of the visual approach in an effort to exploit efficient memory workload. “Apparently when a topic is first taught, a visual presentation often aids visualisers’ understanding, but practice of the procedure of formula may lead to habituation, when an image is no longer necessary” (p. 302).

Like Presmeg (1986), Geary and colleagues were interested in the development of problem solving approaches. Geary and Wiley (1991), in particular, analyzed the use of alternative approaches by two different age groups. The younger group was between 18 and 31 years old, while the older group was between 60 to 82 years old. Each subject was tested individually on a total of 40 basic addition problems. The problems involved two one-digit numbers (e.g., 6+9). For each problem, subjects were asked to explain their thinking processes, which were mainly categorized into: 1) verbal counting (e.g., six plus nine equals fifteen), 2) decomposition (e.g., 6+9=5+1+9=5+10=15), and 3) memory retrieval (e.g., 15). The memory retrieval approach required long-term memory workload. It was therefore considered to be a mathematically more developed approach than the verbal counting and decomposition approaches. The subjects’ performance was assessed based on accuracy and time spent on each problem.

The findings by Geary and Wiley (1991) revealed different choices of approaches by the two age groups. On one hand, the older group performed better in terms of accuracy and favored the use of the more mature approach. They showed more frequent use of the memory retrieval approach, and less frequent use of the decomposition and verbal counting approaches. On the other hand, the younger group performed better in terms of overall time spent on the entire experiment. At the same time, the older group appeared slower than the younger group in retrieving addition facts and producing verbal answers. More interestingly, on more difficult problems, the older group appeared to switch from memory retrieval to decomposition. They did so even when they were aware of the additional time required to solve the problems. Geary and Wiley (1991) maintain that these problem solvers consciously adjusted their choice of approaches according to each problem's difficulty level. In particular, they reserved the use of decomposition as a backup approach when memory retrieval fell short in the first place.

Similarly, Geary and Brown (1991) analyzed children's choices in problem solving approaches and speed of processing information associated with the problems. Their experiments using simple addition problems involved three groups of third and fourth grade students: gifted, normal, and mathematically disabled. They identified children's addition approaches into two main categories. The first category included the memory retrieval approach, which resorted to strong long-term memory and was often associated with quick mental calculation. The second category included the counting approach, which varied from physical use of fingers to audible or indistinct lip movement.

The findings revealed that the gifted group, significantly more than the non-gifted groups, utilized the memory retrieval approach more frequently than the counting approach. The constant and frequent use of the memory retrieval approach by the gifted group was independent of the problem's difficulty level. The non-gifted groups, however, executed retrieval with easier problems and counting with more difficult ones. As for reaction time and error rate, the gifted group performed significantly better than the non-gifted groups. This difference was evidenced in the analysis of the verbal counting rate, but not of the memory retrieval rate. Based on their findings, Geary and colleagues (Geary & Wiley, 1991; Geary & Brown, 1991) have, to some extent, verified Siegler's (1983) view on approaches often used to backup imperfect approaches. A similar observation in the older group (Geary & Wiley, 1991) was once again evident in the non-gifted groups (Geary & Brown, 1991). In these groups, the use of counting on more difficult problems was interpreted as a remedy to compensate for memory retrieval. Problem solvers appeared eager to obtain the correct answers even if they had to suffer through a longer reaction time associated with a less efficient approach.

Furthermore, in a two one-digit addition experiment, Siegler and Robinson (1982) observed that young children could produce four different approaches. First, in the "counting-fingers" approach, children raised their fingers to correspond with each addend and counted them. Second, in the "fingers" approach, children raised their fingers to correspond with each addend without counting them. Next, in the "counting" approach, children counted aloud without an external referent. Finally, in the "retrieval" approach, children performed addition without any visible or audible referent. Children spent less time in problem solving when using the retrieval approach, followed by the fingers, counting, and counting-finger approaches. Other researchers further classified the counting approach into max, min, and sum approaches (Fuson, 1982; Groen & Parkman, 1972). The max approach counted the larger value as an addend to the smaller value, for example, counting 3,4,5,6 to solve 2+4. The min approach counted the smaller value as addend to the larger value, for example, counting 5,6 to solve 2+4. The sum approach counted both addends starting from 1, for example, counting 1,2,3,4,5,6 to solve 2+4. Siegler et al. (1996) maintained that schooling played an important role in the children's acquisition and use of addition approaches. In the earlier grades, schools promoted the frequent use of the min approach as opposed to the max approach. Later on, children were oriented towards the use of the decomposition approach and eventually the retrieval approach. The choice of problem solving approaches for this elementary school mathematical task was influenced developmentally by the instructional settings.

Aesthetic

Aesthetic aspects were also considered in many studies connected with preference in problem solving approaches. Dreyfus and Eisenberg (1986), for instance, were interested in exploring whether students assessed the aesthetic value of mathematical reasoning in problem solutions. Their study involved college-level mathematics students who had been rigorously prepared in advanced mathematics courses. They were tested on several carefully chosen mathematics problems which involved many different approaches not immediately apparent to average students, yet readily accessible with high school mathematics knowledge. After completing the test, students were presented with elegant approaches. They were not able to supply elegant approaches in the test as they had been expected to, and they were not able to recognize the differences between elegant and pedestrian approaches. Furthermore, when presented with elegant approaches, they showed no enthusiasm and found them no more attractive than their own approaches. In other words, they had no sense of aesthetic appreciation. Dreyfus and Eisenberg (1986) concluded that mathematics instruction in classroom settings lacked an emphasis on reflective thinking, especially aesthetic value.

Silver and Metzger (1989) also assessed the role of the aesthetic but at a much higher level of expertise in a study involving university professors in mathematics. They examined the aesthetic influence on mathematical problem solving experience in two assessments. In one assessment, they monitored the role of aesthetic value in the process of problem solving as discussed by Poincare (1946) and Hadamard (1945). In another assessment, they analyzed the sense of aesthetic in the evaluation of the completed solutions as described by Kruteskii (1976) or the problems themselves. Silver and Metzger (1989) found that these expert problem solvers displayed signs of aesthetic emotion. On one occasion, a subject resisted the temptation to resort to the use of calculus in solving a geometry problem, acknowledging the possibility of a “messy equation” (p. 66). Only after some unsuccessful attempts to seek a geometric approach did the subject concede to solving the problem using calculus. Although successful, he felt that “calculus failed to satisfy his personal goal of understanding, as well as his aesthetic desire for ‘harmony’ between the elements of the problem and elegance of solution” (p. 66). On another occasion, having solved another geometry problem algebraically, the same subject appeared unsettled, recognizing that a geometric approach could be “more elegant” (p. 66).

Using a similar scope of analysis as Silver and Metzger (1989), Koichu and Berman (2005) examined how three members of the Israeli team participating in the International Mathematics Olympiad coped with conflict in their conceptions of effectiveness and elegance. An effective approach led directly to a final result in answering a mathematics problem with minimum memory retrieval of concepts and terms and procedural knowledge. An elegant approach was considered to have clarity, simplicity, parsimony, and ingenuity in solving a mathematics problem with minimum intellectual effort and few mathematical tools. In their clinical interview, Koichu and Berman (2005) observed that when solving geometry problems, these mathematically gifted students consistently directed greater aesthetic appreciation towards geometric approaches than algebraic or trigonometric approaches. However, when such a geometric approach was not readily accessible to them, they immediately resorted to algebraic or trigonometric approaches as long as they effectively solved the problems. Only later on when students had built up their confidence could they develop the desired geometric approach to satisfy their need for aesthetic appreciation. This experience marked the point at which students successfully managed to balance the need for elegant approaches with the time constraint requiring effective approaches.

In addition, Sinclair (2004) analyzed the role of aesthetic value from several conceptual insights. She drew examples from existing empirical findings such as those by Dreyfus and Eisenberg (1986) and Silver and Metzger (1989). In one of her interpretations of their work, she suggested that “mathematicians’ aesthetic choices might be at least partially learned from their community as they interact with other mathematicians and seek their approval” (Sinclair, 2004, p. 276). Furthermore, she indicated that mathematical beauty was only feasible in the process “when young mathematicians are having to join the community of professional mathematicians—and when aesthetic considerations are recognized (unlike at high school and undergraduate levels)” (p. 276).

Related to Sinclair’s (2004) interpretations, Karp (2008) conducted a comparative study on the aesthetic aspect of mathematical problem solving. He was fully aware that

Dreyfus and Eisenberg (1986) observed no aesthetic awareness in mathematics among college-level students. Karp's comparative study involved middle and high school mathematics teachers from the U.S. and Russia. In his study, teachers were asked to provide examples and explanations of "beautiful" mathematics problems and approaches in solving those problems.

Karp's (2008) findings confirmed that the curricular system of education had a tremendous impact on students' aesthetic preference in mathematics problem solving. Each group of teachers showed different perspectives on what counted as mathematical "beauty." In particular, these differences stood out from their selections of mathematics topics. American teachers put extra weight on mathematics topics as prescribed by the American curriculum which were typically associated with real-life situations and applications. Russian teachers did likewise as recommended by Russian curriculum with its traditionally heavy emphasis on algebra, number theory, and geometry. Evidently, these Russian problems tended to require longer approaches and were more algebraically demanding than their American counterparts. In their explanations, American teachers described "usefulness in the teaching process," "useful[ness] in practical life or comes the real world," "non-standard and cannot be solved using ordinary methods that are regularly discussed in school," "unexpectedness of the solution," "openness of the problem," and "a combination of methods and knowledge from different fields of mathematics" (Karp, 2008, p. 40). Russian teachers revealed in their choices of problems and solutions the sense of "overcoming of chaos," "non-standard nature," and "traditional fields" in their origins (p. 40). In his conclusion, Karp indicated a relative character of aesthetic preference in mathematics problem solving.

Problem Solving and Gifted Students

As mentioned earlier with regard to choices of approaches, gifted students demonstrate unique problem solving behavior which is atypical of regular students. Gifted students are often characterized by their high abilities and performance. Terman (1925) initiated a longitudinal study on gifted students and promoted the use of the Stanford-Binet Intelligence Scale. Since then, growing interest in psychometric works has placed the field of gifted education in a more quantifiable position, starting with the analysis of IQ tests. Consequently, the identification of gifted students using educational achievement tests has become increasingly popular. The Study of Mathematically Precocious Youth (SMPY) at Johns Hopkins University established in 1971 by Julian C. Stanley was an example of the successful application of the College Board Scholastic Aptitude Test (SAT) (Brody & Stanley, 2005). The test was considered to be highly effective when administered to a younger age group, such as those at middle school level, than originally intended for those in their last years of high school. "Because few seventh- and eighth-graders have formally studied the mathematical content that high school students have, the SAT appeared to be more of a reasoning test for seventh- and eighth-graders than for high school juniors and seniors" (Brody & Stanley, 2005, p. 22). The success of the SMPY gifted education program has influenced various states and even extended its reach to many other countries such as Australia (Kissane, 1986) and Germany (Wagner & Zimmermann, 1986).

Gifted education programs have flourished with the success of identifying gifted students through psychometric tests. However, this use of psychometric tests had had its share of critiques. Borland (2005), for instance, believes in the prospect of gifted education without gifted students. Taking an American perspective, which considers a more heterogeneous group of people, he maintains that the concept of gifted students is "incoherent and untenable" (Borland, 2005, p. 2) because of: 1) the questionable validity of the concept of gifted students, 2) the questionable value and efficacy of gifted education, 3) the inequitable allocation of educational resources, and 4) the questionable need for the construct of gifted students. Therefore, he argues specifically that the identification of gifted students by means of psychometric tests would be useless without gifted students in the first place.

Wu (2005), by contrast, examined giftedness from a Chinese perspective. He specifically considered a Confucian philosophy that associated nurture (encouraging diligent efforts to success), rather than nature (recognizing innate abilities), with talented performance. His interviews with Chinese teachers revealed their firm convictions about the complete dynamic of giftedness and talented performance. Unlike most Western researchers, these Chinese teachers believed that in addition to gifted children, children with average or even low innate abilities had the potential to achieve high performance in the presence of optimal nurturing. This dynamic also made possible the risk that children with high abilities might only achieve low performance if they are given inadequate nurturing. Wu indicated five environmental factors crucial in nurturing the potentials of young children: 1) parental and familial influences; 2) school and teacher influences; 3) specific training and practice; 4) self-effort, motivation, and perseverance; and 5) chance or opportunity.

In the field of mathematics itself, the most systematic study that explored the nature and structure of the mathematical abilities of gifted students was led by Krutetskii

(1976). He argued that although latent talents in various fields were innate in all young children, these talents might not necessarily be uniform for each child across all fields. That is, he believed that some students were more mathematically able than others. In his analysis, Krutetskii named six attributes of mathematically gifted students as in describing their three main stages of mental activity in solving a mathematical problem. In the first stage, mathematically gifted students gathered necessary information in solving the problem. This formalizing perception attribute (first attribute) could be performed analytically by extracting individual elements from the given composite structure in the order of their significance to the alleged problem. It could also be performed synthetically by interpreting them in an integrated arrangement in terms of their mathematical relationship and functional dependencies.

In the second stage, mathematically gifted students processed information to obtain a solution to the problem. First, they demonstrated effectively the ability to generalize mathematical objects, relations, and operations, as well as the ability to retain these mathematical materials. This generalization attribute (second attribute) emerged very naturally “on the spot” as these students performed problem solving tasks with an insignificant amount of training (Krutetskii, 1976). Second, they displayed an ability to curtail the process of mathematical reasoning and the corresponding system of operations. This curtailment attribute (third attribute) was measured by taking into account the number of steps in a typical course of reasoning versus the number of actual steps taken by the students and the time spent on solving the problem. Mathematically gifted students were known to be capable of solving problems using a minimal path and the least amount of time. Third, they revealed the flexibility and reversibility of mental processes in mathematical reasoning. This flexibility attribute (fourth attribute) facilitated students in varying their approaches to solving a problem without being constrained by standard, stereotypical or habitual approaches, as encountered in their previous problem solving experiences. The reversibility of mental processes also allowed students to switch easily and freely from a direct to a reverse train of thought. Moreover, they showed their own striving to achieve an elegant solution. This striving for elegance attribute (fifth attribute) motivated students to avoid settling to merely solve a problem, but rather to search for a solution with such qualities as clarity, simplicity, economy, and originality.

In the third stage, mathematically gifted students retained information about the solution to the problem. They specifically paid more attention to the mathematical relationships in the problem and the principles of the solution to the problem than to the superfluous, unnecessary content of the problem. This strong mathematical memory attribute (sixth attribute) provided them with generalized and operative, as opposed to selective, memory retention.

In examining the relationship between knowledge, talent, and giftedness, Karp (2007a) interviewed 12 teachers in secondary schools specializing in the study of mathematics, as opposed to those in ordinary schools as in the work by Krutetskii (1976). These teachers were distinguished based on the criteria used in selecting them, including: 1) the number of their former students who had participated or won high-level mathematics competitions, such as the International Mathematics Olympiad; 2) the number of their former students who had become prominent mathematicians, such as those holding senior faculty positions in the mathematics department of leading academic institutions; and 3) their professional activity in terms of the number of professional publications. Karp's interviews revealed several important characteristics of mathematically gifted students. First, these students demonstrated success in problem solving as indicated not only by their outstanding speed and genuine interest in problem solving, but also by their exceptional precision and depth of understanding of problem solving approaches. Second, their non-standard problem solving approaches revealed their capacity for independent thinking. Third, their wealth of knowledge reflected their precocity and competence in absorbing easily profound mastery of a mathematics subject that surpassed their age level.

Other studies have considered creativity as another attribute to identify this group of gifted students. In his "three-ring conception" of giftedness, Renzulli (1978) suggested three clusters of traits that characterized gifted students. In addition to above-average ability and task commitment, he discussed the role of creativity in distinguishing the group of gifted students. Apart from conceptualizing mathematical creativity, mathematics education researchers are aware of the need to assess mathematical creativity as part of fulfilling the goal of the gifted education. Haylock (1987) identified a specific feature in his assessment of mathematical creativity in schoolchildren: the capacity to prevail over fixation and rigidity. In particular, the children were able to overcome the Einstellung effect, a phenomenon generally observed when students commit either 1) algorithmic fixation or 2) content universe fixation. An algorithmic fixation occurred when students persisted on applying a previously learned, yet inappropriate, inefficient or unsuccessful approach (Luchins, 1951, as cited in Haylock, 1987). A content universe fixation occurred when students restricted the range of unnecessary elements in a problem solving situation (Haylock, 1984, as cited in Haylock, 1987).

In addition, Silver (1997) valued both problem solving and problem posing as means to developing students' mathematical creativity in terms of fluency, flexibility, and novelty. Leikin and Lev (2007) analyzed the problem solving behavior of students from three different groups. The first group included those with high IQ scores and high mathematical achievement. The second group included those without high IQ scores but with high mathematical achievement. The third group included those without high IQ scores but with average mathematical achievement. In particular, Leikin and Lev explored the many different approaches that each group of students used in solving two mathematics problems which were taken from either a mathematics curriculum textbook or a mathematics Olympiad textbook. The researchers evaluated mathematical creativity using criteria such as number of approaches, originality, and time spent on each approach. The findings revealed students' mathematical creativity in terms of number of approaches and time spent for each approach. The first group performed as well as the second group on mathematics curriculum problems, but performed significantly better than the second group on mathematics Olympiad problems. The first and second groups performed better than the third group on both types of problems. Hence, the researchers recommended non-standard mathematics problems as an effective means of exploring mathematical creativity in connection with mathematically gifted students.

Chapter III: METHODOLOGY

Subjects

Fifty-four mathematically gifted students at the high school level participated in the current study. The term *gifted students* in the current study was understood to mean highly "selected" students who underwent several rounds of selection to be part of this study. The criterion for selecting these gifted students was based on the definition by New York State Education Department stated in New York State Education Law Chapter 740, Article 90, Section 4452.a:

> [T]he term 'gifted pupils' shall mean those pupils who show evidence of high performance capability, and exceptional potential in areas such as general intellectual ability, special academic aptitude and outstanding ability in visual and performing arts. Such definition shall include those pupils who require educational programs or services beyond those normally provided by the regular school program in order to realize their full potential.

The New York City Department of Education functions in accordance with the New York State Education Department. It has consequently designated a few specialized high schools of New York City. Much like the special secondary schools for the mathematically and scientifically talented in Russia from the 1960s (Vogeli, 1997), the specialized high schools in New York City require prospective students to pass an entrance examination, i.e., the Specialized High Schools Admissions Test. These schools are the most selective public high schools available to serve the needs of academically gifted students in New York City. The 54 students in this study were selected from one of these specialized high schools. They were assumed to have met the criteria of gifted students as defined by New York State Education Department. In particular, the 54 students were among many in that high school who were taking an AP Calculus course at the time this study was conducted. In addition, three research mathematicians unaffiliated with Columbia University were selected as expert consultants.

Instruments and Evaluations

The current study utilized five main instruments: 1) students' preliminary survey, 2) students' test, 3) experts' evaluation, 4) students' follow-up interview, and 5) students' validation survey. Each instrument was utilized in a particular phase of the study (see Figure 2).

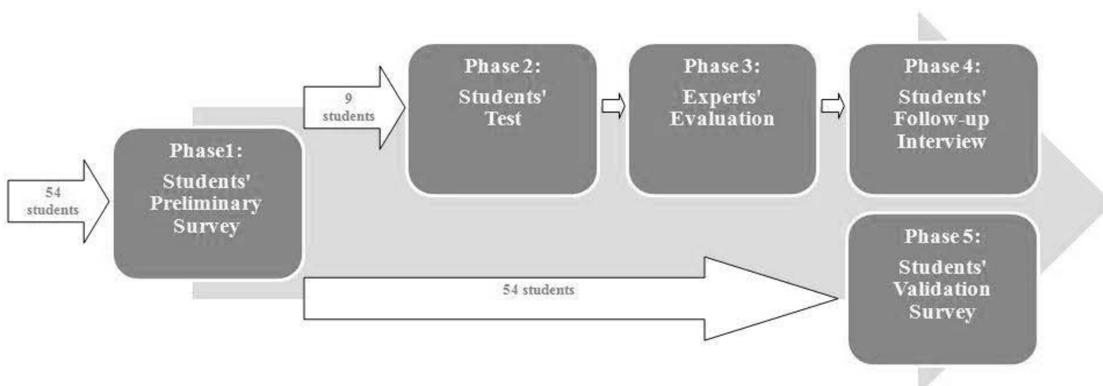

Figure 2. Five Phases of the Study

In Phase 1, the 54 students were asked to fill out a preliminary survey (see Appendix G). These preliminary survey responses provided details about the students' past mathematical experiences, including mathematics courses taken since eighth grade and their grades for each course, the American Mathematics Contest (AMC) 10 and 12 scores, SAT scores (SAT Mathematics Section, SAT Subject Test—Math Level I and II), planned undergraduate major, and favorite mathematics topics.

In Phase 2, the students' AP Calculus teacher was asked by the researcher to make a list of 16 highly recommended students selected from the 54 participating students, based on their performance in their AP Calculus course. From this list, 10 students volunteered to take a test requiring them to solve problems using many different approaches. On the test day, one student did not show up. Therefore, only nine students were included in Phase 2. (The one student who did not show up on the test day was still included in the other phases of the study, i.e., Phases 1 and 5.) The test, proctored by the researcher, was taken by the nine students at the same time in an after-school period. A video recorder was set up to capture each student's process of problem solving as presented by the student in his or her written responses. The test consisted of three non-standard mathematics problems: an arithmetic inequality problem (Problem 1), an algebra problem of two variables (Problem 2), and a geometry problem (Problem 3) (see Appendix H). The researcher selected the three non-standard problems to comprise a standard secondary school mathematics curriculum, which typically included arithmetic, algebra, and geometry. These problems were also particularly chosen because of the many different approaches that students could use to solve them. The correctness of a student's approach was evaluated based on a simple acceptability scoring system: an acceptability score of 1 indicated that a student successfully supplied a correct answer by using an approach in a logical manner to solve the problem; otherwise, an acceptability score of 0 was given (see Appendix I for examples of students' written work for Problem 1). A student's approach was also classified based on a list of approaches for the three problems, henceforth referred to as the collection of approaches (see Appendix J). This list consisted of four different approaches for Problem 1, eight different approaches for

Problem 2, and three different approaches for Problem 3. For the sake of brevity, a coding scheme was utilized. For example, P1A4 indicated Approach 4 for Problem 1. Of the 15 approaches available in the collection of approaches, 13 approaches were prepared by the researcher beforehand and 2 approaches (i.e., P1A4 and P2A8) were added based on students' written work.

In Phase 3, a panel of experts was consulted to evaluate the collection of approaches for their aesthetic value. The panel consisted of three research mathematicians from mathematics research institutes as classified in the Carnegie Research I University. Professor 1 was a Full Professor of Mathematics and has worked in a university for nearly 30 years. Professor 2 was an Assistant Professor of Mathematics who has been working in a university for 8 years. Professor 3 was an Associate Professor of Mathematics who worked in a university for 10 years. The panel of experts was asked to provide two aesthetic evaluations: 1) a five-point scale and 2) an order of preference (see Appendix K).

In the five-point scale evaluation, each expert was to rate each approach in the collection of approaches according to the following rubric. A score of 5 indicated that the student's approach was the most "beautiful" approach ever seen in similar or related problems. A score of 4 indicated that the student's approach was "beautiful," but more "beautiful" approaches in similar or related problems have been observed. A score of 3 indicated that the student's approach was very typical to similar or related problems and was often associated with standard approaches taught or suggested by mathematics teachers or curriculum at the secondary school level. A score of 2 indicated that the student's approach suggested brute-force application of naïve information processing skills relying only on the information explicitly provided in the problem. A score of 1 indicated that the student's approach showed a primitive understanding of basic mathematics skills required to solve similar or related problems.

In the order of preference evaluation, each expert was asked to place all approaches in the collection of approaches for every particular problem in order from most to least preferred approach in terms of aesthetic value. They were also asked to provide careful explanations for why they placed those approaches in such order. The purpose of these two evaluations was simply to mediate possible discrepancies in the three experts' evaluations. (For example, there might have been a case where more than one approach received the higher average five-point scale evaluation.) The more "beautiful" approach for a problem was determined not only by the higher average five-point scale evaluation, but also by the higher rank in the order of preference evaluation. For example, in the case of two or more approaches with similar high average five-point scale evaluations, the most "beautiful" approach was decided according to the majority vote of the experts' first preference in the order of preference evaluation. After the panel of experts determined the most "beautiful" approaches for the three problems, these approaches were presented to the 54 students: 1) via students' follow-up interview and, concurrently, 2) via student's validation survey. In other words, Phase 4 was conducted at the same time as Phase 5.

In Phase 4, a follow-up interview was conducted with each of the nine students who had previously taken the test in Phase 2. It elicited students' explanations for their problem solving approaches and their reactions to the aesthetic view of the panel of experts. Appendix L presents an illustration of the interview questions. The interview was conducted individually for each student and video recorded. Each student was informed of his or her test result during the interview (i.e., the number of correct answers on the test) and of the aesthetic evaluation of his or her approaches according to the panel of experts. Each student was asked for his or her opinions of the approaches considered to be “beautiful” by the panel of experts. Transcripts of these interviews can be found in Appendix M.

In Phase 5, a validation survey was conducted with all 54 students (see Appendix N). This validation survey consisted of two parts. The first part of Phase 5 examined a hypothetical problem solving experience. The 45 students who had not taken the test were asked about their understanding of the 15 approaches for the three problems, their order of presenting approaches, and their preferred approaches. The nine students who had taken the test were asked similar questions, but with the additional rephrasing of “If you had to do these problems all over again, what would you do differently?” In terms of their understanding of the approaches, students were to rate each approach according to the following rubric: A score of 2 indicated that a student understood all of the steps or reasoning behind the particular approach. A score of 1 indicated that a student understood some of the steps or reasoning behind the particular approach. A score of 0 indicated that a student did not understand any of the steps or reasoning behind the particular approach. As they went through all 15 approaches, the students were also advised to keep in mind the question of whether they had previously learned the necessary mathematics knowledge involved in each approach. The following rubric was used: A score of 2 indicated that a student had previously learned all of the necessary mathematics knowledge involved in the particular approach. A score of 1 indicated that a student had previously learned some of the necessary mathematics knowledge involved in the particular approach. A score of 0 indicated that a student had not previously learned any mathematics knowledge involved in the particular approach. In the analysis, the most frequently occurring statistic in the data set (i.e., mode) was utilized.

Students were further asked to provide a self-assessment of their likelihood of using each of the 15 approaches. They were expected to measure their belief in their abilities to solve the problem using a particular approach. The following rubric was used: A score of 2 indicated that a student would be very likely to solve the problem using the particular approach. A score of 1 indicated that a student might be able to solve the problem using the particular approach. A score of 0 indicated that a student would not have thought of solving the problem using the particular approach. In terms of their order of presenting approaches, students were asked to rank three approaches that would most likely come up as their first, second, and third attempts at solving each of the three problems. In terms of their most favorite approaches, students were asked to select only one approach which they considered was the most preferable for each problem. They were also asked to provide justifications for why that particular choice was more preferable than the other available approaches. The second part of Phase 5 examined students' attitudes towards problem solving using many different approaches. In particular, the intention was to draw out students' reactions to statements, recommendations, constraints or concerns asserted by the mathematics education researchers, mathematics teachers, and cognitive psychologists in earlier studies (discussed in the literature review). Students were to rate a total of 25 statements (S1-S25) on a five-point scale: 5, 4, 3, 2, 1 for strongly agree, agree, neither agree nor disagree,

disagree, and strongly disagree, respectively. When making inferences, mean and standard deviation were utilized.

Chapter IV: FINDINGS FROM PHASES 1 AND 3: STUDENTS' PRELIMINARY SURVEY AND EXPERTS' EVALUATION

Findings from Phase 1: Students' Preliminary Survey

The current study included 54 students: 28 female and 26 male students. There were six eleventh graders and 48 twelfth graders. Four of the eleventh graders had not taken the SAT at the time of data collection, whereas the other 50 students had an average score of 747 on the SAT Mathematics Section. This score was in the 97th percentile compared with the national average score of 516 for the class of 2010 (*The College Board*, 2011). Moreover, seven students reported an average score of 716 in the SAT Subject Test—Math Level I. This score was also above the national average score of 605 for the class of 2010 (*The College Board*, 2011). In addition, 38 students reported an average score of 758 in the SAT Subject Test—Math Level II. This score was also higher than the national average score of 649 for the class of 2010 (*The College Board*, 2011).

These 54 students also documented their planned undergraduate majors. Twenty-eight students (52%) planned to major in natural science disciplines such as Electrical Engineering, Computer Science, Biology, Chemistry, and Geological Science. Ten students (18%) planned to major in social science disciplines such as Political Science, Economics, Finance, and History. The rest of the 16 students (30%), including the eleventh graders, had not yet declared their planned undergraduate majors. Previous mathematics knowledge of these students was generally uniform as a result of taking the same mathematics curriculum. All 54 students had been taking at least two mathematics classes per academic year. This homogeneity was the result of all students coming from the same specialized high school. With the exception of one of the six eleventh graders, most students at the time of data collection had already taken approximately nine mathematics classes, including algebra, geometry, trigonometry, pre-calculus, calculus, as well as a mix of statistics and computer science courses. All 54 students were taking an AP Calculus course at the time of data collection.

These 54 students also reported a variety of favorite mathematics topics. Twenty-three students (42%) chose calculus as their favorite mathematics topic, 16 students (30%) chose algebra, 9 students (17%) chose geometry, and 6 students (11%) chose a mix of statistics and trigonometry. A student’s explanation for his or her favorite mathematics topic was categorized based on unifying principles (see Table 1). Eleven students (20%) chose their favorite mathematics topics because it provided practical or real-life applications, 9 students (17%) did so because it made logical sense, 9 students (17%) did so because it integrated ideas from previously taken mathematics courses, 8 students (15%) did so because it promoted powerful tools to solve problems with minimal effort, 7 students (13%) did so because it facilitated spatial visualization ability, and 10 students (18%) did so because of other reasons not directly related to mathematics.

Table 1

Students' Explanations for Their Favorite Mathematics Topic

Unifying Principles	Examples
Provides practical or real-life applications	"It helps me understand the mathematics behind production costs. By maximizing volume and minimizing the surface area of a container, I can find the perfect dimensions for minimizing costs"; "there are many applications in engineering and organizing data. You can even use it in sciences such as chemistry"; "most applicable to real life situations…in card games, or the probability that a guess on a multiple choice test will be correct"; "it can be used to solve real-life problems such as calculating the height of a building if you know the distance from and the angle."
Makes logical sense	"I like how everything is logical and like a puzzle. It's also one of the fundamentals for math so it's obviously extremely important"; "requires you to think logically rather than emphasizing on rote learning."
Integrates ideas from previously taken mathematics courses	"It has an appealing factor of problem solving and it utilizes arithmetic learned early on. It is also a new way to view former concepts"; "a combination of various fields of math study…it integrates everything I've learned in math"; "allows one to use combinations of math previously learned in a useful way"; " allows me to combine all of the past math that I have learned."
Promotes powerful tools to solve problems with minimal effort	"I was astounded at the complexity of problems that can be solved with this technique"; "it gives you the ability to understand or find out a lot about a given problem with minimum effort. The concepts involved are interesting and one does not get bogged down in tedious manual calculations"; "it gives me a set of tools I can use to solve problems that previously would have been lengthier or even impossible"; "allows me to solve even the most complex problems with relative ease. The methods used make it easier to solve problems which would otherwise take much longer if I were to use other method"; "it provides me with the techniques to answer math questions without using the graphing calculator or numerous equations with many unknown variables"; "it teaches valuable shortcuts and problems that would take multiple steps before can now be done fast"; "using little information to find the answer."
Facilitates spatial visualization ability	"I am naturally good with shapes. It is easier than other topics because many questions have visual aids, i.e. shapes with dimensions"; "I like being able to visualize the figures"; "it's easiest to visualize, and is less about memorization and is more about applying concepts."
Other	"It's easy"; "Not sure, I just love it"; "It's cool"; "My teacher was more intriguing."

The findings from Phase 1 revealed that the subjects in the current study included a reasonable mix of female and male students. These students enrolled in a specialized high school in New York City. They consequently had had some of the most rigorous high school mathematics curriculum in the country. They also demonstrated excellent performance in a number of standardized tests. The students had expressed interest in continuing their studies at the undergraduate level, choosing mostly natural science disciplines. Moreover, they were able to provide reasonable explanations for their favorite mathematics topics. In general, by virtue of attending this specialized high school, the subjects in the current study could be considered to be mathematically gifted students.

Findings from Phase 3: Experts' Evaluation

The findings from Phase 3 suggested that not all approaches for the three problems received common agreement from all three experts, either in their five-point scale evaluation or in their order of preference evaluation (see Appendix J for the collection of approaches, and Appendix K for instructions for experts' evaluations). There was, however, only one approach with unanimous assessment, namely P3A2. In general, P1A3, P2A2, and P3A3 were considered to be the most "beautiful" approaches for Problems 1, 2, and 3, respectively. Table 2 presents a summary of the findings from Phase 3. (Note: The highlighted parts in this figure are each expert's preferred approaches as well as the most "beautiful" approaches as determined collectively.) In this section, explanations for each approach were described one after another, from most preferred one to least preferred.

Table 2

Summary of the Findings from Phase 3

Panel of Experts' Mathematics Aesthetic Evaluations:

Five-Point Scale and Order of Preference

Problem	Approach	Professor 1		Professor 2		Professor 3		Overall	
		5-Pt Scale	Rank	5-Pt Scale	Rank	5-Pt Scale	Rank	5-Pt Scale	Rank
P1	A1	3	3	5	2	4	2	4.00	2
	A2	2	4	4	3	5	1	3.67	3
	A3	4	1	5	1	4	3	4.33	1
	A4	4	2	2	4	3	4	3.00	4
P2	A1	4	2	4	3	5	3	4.33	2
	A2	4	1	5	1	4	2	4.33	1
	A3	4	4	5	2	4	6	4.33	3
	A4	4	3	3	8	3	8	3.33	6
	A5	3	5	3	5	5	4	3.67	5
	A6	2	7	3	7	4	5	3.00	8
	A7	2	8	4	4	5	1	3.67	4
	A8	3	6	3	6	3	7	3.00	7
P3	A1	3	1	3	3	3	3	3.00	3
	A2	4	2	4	2	4	2	4.00	2
	A3	4	3	4	1	5	1	4.33	1

<u>Problem 1</u>

Fill in the blank with one of the symbols $<, \leq, =, \geq$, or $>$.

$\sqrt{2009} + \sqrt{2011}$ __________ $2\sqrt{2010}$

The panel of experts generally considered Approach 3 (P1A3) to be the most mathematically “beautiful” approach for Problem 1. Professor 3 maintained that P1A3 required “the knowledge of the more advanced notion of convexity. It is very short and direct, and therefore beautiful.” Still, he had hoped that “the student would have to give another proof if he or she was asked to prove strict concavity of square root.” Professor 2 appreciated the generalizability of P1A3 to more advanced mathematics functions that could be afforded only by "formidable intuition." He argued that this geometric reasoning was easily transferable to a "broader context [that] relates to finite-difference approximation to the Laplacian." Professor 1, on the other hand, did not think that P1A3 was "rigorous" because "the proof that the square root function is concave is algebraic." He believed that "the kid who did this had seen the graph and remembered the shape and understood or figured out that that shape has relevance for algebra." But he agreed that P1A3 was "a beautiful approach because it shows the power of geometry to solve algebra problems."

Approach 1 (P1A1) was ranked second most preferred. Professor 2 thought very highly of P1A1 as much as he did of P1A3. He imagined that P1A1 "hinges on the concavity of $\log(x)$, i.e. the step $(x-1)(x+1) < x^2$ is equivalent to asserting that $\log(x-1) + \log(x+1) < 2\log(x)$, which is similar to the original problem." Nevertheless, he doubted whether the student who supplied P1A1 actually "realizes it." Professor 3 thought of P1A1 as "a standard procedure which involves algebraic operations to simplify the expression." He commented that, unlike P1A3, P1A1 was less preferred "from an aesthetic viewpoint due to its mechanical nature at the beginning." Still, he felt that "the last step was clever in avoiding complicated arithmetic" and that it "shows that 2010 is treated as a symbol rather than a number itself, which demonstrates abstraction." Professor 1 was less enthusiastic about P1A1; it was "a textbook solution," as he put it. He thought that "the kid who did this learned the trick somewhere. It's not the kind of thing a kid would come up with on his or her own."

The panel of experts also considered Approach 1 (P1A1) to be relatively similar to Approach 4 (P1A4), yet contended that P1A4 was generally less appealing than P1A1.

Professor 2 felt a little frustrated with P1A4, stating “I don’t see why the student didn’t simply calculate $\sqrt{2009}+\sqrt{2011}$ to begin with.” He further explained his frustration with P1A4 because “this approach doesn’t seem any more powerful than directly evaluating the original function.” On the other hand, he stated that “I don’t think this is a bad approach, but I’m not sure if it allows for a useful perspective on this problem.” Professor 3 also recognized similarities between P1A4 and P1A1. In particular, he pointed out that P1A4 “starts same as in [P1A1] but the last step requires brute-force multiplication and is therefore naïve.” He said that P1A4 is far less preferred to P1A1 or any other approaches because of “its fully mechanical nature” and lack of presence of “abstraction involved.” Interestingly enough, Professor 1 found P1A4 more preferable than P1A1. He said, “[t]he kid who did this I would call ‘scrappy,’ able to find a solution somehow with minimal tools.” He admired students who provided P1A4 for their “willingness to think” and believed that they “had a bigger toolbox than the others, even if one of the tools is fifth grade math.”

Approach 2 (P1A2) elicited the most varied opinions. Professor 3 valued P1A2 as the most aesthetically appealing among the four approaches. He preferred P1A2 because of “the originality of the first step.” He believed that it was “more original and elegant because the first step is not ‘learned’ hence nontrivial.” Yet, Professor 2 thought that Approach 2 was “convoluted, but perhaps some of the techniques will be useful in solving other problems.” He said, “I can’t come up with a simple geometric interpretation of this approach. That doesn’t mean that there isn’t one!” What seemed to bother him more was that he could not “see more general utility in the intuition offered by this approach.” Nevertheless, Professor 1 thought of P1A2 as the least aesthetically appealing of the four approaches. Connecting it with P1A1, he said, "[h]ere too I would imagine that the student had seen manipulations like this before. This particular manipulation is used in calculus to get the formula for the derivative of the square root function." Moreover, he attributed the reason why P1A2 was "less 'beautiful' than Approach 1" to its unnecessary length as "[i]t does more of the same kind of thing to get to the same result."

Problem 2

Given $x^2 + y^2 = 1$, find maximum of $x + y$.

Approach 2 (P2A2) was considered to be the most "beautiful" by the panel of experts. It received an equal average score as P2A1 and P2A3 in the five-point scale evaluation, but was ranked highest by far in the order of preference evaluation. Professor 1 thought that P2A2 was "beautiful" because it "makes the problem seem so simple that it does not require brilliance to solve it." He admitted that it was "a little long," but explained that "finding the farthest out line that crosses the circle is the geometry of the problem that makes the solution obvious." Professor 2 recognized that P2A2 "clearly points out to Lagrange multipliers." He explained:

> The first two sentences of this approach contain a great deal of structure. First, the student introduces the notion of level sets of the original function, and then the student introduces the notion that the level-curves of the original function should be tangent to the constraint manifold at the critical point. This is already a great deal of structure. Indeed, these two observations encapsulate the basic idea of Lagrange multipliers. Aside from smoothness and boundary considerations, these observations solve the problem. At this point, the student can just draw the constraint manifold (a circle) as well as the level sets of the original function (straight lines with a fixed slope) and pick out the points of tangency. The rest of the student's argument is unnecessary.

Professor 3 recognized Lagrange multipliers in P2A2, but did not consider this approach as highly because it was "almost textbook style."

Approach 1 (P2A1) was the second most favored. Professor 1 saw it as "a thoughtful approach, but with a clever twist at the end." He commented that "[i]t shows the ability to formulate some algebraic expressions based on a picture." Professor 3 liked that this "very elegant symmetry breaking argument" was "original in that it involves a proof by contradiction for a constrained optimization problem." He was impressed by the "efficient way of demonstrating how the constraint cannot be satisfied with a larger value of the objective function." Professor 2 thought P2A1 "heroic, yet flawed" in a more general case. He explained that

> [t]he original assertion (i.e., by symmetry) is not well founded. Consider, for example, the situation where the constraint is, say $\sqrt{x} + \sqrt{y} = 1$. In this situation the maximum of $x + y$ occurs at $x = 1, y = 0$ and $y = 1, x = 0$. The assumption of symmetry might lead the student to believe that $x = y = \frac{1}{4}$. The further attempt to justify such a conclusion will be impossible (as it is not true). In this particular case the constraint happened to be (i.e., $x^2 + y^2 = 1$), and the student's analysis demonstrated that the symmetry assumption was correct.

Therefore, although "[t]he techniques used in this analysis are no easily generalizable to other situations," he believed that "the student's attempt to use symmetry to solve such a problem is commendable."

Approach 3 (P2A3) was ranked third. Professor 1 thought that "[t]his is an example of a 'slick' argument." He explained that "[i]t is very short and gives little hint what is going on." Despite the appeals of "slick" arguments, he said, "I try to avoid them because they make me nervous." Professor 2 thought P2A3 made a good use of "a cute inequality." He added that it was "quite clever" but "otherwise not generalizable to other constrained maximization problems." Professor 3 thought that P2A3 would have been more favorable had it been more complete. He believed that it "requires more explanation on why the stated inequality is true and how it gives a complete solution (i.e., case of equality)."

Approach 7 (P2A7) was ranked fourth. Professor 3 commended its originality. He said that "[i]t is highly original to think of the objective function as an inner product for which the constraint is the norm of one of the vectors." Professor 2 also praised this approach as having "a facility for linear algebra." He added, "[w]hile not generalizable to other functions, this type of geometric reasoning will be useful for other problems." Professor 1 thought P2A7 was comparable to P2A6, but "even clunkier and longer." Still, he imagined that "[t]he person using vectors clearly has lots of math training, which would probably include calculus."

Approach 5 (P2A5) was ranked fifth. Professor 2 thought that "[t]his approach uses techniques for analyzing trinomials, but doesn't generalize." Professor 1 thought P2A5 was comparable to P2A4 in that "it puts the problem into a standard form and applies the standard solution method." He ranked it lower than P2A4 because "the solution method is more special and clunkier." However, he acknowledged that "if the student did not know calculus, it's a good solution." Professor 3 thought that it was "highly original." He believed that it demonstrated "a high level understanding of algebraic facts which are often simply memorized at the secondary school level."

Approach 4 (P2A4) was ranked sixth. Professor 3 found "not much aesthetic" in this approach. He thought that it was "mechanical once the substitution is made and methods of calculus are invoked." Professor 2 liked it the least, saying that "[t]his approach is only viable when the constraint can be incorporated into an explicit description of the original function, which of course is not generally the case." Still, Professor 1 thought that "[t]his approach is 'competent.'" He explained, "[t]he solver calmly applied what he or she knew in a way that was almost guaranteed to find the answer, if there is a simple answer." He added, "Mathematics, in my opinion, is more about avoiding brilliance than using it." P2A4, he believed, was an example of this.

Approach 8 (P2A8) was ranked seventh. Professor 2 saw similarities of it in P2A4 in that "it relies on an explicit characterization of the constraint manifold." Professor 3 also saw the similarities, but did not like it because it was "mechanical." Nonetheless, both professors ranked P2A8 higher than P2A4 because P2A8 involved trigonometry, which made it relatively more "elegant," as Professor 3 put it. Professor 1 saw the similarities as in P2A4 as well, but thought that the use of trigonometry was "unnecessary."

Approach 6 (P2A6) was the least favored of all. Professor 1 thought of it as a "grind out trigonometry," that is, "more complicated than necessary." He explained that "[t]he person who did this could have saved time by using less machinery and finding a more conceptual approach, like Approach 1." Professor 2 also thought "[i]t's good that the student understands some trigonometry, but I don't think these tools will apply to more general problems of this type." Interestingly, Professor 3 thought that this approach was rather "elegant because it solves the maximization problem in one step."

<u>Problem 3</u>

Given triangle ABC with median $\overline{CD}$ and $CD = BD$, find measure angle ACB.

Approach 3 (P3A3) was considered to be the most elegant by the panel of experts. It received a relatively similar score in its five-point scale evaluation compared to P3A2, yet was generally more favored than P3A2. Indeed, all three experts agreed that both P3A3 and P3A2 were somewhat similar and aesthetically pleasing. Professor 3 rated P3A3 the highest on a five-point scale. He was intrigued by “the combination of originality and efficiency of the solution.” He contrasted P3A3 with P3A2, suggesting that P3A3 “seems more original by considering the symmetric extension about D.” Professor 2 deemed that P3A3 went beyond P3A2 by facilitating “an illustration of the location of these points (on a circle), as well as a proof that the inscribed angle (associated with a semicircle) is 90°.” Professor 1 agreed with the earlier observations, adding that P3A3 was “another clever insightful proof” that made good use of parallel lines.

Approach 2 (P3A2) was rated the second most favored. Professor 3 regarded it as “aesthetic because of the simplicity of the solution which can be presented with a single picture.” However, unlike P3A3, he thought that P3A2 was rather “standard to consider the circumcircle in a problem like this, so it is not highly original.” Professor 2 echoed the same opinions, describing that P3A2 “paints a picture wherein points A, B, and C all lie on a circle about D.” Comparing it with P3A3, he favored P3A2 less for lacking “a proof of the fact that the inscribed angle associated with a semicircle is 90°.” Professor 1 thought that “[t]he circle construction is very clever.” He took his view one step further, stating, “[t]he interesting thing about this, however, is that it relies on a theorem about angles in a circle, the proof of which is basically the direct approach to this problem." P3A2 was (curiously) the only one of all 15 approaches that received unanimous assessment from the three experts. Yet, aesthetic appeals in P3A2 could clearly be captured, according to the experts' evaluations, stemming from a sense of harmony once all pieces of information were united within a circle.

Approach 1 (P3A1) was graded the least elegant for being a very typical approach. Professor 2 thought that "[t]his approach isn't bad, but doesn't provide a clear picture associated with the problem." Professor 3 felt that "[i]t is an efficient solution, but not necessarily an aesthetic one" because "[t]his approach is built on a straightforward inspection of relations between the angles based on the given information." Although grading it exactly the same as the other two professors on the five-point scale, however, Professor 1 ranked P3A1 most appealing. He believed that P3A1 was "the one I think is the best, but it is not the most beautiful." He explained that this approach "uses no extra machinery and just competently builds up information." Contrasting it with P3A2 and P3A3, he said:

> Both [P3A2 and P3A3] add extra structure that immediately makes the solution obvious. It is beautiful to see a mathematics problem instantly put into a new light by an insight like that. Some of the solutions to [Problems 1 and 2] also involve extra structure, but in those cases I did not feel that the extra structure put the problem in a different light.

He justified his belief further:

> [P3A1] consists essentially of building up information step by unremarkable systematic step until the solution appears. I am comfortable with this because I can see how to apply this kind of reasoning to many other problems. The other approaches do not go in steps, but just give the answer and the consequence of an insightful picture.

The three research mathematicians evaluated the aesthetic value of the 15 approaches in great detail and they appeared impressed by many of the 15 approaches.

Some Perspectives from the Panel of Experts

Professor 1 took the time to discuss his overall perspective on mathematics aesthetic evaluations. He made some remarks about his experience as a professional mathematician, in particular during his graduate school training. He said:

> In grad school, I had problems with the many beautiful arguments you find in advanced textbooks. They would prove that something is true without really saying why or helping me decide related questions. You could learn a beautiful proof, but could not "understand" it. Often a more clumsy, longer argument gives more insight and therefore is more useful in the long run.

It was obvious that Professor 1 placed as much value on the issue of pedagogy as he did on aesthetic value. He cared about the accessibility of concepts at an earlier stage of one's mathematics learning experience. He believed that students with little formal training ought to be able to understand every line of reasoning in a "beautiful" proof effortlessly. A similar view had been declared by a group of mathematicians who chose a "beautiful" proof of the irrationality of $\sqrt{2}$ because of "[t]heir simplicity and the fact that they can be completely understood with a minimal amount of mathematical background" (Dreyfus & Eisenberg, 1986, p. 8).

Unlike Professor 1 who saw "beauty" in the efficacy for learning purpose, Professor 2 found context of generalization most appealing. He was pleased when an approach could be modified and applied to similar problems in a more advanced mathematical setting. He would appreciate it even more if an approach could be seen as a small-scale reproduction of some grand mathematics propositions. Indeed, this very same orientation to generalizability was observed by Krutetskii (1976), who pointed out that the work of mathematically gifted students was very generalizable to different topics.

Professor 3, on the other hand, appreciated originality in “beautiful” mathematics proofs. He deemed an approach “beautiful” if he could find in it some presence of mathematical ideas, steps or reasoning that were connected in an unexpected way. His view of originality perhaps could be understood better by contrasting it with his opposition to “dull” approaches. He disliked the mechanical use of mathematical machinery similar to the mathematical theme in the surface structure of the problem. For him, there was no “beauty” for anything learned or taught in classroom standards. The sense of “foreignness” of an approach attracted him more.

The findings from Phase 3 showed that the mathematics aesthetic evaluations were indeed subjective, in keeping with the expression “beauty is in the eye of the beholder.” Professor 1 valued the learnability of an approach in establishing his view of mathematical “beauty.” Professor 2 appreciated the transferability of an approach to more general situations. Professor 3 regarded the originality of an approach as a determining factor to impress a well-rounded audience. They nonetheless all agreed that with mathematically “beautiful” work came the power to stand out from the “crowd.”

Chapter V: FINDINGS FROM PHASES 2 AND 4: STUDENTS' TEST AND STUDENTS' FOLLOW-UP INTERVIEW

This chapter is devoted to describing the mathematical problem solving behavior of the nine students who took the test in Phase 2 and were interviewed in Phase 4. It describes the students' test performance, their choices of approaches, and their reactions to the experts' opinions of "beautiful" approaches. Table 3 presents a summary of the findings from Phases 2 and 4. (Note: The maximum score for the AMC-12 test is 150. The maximum score for the SAT tests is 800. The maximum score for the test in Phase 2 was 3. When a student did not report a test score, data were considered not available. See Appendix I for examples of acceptability scores of 1 and 0. A student's successful approach was an approach that the student supplied during the test and was scored 1. When a student was not successful in supplying a score 1 approach, data were considered not applicable. A student's preferred approach was an approach that the student preferred among all approaches from the collection of approaches in Appendix J. An aesthetic score for the student's preferred approach is given according to the experts' evaluations. See Table 3 for aesthetic scoring scheme. The last column indicates a typical student with an average statistic in terms of mean or mode. For example, a typical student reported an average SAT Mathematics Section score of 754, and supplied and preferred P2A4 in solving Problem 2.)

Table 3

Summary of the Findings from Phases 2 and 4

		Student 1	Student 2	Student 3	Student 4	Student 5	Student 6	Student 7	Student 8	Student 9	Typical
Background	Grade	12th	12th	12th	12th	12th	12th	11th	12th	12th	12th
	Current Course	AP Calc	AP Calc	AP Calc	AP Calc	AP Calc	AP Calc	AP Calc	AP Calc	AP Calc	AP Calc
	AMC-12	*	94.5	*	*	*	*	*	*	*	94.5
	SAT-Math Section	760	770	640	770	780	800	*	740	770	754
	SAT Subject-Math I	*	*	*	*	*	*	*	*	750	750
	SAT Subject-Math II	*	800	*	800	800	800	*	770	770	790
	Planned Major	Pre-Med	Undeclared	Biology	Comp Sci	Comp Sci	Undeclared	Undeclared	Undeclared	Comp Sci	Undeclared
	Fav Math Topic	Algebra	Geometry	Geometry	Calculus	Calculus	Calculus	Calculus	Algebra	Algebra	Calculus
Acceptability Score	Problem 1	0	1	1	0	1	0	1	1	1	0.67
	Problem 2	0	1	0	1	1	0	0	0	0	0.33
	Problem 3	1	1	1	0	0	1	1	1	1	0.78
	All Problems	1	3	2	1	2	1	2	2	2	1.78
Successful Approach	Problem 1	*	A1	A4	*	A4	*	A4	A4	A4	A4
	Problem 2	*	A8	*	A4	A4	*	*	*	*	A4
	Problem 3	A1	A1,2	A1	*	*	A1	A1	A1	A1	A1
Preferred Approach	Problem 1	A4	A3	A3	A1	A1	A4	A3	A4	A1	A1,3,4
	Problem 2	A2	A2	A4	A1	A4	A4	A4	A4	A2	A4
	Problem 3	A2	A2	A2	A3	A1	A3	A3	A3	A1	A3
Aesthetic Score	Problem 1	3.00	4.33	4.33	4.00	4.00	3.00	4.33	3.00	4.00	3.78
	Problem 2	4.33	4.33	3.33	4.33	3.33	3.33	3.33	3.33	4.33	3.77
	Problem 3	4.00	4.00	4.00	4.33	3.00	4.33	4.33	4.33	3.00	3.92
	All Problems	3.78	4.22	3.89	4.22	3.44	3.55	4.00	3.55	3.78	3.83

Note: * data not available/not applicable

Student 1

Student 1 was a senior planning to enroll in college with a premedical major. She reported an SAT Mathematics Section score of 760. She chose algebra as her favorite mathematics topic because it was relatively simple and easy for her to understand. Her mathematics background was typical of students from that particular specialized high school. She started off with algebra in eighth grade and, from then on, continued with geometry, trigonometry, pre-calculus, and calculus. Her overall acceptability score was 1 for a correct answer in Problem 3. She started the test with Problem 1, producing two approaches which were very similar and unsuccessful. Her written work showed that she approached the problem using derivatives, but was unsuccessful. Clearly, her reflex had to do with her then-calculus course. After going through the other two problems, she returned to Problem 1. On a separate page, she attempted to produce a different approach. This time it was more arithmetic in nature, possibly as a continuation of her work in her first attempt. It was again not successful. When presented with four approaches from the collection of approaches (see Appendix J), she chose Approach 4 (P1A4) as her preferred approach because in her view, it was the most understandable. The other three, she said, were complicated. When told that P1A3 was considered to be "beautiful" by the panel of experts, she was not able to provide an adequate explanation. She answered quite honestly:

> It's probably because it's the simplest way to solve it, but I don't understand how they solve it, so I can't say, but I agree that's easy if I knew how to solve it…it's simple, if you understand the concept of it, don't go through all these big numbers to solve it.

Clearly, her lack of understanding the mathematical knowledge involved in the process of P1A3 prevented her from appreciating its aesthetic value.

Student 1 continued with Problem 2, supplying only one approach which was considered to be unsuccessful. Her written work again had a number of derivatives. It showed that she was considering a calculus approach, a topic related to the course she was enrolled in at that time. Despite this, when eight approaches from the collection were presented to her, it turned out that the calculus approach, Approach 4 (P2A4), did not appeal to her as strongly as the geometry approach, P2A2. On one hand, her explanation for choosing P2A2 over P2A4 had to do more with her understanding of the two approaches. She reported that she would have chosen P2A2 as her first attempt if she had another chance to do the problem again. On the other hand, her choice of preferred approach had little to do with her ability to identify the aesthetic values of the approach. When informed that P2A2 was considered to be "beautiful" by the panel of experts, she was somewhat indifferent. Asked whether she saw anything appealing in P2A2, she only said that “[i]t doesn’t involve so much numbers in it, not many properties involved.” Clearly, her descriptions of aesthetic appeal did not go beyond the explicit and concrete presentations of the approaches.

At the end of the test, she was able to solve Problem 3. Her written work showed that it was very similar to Approach 1 (P3A1). She explained that P3A1 came out to her in her first attempt because she tried different angles to find the measure of the angle in question. Her choice of approach might be viewed as an instinctive one with the sole intention to find the answer to the problem. After shown the three approaches from the collection, she said that she preferred P3A2 to P3A1. Her explanation was again limited by her mathematical understanding of the approaches. When told that P3A3 was considered to be “beautiful” by the panel of experts, she had a reaction similar to her earlier in Problem 1. She said, “I think it’s the same thing, like the beautiful if you know some mathematical properties, you can apply them the right way, you don’t have to involve like numbers or logic as much, you just use those properties.”

Generally, Student 1’s performance was somewhat unsatisfactory since she provided only one correct approach for Problem 3. Her explanations for this problem solving experience indicated that she was not in her best shape for the test. She said, “I realized I forgot to do some of the math, I haven’t been really doing that much this year, I know that’s probably my biggest issue.” At the same time, her aesthetic feelings did not emerge, probably because of her lack of understanding the necessary mathematical concepts involved in the approaches.

Student 2

Student 2 was a senior planning to continue his academic career in college but he reported an undeclared major. He documented an SAT Mathematics Section score of 770, an SAT Subject Test—Mathematics Level II score of 800, and an AMC 12 score of 94.5. His favorite mathematics topic was geometry, but he gave no specific reason for why. He was the only student to receive a maximum acceptability score of 3, having successfully solved all three problems. He started the test with Problem 1, producing two approaches, only one of which was successful. He was not able to solve Problem 1 on his first try. However, after solving the other two problems successfully, he returned to Problem 1 and managed to obtain a correct answer by supplying Approach 1 (P1A1, see Figure 3).

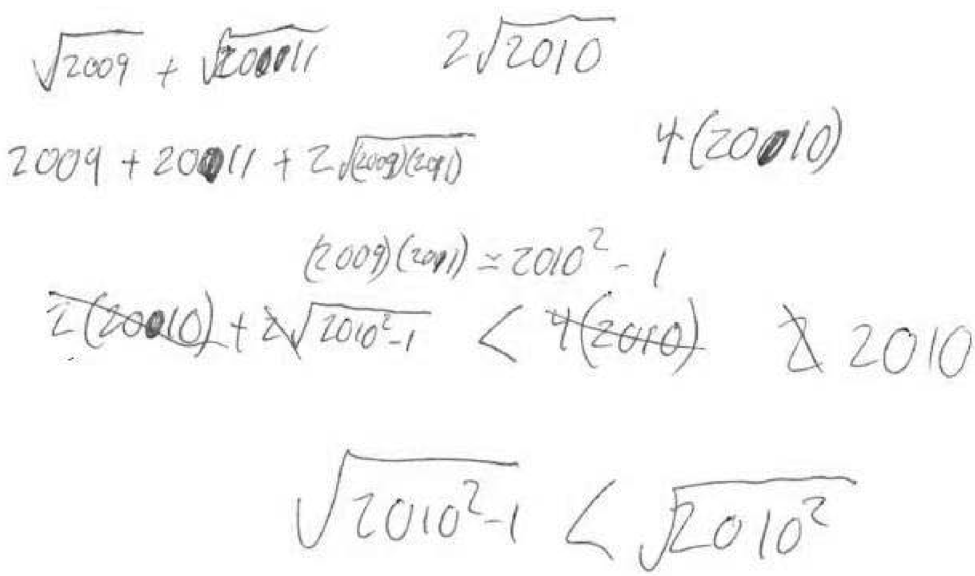

Figure 3. Student 2's Written Work for Problem 1

In his second attempt, Student 2 considered a relationship of arithmetic and geometric means, but it was not quite as successful. He was the only student who successfully solved Problem 1 using an approach other than P1A4. As a matter of fact, he seemed not to bother pursuing P1A4 even after successfully producing P1A1. His choice of P1A1 demonstrated a developed level of awareness of mathematics aesthetic values. When exposed to P1A4 from the collection, he somewhat disliked the mechanical calculation involved. He said:

> It starts off the same way as me, but in the end he just multiplied them together, instead of doing the difference of two squares method. I prefer the first one better, well because of laziness, I don't particularly want to multiply 2009 and 2011, when I could rather abstract it and get $2010^2 - 1$, so yeah.

Provided with all four approaches, he chose P1A3 as his preferred approach. He also stated that it would have been his first attempt had he thought of it. He clearly had an adequate comprehension of this approach. He said:

> Because it's concave, and it's not increasing steadily, steadily in one direction, as you get further out the, intuitively, I know what he's saying, but I can't really put it into words, because the averages, as you get further out, the averages of any two points is going to be less and less, like when you draw a line between the two points and take the midpoint, that'll be it, and you would want that point.

He mentioned that P1A3 was "a quick, smart, logical argument." He clarified his preference for this approach, saying "because it's the simplest, the most abstract, you can generalize it to any function that is strictly concave, or concave over the range or domain that you want to use it." It was obvious that he recognized the value of the generalizability of P1A3. Indeed, when told that it was considered to be "beautiful" by the panel of experts, he was able to offer a justification of it. He said:

> Yeah, it's more aesthetically pleasing, shall we say, it's also a bit more fun in a way, because if I knew this, like I said, I could use it on any other problem that is like this, whereas if I get a bunch of numbers with some function being acted on all of them, and I'm asked to classify them in some way like this, I can just refer back to this and generalize it….

Student 2 continued with Problem 2, producing two approaches, only one of which was successful. He solved the problem successfully on his first attempt, using an approach that had not been in the original collection (it was later added as Approach 8, P2A8, in Appendix J). He explained how seeing the constraint function $x^2 + y^2 = 1$ made him generate the unit circle visualization and connect to it at once with trigonometry aspects. This, he said, enabled him to think of the problem as a combination of calculus, geometry, and trigonometry. Perhaps his strong problem solving ability came about as a result of his access to advanced mathematics education as well as his mathematical aptitude. After his first successful attempt, he continued working on it to furnish another approach. His second attempt appeared incomplete and was considered to be unsuccessful. He said that he was trying to replicate the problem into an optimization of a perimeter of a rectangle type of problem. When shown eight approaches from the collection, he at first identified P2A2 as his second preferred after P1A8. In his explanation, he cleverly made a connection with his favorite mathematics topic, geometry. He said, "Well, this one [P2A2] kind of nicely turns it into a geometry problem, it seems a little roundabout for my taste." When informed that P2A2 was considered to be "beautiful" by the panel of experts, he was not initially completely convinced. Still, he described it as "nice and pleasing." When asked to compare P2A2 with other approaches, he eventually said that the calculus approaches P2A4 and P2A8 were "kind of the brute force" but could be used "for everything, whereas [P2A2] only works for this condition [in Problem 2].” To some extent, he valued the parsimonious aspect of aesthetic values. He said that P2A2 was “kind of low tech, I think. I didn’t have to use more advanced topics in order to solve a problem which you wouldn’t think only requires less sophisticated methods to solve.” In the end, he switched his preference to P2A2 “because of the simplicity and the solving of a complex problem with simple tools.”

Lastly, he produced three approaches for Problem 3. Quite remarkably, when compared with the other eight students taking the test, two of his three approaches were not only distinct, but they were also successful. His first attempt was a similar version of Approach 1 (P3A1) from the collection. He was able to represent the angle measures involved in the triangle using algebraic representations. After successfully solving all three problems in the test, he returned to Problem 3 with his second attempt. He was able to produce P3A2 from the collection. He drew a semicircle instead of a complete circle, and immediately indicated a right angle for the measure angle in question. In comparing the two approaches, he favored P3A2 rather than P3A1. He explained:

> I like it more because it’s kind of more abstract in a way, whereas this one Approach 1 turns it into an algebra problem, even though the proof of inscribed in the semicircle, the simplest proof is actually the same thing, I still like this Approach 2 more, it’s simpler and more abstract, I tend to like more abstract things better.

When shown P3A3, he was a little indifferent at first. He replied, “It’s a rectangle, okay so, what about it?” Then he added, “Yeah, that’s pretty simple, I see how the logic works, I would not have thought about this approach, I usually don’t extend into a parallelogram, I don’t know, it just doesn’t occur to me naturally.” After being informed about P3A3 being considered “beautiful” by the panel of experts, he paused for a while, then said:

> Well, it’s not, it’s a good bit of thinking, I have to think of a reason why you said that, it’s because, the person had to realize that, hey if you extend

> these two lines parallel, you get a parallelogram, you can work with that, because this forms, this also intersects with that one, so you get a diagonal, it's, I guess, it's a nice, I can't really think of a solid reason, it's all opinion really.

When asked whether he would choose P3A3 as his first attempt, he declined and maintained his position with P3A2. He said:

> I'd still say [P3A2], yeah, I don't know why I like using, I like working with triangles and circles better than I do parallelograms, quadrilaterals in general, I can't think of anything, simpler possibly. Of course, maybe [the panel of experts] likes working with parallelograms.

In general, Student 2 was the top performer with a perfect acceptability score of 3. The manner of his mathematical thinking proceeded rather systematically. He said:

> Well, the first solution was really easy, because I just solved the problem however I thought of first, it's just like a normal question, whereas the second approach, for each of them, I was thinking, well, if I couldn't do that, what is some other way I can try and solve it, and I just thought of anything I could to solve it, and after that, I kept just trying other methods, but usually it didn't work, or I find some problem where I thought they could happen.

He did not appear to show much stress during the test. That said, he did not advance further when he thought his subsequent approaches led him nowhere closer to the answer. He was nevertheless able to see clearly the benefits of interacting between these approaches, such as having a stepping stone for another approach, from his experience solving Problem 3. Moreover, his appreciation of the mathematics aesthetic value was outstanding. Unlike other students who generally perceived "beauty" as only the outward appearance of an approach, Student 2 was able to apply a more sophisticated judgment of aesthetic values. This was evident from his accounts on the generalizability, parsimoniousness, and abstraction of the approaches that he considered to be mathematically elegant.

Student 3

Student 3 was a senior planning to major in Biology for her undergraduate study. She reported an SAT Mathematics Section score of 640. Her favorite mathematics topic was geometry because of her fondness for visual reasoning. Her overall acceptability score was 2 for correct answers on Problems 1 and 3. She began the test with Problem 1, supplying four approaches, of which only two were different. Her first attempt was successful and classified as Approach 4 (P1A4). She continued working on Problem 1, looking for different ways to solve it. She wrote down two additional approaches on two different pages. Although she recorded them as different approaches, they both had very much the same ideas as P1A4 and they were all correct. Her fourth attempt started off with the same reasoning as her first attempt, but she quickly switched to using some algebraic notations. Apparently, she was gearing towards a more general version of P1A1, but it proved to be incorrect. In the interview, she said:

> The first three approaches that like came to mind right away, whereas the fourth approach was when I finished the test and I was trying to check if that answer was right. So I tried a different way to just check my answer rather than resolve it. So I'm more confident that the first three approaches the answer I got there was the right answer.

Student 3 made use of her incorrect result from her last attempt to dispel her doubts about her answers earlier on. She wanted to verify that the answers she obtained in her previous three attempts were in fact correct. One might argue that accepting the accuracy of her first three attempts simply because of a conflicting result in her last attempt did not sound reasonably compelling. Nevertheless, for her, this experience signaled a valid check and thus a boost in her confidence. Moreover, she explained that she preferred P1A4 because it was easier and more straightforward, although she mentioned that it was also more computationally involved. P1A1, by contrast, was more logically clear and required less calculation, as she said. When shown four approaches from the collection, she was able to recognize that her four approaches were similar to P1A4 and P1A1. Moreover, she chose P1A3 as her preferred approach. She recognized the value of the concavity of square root function in reducing the amount of work needed to solve the problem. She claimed that it would have become her first attempt if she had to do a similar problem all over again. After being informed that P1A3 was considered to be "beautiful" by the panel of experts, she concurred. This information did not come as a surprise to her. However, one could sense her positive reaction even as she showed only a subtle sign of contentment because of this conformity. Although she indicated P1A3 as her preferred approach before being told about the aesthetic selection by the panel of experts, her explanation for the aesthetic seemed to intensify later on. She revealed a greater conceptual understanding of the geometric properties of the square root function.

Student 3 continued with Problem 2, providing four approaches, all of which were very much alike and unsuccessful. In general, she appeared to apply a guess-and-check approach. She expressed her uncertainty about the problem but did not provide detailed explanations for it, other than "that's all I can think of." When presented with eight approaches from the collection, she pointed out without any hesitation that Approach 4 (P2A4) would have been her first attempt if she had to do a similar problem all over again. She said, "I know when I was first doing the test and I saw the problem, I knew that I needed to use derivatives, but I didn't remember how to use them, so that would probably be the first one." She added that it looked very familiar to her as she connected it with her recent exposure of derivatives in her then-AP Calculus course. Despite her reported favorite mathematics topic, geometry, she did not say much else about the geometric approach P2A2, except that it would have been her second attempt if she had a chance to do it again.

Student 3 successfully solved Problem 3. She provided three approaches, all of which were similar to Approach 1 (P3A1). In some sense, they were simpler than P3A1 in the collection because she did not offer as many geometric arguments as she did algebraic representations to denote her inferences. In the interview, she said that she was more confident about her logical reasoning in her second attempt compared with her first attempt. She added that obtaining the same answer twice in her first and second approaches “solidifies” her conviction of getting the correct answer. Her third attempt was also similar to P3A1. Yet, she ended up with a result that differed from the previous two. She explained in the interview how she felt somehow more relieved about her results in the first two approaches. This was yet another example of a peculiar behavior of rationalizing her problem solving skills, as observed previously in Problem 1. From the collection, she chose P3A2 as her preferred approach. In explaining her choice, she referred to her fondness for circles more than parallelograms. P3A3 was not quite as likeable to her. She discussed how parallelograms were more connected to triangles than circles were to triangles. Unlike a circle, she said, a parallelogram could be constructed using only two congruent triangles. Other than the geometric shapes involved in the three approaches, her responses indicated a disregard for the deeper structure of mathematical proofs behind those approaches.

In general, Student 3 performed adequately on the test by supplying one successful approach each for Problems 1 and 3. She was able to recognize some values of problem solving using many different approaches. She also acknowledged the benefit inherent in taking advantage of a previously obtained result and comparing it with one obtained in a different way. She said:

> I kept skipping back and forth because originally I was able to find like one way to do something and then I moved on and I realized I could've done something else so I returned to the original question. And also a lot of times the second time when I'm doing the question a different way, I would end up with a different answer and then realized that the first way I had done it was completely wrong so like it was just interesting to like be able to think about different ways, instead of just going the one easiest way to do per question.

Student 3's aesthetic choices of approaches indicated her particular inclination towards the more economical mathematical thinking and knowledge required in those approaches. Nevertheless, her choices revealed a literal manner in which the approaches were transcribed, overlooking other critical aspects of aesthetic values such as originality or simplicity.

Student 4

Student 4 was a senior planning to major in Computer Science in college. He recorded an SAT Mathematics Section score of 770 and an SAT Subject Test—Mathematics Level II score of 800. His favorite mathematics topic was calculus because of its relative ease compared to other mathematics topics. His overall acceptability score on the test was 1 for a correct answer in Problem 2. He started with Problem 1, producing two approaches, all of which were similar to Approach 4 (P1A4) but not successful. When shown four approaches from the collection, he said that P1A1 needed fewer computations than P1A4. He added, "The other ones [P1A2 and P1A3] are kind of messy, easy to mess up kind of arithmetic and algebra, this one [P1A1] feels a bit more elegant to me." Thus, he preferred P1A1. He was also not able to comprehend the aesthetic value behind P1A3 as suggested by the panel of experts. Because of his lack of understanding, he did not find P1A3 particularly more appealing than P1A1. He said, "It's hard to tell because I don't totally understand it, I usually don't find graphs beautiful in general, though."

Student 4 did very well on Problem 2. He supplied only one approach, which was similar to Approach 4 (P2A4) and considered to be successful. He chose this calculus approach P2A4 as his first attempt for the same reason as most students did: reflexive short-term memory recall of his most current mathematics course. In the interview, he discussed this: "I guess that just means that I've been doing this kind of [calculus] problem more recently." When informed that P2A2 was considered to be "beautiful" by the panel of experts, he said:

> Like for me, it feels more like algebra and like numbers, I can't make too much of a connection with graph, the problem is more like algebra so I'm having a hard time like making a connection related to it, so I'm not really a big fan of this approach, I don't see anything beautiful in this approach.

He nevertheless chose P2A1 as his preferred approach. He said that P2A1 was "[s]ort of like [to] prove that it cannot be false, it's very like convincing immediately to me, more logical, there's less places where you can get lost." He was clearly intrigued by the proof-by-contradiction approach in P2A1. One might think that this was the first time he had been exposed to this proof-by-contradiction approach. One could also assume that there was an element of originality in his surprise when he perceived something completely different. A proof-by-contradiction approach might have been frequently relied upon by research mathematicians. Still, for Student 4, this mathematical scene did not seem to have ever taken place during his entire learning experience. Because it was such an uncommon approach for him, it was also the most appealing.

Student 4 continued with Problem 3, supplying one approach which was considered to be unsuccessful. He appeared somewhat confused by this problem. Later on in the interview, he revealed that he had trouble understanding the definition of a median of a triangle. Consequently, he could not proceed further. When presented with three approaches from the collection, he first discussed how he somewhat disliked Approach 1 (P3A1). It was clear that he saw the mechanical nature of this approach. He said, "The first approach [P3A1] has too many steps and feels clunky to me. The next two [P3A2 and P3A3] seem easier and are more intuitive." He showed great interest in P3A3, as he said, "Yeah, I like that one, because extending the line completely changes the problem for me but in a way that's more solvable because it's just a parallel line, but by making it a parallelogram makes it much easier to solve the problem." When informed that P3A3 was considered to be "beautiful" by the panel of experts, he was, to some extent, able to see the harmony in unifying randomness in the argument involved in P3A3. In some sense, his choice might be a clue to his awareness of the element of originality in aesthetic values. He said:

> I guess I'd agree because extending the line and making a couple of parallel lines, it seems a bit random at first, but then it makes the angle you're trying to solve part of something that's much easier to solve, it proves it in a very unexpected way.

In general, Student 4 did not perform satisfactorily as he solved only one problem successfully. He showed a certain understanding of mathematics aesthetic values such as originality or unexpectedness in an approach. Nonetheless, he appeared relatively simpleminded to the extent that he understood that his objective of problem solving was mainly to find the answer to the problem. He also showed some sign of fixation in his problem solving behavior. He did not see himself as being flexible enough to view the problem from a different perspective in order to solve it differently from his initial approach. He said:

> Once I solved it in one way, it's hard to not think of that way, I guess, like the second kind of try, like it's hard to try completely different approach starting from the beginning, that part was hard. I don't know, I guess I just like the idea that the first one worked, and it's to me like, you know, it's like you have to get from the question to the answer, so it's not really about the steps or the process, it's like getting from the question to the answer, so once I've gotten to the answer, it's hard to start over and just think a completely different way to do it.

Student 5

Student 5 was a senior planning to major in Computer Science at the college level. He reported an SAT Mathematics Section score of 780 and an SAT Subject Test—Mathematics Level II score of 800. He chose calculus as his favorite mathematics topic but provided no clear reasons for doing so. He did relatively well on the test. His acceptability score was 2 for correct answers in Problems 1 and 2. He began the test with Problem 1, producing three approaches, only two of which were considered to be different. His first two attempts involved approximations using some combinations of decimal and fractional notations, but he was not successful on either. On his third try, he managed to produce a successful approach similar to Approach 4 (P1A4). During the interview, he explained some thoughts about how he worked out Problem 1, particularly how P1A4 came about in his third attempt. He said, "Yeah, I couldn't find anything at that time so I resort to that, because I need to get the answer, because at first, I just played around the first two, I think because I knew it wasn't time constrained, or like, yeah, so I just tried those first until I couldn't then I, I just did that one to get the answer." When presented with four approaches from the collection, he chose P1A1 as his preferred approach. In his explanation, he discussed an aesthetic element of surprise. He said:

> I mean I like [P1A1], it didn't came to me during the test, but I like it because, cause it has different twist in it, but what usually comes to me first was squaring both sides, to solve problems like that, like how do you deal problems with square roots, well, you square it, so I just decided to square both sides, just playing around with the number and see what happened, but [P1A1] seems interesting, like oh, I never looked at it that way, and I just liked it, it looks interesting, it looks obvious right now, but it would never occur to me, it looks surprising you can subtract by 1.

P1A2 and P1A3 did not appeal as much to him. He said that P1A2 seemed a little too long. He also dismissed P1A3 as being "beautiful," as suggested by the panel of experts. From his explanation, one might sense his inability to comprehend the deeper structure of the mathematical proofs in the two approaches.

Student 5 continued with Problem 2. He successfully solved Problem 2, presenting three approaches, only two of which were different. His first two attempts utilized some algebraic manipulations of the constraint function, but were considered to be unsuccessful. His third attempt was similar to Approach 4 (P2A4) and was considered to be successful. During the interview, he mentioned his then-AP Calculus course to justify resorting to the calculus approach, P2A4, as his way of solving the problem. He said, "I'm learning calculus right now, so I figure why not use calculus, which is still fresh, more fresh." He also chose P2A4 as his preferred approach. When informed that P2A2 was considered to be "beautiful" by the panel of experts, he somewhat disagreed. He said:

> I'm not very good at considering hypothetical situation, so like I'm not that type of person to solve the problem with that method, like consider here, I mean, like assuming some line here non-existent before to come up

> here, my geometry is not very good, I mean, it's very logical way of thinking, I mean, yeah.

Problem 3 proved to be a considerable challenge for him. He supplied one approach, which was considered to be unsuccessful. His written work showed that he utilized the law of cosines in a special case of the given triangle, namely an isosceles triangle. When shown three approaches from the collection, he chose Approach 1 (P3A1) as his preferred approach. He said that P3A1 was the most logical of the three. Moreover, he liked P3A1 perhaps because he did not like the other two approaches. He said, "Yeah, I chose this one, because I like the whole logical following it, the other ones were just like consider this, consider that, it's because of this, it's that, therefore it's 90°." When informed that P3A3 was considered to be "beautiful" by the panel of experts, he was rather indifferent. The way he viewed P3A3 simply appeared as a way to show the existence of a rectangle. Unfortunately, he was not able to make a direct connection between this proof and the alleged angle measure.

Overall, Student 5 performed relatively well. He found this problem solving experience very interesting as he had no experience being assessed to supply as many different approaches as possible to solve one problem. During the test, he worked somewhat systematically, producing various approaches until successful, then moving on to the next problem, instead of looking for additional approaches for the one he successfully solved. Evidently, he was used to taking tests, the goals of which were simply to solve the problems. In fact, he explained that he felt a little uncomfortable with the instruction to seek many different ways to solve the problems. He said:

> Yeah, I'm more of a structured person, I like to think that like, oh, I got the answer definitely right, and knowing that there are multiple

> approaches to the problem, especially if you're like, this might be wrong, oh no, this might be wrong.

On the other hand, he understood some of the benefits of problem solving with many different ways, such as "checking my work type of thing." Moreover, his aesthetic feelings were mostly expressed in approaches that were not only unusually different than the ones he normally encountered, but also reasonably understandable by means of his most recent mathematics knowledge. In general, he did not seem to show that much interest in the aesthetic aspect of problem solving approaches.

Student 6

Student 6 was a senior planning to continue his academic career in college but had not yet decided on his major course of study. He reported a perfect score of 800 on both SAT Mathematics Section and SAT Subject Test—Mathematics Level II. His favorite mathematics topic was calculus, but he supplied no clear reasons for why. His overall acceptability score on the test was 1 for a correct answer in Problem 3. He began the test with Problem 3. In the interview, he said that he was more comfortable with that problem than with the other two problems. He produced three approaches, all of which were similar to Approach 1 (P3A1). His first attempt was very much an algebraic variation of P3A1. The next two attempts were very similar to his first approach. He acknowledged this resemblance during the interview. When shown four approaches from the collection, he chose P3A3 as his preferred approach. He seemed attracted to the uncommon geometrical relationship between parallelogram and triangle. He explained that the construction of a parallelogram, unlike that of a circle, was considered to be very closely related to the given triangle in the problem. He said:

> For this one, I chose P3A3 as my favorite because when I looked at the three approaches, well, I only chose it because it's because I wouldn't think of that during the test, it's sort of, something about it really, like I would normally just create the triangle and use that triangle, I mean the angle and like, but the fact that they solved it by creating a different shape, rather than just using a triangle, I found that interesting.

Problem 1 did not interest him that much. He produced only one approach, which made use of approximations to the square roots, but it was considered to be unsuccessful. Among the four approaches from the collection, P1A1 and P1A3 did not attract much of his attention because of his lack of understanding of the reasoning involved. With regard to P1A2 and P1A4, Student 6 said that he had the knowledge necessary to perform those approaches except that he did not think about them during the test. In the beginning, he appeared somehow inconsistent with his explanations for the two latter approaches. He indicated that he would have solved the problem using P1A2 in his first attempt. As he explained, "I use Approach 2 often in solving problems currently in my calculus class so I would be comfortable using this method." However, he chose P1A4 as his preferred approach because "this approach is the most straightforward and I would just be relying on my arithmetic skills." When asked about the seemingly reversed explanations for his first attempt and his preferred approach, he discussed his risk tolerance for the approaches. On one hand, he would have started off the problem using P1A2 because P1A4 was thought to require more arithmetic multiplications and therefore P1A2 would have taken less time on the test. On the other hand, he preferred P1A4 because P1A2 demanded more careful thinking to multiply the binomial conjugates and hence P1A4 was considered to be more error free. Clearly, his choices in his first attempt and his preferred approach were rationalized based on his risk aversion to the other choice instead of his aesthetic inclination to that particular choice.

In the end, he mentioned the originality of P1A4. He said, “I mean [P1A4] is simpler, I’ve never seen that before.” He then went on to make an important comment to position his view on the benefits of problem solving in many different ways, saying that “if I couldn’t solve the problem with [P1A2], I would resort to [P1A4].” Still, he was not able to recognize the aesthetic value of P1A3, even after informed that P1A3 was considered to be “beautiful” by the panel of experts. His inability to appreciate P1A3 was mostly connected with his lack of understanding the mathematical knowledge involved in that approach. At the same time, he indicated that the “beauty” in P1A3 might be perceived by the brevity of the literal lines in its proof. He said, “Well, I’m not confident with graphing, I think, I mean, maybe it’s because it’s short, like the proofs here, I’m not sure.”

He was not as successful in solving Problem 2 as he was in Problem 1. He provided two approaches, which were similar but still considered to be unsuccessful. When shown eight approaches from the collection, he chose Approach 4 (P2A4) as his first attempt and his preferred approach. His rationale for his choices primarily hinted at his most recent mathematics learning experience in an AP Calculus course. When informed that P2A2 was considered to be “beautiful” by the panel of experts, he seemed a little indifferent. His explanations were limited to a comparison of that approach with the others for its noticeable graphic appearance. He said, “Well, that’s the only approach that uses graphs and creates like triangles, and tangents, the other ones are just formulas.”

In general, Student 6 did not do as well as his peers. Despite his performance, he contributed several meaningful accounts describing his thinking process in choosing first attempts and favorite approaches. He occasionally indicated his choices by comparing and contrasting the advantages and disadvantages of other choices. In some instances, his lack of mathematical knowledge impeded his appreciation of mathematical “beauty,” as his descriptions of preferred approaches went only as far as a concrete presentation of the approaches.

Student 7

Student 7 was the only eleventh grader who took the test. He did not document any plan to pursue a postsecondary school career. Because he had not taken any national standardized test, neither an SAT Mathematics Section score nor an SAT Subject Test—Mathematics Level II score was reported. His mathematics curriculum was similar to that of the other students in that particular specialized high school, except that he accelerated it by taking two or three classes per academic semester. He indicated calculus as his favorite mathematics topic because he recognized the power of the tools of calculus for solving problems with minimal effort. His overall acceptability score was 2 for correct answers in Problems 1 and 3. He began with Problem 1. His work showed a very well-written proof of Approach 4 (P1A4) and was considered to be successful. When shown four approaches from the collection, he recognized similarities between P1A4 and his work. In replying to the question of what he thought of P1A4 compared with the other three approaches, he stated that with P1A4, “You just kind of hacked away at it.” He explained, “Okay, I mean like you’re taking the numbers, right? And you’re playing with them, and you do this big multiplication, and you finally get this large number is bigger than that large number.” When informed that P1A3 was considered to be “beautiful” by the panel of experts, he agreed. He said, “I like [P1A3] just because it’s not like plugging away at the numbers, you know, like you’re not working or doing the manipulations, you’re looking at something like in the nature of the square root function to solve the problem.” He was not as successful in Problem 2 as he was in Problem 1. He only produced one approach, which was considered to be unsuccessful. When provided with eight approaches from the collection, he indicated that he would have solved Problem 2 using Approach 4 (P2A4) if he had to do it all over again. He also chose P2A4 as his preferred approach. His rationale for this choice had to do with his recent AP Calculus course. In responding to the question of whether he thought any one of the four approaches was more “beautiful” than the others, he said, “I’m not sure. Nothing strikes me particularly as beautiful.” Furthermore, he could not find any aesthetic appeal in P2A2, even after informed that it was considered to be “beautiful” by the panel of experts.

Student 7 then continued with Problem 3. He provided one approach, which was similar to Approach 1 (P3A1) and was considered to be successful. Unlike six other students, his written work in P3A1 involved more geometrical reasoning than algebraic representations. When presented with three approaches from the collection, he chose P3A3 as his preferred approach. His justification was essentially his dislike of the other approach. He said, “I don’t know, well, [P3A1] seems like what we’ve been describing as ugly, brute force.” Nevertheless, he thought that constructing extra lines to make up a new geometrical figure did not contribute any “beauty” to the problem as it did already in P3A1. More specifically, he considered this extension to be “artificial” to the extent that it “negated” something that was perhaps already more “beautiful” beforehand.

Overall, Student 7 did well on the test. His problem solving experience was simpleminded because he stopped producing approaches once he successfully solved each problem. In the interview, he recognized that problem solving using many different approaches made it possible for him to validate results between approaches, "to check the answer from the previous approach, to see if the other approach is valid." He also appeared to have a sense of a "beautiful" approach in that he considered it only a means to make a relative comparison to an "ugly" approach.

Student 8

Student 8 was a senior planning to continue her education in college, yet had not declared her major. She recorded an SAT Mathematics Section score of 740 and an SAT Subject Test—Mathematics Level II score of 770. Her favorite mathematics topic was algebra because of its relative ease, compared with other mathematics topics. Her acceptability score was 1 as she successfully solved only Problem 1. She began the test with Problem 1. Her written work showed similarities with Approach 4 (P1A4) and was considered to be successful. When shown four approaches from the collection, she chose P1A4 as her preferred approach. She said, "I thought I was more comfortable with adding and multiplying and I thought the others were more abstract." When informed that P1A3 was considered to be "beautiful" by the panel of experts, she viewed it as a consequence of applying more advanced mathematical knowledge. She said:

> I guess they might think that can be shown the other ways, it's something that, it's a method that's more efficient of solving it, it shows a higher level of understanding it, because you're using calculus concept rather than algebra concept, more sophisticated knowledge.

In the end, she affirmed her conviction in choosing P1A4 as her preferred approach. Her explanation established a clear relationship between her preferred approach and her confidence in her mathematical knowledge. She said that "because I know the algebra, I think I've learned this math, long time ago, it's something that's repeating in calculus, pre-calculus and so on, and a lot more comfortable with this than graphing."

Moreover, she was not successful in Problem 2. Using only one approach, she made some inappropriate approximations. When presented with eight approaches from the collection, she realized that she could have solved the problem using the calculus Approach 4 (P2A4). She chose P2A4 as her preferred approach and relied on her then-AP Calculus course to support her explanation. She said, "I would most likely choose Approach 4 because it involves derivatives, which is a topic that I studied very recently. As a result, that knowledge is most accessible to me." P2A2 did not attract her attention, and she was not able to comprehend why P2A2 was considered to be "beautiful" by the panel of experts.

Student 8 then continued with Problem 3. She successfully solved this problem only after her second attempt because her first one was not as effective. The two approaches she supplied were similar to Approach 1 (P3A1). When presented with three approaches from the collection, she repeated her rationale for choosing her preferred approach, just as she did in Problem 1. She preferred P3A1 because "it's less abstract." Her confidence in her mathematical skills prevented her from moving away from this particular selection. She said, "I don't think [P3A1] is the most beautiful necessarily, I just think that's what I would do over again, I wouldn't, I don't have the, I think, knowledge has to do with it maybe." When informed that P3A3 was considered to be "beautiful" by the panel of experts, she made a geometrical comparison of the three approaches. She said, "I like the shape of parallelogram, this reminds me of a diamond, like more of that shape and I would, I mean that would be more aesthetically pleasing than a circle especially with a triangle in the middle." She also mentioned a sense of harmony in P3A3, saying that "there are parallel lines, I think can be aesthetically pleasing, I think has to do with uniformity." In the end, she chose P3A3 as her preferred approach.

In general, Student 8 did satisfactorily on the test. She also provided adequate understanding and reasoning in terms of aesthetic value.

Student 9

Student 9 was a senior planning to major in Computer Science for her undergraduate study. She reported an SAT Mathematics Section score of 770, an SAT Subject Test—Mathematics Level I score of 750, and an SAT Subject Test—Mathematics Level II score of 770. She reported that she liked algebra because it was easy. Her acceptability score was 2 for correct answers in Problems 1 and 3. She started the test with Problem 1. She produced three approaches, all of which were similar to Approach 4 (P1A4). She obtained the correct answer right on her first attempt. Despite this correct answer, she continued with a second attempt on the problem. In her second attempt, she applied some derivatives to some square root functions. It was not as successful as the first one. She then continued with her third attempt, yet it was just another reworking of the first one. In the interview, she described the thinking process of her problem solving experience as instinctive. That is, she was eager to find the answer as quickly as possible in the first place. She said, "So for [P1A4], when you gave me that problem, thing is, because especially it's a square root, your first instinct is like, making them into integers, like squaring." When presented with four approaches from the collection, she said that P1A1 was the most efficient of the four approaches, and thus she chose it as her preferred approach. P1A4 was considered to be very time-consuming in her opinion. She said: “If you asked me to find the most efficient way to do it, then this P1A1 would be better, I think because you can clearly see that this is going to be 1 less, so this is obviously less than that.” Despite the brevity of P1A1, she realized that the time involved in thinking about P1A1 would probably be much longer than the time to work out the calculation involved in P1A4. P1A4 would still have been her first attempt even if she had to do it all over again because:

> You just want to get answer anyway, I guess, when you’re thinking to solve a problem, you don’t like consider many different ways, and think like, oh, this is the most efficient way, let me do it that way, you just think of a way to do it, and then you do it that way.

When asked which among the four, she considered to be “beautiful,” she chose P1A3. Her explanations centered on the outer appearance of the problem nonetheless. She said:

> Well, I kind of think it’s this one [P1A3] just cause it doesn’t have so many, like here [P1A2], it goes down, it has so many radicals, but, I can’t think of the right word, but this one is kind of like a sentence, it invokes, kind of neater approach, I think.

In response to the aesthetic evaluation of the panel of experts, she added that P1A3 “laid down in a very straightforward, a very matter of fact kind of way, and it’s like very short and simple.” In the end, she reiterated her preferred approach of P1A1.

Student 9 continued with Problem 2. She worked out one approach, which was considered to be unsuccessful. When presented with eight approaches from the collection, she chose Approach 2 (P2A2) as her preferred approach. Her rationale for this choice had to do with how she related her confidence in her mathematical skills and how she viewed the written-out reasoning. She said, “Well, I just thought that’ll be the best way to do it and I like that way…. I am most familiar with its concepts and it seems to be the clearest way to solve it." In some sense, she associated P2A2 with a geometric approach and P2A4 with a calculus approach. The equation $x^2 + y^2 = 1$ immediately caused her to draw the geometrical shape of a circle, she said. As for her then-AP Calculus course and P2A4, she said that she felt more confident in her mathematics skills needed in P2A2. She also thought that the knowledge involved in P2A4 was too recent to be devised at that moment since she had not had as much experience with it as she did with P2A2. Likewise, she believed that her favorite mathematics topic, algebra, was irrelevant to her choice of first attempt. She was more focused on finding the answer to the problem, but paid little attention to different possibilities of presenting the answer to the problem. In some sense, she was very used to the usual mathematics tests in which the goal was simply to solve the given problems. She said:

> I don't really think that way, I mean, I like algebra, but I don't, I won't do algebraic approach all the time for all kinds of problems, like this calculus problem, I think it's more like, when you see a problem, you just try to solve it the way any kind of way that's easiest or comes to you first, I think it depends a lot on a problem, because some problems you can solve in multiple ways, different ways to solve it, and then whereas some problems are strictly calculus, if they say specifically find the derivative of this, then you would think of calculus, you wouldn't think of, like oh, I may solve this in algebraic way.

To some extent, she was able to see some aesthetic appeal in P2A2, but it related more to the external attraction rather than to the proof itself. The figure in P2A2 seemed to attract her interest very much. She said:

> I think just the fact that you can graph it and see what you're doing physically, as opposed to like more theoretical proof. Well, if you don't have the graph, then it would be more, it would be more kind of, more jumbled up, or more confusing because you'll have a lot of text and you read through, and you won't be able to have something to attach yourself to, like round yourself, like oh, angle ABO, you're like, I don't know what that is.

Student 9 then continued with Problem 3. She furnished only one approach yet a successful one, which was categorized as Approach 1 (P3A1). When presented with three approaches from the collection, she chose P3A1 as her preferred approach because of its clear logical flow. When informed that P3A3 was considered to be “beautiful” by the panel of experts, she disagreed. She did not find the line extensions in P3A3 particularly appealing. She said, “I feel like constructing two parallel lines makes it more, just like more work that what it’s needed.”

Overall, Student 9 performed satisfactorily. Moreover, she did not feel comfortable with the notion of problem solving in many different approaches. Her test experience during most of her academic career somehow supported her obsession to find only the answer of the problems. Consequently, she could not identify considerable aesthetic appreciation for the problem solving approaches as she was only concerned with those approaches which she found more understandable.

Chapter VI: FINDINGS FROM PHASE 5: STUDENTS' VALIDATION SURVEY

This chapter describes the mathematical problem solving behavior of the 54 students who took the validation survey in Phase 5. It includes two parts. The first part investigates the students' hypothetical problem solving experiences, and the second part considers their attitudes towards problem solving using many different approaches.

Students' Problem Solving Experiences

This section presents the findings from the first part of Phase 5. It describes the relationship between students' prior knowledge and understanding of approaches for the three problems in the test. It also connects these experiences with the likelihood that the students would supply those approaches in a hypothetical test taking experience as well as their choices of first and preferred approaches. Figure 4 presents the first summary of the findings from the first part of Phase 5. (Note: The clustered column chart compares values across categories of approaches from Appendix J. The vertical axis represents the number of students for each category. The horizontal axis represents a score following the rubrics explained in Chapter III: Methodology.)

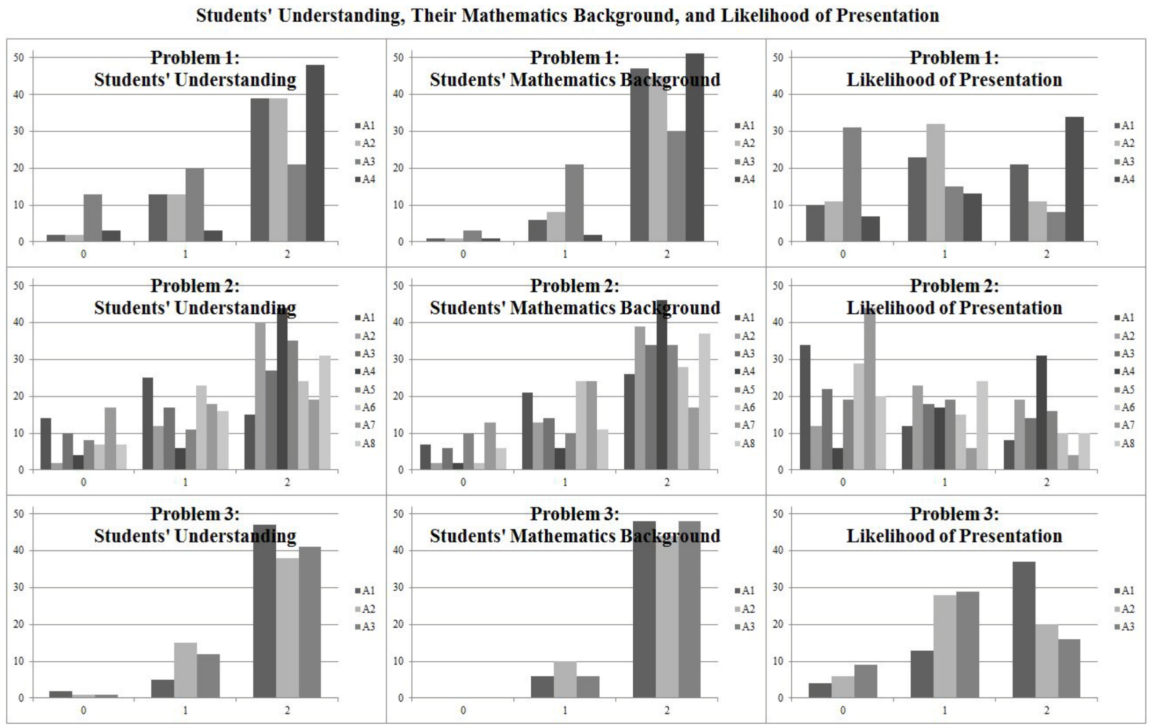

Figure 4. First Summary of the Findings from the First Part of Phase 5

Regardless of their relatively adequate understanding and knowledge of reasoning involving all other approaches in Problems 1 and 3, the students reported that they would use P1A4 and P3A1, respectively. In particular, the first two charts in the last row of Figure 4 indicates that nearly all students assessed their own understanding and mathematics background with respect to all three approaches for Problem 3 with a score of 2 (i.e., students understood all of the steps and had learned previously all of the necessary mathematics knowledge involved in those three approaches). The following chart of the corresponding likelihood of presentation reveals that the number of students assessing their likelihood of presenting P3A1 with a score of 2 was significantly greater than the numbers of students assessing their likelihood of presenting P3A2 and P3A3 with a score of 2 (i.e., P3A1 was more significantly than P3A2 and P3A3 to be chosen by students as their likely approach in solving Problem 3). Regarding Problem 2, the enrollment of all 54 students in an AP Calculus course at the time of the study might have been an important factor in their likelihood of presenting the calculus approach to P2A4. This observation was also found in many responses from the students' validation survey. One student wrote:

> I chose [P2A4] because I have most recently learned that so it's the freshest in my mind. They are related to what I have learned recently in calculus. As a result, that knowledge is most accessible to me. If given this test now, I would most likely resort to [P2A4].

Therefore, the findings suggest some association between students' understanding and mathematics background and the likelihood that students would supply a particular approach.

The next three smaller tables in Table 4 provide the relationship between students' order of presenting an approach, their first approach, their preferred approach, and their favorite mathematics topic. The highlighted areas in the tables are the modes, the most frequently appearing statistic for a particular data set. For example, P1A4 was the predominant first approach for Problem 1 since 24 students reported that they would supply P1A4 as their first approach, compared to 16, 6, and 8 other students who would supply P1A1, P1A2, and P1A3 as their first approach, respectively. Likewise, P1A4 was the predominant first approach for Problem 1 among students who chose algebra as their favorite mathematics topic.

Table 4

Second Summary of the Findings from the First Part of Phase 5

Students' Order of Presenting Approach and Their Preferred Approach

Order/Preferred Approach	Problem 1				Problem 2								Problem 3		
	A1	A2	A3	A4	A1	A2	A3	A4	A5	A6	A7	A8	A1	A2	A3
First Approach	16	6	8	24	3	14	5	28	2	1	0	1	35	12	7
Second Approach	22	13	4	15	4	9	6	10	13	2	1	9	15	25	14
Third Approach	12	26	9	7	9	14	8	9	7	4	1	2	4	17	33
Preferred Approach	14	10	14	16	4	20	6	20	2	0	0	2	29	15	10

Students' Choice of First Approach and Their Favorite Mathematics Topic

Favorite Mathematics Topic	Problem 1				Problem 2								Problem 3		
	A1	A2	A3	A4	A1	A2	A3	A4	A5	A6	A7	A8	A1	A2	A3
Algebra	4	1	2	9	1	5	1	8	1	0	0	0	10	3	3
Calculus	9	4	4	6	1	5	3	14	0	0	0	0	15	6	2
Geometry	2	1	2	4	0	3	1	4	0	0	0	1	7	2	0
Other	1	0	0	5	1	1	0	2	1	1	0	0	3	1	2

Students' Choice of Preferred Approach and Their Favorite Mathematics Topic

Favorite Mathematics Topic	Problem 1				Problem 2								Problem 3		
	A1	A2	A3	A4	A1	A2	A3	A4	A5	A6	A7	A8	A1	A2	A3
Algebra	4	3	4	5	0	8	1	7	0	0	0	0	8	3	5
Calculus	7	3	8	5	3	7	4	9	0	0	0	0	12	7	4
Geometry	1	3	1	4	0	2	2	4	0	0	0	1	6	3	0
Other	2	1	1	2	1	3	0	0	2	0	0	0	3	2	1

According to the experts' evaluations, P1A4, P2A4, and P3A1 were considered to be the least original, the least abstract, and the most mechanistic, and often associated with approaches taught by secondary school mathematics teachers. The three approaches thus could be considered ones that required the least mathematical thinking. According to the students' explanations, less thinking simply meant more straightforward, more familiar, easier, and requiring only the most basic knowledge of mathematics. The first smaller table in Table 4 demonstrated a connection between students' choice of first approaches and the level of mathematical thinking involved in those approaches. Compared to Problems 2 and 3, Problem 1 delivered a clearer confirmation of this general pattern. Twenty-four out of 54 students (44%) declared that they would supply P1A4 as their first approach. Sixteen others (30%) said they would supply P1A1 as their first approach. Obviously, both P1A4 and P1A1 were prominent in students' mind at the outset by taking the majority of the votes (74%). Unlike P1A1, P1A4 entailed less mathematics sophistication; if anything, it obligated mundane calculation. Students recognized the lengthy time involvement for calculating 2009×2011 and 2010×2010. Still, most students preferred P1A4 to P1A1 as their first approach. They were convinced that their labor in machine-like computation would prove more fruitful in the end. In a way, they unconsciously built up their persistence in getting the correct answer to compensate for their lack of advanced mathematics knowledge and, consequently, their lack of willingness to seek a more efficient approach. They did so even by sacrificing their time. One student wrote:

> I would think the brute force approach [P1A4] would bring me an answer first. Then I would try other approaches. [P1A4] is the least effective method, but requires almost no thinking. [P1A4] is the most straightforward way to solve the problem.

Therefore, the findings reveal a general pattern in which the students' order of presenting their first, second, and third approaches was more likely connected with the level of thinking demanded by those approaches. The more cognition an approach demanded, the more likely it would appear later in the order of presentation. The less thinking involved in an approach, the more likely students voted for it as their first approach.

The second smaller table in Table 4 demonstrates that students in general chose P1A4, P2A4, and P3A1 as their first approaches, regardless of whether they chose algebra, calculus, geometry or other as their favorite mathematics topic. This choice of first approach, however, was more connected with students' goal of getting a correct answer at once. Students' choice of a calculus approach for P2A4 was particularly evident, given that when students were posed with a more demanding problem, more recent mathematical experience (i.e., their AP Calculus course) dominated their favorite mathematics topic. This more recent mathematical experience created for the students such an instantaneous reaction in their mathematical thinking process that it facilitated greater assurance in their problem solving ability through a clearer understanding of the problem. One might infer from this observation that when sufficiently challenged in a problem, the students' favorite mathematics topic had little to do with their first approach. This was most probably because the students were very eager to obtain the answer to the problem as quickly as possible. In doing so, recent mathematical experience apparently played a more important role in selecting a more understandable and more familiar problem solving approach. Therefore, the findings suggest no direct relationship between students' choice of first approach and their favorite mathematics topic.

Finally, the findings demonstrate that students' choices of preferred approaches (i.e., P1A4, P2A4, and P3A1) were considered to be among the least preferred by the panel of experts. Students showed little aesthetic appreciation for "beautiful" approaches in their explanations of preferred approaches. Although P2A2 and P2A4 were voted equally preferred, those who chose P2A2 as their preferred approach were not able to offer adequate justification for its aesthetic value. Moreover, a few students found "beautiful" approaches to be rather objectionable. One student, for example, considered P2A2 to be "too showy." Another student said that P3A2 and P3A3 "try too hard to be unique and clever, when they are totally unnecessary."

Students' Attitudes towards Problem Solving Using Many Different Approaches

This section presents the findings from the second part of Phase 5. It describes students' attitudes towards problem solving using many different approaches. Figure 5 presents a summary of the findings from the second part of Phase 5. (Note: The column chart depicts students' responses for each statement [S1-S25] in the second part of Appendix N. The vertical axis represents the number of students. The horizontal axis represents a score on a five-point scale: 5, 4, 3, 2, 1 for strongly agree, agree, neither agree nor disagree, disagree, and strongly disagree, respectively. Each statement is first described or rephrased in the analysis, and then accompanied by its mean and standard deviation. For example, students neither agreed nor disagree with Statement 1, "I learn more from solving one problem in many different ways than I can from solving many different problems, each in only one way," and S1-3.39-1.07 indicates that students assessed Statement 1 with an average score of 3.39 and a standard deviation of 1.07.)

The first seven statements in the list (see Appendix N) were designed to investigate whether students agreed with the benefits of problem solving with many different approaches. The findings show that students were generally able to recognize these benefits. More specifically, they understood that problem solving using many different approaches: 1) provided greater access to a range of representations and solution strategies in a particular instance that could also be useful in future problem solving encounters (S2-4.03-0.79); 2) facilitated connection of a problem at hand to different elements of knowledge with which they might be familiar, thereby strengthening networks of related ideas (S3-4.04-0.75); 3) improved their critical thinking skills by

Students' Attitudes towards Problem Solving Using Many Different Approaches

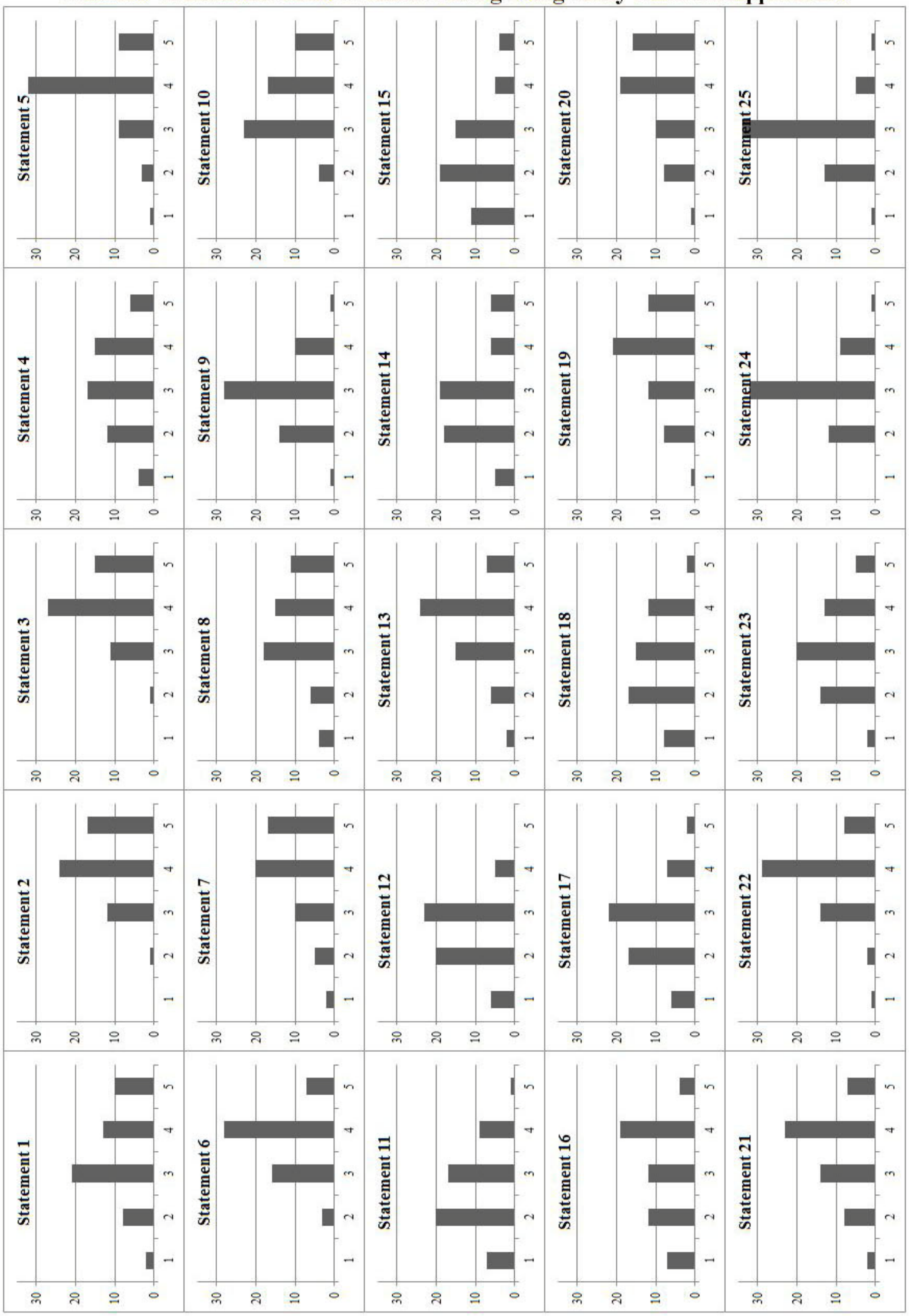

Figure 5. Summary of the Findings from the Second Part of Phase 5

comparing and contrasting the many different ways (S5-3.83-0.84 and S21-3.46-1.02); 4) improved deeper understanding of subject matters by looking at different perspectives (S6-3.72-0.76); and 5) provided them with backup strategies when they could not recall a typical solution to the problem (S7-3.83-1.09).

Students, however, were generally indifferent about the first and fourth statements. While recognizing those benefits, students still believed they would have learned just as well through the experience of problem solving using many different approaches (S1-3.39-1.07). One might also infer from the students' view that the benefits mentioned earlier were still attainable even without the presentation of more than one approach in solving a problem. On the issue of creativity, students appeared somewhat unenthusiastic about coming up with more than one approach to solve a problem. They did not comprehend that such training could prepare them to become more creative problem solvers in the future (S4-3.13-1.12).

The other 18 statements in the list were designed to analyze the students' problem solving experience in a classroom setting. Most students neither agreed nor disagreed when it came to teachers' involvement in assisting them to solve with many different approaches (S8-3.43-1.16). They were nevertheless indifferent in their beliefs that they were capable of coming up with many different approaches to solving a problem (S9-2.93-0.77). In a way, one could determine that they were noticeably apathetic about the idea of embracing problem solving with many different approaches in their classroom practice. Most students indeed were indifferent when thinking of more than one approach to solving a problem (S23-3.09-1.01). One explanation might be that such an experience had already been incorporated into their day-to-day classroom practice. Indeed, based on most students' responses, they did not feel that their mathematics teachers had the time to facilitate students who initiated discussion about solving one problem with many different approaches (S11-2.57-0.98). Still, most students were more in favor of agreeing that their teachers often encouraged them to solve a problem using many different approaches (S10-3.61-0.88). Likewise, a large number of students leaned towards disagreeing with the misconception that they often felt hesitant to share their ways of solving a problem that differed from those that their teachers demonstrated on the board (S12-2.50-0.82).

Furthermore, students valued the advantage of learning from incorrect approaches. Most agreed that flawed approaches would help them avoid making similar mistakes in their future problem solving endeavors (S13-3.52-0.99). More students also tended to oppose the concerns that they would become distracted (S14-2.81-1.12), confused (S15-2.48-1.14) or bored (S17-2.67-0.97) by the many different approaches. Moreover, they did not feel worried about struggling to understand just one approach, even when seeing a new problem for the first time (S18-2.69-1.10). Then again, the students themselves appeared to demand a clearer sense of purpose for why they needed to learn problem solving in many different approaches. Most of them associated this uncertainty with the reality that no standardized test in a school, state or national context involved a particular testing instruction to solve with many different approaches (S16-3.02-1.19). Because of this uncertainty, most students often became unmotivated, agreeing that they would often care to use only one approach as long as they could solve the problem (S19-3.65-1.05). Even when many different approaches were presented to them, they agreed that they would often prefer using only those approaches that were the easiest for them to understand (S20-3.76-1.10). But they agreed that being able to choose one or some approaches helped them better understand the context of a problem (S22-3.76-0.82). Still, the majority of the students were indifferent about wanting either: 1) more occasions where teachers demonstrated problem solving using many different approaches in a classroom setting (S24-2.98-0.69), or 2) more opportunities for themselves to practice problem solving with many different approaches (S25-2.85-0.68).

The findings reveal that students for the most part were able to appreciate the values of problem solving using many different approaches as recommended by researchers in the field of mathematics education and cognitive psychology. More importantly, they managed to a certain extent to dismiss their classroom teachers' constraints and concerns about the practice of problem solving using many different approaches. Nevertheless, students appeared rather unenthusiastic about embracing the idea of problem solving using many different approaches in an actual classroom practice.

Chapter VII: SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

Summary and Conclusions

This study sought to analyze the mathematical problem solving behavior of mathematically gifted students at the high school level. In particular, it explored influences on choice of approach, with an emphasis on the aesthetic value. Fifty-four students from a specialized high school in New York City participated in this research. The study began with collecting students' past mathematical experiences by means of a preliminary survey. Nine of the 54 students took a test requiring them to solve three non-standard mathematics problems using many different approaches. A panel of three research mathematicians was consulted to evaluate the aesthetic value of mathematical reasoning in those approaches. The nine students were then interviewed. All 54 students also took a validation survey to support inferences made from the problem solving behavior of the nine students. In general, the current study revealed that students were not very used to the practice of a specific fourth step of Polya's (1945) problem solving process, namely, looking back to find alternative approaches to solving the same problem. Indeed, students generally chose to supply workable, yet mechanistic approaches as long as they obtained a correct answer, but were not successful in looking back to find other approaches afterwards.

This section describes the main findings of the current study in connection with the research questions proposed earlier. Answers to the first and second research questions were based only on the work of the nine students who took the test. This fact excluded the possibility of using statistical methodology. The results, therefore, should be viewed as suggesting some conclusions rather than clearly proving them.

Research question 1: *How do gifted students' past mathematical experiences affect the number of approaches used by them when solving non-standard mathematics problems?*

The nine students, whose test performances were analyzed, were recruited from the same specialized high school. The school utilizes a uniform mathematics curriculum, which makes the analysis of differences in past mathematical experiences of the nine students fairly limited. It would be possible, however, to categorize these nine students into two "groups" of unequal size based on those past mathematical experiences: 1) the first "group" consists of Student 2, and 2) the second "group" consists of the other eight students. Indeed, unlike the eight students in the second "group," Student 2 had a greater past mathematical experience. First, unlike the eight students, he had taken the AMC-12 test. Second, unlike the eight students, he was a member of the mathematics team in that particular specialized high school. Not surprisingly, he simply had better mathematical knowledge. He himself mentioned in the interview that he had seen a mathematical fact similar to Problem 3.

Moreover, the test results of Student 2 differed substantially from those of the eight students. He was the only student who was able to supply two different and successful approaches for Problem 3, whereas none of the eight students were able to supply more than one successful approach for that problem. He was also the only student to solve all three problems successfully, whereas none of the eight students were able to solve more than two problems successfully.

This finding suggests that students' past mathematical experiences, to some extent, are connected with the number of approaches they use when solving non-standard mathematics problems (nevertheless, further study is necessary to explore whether, indeed, the number of courses, tests, and so on was directly responsible for the better test performance in finding different approaches for problem solving, or whether both were influenced by some other characteristic, for example, abilities.)

Research question 2: *How are gifted students' past mathematical experiences connected with the mathematics topics involved in their solutions?*

The findings in the current study are unique to each case. As mentioned earlier, Student 2 had a relatively greater past mathematical experience than the other eight students who took the test. As a result, Student 2 chose to solve Problem 1 using an algebraic-like Approach 1 (P1A1), while five students (from the second "group" of eight students) solved it using an arithmetic-like Approach 4 (P1A4). Student 2's mastery of algebra appeared to have placed him at a different level of conceptual understanding, compared with the other eight students.

Problem 2 could be analyzed differently. Student 2 chose to solve Problem 2 using a calculus-in-polar-coordinate Approach 8 (P2A8), while two students (from the second "group" of eight students) solved it using a regular calculus Approach 4 (P2A4). This difference could be explained by the fact that Student 2 had greater past mathematical experience (particularly in trigonometry, in addition to calculus) than the other eight students who took the test. Despite this, P2A8 and P2A4 were both calculus approaches in nature. Indeed, most students viewed Problem 2 as a calculus problem instead of an algebra problem, as it was intended. Given that all nine students were enrolled in an AP Calculus course, one could infer that the students' recent mathematics course played an important role in their decisions to use calculus approaches to solve a calculus-related problem.

Problem 3 could be analyzed primarily using the fact that Student 2 knew a mathematical fact similar to Problem 3. In solving Problem 3, Student 2 discussed how he simply attempted to prove a known fact instead of figuring out an answer to the problem. As a result, he chose to solve Problem 3 using a circle-construction Approach 2 (P3A2) in addition to a direct Approach 1 (P3A1), which was the only approach supplied by six students (from the second "group" of eight students). Then again, one could analyze the way in which these students solved Problem 3. Despite the mediocrity of the test results of the second "group" (of eight students) compared with the superiority of the test result of the first "group" (Student 2), there appeared to be some differences (insignificant but noticeable) in the way students presented P3A1. Specifically, Student 7 (who was the only eleventh grader taking the test) used a formal geometric type of proof in his reasoning, similar to Approach 1 (P3A1), whereas the other six students, including Student 2 from the first "group," used some algebraic notations to represent the angles of the triangle. Apparently, even though all nine students had taken a geometry course as part of their uniform mathematics curriculum, this finding suggests that this particular mathematics course, to some extent, was not clearly connected with the students' approaches to solving this particular problem.

Research question 3: *To what extent are gifted students' approaches considered "beautiful" by experts?*

The findings in the current study reveal that the gifted students' approaches were not considered to be the most "beautiful" by the panel of experts. In general, a typical student who took the test was only able to supply the least preferred approach according to the panel of experts. A typical student solved Problems 1, 2, and 3 using P1A4, P2A4, and P3A1, respectively (see Table 3). Likewise, the 54 students in the validation survey generally chose P1A4, P2A4, and P3A1 as their first approach in solving Problems 1, 2, and 3, respectively (see Table 4). According to experts' aesthetic evaluations, P1A4, P2A4, and P3A1 were scored 3.00, 3.33, and 3.00, and were ranked the fourth of four approaches, the sixth of eight, and the third of three, respectively (see Table 2). Apparently, the students' approaches were considered to be the least "beautiful" by the panel of experts and were often associated with standard approaches taught by mathematics teachers at the secondary school level.

Research question 4: *To what extent are experts' preferred approaches considered "beautiful" by gifted students?*

The findings in the current study show that the experts' preferred approaches were not considered to be "beautiful" by gifted students. When informed that P1A3, P2A2, and P3A3 were considered to be "beautiful" by the panel of experts, students in their interviews generally disagreed. Even those who agreed in the interviews, or those of the 54 students who chose them as their preferred approaches in the validation survey, were for the most part not able to provide adequate explanations for the aesthetic value of those approaches. They were only able to see the outward appearance of those "beautiful" approaches. For instance, P1A3 was considered to be "beautiful" because of the relatively shorter lines of argument, P2A2 was considered to be "beautiful" because of the presence of the graph accompanying the solution, and P3A3 was considered to be “beautiful” because of the physical shape of the parallelogram. Clearly, the experts’ preferred approaches did not appeal to the students as “beautiful” in the sense of the deeper structure of the mathematical arguments involved in those approaches, as discussed by the experts.

It can be concluded that this study confirmed the results of Dreyfus and Eisenberg (1986), who emphasized that students are not very interested in aesthetic values in problem solving and that they actually do not demonstrate these aesthetic values in their problem solving experiences. These findings seem to be important because both the methodology of the study and the group of subjects in this study were substantially different from those in the study by Dreyfus and Eisenberg (1986).

Limitations

This study has some limitations. First, the issue of giftedness itself can be interpreted in different ways. The subjects in this study were considered to be gifted to the extent that they were enrolled in a specialized high school. However, using a different conception or definition of giftedness, these subjects might not be considered to be gifted. For example, none of the 54 students was a winner in the International Mathematical Olympiad. It may be that by selecting another group of (more or less gifted) students, the researcher would obtain another result. Theoretically, one can hypothesize that the effect of the mathematical experience differs across different groups.

Second, the subjects in this study did not prove to be sufficiently diverse in their mathematical background. All students had more or less the same teachers and took the same courses and tests. As a consequence, it was problematic to identify differences in students' past mathematical experiences in a meaningful way, thereby decreasing the sample for comparing past experiences with the results of the problem solving sessions.

Third, the current study involved only a relatively small number of subjects. The goal of this study was to identify possible patterns and collect some evidence which could later be verified more rigorously. At this point, several studies in the development of the current one can be suggested.

Recommendations for Future Research

First, it would be of interest to conduct similar studies across a range of mathematical abilities, starting with a highly gifted population, such as winners of the International Mathematical Olympiad, and continuing with a population not characterized as gifted at all. Second, it would be of interest to have a study with a larger group which would permit the use of statistical methodology. It would also be helpful to explore a more diverse group in terms of the mathematical background; such a study also seems to be of interest. Finally, it would be of interest to conduct a study with another set of problems.

Recommendations for Classroom Practice

In addition to recommendations for research, some recommendations for classroom practice for teaching mathematically gifted students could be suggested based on this study. In particular, this study demonstrated that classroom practice paid little attention to the fourth step of Polya's (1945) problem solving process. Therefore, one could recommend presenting many different approaches for solving problems as well as discussing their advantages or disadvantages. Sinclair (2004) inferred, from the findings of Dreyfus and Eisenberg (1986) and Silver and Metzger (1989), that the process in which aesthetic became a motivation for mathematical problem solving "only gains momentum in and after graduate school, when young mathematicians are having to join the community of professional mathematicians–and when aesthetic considerations are recognized (unlike at high school and undergraduate levels)" (p. 276). Still, young children could and should have practice in adapting to and adopting the mathematical culture and its values, even if they will not ultimately join the community of professional mathematicians and even if they do not view their careers as directly connected with mathematics.

Furthermore, one could recommend that students be encouraged to contrive, on their own, the possibilities of novel approaches. This self-discovery learning approach could empower students to think beyond the norm skills-and-drills mathematical routines. The instrument which was used in the study for research purposes could also be used for the purpose of education: teachers may present many different approaches to finding the solution of the problem. Such a presentation would allow students to compare and contrast the approaches presented by the teachers. Given this frequent accumulation of different approaches, students could begin to grow their sense of mathematics aesthetic appreciation.

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APPENDIX A

Example of Connections Based on Similarities and Differences between Various Representations of the Same Concept (Leikin & Levav-Waynberg, 2007)

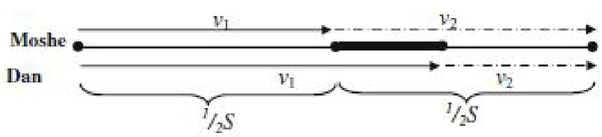
Problem 2: Dan and Moshe walk from the train station to the hotel. They start out at the same time. Dan walks half the time at speed v_1 and half the time at speed v_2. Moshe walks half way at speed v_1 and half way at speed v_2. Who gets to the hotel first: Dan or Moshe?

Without loss of generality assume $v_1 > v_2$

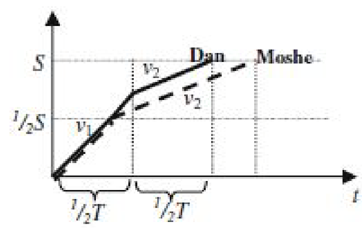
Solution 1: Logical (verbal) solution

If Dan walks half the time at speed v_1 and half the time at speed v_2 and $v_1 > v_2$ then during the first half of the time he walks a longer distance that during the second half of the time. Thus he walks at the faster speed v_1 a longer distance than Moshe. **Dan** gets to the hotel **first**.

Solution 2: Pictorial solution (1D): This solution is a pictorial representation of the logical solution.



Solution 3: Graphic solution (2D):



Solution 4: Algebraic solution:

Table:

	time	speed	way
Moshe Half way	$\frac{s}{2v_1}$	v_1	$\frac{s}{2}$
Half way	$\frac{s}{2v_2}$	v_2	$\frac{s}{2}$
Dan Half time	$\frac{y}{2}$	v_1	$v_1 \cdot \frac{y}{2}$
Half time	$\frac{y}{2}$	v_2	$v_2 \cdot \frac{y}{2}$

$x = \frac{s}{2v_1} + \frac{s}{2v_2} \Rightarrow x = \frac{s \cdot (v_1 + v_2)}{2v_1 \cdot v_2}$: x- time spent by Moshe

$s = v_1 \cdot \frac{y}{2} + v_2 \cdot \frac{y}{2} \Rightarrow y = \frac{2s}{(v_1 + v_2)}$: y – time spent by Dan

Let's compare $\frac{s \cdot (v_1 + v_2)}{2v_1 \cdot v_2}$ and $\frac{2s}{(v_1 + v_2)}$.

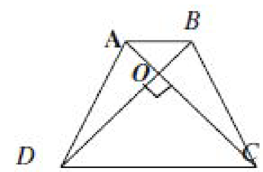
$$0 \le (v_1 - v_2)^2 \Leftrightarrow 4v_1 \cdot v_2 \le (v_1 + v_2)^2 \Leftrightarrow \frac{2s}{(v_1 + v_2)} \le \frac{s \cdot (v_1 + v_2)}{2v_1 \cdot v_2}$$

Dan gets to the hotel **first**.

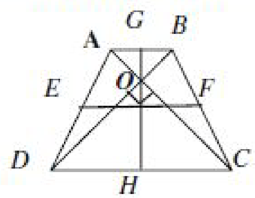
APPENDIX B

Example of Connections between Different Mathematical Concepts and Procedures (Leikin & Levav-Waynberg, 2007)

Problem 3: In an isosceles trapezoid ABCD the diagonals are perpendicular. Prove that the height of the trapezoid equals its midline.

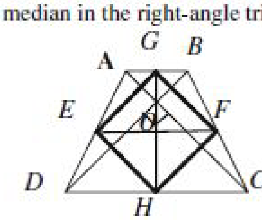


For all the following solutions we construct the height through O: the point of intersection of the two diagonals



Solution 1: Midline in a trapezoid theorem and right-angle isosceles triangle

Solution 2: Midline in a trapezoid theorem and median in the right-angle triangle



Solution 3: Four midlines in the quadrilateral theorem and diagonals in a square.

APPENDIX C

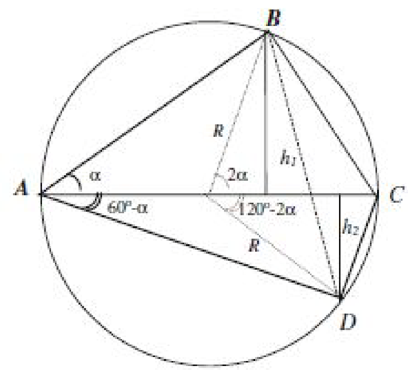
Example of Connections between Different Branches of Mathematics (Leikin & Levav-Waynberg, 2007)

Problem 5: Quadrilateral ABCD is inscribed in a circle. The diagonal AC is the diameter of the circle. Angle A is 60 degrees. We mark angle BAC as α. Which values of α give the maximum area of quadrilateral ABCD?

Observation for solutions 1, 2, and 4

$$S_{ABCD} = S_{ABC} + S_{ADC}$$
$$S_{ABCD} = R \cdot h_1 + R \cdot h_2 = R \cdot (h_1 + h_2)$$

S_{ABCD} is maximum when h_1+h_2 is maximum



Solution 1: Calculus-based solution of the minima-maxima problem

$$h_1 = R\sin 2\alpha; \quad h_2 = R\sin(120^\circ - 2\alpha)$$

$$h_1 + h_2 = R(\sin 2\alpha + \sin(120^\circ - 2\alpha)), \quad 0^\circ < \alpha < 60^\circ$$

$$(h_1 + h_2)' = 0 \Leftrightarrow 2\sqrt{3} \cdot R \cdot \sin(\frac{\pi}{3} - 2\alpha) = 0 \Rightarrow \alpha = \frac{\pi}{6} \Rightarrow \alpha = 30^\circ$$

Solution 2: Properties of trigonometric functions

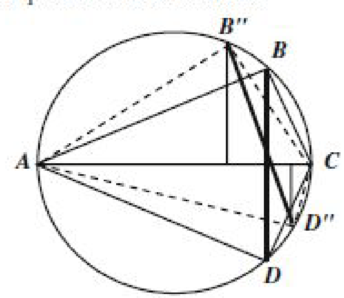
As in solution 1: $h_1 + h_2 = R\sqrt{3}\cos(60^\circ - 2\alpha), \quad 0 < \alpha < 60^\circ$

$\cos(60^\circ - 2\alpha) = 1$ is the maximum value of the function $\Rightarrow \alpha = 30^\circ$

Solution 3: Geometric solution (circle properties)

Angle *BAD*, a constant angle (60°), relies on the cord *BD*. Hence, the two diagonals of the quadrilateral are constant.

The area of such a quadrilateral is maximum when the diagonals are perpendicular, in other words, when the AC bisects angle *BAD*. Thus $\alpha = 30^\circ$



Solution 4: Symmetry considerations

As in solution 1: segment *BD* is constant for all values of α.

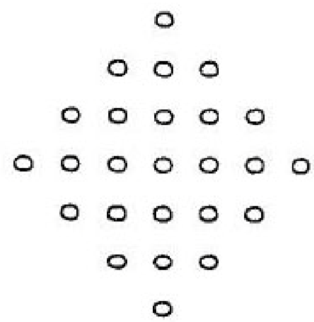
In a symmetric situation, BD is perpendicular to AC and the area of the quadrilateral equals $^1/_2$BD·AC.

If we break the symmetrical situation we decrease the sum of the heights and therefore reduce the area of the quadrilateral. Thus $\alpha = 30^\circ$

APPENDIX D

Marble Arrangement Problem
(Silver et al., 1995)

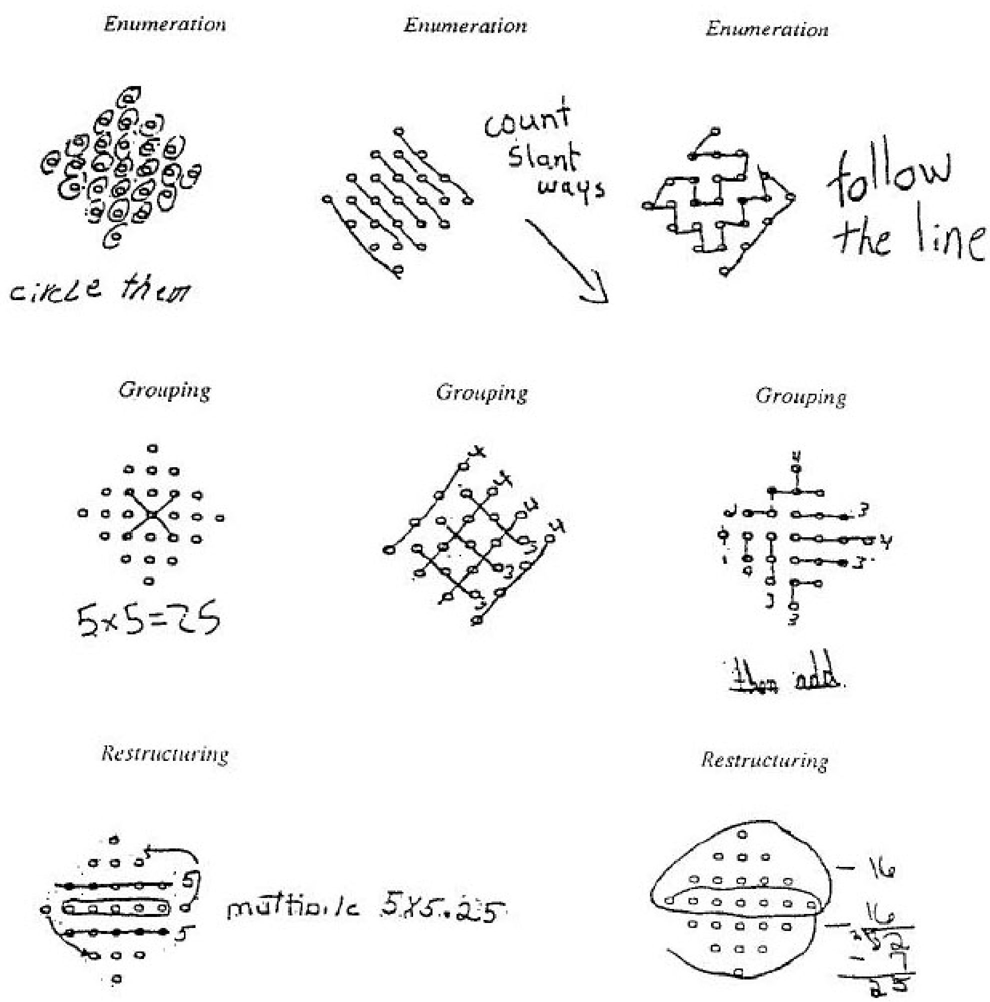
How many marbles are there in the picture below?



FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN. Write your ways of finding the answer and write your answer.

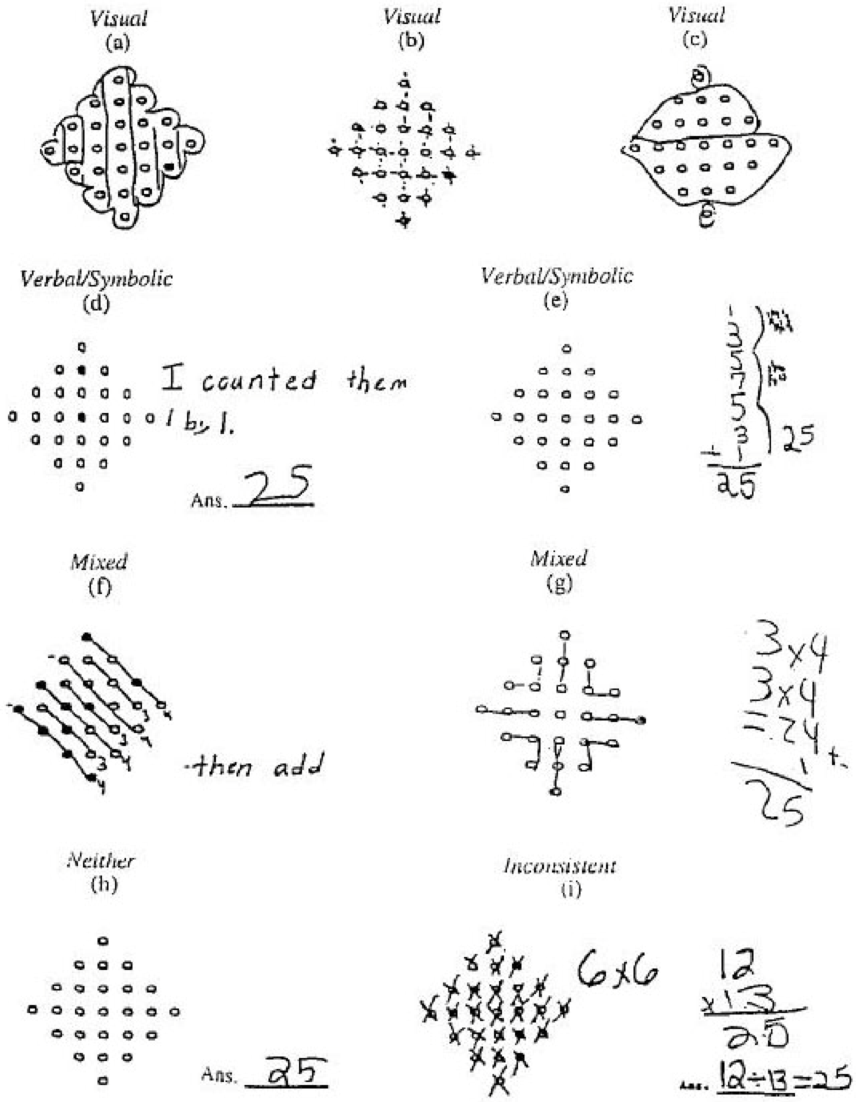
APPENDIX E

"Multiple Solution Strategies" for Marble Arrangement Problem
(Silver et al., 1995)



APPENDIX F

"Modes of Explanation" for Marble Arrangement Problem (Silver et al., 1995)



APPENDIX G

Students' Preliminary Survey

PRELIMINARY SURVEY

Name: ______________________________ Today's Date: ____________

School: ______________________________ Date of Birth: ____________

Grade Level: ______________________________ Gender: ____________

AMC 10: ______________________________ (e.g. 100 out of 150.)

AMC 12: ______________________________ (e.g. 100 out of 150.)

SAT – Math Section: ________________________ (e.g. 700 out of 800.)

SAT Subject Test – Math Level I: ____________ (e.g. 700 out of 800.)

SAT Subject Test – Math Level II: ____________ (e.g. 700 out of 800.)

Planned Undergraduate Major: ____________ (e.g. Undeclared.)

Favorite Mathematics Topic: (e.g. Geometry.)

I choose this Favorite Mathematics Topic for the following reasons: ____________

____

____

____

____

____

Please list all mathematics courses in the order in which they have been taken since eighth grade along with corresponding information such as grade level, school year, grade received, and school name.

	Grade Level	School Year	Mathematics Course	Grade Received	School
e.g. 1	8	Fall 2006	Algebra	91	ABC Middle School
e.g. 2	10	Fall 2008	Pre-calculus	92	DEF High School
e.g. 3	11	Summer 2010	Complex analysis	A-	GHI College
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					
11					
12					

APPENDIX H

Students' Test

INSTRUCTIONS

1. This test consists of three problems.
2. Please solve each problem in **as many different approaches as possible**. You may feel free to solve the problem in any order.
3. However, on the sheets provided, please mark the order in which you present each approach for each problem.
4. Please refer to Sample Problem and Sample Answer for explanations of instructions 2-3.
5. You have a maximum of 60 minutes to complete this test.
6. Write clearly. Do not scribble, erase, or discard any of your work that you feel is not part of your responses. Instead, you may cross them out lightly.

SAMPLE PROBLEM

Sample Problem 0. Solve: $7 \times 21 + 79 \times 7$

SAMPLE ANSWER

1st approach: $7 \times 21 + 79 \times 7 = 147 + 543 = 700$

2nd approach: $7 \times 21 + 79 \times 7 = 7 \times (21 + 79) = 7 \times 100 = 700$

TEST

Name: ________________________ Date: ________________

School: ________________________

1. Fill in the blank with one of the symbols $<, \leq, =, \geq$, or $>$.
   $\sqrt{2009} + \sqrt{2011}$ __________ $2\sqrt{2010}$
2. Given $x^2 + y^2 = 1$, find maximum of $x + y$.
3. Given triangle ABC with median $\overline{CD}$ and $CD = BD$, find measure angle ACB.

Reminder – Please avoid discussing questions involved in this test with other students. Thank you.

APPENDIX I

Examples of Students' Written Work for Problem 1 and Their Acceptability Scores

<table>
<tr><th>Examples of an Acceptability Score of 1</th><th>Examples of an Acceptability Score of 0</th></tr>
<tr><td>$\sqrt{2009} + \sqrt{2011}$ __ $2\sqrt{2010}$
$\sqrt{2009} + \sqrt{2011}$ __ $\sqrt{8040}$
$2009 + 2\sqrt{2009}\cdot\sqrt{2011} + 2011$ __ 8040
$2\sqrt{2009}\cdot\sqrt{2011} + 4020$ __ 8040
$2\sqrt{2009}\cdot\sqrt{2011}$ __ 4020
$\sqrt{2009}\cdot\sqrt{2011}$ __ 2010
$\sqrt{4040099}$ __ 2010
$4040099 \le 4040100$
$\sqrt{2009} + \sqrt{2011} < 2\sqrt{2010}$

2009×2011: 2009, 20090, 000000, 4018000, 4040099
2010×2010: 0000, 20100, 000000, 4020000, 4040100</td><td>$(\sqrt{2009} + \sqrt{2011}$ __ $2\sqrt{2010})$
$2009 + \sqrt{2009\cdot 2011} + 2011 \overset{?}{=} 4\cdot 2010$
$\sqrt{2009\cdot 2011} \overset{?}{=} 2\cdot 2010 = 4040$
$2009 \times 2011 = 404009$ $<$ 4040×4040: 161600, 16160000, 16321600
$\sqrt{404009} \overset{?}{=}$
$\sqrt{404009}$
2009×2011: 2009, 20090, 4018000</td></tr>
<tr><td>$(\sqrt{2009} + \sqrt{2011})^2$ __ $(2\sqrt{2010})^2$
$\dfrac{2009 + 2011 + 2\sqrt{2009}\sqrt{2011}}{2}$ __ $\dfrac{8040}{2}$ ($\cancel{2\sqrt{2009}\sqrt{2011}}$)
$2010 + \sqrt{2009}\sqrt{2011}$ __ 4020
$(\sqrt{4{,}040{,}099})^2$ __ $(2010)^2$
$4{,}040{,}099 < 4{,}040{,}100$

$2009 + 2011 = 4020$
$2009 \times 2011 = 4{,}040{,}099$
2010×2010: 20100, 4020000, $4{,}040{,}100$</td><td>$\sqrt{2009} + \sqrt{2011}$ __ $2\sqrt{2010}$
$\sqrt{2000} + \sqrt{2000}$
$\cancel{\sqrt{4000}}$
$\sqrt{100\cdot 20}$
$10\sqrt{20}$
$10\cdot 2\sqrt{5}$
$20\sqrt{5} + 20\sqrt{5}$ $2\sqrt{2010}$
$2(20\sqrt{5})$
$40\sqrt{5} < 40\cdot 2\sqrt{5}$
$15 \times 15 = 225$; $20 \times 20 = 400$
$\sqrt{2009} + \sqrt{2011} < 2\sqrt{2010}$</td></tr>
</table>

APPENDIX J

Collection of Approaches

P1A1: Approach 1 for Problem 1

$$\begin{array}{ccc}
\sqrt{2009}+\sqrt{2011} & ? & 2\sqrt{2010} \\
\left(\sqrt{2009}+\sqrt{2011}\right)^2 & ? & \left(2\sqrt{2010}\right)^2 \\
2009+2011+2\sqrt{2009}\sqrt{2011} & ? & 4\cdot 2010 \\
\sqrt{2009}\sqrt{2011} & ? & \sqrt{2010}\sqrt{2010} \\
\sqrt{(2010-1)}\sqrt{(2010+1)} & ? & \sqrt{2010}\sqrt{2010} \\
\sqrt{2010^2-1} & < & \sqrt{2010^2}
\end{array}$$

Therefore, $\sqrt{2009}+\sqrt{2011} < 2\sqrt{2010}$.

P1A2: Approach 2 for Problem 1

$$\begin{array}{ccc}
\sqrt{2009}+\sqrt{2011} & ? & 2\sqrt{2010} \\
\sqrt{2011}-\sqrt{2010} & ? & \sqrt{2010}-\sqrt{2009} \\
\sqrt{2011}-\sqrt{2010}\cdot\dfrac{\sqrt{2011}+\sqrt{2010}}{\sqrt{2011}+\sqrt{2010}} & ? & \sqrt{2010}-\sqrt{2009}\cdot\dfrac{\sqrt{2010}+\sqrt{2009}}{\sqrt{2010}+\sqrt{2009}} \\
\dfrac{2011-2010}{\sqrt{2011}+\sqrt{2010}} & ? & \dfrac{2010-2009}{\sqrt{2010}+\sqrt{2009}} \\
\dfrac{1}{\sqrt{2011}+\sqrt{2010}} & < & \dfrac{1}{\sqrt{2010}+\sqrt{2009}}
\end{array}$$

Therefore, $\sqrt{2009}+\sqrt{2011} < 2\sqrt{2010}$.

P1A3: Approach 3 for Problem 1

Because the graph of square root function is strictly concave, it follows that $\frac{\sqrt{2009}+\sqrt{2011}}{2} < \sqrt{2010}$. Therefore, $\sqrt{2009}+\sqrt{2011} < 2\sqrt{2010}$.

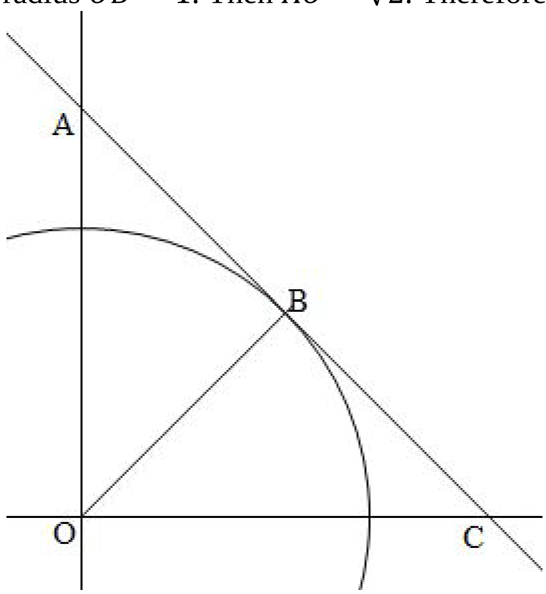
P1A4: Approach 4 for Problem 1

$$\begin{array}{ccc}
\sqrt{2009}+\sqrt{2011} & ? & 2\sqrt{2010} \\
\left(\sqrt{2009}+\sqrt{2011}\right)^2 & ? & \left(2\sqrt{2010}\right)^2 \\
2009+2011+2\sqrt{2009}\sqrt{2011} & ? & 4\cdot 2010 \\
\sqrt{2009}\sqrt{2011} & ? & \sqrt{2010}\sqrt{2010} \\
\sqrt{4040099} & ? & \sqrt{4040100} \\
4040099 & < & 4040100
\end{array}$$

Therefore, $\sqrt{2009}+\sqrt{2011} < 2\sqrt{2010}$.

P2A1: Approach 1 for Problem 2
By symmetry, $x = y = \frac{1}{\sqrt{2}}$. A proof by contradiction is as follow. Without loss of generality, consider $a < b$ such that $x = \frac{1}{\sqrt{2}} - a$ and $y = \frac{1}{\sqrt{2}} + b$. It means that $x^2 + y^2 = 1 + a^2 + b^2 + \sqrt{2}(b - a) > 1$, which contradicts $x^2 + y^2 = 1$. Therefore, maximum of $x + y = \sqrt{2}$.

P2A2: Approach 2 for Problem 2
Consider a straight line $x + y = a$. In order to satisfy the condition $x^2 + y^2 = 1$, it follows that this line $x + y = a$ is a tangent line to the circle $x^2 + y^2 = 1$. Consider triangle AOC (see figure below). Because AC is tangent to the circle, it follows that $m\angle ABO = 90°$. Because of the straight line $x + y = a$, it follows that $AO = CO$, that $\angle CAO \cong \angle ACO$, that triangles ABO and CBO are congruent, and that they are $45°: 45°: 90°$ special right triangles. Because of the circle $x^2 + y^2 = 1$, it follows that radius $OB = 1$. Then $AO = \sqrt{2}$. Therefore, maximum of $x + y = \sqrt{2}$.



P2A3: Approach 3 for Problem 2
Given $x^2 + y^2 = 1$, it follows that $(x + y)^2 \leq 2(x^2 + y^2) = 2$. Therefore, maximum of $x + y = \sqrt{2}$.

P2A4: Approach 4 for Problem 2
This problem can be related to a maximization of the function $f(x) = x + \sqrt{1 - x^2}$ whose derivative with respect to x is $f'(x) = 1 - \frac{x}{\sqrt{1-x^2}}$. For $f'(x) = 0$, it means $x = \frac{1}{\sqrt{2}}$. Because $f'(x) > 0$ for $x < \frac{1}{\sqrt{2}}$ and $f'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$, it follows that $x = \frac{1}{\sqrt{2}}$ provides a maximum value for $f(x)$. Therefore, maximum of $x + y = \sqrt{2}$.

P2A5: Approach 5 for Problem 2
Consider $x + y = A$, or $y = A - x$. Given $x^2 + y^2 = 1$, it follows that $x^2 + (A - x)^2 = 1$. This implies a quadratic trinomial $2x^2 - 2Ax + (A^2 - 1) = 0$ whose determinant is $8 - 4A^2$. Then, A is maximum when the determinant is zero. Therefore, maximum of $x + y = \sqrt{2}$.

P2A6: Approach 6 for Problem 2
Given $x^2 + y^2 = 1$, it follows from trigonometric identity $\sin(a + b) = \sin a \cos b + \cos a \sin b$ that $x + y = \cos\theta + \sin\theta = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) \leq \sqrt{2}$ for all values of θ. Therefore, maximum of $x + y = \sqrt{2}$.

P2A7: Approach 7 for Problem 2
Given $x^2 + y^2 = 1$ and considering vectors $\vec{a}(x, y)$ and $\vec{b}(1,1)$ with θ being the angle between the two so that $\|\vec{a}\| = \sqrt{x^2 + y^2} = 1$ and $\|\vec{b}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$, it follows that $\vec{a} \cdot \vec{b} = (x)(1) + (y)(1) = \|\vec{a}\| \|\vec{b}\| \cos\theta = (1)(\sqrt{2})\cos\theta \leq \sqrt{2}$ for all values of θ. Therefore, maximum of $x + y = \sqrt{2}$.

P2A8: Approach 8 for Problem 2
Given $x^2 + y^2 = 1$, it follows that $x = \cos\theta$ and $y = \sin\theta$. Then, this problem can be related to a maximization of the function $f(\theta) = \cos\theta + \sin\theta$ whose derivative with respect to θ is $f'(\theta) = -\sin\theta + \cos\theta$. For $f'(\theta) = 0$, it means that $\sin\theta = \cos\theta$ and that $\theta = \frac{\pi}{4}$. Because $f'(\theta) > 0$ for $0 < \theta < \frac{\pi}{4}$ and $f'(\theta) < 0$ for $\frac{\pi}{4} < \theta < 0$, it follows that $\theta = \frac{\pi}{4}$ provides a maximum value for $f(\theta)$. Then $x = y = \frac{1}{\sqrt{2}}$. Therefore, maximum of $x + y = \sqrt{2}$.

P3A1: Approach 1 for Problem 3
Because of median $\overline{CD}$, it follows that $AD = BD$. Then, because $CD = BD$, it follows that $\angle CBD \cong \angle BCD$. Likewise, because $CD = AD$, it follows that $\angle ACD \cong \angle CAD$. Since the sum of measures of all three angles in a triangle is 180°, then $m\angle CBD + m\angle BCD + m\angle ACD + m\angle CAD = 180°$ and $m\angle BCD + m\angle ACD = 90°$. Therefore, $m\angle ACB = 90°$.

P3A2: Approach 2 for Problem 3
Consider a circle centered at D with radius DC. Then AB is a diameter of the circle because median $\overline{CD}$ implies $AD = BD$ and given that $CD = BD$. So $\angle ACB$ is an inscribed angle of the alleged circle. Therefore, $m\angle ACB = 90°$.

P3A3: Approach 3 for Problem 3
Extend CD. Construct a line parallel to BC from A and another line parallel to AC from B, both of which intersect CD at E. Then $ACBE$ is a parallelogram with congruent diagonals AB and CE because median $\overline{CD}$ implies $AD = BD$ and given that $CD = BD$. Specifically, $ACBE$ is a rectangle. Therefore, $m\angle ACB = 90°$.

APPENDIX K

Materials for Experts' Evaluations

INSTRUCTIONS

1. Please find attached a copy of students' test, a list of students' approaches, and an Excel file of evaluation template. Please use the Excel file for your evaluations.
2. In regards to the aesthetic aspect of students' approaches, there are two kinds of evaluations that are expected here:
   a. Five-point scale:
      For each of the three problems, please rate each approach in the attached compiled list according to the following rubric:
      5 – Student's approach is the most beautiful approach that I have ever seen in similar or related problems.
      4 – Student's approach is beautiful, but I have seen more beautiful approaches in similar or related problems.
      3 – Student's approach is very typical to similar or related problems and is often associated with standard approaches taught or suggested by mathematics teachers or curriculum at the secondary school level.
      2 – Student's approach suggests brute force application of naïve information processing skills relying only on the information explicitly provided in the problem.
      1 – Student's approach indicates primitive understanding of basic mathematics skills required to solve similar or related problems.
   b. Order of preference:
      For each of the three problems, please put students' approaches in order from the most preferred approach to the least preferred approach. Please provide careful explanations for your preference in regards to aesthetic aspect. Please limit each explanation to a maximum of 200 words.
3. After evaluating students' approaches, you are welcome to provide your own approaches. If you decide to do so, please include your own approaches during your evaluations of both the five point scale and the order of preference. (You may scan your own handwritten solutions and send them as a PDF attachment.)
4. Please categorize students' approaches into mathematics subject classification with a brief description of your own interpretation.
5. Please email all your responses to hht2105@columbia.edu.

EXPERTS' EVALUATIONS

Name: ________________________________ Date: ____________________

University: ________________________________

Problem	Approach	Ratings for Five-Point Scale	Brief Note	Mathematics Subject Classification	Explanation	Order of Preference
e.g. 0	1	3		Order of operation	Student applied a standard procedure of order of operations.	2nd
e.g. 0	2	4		Factorization	A rather more elegant approach which made use of sophisticated knowledge of factorization and resulted in an efficient mental arithmetic: less time and memory workload spent on solving the problem.	1st
1	2					
1	2					
1	3					
1	4					
2	1					
2	2					
2	3					
2	4					
2	5					
2	6					
2	7					
2	8					
3	1					
3	2					
3	3					

APPENDIX L

Students' Follow-up Interview

INTERVIEW

Name: ________________________________ Date: ___________________

School: ________________________________

Part 1 – Pre demonstration of the collection of approaches:

1. How do you feel about the test? Have you had any experience in taking a test requiring you to solve in many different approaches?
2. How did the instruction to look for many different approaches affect your overall performance in this test? (Positively or negatively?)
3. Have you seen any of these problems before? (In school context? Or outside school context?)
4. What about the approaches you have provided here? (Did you learn this in school? Outside school? Or did you invent it on your own?)
5. What is similar about all approaches you have provided here? What about differences?
6. What influences your decision to solve a problem in a particular approach?
7. For each particular problem [e.g. Problems 1, 2, and 3], do you see any advantages or disadvantages using any one of your approaches here?
8. Among all of those approaches that you have provided here, which one do you prefer? [e.g. Approach 1]. What influences your decision to prefer a particular approach among all different approaches that you have provided here?
9. You mentioned in your take-home survey that your favorite mathematics subject was [e.g. geometry]. There was no [e.g. geometric] approach in this particular problem [e.g. Problems 1, 2, and 3] that you have solved. Do you have any comment on that?
10. You mentioned in your take-home survey that you have taken a mathematics course in [e.g. algebra]. There was no [e.g. algebra] approach in this particular problem [e.g. Problems 1, 2, and 3] that you have solved. Do you have any comment on that?
11. Is there a connection or interaction among all different approaches that you have provided here? How would you explain the sequencing of approaches that you have provided here? [e.g. In problem number 1, why did you come up with geometric approach as Approach 1? After that, why did you follow up with algebraic approach as Approach 2? Why not Approach 3 after Approach 1?]
12. Can you relate your previous explanations to the time you spent on each approach? [e.g. You spend more time on Approach 1 than 2].
13. How do you relate your preferred approach to your favorite mathematics subject?
14. How do you relate your preferred approach to the way you were taught in school?

Part 2 – Post demonstration of the collection of approaches:

1. Please take a look at the collection of approaches that I have prepared here. Which approach do you understand based on the mathematical steps or logical reasoning behind it? Which approach do you not understand?
2. Would you have thought about solving the problem by using these approaches? (Why?)
3. From this collection of approaches, which one do you prefer? (Why?)
4. How does this collection of approaches change your preferred approach as you answered earlier based only on your own approaches?
5. This particular approach [e.g. Approach 4] was considered elegant by a group of Mathematics professors. What do you think about this claim?

Part 3 – Students' attitude towards solving in many different approaches:

1. Think about your mathematics classes. Do you recall any mathematics teacher who has taught to solve in many different approaches?
2. What was the mathematics topic that he or she taught in which problems were solved in many different approaches? [e.g. solving systems of two linear equations with two variables, solving quadratic equations, etc.]
3. Do you recall any classroom experience where you or your classmates attempted to solve a problem in a different approach than the teacher taught? (I will refer 'solving a problem in many different approaches' as 'this experience' henceforth).
    a. What was your teacher's reaction? (Did he or she encourage you or your classmates to seek for different approaches than he or she taught? Or did he or she think that the different approaches were out of context of the mathematics topic that he or she was teaching at that time?)
    b. Did your teacher actively promote this experience in his or her teaching method? Or did he or she react positively (or passively) only when students brought up this experience in the classroom?
    c. What do you think about this experience? (Would this help you learn more mathematics concepts from the problem? Or would this impede your learning experience on the mathematics concept that your teacher taught you?)
    d. Would you attempt more often to solve a problem in different approaches if your teacher encourages you to do so?
4. What are the implications of this test for your future learning experience, especially in mathematics?
5. Is there anything else you want to add in your general opinions about learning mathematics by solving problems in many different approaches?

Reminder – Please avoid discussing questions involved in this interview with other students. Thank you.

APPENDIX M

Transcripts of Students' Follow-up Interview

Interview with Student 1

Interviewer: So how do you feel about the test? You can take a look at this.

Student 1: This test?

Interviewer: Yes, this test.

Student 1: I realized I forgot to do some of the math, I haven't been really doing that much this year, I know that's probably my biggest issue.

Interviewer: Have you had any experience in taking a test requiring you to solve in many different approaches?

Student 1: I think probably when I was younger and the state test sort of you have to do it in two different ways, but it's a lot simpler.

Interviewer: Have you seen any of these three problems before?

Student 1: Not the exact problem.

Interviewer: But it's similar?

Student 1: Yeah, I've seen similar problems.

Interviewer: Which one?

Student 1: All three of them.

Interviewer: Was it in school? In high school?

Student 1: Yeah, and in 8^{th} grade, and I took a prep course for getting, to take SHSAT, and I had stuff like that.

Interviewer: Which ones?

Student 1: Probably the second two more than the first one.

Interviewer: Okay, so you have had all of these approaches before?

Student 1: Yeah, I've seen them before.

Interviewer: I want to ask you to take a look at this table over here, and this is kind of like the guide for you to do these questionnaires, I have four different approaches for Problem 1, you can take a look at these, you can use this, what I want you to do for each problem, whether you understand the steps or not, 2 if you understand all the steps, 1 if you understand some, 0 if you don't understand anything, and then second, whether you have the mathematical tools to be able to perform this, like have you learned this, and tell me what is the essential tools, what is it needed for you to be able to perform the approach?

Student 1: What do you mean?

Interviewer: Like for example like this one, Approach 1, all I need to know is multiplication and addition, for Approach 2, factoring or something like that, so that's what I mean by essential mathematical knowledge, number 4, 5, and 6 are like hypothetical questions, that means, if you had to do it all over again, if you had to do similar problems, what is the likelihood, what is the chance that you'd use this approach, 2 if you're very likely, 1 if you may be able to solve it like this, 0 if you wouldn't have thought about solving it like this, and then take a look at this, you can think aloud.

Student 1: Oh okay.

Interviewer: You can think aloud as you read the problem.

Student 1: This one, factoring, square roots.

Interviewer: So would you have thought about this?

Student 1: Not sure.

Interviewer: So take a look at the next one, what do you think is the necessary tools? You can just write it out. And this is Approach 3. What about Approach 4? The whole idea of the study is whether or not students can solve a problem in many different ways, so here we have four different approaches, now there are four of these, which approach would you choose in your first attempt to solve this problem? Approach 4?

Student 1: Yeah.

Interviewer: Second and third?

Student 1: Approaches 1 and 2.

Interviewer: Which one among all of these four do you prefer?

Student 1: This one.

Interviewer: Let's try to circle this Approach 4. So I have consulted with the university professors, I cannot name the name of the school, they said that Approach 3, they said, it's considered to be beautiful, can you comment on that?

Student 1: It's probably because it's the simplest way to solve it, but I don't understand how they solve it, so I can't say, but I agree that's easy if I knew how to solve it.

Interviewer: It's basically about being very easy?

Student 1: Yeah, it's simple, if you understand the concept of it, don't go through all these big numbers to solve it.

Interviewer: Let's move on to the second problem here, we have eight different approaches for Problem 2, same thing here, whether you understand the steps, whether you have the ability to perform like this, do you understand this?

Student 1: Okay.

Interviewer: So this is the second one, you think aloud as you read this.

Student 1: Well, I understand it logically, but it probably won't come to mind, so unit circle and basic triangles.

Interviewer: And then this is the third approach, see if you can understand this.

Student 1: Yeah, I don't understand how to do this, like I understand the rest but I don't understand this one.

Interviewer: This is Approach 4.

Student 1: Derivatives, oh okay, I get it.

Interviewer: So that would be Approach 5, then Approach 6, this is seven.

Student 1: What is that symbol? Like a line segment?

Interviewer: If you don't understand, maybe you can put 0 or 1. This is the last one, Approach 8.

Student 1: I understand this one cause I get it.

Interviewer: So what do you think is the tools?

Student 1: Trigonometry, reasoning.

Interviewer: Alright, now all of these eight, among all of these eight, do you see anything similar or anything different?

Student 1: Like 6 and 8 similar.

Interviewer: Among of these eight which one is your favorite?

Student 1: Approach 8.

Interviewer: Which one would you attempt the first if you had to do it all over again? Which approach would you start with?

Student 1: Approach 2.

Interviewer: What about your second attempt? You can take a look at them one more time.

Student 1: Okay.

Interviewer: Now this is the third problem and the same thing again.

Student 1: Can I draw the triangle?

Interviewer: Yeah, sure, you can write whatever you want. So you fully understand the steps here, so what are the tools here?

Student 1: Triangle properties.

Interviewer: Anything else you want to add?

Student 1: No.

Interviewer: Let's go to the second approach, see if you understand the steps.

Student 1: Yeah, now I get it, I think it's just angles in relation to circles.

Interviewer: Anything else?

Student 1: No.

Interviewer: This is the last one.

Student 1: So I understand the mathematics, but I don't understand the steps, like I don't know.

Interviewer: What do you think is the tools here?

Student 1: Properties of angles.

Interviewer: Now which one do you prefer among all of these three?

Student 1: This one.

Interviewer: The second one?

Student 1: Yeah.

Interviewer: By the way, I asked the university professors, and they said, approach 3 is considered to be beautiful, can you take a look at this and comment on that?

Student 1: I think it's the same thing, like the beautiful if you know some mathematical properties, you can apply them the right way, you don't have to involve like numbers or logic as much, you just use those properties.

Interviewer: What about this one Problem 2? They said, Approach 2 was considered beautiful, can you comment on that? And that's your favorite.

Student 1: It doesn't involve so much numbers in it, not many properties involved.

Interviewer: Okay. Thank you for your time.

<u>Interview with Student 2</u>

Interviewer: How do you feel about the test?

Student 2: It was easy, it's fine, the problems seem simple, I knew some shortcuts beforehand so I could just look at it and tell the answer was, and then I had to figure out how I knew the answer.

Interviewer: When you said you knew the shortcuts, do you mean that you have seen these problems before?

Student 2: No, I've never seen the problems themselves before but I've seen something similar, like the triangle one, I saw that this was, I've seen this property of triangle before where the median is equal to the two sides that are, or the two line segments that are formed by it, I've seen before that that's always a right triangle so I knew that was, that angle measure was always 90 degrees, so I just need to figure out which one, rather how I did that, how to figure that out, so I just went about proving it.

Interviewer: So you knew the fact, you knew the answer already, and you just had to prove it, you just had to show that that fact is correct.

Student 2: Yeah, I just had, I just went about proving it.

Interviewer: Have you had a test requiring you to solve in many different ways?

Student 2: Occasionally, my teacher will tell us to solve a problem in a way that, like for example, in calculus, there's the limit version of finding the area under the curve, and there's just taking integral version of finding the area under the curve, and occasionally, on the test, she'll test us to find, even though we know how to do the integral method, she'll tell us to limit definition for the area under the curve, to find the area just to see if we still remember it, not often though, it's relatively new.

Interviewer: So how did the instruction to look for many different approaches affect your overall performance in this test?

Student 2: Well, the first solution was really easy, because I just solved the problem however I thought of first, it's just like a normal question, whereas the second approach, for each of them, I was thinking, well, if I couldn't do that, what is some other way I can try and solve it, and I just thought of anything I could to solve it, and after that, I kept just trying other methods, but usually it didn't work, or I find some problem where I thought they could happen.

Interviewer: Right, can we take a look at like your Problem 2 Approach 2 on page 5?

Student 2: I was thinking how I knew that, I was thinking of the perimeter problem that like we've done maximin problems before, so I was thinking of how the perimeter is, or rather the area is maximized when it's a square, but I realized that had, that didn't actually have to do with the problem, and so yeah, I cut it off at that point, when I realized it took a lot longer to go through that, it wasn't exactly relevant, I guess this one was an example of that too, I forgot what I was thinking, oh I was thinking of arithmetic versus geometric mean, it looked like it, but didn't exactly work.

Interviewer: Let's look at Problem 3, I see that you have two different approaches here, I don't know if you realized over here, this is the first approach for Problem 3 is kind of similar to what I have here on the survey, Approach 1, and the other one, you in fact mentioned about inscribed angle in the semicircle, like Approach 2 here, so I want to ask you, for the two you had here, which one do you prefer?

Student 2: Well, I think it would be this one Approach 2 on page 8, I like it more because it's kind of more abstract in a way, whereas this one Approach 1 turns it into an algebra problem, even though the proof of inscribed in the semicircle, the simplest proof is actually the same thing, I still like this Approach 2 more, it's simpler and more abstract, I tend to like more abstract things better.

Interviewer: Have you looked at Approach 3 for Problem 3?

Student 2: Do you mind if I draw so I can visualize it?

Interviewer: Sure, sure.

Student 2: It's a rectangle, okay so, what about it?

Interviewer: You see how it works.

Student 2: Yeah, that's pretty simple, I see how the logic works, I would not have thought about this approach, I usually don't extend into a parallelogram, I don't know, it just doesn't occur to me naturally.

Interviewer: When you solve Problem 3, can you tell me why you started off with Approach 1 first, instead of Approach 2?

Student 2: Well, it came to my mind first because at first I just drew it out and then I realized that these both are isosceles triangles and so these two angles, I have seen that idea, this thing before, like I said, I've seen the right triangle that halfway to midpoint of the hypotenuse equidistant long.

Interviewer: So why algebraic approach first, then the inscribed angle approach later?

Student 2: I guess I'm just more used to doing algebra problems to doing geometry problems, it's been a long time since I took geometry.

Interviewer: Actually, you're pretty good with geometry, in fact, in your survey, your favorite subject is geometry.

Student 2: Yeah, my favorite subject is geometry, just haven't worked with it for a long time, I'm not in the state of thinking, I guess.

Interviewer: So after I presented to you Approach 3, which one do you now prefer?

Student 2: I'd still say Approach 2, yeah, I don't know why I like using, I like working with triangles and circles better than I do parallelograms, quadrilaterals in general, I can't think of anything, simpler possibly.

Interviewer: What is simpler about triangles and circles than parallelograms?

Student 2: Well, they have less sides.

Interviewer: Circle has infinite number of sides.

Student 2: Yeah, but I think of it as just one, I don’t know, it’s ingrained prejudices, I’m not exactly sure why they’re there.

Interviewer: So according to the university professors, Approach 3 was considered beautiful, can you comment on that?

Student 2: Well, it’s not, it’s a good bit of thinking, I have to think of a reason why you said that, it’s because, the person had to realize that, hey if you extend these two lines parallel, you get a parallelogram, you can work with that, because this forms, this also intersects with that one, so you get a diagonal, it’s, I guess, it’s a nice, I can’t really think of a solid reason, it’s all opinion really.

Interviewer: Right, something beautiful is subjective, I should say.

Student 2: Of course, maybe he likes working with parallelograms.

Interviewer: Okay, I don’t know if you read the sample problem, it’s about a kid solving this problem and there’s his response to the questions in the survey, he said, if he would choose Approach 1 to get the correct answer, not so much about Approach 2, although Approach 2 is the one I prefer the most, do you have anything to say about this?

Student 2: He’s, the reason he said that is because, I guess, he had problems with using abstract ideas, like abstracting things.

Interviewer: Can you comment on this phrase, to obtain the correct answer?

Student 2: I think he believes his abilities to do computation is better than his ability to abstract things, for example, the actual multiplication of a 7 and 21, and adding that to multiplication of 79 and 7 is easier, or he’s more confident to get it correct than abstracting it that actually if you use the distributive property that up to 100, it’d be much easier.

Interviewer: He actually realized that.

Student 2: He did realize that, the thing is he first thought of, or, well, the thing is, I believe his goal was to get the answer correct rather than to solve it in the easiest fashion, it may have been harder for him to, or more tedious or whatever it was to multiply 7 by 21 and 79 by 7 and add them together, but he doesn’t believe, he thought he would get an incorrect if he had tried to abstract, and then add the 21 and 79.

Interviewer: Good, have you looked at all four approaches I have here for Problem 1?

Student 2: Yeah.

Interviewer: You solved this problem using Approach 1.

Student 2: Yeah, so last time I worked with square roots, I squared both sides to reduce the amount of radicals you have and then solving for the radicals, keep squaring it, so first thing came to mind was squaring both sides and so I did, and then I just simplified it a little bit, but I had that, you could subtract two of these and get two these, and so this is the square root, it's pretty much the same but it ends up to be the difference of two squares.

Interviewer: Okay, so since you did Approach 1, let's take a look at the other approaches here, do we understand the second approach?

Student 2: Yeah, he was multiplying by conjugates, which is another way of dealing with radicals, so he just got, split it into two separates things that have conjugates and then multiply both sides by the conjugates from there.

Interviewer: What about Approach 3 here?

Student 2: Yeah, because it's concave, and it's not increasing steadily, steadily in one direction, as you get further out the, intuitively, I know what he's saying, but I can't really put it into words, because the averages, as you get further out, the averages of any two points is going to be less and less, like when you draw a line between the two points and take the midpoint, that'll be it, and you would want that point.

Interviewer: Good, what about the fourth one?

Student 2: It starts off the same way as me, but in the end he just multiplied them together, instead of doing the difference of two squares method.

Interviewer: Okay, can you compare the two, I mean, which one do you prefer?

Student 2: I prefer the first one better, well because of laziness, I don't particularly want to multiply 2009 and 2011, when I could rather abstract it and get 2010^2 - 1, so yeah.

Interviewer: Now among all of these four, which one is your favorite?

Student 2: I don't know, but if I would've thought of three, I would have like, yeah, three is the best, because it's the simplest, the most abstract, you can generalize it to any function that is strictly concave, or concave over the range or domain that you want to use it.

Interviewer: When we consulted with university professors, they also said that Approach 3 was considered beautiful, can you comment on that, can you see what they're trying to say there?

Student 2: Yeah, because you can abstract, it's most easily abstracted to whole bunch of other applications, for example, you could take pretty much any curve that's concave or convex in any way, you can change it to convex, then it'd become greater than over that.

Interviewer: Okay, I want to go back to the sample problem, remember the guy said that the first approach is to get the answer right away, then probably he might be able to try Approach 2, can you make some connection with this Problem 1, I mean you actually did Approach 1 on your first attempt, and that was successful, I was curious whether you're at that time experiencing what the guy was talking about?

Student 2: That's true, this Approach 1 is the first thing I thought of, but I would rather have used this Approach 3 as my first attempt if I had thought of it, but I didn't.

Interviewer: So you actually, can we say that even as your objective is to get the answer as quickly as possible, you still prefer to use that elegant Approach 3 as your first attempt?

Student 2: Yeah, it's more aesthetically pleasing, shall we say, it's also a bit more fun in a way, because if I knew this, like I said, I could use it on any other problem that is like this, whereas if I get a bunch of numbers with some function being acted on all of them, and I'm asked to classify them in some way like this, I can just refer back to this and generalize it, so I'll look at it and say, oh, it's convex or it's concave in that area, I'll just arrange them in such a way that gives me another way of looking at it, or another equation to use to help me find the answer, it's also the more elegant solutions tend to be the simpler ones, which is nice.

Interviewer: I see, sorry, I'm going back to the sample problem again here, and you said you prefer the second approach, you said Approach 1 is more complicated because it requires more computations, can we say that Approach 1 more brute force approach and Approach 2 more elegant approach?

Student 2: Yeah.

Interviewer: So the next question is looking at these four approaches, I mean, without me showing you Approaches 2 and 4, do you think you'd say the same about Approach 3 as being aesthetically appealing as you said before?

Student 2: Yeah, it's just the same.

Interviewer: I mean, will your point of view looking at Approach 3 in terms of aesthetic value be changed without looking at the sort of ugly approaches, like you said 4 since it's more computational?

Student 2: Yeah, again, because it can be generalized to a bunch of other things, for a lot of different problems, it's nice and it's simple, it's a good logical idea.

Interviewer: Say, a teacher shows you a new problem like this and he just shows you this one only approach, Approach 3, would you think you'd be able to appreciate the aesthetic appeals of this even if he didn't show you the uglier approaches?

Student 2: Yeah, the contrast doesn’t exactly necessarily for, to see something beautiful, to see how the structures, how pleasing it is, nice simple thing, it just looks good.

Interviewer: Good, take a look Problem 2, here you solved it using Approach 8, and you got the right answer here, now given you know all of the eight approaches, which one do you prefer?

Student 2: Well, this one Approach 2 kindly of nice turns it into a geometry problem, it seems a little roundabout for my taste, the only one, it’s, well, no, it seems roundabout for my taste, whereas the one I thought of was rather, or I think is very simpler.

Interviewer: So you prefer Approach 2 among all of the eight?

Student 2: No, I prefer mine, Approach 8, because how I wrote it is simpler at least, I just saw x^2 and y^2, well they’re equal one, well, there’s the equation that is like that I’ve seen before, that sine square and cosine square, so at that point I thought cosine and sine, well the point where you get the, I think it had to do, why did I think of the square, oh I think it has to do with the maximin problems, previous experience has told me that I was going to end up with the squares no matter what I did, well, what do you know, it worked, and I went about proving it.

Interviewer: Okay, so university professors said Approach 2 was considered beautiful, can you comment on that?

Student 2: Yeah, well, oh, well, since we’re thinking of, since this graph illustrates it quite well, for x+y=A the maximum you’re going to get is when this point is high as you can possibly get it, because that’s representation of A on the y-axis, so then you just draw the circle as defined by x^2+y^2=1 and you move A as high up you can while still having that line, the line like that, straight line tangent to the circle, which is the furthest point you can get, the highest out you can get A without completely leaving the domain.

Interviewer: I see you understand the problem, now can we see why they consider this beautiful?

Student 2: No, I can’t really think about it, it’s not that I don’t see that this is nice and pleasing, I can’t really explain why it is, it’s again one of those nice and simple, good looking solutions, I think Approach 2 is simpler than Approach 8 because, if you take out, it’s just, well, it’s just you can visualize it in your head a lot easier than say I don’t know, Approach 8, or how I did it, it’s much easier to see it in your head I guess.

Interviewer: I remember your favorite subject is geometry.

Student 2: Yeah, I can see Approach 2 is geometric, and Approach 8 had to do more with calculus, I tend to think of calculus just as extension of geometry in that it’s still working figures on the Cartesian plane, or well, and then once you get into multivariable one, vector calculus, you’re working solid, three dimensional space.

Interviewer: If you take a look at the problem by itself, what type of problem does Problem 2 look like to you?

Student 2: From that, I think it's geometry problem, yeah, just looking at it, oh sorry, finding the maximum thing makes it look like calculus problem, the whole thing.

Interviewer: So you see that the university professors think that this geometric approach for this calculus problem is more aesthetically pleasing that the calculus approach?

Student 2: Well, how do I explain that, I guess, in that context, calculus approach is kind of the brute force approach, it works for everything, whereas this only works for this condition, like you can't do this with any maximize x^3+y^3, it's kind of low tech I think.

Interviewer: Can you explain what you mean by "low tech"?

Student 2: Well, he didn't have to use more advanced topics in order to solve a problem which you wouldn't think only requires less sophisticated methods to solve.

Interviewer: Okay, now that you know all of the 8 approaches, which one would you use as your first attempt?

Student 2: Approach 8, because, I guess, it's something I've done more recently, but Approach 2, it's just starting that out I'm not sure if I can solve it that way, whereas with 8, I know that I could solve it that way, you notice that I tried a couple here and there, if I want to solve the problem first, then I'll go with 8.

Interviewer: So is your experience here the same as this student in the sample problem?

Student 2: Yeah, it's pretty similar, I guess, it's getting the solution as fast as possible is the thing that I was trying to do here.

Interviewer: Okay, now regular tests don't require you to solve in many different ways, but if you're assessed in tests requiring you to solve in many different ways, how would you think of that?

Student 2: I could, but it'd probably be harder for me than most regular tests.

Interviewer: So our learning behavior depends on the assessment of the test?

Student 2: So how we learn is based on how we're tested, maybe it would be true for many people, for me, I learn as many different methods as I possibly can, and do the ones that I like the best, or the easiest.

Interviewer: Let's make a connection with creativity in this context, will solving one problem in many different ways help you be more creative in problem solving in general?

Student 2: Yeah, I think so, the ability to think in different ways, to see problems from different perspectives is very helpful to find solutions that you wouldn't think you'd be able to solve normally.

Interviewer: Okay, what about the issue of fixation, one student told me that he couldn't switch into a different mode of thinking once he tried his first approach or even after he successfully got the answer, so his approaches after the first one would be similar to that first one, can you comment on that?

Student 2: Well, I can see that happening, you're so used to working, having to solve the problem in that way, or the first method for him, for him, he's been working like that for a long time so he's not used to working in a different way, I guess.

Interviewer: Have you had any experience like this before?

Student 2: No, I don't remember it ever happening to me, I mean, I could solve it in this way, then let's try it that way, and I did it, it's not hard to switch in the way I'm thinking.

Interviewer: Thank you.

<u>Interview with Student 3</u>

Interviewer: So I've taken a look at your test here and I'm going to ask you these questions. How do you feel about the test?

Student 3: Well, I mean the questions, the questions, I kept skipping back and forth because originally I was able to find like one way to do something and then I moved on and I realized I could've done something else so I returned to the original question. And also a lot of times the second time when I'm doing the question a different way, I would end up with a different answer and then realized that the first way I had done it was completely wrong so like it was just interesting to like be able to think about different ways, instead of just going the one easiest way to do per question.

Interviewer: Have you had any experience taking a test that required you to solve in many different approaches?

Student 3: Well, I mean generally like most math classes they show you like more than one way to do something but then they tell you that on the test, you can use whichever way you're more comfortable with. So I was never really put in that situation.

Interviewer: When you said, so you had this experience before but then the test only requires you to solve it in one way to get the correct answer. In which subject is it? I mean I know it's math, but in which topic?

Student 3: Yeah. Um, pre-calculus, that happened a lot like, and now calculus like especially when we're doing like limits and like even the upcoming test, we were told that you can the definition to solve it, but then there're all of these shortcuts that you can use to get around and you still get the same answer.

Interviewer: Okay. So these three problems, have you seen any of these three problems? Have you seen any of these three before?

Student 3: No.

Interviewer: So these problems are pretty new. So I have these questions here. For the first problem, do you feel that you actually solved this problem? Let's see. I recorded four. One, two, three, and four. Do you feel that you actually solved, successfully solved this problem?

Student 3: The right answer?

Interviewer: Right.

Student 3: The first three approaches I got consistently greater than, I mean less than, and then the last approach I tried I got equals to for some reason. But I felt more confident with the first three approaches cause I felt like the fourth approach was just testing something out, see if it would work again.

Interviewer: Can you repeat the last sentence and can you elaborate on that?

Student 3: Okay, well, the first three times I did it, the first three approaches that like came to mind right away, whereas the fourth approach was when I finished the test and I was trying to check if that answer was right. So I tried a different way to just check my answer rather than resolve it. So I'm more confident that the first three approaches the answer I got there was the right answer.

Interviewer: So when you come up a different result for the fourth approach, how do you feel about your first three approaches?

Student 3: I'm more confident about those, because those are the ones that came to mind right away, whereas the fourth approach I had to like really think about another way to do it.

Interviewer: Okay. What about Problem 2?

Student 3: That I wasn't confident at all actually because I mean I couldn't understand the question right away and then I tried solving it so many different ways, I mean I'm sure my answer was wrong because that's all I can think of.

Interviewer: It's okay. So you have two, so you have, you also have four, one, two, three, four here. What about Problem 3?

Student 3: I think a lot of how I solved it, I work a lot better with visual representations and if I see something, it makes it easier for me, but in that problem, a lot of my information there was based on assumption, or like I was trying to connect what I was presented with to come up with a solution that would work.

Interviewer: What do you think about the correctness of this problem? Your solution? In two approaches? I'm sorry, three approaches. Do you feel confident about the solution?

Student 3: No, not really. This is strange, but after the test that I realized that could've been a different measure and so I was thinking about different answers after the test as well.

Interviewer: Okay, that's good.

Student 3: So that definitely puts me like, as long as, whenever I know that I'm questioning all these different answers, then I know that I'm not sure about the answers.

Interviewer: Okay. I just have a quick question about this, the one you've written here. I'm wondering how you know that CD is perpendicular to AB over here?

Student 3: Okay, well, yeah, that was part of the difficulty in solving that one. I almost forgot like that, I realized that the median would split them in half.

Interviewer: What do you mean "split them"? What is them?

Student 3: Like the sides, the one side if it's a median, it would split, divide them in half.

Interviewer: The length of the sides?

Student 3: Yeah.

Interviewer: This AB.

Student 3: Yeah. So I tried thinking that if this was equal, that was equal, that was equal, and so I ended up saying that it would have to be 90, then I realized that it could have been, sorry, 30-30-60 triangle, no, my bad, 60-60-30 and then it would have been, what am I saying, 40…

Interviewer: 30-60-90?

Student 3: 30-60-90, yeah, and then that was when I started questioning, that probably wasn't right. At the beginning, I just assumed right away that if those were equal, it would have to be 45-45-90.

Interviewer: So you assumed that it was a 45-45-90 triangle?

Student 3: Yeah.

Interviewer: So actually, 90 is the correct answer.

Student 3: Oh okay.

Interviewer: Just curious how you get this inference from, so I guess you're guessing?

Student 3: It's not even guessing, I can't even remember what information we're given.

Interviewer: Do you want to see the problem?

Student 3: Right, find the measure, I separated it and I said that if this one is equal and that would mean that it's reflexive and so that angle and that angle have to be the same and that must have been the different angle. That's why I said that.

Interviewer: Okay, good. So looking at these approaches, do you feel about all of these approaches? For example, for Problem 1, you have four different approaches. What do you think about these four approaches? Are they similar? Are they different? Why do you call them Approach 1, Approach 2, Approach 3, Approach 4? What is different about them?

Student 3: They were all generally similar, a lot of times I was switching steps back and forth, like I know for Problem 1, there was one approach where I just tried to like factor out the radical and then do it that way, and then there was another approach where I realized I didn't want to use the radicals, so I used a different way to figure it out.

Interviewer: What about Problem 2? What about your approach here and this approach here?

Student 3: That was just, I mean, I tried first to get rid of the radicals, and then I don't know how that worked out in getting the answer, and then I tried to use the radicals to solve it, I tried setting it equal to zero and solving that.

Interviewer: Okay, I see what you did here. And then these three approaches for Problem 3.

Student 3: Yeah, the second approach for Problem 3 was the more logical one where I actually wrote down what I was thinking about the problem.

Interviewer: Right. And do you see anything different about page 9 and page 10 here?

Student 3: Well, for me page 9 feels like it's more of the inferences and assumptions I'm just trying to connect everything in my head and then Approach 2 solidifies everything and that's when I was thinking about it mathematically.

Interviewer: I like that word "solidifies." Okay, I'm going to show, these are the approaches that they gave me, they meaning the professors in a university, sorry I cannot tell the name the school or anything like that. But for Problem 1, I want to show there are four different approaches so take a look at these one, two, three, four and tell me what you think about this as you read this. Actually can you just put it here? Do you need a pen?

Student 3: Thank you.

Interviewer: So as you read these solutions, keep in mind these four questions, in terms of understanding of these approaches, rate each approach according to these rubric, you put 2 if you understand all the necessary steps, 1 if you understand some of the steps, 0 if you don't understand anything at all, and then this the mathematical knowledge, if you think you have all the necessary mathematics knowledge, that means you have the tools, mathematical tools for this, you put 2, if you don't have the essential mathematics tools, you put 1, if you don't have anything, you put 0, and you have to identify what is the essential mathematics knowledge that is required to be able to perform this solution, and then, alright, let's try to focus on these, and here's a table to write your answer, you can talk as you read the problem, tell me what you're thinking.

Student 3: This approach I get because, I wouldn't have thought of it right away, but I get that they just multiplied it out to realize, to figure out which one was greater than, so this one I feel like I get the steps, I'd be able to do it right based on the knowledge I have, what's the third question?

Interviewer: What do you think is required to perform this approach?

Student 3: Just multiplying out binomials.

Interviewer: Okay, you can write it out. The next question is, how likely you're going to present this? If you had to do it all over again with similar problem, what is the chance that you'd attempt to solve this problem using this approach? 2 if it's very likely, 1 if you might be able to solve it, 0 if you would not have thought about this.

Student 3: I probably would not have thought about this.

Interviewer: Next one.

Student 3: I mean, I get the reasoning behind this, but right away, it really confuses me, like the jumps back and forth, I know that they write it out in different way and instead of plus, they're subtracting and just writing it out, instead of just doing radical 2010 but I wouldn't be able to follow these steps unless someone was actually explaining it to me.

Interviewer: But looking at this approach, this solution, do you understand all of these steps? Okay, so that you can write it there, and so what do you think is the necessary mathematical tools here?

Student 3: I mean it’s a lot of arithmetic and logical reasoning with the problem, and I definitely wouldn’t use it.

Interviewer: That’s okay. Then Approach 3?

Student 3: Alright, I see where they’re getting it from, like I know this information, that square root function is strictly concave, and I follow what they were doing, yeah, I mean, that’s one way I might use, if I would’ve thought about using that rule, it would’ve made it easier.

Interviewer: Would you be able to graph it?

Student 3: I wouldn’t be able to graph it, like right away, I wouldn’t be. Do you want me to?

Interviewer: No, that’s fine.

Student 3: Okay.

Interviewer: So that means you might be able to do it. So this is the fourth approach.

Student 3: This is one of the ways I tried using it, yeah, it looks similar to approach one, here they actually multiply, I definitely understand that and have the knowledge to be able to do that, and I probably use that.

Interviewer: Just want to focus on the first problem first, we have four approaches, do you feel they are different?

Student 3: Yeah, because I saw similarities only between Approach 1 and Approach 4, but then halfway through instead of like here they cancel out and then multiply to get the actual value but then here they try to, Approach 1 is more logical, like I feel like you have to think about this way, whereas Approach 4 has a lot of arithmetic approach, Approach 4 is straightforward in the way you would solve it, but it’s more actual arithmetic like multiplying it out.

Interviewer: Okay, so all of these four approaches, which one do you prefer?

Student 3: I prefer Approach 4.

Interviewer: Approach 4, so I’m going to circle this, and the next question is if you have to do it all over again, which approach would you attempt using first?

Student 3: Knowing that all of these exist, probably Approach 3.

Interviewer: So what would be the hypothetical sequence of future presentation? What about the second one? Help me fill this out.

Student 3: It would probably Approach 1, then here, the last one I would try would be Approach 2.

Interviewer: Can you explain the reasoning behind the sequencing? Why is this the first one? Why is this the second one? And so on.

Student 3: Approach 3 is a lot quicker, if I would've thought knowing the square root function strictly concave, so that would've been quick to solve. Approach 1 is more of the way I would've thought to solve it then there's like less multiplying number out and then Approach 4 is more straightforward, but it has more math involved, and then Approach 2 is the last one I would've thought of, it gave me a hard time like understanding it right away.

Interviewer: So earlier you said, Approach 4 is your favorite approach, right? I want to tell you that Approach 3 was said to be a beautiful approach by the university professors. Do you have any comment on that?

Student 3: I agree with them, because all you need to know is the square root function is concave and then you would've easily been able to apply that knowledge and figure out the problem, but all the other ways like require you to actually try either solve the square roots and get the values and simplify in some other ways, like this is the one with the knowledge right way you know the answer so it makes sense they said it's the best one.

Interviewer: So let's try to do it with the second one, so I have eight approaches for number two. Do you understand the steps here?

Student 3: So I get that they're solving, because you don't really use x and y, the two different variables so they try to set a and b and set to x^2 + y^2, but I don't really get it.

Interviewer: Do you think you have the tools to understand this?

Student 3: Yeah.

Interviewer: And what are the necessary tools to be able to solve this problem?

Student 3: Well, there's substitution, and you need to know symmetry and radicals.

Interviewer: Will you be likely that you solve this problem this way?

Student 3: No probably not.

Interviewer: Okay, here's the next one.

Student 3: This is probably the way I would do it if I had a second chance, like I just like using geometry and I see the picture in front of me and their reasoning here because x^2 +y^2 = 1 just seems abstract to me and when it's put into a diagram and it's explained, you say there's a tangent line to the circle, it just makes it a lot easier to understand.

Interviewer: What is the diagram for x^2 +y^2 = 1?

Student 3: It's a straight line, sorry, they're factoring it out, instead of using x^2 +y^2, they're using x+y, so they graph it as a line and then x^2 +y^2 = 1 is a circle, they're telling you that the line is tangent to the circle they give you, and yeah, this is definitely a lot easier to understand.

Interviewer: So you can write here. Then this is Approach 3. Tell me if you understand the steps.

Student 3: Okay, so I know that they're factoring that out and then they're replacing it with x^2 +y^2 but I get a little bit lost over the 2, like I feel like there could have been a step included that may be left out, oh okay, I see, alright, so they add a one here, they multiply by 2 so they have to multiply by 2 on both sides and they factor that out, yeah, I mean I get that, and I have the knowledge to use it but I probably wouldn't use that way.

Interviewer: So what are the tools?

Student 3: Factoring and binomials.

Interviewer: This is Approach 4.

Student 3: Yeah, this is the way we learned last term in AP Calculus, I definitely understand what they're doing, have the knowledge to use it, and they use derivatives, yeah, just max/min problem, and I probably would have thought of this in the future.

Interviewer: So this is Approach 5.

Student 3: Okay, I get how that works and I have the reasoning, a lot of substitutions, if I'm graphing, it probably helps just to picture and I might consider using that one.

Interviewer: And this is Approach 6.

Student 3: I understand it, I have the material, but I probably wouldn't use it, you need to use a lot of trigs, you probably have to pick up on that right away, without reading this explanation, I would never think to use that method to solve for it.

Interviewer: Anything else you want to add?

Student 3: No.

Interviewer: This is Approach 7.

Student 3: I sort of get it, I know I have the knowledge to use it, but it's still really hard to follow, and I probably wouldn't consider using it, possibly because I have a hard time just understanding how they apply the vector with trigonometry.

Interviewer: Anything else?

Student 3: No.

Interviewer: So that's Approach 8, that's the last one.

Student 3: This one reminds me of the other approach using derivatives, Approach 4, Approach 8 looks like Approach 4, except using just x and y, they use cosine theta and sine theta, so that I understand, I might consider using it, but it just seems easier to keep the x and y and solving for the derivatives.

Interviewer: Anything else you want to add?

Student 3: No.

Interviewer: Okay, let's do hypothetical question again, if you had to do it all over again, which one would you do first?

Student 3: Probably Approach 4 because I know when I was first doing the test and I saw the problem, I knew that I needed to use derivatives, but I didn't remember how to use them, so that would probably be the first one. Approach 2 would be the second one just because it's easier for me to follow, probably also the last one will be Approach 7.

Interviewer: So among all of these eight approaches, which one do you prefer?

Student 3: Approach 4.

Interviewer: Why is that?

Student 3: I'm more familiar with it and it's easy to follow if you have the mathematical knowledge to be able to do it.

Interviewer: Okay, the professors said that Approach 2 is considered to be beautiful, can you take a look at that and comment on that?

Student 3: Oh yeah, I like this one a lot just because the diagram helped out and you don't really need much mathematical knowledge to be able to figure it out because as long as you know how the geometric rules work, you'd be able to solve it.

Interviewer: Let’s speed up a bit. Problem 3 here. Problem 3 has three different approaches.

Student 3: I get this one [Approach 1] because it’s probably one of the ones that I attempted.

Interviewer: Here is Approach 2.

Student 3: I also understand now that I have the knowledge to use it, I might have considered using it just because I wouldn’t have put it in the circle.

Interviewer: And then the last one here.

Student 3: I also understand that one, and I have the knowledge, I might have used it because it wouldn’t occur to me to use a parallelogram and just thought it would be easier to draw with the triangle.

Interviewer: Just really quickly, can you order this one and give me the favorite.

Student 3: Okay.

Interviewer: Thank you.

<u>Interview with Student 4</u>

Interviewer: So how do you feel about the test?

Student 4: Well, in the second question, I don’t really know one of the words, what the term meant, the triangle one.

Interviewer: The second question or the third question, the third question was the triangle one.

Student 4: Yeah, alright.

Interviewer: You don’t understand the term?

Student 4: I guess, median. The first one I tried approximating I guess I didn’t realize how close the answer should be, so I guess that was a bit of a problem, the second one I had only one way to do it really.

Interviewer: Have you had any experience taking a test requiring you to solve in many different approaches?

Student 4: No.

Interviewer: So this would be the first one?

Student 4: Yeah.

Interviewer: How do you feel about the instruction to solve in many different ways?

Student 4: Once I solved it in one way, it's hard to not think of that way, I guess, like the second kind of try, like it's hard to try completely different approach starting from the beginning, that part was hard.

Interviewer: So in other words, when you solved the problem using first approach, you tend to keep thinking about that approach as go with the second approach, can you explain why?

Student 4: I don't know, I guess I just like the idea that the first one worked, and it's to me like, you know, it's like you have to get from the question to the answer, so it's not really about the steps or the process, it's like getting from the question to the answer, so once I've gotten to the answer, it's hard to start over and just think a completely different way to do it.

Interviewer: Let me try to understand you, the first attempt seems to me that you're trying to get the answer right away, and then the second attempt tends to mimic the first approach, right?

Student 4: Yeah.

Interviewer: Does the instruction affect your performance in any way?

Student 4: I think it's slowing me down because there's like, knowing that there are many options that can solve it, I guess, like in regular math class, they're testing you on like specific math concepts you just learn, but here it's, you know, just give you this problem and think of ways to do it, I guess, it slows me down, because it's more general, and not like I have to start with more basic I guess.

Interviewer: Have you seen any of these three problems before? Anywhere in the school context, outside school context, or like reading a book?

Student 4: The second one, I feel like I've seen that type of problem, I'm not sure if I've seen the exact one.

Interviewer: The next few questions will be to figure out the factors influencing you when solving a problem in particular approaches, so the first problem over here has four different approaches, tell me what you're thinking as you read these approaches, so like, do you understand the steps, the logical reasoning here?

Student 4: Like do I understand this step?

Interviewer: Yeah, how do you feel about this approach?

Student 4: Up to here, it's pretty basic, I'm pretty sure about this, but I didn't think to do that.

Interviewer: Would you have thought about doing this?

Student 4: Maybe given enough time, I'm not sure.

Interviewer: What about your skills, are you confident that you have the knowledge to do this approach?

Student 4: Yeah.

Interviewer: It's just you didn't come up with this, so what would be a factor triggering this approach, coming up with this approach, given that you know all the knowledge here?

Student 4: I don't totally understand.

Interviewer: You just told me that you have all the knowledge here, but it's just that this approach did not come up right away, so I'm wondering how I can give you some hints so that you'd be able to come up with this approach, what would that hint be?

Student 4: I'm not sure, like I know if I'm doing a similar problem recently, I may be able to do that, I'm not sure what in the problem to get me to think of doing that.

Interviewer: Okay, let's take a look at the second approach here, tell me if you get it or not.

Student 4: I think I get it, I think it's just rationalizing, this was actually I was thinking of trying, but I don't know, for some reason, it didn't work out, but I think I would've tried it eventually.

Interviewer: Let's take a look at the third one here, tell me if you get it.

Student 4: I don't think I totally understand that.

Interviewer: That's okay, now let's go to the next one.

Student 4: Yeah, that's similar to what I did, I didn't complete it that one, I guess if I had a calculator, then I would probably do that.

Interviewer: Okay, among all of these four approaches, can you tell me which one you prefer?

Student 4: I think I prefer Approach 1 because the other ones are kind of messy, easy to mess up kind of arithmetic and algebra, this one feels a bit more elegant to me.

Interviewer: The university professors actually thought that Approach 3 was considered beautiful, can you comment on that?

Student 4: It's hard to tell because I don't totally understand it, I usually don't find graphs beautiful in general though.

Interviewer: That's fine, here is the second problem with eight different approaches, so we'll do the same thing here.

Student 4: I guess I don't get this part in Approach 1.

Interviewer: What about the second one?

Student 4: Yeah, I understand the steps, I think I could probably have done this.

Interviewer: What about this third approach?

Student 4: I don't totally understand this part.

Interviewer: That's fine, what about this Approach 4?

Student 4: I think this is what I did actually, I understand this.

Interviewer: What about Approach 5?

Student 4: I don't know what a quadratic trinomial is, oh, okay, I get it, I guess I understand this one.

Interviewer: What about this one?

Student 4: Yeah, I understand this.

Interviewer: Approach 7?

Student 4: I don't really understand vectors.

Interviewer: That's okay, last one, 8.

Student 4: Yeah, I get this, this is like similar to what I had.

Interviewer: So we had eight approaches for Problem 2, can you choose one that you prefer?

Student 4: It's hard because actually I did one of them, Approach 4.

Interviewer: Does that mean you prefer 4 to the others?

Student 4: I guess that just means that I've been doing this kind of problem more recently, and there's not too many logical leaps, it's just pretty straightforward step of what to do.

Interviewer: So the university professors said that Approach 2 is beautiful, can you comment on that?

Student 4: Like for me, it feels more like algebra and like numbers, I can't make too much of a connection with graph, the problem is more like algebra so I'm having a hard time like making a connection related to it, so I'm not really a big fan of this approach, I don't see anything beautiful in this approach.

Interviewer: Do you see anything beautiful in the 8 approaches?

Student 4: I guess, the first one, proof by contradiction.

Interviewer: What is appealing about it?

Student 4: Sort of like prove that it cannot be false, it's very like convincing immediately to me, more logical, there's less places where you can get lost.

Interviewer: Okay, let's look at Problem 3, it has three approaches, you can take a look at it one by one.

Student 4: Yeah, yeah, I understand.

Interviewer: Let's take a look at the second one.

Student 4: Oh, I get it.

Interviewer: What about this one, Approach 3?

Student 4: Yeah, I like that one, because extending the line completely changes the problem for me but in a way that's more solvable because it's just a parallel line, but by making it a parallelogram makes it much easier to solve the problem.

Interviewer: So actually the university professors said that Approach 3 was beautiful, do you see anything that they say it's beautiful?

Student 4: I guess I'd agree because extending the line and making a couple of parallel lines, it seems a bit random at first, but then it makes the angle you're trying to solve part of something that's much easier to solve, it proves it in a very unexpected way.

Interviewer: Okay, so let's do hypothetical questions, if you had to do this all over again, which approach would you try first?

Student 4: I guess with the knowledge I have now, I would try Approach 3 first, but I think I would like, without any of this knowledge, I would do Approach 1 first, I don't think it would occur to me to try another shape to find the angle, I wouldn't try to draw more lines to solve the problem, I think that would mess up the problem.

Interviewer: Okay, would you think you would get the answer faster using Approach 1 or 3?

Student 4: Probably Approach 3.

Interviewer: Have you looked at the sample problem?

Student 4: Yeah.

Interviewer: Here's a response to that, can you relate to your experience in Problem 3?

Student 4: I think that's how I felt about the triangle problem, I would feel more confident to use what I know about triangles, but I definitely prefer the one making parallelogram, I guess it's kind of make sense to just improvise based on what you're confident in and it reminds me when you're trying to say something in foreign language like you're trying to awkwardly form sentences based on the few words that you do know, even though you obviously prefer a sentence with more complex words but if you're taking a test, you work from what you know and try to find the answer from what you know, from what you're confident more.

Interviewer: Take a look at this response, he said, so as to obtain the correct answer, so he thought the first attempt is the first approach, in other words, he wants to get the answer as quickly as possible, even though he prefers the second approach here, can you comment on that?

Student 4: I guess, I often feel the same way about getting the answer as quickly as possible, I guess it depends on the context of answering the question, like taking the test obviously, you want to get the answer as soon as possible, I guess, I usually would want to get the answer as soon as possible in the school context.

<u>Interview with Student 5</u>

Interviewer: How do you feel about the test?

Student 5: I thought that was interesting, there were three problems, and how you mentioned that there’re actually many different ways to solve them.

Interviewer: Have you had any experience taking a test that requires you to solve in many different ways?

Student 5: No, this was the first one.

Interviewer: How do you feel about the instruction to solve in many different ways, did that affect your overall performance?

Student 5: Yeah, it requires me to think outside the box, how can I solve this in another way, cause there’s no one way to solve the problem.

Interviewer: So is this a positive or a negative effect on your performance?

Student 5: Yeah, I’m more of a structured person, I like to think that like, oh, I got the answer definitely right, and knowing that there are multiple approaches to the problem, especially if you’re like, this might be wrong, oh no, this might be wrong.

Interviewer: Usually when you solve a math problem, do you just solve the problem and stop thinking after you found the answer?

Student 5: I mean when I say different approach, I’m just saying like a whole different mathematical concept, if you see oh, you could solve it using proofs, you could solve it using geometry, you could solve it using calculus, and it’s just like, it makes me feel a little uncomfortable taking the test, but like it feels like, solve this using calculus, but there’re multiple ways to solve it using calculus, there’s more a checking my work type of thing, but like using a different mathematical concept, that I’m not 100% recalling it well, it makes me feel oh, I mean, I would feel more comfortable if it’s the one like more checking my work type of thing than looking for so many different ways from different concepts.

Interviewer: Okay, now let’s take a look at Problem 1, your response here, you said your first attempt would be Approach 4 and your favorite was Approach 1.

Student 5: Yeah.

Interviewer: That’s in fact what you’re doing the test, you started off with an approach, actually two different approaches according to you for Problem 1 before you, before you finally solved the problem using your third approach or Approach 4 here for Problem 1.

Student 5: Yeah, I couldn’t find anything at that time so I resort to that, because I need to get the answer, because at first, I just played around the first two, I think because I knew it wasn’t time constrained, or like, yeah, so I just tried those first until I couldn’t then I, I just did that one to get the answer.

Interviewer: Did you ever think that you could use your answer you get from Approach 4 to probably get, figure out your previous approaches, the ones that didn’t work?

Student 5: I mean I like Approach 1, it didn’t came to me during the test, but I like it because, cause it has different twist in it, but what usually comes to me first was squaring both sides, to solve problems like that, like how do you deal problems with square roots, well, you square it, so I just decided to square both sides, just playing around with the number and see what happened, but Approach 1 seems interesting, like oh, I never looked at it that way, and I just liked it, it looks interesting, it looks obvious right now, but it would never occur to me, it looks surprising you can subtract by 1, it would never come to me on test, and here I would, I like Approach 2 better than 3 because of the graph thing.

Interviewer: What about Problem 2, what were you thinking as you solved the problem, I mean Approach 4 is your favorite approach and also your first attempt, in fact you solve the problem like that, using Approach 4.

Student 5: This could be solved in two different ways, either calculus or geometry, my geometry is very very rusty, but I’m learning calculus right now, so I figure why not use calculus, which is still fresh, more fresh, the other thing like Trigonometry, I mean I understand how it works, but I would never never in a million years I would’ve like, wow.

Interviewer: So were you surprised by this Approach 6 as much as you were surprised by Approach 1 in Problem 1, like element of surprise?

Student 5: Well, element of surprise, yeah, it’s very viable method of solving a problem, it just would never come up to me.

Interviewer: So would you change your favorite approach then?

Student 5: No, I think I would still like Approach 4, because I just learned that, it’s still fresh to me, that’s exactly why I chose that method.

Interviewer: Can we take a look at Approach 2 and can you comment on that?

Student 5: I’m not very good at considering hypothetical situation, so like I’m not that type of person to solve the problem with that method, like consider here, I mean, like assuming some line here non-existent before to come up here, my geometry is not very good, I mean, it’s very logical way of thinking, I mean, yeah.

Interviewer: So some university professors say that this Approach 2 is considered beautiful, can you comment on that?

Student 5: Well, the whole concept of math works out like, no matter how you look at things, they always turn out to be the same way, and this is exactly that, like 7 times 3 is the same as if you're adding 7 three times, so same answer no matter, but here it's just like, it's no dispute, it's just logic, what is it, first you have straight line, and then circle here, tangent, and then triangle, yeah, it just follows, it's like a logic proof basically, I mean, maybe the logic concept of it that makes it beautiful, I don't know.

Interviewer: Okay, how about Problem 3 here?

Student 5: I didn't get this problem, like I mean, my geometry is horrendous.

Interviewer: That's okay, but you do understand the steps here in Approach 1?

Student 5: Yeah, this is the stuff we learned in 9th grade, the geometry.

Interviewer: Approach 2?

Student 5: Oh, I see, this makes sense, that makes sense, that's the whole inscribed angle of the circle here.

Interviewer: What about this one?

Student 5: That's a rectangle, yeah, it's proving it's a rectangle.

Interviewer: Okay, you said here your favorite is Approach 1.

Student 5: Yeah, I chose this one, because I like the whole logical following it, the other ones were just like consider this, consider that, it's because of this, it's that, therefore it's 90 degree.

Interviewer: Okay, here's a classroom situation, a teacher gives you a new problem, and he shows you 8 different approaches at the same time, using totally different mathematical concepts.

Student 5: I think it'd be interesting, but it will feel a little, well, a little, you have to know all of those methods, because all they sound very interesting.

Interviewer: Can you say something about your ability to recall those many different mathematical concepts?

Student 5: Yeah, I have to recall all of those concepts, but I'm quick to recall them, so it wouldn't be too much of a problem to recall, maybe a little bit overwhelmed, it's like, I mean, it never happened to me before, it's like, well, do you remember back in 8th grade when you were learning factoring, oh, you can use it here right now, well, that was a bad example, because factoring is always used in high school, if right now my teacher asks me to recall something from 9th grade like geometry where we're learning about logic and proofs and stuff, and he or she expects me to put that on the test, I'll just be like, excuse me, but I mean, it's pretty cool to know that you're not restricted to one way of solving a problem.

Interviewer: Okay, that'll be it, thanks so much.

Interview with Student 6

Interviewer: So let's start, how do you feel about the test?

Student 6: I thought, I thought when I first saw it, it would be easy, but when I started it, I found that I couldn't find many ways.

Interviewer: So was it the instruction that affected your overall performance in the test?

Student 6: Because usually in class, we just solve the problems that is easiest to each person usually.

Interviewer: So you never had any experience taking a test that requires you to solve in many different ways before?

Student 6: No, this is the first time.

Interviewer: Okay, have you seen any of these three problems before?

Student 6: You mean the actual problems?

Interviewer: Similar ones maybe?

Student 6: I've probably seen similar ones like number 2, maybe number 3.

Interviewer: How do you describe number 2?

Student 6: Not with exponents, maybe like x times y equals some number, so what is x+y, something like that.

Interviewer: What about the third one?

Student 6: Third one, well, I've done a lot of those kinds of problems where they give you line segments and stuff, and you have to figure out the angle.

Interviewer: I want to ask the sequencing, what were you thinking as you present your approaches as you worked on the problems?

Student 6: Well, I first just tried to solve the problem in a way that's most comfortable for me, and then go try to figure out different approaches, whether it's harder, but usually I would first start off with the easier approach.

Interviewer: So here, I was looking at your test, you actually started with the third problem, instead of the first problem, and you did get the correct answer on your first attempt here for Problem 3, do you think third problem is easier than first problem?

Student 6: Because number three seems more familiar to me than the other ones to me so I just started off with that one.

Interviewer: In the first survey, you mentioned that your favorite topic was derivatives, so it's like calculus, right, but it seems like you didn't start with Problem 2 which is more like calculus problem, can you say anything about that?

Student 6: Well, I think it's because I'm learning calculus now, but it's just at that time number 3 looked easier to me.

Interviewer: For Problem 1, you said you prefer Approach 4, can you explain more of your response here?

Student 6: Well, Approach 4 would be like, basically if I didn't know any more complicated, like more advanced math, that would be the best way to solve it, just multiply out or square, get rid of radicals to figure out less than, greater than, or perhaps equal, but the other ones, I have to use some of formulas, or some of more recent knowledge maybe.

Interviewer: Here you said your first attempt would be Approach 2, I am wondering why not your favorite Approach 4, what is the relationship between your first attempt and your favorite approach?

Student 6: Well, Approach 4 is straightforward, just multiplying, but then, I wouldn't start off with that because I think that would take longer than some other approach that I would take so I chose the second approach.

Interviewer: But if you know Approach 4 takes longer time than Approach 2, why do you prefer Approach 4 to Approach 2?

Student 6: Well, there I just need to multiply and not to make any mistake, but Approach 2 I have to multiply square roots, there's more room for error I guess for Approach 2.

Interviewer: So you are risking yourself into more error by taking Approach 2 at first, no?

Student 6: Yeah, I know, but I think I'm better at just regular multiplying number like in Approach 4.

Interviewer: So why not try with Approach 4 first then?

Student 6: But I think Approach 2 is faster if I'm taking the test, because it's more recent, more like a recent concept, like what I just learned, well, multiplying 2009 and 2011 might take longer time, but the other method is faster, I mean Approach 4 is simpler, I've never seen that before, like if I couldn't solve the problem with Approach 2, I would resort to Approach 4.

Interviewer: Okay, let's see the sample problem and the example of the response from a student here, can you relate your experience with this?

Student 6: Well, when I see the problem at first, it doesn't seem like a simple problem, I would probably think of a more complicated way of solving it rather than like just a multiplying straight out and getting big number to compare, I think because I can find the answer faster, to do it with more complicated way.

Interviewer: So your experience is quite the opposite of the one in sample problem?

Student 6: Well, the sample problem was a lot easier, so it's kind of different than this one where the number is a lot bigger, so just multiplying, using the order of operations, it's different from here, so it takes longer, I mean I am very sure to get the answer right out using Approach 4 in Problem 1, but I just want to try, like, find a faster way to do it, but maybe see, use answer from Approach 4 to go back to the other approaches that I didn't finish.

Interviewer: Okay, now some university professors think that Approach 3 is beautiful, can you comment on that?

Student 6: Well, I'm not confident with graphing, I think, I mean, maybe it's because it's short, like the proofs here, I'm not sure.

Interviewer: Let's see, for Problem 2, your favorite approach here is your first attempt, like you have here for Approach 4.

Student 6: Well, because I'm currently taking calculus, more comfortable I guess.

Interviewer: What about your second attempt?

Student 6: Well, truthfully, I couldn't think of many different methods, but when I looked at all of these, I found Approach 2 is simpler than the other ones, in terms of understanding.

Interviewer: So university professors said also Approach 2 is beautiful, do you see anything beautiful?

Student 6: Well, that's only approach that uses graphs and creates like triangles, and tangents, the other ones are just formulas.

Interviewer: What about Problem 3, here you said, your favorite is Approach 3 and your first attempt would be Approach 1, in fact that what you did in your actual test, and you did that well, you get the correct answer, can you comment on that, your favorite approach and your first attempt?

Student 6: For this one, I chose Approach 3 as my favorite because when I looked at the three approaches, well, I only chose it it's because I wouldn't think of that during the test, it's sort of, something about it really, like I would normally just create the triangle and use that triangle, I mean the angle and like, but the fact that they solved it by creating a different shape, rather than just using a triangle, I found that interesting.

Interviewer: Say now, if a teacher shows you 8 different approaches on the board, would you get confused by these approaches?

Student 6: I would probably be a little confused, but if I were in that situation, I would just try to understand, absorb the ones that I understand the most, so I'd probably get two or three of them.

Interviewer: What about calculus, I mean geometric approach like here in Problem 2, would you be able to recall that geometry that you probably learned like two years ago?

Student 6: Probably I won't remember everything, but I would be able to recall a lot of them, since it incorporates into the problem itself, but once I see it, I'd probably remember.

Interviewer: Okay, thank you.

<u>Interview with Student 7</u>

Interviewer: So how do you feel about the test?

Student 7: No, not too great actually, but then if you notice the STEP crews outside were a little bit loud, so I was having a tough time concentrating.

Interviewer: So basically the classroom testing environment?

Student 7: Yeah, I think so.

Interviewer: What about the instruction? Have you had any experience in taking a test requiring you to solve in many different approaches?

Student 7: No, I don’t think so, I can’t recall anything.

Interviewer: Does the instruction to solve in many different ways affect your performance in any way?

Student 7: No, I don’t think so, because just the way I looked at it, I tried to solve the problem first then looked for other ways to do it.

Interviewer: Have you seen any of these problems before?

Student 7: No.

Interviewer: I want to take a look at Problem 2 here.

Student 7: I think I said something a bit strange for that.

Interviewer: By the way, I should tell you that there’s no right or wrong answer. So for the second problem, you said you have a total of, you have one approach here?

Student 7: Yeah, right there I started a second one.

Interviewer: You were doing some derivatives here.

Student 7: Yeah, then I saw the survey you handed out, the way I would have liked to do it.

Interviewer: I see you were doing derivatives here so basically you’re trying to do calculus approach for this problem. What about this? Can you tell me what you’re doing here?

Student 7: I think for that part, I don’t really remember, I’m sorry.

Interviewer: That’s okay. You mentioned that in your first survey, your favorite math topic is calculus and you wrote, I chose calculus because it gives me a set of tools I can use to solve problems that previously would have been lengthier or even impossible, can you explain that?

Student 7: Such as minimization and maximization problems, like simple ones we have some, cause I’m also taking pre-calculus, so we were learning, so I was learning two methods for those at the same time, so with pre-calculus, it seems what I thought as very ineffective method of finding minimum or maximum as opposed with calculus.

Interviewer: Can you explain what you mean by lengthier?

Student 7: Like an indirect way.

Interviewer: You mean the logical steps?

Student 7: Right.

Interviewer: You have the survey that I gave you, the one you get earlier today, the second one, wait hold on, good, so if you take a look at Problem 2 Approach 4, so this is basically calculus approach, I mean, have you learned anything like this in class?

Student 7: Yeah, just with two formulas and putting it in terms of one variable and taking the derivatives.

Interviewer: So you've seen something like this before?

Student 7: No, not that problem, but that method of learning I mean.

Interviewer: You said you never had any experience in taking test requiring you to solve in many different ways before.

Student 7: No, I don't recall.

Interviewer: Do you have any experience in the classroom where you want to shout out a new different way, maybe the one different from the teacher's?

Student 7: Well, just in class today we were looking at problem, my teacher asked, you know, is there another way to do this? So there were three methods on the board.

Interviewer: How do you feel about that? Do you get confused many different ways on the board at the same time?

Student 7: No, I think it's interesting, like there isn't just one method to look about it, but there's like different ways of approaching the problem.

Interviewer: Say, if you haven't seen the problem before, say this problem is very new, would you be confused to understand these?

Student 7: Like what my teacher did today was we've been looking at a style of problem before, finding the area between two curves, and she gave us a particular instance or a specific case where there're multiple ways to do it, so that's a topic that we're already familiar with.

Interviewer: Okay, so say you had to do this all over again, you want to solve in many different ways, how would you think of your first approach to solve the problem?

Student 7: You mean what's my rationale for choosing this approach?

Interviewer: Right.

Student 7: I think like I mentioned before, it was, I was trying to solve the problem first.

Interviewer: By solving first, you mean to get the answer?

Student 7: Yes, to get the answer first, then I suppose I would use the other approaches to check it.

Interviewer: What do you mean to check?

Student 7: To check the answer from the previous approach, to see if the other approach is valid.

Interviewer: Do you often do that in your learning experience?

Student 7: Yeah, sometimes, if there's two ways of going about it, like a lot of these seem to be connected to what we've been doing in the past few weeks in class, such as on the test yesterday, like to find the limit definition, I mean to use the limit definition to find the definite integral, which was an example of a rather lengthier method, and part B of that question was to check our answer using the fundamental theorem of calculus.

Interviewer: So you had this kind of test before as a way to check your previous answer? Let me try to understand. You have a test where you have two questions, basically the first one to get the answer and the second one to check the answer you get in the previous question.

Student 7: Right.

Interviewer: How do you feel about that? Say you don't get the answer in the first one, how would you check your answer in the second one?

Student 7: Well, the second one was a different approach, it's a different way of finding the answer, if you get the same one as the first one.

Interviewer: Okay, what if you forget how to do the first question, say you cannot recall the first approach, so same problem, what would you do in the second part of the question?

Student 7: Well, if I don't remember the first approach, I would probably resort to the second approach as a means to solve the problem.

Interviewer: So that means, you're using the second approach as a backup to the first approach?

Student 7: Well, in this case, the first approach that was being tested as the lengthier way of doing it and I would rather not do, I don't know if it's a backup, it's just this specific situation, but I guess you could view the second method as a backup.

Interviewer: I was trying to extract from you whether you actually yourself independently would find to find the second approach if the teacher or test does not require you to find the second approach?

Student 7: I doubt it.

Interviewer: So in the testing situation if the test doesn't require you to solve in many different ways, you would just solve the problem and that's the end of it?

Student 7: Yes.

Interviewer: Okay, now would you have attempted more if the teacher encouraged you to do so? This is different from the testing setting, I mean some teachers would show you many different ways to solve some problem but in the test they probably don't require you to do so, so there's no assessment sort to say.

Student 7: I think in the classroom setting it would be about exploring the topic, like finding different ways of understanding the problem, and in the testing situation, it'd be like, can you do this, can you use any method to find the answer.

Interviewer: So if the method of assessment is changed, in other words, if the teacher requires you to solve in many different ways in test, would you be more likely say at home to study about solving the problem in many different ways?

Student 7: Oh yes, if I knew that would likely to be tested.

Interviewer: Okay so let me get this, so your learning behavior depends on assessment process.

Student 7: I suppose so.

Interviewer: Have you looked at the four approaches for Problem 1?

Student 7: Not really, not in depth.

Interviewer: Let's take a look at this and tell me what you're thinking.

Student 7: I think this is what I did here in Approach 4.

Interviewer: Okay. What about Approach 1? Do you understand this?

Student 7: Not quite, in between those steps, oh okay I see.

Interviewer: What about this one?

Student 7: Okay.

Interviewer: Let’s take a look at the third one. Tell me if it’s okay, if you get it.

Student 7: I’m wondering what you mean by the graph of the square root function is strictly concave, oh okay, oh yeah, I see.

Interviewer: This of course you know, because you did this one. Now tell me which one you prefer.

Student 7: I like number 3 just because it’s not like plugging away at the numbers, you know, like you’re not working or doing the manipulations, you’re looking at something like in the nature of the square root function to solve the problem.

Interviewer: Okay, say you had to do it all over again, which approach would you attempt first? Which one comes up to your mind first?

Student 7: Approach 4.

Interviewer: Second?

Student 7: Approach 1.

Interviewer: How about third?

Student 7: I’m not sure about Approaches 2 and 3.

Interviewer: What about this, would Approach 3 come up on the list?

Student 7: No, I don’t think I would think of it.

Interviewer: Some university professors, and I don’t name the school they’re from, they said that Approach 3 was considered to be beautiful, can you comment on that?

Student 7: I mean, it was like, like I said before, okay, this one, Approach 4, you just kind of hacked away at it.

Interviewer: Can you explain that word?

Student 7: Okay, I mean like you’re taking the numbers, right? And you’re playing with them, and you do this big multiplication, and you finally get this large number is bigger than that large number.

Interviewer: Brute force?

Student 7: Yeah.

Interviewer: So take a look at the second problem and tell me what you're thinking.

Student 7: Okay, I'm not really following Approach 1.

Interviewer: That's okay. We're kind of running out of time. How about Approach 2?

Student 7: Okay.

Interviewer: 3?

Student 7: No.

Interviewer: 4?

Student 7: Yeah, I follow that one.

Interviewer: 5?

Student 7: Yeah.

Interviewer: 6?

Student 7: Yeah.

Interviewer: 7?

Student 7: No, I don't think.

Interviewer: 8?

Student 7: I think so.

Interviewer: Alright, let's do this again, hypothetically, which one would you do first?

Student 7: Approach 4

Interviewer: Can you tell me why?

Student 7: That one makes more sense to me, I don't know it just seems logical to me, you have two equations and you put them together into one.

Interviewer: So your decision is based on your knowledge?

Student 7: Yes, I learned about this.

Interviewer: So it has nothing to do with getting the answer right away?

Student 7: I wouldn't say that.

Interviewer: So your decision, was it because of your knowledge or because you want to get the answer right away?

Student 7: Both, because I can get the answer using that knowledge comfortably.

Interviewer: Okay, then what about the second attempt?

Student 7: Approach 2.

Interviewer: Because you have knowledge of this?

Student 7: Yes.

Interviewer: What about the third one?

Student 7: Nothing really strikes me, probably this one, Approach 5, again it's my knowledge.

Interviewer: So now among all of these eight, which one do you prefer?

Student 7: The one that I like is number 4.

Interviewer: Which one do you think is most beautiful?

Student 7: I'm not sure. Nothing strikes me particularly as beautiful.

Interviewer: Okay, the university professors said Approach 2 was beautiful, can you comment on that or do you see anything beautiful about Approach 2?

Student 7: No, I don't think so, not beauty in the way that like the Approach 3 for Problem 1. This Approach 2 for Problem 2 not so, it seems like a good way to go about it, but it doesn't strike me as particularly beautiful, I mean, I suppose it's similar to the other beautiful problem, it's not so much showing that it's not so much brute force.

Interviewer: Hold on, so it seems to me that you can see something beautiful only if there's other thing that's ugly, you said in the first problem, Approach 3 is beautiful because Approach 4 is brute force?

Student 7: Yes.

Interviewer: But in Problem 2, you don't see anything particularly strikes you as beautiful because you don't see anything ugly or anything with brute force?

Student 7: Right, I see, none of these particularly strikes me as ugly, so I suppose because there's no contrast between the two.

Interviewer: Interesting, okay. Let's take a look at Problem 3. I think you did Approach 1 in the test.

Student 7: Right, I know that.

Interviewer: This is Approach 2.

Student 7: I see.

Interviewer: What about Approach 3?

Student 7: Yeah.

Interviewer: Same question, which one would you attempt first if you had to do it all over again?

Student 7: Approach 1.

Interviewer: Second attempt?

Student 7: Approach 3.

Interviewer: Which one do you prefer among all of the three?

Student 7: I prefer Approach 3.

Interviewer: So why do you choose this one?

Student 7: I don't know, well, Approach 1 seems like what we've been describing as ugly, brute force.

Interviewer: I see, do you see anything beautiful one?

Student 7: If I have to choose, then I would choose Approach 3.

Interviewer: Okay, so the university professors also chose Approach 3 as the beautiful one, can you comment on that?

Student 7: I don't see anything beautiful here, it seems artificial to me, in the other two problems, like you're referring to something about what you, like what you're given, an aspect in order to solve the problem, in this one, you're creating something around it and then using an aspect of that to solve it, the method is, constructing additional information in the process seems artificial, it's not like adding things in general, I'm not sure I would say that negate beauty but in this particular instance, it strikes me as artificial.

Interviewer: Let's go back to the second problem, when you say an approach being beautiful, and the fact that your favorite Approach 4 doesn't look beautiful to you, could it be because you have little knowledge when comparing the eight approaches here?

Student 7: I think beauty has something to do more with experience rather than knowledge, so as I see many different approaches, experience many different approaches, then I will grow sense of something being beautiful.

Interviewer: Okay. Thank you.

Interview with Student 8

Interviewer: How do you feel about the test?

Student 8: I didn't know any of the questions.

Interviewer: Actually, you solved Problem 1, this was the correct answer.

Student 8: Okay.

Interviewer: Have you read this survey?

Student 8: No, I haven't.

Interviewer: Can you flip through these four approaches for Problem 1 and tell me what you think, maybe we can see Approach 4 first because that's the one you did there.

Student 8: Yeah, I understand this.

Interviewer: How about Approach 1?

Student 8: I don't understand how to get from this step to this step, oh, okay, I see, I get it.

Interviewer: Next approach here?

Student 8: Kind of multiplying by conjugates, same thing here, yeah, I get it.

Interviewer: What about this one, this is the third approach.

Student 8: I kind of get it, I guess, I don't really get this part, oh, I see, I get that part.

Interviewer: So then among all of the four, which one do you prefer?

Student 8: The way that I did it, Approach 4, because I thought I was more comfortable with adding and multiplying and I thought the others were more abstract.

Interviewer: Some university professors said Approach 3 is beautiful, can you comment on that?

Student 8: I guess they might think that can be shown the other ways, it's something that, it's a method that's more efficient of solving it, it shows a higher level of understanding it, because you're using calculus concept rather than algebra concept, more sophisticated knowledge.

Interviewer: Remember the sample problem, you can read it, can we compare your experience there with this student's response?

Student 8: I think he's thinking that 100 is an easier number, because it's a multiple of 10 rather than, as opposed to 21 and 79, so it makes it easier just to multiply 7 by 100, very very simple, you're combining two not beautiful numbers 21 and 79, then from there, you get a beautiful number 100, because it's easier to multiply by 100.

Interviewer: So is this a similar situation as your experience for Problem 1?

Student 8: I think that sometimes we want to do things efficiently so we don't even though I realized this approach is more, well, I want to get the correct answer, I think, yeah.

Interviewer: If you had to do this all over again, which one do you prefer?

Student 8: Approach 4 maybe because I know the algebra, I think I've learned this math, long time ago, it's something that's repeating in calculus, pre-calculus and so on, and a lot more comfortable with this than graphing.

Interviewer: So your first attempt will be Approach 4, okay, so here's next Problem 3, and you tried Approach 1 and the answer is 90 degree.

Student 8: Yeah.

Interviewer: Let's take a look at Approach 2 then.

Student 8: I'm not sure, oh, okay, right, okay.

Interviewer: Next Approach 3.

Student 8: Oh okay, oh okay, yeah, okay so, it’s a parallelogram, oh okay, alright, I get it.

Interviewer: Which one do you prefer?

Student 8: The first one because it’s less abstract.

Interviewer: What about your first attempt say if you had to do it all over again?

Student 8: This Approach 1 would be my first attempt.

Interviewer: University professors said that Approach 3 is beautiful, can you comment on that, or do you see anything aesthetically pleasing?

Student 8: I like the shape of parallelogram, this reminds me of a diamond, like more of that shape and I would, I mean that would be more aesthetically pleasing than a circle especially with a triangle in the middle, and there are parallel lines, I think can be aesthetically pleasing, I think has to do with uniformity.

Interviewer: Do you see that same beauty in Approach 1?

Student 8: I don’t think Approach 1 is the most beautiful necessarily, I just think that’s what I would do over again, I wouldn’t, I don’t have the, I think, knowledge has to do with it maybe.

Interviewer: Okay, let’s get back to sample problem, sorry, say you’re learning this problem, or you’re seeing this problem for the first time, and teacher just gives you Approach 2, would you still think that Approach 2 is beautiful without comparing it with Approach 1, because you said it has too much calculation?

Student 8: No, because I don’t consider mathematical expressions to be beautiful, if this were fresh, if I didn’t know about Approach 1, if I wouldn’t be able to compare them at all, I don’t.

Interviewer: So can we say that you have to see an uglier approach to be able to appreciate a beautiful approach?

Student 8: Yeah, I mean I just said that this one is more beautiful because it’s a comparison, comparatively I think the second one is more beautiful than the first approach, however, if you just asked me if I thought either of them would be beautiful, I don’t think that they’re beautiful.

Interviewer: So beauty is only observable by comparison?

Student 8: Yeah, in this case, mathematically beautiful, I mean.

Interviewer: Okay, what would you do if a teacher shows you 8 different approaches for a new problem?

Student 8: Well, I would look for the one that seems easier, and that the one I would do the next time.

Interviewer: What about the rest of the approaches?

Student 8: I would only learn them if they are easier, but I wouldn't be that interested mastering them if I still find them more difficult.

Interviewer: Would you be confused about the purpose of the lesson, whether it was to learn the different approaches or whether it was to solve the problem itself?

Student 8: I wouldn't be concerned, because I'm assuming that they're showing us different approaches so that we can just gain greater understanding than just to trick us out.

Interviewer: Okay, in the testing environment, do you have any experience taking a test that requires you to solve in many different approaches?

Student 8: Not really, well, maybe in only two different ways, but that's it.

Interviewer: Okay, I want you to see your two approaches for Problem 1, I actually think that they look the same, do you feel that too, or do you feel that at that time you're actually doing two completely different approaches?

Student 8: Yeah, I was having trouble at that time, I guess, being flexible in solving the problem, or looking at it in a different way, but I think I get the answer right in the end.

Interviewer: When we talked about beautiful approaches, you said you need more than one to compare, do you have any classroom experience in this aspect?

Student 8: I think often teachers refer to messy numbers and that's often like mixed numbers or maybe sometimes odd can be quote and quote messy numbers, so when you have nice numbers like whole numbers, they'll be nice.

Interviewer: Let's say you haven't seen Approach 2 in sample problem, and say you only know about Approach 1, do you think you get that Approach 2 somehow?

Student 8: Yeah, I think if I keep doing the same kind of problems over and over again, I would realize that if I keep doing Approach 1, it would be tedious and somehow I would probably realize that there must be a shortcut to that, so I mean, maybe I will get that after a while but a lot longer if I don't get to do Approach 1 repeatedly.

Interviewer: Wouldn't you be bored if, or if your teacher keeps drilling you with the same problem like that and you keep doing Approach 1?

Student 8: Yeah, I mean after a while, maybe a good learning experience, drills are helpful even though it's boring, I mean, I think math can be kind of boring sometimes.

Interviewer: What about the aesthetic appreciation, what if you're given the Approach 2 first without Approach 1?

Student 8: I still think that, wait, what, well, I think you need to see the ugly approach first before the beautiful approach, otherwise it wouldn't be known that it's a shortcut, I mean, yeah, like it's actually shorter because the other one is longer, but if it's shorter than the one after that, I think, it's just like that, that one wouldn't be looked as much anyway, so there's no need to do too much comparison in that case.

Interviewer: So boring is good?

Student 8: Sometimes, but I kind of think, I would rather see the more beautiful one rather immediately, even though I might not realize there is the beauty in it rather than just see it the next one later, I would rather take the easy way out sort to speak.

Interviewer: Okay, thank you.

Interview with Student 9

Interviewer: How do you feel about the test?

Student 9: Ah, it's kind of hard, it's also like, it's kind of hard to think of different ways and I couldn't really think of many different ways.

Interviewer: So was this the first time you're taking test requiring you to solve in many different ways?

Student 9: Well, I mean, most of the tests I take is just like get an answer.

Interviewer: How did the instruction to solve in many different ways affect your overall performance?

Student 9: I guess, maybe a bit negatively, but it was just like a different kind of questions, I guess.

Interviewer: Different, so you haven't seen any of these questions before?

Student 9: Well, well, these questions aren’t really like the things in my math test, they’re more like questions in AMC kind of thing.

Interviewer: I see, let’s take a look at Problem 1, here you did Approach 4 first, and you got the answer right on, and the second one you tried but wasn’t successful, and then you had a third one that looks like the first one, here in the survey you said your favorite approach is Approach 1, so can tell me what were you thinking at that time?

Student 9: So for Approach 4, when you gave me that problem, thing is, because especially it’s a square root, your first instinct is like, making them into integers, like squaring, but if you asked me to find the most efficient way to do it, then this Approach 1 would be better, I think because you can clearly see that this is going to be 1 less, so this is obviously less than that.

Interviewer: So your first attempt would be to get the answer as quickly as possible?

Student 9: Yeah, like instinct tells you to do that right away just like that squaring thing.

Interviewer: So if you think Approach 1 is more efficient, why not come up with the more efficient approach on your first attempt because you’ll spend less time on that, no?

Student 9: Oh, no, I think, it will take more time to think about the more efficient way to do it, I think, whereas me facing the problem, you just start doing it, then you feel like actively working on it, so that’ll help you try to get the answer quicker, cause like when you’re like doing something, as opposed to like sit there and doing nothing.

Interviewer: So Approach 1 takes longer time to think about the concepts, Approach 4 takes longer time to do the calculations, so you’re compensating on the time to think more by doing the calculations, because time to do calculations is still less than time to think about the concepts?

Student 9: Yeah, it makes sense, because you just want to get answer anyway, I guess, when you’re thinking to solve a problem, you don’t like consider many different ways, and think like, oh, this is the most efficient way, let me do it that way, you just think of a way to do it, and then you do it that way.

Interviewer: I see, let’s talk about a situation where a teacher’s showing you 8 different approaches at the same time to solve one problem like the one here.

Student 9: Well, I think, it’ll be a bit overwhelming, if he shows me three or four different ways to solve it, then that will be helpful because well you can pick which, which way to solve it, you think best for you to do it, and you just do it that way, I mean, it depends, like who it is, some people might find one way easier than another way, or like, if you, I mean some people are like good doing it one way, then it doesn’t matter like, then they will be more efficient and faster in doing it that way, than like forcing themselves to do it their way.

Interviewer: How about in terms of recalling the concepts?

Student 9: I think the geometry will be easier to recall, because the Trigonometric identities are like, they're very precise, like you have to know them, like very, you have to have those specific things memorized, whereas for the geometric one, it's like graphical and you can see what you're doing.

Interviewer: Your favorite approach you said here is Approach 2 and it's also your first attempt so to speak, can you tell me more about that?

Student 9: Well, I just thought that'll be the best way to do it and I like that way.

Interviewer: What about Approach 4, I mean because you're in calculus class, you put that only as your second attempt?

Student 9: I guess I'm just more comfortable with geometry, especially because you have many experience with geometry, whereas calculus I'm just learning right now.

Interviewer: Your favorite mathematics topic as you said in the first survey was algebra, actually, there's an algebraic approach here, Approach 5 you put that as your third attempt, can you explain?

Student 9: I don't really think that way, I mean, I like algebra, but I don't, I won't do algebraic approach all the time for all kinds of problems, like this calculus problem, I think it's more like, when you see a problem, you just try to solve it the way any kind of way that's easiest or comes to you first, I think it depends a lot on a problem, because some problems you can solve in multiple ways, different ways to solve it, and then whereas some problems are strictly calculus, if they say specifically find the derivative of this, then you would think of calculus, you wouldn't think of, like oh, I may solve this in algebraic way.

Interviewer: So you think there's a clue in the problems, what about Problem 2 here, do you see explicit clue that makes you think you need to solve it this way or that way?

Student 9: Well, x^2+y^2=1 and my first thought it's like a circle, so that what I thought, so I go with that.

Interviewer: What about Problem 3, in test you only solved it one way, but it's correct, Approach 1, you have here, also in your survey you said you like Approach 1, it's your favorite, can you tell me more about it?

Student 9: Yeah, cause it's easier for me to understand, to follow, I was able to do it that way, faster, I think, to get the right answer.

Interviewer: I see, can we go back to Problem 1, can you tell me which one do you think is beautiful?

Student 9: Well, I kind of think it's this one, Approach 3, just cause it doesn't have so many, like here Approach 2, it goes down, it has so many radicals, but, I can't think of the right word, but this one is kind of like a sentence, it invokes, kind of neater approach, I think.

Interviewer: So the one that's aesthetically pleasing is the one that has fewer lines in it?

Student 9: Oh, no, I don't think so, I think it's like, I like this one, Approach 3, because it's kind of clear that, it's done in a logical way, like people are able to follow along, and fairly straightforward, you can clearly tell that, oh because of this, this is less than that, and I don't know, just more logical, but Approach 1 is easier to follow, I don't know.

Interviewer: That's okay, some university professors said that Approach 3 was considered beautiful, can you comment on that?

Student 9: Well, I guess, because it's laid down in a very straightforward, a very matter of fact kind of way, and it's like very short and simple.

Interviewer: Short, does that have anything to do with the number of lines of arguments there?

Student 9: Maybe, I guess.

Interviewer: What about Problem 2?

Student 9: Well, like I said, my first thought, it's a circle, I like this one the best, it's kind of aligned with my way of thinking, it's just, it's just, I think geometry lays out very graphically, so you can see like how it works, and then for this problem, you see the circle, you see the triangle and the congruency.

Interviewer: Do you see anything beautiful in this approach?

Student 9: I think just the fact that you can graph it and see what you're doing physically, as opposed to like more theoretical proof.

Interviewer: What if I didn't give you this graph, I mean, what if I just gave you these lines, explanations without the picture here?

Student 9: Well, if you don't have the graph, then it would be more, it would be more kind of, more jumbled up, or more confusing because you'll have a lot of text and you read through, and you won't be able to have something to attach yourself to, like round yourself, like oh, angle ABO, you're like, I don't know what that is.

Interviewer: Approach 2 was considered beautiful by university professors, can you comment on that?

Student 9: I guess in a way that the arguments follow each other, and how it's proved, the maximum is proved in a very logical way, like because this is this, then it must follow, then I guess I can see what they mean, but I guess I consider this stuff to be different from what they think.

Interviewer: That's fine, what about Problem 3?

Student 9: I like Approach 1 because you can see the triangle, with different parameters they give you, it's just like logical kind of approach, it's just because of this, because of this, it follows like this, I don't know, it's easier to understand.

Interviewer: What about Approach 3, it's also considered beautiful by university professors, can you comment on that?

Student 9: I see what they're doing, but I feel like constructing two parallel lines makes it more, just like more work that what it's needed.

Interviewer: So you don't think that Approach 3 is efficient?

Student 9: No, I don't think it's efficient, sorry.

Interviewer: That's okay, let's talk about your classroom experience, have you had teachers showing you many different ways to solve a problem?

Student 9: I have many teachers who mentioned alternate ways of solving things, I think, one or two may have accept alternate ways of solving it on the tests, but for the most part, when we're going over one problem, I usually go for one way to do it.

Interviewer: Say, have you had someone in your class showing teacher a different way of solving a problem than the one the teacher had, what was the teacher's reaction?

Student 9: Yeah, if it's right, they'll be okay with that, they kind of encourage us to think different ways, but on the test, they want it to be done a certain way on the test.

Interviewer: Okay, now I want to ask whether you need to see uglier approach, brute force maybe, in order to appreciate beautiful approach, can you say anything about this?

Student 9: I think when you have solution that goes like a brute force method, there's a reason why it's called brute force, it's not like a pretty picture that paints in your head, but I don't really think in terms of what's beautiful, what's ugly in this kind of situation, as more as like what's efficient, or what's not efficient.

Interviewer: Okay, thank you, that's good.

APPENDIX N

Students' Validation Survey

Name: ________________________________ Date: ___________________

There is no right or wrong answer. Your responses will not influence your school grade in any way. Please be as honest as you can in filling out this survey.

Reminder – Please avoid discussing questions involved in this survey with other students. Thank you.

Part 1

Please refer to the attachment. There are a total of 15 approaches for three problems: 4 approaches for Problem 1; 8 for Problem 2; and 3 for Problem 3.

Q1. In terms of your understanding of these approaches, please rate each approach according to the following rubric:
2 – I understand all of the steps/reasoning behind this approach.
1 – I understand some of the steps/reasoning behind this approach.
0 – I do not understand any of the steps/reasoning behind this approach.

Q2. In terms of your mathematics knowledge, please rate each approach according to the following rubric:
2 – I have previously learned all of the necessary mathematics knowledge involved in this approach.
1 – I have previously learned some of the necessary mathematics knowledge involved in this approach.
0 – I have not previously learned any mathematical mathematics involved in this approach.

Q3. What do you think is the necessary mathematics knowledge involved in this approach?

Q4. If you had to do this problem (hypothetically speaking), what is the chance that you would attempt to solve the problem using this approach?
2 – I would be very likely to solve this problem using this approach.
1 – I might be able solve this problem using this approach.
0 – I would not have thought of solving this problem using this approach.

Q5. If you had to do this problem (hypothetically speaking), which approach would you choose in your 1st attempt to solve this problem? Why? What would be your 2nd, 3rd? (Choose only up to 3 approaches if more than 3 approaches are involved.)

Q6. Among all approaches, please choose your most favorite approach and explain why.

Example of student's response for Sample Problem 0:

Problem	Approach	Q1. Understanding of Logical Reasoning (0/1/2)	Q2. Necessary Mathematics Knowledge (0/1/2)	Q3. Necessary Mathematics Knowledge	Q4. Likelihood of Future Presentation (0/1/2)	Q5. Hypothetical Sequence of Future Presentation
e.g. 0	1	2	2	Basic multiplication and addition	2	1^{st}
e.g. 0	2	2	2	Factorization and number sense that 21&79 make up 100 which is easier to multiply than 21 or 79 alone	1	2^{nd}

Q5. Explanations for hypothetical sequence of future presentation: *I choose Approach 1 as my 1^{st} attempt because I am more confident with my ability in solving the problem using Approach 1 so as to obtain the correct answer but not so much using Approach 2, although Approach 2 is the one I prefer the most. If I had enough time taking the test, I would try to think of Approach 2 as my 2^{nd} attempt.*

Q6. My most favorite approach is Approach *2 because I am too lazy to multiply 21 by 7 and 79 by 7 and add them up together. I would rather multiply 100 by 7.*

Student's response for Problem 1:

Problem	Approach	Q1. Understanding of Logical Reasoning (0/1/2)	Q2. Necessary Mathematics Knowledge (0/1/2)	Q3. Necessary Mathematics Knowledge	Q4. Likelihood of Future Presentation (0/1/2)	Q5. Hypothetical Sequence of Future Presentation
1	1					
1	2					
1	3					
1	4					

Q5. Explanations for hypothetical sequence of future presentation: ___________________

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Q6. My most favorite approach is Approach ______ because ________________________

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Student's response for Problem 2:

Problem	Approach	Q1. Understanding of Logical Reasoning (0/1/2)	Q2. Necessary Mathematics Knowledge (0/1/2)	Q3. Necessary Mathematics Knowledge	Q4. Likelihood of Future Presentation (0/1/2)	Q5. Hypothetical Sequence of Future Presentation
2	1					
2	2					
2	3					
2	4					
2	5					
2	6					
2	7					
2	8					

Q5. Explanations for hypothetical sequence of future presentation: ___________________

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Q6. My most favorite approach is Approach ______ because _______________________

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Student's response for Problem 3:

Problem	Approach	Q1. Understanding of Logical Reasoning (0/1/2)	Q2. Necessary Mathematics Knowledge (0/1/2)	Q3. Necessary Mathematics Knowledge	Q4. Likelihood of Future Presentation (0/1/2)	Q5. Hypothetical Sequence of Future Presentation
3	1					
3	2					
3	3					

Q5. Explanations for hypothetical sequence of future presentation: ___________________

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Q6. My most favorite approach is Approach ______ because ________________________

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<u>Part 2</u>

Please rate each statement below according to the following rubric:
5 – I strongly agree with this statement.
4 – I agree with this statement.
3 – I neither agree nor disagree with this statement.
2 – I disagree with this statement.
1 – I strongly disagree with this statement.

S1	I learn more from solving one problem in many different ways than I can from solving many different problems, each in only one way.	
S2	Learning how to solve one problem in many different ways provides me with access to a range of representations and solution strategies in a particular instance that can be useful in future problem solving encounters.	
S3	Learning how to solve one problem in many different ways facilitates connection of a problem at hand to different elements of knowledge with which I may be familiar, thereby strengthening networks of related ideas.	
S4	Learning how to solve one problem in many different ways motivates me to be more creative in seeking the solution of the problem.	
S5	Learning how to solve one problem in many different ways improves my critical thinking skills by comparing and contrasting the many different ways.	
S6	Learning how to solve one problem in many different ways improves a deeper understanding of subject matters by looking at different perspectives.	
S7	Learning how to solve one problem in many different ways provides me with backup strategies when I could not recall a typical solution to the problem.	
S8	I need guidance from my mathematics teachers in order to learn how to solve one problem in many different ways.	
S9	In a classroom setting, I often have many ideas of solving one problem in many different ways.	
S10	In a classroom setting, my mathematics teachers often encourage students to solve one problem in many different ways.	
S11	In a classroom setting, my mathematics teachers often do not have time to facilitate students who initiate discussion of solving one problem in many different ways.	
S12	In a classroom setting, I often feel hesitant to share my ways of solving a problem that are different from the ones my mathematics teachers demonstrate on the board.	
S13	If my mathematics teachers demonstrate incorrect approaches to a problem, I will learn to avoid such mistakes in my future problem solving encounters.	
S14	If my mathematics teachers demonstrate many different approaches to solve one problem, I will often become distracted with those many different approaches.	

S15	If my mathematics teachers demonstrate many different approaches to solve one problem, I will often become confused whether the goal of the lesson is to solve the problem or to learn those many different approaches.	
S16	If my mathematics teachers demonstrate many different approaches to solve one problem, I will often become indifferent since there has not been a standardized test (in school, state, or national context) that involves a particular testing instruction to solve in many different approaches.	
S17	If my mathematics teachers demonstrate many different approaches to solve one problem, I will often become bored by those many different approaches.	
S18	If my mathematics teachers demonstrate many different approaches to solve one problem, I will often become worried that I will have to struggle to understand barely one approach, especially when seeing a new problem for the first time.	
S19	If my mathematics teachers demonstrate many different approaches to solve one problem, I will often care for only one approach so long as I can solve the problem.	
S20	If my mathematics teachers demonstrate many different approaches to solve one problem, I will often care for only approaches that are easiest for me to understand.	
S21	If my mathematics teachers demonstrate many different approaches to solve one problem, I will often compare and contrast some of the advantages and disadvantages of these approaches.	
S22	If my mathematics teachers demonstrate many different approaches to solve one problem, I will be able to choose one or some approaches (among those many different approaches) that will help me understand better the context of the problem.	
S23	If my mathematics teachers ask me to solve one problem in many different approaches in a classroom setting, I will come up with at most one approach or will be most likely to sit and wait until all approaches are presented on the board.	
S24	I really wish that there could be more occasions where my mathematics teachers demonstrate many different approaches to solve one problem.	
S25	I really wish that there could be more opportunities for me to solve one problem in many different approaches in a classroom setting.	