

THE MATHEMATICAL CONTENT KNOWLEDGE OF PROSPECTIVE TEACHERS  
IN ICELAND

Björg Jóhannsdóttir

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## ABSTRACT

### THE MATHEMATICAL CONTENT KNOWLEDGE OF PROSPECTIVE TEACHERS IN ICELAND

Björg Jóhannsdóttir

This study focused on the mathematical content knowledge of prospective teachers in Iceland. The sample was 38 students in the School of Education at the University of Iceland, both graduate and undergraduate students. All of the participants in the study completed a questionnaire survey and 10 were interviewed.

The choice of ways to measure the mathematical content knowledge of prospective teachers was grounded in the work of Ball and the research team at the University of Michigan (Delaney, Ball, Hill, Schilling, & Zopf, 2008; Hill, Ball, & Schilling, 2008; Hill, Schilling, & Ball, 2004), and their definition of common content knowledge (knowledge held by people outside the teaching profession) and specialized content knowledge (knowledge used in teaching) (Ball, Thames, & Phelps, 2008).

This study employed a mixed methods approach, including both a questionnaire survey and interviews to assess prospective teachers' mathematical knowledge on the mathematical topics numbers and operations and patterns, functions, and algebra.

Findings, both from the questionnaire survey and the interviews, indicated that prospective teachers' knowledge was procedural and related to the "standard algorithms" they had learned in elementary school. Also, findings indicated that prospective teachers had difficulties evaluating alternative solution methods, and a common denominator for a difficult topic within both knowledge domains, common content knowledge and

specialized content knowledge, was fractions. During the interviews, the most common answer for why a certain way was chosen to solve a problem or a certain step was taken in the solution process, was “because that is the way I learned to do it.”

Prospective teachers’ age did neither significantly influence their test scores, nor their approach to solving problems during the interviews. Supplementary analysis revealed that number of mathematics courses completed prior to entering the teacher education program significantly predicted prospective teachers’ outcome on the questionnaire survey.

Comparison of the findings from this study to findings from similar studies carried out in the US indicated that there was a wide gap in prospective teachers’ ability in mathematics in both countries, and that they struggled with similar topics within mathematics.

In general, the results from this study were in line with prior findings, showing, that prospective elementary teachers relied on memory for particular rules in mathematics, their knowledge was procedural and they did not have an underlying understanding of mathematical concepts or procedures (Ball, 1990; Tirosh & Graeber, 1989; Tirosh & Graeber, 1990; Simon, 1993; Mewborn, 2003; Hill, Sleep, Lewis, & Ball, 2007).

The findings of this study highlight the need for a more in-depth mathematics education for prospective teachers in the School of Education at the University of Iceland. It is not enough to offer a variety of courses to those specializing in the field of mathematics education. It is also important to offer in-depth mathematics education for those prospective teachers focusing on general education. If those prospective teachers

teach mathematics, they will do so in elementary school where students are forming their identity as mathematics students.

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## I Introduction

### Need for the study

*“Students learn mathematics through the experiences that teachers provide. Thus, students’ understanding of mathematics, their ability to use it to solve problems, and their confidence in, and disposition toward, mathematics are all shaped by the teaching they encounter in school” (NCTM, 2000, p.16).*

Elementary school teachers play an important role in mathematics education. They provide the foundation of computation and mathematical reasoning needed for further mathematics education. A teacher with little grasp of mathematics or/and prior experience has little room for progress or novelty in the classroom or the ability to fuel students’ interest in the subject.

The knowledge mathematics teachers possess, or should possess, has been of interest within the scholarly community for some time. The kind of knowledge desirable for mathematics teachers has interested researchers as well (Hill et al., 2007). Many studies sought but failed to find a strong relationship between teachers’ mathematical content knowledge and student achievement. However, studies have shown a relationship between student achievement and teachers’ performance on mathematical tasks in the context of teaching (Mewborn, 2003; National Research Council, 2001).

More than 25 years ago Shulman (1986) introduced the term “pedagogical content knowledge”, drawing attention to the special kind of knowledge teachers need.

“Pedagogical content knowledge” in mathematics is a combination of mathematical

content and pedagogy (Hill et al., 2007). This special knowledge mathematics teachers need has also been stressed by The National Research Council (2001) as they describe how teachers' knowledge of the content entails choosing the appropriate content, deciding its representation and determining how much time to spend on it, and how teachers' knowledge of the students involves assessing students' current mathematical thinking, strategies for representing the material and meeting their students' learning needs.

Building on Shulman's categories of teachers' knowledge, Ball and her colleagues at the University of Michigan have, as a result of their research, introduced four domains of teachers' mathematical knowledge. These domains are: common content knowledge (CCK), specialized content knowledge (SCK), knowledge of content and students (KCS), and knowledge of content and teaching (KCT). Common content knowledge is the mathematical knowledge and skills used in settings other than teaching (e.g. correctly solving mathematics problems, using terms and notations). Specialized content knowledge is the mathematical knowledge and skills unique to teaching (e.g. looking for patterns in errors, assessing generalizability of nonstandard solution methods). Knowledge of content and students is the awareness of the interaction between the students and the content (e.g. what topics or concepts confuse, interest or motivate students, what they find hard/easy), while knowledge of content and teaching is the merge of mathematics and teaching (e.g. in what order to teach content, how to represent it, what examples to use) (Ball et al., 2008).

Research on mathematics teaching in the US, especially those studies stressing the importance of teachers' knowledge of mathematics and how children learn mathematics,



have increased in this decade. Findings indicate that some US teachers may not have deep and rich understanding of mathematics, or “what it takes” to educate students in the 21<sup>st</sup> century (Brown & Borko, 1992; Mewborn, 2003).

With respect to prospective teachers’ mathematical knowledge, studies have shown that it is fragmented and procedural (Hill et al., 2007). Prospective teachers rely on memory for particular rules in mathematics and do not have the underlying understanding of concepts and procedures (Borko & Brown, 1992). For example, only 27% of the 600 elementary school teachers taking the Massachusetts states licensing exam in 2009 passed the mathematics section the first time round (Vaznis, 2009). No similar studies have been completed in Iceland, leaving the mathematical content knowledge of prospective teachers unknown.

Research has shown that with increased and deepened understanding of mathematics, teachers’ practices change (Mewborn, 2003), and that mathematics methods courses can serve as a catalyst for prospective teachers’ interest in mathematics (Macnab & Payne, 2003). Caswell’s (2009) findings indicate that in order for students to have a positive attitude towards mathematics and to value the subject, they must be given quality mathematics instructions from early age. When that is done a positive attitude towards mathematics is shaped for the next generation of teachers.

## **Purpose of the study**

Students preparing to become teachers in Iceland are a diverse group. Some enter the teaching preparation program directly after high school/college, while others enter several years later, sometimes after teaching for a while as instructors<sup>1</sup>.

The purpose of this study was threefold: first to determine the level of elementary mathematical content knowledge among prospective teachers in Iceland; second, to examine the influence of age and teaching experience on prior mentioned knowledge, and third, to see how the results of this study compared to results from similar studies conducted in the US.

Research questions to be answered by the study included:

- I. What is the level of elementary mathematical content knowledge among prospective Icelandic teachers? More precisely, (a) What is the level of their common content knowledge?, and (b) What is the level of their specialized content knowledge?
- II. Does age and/or teaching experience prior to entering teacher education program influence levels of knowledge reported in question I?
- III. How does the level of elementary mathematical content knowledge reported in question I compare to findings from similar studies using the MKT measures carried out in the US?

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<sup>1</sup> An instructor in an Icelandic school is an employee hired to teach without having the Ministry's of Education, Science and Culture right to use the job title elementary teacher (Ministry of Education, 2009).

## **Procedures of the study**

The participants in the study were 38 students preparing to become teachers at The School of Education, University of Iceland. The study utilized mixed data-collection methods, including printed questionnaires and interviews.

Ball and her colleagues, through the Learning Mathematics for Teaching project (LMT project<sup>2</sup>), have developed an instrument to measure the mathematical knowledge needed for teaching, the MKT measures. This instrument focuses on the mathematics problems encountered in teaching, such as teachers' representations of content, explanations, teaching moves and students' solutions methods and errors (Hill et al., 2007). The researcher got permission to use the MKT measures, by attending a required workshop (LMT, 2012), in April 2011.

For the present study, a questionnaire survey was developed by translating and adapting selected items from the MKT measures. The items were selected and adapted to be in line with the Icelandic school community and The Icelandic National Curriculum for Mathematics. The questionnaire survey consisted of items from two mathematical categories: numbers and operation, and patterns, function and algebra. The items in the questionnaire survey reflected both whether prospective teachers could answer plain mathematics problems and how they solved special mathematical tasks that could arise in teaching the subject.

The questionnaire survey was administered to all 38 prospective teachers to assess their knowledge of elementary mathematics. A sub-sample of ten prospective teachers was selected for interviews to elicit their way of thinking while solving problems as well

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<sup>2</sup> LMT: Learning mathematics for teaching, see: <http://sitemaker.umich.edu/lmt/home>

as to probe their mathematical explanations. The age as well as teaching experience of interviewees varied. Of the interviewees, two came from each end of the scale (high mathematical knowledge – low mathematical knowledge) and six came from the middle.

Using transcripts from the interviews and data from the questionnaires, both qualitative and quantitative analysis was applied to answer the research questions. The combination of quantitative and qualitative analyses was used in an attempt to enhance the strength in which conclusions could be reached, when patterns from one method repeated themselves in the other method.

Statistical analysis of questionnaire responses and transcripts from interviews provided answers to the first research question. To answer question two, regression analysis using the questionnaire data collected was used to compare participants with regards to their teaching experience and age. Also, the correlation between the variables and test score was examined. Transcripts from interviews were further used to evaluate participants and see if their solution methods and explanations varied according to their age and/ or teaching experience. Results from the questionnaire survey were compared to results from similar studies carried out in the US to provide answer to the third research question.

## **II Education in Iceland**

### **Educational system**

Iceland is a republic with a population around 320,000. The majority of the nation, or around 65%, lives in the capital Reykjavik and the surrounding area (Statistics Iceland, 2013).

The educational system in Iceland is divided into preschool, compulsory, upper secondary and higher education. The compulsory education spans from 1<sup>st</sup> to 10<sup>th</sup> grade and is mainly funded by the state. The Ministry of Education, Science and Culture is responsible for education in Iceland and issues the National Curriculum Guidelines. The National Centre for Educational Materials is in charge of publishing and distributing teaching materials and the Educational Testing Institute oversees all implementation and grading of national assessments. During primary education, grade 1 to 7, it is common for the same teacher to teach all academic subjects to students. From 8<sup>th</sup> grade and on, there are usually specialized teachers for each subject. Students in Iceland take standardized tests at the end of 4<sup>th</sup>, 7<sup>th</sup>, and 10<sup>th</sup> grade (Namsmatsstofnun).

### **Numbers in education**

There are 173 schools in Iceland for grades 1-10 (Ministry of Education, 2009). In the year 2011 there were 42,364 students in classes 1-10 in Iceland, of those 25,904 students resided in or around the capital, Reykjavik. Out of 4,743 people that taught in 2011, 4,531 were certified teachers and 212 were instructors. Around 70% of the instructors worked in schools outside the capital area. This ratio between instructors and certified teachers has decreased since the fall of the economy in 2008. In the years 2004-

2007, around 14% of those teaching, were not certified teachers (Statistics Iceland, 2012; Ministry of Education, 2009). Of people teaching in 2011, 80.1% were women (Statistics Iceland, 2012).

### **Teacher education**

New laws regarding teacher education in Iceland were passed July 1<sup>st</sup> 2008 and came into effect on July 1<sup>st</sup> 2011. The year 2011 was the last year where three years of preparation was needed to become a certified teacher. Now, becoming a certified teacher requires a master's degree and 5 years of preparation at a teachers college. With the new laws in effect, the numbers of applicants for teacher education have dropped. This school year, 2012-2013, 397 students are studying for a B.Ed degree at the School of Education, University of Iceland. Of those 397 students, 194 began their studies this fall, and 34 are specializing in elementary education. The M.Ed department at the School of Education, University of Iceland, has 61 registered students (Gudmundsdottir, 2012).

To become a teacher, a three-year theoretical and professional undergraduate program (180 ECTS credits) and a graduate program (120 ECTS credit) are needed. Admission requirements for the undergraduate program are a matriculation examination or equivalent education.

At the University of Iceland, The School of Education offers three lines in the undergraduate program for prospective teachers. The lines are: General teacher education, Subject teacher education and Early childhood education. The General teacher education line is meant for those who want to teach in grades 1-10 and specialize in two subjects. The Subject teacher education is a line for those who wish to specialize in one subject and get qualifications to teach both in primary school and beginners' courses in

upper secondary schools, grades 1 – 12. The line Early childhood education is intended for those wanting to teach grades 1 – 4.

All three lines share core courses that add up to 100 ECTS credits. These core courses consist of courses in the foundation of education and pedagogy, developmental psychology, sociology, and philosophy and preparation courses for academic studies. One of the core courses is *Icelandic and Mathematics in compulsory education*, where the focus is on these subjects' place in the curriculum and their connection to other subjects.

Within the General teacher education line prospective teachers choose two elective fields<sup>3</sup>, each worth 40 ECTS credits. Two of the 12 elective fields contain mathematics course(s), the fields Mathematics and Early basic school teaching. The mathematics course in the Early basic school teaching field is called *Teaching mathematics to young students* and focuses on how to build a continuance in students' education and to strengthen the mathematical foundation of the prospective teachers. The mathematics courses in the Mathematic elective field for 40 ECTS credits are: *Algebra and functions, Numbers, logic and arithmetic, Teaching mathematics to teenagers, and Geometry*.

Within the Subject teacher education line prospective teachers choose one elective field<sup>4</sup> worth 80 ECTS credits. On this line, the elective field Mathematics is the only field containing mathematics. All the students that choose mathematics take the courses:

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<sup>3</sup> The elective fields are: Early basic school teaching, Foreign languages (English, Danish), Design and woodwork, Icelandic, Food-culture-health, Arts and crafts, Natural sciences (biology, physics, chemistry, geography), Social studies (sociology, history, Christianity studies, life skills, geography), Mathematics, Music, theatre and dance, Textiles, Information technology and media.

<sup>4</sup> The elective fields are: Foreign languages (English, Danish), Design and woodwork, Icelandic, Food-culture-health, Arts and crafts, Natural sciences (biology, physics, chemistry, geography), Social studies (sociology, history, Christianity studies, life skills, geography), Mathematics, Music, theatre and dance, Textiles, Information technology and media.

*Algebra and functions, Numbers, logic and arithmetic, Geometry, Teaching mathematics to teenagers, and Teaching mathematics.* Additionally they can choose from a variety of mathematics courses including *Calculus, Discrete mathematics, Linear algebra, Number theory* and more.

The line Early childhood education is designed for those wanting to teach all academic subjects to young learners. Their core course, *The development of language and education of young children*, involves theories of how children learn mathematics. One of the elective courses within the Early childhood education line is *Teaching mathematics to young students*. This is the same course that is offered in the elective field Early basic school teaching in General teaching education.

Prospective teachers receiving first grade from the undergraduate program qualify for graduate studies. The graduate studies leading to a Masters degree are intended to strengthen students' knowledge in their field of specialization as well as to reinforce them as professionals and researchers. Pursuant to the undergraduate studies, the graduate programs are Early childhood in pre- and primary schools (17 students the academic year 2012-2013), General teacher education (8 students the academic year 2012-2013), and Teacher education – subject specialization (36 students the academic year 2012-2013) (Gudmundsdottir, 2012). The graduate programs are 120 ECTS units. All the graduate programs offer a line, pedagogy, where students can choose mathematics education for 40 ECTS units. Alternatively, if the graduate line, Teaching young children is chosen, there is a course, *The development of children's mathematical ideas*. Other graduate routes do have mathematics education as an elective (University of Iceland, 2012).



## **National curriculum in mathematics**

In Iceland there is a national curriculum in mathematics composed and published under the supervision of The Ministry of Education, Science and Culture. The current curriculum in mathematics, published in 2007, is based on a curriculum from 1999 with only minor changes in the form of increased emphasis on problem solving, reasoning and the use of mathematics in everyday life (Ministry of Education, 2007). In the curriculum, the subject is divided into six categories: Numbers, Arithmetic, procedures and estimation, Proportions and percentages, Patterns and algebra, Geometry, Statistics and probability (Ministry of Education, 2007). The curriculum emphasizes that students should be taught in such a way that mathematics is perceived as a whole, and doing mathematics is a process and a creative activity. It also emphasizes the importance of the co-existence of understanding and skill or proficiency. According to the curriculum it is important to connect students' tasks to their environment and use visual aid to support students' understanding in elementary school (Ministry of Education, 2007).

New laws regarding education at all levels were passed in Iceland in 2008. As a result, curriculum within each field has been reviewed, including the curriculum for mathematics (The Ministry of Education, Science and Culture, 2011).

In 2012 the Ministry of Education, Science and Culture published a draft for a curriculum in mathematics. It is for the most part similar to its predecessor, but more emphasis is now placed on different solution methods, including students own solution methods. Students are supposed to be creative in mathematics and encouraged to develop or come up with their own methods, discuss them and study their fellow students' methods (The Ministry of Education, Science and Culture, 2012). With regards to

teaching methods, the new draft neither specifies the way lessons should be structured as the previous one did, nor does it emphasize group work as was done in the 2007 version. Instead the emphasis is on students' ideas, methods and mistakes and the use of those as a base for discourse and understanding of mathematics. In this curriculum draft the subject is divided into four categories instead of six, the four categories being: Numbers and operations, Algebra, Geometry and measurement, and Statistics and probability. Another notable difference is that the goals at the end of each period are fewer and not as specific as before (The Ministry of Education, Science and Culture, 2012).

Following are the subject goals students should have mastered at the end of fourth grade according to the current curriculum in the subject categories of numbers and operations and patterns and algebra. *Italic* are additional goals and changes made in the draft for a new curriculum.

According to the National Curriculum in Mathematics (Ministry of Education, 2007), students should at the end of fourth grade:

- Numbers:** ➤ **Know natural numbers and have been introduced to rational numbers and integers, more precisely:**
- Show understanding of the natural numbers 1 – 1000
  - Be able to read, write, show on a calculator and use in discourse and other settings numbers as big as 10,000
  - Be able to order and compare natural numbers, both with words and symbols
  - Know a way to count items in a set as well as having the ability to estimate items in a set and compare estimation with real number
  - Know negative numbers in natural settings, e.g. on a thermometer and a calculator
  - Be able to use decimals with one or two digits
  - Show understanding of a half, third, quarter, fifth, tenth and

hundredth of a whole

➤ **Have been introduced to simple concepts of number theory, more precisely:**

- Understand what it means that one number is divisible by another
- Know even and odd numbers
- Be able to skip count
- Have used a calculator to look at numbers' attributes, e.g.

divisibility

➤ **Have an understanding of the representation of numbers in the decimal system, more precisely:**

- Understand the decimal system as place value, e.g. by using manipulatives and pictures
- Show understanding and knowledge in the basics with regards to writing natural numbers in the decimal system

✓ *Be able to give examples and show how simple fractions and ratios are used in daily life*

**Arithmetic,  
operations,  
and  
estimation**

➤ **Have a good understanding and skills in simple calculations with natural numbers and decimals, more precisely:**

- Be able to use a variety of methods to add and subtract natural numbers (at least for numbers 1-1000), both mentally and on paper
- Be able to set up the multiplication table up to  $10 \cdot 10$
- Be able to multiply a three digit number with a one digit number and solve easy division problems
- Understand that multiplication is repeated addition
- Understand that division can be seen as either the process of dividing or as repeated subtraction
- Have used a calculator to gain better understanding of mathematical operations and be able to use a calculator to add, subtract, multiply and divide.

- Have a sense for numbers and quantities such that they can estimate whether an answer from a problem is plausible or not
- Be able to use the fact that addition and subtraction on one hand and multiplication and division on the other are inverse operations to check answers to problems
- Show understanding of addition and subtraction of decimals (tenths, and hundredths) by using manipulatives and figures or diagrams
- **Have gotten to know methods for mental mathematics and estimation, more precisely:**
  - Show skills in mental mathematics dealing with whole tens, hundreds, and thousands and know methods to solve mentally easy and accessible problems with two digit numbers
  - Be able to use estimation to next ten to simplify calculations
- ✓ *Take part in creating suitable methods, built on student's own understanding, to add, subtract, multiply and divide*
- ✓ *Be able to use mental mathematics, calculator, software and pen and paper to solve problems from daily lives*

**Patterns and algebra**

- **Have explored patterns in order to find what comes next and form a general rule, more precisely:**
  - Be able to explore, make and express themselves about rules in patterns (including number patterns) and predict the continuation, e.g. by using models or manipulatives
  - Be able to change the pattern from one form to another, by using things, datasets, graphs, calculators, spoken or written language and symbols
  - Be able to express relations between factors they know from their daily lives, e.g. family relations
- **Realize that other symbols can substitute numbers in**

**mathematics, more precisely:**

- Be able to solve simple equations where blanks are used to represent an unknown variable
  - Be able to reason that when the same number is added to each side of an equation, the equal sign still holds, e.g. by using manipulatives
- ✓ *Be able to solve equations using non-standard methods and rationalize their solutions, e.g. by using manipulatives*

**Iceland's performance on international studies**

Iceland has participated in international studies on students' mathematical knowledge. They participated once in TIMSS, but have taken part in PISA since the year 2000.

**TIMSS**

The Third International Mathematics and Science Study (TIMSS) took place in 1995. TIMSS was conducted at five grade levels and included more than 40 countries. Icelandic third and fourth graders scored significantly lower than the international level in the mathematics section of the study. Iceland's third grade score was 474 and the international average was 529. In third grade, Icelandic students ranked 24<sup>th</sup> of 26 nations. In fourth grade the international average was 470. Icelandic fourth graders scored 410 on average and ranked 23<sup>rd</sup> of the 24 participating nations (Mullis, Martin, Beaton, Gonzales, Kelly, & Smith, 1997).

Seventh and eight grade students scored significantly below the international average in algebra, but at or above the average in other content areas measured (Beaton, Mullis, Martin, Gonzalez, & Smith, 1997).

At the final year of secondary school, Icelandic students' mean score was significantly higher than the international mean in mathematics literacy. Iceland's mean score was significantly higher than 15 of the 21 participating nations in 1995 (Mullis, Martin, Beaton, Gonzalez, Kelly, & Smith, 1998).

### **PISA**

The Programme for International Student Assessment (PISA) is a collaborative effort on behalf of the OECD countries to assess the knowledge and skills of 15 year old students in the key subjects; reading, mathematics and science. In addition, the study processes data on factors that influence the development of knowledge and skills, such as the students' home background and attitudes towards school and learning (OECD, 2003). The study is conducted every three years and began in 2000. Each circle of the study covers the domains of reading, mathematical and scientific literacy but with emphasis on one of the subjects in each round. In 2000 the emphasis was on reading literacy, 2003 on mathematics literacy, 2006 on science literacy and 2009 again on reading literacy. In PISA mathematics literacy is defined as: *“An individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen”* (OECD, 2003).

Due to Iceland's small population, all Icelandic students in 10<sup>th</sup> grade (the final year of compulsory education) participate in the PISA study, except if prevented by severe mental or physical disability. This fact makes the Icelandic findings are very reliable since 95% of 15 year old students participate in the study (Mejding & Roe, 2006).

Iceland has participated in the PISA study from the beginning. Every cycle Iceland scores statistically significantly above the OECD mean in mathematics literacy. (OECD/UNESCO-UIS, 2003; OECD, 2004; OECD, 2007; OECD, 2010).

### III Literature review

#### History of teacher assessment

The knowledge teachers should or ought to possess in order to teach mathematics has been of interest for a long time. People have differed in their opinion of the extent of that knowledge, but most agree that it is preferable for teachers to know what they are supposed to teach the students and perhaps something more.

The reason for measuring teachers' knowledge has two roots. First, there is the need to make sure teachers actually know what they are supposed to teach; that is, that they qualify for their job. Second, there is the scholarly interest in the extent of knowledge and the nature of the knowledge needed for teaching.

For at least two centuries, teachers' knowledge has been measured and assessed. The main purpose of these assessments has been the certification process, to evaluate individual teacher's knowledge or performance. The management of these assessments has for the most part been in the hands of local officials, state legislators and testing firms, parties outside the educational field (Hill et al., 2007). Since the 1960's a number of researches have aimed at measuring teachers' knowledge, and investigate its relationship to teachers' professional training and instructional effectiveness. Variables like teacher preparation, coursework and experience have been used as proxy measures to predict student achievement. These variables were assumed to be what came closest to the teachers' knowledge that produced student outcomes. These measures make sense, given that formal teaching education is supposed to supply the knowledge needed for teaching. Another way to determine teachers' knowledge is using subject matter based tests (Hill et al., 2007).



The National Teacher Examination (NTE) also known as the Praxis series is the most widely used and studied teacher examination in the US (Hill et al., 2007). Many states use the NTE as a part of their licensure and certification process (Sasson, 2008). The NTE first appeared over 60 years ago. Many school authorities supported its creation and welcomed an instrument that was meant to facilitate their teacher selection. It was even anticipated that teachers' colleges would alter their education in order to prepare students for this examination (Pilley, 1941). Even though high hopes of finding "better" teachers were tied to the NTE, researchers have failed to find a positive relationship between teachers' NTE scores and students' achievement.

The search for what promoted students gains in mathematics continued and was directed at teachers' mathematical knowledge, as evidence pointed to subject matter knowledge as the source of teachers' effectiveness, rather than general pedagogical knowledge. Studies aimed at directly measuring teachers' mathematical or verbal knowledge usually have shown a positive relationship between that knowledge and students mathematics achievement (Hill, et al., 2007).

### **What kind of knowledge is needed in mathematics**

Children get acquainted with formal concepts of mathematics in elementary school, and that is where their understanding of mathematics is shaped. According to research, the following beliefs apply to the mathematics classroom (Conference Board of the Mathematical Sciences, 2012, p. 10):

- Doing mathematics means following the rules laid down by the teacher.
- Knowing mathematics means remembering and applying the correct rule when the teacher asks a question.

- Mathematical truth is determined when the teacher ratifies the answer.

For the most part of the 20<sup>th</sup> century primary focus was on students' computational skills in mathematics, even though there were periods where emphasis on meaningful mathematical learning was included. Throughout such periods, balance between skill and practice, or procedural and conceptual understanding was sought (Langrall, Mooney, Nisbet, & Jones, 2008).

During the last decades researchers interested in students' mathematical learning have paid a lot of attention to students' conceptions and different forms of mathematical knowledge. They have distinguished between mathematical knowledge; knowledge that scratches the surface and enables students to use certain algorithms and a deeper understanding where the use of the algorithms is accompanied by the knowing of why and how they work, as well as connection between topics in mathematics (Even & Tirosh, 2008). Skemp's (1978) instrumental knowledge and relational knowledge are examples of these types of knowledge. Skemp (1978) differentiates between what he calls relational understanding and instrumental understanding of mathematics. Relational understanding means knowing both what to do and why, while instrumental understanding refers to "rules without reasons", and not really much understanding. As an example of instrumental understanding, Skemp (1978) mentions 'borrowing' in subtraction and 'turn the second fraction upside down' while dividing fractions. Closely related to Skemp's (1978) understanding are procedural understanding and conceptual understanding. Procedural understanding of mathematics is the mastery of computational skills, and knowledge of definitions and algorithms. Conceptual understanding, on the

other hand, is the knowledge of the underlying structure of mathematics (Eisenhart, Borko, Underhill, Brown, Jones, & Agard, 1993).

Hiebert and colleagues (1986; 1992) distinguish between procedural knowledge and conceptual knowledge, claiming that both are needed for mathematical expertise. Conceptual knowledge relates to something already known, while procedural knowledge is a chain of operations to be performed with or without understanding. Fischbein (1983) suggests three correlated dimensions of mathematical knowledge; algorithmic, formal and intuitive knowledge. Algorithmic knowledge consists of rules, procedures and their theoretical justifications. Formal knowledge involves axioms, definitions, theorems and proofs and intuitive knowledge is the type of knowledge usually accepted as being obvious, like mental models used for representing mathematical concepts and operations. Ma (1999) introduces a 'profound understanding' of mathematics when she describes the knowledge of the Chinese teachers in her study as "...an understanding of the terrain of fundamental mathematics that is deep, broad, and thorough" (p. 120).

Krauss, Baumert, and Blum (2008) classify mathematical content knowledge, from simple to more complex, as follows: "(1) the everyday mathematical knowledge that all adults should have, (2) the school-level mathematical knowledge that good school students have, and (3) the university-level mathematical knowledge that does not overlap with the content of the school curriculum (e.g., Galois theory or functional analysis)." Additionally they classify mathematical content knowledge for teachers as a deep understanding of the content of the mathematics curriculum to be taught (Baumert, et al., 2010).

Institutions and organizations concerned with mathematics education also have

their ideas regarding mathematical knowledge and understanding. The Common Core State Standards (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2012) state what students should understand in mathematics.

They describe mathematical understanding as:

One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, *why* a particular mathematical statement is true or where a mathematical rule comes from. There is a world of difference between a student who can summon a mnemonic device to expand a product such as  $(a + b)(x + y)$  and a student who can explain where the mnemonic comes from. The student who can explain the rule understands the mathematics, and may have a better chance to succeed at a less familiar task such as expanding  $(a + b + c)(x + y)$ . Mathematical understanding and procedural skill are equally important, and both are assessable using mathematical tasks of sufficient richness. (p. 4)

From the above one can see that students' mathematical knowledge and understanding has been under the microscope of scholars for several years. The same goes for their teachers' understanding and knowledge, which has been thoroughly examined.

### **Teachers' knowledge of mathematics**

When it comes to teachers' knowledge of mathematics scholars agree that teachers' own experience as students of elementary mathematics is not sufficient as a base to teach it (Conference Board of the Mathematical Sciences, 2012). Studies aimed at teachers' mathematical knowledge mainly have focused on two topics: first, the teachers' characteristics, e.g. the number of mathematics courses they have completed, and second, the nature of teachers' mathematical knowledge (Ponte & Chapman, 2008).

***Impact of teachers' knowledge.*** The way teachers perceive and implement the curriculum depends on their knowledge and beliefs (Shulman, 1987; Thompson, 1992).

Skemp (2006) differentiates between “instrumental mathematics” (rules without reason) and “relational mathematics” (knowing both what to do and why) when talking about mathematics teaching. He says: “For we are not talking about better and worse teaching of the same kind of mathematics.... I now believe that there are two effectively different subjects being taught under the same name “mathematics” (p. 91). Skemp (2006) mentions advantages that might cause teachers to teach instrumental mathematics. Instrumental mathematics is usually easier to understand, the rewards are more immediate and apparent, and the path to the right answer is shorter. Advantages of relational mathematics, according to Skemp (2006), include its adaptability to new tasks and easiness to remember, as well as the fact that relational knowledge can be a goal in itself.

Research has shown that teachers’ content knowledge is related to the mathematical quality of teachers’ instructions and teaching style (Baumert, et al., 2010; Charalambous, 2010; Hill & Ball, 2009; Hill, Blunk, Lewis, Phelps, Sleep, & Ball, 2008; Shulman, 1987). Teachers’ knowledge of mathematics is important for utilizing instructional materials in most productive way, assessing students’ progress and finding the most effective presentation and sequence of the subject, (Ball, Hill, & Bass, 2005; Baumert, et al., 2010). Research has shown that when teachers’ understanding of mathematics increases, their teaching changes. Sowder, Philipp, Armstrong, and Schapelle (1998) found that once teachers had increased their knowledge, they depended less on their material and were more willing to try new things with their students.

Teachers’ understanding of mathematics must be solid, because for mathematics to be meaningful to students the teaching has to be effective. Effective teaching requires

an understanding of the underlying meaning of concepts and procedures, as well as justifications for the ideas and procedures presented and the ability to make connections among topics (Ball, Ferrini-Mundy, Kilpatrick, Milgram, Schmid, & Schaar, 2005; Oakes & Lipton, 2002). Teachers' understanding of mathematics therefore plays an important role in the real mathematical thinking that occurs in the classroom (Ma, 1999). According to the before mentioned sources, teachers' content knowledge touches upon almost every aspect of mathematics teaching and learning.

*Nature of teachers' knowledge.* The notion that teachers needed some special kind of mathematical knowledge has interested researchers for over 25 years. Scholars interested in teachers' knowledge have made the assumption that inside the classroom teachers use subject matter knowledge in a unique way. They have wondered whether the knowledge of someone majoring in mathematics is sufficient for teaching, or whether some other kind of knowledge is needed (Hill et al., 2007).

Research has often failed to find a strong relationship between elementary teachers' mathematics course completion and student achievement. What seems to be relevant is the knowledge and use of the mathematics needed in the work of teaching. Hence, studies focusing on teachers' performance on mathematical tasks in the context of teaching have found a relationship between performance and student achievement (Kilpatrick, Swafford, & Findell, 2001; Mewborn, 2003; National Mathematics Advisory Panel, 2008).

Hill, Rowan and Ball (2005) investigated whether and how teachers' mathematical knowledge for teaching added to students' gain in mathematics achievement. Their results suggested that teachers' mathematical knowledge for teaching

positively affected students' gain in mathematics achievement during first and third grade (Hill, Rowan, & Ball, 2005). These results indicated that teachers' knowledge is important, even when teaching elementary mathematics.

Institutions and organizations interested in mathematics education have described the desirable knowledge and qualities mathematics teachers should possess. In 1991 The National Council of Teachers of Mathematics portrayed the knowledge needed for teaching as:

The content and discourse of mathematics, including mathematical concepts and procedures and the connections among them; multiple representations of mathematical concepts and procedures; ways to reason mathematically, solve problems, and communicate mathematics effectively at different levels of formality. (p. 132)

Ten years later, Kilpatrick and his colleagues (2001) put it:

Knowledge of mathematical facts, concepts, procedures, and the relationships among them; knowledge of the ways that mathematical ideas can be represented; and the knowledge of mathematics as a discipline - in particular, how mathematical knowledge is produced, the nature of discourse in mathematics and the norms and standards of evidence that guide argument and proof. (p. 371)

There seems to be an agreement that teachers need to have a deep understanding and knowledge of the mathematics they teach and they need to be able to use this knowledge in their teachings. Teachers need to know how their teaching material is connected to other mathematical topics, both prior and beyond their teaching level (National Council of Teachers of Mathematics, 2000; National Mathematics Advisory Panel, 2008; Mewborn, 2003; Conference Board of the Mathematical Sciences, 2012).

The NCTM standards (2000) describe effective teaching of mathematics and the knowledge such teaching involves: "Effective mathematics teaching requires

understanding what students know and need to learn and then challenging and supporting them to learn it well...Effective teaching requires knowing and understanding mathematics, students as learners, and pedagogical strategies” (pp. 16-17). The NCTM standards (2000) go on and claim that teachers need various kinds of mathematical knowledge; knowledge about the subject, the curriculum, and the challenges students may face within the subject. Teachers need knowledge about the teaching of mathematics, how to teach the subject effectively, and how to assess students’ understanding in addition to an understanding of their students as learners of mathematics.

Educational research has distinguished between three types of teachers’ knowledge: content knowledge, pedagogical content knowledge and basic pedagogical knowledge. Both general and pedagogical content knowledge are considered important factors in the quality of teaching. They also affect students’ learning and motivational development. Studies have indicated that content knowledge alone is not sufficient to guarantee quality teaching, but they also have implied that teachers lacking in mathematics content knowledge are less equipped to explain and represent topics in such ways that makes sense to the students. This lack of conceptual understanding cannot be compensated with general pedagogical skills. Pedagogical content knowledge cannot exist without content knowledge, but pedagogical content knowledge is needed to facilitate learning. Teachers with similar level of content knowledge can differ in their pedagogical knowledge and thus create a different mathematical experience for students (Baumert, et al., 2010).

The nature of teachers’ mathematical knowledge has interested scholars for a long



time and some researchers have dedicated their carrier to the study of teachers' mathematical knowledge.

*Shulman's contribution.* Lee Shulman, an educational psychologist, was interested in teacher education. He argued that teacher education programs should not develop either general pedagogical skills or content knowledge, but focus on knowledge at the intersection of both. In 1986 Shulman and his colleagues introduced the term pedagogical content knowledge, a combination of subject matter and pedagogical knowledge. By doing so they drew attention to the specific ways of which teachers must know and use content knowledge in their teaching (Hill et al., 2007).

Shulman (1987) claimed that general teachers' knowledge comprised at least the following categories:

- General pedagogical knowledge, with special reference to those broad principles and strategies of classroom management and organization that appear to transcend subject matter;
- Knowledge of learners and their characteristics;
- Knowledge of educational contexts, ranging from workings of the group or classroom, the governance and financing of school districts, to the character of communities and cultures;
- Knowledge of educational ends, purposes, and values, and their philosophical and historical grounds;
- Content knowledge;
- Curriculum knowledge, with particular grasp of the materials and programs that serve as “tools of the trade ” for teachers;

- Pedagogical content knowledge, that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding. (p. 8)

The last three categories are content specific. Content knowledge is the amount and organization of knowledge the teacher possesses. This knowledge goes beyond the knowledge of facts and concepts; it requires an understanding of the structures of the subject. As Shulman put it: “The teacher need not only understand *that* something is so; the teacher must further understand *why* it is so, on what grounds its warrant can be asserted, and under what circumstances our belief in its justification can be weakened and even denied” (1986, p. 9). Curricular knowledge consists of knowledge of the curriculum and its associated material. Last but not least there is Pedagogical content knowledge, the subject matter knowledge for teaching, described by Shulman (1986, p. 9) as “...[a] particular form of content knowledge that embodies the aspects of content most germane to its teachability”, including:

...the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the most useful ways of representing and formulating the subject that make it comprehensible to others. . . . Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. (p. 9)

According to Shulman (1986), when these three categories of knowledge are included in a measurement tool there is a way to assess a professional. Such examination could distinguish between a mathematician and a mathematics teacher, the key ingredient

discriminating between the two being the pedagogical content knowledge, or as Shulman put it: “Pedagogical content knowledge is the category most likely to distinguish the understanding of the content specialist from the pedagogue” (1987, p. 8).

Shulman criticized politicians and teacher educators in the past for seeing teaching needing only basic skills, content knowledge and general pedagogical skills. By doing so, Shulman claimed they simplified teaching and diminished its demands.

Shulman described teaching:

... teaching necessarily begins with a teacher’s understanding of what is to be learned and how it is to be taught. It proceeds through a series of activities during which the students are provided specific instruction and opportunities for learning, though the learning itself ultimately remains the responsibility of the students. Teaching ends with new comprehension of both the teacher and the student. (1987, p. 7)

Shulman (1986) also found fault with researchers of his time, claiming they did not take into account the subject matter and made too sharp of a distinction between content and pedagogical process. He pointed out that as much attention needed to be paid to the content aspect of knowledge as to the pedagogical aspect.

Even though Shulman placed great emphasis on content knowledge, he by no means thought that pedagogy was redundant, as he argued “mere content knowledge is likely to be as useless pedagogically as content-free skill” (1986, p. 8).

Shulman and his colleagues pursued through their work to direct the research and policy communities towards the nature and types of knowledge needed for teaching a subject (Ball, Thames, & Phelps, 2008).

With Shulman’s emphasis on content knowledge and introduction of pedagogical content knowledge there was an awakening among researchers in the field. They began to

investigate the mathematical knowledge held by teachers in order to map it and hopefully measure it, under the assumption that inside the classroom teachers used subject matter knowledge a unique way (Hill et al., 2007). They began to measure or document teachers' and prospective teachers' level of content and pedagogical content knowledge (Ball et al., 2008).

*The contribution of Ball and the research team at the University of Michigan.*

Deborah Ball and her colleagues at the University of Michigan have been working on a practice-based theory of content knowledge for teaching based on Shulman's concept of pedagogical content knowledge (Ball, Thames, & Phelps, 2008). They have focused on teachers' mathematical knowledge, trying to map it and develop a measurement tool capable of reliably measuring that knowledge (Hill, Schilling, & Ball, 2004). During the development, Ball and her colleagues, wondered if there was one construct that could be called "mathematics knowledge for teaching", or if teachers' mathematical knowledge was made up by several constructs, which combined, made up the mathematical knowledge teachers used while teaching.

Ball and her colleagues focused on how teachers needed to know their content. Instead of placing emphasis on teachers and their individual knowledge, they placed it on the use of knowledge in and for teaching. In their study of mathematical knowledge needed for teaching, Ball and her colleagues noticed how much special mathematical knowledge was needed in everyday teaching, where "teaching" refers to everything that teachers have to do to facilitate students' learning. According to Ball and colleagues the mathematical tasks of teaching include:

Presenting mathematical ideas

Responding to students' "why" questions

Finding an example to make a specific mathematical point

Recognizing what is involved in using a particular representation

Linking representations to underlying ideas and to other representations

Connecting a topic being taught to topics from prior or future years

Explaining mathematical goals and purposes to parents

Appraising and adapting the mathematical content of textbooks

Modifying tasks to be either easier or harder

Evaluating the plausibility of students' claims (often quickly)

Giving or evaluating mathematical explanations

Choosing and developing useable definitions

Using mathematical notation and language and critiquing its use

Asking productive mathematical questions

Selecting representations for particular purposes

Inspecting equivalencies (Ball et al., 2008 p. 400)

When teachers are faced with creative students' solutions, they need to figure out if these solutions are legitimate, and if they work in general or only for a particular problem, or a group of problems. As Ball et al. put it, "Being able to engage in this sort of mathematical inner dialogue and to provide mathematically sound answers to these questions is a crucial foundation for determining what to do in teaching this mathematics" (2008, p. 398).

Ball and colleagues claim that the mathematical knowledge needed for teaching is

multidimensional and find it important to identify, isolate and measure it. Through their research, they have found evidence of a specialized content knowledge for teaching, implying that teachers have a different structure of mathematical knowledge than does a well-educated adult (Ball, Thames, & Phelps, 2008). Teachers explaining mathematical procedures or ideas, exemplifying mathematical wonders and working with non-standard solution methods make use of this specialized knowledge (Hill et al., 2007).

*Knowledge domains.* Through their work, Ball and colleagues have proposed that Shulman's (1986) content knowledge could be divided in two; common content knowledge (CCK) and specialized content knowledge (SPK), and his pedagogical content knowledge divided into knowledge of content and students (KCS) and knowledge of content teaching (KCT) (Ball et al., 2008).

Common content knowledge (CCK) is defined "as the mathematical knowledge and skill used in settings other than teaching" (Ball et al., 2008, p. 399). The common content knowledge, the knowledge needed to correctly solve mathematics problems, is used in a variety of settings, not just teaching. Teachers make use of this common content knowledge when they use terms and notations in mathematics, and calculate. When teachers lack in this domain of knowledge instruction suffers and valuable time is lost when teachers struggle solving a problem or answering a question (Ball et al., 2008)

The second domain of content knowledge is the specialized content knowledge (SPK), the content knowledge unique to teaching. Teachers make use of this knowledge, when for example, analyzing students' errors, creative solutions, and when they explain and justify reasons behind mathematical procedures and algorithms. Teachers use SPK when dressing up mathematics in a way, not needed in other situations, to facilitate and

encourage students' learning (Ball et al., 2008).

Ball and colleagues (1998) divided Shulman's pedagogical knowledge in to knowledge of content and students (KCS) and knowledge of content and teaching (KCT). Knowledge of content and students is the knowledge of the interaction between the students and the content. Knowing what topics or concepts confuse, interest or motivate students falls under the KCS category (Ball et al., 2008).

Knowledge of content and teaching merges the knowledge of mathematics and teaching. Teachers make use of this knowledge when determining in what order to teach, or represent topics. KCT is used when teachers decide what instructional method to apply with a specific content, which examples to use in the introduction and which to use to engage the students in the content. When deciding to take part in a mathematical discourse, when to ask a question or give a task teachers link their knowledge of mathematics and pedagogy and apply KCT (Ball et al., 2008).

*Interaction between knowledge domains.* There is by no means a clear-cut distinction between different knowledge domains. The underlying theory is based on teaching and therefore brings with it all the things that can occur in teaching and learning. When looking at a student's error a teacher might figure out mathematically what went wrong in the student's assumptions or calculations, and apply SCK, while another teacher doing the same might recognize this mistake as common among students by applying KCS. When teachers recognize a mistake student made, they use CCK. When they analyze the nature of the error they use SCK. Knowing if this is a common mistake, or why the student made this mistake, KCS comes in play. A "simple" teacher's task of making a decimal problem for students can touch up on all the domains of knowledge.

Making a list of decimals to be ordered, that reveal mathematical issues is SCK, ordering the decimals is CCK, recognizing which decimals confuse students is KCS, while deciding what to do about it is KCT (Ball et al., 2008).

The definition and categorizing of the knowledge teachers need in order to teach mathematics can be useful when studying what part of teachers' content knowledge plays the greatest part in students' achievement. This information about the domains of teachers' knowledge can advise designers of teachers' materials as well as designers of teacher education programs and professional development for teachers (Ball et al., 2008).

*Contribution of others.* Leinhardt and Smith (1985) studied the difference between expert and novice teachers and identified two attributes of teachers' knowledge; lesson structure knowledge and subject matter knowledge. Lesson structure knowledge involves planning and conducting a lesson while subject matter knowledge involves concepts, algorithmic operations and procedures, classes of students' errors and curriculum presentation. Their results suggested that common and specialized mathematical knowledge were related, but not completely equivalent and teachers' knowledge of mathematics for teaching was at least partly domain-specific, rather than just related to overall intelligence, mathematical or teaching ability. Their findings supported Shulman and others' assertion that knowledge for teaching is made of general content knowledge and more specific domain knowledge (Leinhardt & Smith, 1985).

Kilpatrick (2001) and colleagues claimed three kinds of knowledge were crucial for teaching school mathematics: knowledge of mathematics, knowledge of students and knowledge of instructional practices. Knowledge of mathematics is the basic content knowledge. Knowledge of students involves knowing about students and their stance



towards mathematic as well as their common conceptions and misconceptions.

Knowledge of practice has to do with planning and executing a fruitful lesson, create a community of learners and lively mathematical discourse.

Building on Shulmans work, Krauss et al. (2008) talked about three elements of teachers' pedagogical knowledge; knowledge of mathematical tasks for learning, knowledge of students' conceptions and prior knowledge, and knowledge of mathematics-specific instructional methods.

***Defining mathematical content knowledge in teaching.*** Developing a theoretical framework for content knowledge used in teaching is an ongoing project. Even though scholars agree on the existence of a specialized knowledge needed by teachers, the term still needs a clear definition, and is often not distinguishable from other kinds of teachers' knowledge. Following are descriptions of pedagogical knowledge from Shulman (1986; 1987)

A second kind of content knowledge is pedagogical knowledge, which goes beyond knowledge the subject matter per se to the dimension of subject matter *for teaching*.... Within the category of pedagogical content knowledge I include for the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstration – in a word, the ways of representing and formulation the subject that make it comprehensible to others. (Shulman, 1986, p. 9)

Pedagogical content knowledge identifies the distinctive bodies of knowledge for teaching. It represents the blending of content and pedagogy into an understanding of how particular topics, problems or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction. Pedagogical content knowledge is the category most likely to distinguish the understanding of the content specialist from that of the pedagogue. (Shulman, 1987, p. 4)

Some definitions of pedagogical content knowledge can be broad and unclear, like these definitions: “An integration of teacher understanding that combines content (subject matter), pedagogy (instructional methods), and learner characteristics” (Glossary of Education, 2013), “...the intersection of knowledge of the subject matter with knowledge of teaching and learning” (Niess, 2005, p. 510). “PCK, which forms a distinct body of instruction- and student-related mathematical knowledge and skills—the knowledge that makes mathematics accessible to students (Baumert, et al., 2010, p. 142).

Definitions can also be very specific and include just about everything a teacher knows, feels and beliefs about a topic:

Pedagogical content knowledge is a teacher’s understanding of how to help students understand specific subject matter. It includes knowledge of how particular subject matter topics, problems, and issues can be organized, represented and adapted to the diverse interests and abilities of learners, and then presented for instruction . . . The defining feature of pedagogical content knowledge is its conceptualization as the result of a transformation of knowledge from other domains. (Magnusson, Krajcik, & Borko, 1999, p. 96)

The mathematical knowledge for teaching, identified by Ball and her colleagues is defined as: “the mathematical knowledge needed to carry out the work of teaching mathematics” (Hill, Rowan, & Ball, 2005, p. 373), and teaching is “everything that teachers must do to support the learning of their students” (Ball, Thames, & Phelps, 2008, p. 395). “Examples of this “work of teaching” include explaining terms and concepts to students, interpreting students’ statements and solutions, judging and correcting textbook treatments of particular topics, using representation accurately in the classroom, and providing students with examples of mathematical concepts, algorithms, or proofs” (Hill, Rowan, & Ball, 2005, p. 373).

*Methods of determining teachers' knowledge.* Whatever the nature of teachers' knowledge has been thought to be, the need to measure it has been stable. To find out how teachers teach, the most logical way seems to be to watch them while doing so. Observations of teachers (direct or recorded) might be the earliest and most widely used method of assessing teachers' mathematical knowledge. The disadvantages of observations include difficulty to generalize, subject to interpretation, and often restricted to the topic being observed at a specific time. Therefore observational data is often combined with other kinds of data. The easiest and most efficient way of determining teachers' knowledge at scale seems to be written tests. The National Teacher Examination (NTE) is the most widely used and studied teacher examination in the US. Studies have sought to find a relationship between NTE scores and student achievement, but without much success (Hill et al., 2007).

Another method widely used to explore teachers' knowledge, is the combination of mathematical tasks and interviews. The tasks used, often have a narrow focus on a specific content area within mathematics. Some of the tasks are supposed to measure general mathematical knowledge, while others focus on situations that might arise in the classroom. The combination of tasks and interviews allows for the researcher to adjust questions to each respondent based on their previous answers or solutions. By doing so the researcher gains a better understanding of the respondents' way of thinking (Hill et al., 2007).

Methods of evaluating teachers' knowledge have their shortcomings. Focusing on a specific topic within mathematics makes it difficult to generalize and open-ended tasks

and interviews are not always reliable. Tasks intended to simulate what happens in the classroom may not necessarily reflect teachers' knowledge in action (Hill et al., 2007).

*Measurement tools.* There has been a debate regarding what teachers should know in order to teach mathematics. Some have argued that strong content knowledge is what is needed to teach mathematics, while others argued that professional knowledge, like knowledge of students' thinking and tasks was the most important factor of teachers' knowledge (Hill, Schilling, & Ball, 2004).

The educational production function literature suggests that teachers' intellectual resources have impact on students' learning (Hill et al., 2005). Where intellectual resources, measured with variables like courses taken, degrees attained and results from basic skills test, have been used as referent. This view on teachers' knowledge differs from those who see teachers' knowledge as their ability to understand and use subject matter knowledge to teach, a view that goes beyond courses taken or degrees attained.

Once the notion of special mathematical knowledge unique to teaching was accepted, the need to measure this knowledge emerged. Tools were needed, both for those concerned with teachers' knowledge and student achievement, and those interested in the change in teachers' knowledge over time. Researchers needed tools to reliably and validly make inference about one or a group of teachers. As a respond to this need series of paper and pen based measurements emerged, intended to measure teachers' knowledge. Experts in the field of mathematics, teaching mathematics and psychometrics began creating measuring instruments. These experts worked with the idea that teachers' knowledge was complex and strove to include measures of all its aspects (Hill et al., 2007).

*Written assessment.* As a result of this work, many paper and pen based measurements exist, but four have come the furthest in instrument development. Their creators have developed special domain maps, completed pilot tests, conducted psychometric analyses and worked on validation (Hill et al., 2007).

The following tests allow for teachers' knowledge to be measured at scale with known reliability and validity. The MKT measures (also known as The Study of Instructional Improvement/Learning Mathematics for Teaching (SII/LMT)) (LMT, 2012) and the SimCalc rate and proportionality teaching survey (SRI International, 2011), are multiple choice tests, while the Knowledge for Algebra Teaching (KAT) measures (KAT, 2012) and the Diagnostic Teacher Assessments in Mathematics and Science (DTAMS) measures (Bush & Grimes, 2006) use both multiple choice and short open response answers (Hill et al., 2007).

These tests have common characteristics. They include items made to represent problems and situations likely to occur in the classroom, and focus on a specific area within mathematics. Though these tests have similar features, they do differ in important aspects, like the goals of the assessment and the interpretation of scores. The KAT and DTMAS assessments are based on a fixed criterion so that scores on these measures can be concretely interpreted in relation to the criterion. The MKT measures were intended to order teachers relative to one another and to the mathematical knowledge for teaching being measured. SimCalc, the fourth project was arranged to measure curriculum specific mastery of the content needed to teach the SimCalc Curriculum. When it comes to the definition of mathematical knowledge for teaching, another difference among the measures emerges. The MKT measures have four knowledge domains, SimCalc has two

and the KAT and DTMAS projects address three aspects of knowledge for teaching and four mathematics domains. None of these approaches to mathematical knowledge for teaching match Shulman's (1986) original description of pedagogical content knowledge, and they don't match each other (Hill et al., 2007).

Assessments with short answer responses and multiple-choice questions have their shortcomings. Is it possible to measure to any depth the mathematical knowledge teachers use in their work in such a form? Situations arise in teaching needing professional judgments that are hard to answer with a "correct" answer. Another known problem is teacher resistance to multiple-choice assessments. Instruments with a narrow focus on specific mathematics content make it hard to generalize about overall teacher knowledge. Open-ended tasks and interviews are not always reliable. Some of the assignments are supposed to replace real life teaching experience, but they do not necessarily paint an accurate portrait of the teacher's mathematical knowledge in action (Hill et al., 2007). A teacher may perform well on these assessments but fail to show the same level of knowledge in the classroom (Borko, Eisenhart, Brown, Underhill, & Jones, 1992). On the other hand, the opposite is also plausible, teachers might fail to express fully their knowledge in a clinical setting due to stress, anxiety or other factors, while in their classroom, where they feel more comfortable, their knowledge is more apparent.

*The MKT measures.* Building on Shulman's (1986) notion of pedagogical content knowledge and other research on teachers' mathematical knowledge, Ball and her colleagues at the University of Michigan have been working on a theory on the mathematical knowledge of elementary teachers (Delaney, Ball, Hill, Schilling, & Zopf, 2008; Hill, Schilling, & Ball, 2004). They seek to understand and measure the

mathematical knowledge teachers use in the classroom (Hill, Ball, & Schilling, 2008). They call this knowledge the mathematical knowledge for teaching, or MKT. The concept of MKT surfaced during research on the re-occurring tasks in mathematics in the classrooms (Delaney et al., 2008).

Psychometric analyses support the existence of MKT and suggest that it can be divided into sub divisions or domains. The suggested domains are: common content knowledge (CCK), specialized content knowledge (SCK), knowledge of content and students (KCS), and knowledge of content and teaching (KCT) (Delaney et al., 2008). Knowledge of content and students (KCS) is closely related to Shulman's (1986) pedagogical content knowledge, since both focus on teachers' understanding of what makes a specific topic easy or difficult for students and the conceptions and misconceptions students bring with them to the classroom (Hill et al., 2008).

Along with theory building, the research team in Michigan has developed a questionnaire to measure MKT. The purpose of the measures is to research the nature, role and different types of mathematical knowledge for teaching (Ball et al., 2008).

The MKT measures focus on three domains, with one of the domains divided into two components. The three domains are:

- Content knowledge, divided into:
  - Common content knowledge: The knowledge teachers are supposed to evolve in students. This category measures mathematical knowledge not specific to teaching.
  - Specialized content knowledge: The mathematical knowledge used in teaching, but not directly taught to students.

- Knowledge of content and students: The knowledge of how students learn content corresponds to Shulman's (1986; 1987) thoughts regarding students' conceptions and misconceptions.
- Knowledge of content and teaching: The knowledge of how to design a lesson, including choosing examples, representation and guiding students toward accurate mathematical ideas. This knowledge corresponds to Shulman's (1986) idea of "most useful form of representation" (Hill et al., 2007).

Constructing items to measure the mathematical knowledge for teaching can be problematic. Teaching is so connected to different students and material, making it hard to construct items with only one correct answer. Another problem is making sure that items actually do require special professional knowledge. Third, is keeping items in line with professional skills instead of ideology. It is difficult to design an instrument to fully measure the complex knowledge and reasoning skills and the relation between them without a better theoretical mapping of this area of expertise. Valid teachers' assessments should build on teachers' actions in the classroom with real students, content and material. Mathematical competence is not established by quickly solving routine mathematical problems. The mapping of teachers' knowledge for teaching is a work in progress (Hill et al., 2007).

*MKT items.* The MKT measures have been developed and used for over 10 years. Extensive measures have been taken to ensure reliability and validity, piloting each item within the measures with over 600 elementary teachers. The purpose of the measures is



the research of the nature, role and different types of mathematical knowledge for teaching (Ball et al., 2008). Since elementary teachers teach elementary mathematics, the authors used similar material as is used in K-12 schools when developing the items for the MKT measures. They also kept in mind what teachers would know about mathematics that other adults would not, and strove for items that distinguished between teachers (Hill et al., 2008; Hill et al., 2007). As a result, the tasks in MKT aim to reach mathematical knowledge beyond solving students' tasks; the knowledge of why algorithms work the way they do, classroom tasks like choosing representations, explaining, interpreting students responses, assessing students' understanding and their difficulties, evaluating teaching material and the best way of presenting it to students (Hill et al., 2007; Hill et al., 2004).

The MKT items used in the present study included two groups of items; one intended to investigate common content knowledge; the knowledge adults familiar with mathematics should know. The other group consisted of items aimed to measure the knowledge that is acquired by teaching mathematics (Hill et al., 2004). The items in the latter group can be roughly divided into four categories:

- Common student errors: identifying and providing explanations for errors, having a sense for what errors arise with what content, etc.
- Student's understanding of content: interpreting student productions as sufficient to show understanding, deciding which student productions indicate better understanding, etc.
- Student developmental sequences: identifying the problem types, topics, or mathematical activities that are easier/more difficult at particular ages,

knowing what students typically learn “first,” having a sense for what third graders might be able to do, etc.

➤ Common student computational strategies: being familiar with landmark numbers, fact families, etc.

(Hill et al., 2008, p. 380)

Topic wise the items in the MKT measures are divided into numbers and operations (two groups, for grades K-6, and 6-8), patterns, functions and algebra (two groups, grades K-6, and 6-8), geometry (grades 3-8) as well as topic specific measures for grades 4-8 in rational numbers, proportional reasoning, geometry and data, probability and statistics.

*The MKT measures outside the US.* The MKT measures were originally developed to measure and research the mathematical knowledge for teaching held by US teachers. The fact that the measures relate to the task of teaching, not teaching practice, makes them more universal and more suitable for translation (Ng, Mosvold, & Fauskanger, 2012). The International Test Commission (2010) stresses the importance of minimizing effects of cultural differences when translating tests. Culture has influence on many aspects of life, so a careful consideration needs to be taken to potential cultural differences related to teachers, students, mathematics, and teaching material when attempting to translate and adapt the MKT measures (Delaney et al., 2008; Ng et al, 2012; Stylianides & Delaney, 2011). Literal translation is not sufficient in order for the items to reliably measure mathematical knowledge (Delaney et al., 2008); the translation

needs to be more focused on preserving “the functional equivalence of a text” (Ng, Mosvold, & Fauskanger, 2012, p. 155).

Despite possible differences in teaching between countries, it is plausible that the mathematical knowledge for teaching has some common characteristics in different countries. The answer to the question “Why can’t you divide by 0?”, a content knowledge question, is based on the definition of division and seems to be universal, and should be known by all teachers (Delaney et al., 2008).

The MKT measures have been used to measure teachers and prospective teachers’ mathematical knowledge for teaching outside the US. They have been adapted and translated for use in Ghana, Indonesia, Ireland, Korea, and Norway (Hambleton, 2012). The first ones to take on the task were Delaney and his colleagues who adapted the measures to use in Ireland. By doing so they set the guidelines for others interested in using the MKT measures outside the US. According to Delaney and colleagues (2008) changes made during the adaptation of the MKT items from American English to Irish English fell into four categories, (1) Changes related to the general cultural context, (2) Changes related to the school cultural context, (3) Changes related to mathematical substance, and (4) Other changes.

A careful consideration is needed before taking on the task of translating and adapting test items. The reason why some test items do not work the way they are supposed to on international test can be attributed to translation errors (Mosvold, Fauskanger, Jakobsen, & Melhus, 2009), and the slightest misunderstanding of terms can change the way the test distinguishes between teachers (Delaney et al., 2008).

The investigation of the MKT measures’ adaptability can be useful for future

studies because it can enable comparison of teachers' knowledge across nations (Ng, Mosvold, & Fauskanger, 2012). Even though adjusting and translating the MKT measures involve challenges and cost, it is worthwhile since the MKT measures give an opportunity to measure teachers' knowledge at scale with quality (Blömeke & Delaney, 2012).

*Numbers and operations.* In the current study, the main focus was on teachers' knowledge of numbers, operations and algebra. Looking back through history, numbers have been the foundation of the mathematical curriculum with the development of number sense as one of its essentials. Also, the mathematics learned through grades 1 – 10 has had a solid base in numbers and operations (National Council of Teachers of Mathematics, 2000). Most teachers know how to perform the algorithms associated with the four arithmetic operations and many of them equate the operations with these algorithms and their notation (Conference Board of the Mathematical Sciences, 2001). But being able to use one standard algorithm to solve a problem is not what is sought for in mathematics education. Students should be able to use the basic algorithms of whole number arithmetic fluently, and they should understand how and why the algorithms work (Ball, Ferrini-Mundy, Kilpatrick, Milgram, Schmid, & Schaar, 2005). Teachers need "...in order to interpret and assess the reasoning of children learning to perform arithmetic operations, [to] be able to call upon a richly integrated understanding of operations, place value, and computation in the domains of whole numbers, integers, and rationals" (Conference Board of the Mathematical Sciences, 2001, p. 58).

Elementary students' experience of mathematics centers on whole numbers and their operations. How they do in the subject has a lot to do with how they build their self-

image as mathematics learners and the mental image they build of the subject mathematics. Elementary teachers therefore not only build the foundation for further mathematics, but they also shape students' disposition toward the subject (Russell, 2010).

In the recently published Common Core Standards (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2012), one of the key points in mathematics is: "The K-5 standards provide students with a solid foundation in whole numbers, addition, subtraction, multiplication, division, fractions and decimals—which help young students build the foundation to successfully apply more demanding math concepts and procedures, and move into applications."

In the Common Core Standards for first and second grade, two of the four knowledge domains in mathematics involve numbers and operations and algebraic thinking. In grade three, four and five, the fraction domain is added (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2012).

The understanding of numbers develops significantly through children's first years, so during their first years in school their sense of numbers needs to be strengthened. Students should come across different meanings of addition and subtraction of whole numbers and be able to explain their methods and be aware of that there may be many methods and approaches towards solving a problem. The concepts of multiplication and division begin to form in the early grades. It is important for students to grasp the meaning of whole number multiplication and division, their relationship, and compare different solution strategies (National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices; Council of Chief State School Officers, 2012).

Research has indicated that when young students are encouraged to develop and explain their own arithmetic methods, as well as to listen to others and examine their methods, important learning can occur. Also, research has indicated that students who invent their own strategies have a better understanding of the base ten number system, in addition to better abilities to transfer their knowledge to new situations (National Council of Teachers of Mathematics, 2000).

*Algebra.* In the NCTM's standards (2000) it says: "All students should learn algebra" (p.37), but they do, and the learning begins in the elementary classroom. The significance of algebra is articulated in the National Mathematics Advisory Panel's report. There the importance of properly preparing students for algebra is stressed, since algebra is seen as a "demonstrable gateway to later achievement" (2008, p. xiii).

Numbers and operations are closely related to algebra, the same principles apply to equation solving and the structural properties of system of numbers (National Council of Teachers of Mathematics, 2000). The processes of generalizing and formalizing are considered to contribute to the understanding of the nature of mathematics. When instruction in elementary school emphasizes the underlying properties and structure of numbers and operation, research has shown that elementary students are capable of developing these processes, giving them solid foundation for geometric and algebraic reasoning (Langrall, Mooney, Nisbet, & Jones, 2008). Elementary teachers should prepare students for more complex mathematical work in the field of algebra. They can do so by helping students gain understanding and experience by working with patterns and number properties. Such work can prepare students for future work with functions and symbols and algebraic expressions. Elementary work with patterns develops

recursive thinking and is the prelude to functions while numbers and their properties precede symbols and algebraic expressions (National Council of Teachers of Mathematics, 2000). Elementary teachers can contribute to the development of basic mathematical concepts with their students. The symmetrical meaning of the equal sign, as well as the commutative and associative properties, are all foundations of algebraic thinking (Van Dooren, Verschaffel, & Onghena, 2003). Even though the proper vocabulary is not introduced at such early age, the algebraic properties students use, like the commutative and associative properties, should be promoted.

Researches indicate that students have a variety of difficulties with the variable concept so fostering understanding from early on is important. In early grades the notion of a variable as a placeholder for number is appropriate (National Council of Teachers of Mathematics, 2000). The algebraic character of early mathematics can be introduced to students through various activities. Research indicates that students can benefit from early exposure to letters being used to represent variables (Carraher & Schliemann, 2010). The meaning of the equation sign as a sign of equivalence and balance needs to be promoted from the beginning since young students tend to interpret the equal sign as an operational symbol (National Council of Teachers of Mathematics, 2000).

As a preparation for algebra, a major goal for mathematics in grades K-8 should be proficiency with fractions and whole numbers as well as particular aspects of geometry and measurement (National Mathematics Advisory Panel, 2008). The National Mathematics Advisory Panel (2008) recommends that teacher education programs and licensure tests for elementary teachers should attend to these topics, fractions, whole numbers and the appropriate geometry and measurement topics.

*Division and fractions.* Division and fractions are related. They can even be seen as the same thing, where the fraction bar represents the division sign.

When first introduced to fractions students have a hard time seeing them as a single number. They tend to treat them as two separate numbers and use whole-number thinking in order to try to solve problems concerning fractions. Such doing strongly indicates that students do not have the basic understanding of what a fraction is. This indication makes it important to focus on developing the understanding of what a fraction is from the beginning (Cramer & Whitney, 2010). Fractions lay the foundation for studies of ratios, proportions, and percentages, which cannot be properly understood without fractions. The arithmetic of fractions is important as a foundation for algebra (Ball et al., 2005).

The National Mathematics Advisory Panel (2008) recommends proficiency with fractions as a major goal in mathematics education. They also recommend that teacher education programs focus on fractions and whole numbers. The importance of fractions in early mathematics is supported by Siegler et al. (2012) study. Their results indicated that 10 years old elementary school students' knowledge of fractions and whole number division, could predict those students' knowledge of algebra and overall mathematical achievement in high school.

Leinhardt and Smith (1985) chose fractions as the topic to study when examining the relationship between expert teachers' classroom behavior and their subject matter knowledge. Their reason for the choice of fractions was that they are one of the more difficult topics in elementary arithmetic. Leinhardt and Smith (1985) noticed



considerable variability in knowledge of fundamental fractions concepts among the teachers, where 25% of them had relatively high mathematical knowledge.

Ma (1999) studied teachers' understanding of fundamental mathematics in China and the US. One of the topics she investigated was division by fractions. Her results showed that 43% of US teachers succeeded in the calculations, but only 4% of them could construct a conceptually correct story problem regarding division by fractions. 4% of the US teachers could display a correct approach to the relation between perimeter and area of a rectangle. In short, the more conceptually demanding the questions were, the worse the US teachers did (Ma, 1999).

Tirosh (2000) studied prospective elementary teachers' knowledge of division of fractions, both with regards to content and students' approach to the topic. Findings indicated that prospective elementary teachers were aware of arithmetical and reading comprehension mistakes, but unaware of conceptual mistakes students, for example mistakes contributed to students' transfer of properties of whole number division to fractions.

Ball (1990) investigated knowledge of division among prospective elementary and secondary teachers, focusing on division by fractions, division by zero, and algebraic equations. Her findings indicated that prospective teachers' knowledge of division was procedural and had gaps in it.

Tirosh and Graeber (1989, 1990) examined prospective elementary teachers' beliefs regarding division. Their results implied that a substantial portion of their participants were influenced by the misconceptions that division always made smaller, and that the quotient had to be less than the dividend.

Simon (1993) researched prospective elementary teachers' knowledge of division. His findings implied that their knowledge was mainly procedural and that they had weak conceptual knowledge, including of the foundations of familiar algorithms.

These research findings indicate that prospective teachers are ill prepared to see division as a conceptual object and therefore will have problems guiding their students in their conceptual understanding of division.

### **Mathematics teacher education**

Studies of mathematical knowledge have brought attention to the mathematics instruction in the US and can be used in the construction of teacher education courses (Hill, Sleep, Lewis, & Ball, 2007). Two decades ago Catharine Brown and Hilda Borko (1992) wrote a chapter on becoming a mathematics teacher. One of the assumptions guiding their work was that becoming a teacher was a lifelong process; beginning long before formal teacher education, and continuing all through the teacher's career. They were on the same page as Shulman (1987) who said: "A knowledge base for teaching is not fixed and final" (p. 12). Brown and Borko (1992) cited research whose results indicated that prospective teachers formed definite conceptions of the nature of teaching prior to entering the teaching profession, and that unless teacher education could change these conceptions, teachers would teach similar to the way they were taught (Borko & Brown, 1992). Ball (1990) supported that when she stressed the importance of teacher educators taking in to account what prospective teachers already knew, and what they would learn in the field.

Investigators claim that content knowledge should be the core of teachers' education (Brown & Borko, 1992). Research findings confirm the importance of strong

content knowledge, finding a relationship between the comprehension of new teachers and the teaching style they employ (Shulman, 1987). Prospective teachers with strong content knowledge also appear to be more flexible in their teaching and provide more conceptual explanations (Brown & Borko, 1992). The content knowledge has to be both theoretical and pedagogical since research has shown that teachers' coursework in mathematics education is more influential on student achievement than their coursework in mathematics (Mewborn, 2003).

Mathematics educators seem to agree that "teachers must know in detail and from a more advanced perspective the mathematical content they are responsible for teaching...both prior to and beyond the level they are assigned to teach" (National Mathematics Advisory Panel, 2008, p. 37). When it comes to the actual mathematics preparation of prospective teachers it seems to be deficient. When entering teacher education program many prospective teachers find it difficult to get the meaning behind the mathematics they are taught (Ball & Bass, 2000), which might be attributed to the fact that subject matter courses in teacher preparation programs tend to be scholarly and irrelevant to classroom teaching (Ball, Thames, & Phelps, 2008). Also, even though teacher educators and prospective teachers strive to teach for conceptual understanding their lessons for the most part consist of methods and examples that encourage procedural understanding (Eisenhart, Borko, Underhill, Brown, Jones, & Agard, 1993). The importance of mathematical content knowledge cannot be trivialized. Teacher educational programs that compromise on subject matter knowledge, leaving their prospective teachers with limited content knowledge of the mathematics they are supposed to teach, have negative effects on their pedagogical content knowledge leading

to unfavorable effects on their instructional quality and student progress (Baumert, et al., 2010). The National Mathematics Advisory Panel (2008) recommends that the mathematics preparation of elementary and middle school teachers must be strengthened as a mean for improving teachers' effectiveness. These mathematic preparations include giving prospective teachers plenty of opportunities to learn mathematics for teaching. It is important to identify the aspects of teaching that work and are most beneficial for beginning teachers. According to Ball, Sleep, Boerst, and Bass (2009) teaching includes "planning, choosing and using representations, conducting discussions of mathematics problems - and then analyze and decompose these domains into teachable components." Since it is unlikely that all the arts and crafts of teaching can fit into one course or even a lot of courses, teacher education courses should focus on "practices most likely to equip beginners with capabilities for the fundamental elements of professional work and that are unlikely to be learned on one's own through experience" (Ball et al., 2009).

Studies have shown the importance and effect mathematics education courses can have for prospective teachers. These courses can cause growth in prospective teachers' interest in mathematics (Macnab & Payne, 2003).

A study of Cypriot prospective teachers implied that they had misconceptions and negative attitude towards mathematics. Of the participants, 24% agreed with the statement "*I detest mathematics and avoid using it at all times*". This attitude seems to be subjective to change, since there was detected a positive shift in attitude through the study, in particular regarding the satisfaction from and the usefulness of mathematics (Philippou & Christou, 1998).

Burton (2012) studied prospective teachers' perceptions of mathematics before and after they attended a mathematics methods course and had field experience. The findings indicated that even though the majority of prospective teachers expressed negative experiences and impressions of mathematics in the beginning, their disposition changed after their experience with the mathematics methods course and field experience.

The results of Frykholm's (1999) study of 63 prospective mathematics teachers indicated that prospective teachers needed assistance in implementing the theory they learned in teacher education programs. In the study the NCTM standards were seen by prospective teachers as material to be learned rather than to be integrated into their teaching style. Their teaching style seemed to be adopted from their cooperating teachers, many of whom still followed the traditional model.

In their report, meant for those who teach mathematics to prospective teachers, The Conference Board of the Mathematical Sciences (2012) had recommendations regarding how to go about the mathematics education of prospective teachers. The first recommendation seems obvious; prospective teachers need courses to enable them to understand the mathematics they will teach. The Conference Board of the Mathematical Sciences further elaborated:

Prospective teachers need to understand the fundamental principles that underlie school mathematics, so that they can teach it to diverse groups of students as a coherent, reasoned activity and communicate an appreciation of the elegance and power of the subject. Thus, coursework for prospective teachers should examine the mathematics they will teach in depth, from a teacher's perspective. (p. 17)

The Conference Board (2012) also suggested that prospective teachers get ample time to reason, explain and make sense of the mathematics they will teach, and that throughout their careers, teachers will have time to attend to professional development in

their mathematics. Cooperation between teacher education programs and mathematics faculty was another action The Conference Board (2012) suggested to ensure that prospective elementary teachers had sufficient knowledge and skills when certified. As a knowledge base for teaching elementary mathematics, The Conference Board (2012) recommended as a prerequisite to teaching: "...an elementary teacher should study in depth, and from a teacher's perspective, the vast majority of K–5 mathematics, its connections to prekindergarten mathematics, and its connections to grades 6–8 mathematics" (p. 23).

Ball et al. (2009) recommended ways to improve the mathematics education of teachers. First, prospective teachers' education is based on their instructors' expertise, so a shared professional curriculum is needed to prepare teachers to teach mathematics. Second, instructors in teacher education need support. They do not have a curriculum either, and there is little established pedagogy on teaching practice. Teaching the craft of teaching is different from teaching a specific subject, so even if instructors are experienced K-12 teachers, it is not equivalent. Last, but not least, Ball et al. (2009), in line with Frykholm's (1999) findings, stressed the importance of student teachers' practical experience, claiming it was not enough to read, hear and talk about teaching, there had to be some doing.

### **Research on teacher education in mathematics**

Most studies regarding mathematics teacher education, mathematical teaching or student performance, state mathematics teachers' content knowledge as one of the big influences.

Ponte & Chapman (2006) studied researches on prospective elementary teachers' knowledge. They found issues of concerns both in regard to what prospective elementary teachers knew and how they knew it. Ponte & Chapman (2008) listed the following issues with prospective elementary teachers' knowledge of mathematics:

- Procedural attachments that inhibit development of a deeper understanding of concepts related to the multiplicative structure of whole numbers
- Influence of primitive, behavioral models for multiplication and division
- Adequate procedural knowledge but inadequate conceptual knowledge of division and sparse connections between the two
- Incomplete representations and narrow understanding of fractions
- Distorted definitions and images of rational numbers
- Lack of ability to connect real-world situations and symbolic computations
- Serious difficulties with algebra
- Difficulty in processing geometrical information and lack of basic geometrical knowledge, skills and analytical thinking ability
- Inadequate logical reasoning.

Research has indicated that elementary teachers and prospective elementary teachers rely on memory for particular rules in mathematics, their knowledge is procedural and they do not have an underlying understanding of mathematical concepts or procedures (Ball, 1990; Tirosh & Graeber, 1989; Tirosh & Graeber, 1990; Simon, 1993; Mewborn, 2003; Hill et al., 2007). Prospective teachers seem to believe that they

will and can be effective elementary mathematics teachers, even though they demonstrate low mathematical understanding (Stevens & Wenner, 1996; Macnab & Payne, 2003), and dislike the subject (Ball, 1990; Philippou & Christou, 1998; Burton, 2012).

Over 20 years ago Ball (1990) investigated the mathematical understanding of 252 prospective teachers. Her results indicated that the prospective teachers had difficulties explaining the meaning of division by fractions. They were able to do the calculations, but not to come up with a mathematically appropriate story of the division. Her results also indicated that prospective teachers focused on procedures and rules in mathematics. Tirosh and Graeber (1989, 1990) examined prospective elementary teachers' beliefs regarding multiplication and division. Their results implied that a substantial portion of their participants were influenced by the misconceptions that multiplication always made bigger, division always made smaller, and that the quotient had to be less than the dividend. Simon (1993) also investigated prospective teachers' knowledge of division using open response items and interviews. His findings were in line with Ball's (1990) that in general, prospective teachers' knowledge was procedural and fragmented; their conceptual knowledge was weak, including the foundations of familiar algorithms.

Stacey, Helme, Steinle, Baturu, Irwin, & Bana (2001) researched prospective elementary teachers' decimal numeration understanding. Their findings indicated that a significant proportion of prospective elementary teachers lacked in knowledge of decimals. The prospective elementary teachers had problems with the relationship between decimals, whole numbers, fractions, zero and negative numbers.



Van Dooren, Verschaffel and Onghena (2003) studied prospective teachers' arithmetic and algebra word problem solving skills and strategies. Their results indicated that some prospective elementary teachers had serious problems with algebra and were unable to correctly apply an arithmetic method to solve a problem at hand. Their results also implied that prospective teachers needed guidance to transform their thinking from arithmetical to algebraic thinking.

McCoy (2011) examined the relationship between mathematics teacher efficacy and the growth in specialized mathematical knowledge among prospective elementary teachers. Her findings indicated that prospective teachers' specialized knowledge grew significantly during mathematics methods/content course. Findings also indicated that common content knowledge, personal mathematics teacher efficacy and mathematics teaching outcome expectancy increased as a result of the course.

Johnson (2011) investigated the development of mathematical knowledge for teaching among 35 senior elementary education majors during a methods course in mathematics education. Her results indicated that prospective elementary teachers developed their own conceptions of MKT, different from the original definition, though there was some overlapping. She also found evidence of growth in MKT, though not as much as McCoy (2011).

Macnab and Payne (2003) conducted a study of attitudes and feelings regarding mathematics and mathematics teaching among prospective primary school teachers in Scotland. Results from their study indicated that for many prospective primary school teachers, secondary school mathematics was boring and difficult. The prospective teachers anticipated mathematics teaching to be the most challenging and least exciting of

all the main curricular areas, even though they took confident and optimistic view of their mathematical skills in regard to the primary school curriculum.

This chapter introduced the ideas scholars and institutions interested in mathematics education have about the mathematical knowledge students and teachers should possess. Research cited in the chapter indicated that the reality is not in line with these ideas and expectations; teachers' mathematical knowledge appears to be shallow and procedural.

## IV Methodology

### Participants

The School of Education at the University of Iceland offers a five year teacher education program, ending with a master's degree in education, which is needed to teach in K-10 schools.

The sample of students obtained for this study was a convenience sample. The initial idea with the study was to investigate prospective teachers on the elementary line in their final year of studies. Because of the change in teacher education in Iceland (the requirement of master's degree, adding two years to the studies), that was not a viable option. After consulting with the faculty at the School of Education at the University of Iceland, a decision was made to include in the study prospective elementary teachers in their second and third year, and masters students, both in elementary education and general education. The prospective elementary teachers were all enrolled in the course *Teaching mathematics to young students*. The researcher visited their class and invited them to take part in the study. All of the prospective elementary teachers in the course agreed to take part. All masters' candidates attending a compulsory course called *Subject Teaching* (a total of 50) were invited to take part in the study, 23 of them accepted.

The study was introduced to potential participants; its purpose, procedures, expected time commitment, as well as the measures taken to protect their identity were explained. Participants were made aware that participation was voluntary and that there were no benefits from taking part in the study. When participants showed up to take part in the study they signed a copy of an informed consent document. Copies of the

recruitment letter and the informed consent document can be found in Appendices B and C.

The participants in the study were 38 prospective teachers in the teacher education program within the School of Education at the University of Iceland. There was a great variety among the participants. Of the participants, 15 were in their second or third year towards a B.Ed degree in elementary education, and 23 in their fourth year towards a master's degree, four of them in elementary education, and the others in different subjects. The participants in this study accounted for 44% of B.Ed students in elementary education, and 38% of M.Ed students at the School of Education, University of Iceland, at the time of the study (Gudmundsdottir, 2012).

Of the participants, eight had the subject mathematics as their specialty at the School of Education, and one had a BS degree in mathematics. Nineteen of the participants had elementary education as their major while 11 had another major. For example, one participant had science education as a major. Seven of the participants had teaching experience (some currently teaching) and five of those had taught mathematics, for up to 20 years. The oldest participant in the study was born in 1960 and the youngest in 1991, leaving the age difference between the oldest and the youngest participant 31 year. Four of the participants were males. There was a great difference in the mathematics education of the participants, both at the School of Education and prior to entering the teacher education program. Some of the participants had completed 10 courses in mathematics in their high school/college while others had completed only two.

Out of the 38 participants, 10 were interviewed, two males and eight females. Of the interviewees, five had a strong mathematics background (7 or more courses in high

school/college), three were prospective elementary teachers, and nine were in the 4<sup>th</sup> year towards their masters. Two of the interviewees ranked in the 90<sup>th</sup> percentile and 2 below the 30<sup>th</sup> percentile in the questionnaire survey, the others between them.

## **Procedures**

This study utilized mixed methods design. A mixed methods research study is a study "...where the researcher mixes or combines quantitative and qualitative research techniques, methods, approaches, concepts or language into a single study" (Johnson & Onwuegbuzie, 2004, p. 17).

The combination of mathematical tasks and interviews is a widely used method to explore teachers' mathematical knowledge. The method allows for the researcher to adjust questions to each respondent based on their previous answers or solutions. By doing so the researcher can gain a better understanding of the respondents' way of thinking (Hill et al., 2007), and gives the interviewer an opportunity to probe the interviewee when further information or clarity is needed (Johnson & Turner, 2003).

### **Quantitative procedures**

The quantitative measurement tool used in this study was a questionnaire survey consisting of translated and adapted items from the MKT measures. The MKT measures consist of multiple-choice items. Each item either stands alone, stem item, or has other problems attached to it, leaves. Figure 1 shows examples of items, with and without leaves.

7. Which of the following story problems could be used to illustrate  $1\frac{1}{4}$  divided by  $\frac{1}{2}$ ? (Mark YES, NO, or I'M NOT SURE for each possibility.)

	Yes	No	I'm not sure
a) You want to split $1\frac{1}{4}$ pies evenly between two families. How much should each family get?	1	2	3
b) You have \$1.25 and may soon double your money. How much money would you end up with?	1	2	3
c) You are making some homemade taffy and the recipe calls for $1\frac{1}{4}$ cups of butter. How many sticks of butter (each stick = $\frac{1}{2}$ cup) will you need?	1	2	3

8. As Mr. Callahan was reviewing his students' work from the day's lesson on multiplication, he noticed that Todd had invented an algorithm that was different from the one taught in class. Todd's work looked like this:

$$\begin{array}{r} 983 \\ \times 6 \\ 488 \\ +5410 \\ \hline 5898 \end{array}$$

What is Todd doing here? (Mark ONE answer.)

- a) Todd is regrouping ("carrying") tens and ones, but his work does not record the regrouping.
- b) Todd is using the traditional multiplication algorithm but working from left to right.
- c) Todd has developed a method for keeping track of place value in the answer that is different from the conventional algorithm.
- d) Todd is not doing anything systematic. He just got lucky – what he has done here will not work in most cases.

Figure 1. An example of MKT items, with and without leaves.

In order to use the MKT measures, a participation in a training workshop is required (School of Education, University of Michigan). The researcher attended this workshop in the spring of 2011, where the history of the project, technical information regarding validity and reliability of the measures, and how to go about administering the measures was introduced.

During the initial screening of the MKT measures the researcher decided to focus on the most recent scales in the targeted mathematical topic domains, numbers and operations and patterns, functions and algebra. The most recent scales for numbers and operations were from 2008 while scales from 2006 were the most recent for patterns, functions and algebra. There were two scales for each topic, and each scale consisted of a set of items. The researcher chose the elementary level because of her initial interest in elementary teachers, and also because if general teachers teach mathematics, they usually do so at the lower levels of school. The researcher started with two complete questionnaires from each topic, numbers and operations, and patterns, functions and algebra. Throughout the first screening, items regarding topics not covered in Icelandic

elementary schools were omitted (including items on ratio, proportions, and cross multiplication).

The number of items on an MKT assessment should be at least 15-20, since longer tests are more reliable. For items to discriminate among participants their slope should be above .5 and item difficulty should be well targeted (Learning Mathematics for Teaching, 2011A). Following these guidelines the remaining items were translated and adapted.

*The process of translating and adapting the measures.* In spite of difference between the teaching culture in the US and Iceland, it seems plausible that there is some overlapping in the mathematical knowledge needed for teaching in both countries, as well as in the understanding and skills students are expected to show. The national curriculum in Iceland states the knowledge that students are supposed to have in mathematics and the nature of that knowledge. There is no national curriculum in the US, but there are the NCTM standards and the Common Core Standards, which state, like the national curriculum in Iceland, what students should know in mathematics. A comparison of these documents gave ideas about similarities in emphasis in mathematics education in both countries. Similarities included for example: Solid understanding of the base 10 number system, fluency with arithmetic operations, students' opportunity to develop their own solution methods as well as the chance to discuss and evaluate the solution strategies of others, understanding of the relationship between multiplication and division, and students' ability to estimate answers.

The MKT measures have been translated and adapted for use in countries outside the US. The researcher contacted Sean Delany, who adapted the measures for use in

Ireland, and Reidar Mosvold, who translated and adapted the measures for use in Norway, for advice on how to go about translating and adapting the measures.

According to literature regarding test item translation and adaptation, it is preferable that the translator is a native speaker of the target language, knows the target culture and is an expert in the subject matter (Delaney et al., 2008). The researcher in this study met these conditions. To gain insight into elementary mathematics education in the US, the researcher taught for a year as an elementary teacher in New York City. During that time she had the opportunity to familiarize herself with teaching strategies, material and curricular emphasis.

When deciding which items to use and to determine the appropriateness of the translation and the adaptation, the researcher sought professional opinion from practicing elementary teachers, a professor in mathematics teacher education, a professor in mathematics, and doctoral students in mathematics education. These professionals received instructions to evaluate whether the wording of the problems was suitable for elementary teachers and the topics appropriate. They were also urged to comment on whatever they thought was important to increase the validity of the items. The mathematician was also asked for opinion on the mathematical suitability of the items as well as the proper use of mathematical language. The most important factor for the choice of items for use in the survey was the items' relevance for measuring knowledge stipulated in the Icelandic national curriculum in mathematics.

The items were translated and adapted using a framework developed by Delaney et al. (2008), with the addition of one category (the fourth) from Mosvold et al. (2009). During the translation and adaptation process, changes made to the items were sorted into



the following categories:

- Changes related to the general cultural context; changing people's names, changing culturally specific activities.
- Changes related to the school cultural context; changing context so that it was familiar to Icelandic teachers and according to the Icelandic school and educational system.
- Changes related to mathematical substance; changing substance so that it was familiar to Icelandic teachers. This involved for example changes of measurement units and currency, and changes in representation so it resembled the representation used in Icelandic schools.
- Changes related to the translation from American English to Icelandic; structural aspects of the languages made certain changes necessary, for example the use of gender in Icelandic words.
- Other changes; changes not necessitated by cultural requirements. These changes include alterations to visual appearance and further clarification of an item.

11. Students in Mr. Hayes' class have been working on putting decimals in order. Three students — Andy, Clara, and Keisha — presented 1.1, 12, 48, 102, 31.3, .676 as decimals ordered from least to greatest. What error are these students making? (Mark ONE answer.)

- a) They are ignoring place value.
- b) They are ignoring the decimal point.
- c) They are guessing.
- d) They have forgotten their numbers between 0 and 1.
- e) They are making all of the above errors.

Figure 2. A published MKT item.

To better understand the process of translating and adapting an MKT item, a published item (figure 2) is used as an example. The first thing that needs to be adapted in this item is the teachers' name. In Iceland people go by their first name, so the teacher would be referred to by his first name, say Halldór. In Icelandic it is more common to talk about ordering decimals rather than put them in order, so this would translate to order decimals according to size. The students in the problem get Icelandic names, Andri, Klara and Erla. In Iceland the decimal comma is used, not the decimal point, and a zero is put before the comma in the absence of another whole number. The series of numbers, 1,1, 12, 48, 102, 31.3, .676 would look confusing if the only change made was the exchange of points and commas, and the addition of zero, 1,1, 12, 48, 102, 31,3, 0,676, so another visual representation would be needed:

$$1,1 \quad 12 \quad 48 \quad 102 \quad 31,3 \quad 0,676$$

The literal translation of the word ignore is rather harsh in Icelandic so “don't take into account” would be more appropriate in this context. Since the decimal point is not used in Iceland, answer possibility b) becomes questionable. If you skip the decimal comma, .676 looks like 0676, which is not a likely possibility. The wording of possibility d) would not be used in Icelandic. Instead of using “their numbers between 0 and 1”, “what applies to numbers between 0 and 1” sounds more natural.

The changes made to this item fall into the framework categories as follows:

- Changes related to the general cultural context; the teacher's and students' names.
- Changes related to the school cultural context; nothing fell into that category this time.
- Changes related to mathematical substance; decimal point changed to decimal

- comma, and the addition of zero.
- Changes related to translating from American English to Icelandic; “put in order” changed to “order according to size”, “ignore” changed to “don’t take into account”. “Their numbers between 0 and 1” changed to “what applies to numbers between 0 and 1”.
  - Other changes; visual changes            1,1    12    48    102    31,3    0,676

***The final questionnaire survey.*** Teachers show their knowledge by solving mathematical problems, evaluating non-standard solution methods, and by correctly using mathematical notations and definitions (Learning Mathematics for Teaching, 2004). The final set of items selected for the Icelandic survey contained 24 items (51 counting the leaves), 12 from each topic; number and operations, and patterns, functions and algebra. Most of the items dealt with mathematical topics usually covered in grades 1-5. Topics covered included: subtraction, division, fractions (multiplying, dividing, simplifying), alternative algorithms, positive and negative numbers, perimeter, area, patterns, writing equations, and functions. When lining up the items within the questionnaire, the researcher strove to put relatively easier items in the beginning in order not to repel more insecure participants.

It is not possible to give an example of the actual questionnaire used, since the MKT items are released only to those who have attended a special training workshop (Learning Mathematics for Teaching, 2004). Instead a sample questionnaire with published items is in Appendix A.

***Reliability and validity.*** The MKT elementary items were piloted with over 600 elementary teachers (School of Education, University of Michigan). Extensive research

has been conducted to investigate whether the MKT items reliably and validly measure teachers' mathematical knowledge for teaching (Hill, 2007). The reliabilities reported for the numbers and operations, and patterns, functions and algebra scales are good, ranging from 0.71 to 0.84 in coefficient alphas for a classical test theory measure of reliability (Hill, Schilling, & Ball, 2004).

Since the MKT items used in the survey were translated and adapted, the reported reliability and validity of the original items were compromised. Reliability of a test deals with the test's ability to measure consistently. Cronbach's alpha provides a measure of internal consistency of a test, which is whether the test items are measuring the same concept or construct, or traits that are highly correlated. Calculating Cronbach's alpha for a test is convenient when there is only one test administration (Allen & Yen, 1979; Tavakol & Dennick, 2011). Number of items on a test affects the alpha; larger number of items yields a larger alpha. Acceptable values of alpha range from 0.70 to 0.95 (Tavakol & Dennick, 2011).

If a test is supposed to measure more than one concept or trait, it makes sense to calculate alpha for each of the concepts (Tavakol & Dennick, 2011). In order to check reliability of the questionnaire used in this study, alpha was calculated for its different parts. Knowledge wise, the items fell into two categories, common content knowledge (CCK) and specialized content knowledge (SCK). Topic wise the items also fell into two groups, numbers and operations (NOP), and patterns functions and algebra (PFA).

Table 4.1

*Scale reliability*

	Number of items, including the leaves	Reliability (Cronbach's alpha)	Maximum Information <sup>5</sup>	Number of items over 1 standard deviation
Number and operations	21	0.789	0.068	5
Patterns, functions and algebra	30	0.862	-0.107	2
Common content knowledge	25	0.819	-0.191	2
Specialized content knowledge	26	0.846	0.211	5
All items	51	0.903	0.014	7

Table 4.1 shows the calculated alpha for each of the topics, numbers and operations and patterns, functions and algebra, as well as for each of the knowledge domains, common content knowledge and specialized content knowledge. The values of alpha for these groups of items were well within the range of acceptable values (Tavakol & Dennick, 2011).

When further divided, the items groups got the following alpha; 0.59 CCK & NOP (7 items), 0.77 CCK & PFA (18 items), 0.72 SCK & NOP (14 items), and 0.78 SCK & PFA (12 items). These statistics show that the reliability of the questionnaire was acceptable, with the exception of the CCK & NOP items, where the reliability is rather low, due to few items.

<sup>5</sup> Item measures ability with the greatest precision at this ability level.

A test has validity if it measures what it is supposed to measure (Allen & Yen, 1979). Throughout the whole process of translating and adapting the test items, opinion was sought from professionals in the mathematics education field. Working elementary teachers, mathematics teacher educator and doctoral students in mathematics education were consulted at various stages of the translation and adaptation process to ensure content validity.

### **Qualitative procedures**

The researcher is an instrument of the research in qualitative studies (Maxwell, 2005). The researcher in this study was an experienced mathematics teacher at all levels, both in Icelandic and US schools and had a good knowledge of both the Icelandic national curriculum in mathematics and the Common Core State Standards in mathematics. The researcher had an M.Ed. in mathematics education and was working towards a Ph.D. in mathematics education. Some of the interviewees constantly sought approval from the researcher while solving the mathematics problems and answering questions. That might result from their insecurity in mathematics accompanied by the researchers position. This might have caused respondents to shy away from taking a guess or trying to answer a question they were not sure about the answer of.

Once participants had answered the survey, they were asked if they would participate in an interview. Of the 38 survey participants, 10 agreed to be interviewed. Some of the survey participants gave lack of time as an answer to why they could not take part in an interview, while others bluntly said their lack of ability in mathematics caused them to have no interest what so ever in being interviewed.

The interviews were semi-structured and lasted from 30 to 50 minutes. The interviews were audio recorded. The semi-structured approach was chosen since it gave the interviewer more flexibility in the use of the interview guide. For example, the questions could be ordered in such a way that they aligned with the natural course of the interview. Also, the semi-structured approach allowed for the interviewer to probe and dig deeper for more information and clarity when needed (Gibson & Brown, 2009). Structured interview approaches enable the researchers to compare data from different individuals (Maxwell, 2005).

The main intent of the interviews was to gather data for research question one (What is the level of elementary mathematical content knowledge among prospective Icelandic teachers?) and two (Does age and/or teaching experience prior to entering teacher education program influence levels of knowledge reported in question 1?)

The interviews were composed of two parts; background information on the participants and their mathematical preparation to date and mathematical problems. In order to gain an understanding of how the prospective teachers used their knowledge and of the strategies they employed while solving the problems, they were asked to ‘think aloud’ during their solutions of the problems. The ‘think aloud’ method is considered a good method to gain access to participants thinking process while solving problems (van Someren, Barnard, & Sandberg, 1994).

The first part of the interview focused on the background and education of interviewees prior to entering the teacher education program. Participants were asked about their major in high school/college and their mathematical education during that time, how many mathematics courses they completed and the content of the courses.

They were also asked about their interest in the mathematics they learned before entering the teacher education program. This part of the interview served to gather data for the second research question; Does age and/or teaching experience prior to entering teacher education program influence levels of knowledge?

The second part of the interview aimed at shedding light on interviewees' mathematical content knowledge. In this part of the interview, participants were asked to solve four mathematical problems, similar to those Ball (1990) and Ma (1999) used in their studies. In the instructions prior to this part of the interview, participants were asked to solve the problems, as well as to think of story problems that could be used to accompany the problems. They were also asked to think aloud while solving the problems, that is, to explain the steps they took during the solutions. They were told that afterwards they would be asked questions regarding their solutions, questions similar to those they might get from their students. The mathematical problems were: subtraction with regrouping, double-digit multiplication, division without remainder, and division of fractions. Once the participants had solved each problem, they were asked about their use and choice of methods, words, and if they could come up with an appropriate story in context with the problem. They were also asked if they could think of other ways of solving each problem.

The problems and questions used in the interviews were chosen to investigate the prospective teachers' use of mathematical methods and language, and how they rationalized the use of certain algorithms and the reasons why they worked. These questions were in line with the focus of the Icelandic national curriculum and the new curriculum draft, where the proper use of mathematical language and being able to use



multiple solution methods is stressed. Being able to come up with a story to accompany a problem helps connect mathematics to everyday life, which is considered an important factor in mathematics education stressed in the Icelandic national curriculum in mathematics.

The interview was pilot-tested with working elementary teachers as well as a colleague in mathematics education to ensure the appropriateness of the questions and problems, as suggested by Maxwell (2005), in addition to establish a timeframe for potential interviewees. The interview guide (Appendix D) was created in such a way that there was plenty of air between the questions for notes and comments from the researcher.

### **The data**

The data from the questionnaire surveys and transcripts from the interviews were analyzed in order to answer the research questions of this study. Once the questionnaires had been gathered, the problems were graded and the outcomes were inserted into an Excel file. The initial recording of the data counted responses to each possible answer since the wrong answers could give important information regarding misconceptions or point to a flawed item. The scores were all normalized and basic descriptive statistics were gathered. A test for normality for the whole questionnaire survey revealed that skewness was 0.229 and Kurtosis -0.664. Skewness between 0 and .5 is fairly symmetrical (Bulmer, 1979), and Kurtosis -0.664 indicates that the distribution is flatter than the normal distribution and with light tails (DeCarlo, 1997). These statistics show that the distribution of scores was approximately symmetric.

**Item response theory.**

In order to utilize item response theory (IRT), the participants' answers were transformed to binary format. IRT was used to find each participant's trait level and each item's difficulty. Participants' trait level is their level on the psychological trait being assessed by the test items. For example, a person with high mathematical knowledge is more likely to answer a mathematics question correctly than is a person with low mathematical knowledge. Participants' trait level is one of the factors affecting how they answer a particular item (Furr & Bacharach, 2008). Another factor influencing participant's probability of answering a certain way is item difficulty. For example, an item with high level of difficulty is less likely to be answered correctly than is an item with low level of difficulty. Trait level and item difficulty are related concepts in IRT. Item difficulty is calculated based on trait level (Furr & Bacharach, 2008). When the items difficulty matches the participant's trait level, the participant has a 50% chance of answering the item correctly (Allen & Yen, 1979). In IRT analysis trait level scores and difficulty scores are usually standardized with means 0 and standard deviation of 1. A person with trait level 0 has an average level of the trait being measured and an item with difficulty level 0 is an average item. An item with difficulty level 2 is a difficult item and an item with difficulty level -2 is an easy item. Test items vary in their ability to differentiate between persons with different trait levels. This ability is called item discrimination. The discrimination value implies the connection between the item and the trait being measured by the test (Furr & Bacharach, 2008).

Item characteristics-curves were created to examine each item, such as to show the probabilities of persons over the range of trait values to answer the item correctly and

to see the items discrimination ability. At the trait level where the slope of the item-characteristic curve is steepest, the item discriminates most effectively among participants in the test (Allen & Yen, 1979).

## **Interviews**

In order to get deeper insight into the prospective teachers' thinking process and more thorough answers to the research questions, information gathered from the interviews was investigated.

The analysis of the interviews began with the researcher listening to the interview tapes and taking notes and looking for patterns and themes. The researcher transcribed the interviews verbatim and word-by-word, including emotional expressions, resulting in the analysis of what was said to begin right away (Kvale & Brinkmann, 2009). In order to ensure accuracy of the transcripts, the researcher listened to the interviews again while reading the transcripts. Once that was done the interviews were coded. Some of the coding categories had been identified prior to the coding process, while others emerged during the process when important themes and patterns revealed themselves. The coding sheet can be found in Appendix E.

## **Research question I**

To answer the first research question, the first thing that was done was to examine responses from the whole questionnaire survey to see if there was a significant difference in ability for the mathematics topics numbers and operations and patterns, function and

algebra. A t-test was used to examine if there was a significant difference in responses between prospective teachers based on their major in the School of Education.

### **Research question I (a)**

To answer part (a) of the first research question regarding Icelandic prospective teachers' level of common content knowledge in elementary mathematics, responses from the questionnaire survey were graded, with a special focus on items having to do with common content knowledge. Common content knowledge (CCK) is “ the mathematical knowledge and skill used in settings other than teaching” (Ball et al., 2008, p. 399). The items in this category dealt with the following subjects: fractions, division, percentages, perimeter, area, patterns, formulas, functions, expressions, system of equations, rules in mathematics, mathematical facts, and mathematical concepts. Descriptive statistics were gathered for the common content knowledge items, and information from the IRT analysis was used to study the level of ability within the group of prospective teachers. The group was examined as a whole, and t-tests were conducted to see if there was a significant difference between participants based on their major.

The item difficulty analysis was used to identify strengths and weaknesses within the prospective teachers' common content knowledge. Item difficulty is connected to participants' trait level and is the trait level needed to have a 50% chance of answering the item correctly (Furr, 2008). Once item difficulty had been calculated, items were grouped into easy, medium and difficult, based on their difficulty. For this purpose, items with a difficulty level at or below -0.43 were considered easy, and items at or above difficulty level 0.43 were considered difficult, items between them medium.

Data from the interviews were used to further shed a light on prospective teachers' level of common content knowledge; in particular information regarding the proper use of mathematical language, notations and if interviewees could solve the mathematics problems given to them in the interview.

### **Research question I (b)**

In order to answer part (b) of research question one, regarding Icelandic prospective teachers' level of specialized content knowledge in mathematics, responses from the questionnaire were graded with a special focus on items regarding specialized content knowledge. Specialized content knowledge in mathematics is the content knowledge used in the teaching of mathematics (Ball et al., 2008). Items within this knowledge domain included the following topics: Rules in mathematics, alternative solution methods, mathematical explanations, the making of story problems, use of visual aids and models, and mathematical definitions. Descriptive statistics were gathered for the specialized content knowledge items. Item difficulty was examined to identify topics relatively easy and difficult for participants. Trait level scores were examined for participants based on their major, and t-tests conducted to investigate possible difference between groups.

Results from the interview coding with similar subject themes were compared to results from the questionnaire survey to provide a better understanding of the prospective teachers' specialized content knowledge.

## **Research question II**

To answer the second research question, Does age and/or teaching experience prior to entering teacher education program influence levels of knowledge reported in question I?, the relationship between prospective teachers' age and scores on the questionnaire survey was examined. Correlation between age and overall survey scores was calculated, as well as scores from the common content knowledge part of the test and the specialized content knowledge part.

The prospective teachers were divided into two groups based on whether they had teaching experience in mathematics or not. Simple regression was used to see if and how age and teaching experience affected the overall score from the questionnaire as well as specific parts of it, the common content knowledge part and the specialized content knowledge part.

Data from the interviews were used to further investigate potential difference in knowledge depending on interviewee's age and teaching experience. Data regarding the use of mathematical language, approach to solution methods and algorithms were especially examined.

## **Research question III**

To answer the third research question regarding the comparison of findings from research question I and findings from similar studies using the MKT items carried out in the US, a review of study findings was conducted. Also, the item difficulty calculated in the current study was compared to the item difficulty reports from the publishers of the MKT items (Learning Mathematics for Teaching, 2011B; 2011C). The purpose was to

see where the prospective students had similar ease/difficulty with items as did US elementary teachers, and where it differed. Item difficulty had been reported for 42 of the 51 items used in the study and was based on the elementary teachers participating in the pilot testing of the MKT measures. When an item had been pilot tested with more than one group of teachers, and had more than one reported item difficulty, the average difficulty level for that item was calculated.

## V Results

The purpose of this study was to investigate the mathematical content knowledge of prospective teachers in Iceland. The study consisted of quantitative and qualitative methods. The quantitative part was a questionnaire survey answered by all the 38 participants. The qualitative part of the study was interviews with 10 of the prospective teachers that prior had answered the questionnaire survey. Following data collection, the data was analyzed in order to answer these three research questions:

- I. What is the level of elementary mathematical content knowledge among prospective Icelandic teachers? More precisely, (a) What is the level of their common content knowledge?, and (b) What is the level of their specialized content knowledge?
- II. Does age and/or teaching experience prior to entering teacher education program influence levels of knowledge reported in question I?
- III. How does the level of elementary mathematical content knowledge reported in question I compare to findings from similar studies using the MKT measures carried out in the US?

### Demographics

Demographic data were collected on a variety of variables, and are summarized in table 5.1. These data were analyzed in order to get a clearer picture of the participants in the study.



Table 5.1

*Demographics (N = 38)*

	N	%
<b>Gender</b>		
Females	34	89.5
Males	4	10.5
<b>Age</b>		
20 – 29	21	55.2
30 – 39	12	31.6
40+	5	13.2
<b>Level of studies</b>		
B.Ed	15	39.5
M.Ed	23	60.5
<b>Major in School of Education</b>		
Elementary education	19	50.0
Mathematics	8	21.1
Other	11	28.9
<b>Mathematics teaching experience</b>		
Yes	5	13.2
No	33	86.8
<b>Mathematics courses in High school/college<sup>1</sup></b>		
2 – 3	10	27.8
4 – 6	13	36.1
7+ <sup>2</sup>	13	36.1
<b>Location of High school/college</b>		
Reykjavik and surrounding area	21	55.3
Outside Reykjavik	17	44.7

<sup>1</sup> A student has to complete at least two mathematics courses in a high school/college (Ministry of Education, 1999).

<sup>2</sup> Two of the participants did not recall how many mathematics courses they had completed in high school/college.

As expected, the majority (89.5%) of participants were women. The average age of participants was 32 years ( $SD = 8.00$ ). Of the participants, 26 (68.4%) had taken more than three mathematics courses in high school/college, and 17 (44.7%) of them had attended high school outside of Reykjavik. Eight (21.1%) participants had some teaching experience, but 5 (13.8%) had teaching experience in mathematics.

### The interviewees

Ten of the participants in the questionnaire survey agreed to be interviewed. To protect the identity of the interviewees they were given pseudonyms. Table 5.2 gives information on interviewee's major in the School of Education, whether or not they have experience teaching mathematics and their mathematical preparation. Interviewee's mathematical preparation was categorized based on numbers of mathematical courses completed in high school and college; the categories were, little (2-4 courses), medium (5-6 courses), and much (7+ courses).

Table 5.2

*Interviewees (N = 10)*

Interviewee	Major in School of Education	Mathematical Preparation	Teaching Experience
Allison	Elementary Education	Little	No
Beatrix	Elementary Education	Medium	No
Claudia	Elementary Education	Little	No
Debra	Elementary Education	Medium	No
Elyse	Other	Much	No
Fiona	Other	Little	No
Gina	Mathematics	Much	No
Heather	Mathematics	Much	Yes (much)
Ida	Mathematics	Much	No
Johnna	Mathematics	Much	Yes

### The questionnaire survey

Results from the questionnaire survey are discussed in terms of trait levels, and standard deviations, as outlined in the Terms of Use for the MKT instrument (Learning Mathematics for Teaching, 2011D). No raw scores or percentages are mentioned in

relation to results from the mathematical content knowledge part of the questionnaire survey.

The results from the questionnaire survey reflected a great variety in the ability level of the prospective teachers participating in this study. The test scores were approximately normally distributed, and the range of standardized scores was 3.68.

There was not a statistically significant difference between scores from the two topics, numbers and operations (NOP) and patterns, functions and algebra (PFA). None the less, a lot more items in the PFA section of the questionnaire survey were answered as: “I don’t know” or skipped, than were in the NOP section. In the PFA section 23.4% of items fell into that category, while only 5.5% of items in NOP did so. It could not be determined whether this was due to the fact that the NOP section came first in the survey, leaving participants tired when it came to the PFA part or towards the end of the survey, or whether some of that difference could be explained by different level of participants’ ability in those two subject categories.

When examining the data from the questionnaire survey, it was evident that there was a difference in score between prospective teachers majoring in mathematics and prospective teachers majoring in other subjects. Because of that difference prospective teachers were grouped according to their major: mathematics majors, elementary education majors, and other majors. When comparing mean trait levels<sup>6</sup> from these groups, a significant statistical difference was found between the trait levels of prospective teachers majoring in mathematics and prospective teachers majoring in other areas.

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<sup>6</sup> Trait level is participants’ level on the psychological trait being assessed by the test items (Furr & Bacharach, 2008).

Table 5.3

*Comparing means of mathematics majors and other majors*

	N	Mean	SD	t	df	p	d
The whole survey				5.67	25	< 0.001	2.39
Mathematics Majors	8	1.23	0.56				
Elementary Majors	19	-0.25	0.64				
The whole survey				5.95	17	< 0.001	2.76
Mathematics Majors	8	1.23	0.56				
Other Majors	11	-0.47	0.65				

Table 5.3 shows that the scores of mathematics majors were significantly different from both elementary education majors ( $p < 0.001$ ) and other majors ( $p < 0.001$ ).

Inspection of the groups' means indicated that the average score of mathematics majors ( $M = 1.23$ ) was significantly higher than the score of elementary education majors ( $M = -0.25$ ) and other majors ( $M = -0.47$ ). The effect size,  $d$ , was very large, at 2.39 and 2.76 respectively (Morgan, Leech, Gloeckner, & Barrett, 2011).

### **Research question I**

When analyzing the questionnaire survey in order to answer research question I, the researcher decided to take a look at what items appeared to be fairly easy or difficult for the participants. During the IRT analysis of the data, item difficulty had been calculated. Item difficulty is connected to participants' trait level, and is the trait level needed to have a 50% chance of answering the item correctly. Trait level is participants' level on the trait being assessed by the test items (Furr & Bacharach, 2008). Items were grouped based on their difficulty level, items with a difficulty level at or below -0.43 were considered easy, and items at or above difficulty level 0.43 were considered difficult.

### Common content knowledge

In order to answer research question I (a), data from the questionnaire survey and interviews were analyzed.

*The questionnaire survey.* Twenty-five of the items on the questionnaire survey measured common content knowledge. The average item difficulty calculated for the items was  $-0.19$  ( $SD = 0.95$ ). There was a great variance in participants' trait levels on the common content knowledge items, the scores ranging from  $-1.66$  to  $4.00$ , ( $M = 0.23$ ,  $SD = 1.08$ ). The distribution of trait levels is demonstrated in figure 3. One participant's trait level of 4 skewed the distribution, but without it the distribution was approximately symmetrical.

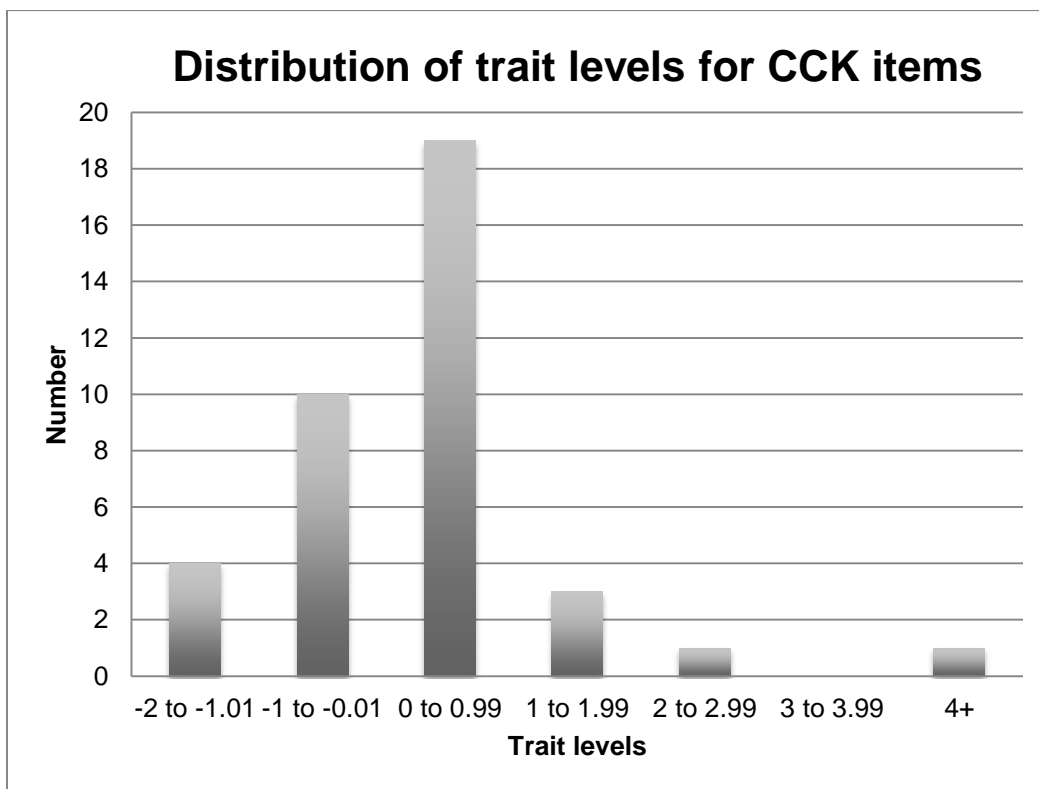


Figure 3. Distribution of trait levels on common content knowledge items.

There was a significant difference in trait levels between the mathematics majors and the other groups of prospective teachers. Between elementary education majors and prospective teachers majoring in other subjects than mathematics, there was no significant statistical difference.

Table 5.4

*Comparison of mathematics majors and elementary education majors on common content knowledge items*

	Mean	SD	t	df	p	d
CCK items			4.15	25	< 0.001	1.74
Mathematics Majors	1.19	0.73				
Elementary Majors	-0.20	0.82				

Table 5.4 shows that mathematics majors significantly differed from elementary education majors in their trait levels on common content knowledge items ( $p < 0.001$ ). Inspection of the two groups means indicated that the average trait level of mathematics majors ( $M = 1.19$ ) was significantly higher than the average trait level of elementary education majors ( $M = -0.20$ ), and the effect size,  $d$ , was very large, at 1.74 (Morgan et al., 2011)

Table 5.5

*Comparison of mathematics majors and majors in other subjects than mathematics and elementary education on common content knowledge items*

	Mean	SD	t	df	p	d
CCK items			4.88	17	< 0.001	2.27
Mathematics Majors	1.19	0.73				
Other Majors	-0.52	0.77				

Table 5.5 shows that mathematics majors were significantly different from prospective teachers majoring in other subjects than mathematics and elementary

education in their mean trait level on common content knowledge items ( $p < 0.001$ ).

Examination of the two groups means implied that the average trait level of mathematics majors ( $M = 1.19$ ) was significantly higher than the average trait level of other majors ( $M = -0.52$ ), and the effect size,  $d$ , was very large at 2.27 (Morgan et al., 2011).

Results from the questionnaire survey indicated that the topics described in table 5.6 were relatively difficult for participants, item difficulty ranging from 0.43 to 2.14.

Table 5.6

*Difficult common content knowledge topics*

Topic	Item difficulty
Identifying surjective function	2.14
Statement about multiplication	1.30
Properties of positive and negative numbers	0.65
Multiplying fractions	0.54
Algebra problem, needing a system of equations to solve	0.43

The questionnaire survey's results implied that the topics described in table 5.7 were fairly easy for participants, item difficulty ranging from -2.46 to -0.43.

Table 5.7

*Easy common content knowledge topics*

Topic	Item difficulty
Formula for perimeter	-2.46
Visual representation of a percentage of an area	-2.14
Patterns (forms)	-1.32
Statement regarding subtraction	-1.17
Bijjective functions	-0.44
Patterns	-0.43

**Interviews.** Participants were asked to solve four problems in the interviews, and to think aloud while doing so. Table 5.8 shows the problems as well as interviewees' performance in solving them.

Table 5.8

*Problems posed in interview*

Problem	Right	Wrong	Tried but didn't finish	Didn't try
74-26	9	1		
79×48	8	2		
1035÷5	9	1		
$2\frac{1}{4} \div \frac{1}{2}$	6	2	1	1

Most of the interviewees could solve the mathematical problems posed, with the exception of division of fractions. Six of the interviewees solved that problem correctly. One said: “Are you kidding me!” when showed the problem and said (s)he honestly did not have a clue about how to solve it. The remaining three interviewees tried to solve the problem but could not. Two of them converted the fractions to decimals while trying to solve the problem, and two found a common denominator for the fractions in the solution process.

**Specialized content knowledge**

Data from the questionnaire survey and interviews were analyzed to answer research question I (b).

**The questionnaire survey.** Specialized content knowledge items were 26 in the questionnaire survey. Their average item difficulty was calculated 0.211 ( $SD = 0.84$ ). The distribution of participants' trait levels in the specialized content knowledge part of the



questionnaire survey was approximately symmetric, as portrayed in figure 4. Participants' trait levels ranged from -2.48 to 2.04, or a difference of 4.52 and the mean was -0.21 ( $SD = 1.04$ ).

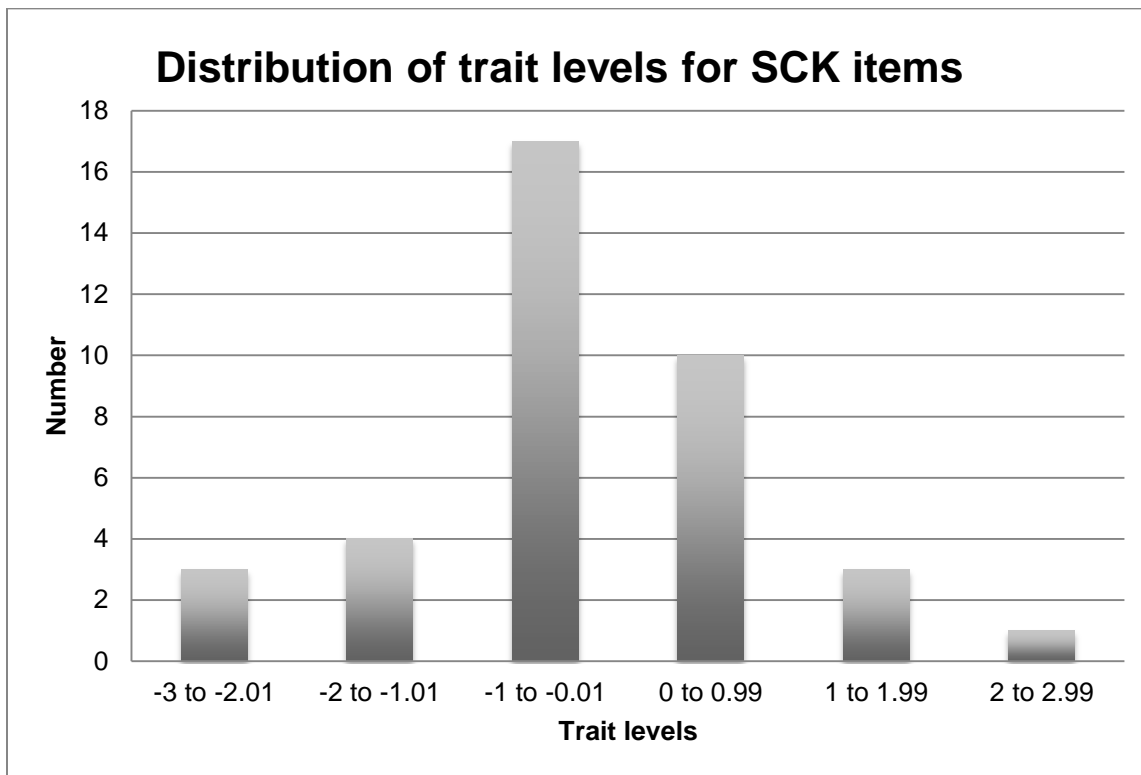


Figure 4. Distribution of trait levels on specialized content knowledge items.

There was a significant difference in test scores on the specialized content knowledge items between the mathematics majors and the other majors, as can be seen in tables 5.9 and 5.10.

Table 5.9

*Comparison of mathematics majors and elementary education majors on specialized content knowledge items*

	Mean	SD	t	df	p	d
SCK items			5.38	25	< 0.001	2.26
Mathematics Majors	1.35	0.56				
Elementary Majors	-0.29	0.78				

Table 5.9 shows that mathematics majors were significantly different from elementary education majors in their trait levels on specialized content knowledge items ( $p < 0.001$ ). Inspection of the means of the two groups indicated that the average score for mathematics majors ( $M = 1.35$ ) was significantly higher than the score for elementary education majors ( $M = -0.29$ ), and the effect size,  $d$ , was very large at 2.26 (Morgan et al, 2011)

Table 5.10

*Comparison of mathematics majors and majors in other subjects than mathematics and elementary education on specialized content knowledge items*

	Mean	SD	t	df	p	$d$
SCK items			6.05	17	< 0.001	2.82
Mathematics Majors	1.35	0.56				
Other Majors	-0.48	0.71				

Table 5.10 shows that mathematics majors were also significantly different from prospective teachers majoring in other subjects than mathematics and elementary education in their mean trait level on specialized content knowledge items ( $p < 0.001$ ). Examination of the groups' means implied that the average score for mathematics majors ( $M = 1.35$ ) was significantly higher than the score for the other majors ( $M = -0.48$ ), and the effect size was very large at 2.82 (Morgan et al., 2011). No significant statistical difference was detected between the elementary education majors and prospective teachers majoring in other subjects than mathematics.

Research question I b dealt with the specialized mathematical content knowledge of participants. Results from the questionnaire survey indicated that the following topics, listed in table 5.11, were rather difficult for participants.

*Table 5.11**Difficult specialized content knowledge topics*

Topic	Item difficulty
Alternative method to divide fractions <sup>7</sup>	1.89
Explanation for equivalent fractions	1.67
Division rules	1.67
Visual model for multiplication	1.17
Alternative subtraction method	1.03

The topics described in table 5.12 seemed to be fairly easy for participants, their item difficulty ranging from -1.10 to -0.43.

*Table 5.12**Easy specialized content knowledge topics*

Topic	Item difficulty
Evaluating different expressions for area	-1.10
Decomposing number into ten and ones	-0.90
Describing a situation with an equation <sup>8</sup>	-0.90
Finding a story to fit a model of a whole number divided by proper fraction	-0.60
Evaluating partial division method	-0.43

A common denominator for difficult items within each knowledge domain, common content knowledge and specialized content knowledge, was fractions.

<sup>7</sup> 5 participants skipped that item.

<sup>8</sup> 8 participants chose the “I don’t know” answer possibility for this item.

*The interviews.* The participants' ability to make word problems, or story problems, to go with the problems they solved and their explanations of mathematical phenomena were used to examine interviewees' specialized mathematical knowledge.

*Subtraction,  $74 - 26$ .* Half of the interviewees said that 6 could not be subtracted from 4; three of them had much mathematical preparation. When asked to further elaborate on that statement, two of the interviewees with much mathematical preparation, gave explanations without referring to mathematics, for example "you have to borrow from the neighbor next door".

Six of the interviewees used "standard algorithm with borrowing" to solve the problem  $74 - 26$ . Of the five interviewees with much mathematical preparation, four chose to solve the problem using "standard algorithm". Three of the four participants that initially used an alternative method to solve the problem, said they would use the "standard algorithm" when teaching others.

All of the interviewees could think of more than one method to solve the subtraction problem, but two of them ran into trouble while trying to use their alternative algorithm to solve the problem. Four of the interviewees mentioned manipulatives or some kind of visual representation of the problem to facilitate students' understanding. Abacus, blocks and the number line were mentioned as such.

*Multiplication,  $79 \times 48$ .* Eight of the participants began using "standard algorithm" to solve the problem. Seven of the participants gave examples of another possible solution method, but two of them ran into trouble while trying to apply that method. One interviewee was able to connect the problem  $79 \times 48$  to binomial multiplication.

Three interviewees had a hard time finding a story problem to go with the multiplication problem. Two of them found a story in the end, but one could not. Apart from one area model story, all of the first stories produced by each interviewee represented repeated addition model. One interviewee came up with a multiplication story regarding area when first asked to find a story problem, but when prompted, five other interviewees came up with an area story.

*Division, 1035/5.* Everyone could come up with a story/stories for the division problem. All of the interviewees made up stories about equal sharing and seven of the stories had to do with dividing 1035 kronas equally between five people. One interviewee came up with a division problem representing repeated subtraction model. This same interviewee came up with a story problem regarding area, where division could be used to find a missing side of a rectangle given the area and the other side. When asked about geometry/area problem in connection with the division problem, three interviewees made up story problems where the area was known and it was supposed to be divided in to five parts. The remaining six interviewees could not think of a story problem.

All of the interviewees mentioned the “standard algorithm” using the division bracket as a way of solving the problem. Half of them mentioned more than one way of solving the problem and usually mentioned partial division as an alternative to the “standard algorithm”. Only one interviewee mentioned a visual representation of the problem. Two of the interviewees used mathematics to explain why they began working from left when solving the division problem using the division bracket. One interviewee began from the right when dividing using the division bracket. The other interviewees could not explain the reasoning behind the algorithm, or tried to explain it without

referring to mathematics. Examples of explanations for why you start from the left when using the division bracket are shown in table 5.13.

Table 5.13

*Explaining division algorithm, interview excerpts*

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<p>Why do you start from the left when doing division with the bracket, but from the right when you do addition, subtraction and multiplication?</p>	<p>“I guess it is because they are opposites. To multiply is the opposite to dividing and that’s why you start at the different place”. [Claudia]</p> <p>“It might not work to explain it like this, but I see fractions and move over the line then multiplication becomes division, or the effect reverses. I can see this working for multiplication where we go this way but the other way in division. I can’t see this reasoning hold for addition and subtraction though”. [Fiona]</p> <p>“Because I’m dividing. Different from the other problems, you run the risk of the number not being divisible by the number you are dividing by. To find out if that is the case, we don’t have to break down some number, so you start at the reverse (wrong) end to fool the problem... So you won’t get into trouble. If you begin at the front, you either end up with a remainder or you don’t. If you start from the end you can’t find it out”. [Elyse]</p> <p>“It is just supposed to be that way. It is surely a little difficult to understand unless your told that, that is just the way it is.” [Debra]</p> <p>“Because that’s the way I learned it [laugh], you probably couldn’t do it the other way.”[Heather]</p>
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One of the interviewees could explain why dividing by zero was undefined and used the definition of division to do so. Two of them gave wrong mathematical explanations. One interviewee said that when dividing by zero the answer would be zero, and another explained that when dividing by zero nothing happened to the other number (it remained the same), which explained why you did not divide by zero. One of the

interviewees gave no explanation and the remaining five explained that dividing by zero was not possible, by some version of “you cannot split between no one”. Table 5.14 gives examples of answers to why division by zero is undefined.

Table 5.14

*Why can't you divide by zero?, interview excerpts*

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Now a student asks you:	“Because zero isn't anything, and you cant divide anything into more parts”. [Allison]
“ <i>Why can't you divide by zero?</i> ”	“Because zero doesn't represent anything, unless it is between other numbers”. [Beatrix]
How do you answer?	<p>“I would answer it like: If you have three pizzas and are going to divide them by nothing, then it is nothing when there is no one to divide it by. You need some presumptions. If I'm alone with the pizzas then the pizzas would be divided by one, but if I'm not there the pizzas aren't either”. [Elyse]</p> <p>“Because a number multiplied with zero is always zero.” [Fiona]</p> <p>“Nothing is happening with zero, you aren't about to divide anything, so nothing changes. So dividing by zero really just means I'm not going to do anything to the number, I don't divide it”. [Debra]</p> <p>“Because you can't let something become nothing. You can't divide something you have into nothing, when you're dividing by zero. You can't divide by zero because zero isn't anything. You can't divide some part or quantity or something like that into nothing”. [Heather]</p> <p>“If I'm dividing something into 0 parts, I only get 0 parts” [Ida]</p>

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*Division of fractions,  $2\frac{1}{4} \div \frac{1}{2}$ .* Three of the interviewees came up with a proper story problem for the division of fractions problem. One came up with a story problem fitting  $2\frac{1}{4} \div 2$ , and one tried to find a story but gave up. The rest of the interviewees did not try to find a story problem. Five of the participants could solve the division of

fractions problem. Two of them could explain both how and why they solved the problem the way they did, while the others referred to their own studies as a reason for their way of solution. Table 5.15 shows examples of how interviewees explained the reason behind the algorithm used to solve division of fractions.

Table 5.15

*Why do you “flip the second fraction and multiply?”, interview excerpts*

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Why do you “ <i>flip the second fraction and multiply</i> ”, and not the first one?	<p>“I can’t really change the former one because that is the number I want the solution for, I’m not allowed to change the size. To find the solution I can move over instead of dividing, so I can use multiplication. And to be able to switch between the operations [division and multiplication], I can do it like that, by switching the numbers and then the sign changes to multiplication.”[Debra]</p> <p>“Because the second part is what we are dividing by and the first part is what we have. If we had a pizza and were dividing by 2, we have to reverse ourselves not the pizza slices. We can move ourselves, but not the pizzas.” [Elyse]</p> <p>“Because that is the way I learned it. If I turned the other one around I would get a totally different result [solves the problem <math>\frac{4}{9} \times \frac{1}{2}</math>, to make sure]. I need to think about this, I need to go home and think about how to explain this, you need to know those things.”[Ida]</p>
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### Interesting mistakes

When explaining a mathematical method, more than half of the participants referred to the algorithm for explanation, and accepted another example of how the method worked as evidence of understanding the method. Almost half of the participants did not accept “adding on to the subtrahend until you reach the minuend”<sup>9</sup> as a valid

<sup>9</sup> For example:  $72 - 28$ :  $28 + 2 = 30$ ,  $30 + 40 = 70$ ,  $70 + 2 = 72$ , so  $72 - 28 = 2 + 40 + 2 = 44$ .



subtraction method, and about a third of them did not accept “equally adding on”<sup>10</sup> as a valid subtraction method. Of the participants, almost three quarters accepted that it was always appropriate to add a zero at the end of a number when multiplying it by 10, and half of them agreed with the statement that dividing a number always made it smaller. Almost one fifth of the participants consented that a larger number could never be subtracted from a smaller one.

## Research question II

Does age and/or teaching experience affect the level of knowledge reported in question I? As previously stated, the student body at the School of Education, University of Iceland is a diverse group with regards to age, experience and mathematical preparation before entering the teacher education program.

### Age

The average age of participants in the study was 32 years ( $SD = 8.00$ ), with both median and mode at 30. The range in age was 31 years. The age distribution was moderately skewed to the right (skewness= 0.88).

To investigate if there was a statistically significant relationship between age and test scores, a correlation was computed. Because the age distribution was skewed, the Spearman’s rho statistic was calculated for the whole questionnaire as well as for the two knowledge domains. The only meaningful correlation was found to be between age and test scores from the SCK part of the test,  $r(36) = 0.26$  which is considered to be a low correlation (Morgan et al., 2011). The direction of the correlation was positive, suggesting that older students had higher test scores and vice versa. The  $R^2$  (0.068)

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<sup>10</sup> For example:  $32 - 17 = 35 - 20 = 15$ .

indicated that approximately 7% of the variance in test scores on the SCK part of the survey could be predicted from age. The correlation between age and test scores from the whole survey, and between age and scores from the CKK part of the survey was too low to be meaningful.

### **Mathematics teaching experience**

Five of the participants in the study had experience teaching mathematics, the experience ranging from one to twenty years. Simple regression was conducted to investigate how well participants' mathematics teaching experience predicted their standardized scores on the whole questionnaire survey, as well as for each part of the knowledge domains, CCK and SCK. The results were statistically significant for the whole questionnaire survey and the SCK part of the survey. Teaching experience significantly predicted scores for the whole survey,  $\beta=0.09$ ,  $t(36) = 2.35$ ,  $p < 0.05$ . The  $R^2$  value was 0.13,  $F(1,36) = 5.55$ , indicating that 13% of the variance in test scores was explained by teaching experience. The results for the SCK part survey were  $\beta = 0.11$ ,  $t(36) = 2.85$ ,  $p < 0.01$ . The  $R^2$  value was 0.18,  $F(1,36) = 8.15$ . The  $R^2$  value indicated that 18% of the variance in test scores was explained by teaching experience.

The combination of age and teaching experience significantly predicted test scores for the SCK part of the questionnaire survey,  $\beta=0.01$  for age and  $\beta=0.10$  for experience,  $t(35) = 0.71$ (age) and 2.24 (experience),  $p < 0.05$ . The  $R^2$  value was 0.20,  $F(2,35) = 4.27$ , indicating that 20% of the variance in test scores was explained by the combination of age and experience.

Statistically significant correlation was found between age and experience,  $r(36) = 0.435$ ,  $p = 0.006$ . The correlation was positive, indicating that people with experience tended to be older and vice versa.

### **The interviews**

The average age of interviewees was 36 years ( $SD = 7.9$ ), and the range was 27 years. Two of the interviewees had teaching experience in mathematics, one 5 years and the other 13 years. Age did not appear to influence how interviewees approached the problems or their explanations. The interviewees with teaching experience also had a strong background in mathematics. They more frequently referred to mathematical properties in their explanations, and appeared more comfortable coming up with story problems. The interviewees with experience and strong background in mathematics were the ones that could right away figure out a story for the division of fractions problem.

### **Research question III**

Item difficulty is related to the trait level of test takers (Furr & Bacharach, 2008), resulting in different item difficulty for the same item when solved by different groups. The item difficulty was calculated for all the items in the current study. The item difficulty of most of the items translated and adapted for the Icelandic questionnaire survey had been reported (Learning Mathematics for Teaching, 2011B; 2011C). Of the 51 items on the questionnaire, 42 had reported item difficulty. For some items, item difficulty was reported from more than one group. In such cases the average item difficulty was calculated in order to compare item difficulty based on US elementary teachers to the item difficulty obtained in the current study.

T-tests were conducted to compare average item difficulty for the both groups, the Icelandic one and the US one, for the whole questionnaire survey and each knowledge domain. Table 5.16 shows the results.

Table 5.16

*Comparison of item difficulty*

	Mean	SD	T	df	p	<i>d</i>
The whole survey			-3.60	82	< 0.001	-0.787
US	-0.67	0.885				
Iceland	0.05	0.944				
SCK items			-3.16	34	0.003	-1.06
US	-0.49	0.94				
Iceland	0.42	0.79				
CCK items			-2.19	46	0.034	-0.63
US	-0.80	0.84				
Iceland	-0.23	0.96				

For the whole questionnaire survey, there was a statistically significant difference between the average item difficulty in the US and Iceland,  $t(82) = -3.60$ ,  $p = 0.05$ ,  $d = -0.787$ . Average item difficulty was lower in the US (-0.67) than in Iceland (0.05), and the effect size was large (Morgan et al., 2011).

For the specialized content knowledge items in the questionnaire survey, there was a statistically significant difference between item difficulty in the US and Iceland,  $t(34) = -3.16$ ,  $p = .05$ ,  $d = -1.06$ . Average item difficulty was lower in the US (-0.49) than in Iceland (0.42), and the effect size was very large (Morgan et al., 2011).

For the common content knowledge items in the questionnaire survey, there was a statistically significant difference between item difficulty in the US and Iceland,  $t(46) = -2.19$ ,  $p = .05$ ,  $d = -0.63$ . Average items difficulty was lower in the US (-0.80) than in Iceland (-0.23), and the effect size was typical (Morgan et al., 2011).

To examine if there was a statistically significant relationship between calculated item difficulty in Iceland and reported item difficulty in the US, a correlation was computed. Correlation was computed for the whole questionnaire survey as well as for each knowledge domain. For the whole questionnaire survey, correlation was  $r(40) = 0.69$ ,  $p < 0.001$  which is a strong, positive linear relationship (Healey, 2009).

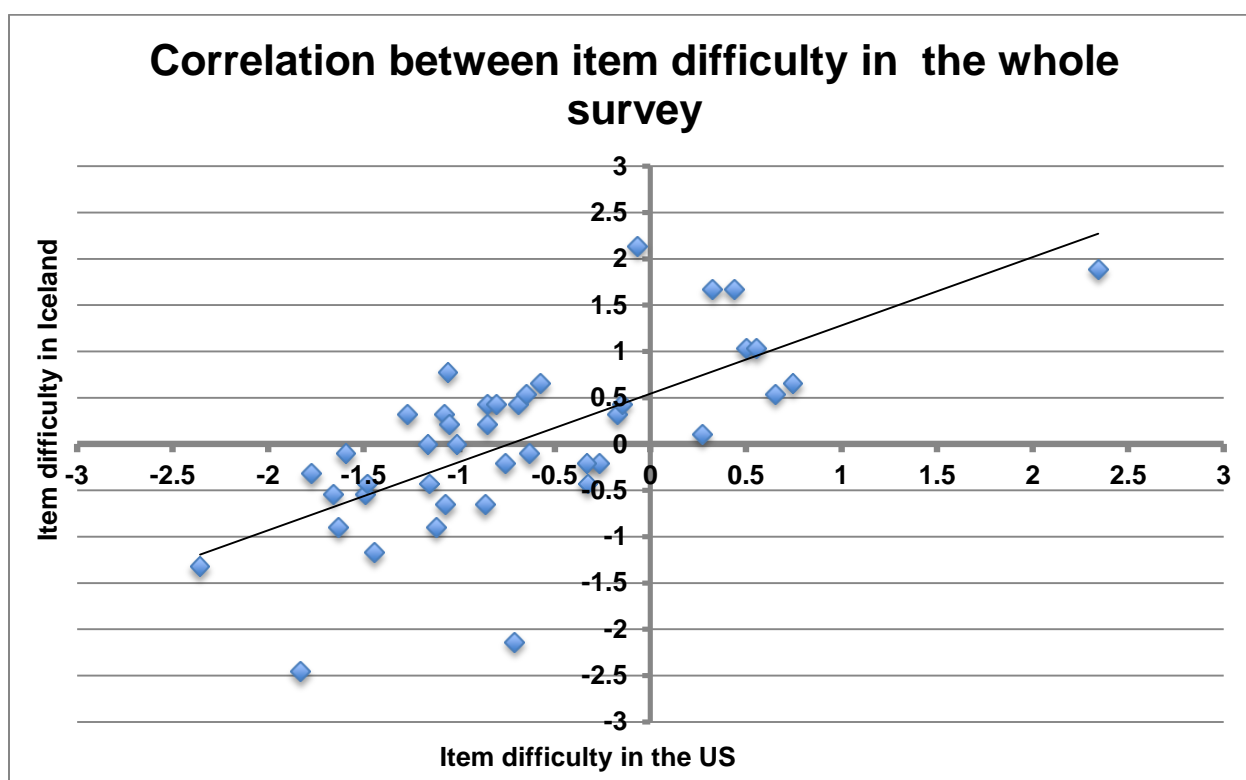


Figure 5. Correlation between item difficulty in Iceland and the US for the whole questionnaire survey.

In figure 5, item difficulty is plotted. In the first quadrant of the graph items are plotted that appeared to be difficult for both groups, the Icelandic one and the US one. In the second quadrant are items difficult for the Icelandic participants but easy for the US teachers. The third quadrant shows items easy for both groups, and the fourth, the empty one, is where items easy for the Icelandic group and difficult for the US group should be.

Points plotted below the X-axis represent items easy for the Icelandic participants, and points above the X-axis represent items difficult for them. Similarly, points plotted to the left of the Y-axis represent items easy for the US participants, while items to the right of the Y-axis represent items difficult for them.

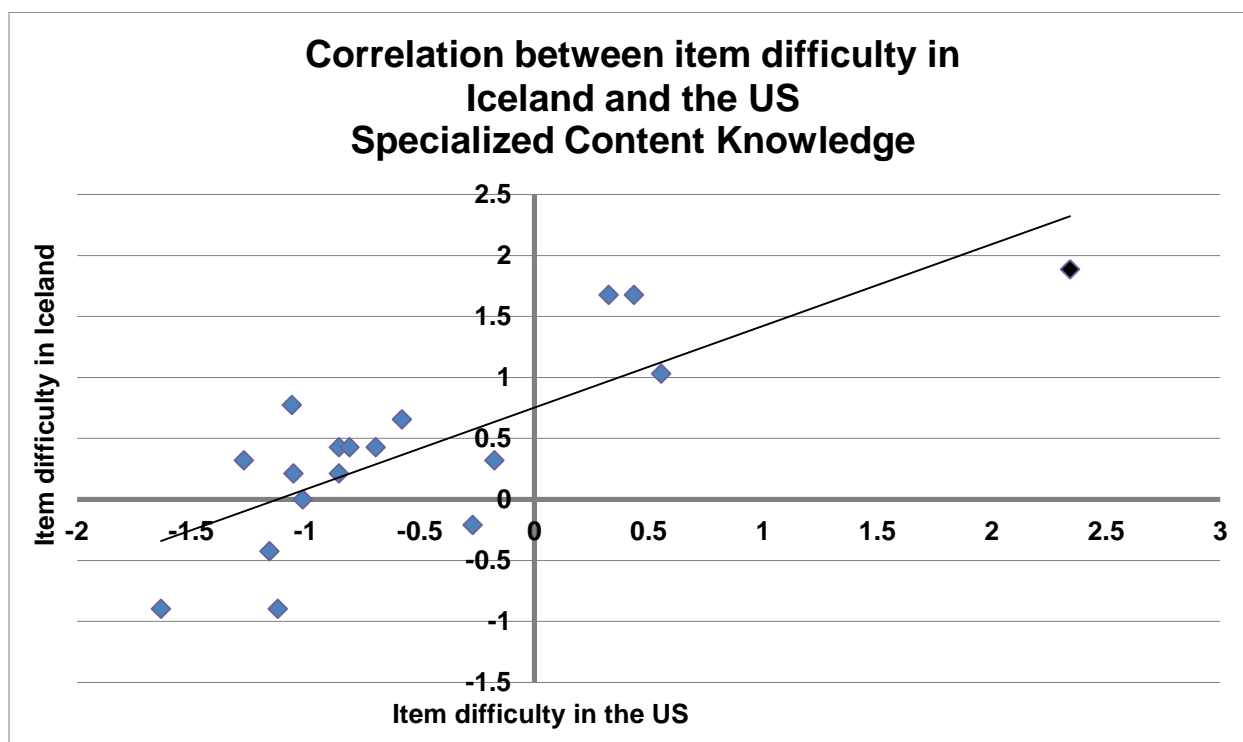


Figure 6. Correlation between item difficulty in Iceland and the US for SCK items.

The calculated correlation between item difficulty of specialized content knowledge items in Iceland and the US was very strong and positive (Healey, 2009),  $r(16) = 0.79$ ,  $p < 0.001$ , and is plotted in figure 6. The black point (2.343, 1.887) represents an item very difficult for both groups, evaluating an alternative method to divide fractions. Other topics appearing difficult for both groups included alternative methods for subtraction and explaining division rules. Topics relatively easy for both groups were for example, breaking numbers into tens and ones and evaluating alternative methods for division.

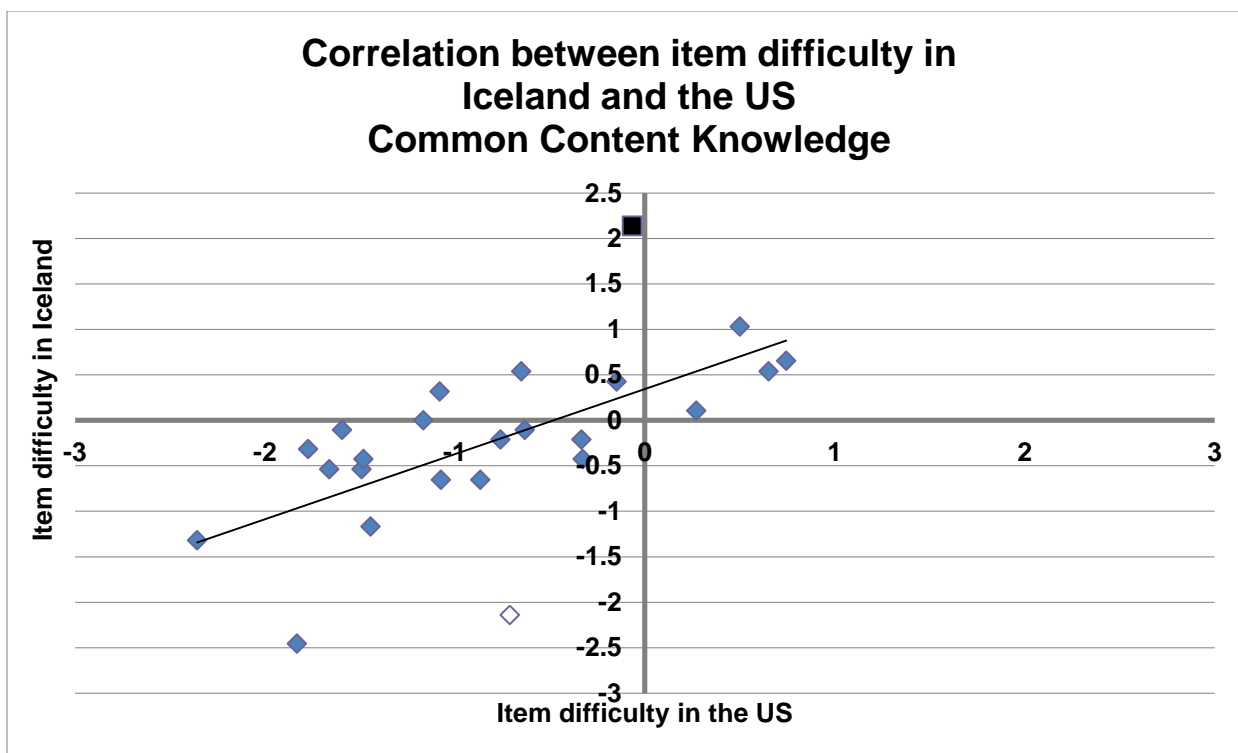


Figure 7. Correlation between item difficulty in Iceland and the US for CCK items.

The calculated correlation between item difficulty of common content knowledge items in Iceland and the US was strong and positive (Healey, 2009),  $r(22) = 0.62$ ,  $p < 0.001$ , and is plotted in figure 7.

The black rectangular point in figure 7 (-0.067, 2.14) represents an item seemingly easy for the US group but difficult for the Icelandic group. This item dealt with surjective functions. The white point (-0.711, -2.14) represents an item appearing easy for both groups, but that had lower item difficulty for the Icelandic group than the US group. This item dealt with recognizing a percentage of an area. Other topics relatively easy for both groups included dividing a whole number by a proper fraction, general rules for addition, subtraction, multiplication and division, perimeter, and working with patterns. Topics appearing difficult for both groups were for example properties of positive and negative numbers.

Some items had a difference greater than 0.5 in item difficulty, between the two groups. Multiplication of fractions and equivalent fractions seemed difficult for the Icelandic group (item difficulty 0.54 and 1.67) but easy for the US group (item difficulty -0.94 and -0.291), while evaluating an alternative method for dividing fractions was more difficult for the US group (2.34) than the Icelandic one (1.89). Evaluating a fact regarding a rectangle was rather difficult for the Icelandic group (0.32) but easy for the US one (-1.08). Modeling an expression with algebra tiles was easy for the US group, item difficulty of the leaves ranging from -1.060 to -0.578 while it was difficult for the Icelandic participants, most likely a result of Icelanders' unfamiliarity with this sort of representation. There was a great difference in item difficulty of items dealing with functions. Bijective functions were easy for both groups, but easier for the US group. Difficulty of those items ranged from -0.539 to -0.318 for the Icelandic group, but from -1.770 to -1.480 for the US group. When it came to surjective functions the difference was even more visible, that item was very difficult for the Icelandic group, item difficulty 2.140 while item difficulty for the US group was -0.067.

Johnson (2011) investigated the development of mathematical knowledge for teaching among 35 senior elementary education majors. She was interested in the growth of knowledge during a methods course in mathematics education. She focused on numbers and operations (NOP) and used patterns, functions and algebra (PFA) as an internal control, since that topic was not covered in the methods course. Johnson (2011) used complete MKT scales for both topics and reported descriptive statistics in terms of standard deviation, and based on the normalized scores from the original MKT scales (Johnson, 2011). Since the current study used translated and adapted items from different



MKT scales, direct comparison of statistics was not possible. Johnson's (2011) participants were prospective elementary teachers, so scores from prospective elementary teachers in the current study were used to compare results. Results from both studies indicated that there was a great difference in test scores for both topics, patterns, functions and algebra and numbers and operations. In patterns, functions and algebra, the Icelandic prospective elementary teachers had a less difference, 2.5 standard deviation ( $SD = 0.692$ ) separating the highest scoring participant and the lowest scoring participant. The least difference between participants in the Johnson study, in patterns, functions and algebra, was 3.724 ( $SD = 0.844$ ). The greater difference along with higher standard deviations indicated more variety in mathematics ability among participants in her study.

In numbers and operations, there was less difference between the groups. The difference between the highest and lowest scoring participant in the current study was 3.062 ( $SD = 0.879$ ) and 3.455 ( $SD = 0.795$ ) in the Johnson study.

## VI Summary, Recommendations and Conclusions

### Summary & Discussion

The sample in this study consisted of 38 prospective teachers in the School of Education at the University of Iceland, both graduate and undergraduate students. All of the participants in the study completed a questionnaire survey and were invited to take part in an interview. Ten of the prospective teachers accepted the invitation and were interviewed. Nine of the interviewees were graduate students.

The choice of ways to measure prospective teachers mathematical knowledge was grounded in the work of Ball and the research team at the University of Michigan, and their theory of the division of content knowledge into common content knowledge and specialized content knowledge.

The questionnaire survey used in this study comprised of translated and adapted MKT items from the topics numbers and operations, and patterns, functions and algebra. Experts in the field of elementary education and mathematics helped validate the clarity and comprehensiveness of items during the translation and adaptation process. Interviews were used to further explore the prospective teachers' way of thinking, reasoning and justifications for methods and rules in mathematics. The Icelandic national curriculum in mathematics was the main reference for choice of items on the questionnaire survey, as well as for problems and questions in the interviews.

Results both from the questionnaire survey and the interviews indicated that the prospective teachers' knowledge was procedural and related to the "standard algorithms" they had learned in elementary school. Prospective teachers appeared to have difficulties evaluating alternative solution methods, and dealing with fractions. During the

interviews, the most common answer for why a certain way was chosen to solve a problem or a certain step was taken in the solution process, was “Because that is the way I learned to do it.”

## **Discussion**

### **Research question I**

Research question I dealt with the elementary mathematical knowledge of prospective teachers. It came as no surprise that prospective teachers majoring in mathematics scored significantly higher on the questionnaire survey than did prospective teachers majoring in other subjects. This statistical difference was detected both for the common content knowledge part and the specialized content knowledge part of the survey.

Three problems stood out as very difficult for the participants in the questionnaire survey. The first dealt with an alternative subtraction method. In order to have a 50% chance of solving that correctly, teachers' trait level had to be more than one standard deviation above the average. The second item dealt with an alternative method of dividing fractions and the third with an explanation for a division rule. To have a 50% chance of answering those items correctly, teacher's trait level had to be nearly 2 standard deviations above the average. The high item difficulty calculated for these items could not be explained by participants' unfamiliarity with the representation of the problems.

The difficulty with fractions was in line with results from the interviews. Four of the interviewees could not solve the division of fractions problem, and were confused about the use of the common denominator, language (numerator, denominator) and the

relationship between fractions and decimals. Of the six interviewees able to solve the problem, three could not come up with a story in connection to the problem. They tried to find a story where pizza was supposed to be divided between people, but could not fit  $\frac{1}{2}$  into the story, “It is difficult to find a story when you can’t visualize the problem” [Elyse]. These results indicated that these prospective teachers were confusing division by fractions with whole-number division and did not have a deep understanding of what a fraction was (Cramer & Whiney 2010).

Three interviewees were able to come up with a story to accompany the division of fractions problem and did so without much effort. These three were Gina, Heather and Johnna, all majors in mathematics in the School of Education. Gina explained how she had just been working with the division of fractions in a course at the School of Education, and student teaching the subject. When asked why she flipped the second fraction and multiplied, and not the former one, she did not know. Johnna explained the flipping of the fraction with the multiplicative inverse, while Heather did not take the second fraction and flip, but put  $\frac{9}{4}$  as a numerator and  $\frac{1}{2}$  as a denominator and solved it by multiplying both the numerator and denominator by the multiplicative inverse of  $\frac{1}{2}$ .

Debra, Elyse and Ida could solve the problem by flipping the second fraction and multiply. When asked why they did so, Ida said this was the way she learned how to do it. Debra referred to a rule she remembered and explained: “I can’t really change the former one because that is the number I want the solution for, I’m not allowed to change the size. To find the solution I can move over instead of dividing, so I can use multiplication. And to be able to switch between the operations [division and

multiplication], I can do it like that, by switching the numbers and then the sign changes to multiplication.”

Elyse’s story represented  $2\frac{1}{4} \div 2$ , dividing pizzas between two people. When asked to solve the problem, she did so correctly and got  $4\frac{1}{2}$ , but did not connect that answer to the story she came up with few minutes earlier. Elyse did not refer to mathematical rules or properties when asked to explain why she solved the problem by flipping the second fraction and multiply. She said: “Because the second part is what we are dividing by and the first part is what we have. If we had a pizza and were dividing by 2, we have to reverse ourselves not the pizza slices. We can move ourselves, but not the pizzas.”

The findings regarding the prospective teachers’ approach to fractions are in line with Ball’s (1990) findings. Her results indicated that the prospective teachers had difficulties explaining the meaning of division by fractions. They were able to do the calculations, but not to come up with a mathematically appropriate story of the division.

Operations with fractions are not a part of the elementary school curriculum. Still, items in the questionnaire survey and one of the problems in the interviews dealt with fractions. One of the things worth pondering over is whether teachers should or should not be able to answer students’ mathematics questions not covered by the curriculum. Mathematics educators have seemed to agree on that “teachers must know in detail and from a more advanced perspective the mathematical content they are responsible for teaching . . . both prior to and beyond the level they are assigned to teach” (National Mathematics Advisory Panel, 2008, p. 37).

In discussion on the four operations in the Icelandic national curriculum in

mathematics (2007, P. 18) it says: “When practicing these mathematical operations emphasis has to be placed on the children’s understanding of the methods, among other things, by letting them develop their own solution methods.” Research has indicated that students who invent their own strategies have a better understanding of the base ten number system as well as better abilities to transfer their knowledge to new situations (National Council of Teachers of Mathematics, 2000)

Results from the questionnaire survey indicated that evaluating alternative solution methods was difficult for participants in this study. These results were also visible in the interviews. For all of problems posed in the interview, interviewees were asked to think of another method of solving the problem. With the exception of subtractions, most of the interviewees could not think of another way of solving the problems apart from the “standard algorithm”.

Heather and Elyse thought  $70 \times 40 + 9 \times 8$  was a correct way of solving  $79 \times 48$ , but realized when they saw that the product did not match their previous answer, that it did not work. Two interviewees suggested  $7 \times 4 + 9 \times 8$  as a way to solve the problem, but realized soon that they did something wrong. One was able to correct the mistake, the other, looking at  $28 + 72$ , decided that a zero was missing at the end of 28, changed it to 280 and got 352 as a final result. Elyse suggested  $(70+40)(9+8)$  as a way of using binomial multiplication to solve the multiplication problem. Johnna was the only one able to correctly connect the problem to binomial multiplication, and suggested  $(80-1)(50-2)$ . Ida mentioned the “Chinese” and the “Russian” method of multiplying as an alternative to the “standard algorithm”.

Those of the interviewees that used alternative approach to solve a problem turned to a traditional approach, or the “standard algorithm” when asked how they would teach others to solve it, even after saying that the alternative approach fostered understanding.

Results from the interviews indicated that the relationship between multiplication and division was unclear for the prospective teachers. Six of them came up with a story problem regarding area for multiplication, where two sides of a rectangle were known and the area needed to be found. Only one interviewee was able to come up with a story problem for area and division, where the area and one side of a rectangle were known and the other side needed to be found. The other interviewees were not able to make the connection between the division problem and the previous multiplication area problem.

Zero is a special number and children often wonder about zero (Ball & Bass, 2000). Two interviewees could adequately explain why division with zero was undefined. One used the definition of division to do so and the other simply said: “Because a number multiplied with zero is always zero”.

Three of the explanations given were mathematically wrong. One of the interviewee was confused about dividing by and dividing into and said: “Because zero isn’t anything, and you can’t divide anything into more parts”. She explained that you could not divide zero by a number, which you surely can. Another explained it like she was dividing by one: “Nothing is happening with zero, you aren’t about to divide anything, so nothing changes. So dividing by zero really just means I’m not going to do anything to the number, I don’t divide it.” And the third explained it as if dividing by zero resulted in zero ( $\frac{x}{0} = 0$ ): “If I’m dividing something into zero parts I only get zero

parts. If I'm dividing something into two parts I need to divide in two. If I'm dividing something between nobody I don't get anything from it."

**Mathematical language.** The importance of mathematical language is stressed in the National curriculum in mathematics (2007, p. 6), "the precise use of language and symbols and the ability to communicate both verbally and in writing supports deeper understanding of mathematical concepts and procedures." In the interview some confusion in mathematical language was detected. One interviewee always mentioned the subtrahend before the minuend when subtracting. The problem  $74 - 26$  became, "twenty six minus seventy four". When describing that there were not enough units to subtract six from four the interviewee used the Icelandic term "gengur ekki upp í" which translates to six does not go into four. When dividing  $1035 \div 5$  one interviewee was describing the method she used (which was the division bracket): "It's the, what do we call it, the bowl? I make a bowl and put five in the *numerator's* place and 1035 as the *denominator*." Another interviewee confused dividing by and dividing into.

The findings from this study regarding prospective teachers' mathematical content knowledge are in line with prior research that has indicated that people can perform mathematical calculations without the understanding of the underlying principle (Ball, 1990). And, even though prospective teachers can perform mathematical operations, they do not have specific apprehension of concepts and principles (Ball, 1990; Tirosh & Graeber, 1989, 1990).

### **Research question II**

Research question II called for an examination of the relationship between participants' age and teaching experience and their performance in the study. There was a



great age difference between participants in the study, 31 years. Results from the study indicated that age did not affect over all test scores from the questionnaire survey, and had a low positive correlation with specialized content knowledge. As expected, age and experience were positively correlated indicating that older participants were more likely to have experience teaching mathematics. Only five participants in the study had experience teaching mathematics, making it difficult to draw conclusive assumptions on the effect of teaching experience on prospective teachers' mathematical content knowledge.

Examination of the interview data did not indicate difference in interviewees approach to problems, mathematical language or explanations based on their age.

Further analysis of data from the study indicated that number of mathematics courses completed in high school/college had the strongest influence on the participants' scores on the questionnaire survey. Mathematics courses completed in high school/college significantly predicted test scores,  $\beta = 0.22$ ,  $t(36) = 3.67$ ,  $p < 0.001$ . The  $R^2$  value was 0.27,  $F(1,36) = 13.44$ ,  $p < 0.001$ . The  $R^2$  value indicated that 27% of the variance in test scores was explained by mathematics courses completed prior to entering the teacher education program. According to Morgan et al., (2011) this was a large effect.

These findings stress the importance of offering mathematics courses to those students entering the teacher education program without sufficient background in mathematics. For those prospective teachers entering the teacher education program without sufficient knowledge in mathematics a remedial mathematics course should be mandatory. Another idea is to administer an entrance exam in mathematics and offer a remedial mathematics course to those scoring below an acceptable grade.

### **Research question III**

Research question III asked for a comparison of results from the current study and similar studies carried out in the US using the MKT measures to measure teachers' knowledge. Research of the literature indicated that not much had been published in terms of research findings reporting results of such studies.

Comparison of item difficulty in the current study and item difficulty reported for the original MKT items revealed a strong positive linear relationship, indicating that similar items were difficult and easy for participants in both countries. Item difficulty from both sources indicated that fractions, division and the evaluation of certain alternative solution methods were difficult topics for participants.

Comparing item difficulty based on Icelandic prospective teachers to item difficulty based on working elementary teachers in the US might not favor the Icelandic group, since most of the participants did not have experience in teaching mathematics. It was surprising to see that the correlation between item difficulty was stronger for specialized content knowledge items ( $r(16) = 0.79, p < 0.001.$ ), than it was for common content knowledge items ( $r(22) = 0.62, p < 0.001.$ ). This was even more surprising since some of the specialized content knowledge items seemed to be unfamiliar to the participants in the current study, particularly items regarding modeling an expression with algebra tiles and a visual model for multiplication. Many participants skipped those items, 9 and 14 respectively.

As nations, Iceland and the US do not seem to have much in common. Iceland is a small nation with a homogeneous population. The US on the other hand has 1000 times the number of people Iceland has and much variety within its population. Still these nations both struggle with similar problems in the mathematics education of teachers.

## **Recommendations**

This study had some limitations that should be noted. The questionnaire survey used in this study was designed to measure the mathematical knowledge of teachers and prospective teachers in the US. Even though measures were taken to ensure the validity of the translation and adaptation of the items used in this study, there may still be areas where culture differences could have skewed results.

The questionnaire survey used items from two mathematical topics, numbers and operations and patterns, functions, and algebra. In order to adequately measure both numbers and operations, and patterns, functions and algebra, the number of test items was large, so testing fatigue might have influenced results. In further research, measuring one topic at a time could prevent this influence.

The participants in this study were prospective teachers so the results from the study cannot be generalized to practicing teachers. The sample in this study was a small convenience sample, and participation in the study was voluntary. Because the sample consisted of volunteers it might represent prospective teachers with more mathematical content knowledge than those who chose not to take part in the study. For further studies it is recommended to strive for a more representative sample.

## **Suggestions for practice**

When looking at the findings from this study in light of the Icelandic national curriculum in mathematics and the new curriculum draft, it seems like prospective teachers do not have the mathematical knowledge needed to prepare students according to the curriculum. For example, alternative solution methods were a topic appearing to be

difficult for the prospective teachers, both in the questionnaire survey and in the interviews. In the curriculum the following goal regarding alternative solution methods can be found: At the end of fourth grade, students should:

- Be able to use a variety of methods to add and subtract natural numbers (at least for numbers 1 – 1000), both mentally and on paper.

And these are goals from the new curriculum draft:

- Students should take part in creating suitable methods, built on student's own understanding, to add, subtract, multiply and divide.
- Students should be able to solve equations using non-standard methods and rationalize their solutions, e.g. by using manipulatives.

The findings from the study indicate that prospective teachers need to 're-invent' their methods and approaches to solving problems, and they need to separate the concept of the operation from a certain algorithm. They need to broaden their vision of the operations, for example, not only see multiplication as a repeated addition, but also on the number line and as area and array models, and Cartesian products in order to introduce these different images of multiplication to their students. They also need to understand the connections between the operations.

These findings point to the need of offering prospective teachers a course in the foundations of mathematics. A course where the foundations of algorithms, rules and theorems are investigated, as well as ample of opportunities is given to the prospective teachers to discover and come up with their own methods of solving problems.

### **Suggestions for future research**

The purpose of this study was to investigate the mathematical content knowledge of prospective teachers in Iceland. Future research regarding prospective teachers' common content knowledge and specialized content knowledge individually as well as research regarding the relationship between the two would provide additional information to those interested in mathematics education. In addition to questionnaires and interviews, observing prospective teachers during practice teaching could be beneficial to see the implementation of the mathematical content knowledge that prospective teachers possess.

Comparative research between countries, for example the Nordic countries would be interesting as well as taking into account comparison of the mathematics education of prospective teachers in different countries. Longitudinal studies, following prospective teachers through their first years of teaching and examining growth of specialized content knowledge could inform the scholarly community about the nature and development of that knowledge.

The relationship between teachers' mathematical knowledge and student achievement could inspire research on the relationship between different mathematical knowledge domains and student achievement.

### **Conclusions**

The findings from this study imply, in line with prior research, that prospective elementary teachers rely on memory for particular rules in mathematics, their knowledge is procedural and they do not have an underlying understanding of mathematical concepts

or procedures (Ball, 1990; Tirosh & Graeber, 1989; Tirosh & Graeber, 1990; Simon, 1993; Mewborn, 2003; Hill et al., 2007).

Examining prospective elementary teachers' mathematical knowledge provides information regarding their mathematics education (Simon, 1993). These prospective teachers developed their mathematical understanding, procedures and approach to mathematics, throughout their mathematics education prior to entering the School of Education. When asked why she put a zero there before multiplying with 4 ( $79 \times 48$ ) Beatrix said: "Because I was told that each time you go down one line you are suppose to add one zero, if you go down two lines you put two zeros." This explanation given by Beatrix was in line with many of the explanations given by prospective teachers during the interviews, indicating that during their mathematics education emphasis was on procedures rather than conceptual understanding.

In elementary school students form their mathematical self-concept and their beliefs about mathematics, what is mathematics and what it means to do mathematics (National Council of Teachers of Mathematics, 2000). Research has indicated that students believe that doing mathematics means following rules set by the teachers and knowing mathematics means remembering and using the correct rule when answering the teacher (Conference Board of the Mathematical Sciences, 2012). Unless teachers' beliefs differ from those, it is going to be a continuing cycle for the next generation. "Because that is the way I learned it", and "that is the way you are supposed to do it" are not sufficient explanations for students according to the Icelandic curriculum in mathematics. It stresses the importance of multiple solution methods, students' reasoning and the

connection between mathematics and students' daily lives. It also stresses the importance of understanding and skills going hand in hand in mathematics education.

Research has indicated that prospective teachers form definite conceptions of the nature of teaching prior to entering the teaching profession, and that unless teacher education can change these conceptions, teachers will teach their prospective students in a similar to the way they were taught (Borko & Brown, 1992). Research has also shown that when prospective teachers are given the opportunity to reconstruct what they know with more depth and meaning it positively affects their mathematical knowledge (Ponte & Chapman, 2008).

Studies have shown the importance and effect mathematics education courses can have for prospective teachers. These courses can cause growth in prospective teachers' interest in mathematics and are positively linked to their mathematical understanding (Macnab & Payne, 2003; Ponte & Chapman, 2008).

Mathematical knowledge is important in teaching. Its importance touches upon almost every aspect of the mathematics lesson: the use of teaching material, presentation of material, and assessment (Ball, Hill, & Bass, 2005; Baumert, et al., 2010). Therefore it is important that prospective teachers enter the teaching profession equipped with deep conceptual understanding of mathematics

The findings of this study highlight the need for a more in-depth mathematics education for all prospective teachers in the School of Education at the University of Iceland. It is not enough to offer a variety of courses for those specializing in the field of mathematics education. It is of utmost importance to also offer in-depth mathematics education for those prospective teachers focusing on general education. If those

prospective teachers teach mathematics, they will do so in elementary school where students are forming their identity as mathematic students.

Let's give Debra the final words. This is what she said when I was explaining to her my study and concerns for the mathematical education of elementary teachers: "That is exactly what I need, advanced mathematics, because I really enjoy it, but it annoys me so much that I have forgotten a lot, you know, of rules. I have to be able to explain what I am doing and why."



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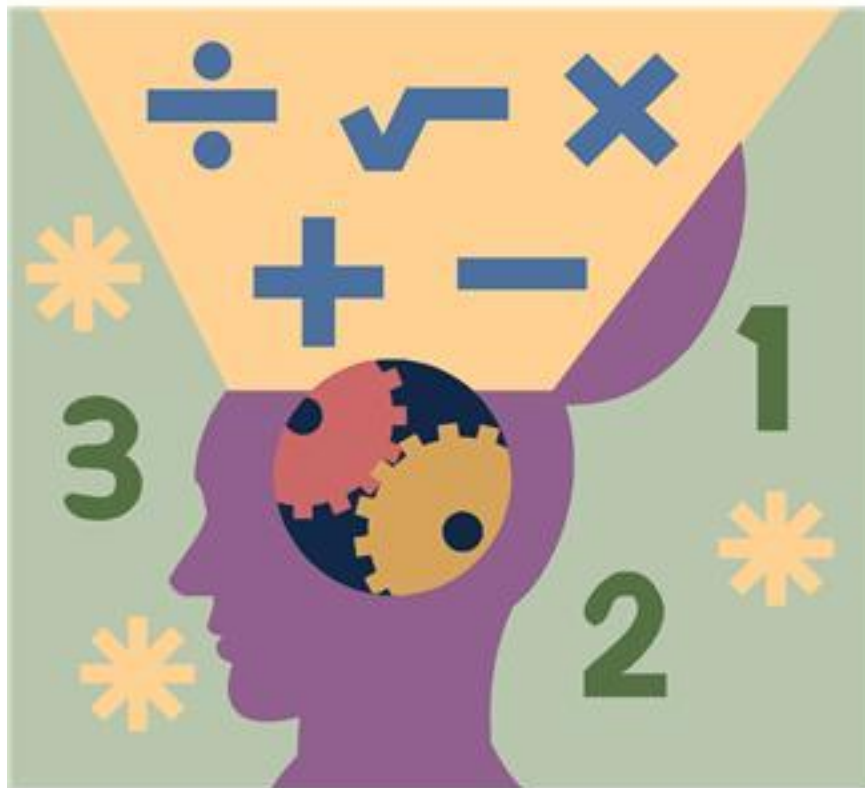


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**NOTE: THE ACTUAL MKT ITEMS ARE NOT DISPLAYED.** The 132 following items are sample items released to demonstrate the kind of items used in the instrument.

**Appendix A: The Questionnaire Survey**

## The Solution Process Prospective Teachers Apply in Mathematics



Copyright © 2006 The Regents of the University of Michigan. For information, questions, or permission requests please contact Merrie Blunk, Learning Mathematics for Teaching, 734-615-7632. Not for reproduction or use without written consent of LMT. Measures development supported by NSF grants REC-9979873, REC-0207649, EHR-0233456 & EHR 0335411, and by a subcontract to CPRE on Department of Education (DOE), Office of Educational Research and Improvement (OERI) award #R308A960003.



Learning Mathematics for Teaching  
University of Michigan  
School of Education  
610 E. University #1600  
Ann Arbor, MI 48109-1259

**NOTE: THE ACTUAL MKT ITEMS ARE NOT DISPLAYED.** The 133 following items are sample items released to demonstrate the kind of items used in the instrument.

1. Ms. Dominguez was working with a new textbook and she noticed that it gave more attention to the number 0 than her old book. She came across a page that asked students to determine if a few statements about 0 were true or false. Intrigued, she showed them to her sister who is also a teacher, and asked her what she thought.

Which statement(s) should the sisters select as being true? (Mark YES, NO, or I'M NOT SURE for each item below.)

	Yes	No	I'm not sure
a) 0 is an even number.	1	2	3
b) 0 is not really a number. It is a placeholder in writing big numbers.	1	2	3
c) The number 8 can be written as 008.	1	2	3

2. Ms. Chambreaux's students are working on the following problem:

*Is 371 a prime number?*

As she walks around the room looking at their papers, she sees many different ways to solve this problem. Which solution method is correct? (Mark ONE answer.)

- a) Check to see whether 371 is divisible by 2, 3, 4, 5, 6, 7, 8, or 9.
- b) Break 371 into 3 and 71; they are both prime, so 371 must also be prime.
- c) Check to see whether 371 is divisible by any prime number less than 20.
- d) Break 371 into 37 and 1; they are both prime, so 371 must also be prime.

**NOTE: THE ACTUAL MKT ITEMS ARE NOT DISPLAYED.** The 134 following items are sample items released to demonstrate the kind of items used in the instrument.

3. Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
$\begin{array}{r} 35 \\ \times 25 \\ \hline 125 \\ +75 \phantom{0} \\ \hline 875 \end{array}$	$\begin{array}{r} 35 \\ \times 25 \\ \hline 175 \\ +700 \phantom{0} \\ \hline 875 \end{array}$	$\begin{array}{r} 35 \\ \times 25 \\ \hline 25 \\ 150 \phantom{0} \\ 100 \phantom{00} \\ +600 \phantom{00} \\ \hline 875 \end{array}$

Which of these students would you judge to be using a method that could be used to multiply any two whole numbers?

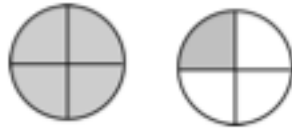
	Method would work for all whole numbers	Method would NOT work for all whole numbers	I'm not sure
a) Method A	1	2	3
b) Method B	1	2	3
c) Method C	1	2	3

4. Ms. Harris was working with her class on divisibility rules. She told her class that a number is divisible by 4 if and only if the last two digits of the number are divisible by 4. One of her students asked her why the rule for 4 worked. She asked the other students if they could come up with a reason, and several possible reasons were proposed. Which of the following statements comes closest to explaining the reason for the divisibility rule for 4? (Mark ONE answer.)

- a) Four is an even number, and odd numbers are not divisible by even numbers.
- b) The number 100 is divisible by 4 (and also 1000, 10,000, etc.).
- c) Every other even number is divisible by 4, for example, 24 and 28 but not 26.
- d) It only works when the sum of the last two digits is an even number.

**NOTE: THE ACTUAL MKT ITEMS ARE NOT DISPLAYED.** The 135 following items are sample items released to demonstrate the kind of items used in the instrument.

5. Mrs. Johnson thinks it is important to vary the whole when she teaches fractions. For example, she might use five dollars to be the whole, or ten students, or a single rectangle. On one particular day, she uses as the whole a picture of two pizzas. What fraction of the two pizzas is she illustrating below? (Mark ONE answer.)

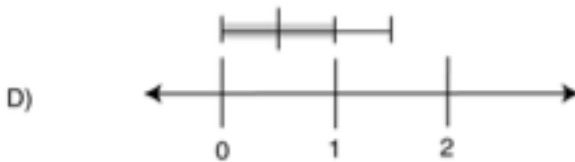
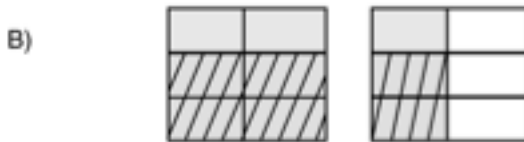


- a)  $5/4$
- b)  $5/3$
- c)  $5/8$
- d)  $1/4$

**NOTE: THE ACTUAL MKT ITEMS ARE NOT DISPLAYED.** The 136 following items are sample items released to demonstrate the kind of items used in the instrument.

6. At a professional development workshop, teachers were learning about different ways to represent multiplication of fractions problems. The leader also helped them to become aware of examples that do not represent multiplication of fractions appropriately.

Which model below cannot be used to show that  $1\frac{1}{2} \times \frac{2}{3} = 1$ ? (Mark ONE answer.)



**NOTE: THE ACTUAL MKT ITEMS ARE NOT DISPLAYED.** The 137 following items are sample items released to demonstrate the kind of items used in the instrument.

7. Which of the following story problems could be used to illustrate  $1\frac{1}{4}$  divided by  $\frac{1}{2}$ ? (Mark YES, NO, or I'M NOT SURE for each possibility.)

	Yes	No	I'm not sure
a) You want to split $1\frac{1}{4}$ pies evenly between two families. How much should each family get?	1	2	3
b) You have \$1.25 and may soon double your money. How much money would you end up with?	1	2	3
c) You are making some homemade taffy and the recipe calls for $1\frac{1}{4}$ cups of butter. How many sticks of butter (each stick = $\frac{1}{2}$ cup) will you need?	1	2	3

**Background Information**

1. Gender:            Male                      Female
  
2. What year were you born?
  
3. What high school/college did you attend?
  
4. What year did you graduate high school/college?
  
5. What was your major in high school/college?
  
6. How many mathematics courses did you complete in high school/college?
  
7. What kind of mathematics courses were they?



8. What is your route at the School of Education?

9. What is/are your major(s)?

10. How many courses have you completed in the following during your studies at the School of Education?

	No courses	One to two courses	Three to five courses	Six or more courses
Mathematics	1	2	3	4
Mathematics Education	1	2	3	4

11. Please indicate the courses you have completed or are currently enrolled in.

	Algebra and Functions
	Numbers, Logic and Arithmetic
	Teaching Mathematics to Teenagers
	Geometry
	Variety in Mathematics Education
	Number Theory and Algebra
	Icelandic and Mathematics in Compulsory Education
	Calculus
	Mathematics Education in Elementary School
	Development of Language and the Education of Young Children
	The Environment as a Foundation of Education
	Teaching Reading and Mathematics
	At the Crossings of Pre School and Elementary School

**Sometimes prospective teachers have experience in teaching.**

12. Did you have experience in teaching before you enrolled in the School of Education?

**If yes:**

Did you teach mathematics?

**If yes:**

i) For how many years did you teach mathematics?

ii) Please indicate which class(es) you taught mathematics:

\_\_\_ 1. Grade

\_\_\_ 5. Grade

\_\_\_ 2. Grade

\_\_\_ 6. Grade

\_\_\_ 3. Grade

\_\_\_ 7. Grade

\_\_\_ 4. Grade

\_\_\_ 8. – 9. Grade

\_\_\_ 10. Grade

13. To what extent do you agree with the following statements?

(1 is strongly disagree and 5 is strongly agree)

	Strongly Disagree			Strongly Agree	
a) In general, I think I know the mathematics I will teach.	1	2	3	4	5
b) I have good knowledge of numbers and operations.	1	2	3	4	5
c) I have good knowlege of <i>all domains</i> in mathematics.	1	2	3	4	5
d) My knowledge of numbers and operations is sufficient to teach the subject.	1	2	3	4	5
e) I consider myself a good mathematics teacher.	1	2	3	4	5

THANK YOU FOR COMPLETING THE QUESTIONNAIARE.

If you have any comments regarding these questions,  
please write them below.

## Appendix B: Recruitment letter

### E-MAIL INVITATION FOR PARTICIPATION IN RESEARCH STUDY

Dear \_\_\_\_\_

I am Björg Jóhannsdóttir, a doctoral student in the Department of Mathematics, Science and Technology at Teachers College, Columbia University. I would like to invite you to participate in my research study "*How prospective teachers handle mathematics and their solution process*". The research study is a part of my doctoral thesis. You are invited to take part in the research study because you have chosen Teaching young children as your line of study.

As a participant in the research study, you will be asked to answer a questionnaire. The anticipated time to answer the questionnaire is about one hour.

The risk of taking part in this research study is similar to the risk of attending a college class or focus groups. You will neither be compensated directly nor indirectly for your participation in the research study.

Information that can be traced to you is confidential. Your name will never appear in any writings or discussions regarding this research study. Once the research study is complete all data will be deleted.

By participating in this research you will be contributing to the understanding of how prospective teachers approach mathematics.

If you would like to participate in this research study, please be in room \_\_\_\_\_, at The School of Education, September, \_\_\_\_ at 3 o'clock. If you have questions, please contact me at -----, or phone number: -----, or you may contact my advisor, Dr.Walker, at -----.

Thank you for your consideration,  
Björg Jóhannsdóttir

## Appendix C: Informed Consent

Teachers College, Columbia University  
 525 West 120<sup>th</sup> Street  
 New York NY 10027  
 212 678 3000  
[www.tc.edu](http://www.tc.edu)

### INFORMED CONSENT

Researcher:  
 Björg Jóhannsdóttir

Faculty Sponsor: Dr. Erika Walker

I, Björg Jóhannsdóttir, invite you to participate in the research study “*How prospective teachers handle mathematics and their solution process*”. The research study is part of my doctoral thesis at Teachers College, Columbia University.

Before you decide whether or not to participate I kindly ask you to read the following information. Don't be afraid to ask if anything is unclear by sending me an email. If you decide to participate, which I hope you do, you can quit whenever you so wish without any repercussions. Should you decide to do so; all your data will be deleted. **By signing this document you give your consent for participation.**

The goal of this research study is to examine how prospective teachers approach mathematical problems and their ways of solving them. This research study utilizes two kinds of data collection, questionnaires and interviews. If you decide to take part you will be asked to answer a questionnaire. You might be asked to take part in an individual interview, but only 10 participants will be asked to do so. The anticipated time to answer the questionnaire is about one hour. If you're asked to take part in an interview it is anticipated to last an hour.

Only Björg Jóhannsdóttir and her faculty sponsor will use the results from this research study in her doctoral thesis and writings of scientific nature.

The risk of taking part in this research study is similar to the risk of attending a college class or focus groups. To talk about and explain one's thinking process can cause discomfort. If you feel any discomfort you can at any point refuse to answer or terminate your participation. You are not likely to directly benefit from your participation, but you will be contributing to the understanding of how prospective teachers approach mathematics and therefore to the improvement of the subject in Iceland. You will neither be compensated directly nor indirectly for your participation in the research study.

Information that can be traced to you is confidential. To ensure confidentiality your answers will be coded with a special number and only I, Björg Jóhannsdóttir, will have access to the file that connects that number to your name. Once participants for the interview have been selected that file will be deleted. Your name will never appear in any writings or discussions regarding this research study. Once the research study is complete all data will be deleted.

As a participant in this research study you can deny to answer any question, whether it is in the questionnaire or the interview. As earlier stated, you can end the interview or your participation at any given point.

Thank you for your assistance.

---

Participant

---

Date

---

Researcher



**Appendix D: Interview Guide**

The elementary mathematical content knowledge of prospective teachers in Iceland

Interview Guide

Björg Jóhannsdóttir

**CASE NO.** \_\_\_\_\_

**INTERVIEWEE NAME:** \_\_\_\_\_

**DATE OF INTERVIEW:** \_\_\_\_\_ **TIME OF INTERVIEW:** \_\_\_\_\_

**LOCATION OF INTERVIEW:** \_\_\_\_\_

## Notes for Identification Issues - Introductory Script

### *Project Overview and Informed Consent*

Project overview and Consent Form

Greetings and Self Introduction...

Hello, I want to thank you for taking time to be a part of my study.

As a volunteer in this study, I understand you might have questions about the process of the interview and/or the questions I might ask. Please feel free to raise any questions you may have; and I will, to my best ability, answer them.

Are there any other questions you have about the overall purpose or process of the study?

As I have stated previously, everything that is discussed during the interview will remain confidential. Your name will never be attached to your interview; instead a pseudonym will be assigned to your interview. During the interview, if there is anything you want to be left out, kept off the record, please inform me during the process or after we finish the interview.

I would also like to take this opportunity to remind you that your participation is voluntary. If, at any time, you do not want to answer a question please inform me and we will skip that particular question.

With your permission, I would like to audio record our interview. By giving your consent, I will be able to document the information you provide accurately and quickly. If you do not consent, I will take notes throughout the conversation. Is it ok if I record our conversation?

Yes \_\_\_\_\_ No \_\_\_\_\_

**YES:** If, at any time, you would like me to turn off the recording device, please just give me the word and I will press the button here (point to button) and it will stop recording our conversation.

**No:** To make sure I record accurate information, I will be taking notes throughout our conversation.

The form that I am about to present you discusses the confidentiality measures we are using in this study. Before we proceed with the interview, I will need you to read and sign this form. As you read through the form, please ask any questions that you may have. If you do not agree with what you read, you do not have to sign, but we will not be able to continue on to the interview portion.

TO DO:

- 1.) Discuss questions and/or comments
- 2.) Request signature on consent form
- 3.) Provide interviewee with consent form
- 4.) Record answer of taping preference.

Do you have any other questions or comments before we start the interview?

## **Background**

Let's start by getting some background information on you.

1. You have already told me about your enrollment status at The School of Education, but I want to confirm a few things.

- a. What department are you enrolled in?
- b. What program are you currently registered under?

2. What year were you born?
3. What high school/college did you attend?
4. What year did you graduate from college?
5. What was your major/minor in college?

**Now I would like to ask you about your studies of mathematics prior to entering The School of Education**

6. How many mathematics courses did you complete in high school/college?
7. What was the nature of these mathematics courses?
  - a. Can you describe the topics of the courses you completed in high school/college?
  - b. What mathematics course or topic did you find most interesting?

**[If a specific course/topic]**

Why did you find that course/topic interesting

**Sometimes students in The School of Education have had experience in teaching prior to entering their program.**

8. Did you have experience in teaching prior to entering your program?

**[If yes...]**

Did you teach any mathematics?

**[If yes...]**

What grade level did you teach mathematics?

### **Math Problems**

**Now I would like for you to take a look at some mathematics problems with me. We will be looking at problems that have to do with mathematical operations and fractions. I would like for you to solve each problem, as well as to think of number stories that could go with each problem. While you are solving the problems, I'd like for you to "think aloud," that is, explain your moves and steps while solving the problem. Once you're done solving the problem I will ask you questions similar to those you might get from students.**

#### **Subtraction, whole numbers:**

We will begin by looking at a topic you will probably encounter in your teaching, subtraction. We have the problem  $74 - 26$ .

9. Could you please solve this problem for me, and remember to "think aloud" when doing so.
10. How do you approach this kind of problem when teaching it for the first time?
11. **[If uses the term "borrowing" ...]** I noticed that you used the term borrowing; can you elaborate on why you use that term?
12. **[If says you can't subtract 6 from 4...]** I noticed that you said that 6 couldn't be subtracted from 4, how do you see that in context with

students' continuing math learning when they encounter negative numbers?

13. Can you think of another way to approach this kind of problems?
14. **[If yes...]** Can you elaborate and show me?

### **Multiplication, whole numbers:**

Now we will look at a multiplication problem.

$$79 \times 48$$

15. Can you think of a number story that goes with that problem?
  - a) Can you think of another one?
  - b) **[If no story regarding area comes up...]** Can you think of a number story that connects to geometry?
    - i) **[If yes...]** can you please share that story?
    - ii) **[If no...]** Ok, let's continue.

Now I would like you to solve the problem.

16. I noticed that you used the \_\_\_\_\_ method for solving the problem. Can you tell me why you put a 0 there when multiplying by 4?
17. Do you know another method to solve the problem?
  - a) **[If yes...]** Can you show me how you would do that?
  - b) **[If no...]** Can you think of another way that might work?
    - i) **[If yes...]** Can you show me?

ii) **[If no...]** Ok, let's continue?

18. Can you think of a way to connect this multiplication problem to binomial multiplication (FOIL)?

**Division, whole numbers:**

**Now let's look at a division problem:**

$$1035/5$$

19. Can you think of a number story that goes with this problem?

a) Can you think of another one?

b) **[If no story regarding area comes up...]** Can you think of a number story that connects to geometry?

i) **[If yes...]** can you please share that story?

ii) **[If no...]** Ok, let's continue.

Now I would like you to solve the problem

20. Why do you choose this method?

21. How would you explain to your students the reason for starting from left, when you start from right when doing addition, subtraction and multiplication?

22. Do you know another method to solve the problem?

a) **[If yes...]** Can you show me how you would do that?

b) **[If no...]** Can you think of another way that might work?

i. **[If yes...]** Can you show me?

ii. **[If no...]** Ok, let's continue.

**Division of fractions:**

Now we are almost done. Last we will look at a problem having to do with division of fractions.

$$2\frac{1}{4} \div \frac{1}{2}$$

23. Can you think of a number story that goes with this problem?
24. Can you think of another one?

Now I would like you to solve the problem.

25. **[If solved correctly...]** How would you explain to your students why you flip the second fraction and not the first one?

**❖ Thank you very much for being part of this study!**

## Appendix E: Coding Sheet

	Activity	Code	Example(s) of	Surprise		
Subtraction – Multiplication – Division - Fractions	Solution Methods	Begins using “standard algorithm.”				
		Begins using alternative algorithm.				
		Gives examples of multiple algorithms.				
		Runs into trouble using/trying an alternative algorithm.				
		Uses a different algorithm to teach others than uses self.				
		Mentions manipulatives or visual representation.				
		Justification for choice of algorithm:	Refers to mathematics			
	Refers to own studies					
	Refers to students					
	Stories	Story represents the problem.				
		Gives more than one story.				
		Story represents:	Multiplication	Repeated addition model <sup>11</sup>		
				Number line model <sup>12</sup>		
Array model <sup>13</sup>						
Area model <sup>14</sup>						
Cartesian product model <sup>15</sup>						
Division	Set (partition) model (sharing) <sup>16</sup>					
	Repeated subtraction model (grouping) <sup>17</sup>					

<sup>11</sup> For example: One can of soda costs 55 cents. How much do 3 cans cost?

<sup>12</sup> For example: A leopard runs 20 miles pr. hour for 2 hours. How far has it gone?

<sup>13</sup> For example: A box of muffins has 3 rows of 4 muffins. How many muffins are in the box?

<sup>14</sup> For example: A sandbox is 3 ft. by 4 ft. What is the area of the sandbox?

<sup>15</sup> For example: Will has 4 pairs of shorts and 6 shirts. How many different outfits can he wear?

<sup>16</sup> For example: We have 6 sweets and 2 children. If we share the sweets equally, how many sweets does each child get?

<sup>17</sup> For example: We have 12 party treats each party bag needs 4 treats. How many bags can we make?



	<b>Mathematical Language</b>	Correctly uses mathematical language/concepts.			
		Incorrectly uses mathematical language/concepts.			
		Makes a false mathematical statement.			
		Does not know/cannot recall a proper concept.			
	<b>Explanations</b>	Refers to mathematical properties.			
		Explains with out referring to mathematics.			
		Refers to algorithm.			
		Cannot explain.			
		Justification of explanation:	Refers to mathematical properties.		
			Refers to students		
Refers to own studies.					
<b>Own Ability</b>	Refers positively to own abilities.				
	Refers negatively to own abilities.				
	Secure.				
	Insecure.				