DEL PEZZO SURFACES WITH IRREGULARITY AND INTERSECTION NUMBERS ON QUOTIENTS IN GEOMETRIC INVARIANT THEORY

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ABSTRACT

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This thesis comprises two parts covering distinct topics in algebraic geometry.

In Part I, we construct the first examples of regular del Pezzo surfaces for which the first cohomology group of the structure sheaf is nonzero. Such surfaces, which only exist over imperfect fields, arise as generic fibres of fibrations of singular del Pezzo surfaces in positive characteristic whose total spaces are smooth, and their study is motivated by the minimal model program. We also find a restriction on the integer pairs that are possible as the irregularity (that is, the dimension of the first cohomology group of the structure sheaf) and anti-canonical degree of regular del Pezzo surfaces with positive irregularity.

In Part II, we consider a connected reductive group acting linearly on a projective variety over an arbitrary field. We prove a formula that compares intersection numbers on the geometric invariant theory quotient of the variety by the reductive group with intersection numbers on the geometric invariant theory quotient of the variety by a maximal torus, in the case where all semi-stable points are properly stable. These latter intersection numbers involve the top equivariant Chern class of the maximal torus representation given by the quotient of the adjoint representation on the Lie algebra of the reductive group by that of the maximal torus.

We provide a purely algebraic proof of the formula when the root system decomposes into irreducible root systems of type A. We are able to remove this restriction on root systems by applying a related result of Shaun Martin from symplectic geometry.

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Notation

We collect here the notation used throughout each part of this thesis.

In Part I:

- All fields are assumed to be of characteristic $p \geq 2$.
- A *variety* over a field k is a finite-type, integral k-scheme.
- $k(X)$ denotes the function field of a k-variety X.
- $k_X := H^0(X, \mathcal{O}_X)$ denotes the field of global functions of a proper k-variety X.
- K_X denotes the canonical divisor associated to the dualizing sheaf ω_X of a Gorenstein variety X.
- $d = K_X^2$ denotes the anti-canonical degree of a del Pezzo surface X, computed as the self-intersection number over the field $k_X = H^0(X, \mathcal{O}_X)$.
- $h^i(X, \mathscr{F}) := \dim_{k_X} H^i(X, \mathscr{F})$ denotes the dimension over the field $k_X = H^0(X, \mathcal{O}_X)$ of the ith cohomology group of a sheaf $\mathscr{F}$ on a proper variety X.
- $q = h^1(X, \mathcal{O}_X)$ denotes the irregularity of a proper surface X.
- $\chi(\mathscr{F}) := \sum_i (-1)^i h^i(X, \mathscr{F})$ denotes the Euler characteristic of the coherent sheaf $\mathscr{F}$ on a proper variety X.
- $\mathbf{F}_X : X \to X$ denotes the absolute Frobenius morphism of a scheme X.
- $\mathbf{F}_{X/S}$ denotes the Frobenius morphism relative to a morphism of schemes $X \to S$.

- $\Omega_{Z/S}$ denotes the sheaf of relative Kähler differentials of an S-scheme Z.

- $T_{Z/S} := \mathscr{H}om(\Omega_{Z/S}, \mathcal{O}_Z)$ denotes the relative tangent bundle of an S-scheme Z.

In Part II:

- k denotes a field and $\bar{k}$ its algebraic closure.

- $e \in T \subseteq B \subseteq G$ denotes a smooth, reductive group G over k with a Borel subgroup B, a maximal torus T, and the identity element e.

- X denotes a variety over k with a *right* G-action and an ample G-linearized line bundle $\mathscr{L}$.

- $[X/G]$ denotes the stack-theoretic quotient of X by a right G-action.

- X^s_G, X^{ss}_G, X^{sss}_G and X^{un}_G denote the loci of stable, semi-stable, strictly semi-stable, and unstable points for the linearized action of G on X.

- $X /\!\!/ G$ the uniform categorical quotient of the semi-stable locus X^{ss}_G by right G-action.

- $N_T \subseteq G$ denotes the normalizer of T in G.

- $W := T \backslash N_T$ denotes the Weyl group of right cosets of T in its normalizer N_T.

- Φ denotes the root system corresponding to $G_{\bar{k}}$ and the maximal torus $T_{\bar{k}}$.

- $\Lambda^*(T)$ denotes the character group of T and $\Lambda_*(T)$ the group of 1-parameter subgroups.

- V denotes a finite dimensional G-representation over k.

- $\mathbb{P}(V)$ denotes the projective space of hyperplanes in V, so that $\Gamma(\mathbb{P}(V), \mathcal{O}(1)) = V$.

- $BT := [\operatorname{Spec} k/T]$ denotes the Artin stack that is the algebraic classifying space of T.

- $A_*(-)$ (resp. $A_*(-)_{\mathbb{Q}}$) denotes the Chow group with coefficients in $\mathbb{Z}$ (resp. $\mathbb{Q}$), graded by dimension.

- $A_*(-)^G$ denotes the G-equivariant Chow group.
- $A^*(-)$ (resp. $A^*(-)_{\mathbb{Q}}$) denotes the operational Chow ring with coefficients in $\mathbb{Z}$ (resp. $\mathbb{Q}$).
- $\mathfrak{g}$, $\mathfrak{b}$, and $\mathfrak{t}$ denote respectively the Lie algebras of G, B, and T.
- $\Delta := c_{\text{top}}(\mathfrak{g}/\mathfrak{b}) \in A^*(BT)$ and $c_{\text{top}}(\mathfrak{g}/\mathfrak{t}) \in A^*(BT)$ are the top Chern classes of the universal T-equivariant vector bundles induced by the adjoint representations.
- $Y \times_G Z$ denotes the uniform categorical quotient of the product $Y \times Z$ of a right G-variety Y with a left G-variety Z by the right G-action $(y, z) \cdot g := (y \cdot g, g^{-1} \cdot z)$.
- $h^g := g^{-1}hg$ denotes the right conjugation by $g \in G$ of an element $h \in G$.
- $\int_Y \sigma$ denotes the degree of a Chow class $\sigma \in A_0(Y)$ on a proper variety Y over k, computed via proper push-forward by the structure morphism.

Part I

Regular del Pezzo surfaces with irregularity

Chapter 1

Introduction

1.1 Regular varieties

Any variety defined over a finitely generated extension k of a perfect (e.g. algebraically closed) field $\mathbb{F}$ can be viewed as the generic fibre of a morphism of $\mathbb{F}$-varieties $\mathcal{X} \to \mathcal{B}$ such that k is the function field of the base $\mathcal{B}$. In this way, the geometry of varieties over imperfect fields is relevant to the understanding of the birational geometry of varieties over algebraically closed fields of positive characteristic. One main difficulty that arises is that, unlike over perfect fields, the notions of smoothness and regularity diverge: a smooth variety is necessarily regular, but a regular variety may not be smooth.

Definition 1.1.1. A variety X is defined to be *regular* provided that the local coordinate ring $\mathcal{O}_{X,x}$ is a regular local ring at all points $x \in X$. A k-variety X is *smooth* over k provided that it is geometrically regular (recalling that a k-variety X is said to satisfy a property *geometrically* if the base change $X_{\bar{k}}$ to the algebraic closure satisfies the given property) .

The notion of smoothness is well-behaved, due largely to the fact that a k-variety X is smooth if and only if the cotangent sheaf $\Omega_{X/k}$ is a vector bundle of rank equal to the dimension of X. Regularity, like smoothness, is a local property, and can be described in terms of the latter as follows: a k-variety X is regular if and only if there exists a smooth $\mathbb{F}$-variety $\mathcal{X}$ and a morphism $\mathcal{X} \to \mathcal{B}$ of which X is the generic fibre. In characteristic 0, a general fibre of a morphism between smooth varieties is smooth, yet in positive characteristic it is common for such morphisms to admit no smooth fibres. In fact, the collection of generic fibres of morphisms between smooth $\mathbb{F}$-varieties that admit no smooth fibres precisely comprises the regular, non-smooth varieties over finitely generated field extensions of $\mathbb{F}$. A standard example is the generic fibre of the family $(y^3 = x^2 + t) \subseteq \mathbb{A}^2 \times \mathbb{A}^1$ of cuspidal plane curves, parameterized by the affine coordinate t over a field of characteristic 2.

1.2 New results

Our study focuses on regular del Pezzo surfaces, a class of varieties that, as we discuss in Chapter 1.3, arises naturally in the context of the minimal model program.

Definition 1.2.1. A *del Pezzo scheme* over a field k is defined to be a 2-dimensional, projective, Gorenstein scheme X of finite-type over $k = H^0(X, \mathcal{O}_X)$ which is *Fano*, that is, for which the inverse of the dualizing sheaf, ω_X^{-1}, is an ample line bundle. A *del Pezzo surface* is a del Pezzo scheme that is an integral scheme.

This thesis answers affirmatively the question of whether there exist regular del Pezzo surfaces X that are geometrically non-normal or geometrically non-reduced by constructing examples of each type which have positive irregularity $h^1(X, \mathcal{O}_X) = 1$. We also find a characteristic-dependent restriction on the anti-canonical degree of regular del Pezzo surfaces that have a given positive irregularity $q := h^1(X, \mathcal{O}_X) > 0$. The main result (represented graphically in Figure 1.1) can be concisely summarized as follows:

Main Theorem.

1. *There exist regular del Pezzo surfaces, X_1 and X_2, with irregularity $h^1(X_i, \mathcal{O}_{X_i}) = 1$ and of degrees $K_{X_1}^2 = 1$ and $K_{X_2}^2 = 2$. The surface X_1 is geometrically integral and defined over the field $\mathbb{F}_2(\alpha_0, \alpha_1, \alpha_2)$ while X_2 is geometrically non-reduced and defined over the index-2 subfield $\mathbb{F}_2(\alpha_i\alpha_j : 0 \leq i, j \leq 3) \subseteq \mathbb{F}_2(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, for elements $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ algebraically independent over $\mathbb{F}_2$.*

2. *If X is a normal, local complete intersection (l.c.i.) del Pezzo surface (e.g. a regular del Pezzo surface) with irregularity $q > 0$ and anti-canonical degree $d = K_X^2$ over a field of characteristic p, then*
$$q \geq \frac{d(p^2-1)}{6}. \tag{1.2.2}$$

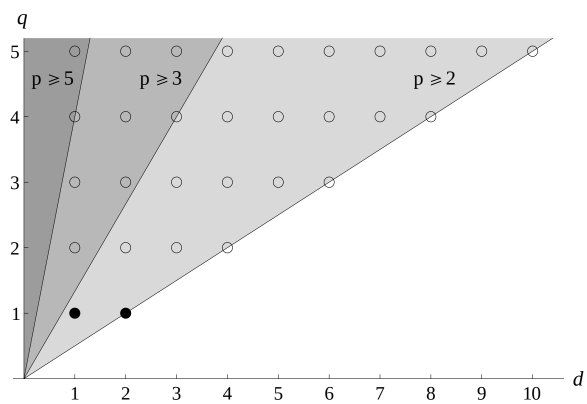

Figure 1.1: Circles represent possible values for the degree d and irregularity q of an l.c.i. and normal del Pezzo surface with positive irregularity $q > 0$. Solid dots represent actual values attained by the regular del Pezzo surfaces constructed in §4.2 and §4.3. Shaded regions demonstrate how the inequality (1.2.2) becomes more restrictive as the characteristic grows.

As our proof of (1) is constructive, it should be possible to obtain concrete descriptions of the geometry in each example. We do so for the degree one surface X_1, proving by explicit computation in Proposition 5.0.1 a detailed version of the following proposition.

Proposition. *There exists a regular form, Z, of a double plane in $\mathbb{P}^3$ and a finite, inseparable morphism $f : Z \to X_1$ of degree $p = 2$. Moreover, if $\bar{Z}$ and $\bar{X}_1$ denote the geometric base changes of Z and X_1, respectively, then this construction has the following properties:*

1. *The induced morphism $f^{red} : \mathbb{P}^2 \cong \bar{Z}^{red} \to \bar{X}_1$ is the normalization of $\bar{X}_1$.*

2. *The singular locus of $\bar{X}_1$ is a rational cuspidal curve C of arithmetic genus one.*

3. The inverse image of C under f^{red} is a non-reduced double line in $\mathbb{P}^2$.

1.3 Motivation from the minimal model program

Among the varieties over function fields, Fano varieties such as del Pezzo surfaces are of particular interest, due to their prominent role in the minimal model program. In brief, the goal of the program is to understand the birational geometry of a variety X by constructing a birational model $\hat{X}$ whose canonical divisor $K_{\hat{X}}$ is a nef divisor; one calls such a variety $\hat{X}$ a *minimal model* of X. If $\hat{X}$ is smooth, then the terminology is justified: $\hat{X}$ is minimal in the sense that any birational morphism $\hat{X} \to X'$ to a smooth variety X' is an isomorphism (cf. [6, Prop. 1.45]).

If X is not itself a minimal model, then there exist effective curves $C \subseteq X$ that pair negatively with the canonical divisor, $C.K_X < 0$. The strategy for constructing $\hat{X}$ is to attempt to contract precisely these negative curves via birational morphisms $f : X \to Y$ and then to partially resolve any serious singularities that were introduced. However, the contraction morphism associated to a certain negative curve may not be birational, and the contracted variety Y may be of lower dimension, as is the case, say, for ruled surfaces. Since the curves contained in fibres of f each pair negatively with K_X, the fibres of f are therefore Fano schemes by Kleiman's criterion for ampleness. In other words, the contraction morphism $f : X \to Y$ realizes X as a Fano fibre-space.

When X is a smooth 3-fold over an algebraically closed field, theorems of Mori [30] and Kollár [25] guarantee that any given extremal ray in the cone of effective curves pairing negatively with K_X can be contracted by a morphism $f : X \to Y$ to a normal variety Y. Furthermore, they classify these contraction morphisms: either f is birational, equal to the inverse of the blowing-up of a point or a smooth curve in Y, or $f : X \to Y$ is a Fano fibration over a smooth variety Y of dimension at most 2. If Y is a point, then X is itself a Fano 3-fold, while if Y is a surface, then X is a conic bundle over Y.

Our case of interest is when Y is a curve, as then $f : X \to Y$ is a del Pezzo fibration. Since X is smooth, the generic fibre of the fibration is a regular del Pezzo surface over the function field of Y. In characteristic 0, regular del Pezzo surfaces are smooth, and there are some results toward a birational classification of these del Pezzo fibrations (cf. [14] for a recent survey). In positive characteristic, however, the generic del Pezzo surface is potentially non-smooth, and the situation is not so clear. Indeed, Kollár asks whether these regular del Pezzo surfaces can be geometrically non-normal, or even geometrically non-reduced, but remarks that understanding this phenomenon seems complicated, especially in characteristic 2 (cf. [25, Rem. 1.2]).

1.4 Regular forms and the classification of del Pezzo surfaces

We can also contextualize our results in terms of the classification of del Pezzo surfaces over an algebraically closed field. In particular, we will see how our Main Theorem makes progress toward determining which singular (possibly non-normal or non-reduced) del Pezzo schemes over algebraically closed fields admit regular k-forms for some subfield k.

Definition 1.4.1. Let K/k be an extension of fields. Given a K-variety $\bar{X}$, one says that a k-variety X is a *(k-)form* of $\bar{X}$ provided that there exists an isomorphism $\bar{X} \cong X \times_k K$.

We recall the classification of del Pezzo surfaces X over an algebraically closed field. When X is normal, Hidaka and Watanabe [18] prove that either X is a rational surface with singularities at worst rational double points or X is a cone over an elliptic curve. Not all of these surfaces admit regular forms, as Hirokado [19] and Schröer [37] show how the existence of a regular form puts restrictions on the possible singularities.

In the course of proving the classification result, Hidaka and Watanabe [18] prove that all normal del Pezzo surfaces over an algebraically closed field satisfy $H^1(X, \mathcal{O}_X) = 0$. Over the complex numbers, this cohomological vanishing $H^1(X, \mathcal{O}_X) = 0$ can be viewed as a consequence of the Kodaira vanishing theorem for normal surfaces (cf. [31]), since $H^1(X, \mathcal{O}_X)$ is Serre-dual to the group $H^1(X, \omega_X)$ and ω_X is the inverse of an ample line bundle.

Reid [35] classifies the non-normal del Pezzo surfaces. He shows that such surfaces X are formed from rational, normal varieties X^ν by collapsing a (possibly non-smooth) conic to a rational curve C that is either smooth or has wildly cuspidal singularities (i.e. cuspidal singular points of order divisible by the prime characteristic $p > 0$). We remark that for these surfaces, the irregularity is equal to the arithmetic genus of the curve of singularities C, that is, $h^1(X, \mathcal{O}_X) = h^1(C, \mathcal{O}_C)$. In particular, when C is smooth, X is a non-normal del Pezzo surface with $H^1(X, \mathcal{O}_X) = 0$.

When C is wildly cuspidal, Reid shows that the normalization X^ν is the cone over a rational curve of degree $d \geq 1$ and the normalization morphism $\phi : X^\nu \to X$ is the contraction to C of the non-reduced double structure D on a ruling D^{red} in X^ν. Moreover, the restriction of ϕ to the ruling D^{red} gives a desingularization of C. This construction requires the cusps of C to be wild, because otherwise the resulting variety X is not Gorenstein (cf. [35, §4.4]). Such examples X are non-normal del Pezzo surfaces of anti-canonical degree $K_X^2 = d$ and irregularity $h^1(X, \mathcal{O}_X) = h^1(C, \mathcal{O}_C) > 0$. Reid constructs explicit surfaces X where the curve C has cusps of arbitrarily large order, showing that the irregularity of a non-normal del Pezzo surface may be arbitrarily large. Such surfaces are arguably the most pathological examples of del Pezzo surfaces.

In light of this classification, a scheme $\bar{X}$ admits a k-form that is a del Pezzo surface over k with irregularity $h^1(X, \mathcal{O}_X) > 0$ only if $\bar{X}$ is a non-normal del Pezzo surface or $\bar{X}$ is a non-reduced del Pezzo scheme. Main Theorem (1) asserts that regular forms can exist in either case, and Main Theorem (2) provides a numerical inequality that, in particular, rules out a large class of non-normal del Pezzo surfaces that could potentially admit regular forms.

1.5 A prior example

Acknowledging Reid's non-normal classification, Kollár remarks in [27, Rem. 5.7.1] on the possibility that regular del Pezzo surfaces could have positive irregularity. He ultimately leaves the issue unresolved, although his question is repeated later by Schröer in [36]. There Schröer constructs an interesting example of a normal del Pezzo surface Y in characteristic 2 that is a local complete intersection (l.c.i.) and regular away from one singular point y_∞, and has irregularity $h^1(Y, \mathcal{O}_Y) = 1$. This variety Y is a form of the example of Reid whose normalization morphism is described as the collapse of a non-reduced double line in $\mathbb{P}^2$ to a reduced cuspidal curve C with arithmetic genus 1. Schröer's method of construction is to begin with any imperfect field k of characteristic 2 along with a non-normal k-form X of the variety constructed by Reid. Schröer then studies actions of the infinitesimal group scheme α_2 on X. He uses one such action to twist the field of definition, thus obtaining the twisted form Y which he proves to be l.c.i. and normal. Schröer shows moreover that no α_2-twisting of the variety X can remove the singularity at y_∞, and hence his surface Y is an optimal one obtainable by this method.

1.6 A brief outline

The numerical bound in Main Theorem (2) is obtained in Chapter 2 by studying the inseparable degree p covers associated to Frobenius-killed classes in the first cohomology group of pluri-canonical line bundles on X. Such covers were studied by Ekedahl in [10] and shown to have peculiar properties, which we interpret to deduce the inequality (1.2.2). The notion of algebraic foliation on a regular (possibly non-smooth) variety is developed in Chapter 3, where we extend results of Ekedahl [9] from the smooth case. The surfaces X_1 and X_2 mentioned in Main Theorem (1) are exhibited as quotients by explicit algebraic foliations on a regular form of a non-reduced double plane in projective 3-space in Chapter 4. We conclude in Chapter 5 with a detailed study of the example X_1, a regular and geometrically integral del Pezzo surface with $h^1(X_1, \mathcal{O}_{X_1}) = 1$.

Chapter 2

Numerical bounds on del Pezzo surfaces with irregularity

The goal of this chapter is to find a restriction on the possible integer pairs (d, q) that exist as the degree $d = K_X^2$ and irregularity $q = h^1(X, \mathcal{O}_X)$ of a normal, l.c.i. del Pezzo surface X over a field k, under the assumption that $q \neq 0$. Our method is to study the torsors, for certain non-reduced group schemes $\alpha_{\mathscr{L}}$, associated to Frobenius-killed classes in the first cohomology group of pluri-canonical line bundles $\mathscr{L} := \omega_X^{\otimes m}$ on X. Originally studied by Ekedahl in [10] and [9], the existence of such torsors are often used as a technique to work around the failing of Kodaira vanishing in characteristic $p > 0$.

2.1 $\alpha_{\mathscr{L}}$-torsors

We briefly summarize here the basic properties of $\alpha_{\mathscr{L}}$-torsors, but we refer the reader to [10] or [26, §II.6.1] for more detailed accounts.

Let $\mathscr{L}$ be a line bundle on a variety X over a field k of characteristic p such that $H^1(X, \mathscr{L}) \neq 0$. We note that if $\mathscr{L}$ is the inverse of an ample line bundle, then this would be an example of the Kodaira non-vanishing phenomenon. Assume as well that pulling-back by the absolute Frobenius morphism $\mathbf{F}_X : X \to X$ does not yield an injective homomorphism from $H^1(X, \mathscr{L})$, that is, there exists a nonzero class $\bar{\xi} \in H^1(X, \mathscr{L})$ for which

$$\mathbf{F}_X^*(\bar{\xi}) = 0 \in H^1(X, \mathscr{L}^{\otimes p}).$$

The Frobenius pull-back defines a surjective homomorphism of group schemes over X from $\mathscr{L}$ to $\mathscr{L}^{\otimes p}$. Let $\alpha_{\mathscr{L}}$ be the group scheme defined as the kernel of this homomorphism, which by definition sits in the short exact sequence of group schemes,

$$0 \to \alpha_{\mathscr{L}} \to \mathscr{L} \xrightarrow{\mathbf{F}_X^*} \mathscr{L}^{\otimes p} \to 0. \tag{2.1.1}$$

Locally the group scheme $\alpha_{\mathscr{L}}$ is isomorphic to the constant non-reduced group scheme α_p, whose fibre over X is the kernel of the pth power endomorphism of the additive group $\mathbb{G}_a$.

The long exact sequence in cohomology associated to (2.1.1) shows that the class $\bar{\xi}$ comes from a nonzero class $\xi \in H^1(X, \alpha_{\mathscr{L}})$ that is determined up to an element of the cokernel of $\mathbf{F}_X^* : H^0(X, \mathscr{L}) \to H^0(X, \mathscr{L}^{\otimes p})$. Via Čech cohomology, one sees that ξ gives rise to a nontrivial $\alpha_{\mathscr{L}}$-torsor $f : Z \to X$. The morphism $f : Z \to X$ is purely inseparable of degree p because $\alpha_{\mathscr{L}}$ is a non-reduced finite group scheme of degree p over X.

To describe this $\alpha_{\mathscr{L}}$-torsor more explicitly, notice that a Frobenius-killed class $\bar{\xi} \in H^1(X, \mathscr{L})$ corresponds to a non-split extension of vector bundles,

$$0 \to \mathcal{O}_X \xrightarrow{i} \mathscr{E} \xrightarrow{\pi} \mathscr{L}^{-1} \to 0,$$

for which there is some splitting $\sigma : \mathscr{L}^{\otimes -p} \to \mathbf{F}_X^* \mathscr{E}$ of the morphism $F_X^* \pi$. We note that the choice of splitting is determined up to an element of $H^0(X, \mathscr{L}^{\otimes p})$. The affine algebra $f_* \mathcal{O}_Z$ is the quotient of the symmetric algebra $\mathrm{Sym}^*(\mathscr{E})$ by the ideal generated by $1 - i(1)$ as well as the image of σ in $\mathbf{F}_X^* \mathscr{E} \subseteq \mathrm{Sym}^p(\mathscr{E}) \subseteq \mathrm{Sym}^*(\mathscr{E})$. Two splittings yield isomorphic quotients precisely when they differ by an element in the image of $\mathbf{F}_X^* : H^0(X, \mathscr{L}) \to H^0(X, \mathscr{L}^{\otimes p})$. Thus we see that this explicit construction is also determined by the data of some class $\xi \in H^1(X, \alpha_{\mathscr{L}})$ lifting $\bar{\xi} \in H^1(X, \mathscr{L})$.

Proposition 2.1.2 (Ekedahl)**.** *If X is a normal, projective, Gorenstein (resp. l.c.i.) variety and $f : Z \to X$ a nontrivial $\alpha_{\mathscr{L}}$-torsor for some line bundle $\mathscr{L}$, then Z is a projective, Gorenstein (resp. l.c.i.) variety satisfying:*

1. $\omega_Z \cong f^*(\omega_X \otimes \mathscr{L}^{\otimes p-1})$,

2. $\chi(f_*\mathcal{O}_Z) = \sum_{i=0}^{p-1} \chi(\mathscr{L}^{\otimes -i})$.

Proof. Showing that Z is integral with $\omega_Z \cong f^*(\omega_X \otimes \mathscr{L}^{\otimes p-1})$ when X is normal can be found in [10, §1] or [26, Prop. II.6.1.7]. From the explicit description of Z given above, we obtain a filtration of $f_*\mathcal{O}_Z$, given by the images of $\mathrm{Sym}^i(\mathscr{E})$, whose successive quotients are isomorphic to $\mathscr{L}^{\otimes -i}$, for $0 \leq i < p$ (cf. [10, Prop. 1.7]); this immediately yields the Euler characteristic formula in (2). Finally, if X is l.c.i., then Z is too as it embeds in the affine bundle **Spec** $\mathrm{Sym}^*(\mathscr{E})/(1 - i(1))$ over X as the Cartier divisor defined locally by $\sigma(s)$, where s is a local generator of $\mathscr{L}^{\otimes -p}$. ■

We intend to use Proposition 2.1.2 (2) to relate the Euler characteristic of the structure sheaf of a normal, l.c.i. del Pezzo surface X to that of a nontrivial $\alpha_{\mathscr{L}}$-torsor $f : Z \to X$. Yet, if the fields $k_Z := H^0(Z, \mathcal{O}_Z)$ and $k_X := H^0(X, \mathcal{O}_X)$ do not coincide, then the Euler characteristics $\chi(\mathcal{O}_Z)$ and $\chi(f_*\mathcal{O}_Z)$ differ by a factor of $[k_Z : k_X]$:

$$\chi(f_*\mathcal{O}_Z) = [k_Z : k_X] \cdot \chi(\mathcal{O}_Z).$$

The following easy lemma controls this factor, showing it is either 1 or p.

Lemma 2.1.3. *If $f : Z \to X$ is a finite dominant morphism of degree d from a proper variety Z to a normal, proper variety X over k, then $k_Z := H^0(Z, \mathcal{O}_Z)$ is a field extension of $k_X := H^0(X, \mathcal{O}_X)$ whose degree divides d, that is,*

$$[k_Z : k_X] \mid d.$$

Proof. There are field extensions $k_X \subseteq k(X) \subseteq k(Z)$ and $k_Z \subseteq k(Z)$. Because X is normal and f is finite, $k_Z \cap k(X) = k_X$. Therefore, $k_Z \otimes_{k_X} k(X)$ is a subfield of $k(Z)$, of degree $[k_Z : k_X]$ over $k(X)$, and hence $[k_Z : k_X]$ divides $[k(Z) : k(X)] = d$. ■

2.2 Normal del Pezzo surfaces of local complete intersection

Let X be a normal, l.c.i. del Pezzo surface over a field k such that for some integer n the cohomology group $H^1(X, nK_X) \neq 0$ (e.g. X is a regular del Pezzo surface with irregularity and $n = 1$). We will see that the construction of the previous subsection can be used to create a degree p inseparable morphism $f : Z \to X$ whose existence puts restrictions on the possible pairs of integers (d, q) that arise as the degree d and irregularity q of such X. The normalcy condition is used to ensure the integrality of Z, and the l.c.i. condition guarantees that we may use the following version of the Riemann-Roch theorem:

Theorem 2.2.1 (Riemann-Roch)**.** *If D be a Cartier divisor on a 2-dimensional variety X of local complete intersection, then*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2}D.(D - K_X).$$

Proof. The Grothendieck-Riemann-Roch theorem asserts for any line bundle $\mathscr{L}$ on X,

$$\chi(\mathscr{L}) = \int_X \mathrm{ch}(\mathscr{L}) \frown (\mathrm{td}(T_{vir}) \frown [X]), \tag{2.2.2}$$

where T_{vir} is the virtual tangent bundle of X (cf. [12, Cor. 18.3.1(b)]). The Todd class is given by $\mathrm{td}(T_{vir}) = 1 + \frac{1}{2}c_1(T_{vir}) + \frac{1}{12}(c_1(T_{vir})^2 + c_2(T_{vir}))$, and the Chern character by $\mathrm{ch}(\mathscr{L}) = 1 + c_1(\mathscr{L}) + \frac{1}{2}c_1(\mathscr{L})^2$. Taking $\mathscr{L} := \mathcal{O}_X$, we see that $\chi(\mathcal{O}_X) = \frac{1}{12}\int_X (c_1(T_{vir})^2 + c_2(T_{vir})) \frown [X]$. Substituting these expressions into (2.2.2) for $\mathscr{L} := \mathcal{O}_X(D)$ results in the formula

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2}\int_X D.(D + c_1(T_{vir})).$$

We finish by noting that $c_1(T_{vir}) = -K_X$, due to the adjunction formula for local complete intersections. ■

The main result of this chapter is the following:

Theorem 2.2.3. *Let X be a normal, l.c.i. del Pezzo surface with irregularity $q_X = h^1(X, \mathcal{O}_X)$.*

1. *If $q_X > 0$ then there exists a positive integer m such that the line bundle $\mathscr{L} := \omega_X^{\otimes m}$ has the following property:*

   $(*)$ *the absolute Frobenius pullback $\mathbf{F}_X^* : H^1(X, \mathscr{L}) \to H^1(X, \mathscr{L}^{\otimes p})$ has a nontrivial kernel.*

2. *If $\mathscr{L}$ is a line bundle that satisfies $(*)$ and is numerically equivalent to $\omega_X^{\otimes m}$ for some integer m, then there exists a nontrivial $\alpha_{\mathscr{L}}$-torsor Z that is an l.c.i. del Pezzo surface of anti-canonical degree*

$$K_Z^2 = p^{1-e}(1+m(p-1))^2 K_X^2. \tag{2.2.4}$$

The field $k_Z := H^0(Z, \mathcal{O}_Z)$ is an extension of $k_X := H^0(X, \mathcal{O}_X)$ of degree p^e with $e \in \{0,1\}$, and if $q_Z := h^1(Z, \mathcal{O}_Z)$ denotes the irregularity of Z, then

$$p^e(1-q_Z) = p - pq_X + \frac{mp(p-1)K_X^2}{12}\left(3 + m(2p-1)\right). \tag{2.2.5}$$

Proof. The existence of an integer m as in (1) is an immediate consequence of Serre's theorems on duality and vanishing of higher cohomology. Let $\mathscr{L}$ be any line bundle satisfying the hypothesis of (2), and let Z be any $\alpha_{\mathscr{L}}$-torsor Z associated to a nonzero Frobenius-killed cohomology class $\bar{\xi} \in H^1(X, \mathscr{L})$. By Proposition 2.1.2, the torsor Z is an l.c.i. variety with dualizing sheaf $\omega_Z \cong f^*(\omega_X \otimes \mathscr{L}^{\otimes p-1})$. Hence, we can compute the anti-canonical degree of Z (over k_Z) as

$$K_Z^2 = \frac{\deg f}{[k_Z : k_X]} \cdot (1+m(p-1))^2 K_X^2.$$

Since $\deg f = p$, Lemma 2.1.3 implies that $[k_Z : k_X] = p^e$ with $e \in \{0,1\}$, which proves (2.2.4).

Since both ω_X^{-1} and $\mathscr{L}^{-1}$ are ample line bundles on X and f is a finite morphism, the line bundle ω_Z^{-1} is ample and Z is therefore an l.c.i. del Pezzo surface. Moreover, Proposition 2.1.2 gives the equality $\chi(f_*\mathcal{O}_Z) = \sum_{i=0}^{p-1} \chi(\mathscr{L}^{\otimes -i})$. The Riemann-Roch theorem shows that $\chi(\mathscr{L}^{\otimes -i})$ is independent of the numerical equivalence class of $\mathscr{L}^{\otimes -i}$ and hence that

$$\begin{aligned}\chi(\mathscr{L}^{\otimes -i}) &= \chi(\omega^{\otimes -mi}) \\ &= \chi(\mathcal{O}_X) + \frac{mi(mi+1)}{2} K_X^2.\end{aligned}$$

We substitute this into our expression for $\chi(f_*\mathcal{O}_Z)$ and use the well-established formulae for summing consecutive integers and their squares to obtain

$$\begin{aligned}\chi(f_*\mathcal{O}_Z) &= p\chi(\mathcal{O}_X) + \frac{K_X^2}{2}\sum_{i=0}^{p-1}(m^2i^2 + mi) \\ &= p\chi(\mathcal{O}_X) + \frac{mp(p-1)K_X^2}{12}\left(3 + m(2p-1)\right).\end{aligned}$$

Because X and Z are each del Pezzo surfaces, Serre duality implies that $H^2(X, \mathcal{O}_X) = 0 = H^2(Z, \mathcal{O}_Z)$. Therefore, $\chi(f_*\mathcal{O}_Z) = p^e(1 - q_Z)$ and $\chi(\mathcal{O}_X) = 1 - q_X$. ■

Main Theorem (2) follows as an immediate corollary:

Corollary 2.2.6. *If X is a normal, l.c.i. del Pezzo surface of degree d and irregularity $q > 0$, then there exists a nontrivial $\alpha_{\omega_X^{\otimes m}}$-torsor, Z, for which $[k_Z : k_X] = p^e$ for some integers $m \geq 1$ and $e \in \{0, 1\}$. Furthermore, any such integers satisfy*

$$\begin{aligned} q &\geq 1 - \frac{1}{p^{1-e}} + \frac{md(p-1)(3 + m(2p-1))}{12} \\ &\geq \frac{d(p^2-1)}{6}, \end{aligned} \tag{2.2.7}$$

with equality in (2.2.7) only if $e = 1$ and $m = 1$.

Proof. For the first inequality, use (2.2.5) of Theorem 2.2.3 and the fact $q_Z = h^1(Z, \mathcal{O}_Z) \geq 0$. For the second inequality, use $e \leq 1$ and $m \geq 1$. ■

In the case when $q_X = 1$, the values of p, m, q_Z, K_X^2, and $h^0(X, \omega_X^{-1})$ are completely determined by that of $e \in \{0, 1\}$. Later we construct examples of regular del Pezzo surfaces exhibiting these values for either choice of $e \in \{0, 1\}$ (cf. Ch. 4).

Corollary 2.2.8. *If X is a normal, l.c.i. del Pezzo surface over a field of characteristic p with irregularity $h^1(X, \mathcal{O}_X) = 1$ and Z is a nontrivial $\alpha_{\omega_X^{\otimes m}}$-torsor for an integer $m \geq 1$, then $m = 1$, $p = 2$, and the anti-canonical degree $K_X^2 = [k_Z : k_X] = 2^e$ for $e \in \{0, 1\}$. Moreover, the cohomology group $H^1(Z, \mathcal{O}_Z) = 0$, and for all $n \geq 1$,*

$$h^0(X, \omega_X^{\otimes -n}) = \frac{n(n+1)}{2^{(1-e)}}.$$

Proof. If $q_X = 1$, then the right-hand side of (2.2.5) is positive, forcing $q_Z = 0$. Thus (2.2.5) simplifies to

$$p^e = \frac{mp(p-1)K_X^2}{12}\left(3 + m(2p-1)\right).$$

As all variables are positive integers, one can quickly solve by brute force. If $e = 0$, then $p = 2, K_X^2 = 1$, and $m = 1$. Similarly, if $e = 1$, then $p = 2, K_X^2 = 2, m = 1$.

If $H^1(X, \omega_X^{\otimes n}) \neq 0$ for some $n \geq 1$, then Serre's theorem on the vanishing of higher cohomology would show the existence of some Frobenius-killed class in $H^1(X, \omega_X^{\otimes N})$, for some $N \geq n$, and then Theorem 2.2.3(2) and our above argument shows that $N = 1$. Thus, $H^1(X, \omega_X^{\otimes n}) = 0$ for all $n > 1$. By Serre duality, $h^1(X, \omega_X^{\otimes -n}) = h^1(X, \omega_X^{\otimes n+1}) = 0$ for any $n \geq 1$, and Riemann-Roch therefore implies $h^0(X, \omega_X^{\otimes -n}) = \chi(\omega_X^{\otimes -n}) = \frac{n(n+1)}{2} K_X^2$. ■

Chapter 3

Algebraic foliations on regular varieties

In contrast to our task in Chapter 2 of finding numerical restrictions on the existence of regular del Pezzo surfaces with irregularity, we begin the dual problem of constructing explicit examples of such surfaces. The $\alpha_{\mathscr{L}}$-torsor construction of the previous chapter will again be important to us, although we shall henceforth view them from an alternative perspective. Beginning with a k-variety Z, equipped with an algebraic foliation $\mathscr{F} \subseteq T_{Z/k}$, one can construct a purely inseparable quotient morphism $f : Z \to Z/\mathscr{F}$ that factors the relative Frobenius morphism $\mathbf{F}_{Z/k} : Z \to Z \times_{k,\mathbf{F}_k} k$. If $Z \to X$ is an $\alpha_{\mathscr{L}}$-torsor, there is a natural rank 1 foliation given by the relative tangent bundle $T_{Z/X}$ that recovers X as the quotient $Z/\mathscr{F}$, for $\mathscr{F} := T_{Z/X}$. The converse does not hold as the quotient morphism $Z \to Z/\mathscr{F}$ for an arbitrary (rank 1) foliation $\mathscr{F}$ is not necessarily an $\alpha_{\mathscr{L}}$-torsor for any choice of line bundle $\mathscr{L}$ on $Z/\mathscr{F}$. However, when $p = 2$, this problem does not arise, and Z may indeed be recovered from the quotient $Z/\mathscr{F}$ as some $\alpha_{\mathscr{L}}$-torsor (cf. §3.2). Ekedahl developed this theory for smooth varieties in [9], and in this chapter we generalize his results to the setting of regular varieties.

Proposition 3.0.1. *Let k be a finitely generated field extension of a perfect field $\mathbb{F}_2$ of characteristic* 2*.*

1. *If Z is a regular k-variety and $\mathscr{F} \subseteq T_{Z/k} \subseteq T_{Z/\mathbb{F}_2}$ is a rank 1 foliation on Z over the extension $k/\mathbb{F}_2$, then the quotient $X := Z/\mathscr{F}$ is a regular k-variety and the quotient morphism $f : Z \to X$ is an $\alpha_{\mathscr{L}}$-torsor for some line bundle $\mathscr{L}$ on X.*

2. *Additionally, if Z is a del Pezzo surface and $\mathscr{F}^{\otimes 2} \cong \omega_Z$, then the quotient X is a regular del Pezzo surface, the sheaf $\mathscr{L}^{-1} \otimes \omega_X$ is a 2-torsion line bundle, and the following equations hold:*

   (a) $[k_Z : k_X] \cdot \chi(\mathcal{O}_Z) = 2\chi(\mathcal{O}_X) + d_X$,

   (b) $K_X^2 = \frac{[k_Z:k_X]\cdot K_Z^2}{8}$.

The fruits of our labor will be harvested in Chapter 4, when we carefully find two such foliations $\mathscr{F}_1$ and $\mathscr{F}_2$ on a specific variety Z. The resulting quotients $X_i = Z/\mathscr{F}_i$ are regular del Pezzo surfaces with irregularity $q = 1$, and in this specific case, the line bundle $\mathscr{L}$ can be identified precisely as the dualizing sheaf ω_X (cf. Cor.3.2.2).

3.1 Quotients by foliations

First we generalize the definition of a foliation on a smooth variety (cf. [9]) to the case of a regular variety over an imperfect field.

Definition 3.1.1. Let Z be a regular variety over a field extension k of a perfect field $\mathbb{F}$ of characteristic p. A *foliation (over the extension $k/\mathbb{F}$)* on Z is a locally free $\mathcal{O}_Z$-submodule $\mathscr{F} \subseteq T_{Z/k} \subseteq T_{Z/\mathbb{F}}$ preserved by the Lie bracket and the p-th power operation (i.e. a sub-p-Lie algebra of $T_{Z/k}$) whose cokernel $T_{Z/\mathbb{F}}/\mathscr{F}$ is locally free. The *rank* of a foliation $\mathscr{F}$ is its rank as a locally free $\mathcal{O}_Z$-module.

This definition recovers the usual notion of a foliation (cf. [9]) in the case where Z is a smooth variety over a perfect field $k = \mathbb{F}$. Our more general definition is contrived so that when $\pi : \mathcal{Z} \to \mathcal{B}$ is a morphism of varieties from a smooth variety $\mathcal{Z}$ over a perfect field $\mathbb{F}$, any foliation $\mathscr{F}$ on $\mathcal{Z}$ (in the sense of [9]) that is vertical with respect to π (i.e. $\mathscr{F} \subseteq T_{\mathcal{Z}/\mathcal{B}}$) will restrict to the generic fibre of π as a foliation (in our sense) over the extension $\mathbb{F}(\mathcal{B})/\mathbb{F}$.

The utility of algebraic foliations comes from the fact that one can use them to quotient varieties to obtain purely inseparable finite morphisms:

Definition/Lemma 3.1.2. Let $k/\mathbb{F}$ be a field extension of a perfect field $\mathbb{F}$ of characteristic p. If $\mathscr{F}$ is a foliation over the extension $k/\mathbb{F}$ on a regular k-variety Z, then there is a k-variety $Z/\mathscr{F}$, which we call the *quotient* of Z by $\mathscr{F}$, along with a purely inseparable morphism $f : Z \to Z/\mathscr{F}$ that factors the relative Frobenius morphism $\mathbf{F}_{Z/k}$ and is given locally by the inclusion of subrings $\mathcal{O}_Z^p \subseteq \mathcal{O}_{Z/\mathscr{F}} \subseteq \mathcal{O}_Z$, where

$$\mathcal{O}_{Z/\mathscr{F}} := \{f \in \mathcal{O}_Z : \delta(f) = 0 \text{ for all local derivations } \ \delta \in \mathscr{F}\}.$$

Proof. The construction of $Z/\mathscr{F}$ is well-defined because the definition of $\mathcal{O}_{Z/\mathscr{F}}$ commutes with localization, a result which ultimately boils down to the fact that the ring of pth powers $\mathcal{O}_Z^p$ is killed by any derivation. Since k, in addition to $\mathcal{O}_Z^p$, is killed by all derivations in $\mathscr{F} \subseteq T_{Z/k}$, the morphism f factors the relative Frobenius morphism $\mathbf{F}_{Z/k} : Z \to Z \times_{k,\mathbf{F}_k} k$. That is, $\mathbf{F}_{Z/k} = g \circ f$ for a unique morphism $g : X \to Z \times_{k,\mathbf{F}_k} k$. In particular, both f and g are purely inseparable morphisms. Moreover, since Z is finite-type over k, the relative Frobenius morphism $\mathbf{F}_{Z/k}$ is a finite morphism, and hence so are the morphisms f and g. Since Z is a finite-type over k, so is the base change $Z \times_{k,\mathbf{F}_k} k$ (with structure morphism given by projection onto the second factor). As X is finite over $Z \times_{k,\mathbf{F}_k} k$, it too is of finite type over k. ∎

For foliations on smooth varieties over a perfect field $k = \mathbb{F}$, the following theorem of Ekedahl provides vital information concerning the structure of the quotient.

Theorem 3.1.3 (Ekedahl)**.** *Let Z be a smooth n-dimensional variety over a perfect field $\mathbb{F}$. Let $\mathscr{F} \subseteq T_{Z/\mathbb{F}}$ be a foliation of rank r and $f : Z \to X := Z/\mathscr{F}$ the quotient of Z by this foliation. Furthermore denote by $g : X \to Z \times_{\mathbb{F},\mathbf{F}_\mathbb{F}} \mathbb{F}$ the morphism so that $g \circ f = \mathbf{F}_{Z/\mathbb{F}}$ is the relative Frobenius morphism. Then the following hold:*

1. *X is a smooth $\mathbb{F}$-variety;*

2. *f and g are finite flat morphisms of degrees p^r and p^{n-r}, respectively;*

3. *there is an exact sequence*

$$0 \to \mathscr{F} \to T_{Z/\mathbb{F}} \to f^*T_{X/\mathbb{F}} \to \mathbf{F}_Z^*\mathscr{F} \to 0,$$

and hence an isomorphism

$$f^*\omega_{X/\mathbb{F}} \cong \omega_{Z/\mathbb{F}} \otimes (\det \mathscr{F})^{\otimes 1-p}.$$

Proof. See [9, §3]. ∎

We now partially extend this result for our applications to regular varieties over finitely generated imperfect fields.

Proposition 3.1.4. *Let Z be a regular variety over a finitely generated field extension k of a perfect field $\mathbb{F}$. Let $\mathscr{F} \subseteq T_{Z/k} \subseteq T_{Z/\mathbb{F}}$ be a foliation of rank r over the extension $k/\mathbb{F}$ and $f : Z \to X$ the quotient of Z by this foliation. Then the following hold:*

1. *X is a regular k-variety;*

2. *f is a flat morphism of degree p^r;*

3. *there is an exact sequence*

$$0 \to \mathscr{F} \to T_{Z/\mathbb{F}} \to f^*T_{X/\mathbb{F}} \to \mathbf{F}_Z^*\mathscr{F} \to 0,$$

and hence an isomorphism

$$f^*\omega_{X/k} \cong \omega_{Z/k} \otimes (\det \mathscr{F})^{\otimes 1-p}. \tag{3.1.5}$$

Proof. Choose a sufficiently large finitely generated sub-$\mathbb{F}$-algebra $A \subseteq k$ so that Z descends to a finite-type integral A-scheme Z_A, $\mathscr{F}$ descends to a subsheaf $\mathscr{F}_A \subseteq T_{Z_A/A} \subseteq T_{Z_A/\mathbb{F}}$, and the fraction field of A equals k. This is possible because Z is of finite-type over k and $\mathscr{F}$ is a submodule of the coherent $\mathcal{O}_Z$-module $T_{Z/\mathbb{F}}$; the $\mathcal{O}_Z$-module $T_{Z/\mathbb{F}}$ is coherent because Z is of finite-type over a finitely generated field extension of $\mathbb{F}$.

It is a classical result that the regular locus of a locally Noetherian scheme is an open locus (cf. [29, Thm. 24.4]). Since $Z = Z_A \times_A k$ is regular, the regular locus on Z_A is a non-empty open neighborhood of the generic fibre Z, and its image in A will be an open neighborhood U of the generic point of A. By replacing $\operatorname{Spec} A$ by a sufficiently small affine subset of U, we may assume that both Z_A and $\operatorname{Spec} A$ are regular $\mathbb{F}$-varieties. Consequently, both Z_A and $\operatorname{Spec} A$ are smooth over $\mathbb{F}$ since $\mathbb{F}$ is a perfect field (cf. [39, Lem. 038V]).

Because $\mathscr{F}$ is a foliation, $\mathscr{F} \cong \mathscr{F}_A \otimes_A k$ and $T_{Z/\mathbb{F}}/\mathscr{F} \cong (T_{Z_A/\mathbb{F}}/\mathscr{F}_A) \otimes_A k$ are locally free. Therefore, by replacing $\operatorname{Spec} A$ by an even smaller open subscheme, we may assume that both $\mathscr{F}_A$ and $T_{Z_A/\mathbb{F}}/\mathscr{F}_A$ are finite locally free $\mathcal{O}_A$-modules. Consider the $\mathcal{O}_A$-module homomorphism $\mathscr{F}_A \otimes \mathscr{F}_A \to T_{Z_A/\mathbb{F}}/\mathscr{F}_A$ induced by the Lie bracket and the $\mathcal{O}_A$-module homomorphism $\mathscr{F}_A \to T_{Z_A/\mathbb{F}}/\mathscr{F}_A$ induced by the pth power operation. Notice that both of these homomorphisms are 0 when localized at the generic point of $\operatorname{Spec} A$ precisely because of our hypothesis that $\mathscr{F}$ is a foliation on the generic fibre Z. By the upper semi-continuity of rank, we may restrict $\operatorname{Spec} A$ even further so that these morphisms are 0 over all of $\operatorname{Spec} A$, which is equivalent to $\mathscr{F}_A$ being a foliation on the smooth variety Z_A over $\mathbb{F}$.

Now, we may apply Theorem 3.1.3 to Z_A and $\mathscr{F}_A \subseteq T_{Z_A/A} \subseteq T_{Z_A/\mathbb{F}}$ to obtain a smooth quotient $X_A := Z_A/\mathscr{F}_A$. The generic fibre $X_A \times_A k$ is therefore a regular variety. Because taking quotients by foliations is a local operation, $Z/\mathscr{F} \cong X_A \times_A k$, proving (1). Assertion (2) holds by localizing the morphism $f : Z_A \to X_A$, which is finite and flat of degree r by Theorem 3.1.3. The exact sequence in (3) follows by localizing that of Theorem 3.1.3(3). The isomorphism in (3) follows by taking the determinant of this sequence, which yields

$$f^*\omega_{X_A/\mathbb{F}}|_Z \cong \omega_{Z_A/\mathbb{F}}|_Z \otimes (\det \mathscr{F})^{\otimes 1-p},$$

and then applying Lemma 3.1.6 to each of the morphisms $Z_A \to A$ and $X_A \to A$. ∎

Lemma 3.1.6. *Let $\pi : \mathcal{X} \to \mathcal{B}$ be an l.c.i. morphism of l.c.i. varieties over a field $\mathbb{F}$. Let $k := \mathbb{F}(\mathcal{B})$ denote the function field of $\mathcal{B}$. Then the dualizing sheaf of the generic fibre $X := \mathcal{X} \times_{\mathcal{B}} k$ is just the restriction of the dualizing sheaf of $\mathcal{X}$:*

$$\omega_{X/k} = \omega_{\mathcal{X}/\mathbb{F}}|_X.$$

Proof. By [16, Def. 1.5], we have $\omega_\pi := \omega_{\mathcal{X}/\mathbb{F}} \otimes \pi^*\omega_{\mathcal{B}/\mathbb{F}}^{-1}$. As ω_π commutes with arbitrary base changes,

$$\omega_{X/k} = \omega_\pi|_X = (\omega_{\mathcal{X}/\mathbb{F}} \otimes \pi^*\omega_{\mathcal{B}/\mathbb{F}}^{-1})|_X = \omega_{\mathcal{X}/\mathbb{F}}|_X,$$

with the last equality justified by $\omega_{\mathcal{B}/\mathbb{F}}$ being locally trivial on $\mathcal{B}$. ■

3.2 Foliations in characteristic two

A degree p inseparable morphism $f : Z \to X$ is not generally an $\alpha_{\mathscr{L}}$-torsor for any line bundle $\mathscr{L}$, even when f is the morphism arising from the quotient by a foliation on a variety Z. Luckily, when $p = 2$ this difficulty does not arise, which allows us to apply Theorem 2.2.3 to the proof of Proposition 3.0.1, the key result used in Chapter 4 to construct regular del Pezzo surfaces with irregularity.

Proposition 3.2.1 (Ekedahl). *Let $f : Z \to X$ be a finite morphism of degree $p = 2$ from a Cohen-Macaulay scheme Z to a regular variety X. Let $\mathscr{L}$ be the line bundle satisfying*

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_Z \to \mathscr{L}^{-1} \to 0.$$

Then $Z \to X$ is an α_s torsor for some $s \in \Gamma(X, \mathscr{L})$, viewed as a section $s \in \mathrm{Hom}(\mathscr{L}, \mathscr{L}^{\otimes 2})$, where α_s is the group scheme kernel of $\mathbf{F}_{\mathscr{L}/X} - s : \mathscr{L} \to \mathscr{L}^{\otimes 2}$. Moreover, if f is a purely inseparable map, then $s = 0$ and hence $f : Z \to X$ is an $\alpha_{\mathscr{L}}$-torsor.

Proof. This is proven in [10, Prop. 1.11] for smooth X, although the proof only requires X to be regular (to guarantee that $f_*\mathcal{O}_Z$ is a locally free $\mathcal{O}_X$-module). ■

We now prove the result advertised at the beginning of this chapter:

Proof of Proposition 3.0.1. Proposition 3.1.4 (1) proves that X is regular, and then Theorem 3.2.1 shows that $f : Z \to X$ indeed arises as an $\alpha_{\mathscr{L}}$-torsor. Proposition 3.1.4(3) proves $f^*\omega_X \cong \omega_Z \otimes \mathscr{F}^{\otimes -1}$ and Proposition 2.1.2 gives $f^*\mathscr{L} \cong \omega_Z \otimes f^*\omega_X^{-1}$. It immediately follows $\mathscr{F} \cong f^*\mathscr{L}$, and also $f^*\omega_X \cong \mathscr{F}$, due to the hypothesis $\omega_Z \cong \mathscr{F}^{\otimes 2}$. Combining these isomorphisms, we obtain $f^*\omega_X \cong f^*\mathscr{L}$. Since f is a finite, flat surjective map of degree 2, the line bundles ω_X and $\mathscr{L}$ differ by a 2-torsion line bundle, and hence are $\mathbb{Q}$-linearly equivalent. If Z is a del Pezzo surface, then X is as well because f is a finite, flat surjective map and so ω_X^{-1} is ample if and only if $\omega_Z^{-1} \cong f^*\omega_X^{\otimes -2}$ is ample. A straight-forward application of Theorem 2.2.3 (2) gives the last two claims. ■

Corollary 3.2.2. *If $Z \to X$ is the quotient of a regular del Pezzo surface Z by a rank 1 foliation $\mathscr{F}$ over $k/\mathbb{F}$, a finitely generated field extension of a perfect field, such that $\mathscr{F}^{\otimes 2} \cong \omega_Z$ and $h^1(X, \mathcal{O}_X) = 1$, then Z is an $\alpha_{\mathscr{L}}$-torsor for $\mathscr{L} \cong \omega_X$.*

Proof. Corollary 2.2.8 guarantees that $\mathbb{F}$ is of characteristic 2. By Proposition 3.0.1, Z is a nontrivial $\alpha_{\mathscr{L}}$-torsor for some line bundle $\mathscr{L}$ that differs from ω_X by a 2-torsion line bundle. In particular, $\mathscr{L}$ and ω_X are numerically equivalent, and therefore by the Riemann-Roch theorem, $\chi(\mathscr{L}) = \chi(\omega_X)$. Serre duality implies $\chi(\omega_X) = \chi(\mathcal{O}_X) = 0$, because $h^1(X, \mathcal{O}_X) = 1$. The groups $H^0(X, \mathscr{L})$ and $H^0(X, \mathscr{L}^{\otimes 2})$ are 0 because $\mathscr{L}^{-1}$ is ample. Therefore, $h^1(X, \mathscr{L}) = h^2(X, \mathscr{L})$, and by Serre duality, $h^2(X, \mathscr{L}) = h^0(X, \mathscr{L}^{-1} \otimes \omega_X)$.

If we assume $\mathscr{L}^{-1} \otimes \omega_X$ is a nontrivial line bundle, then it follows that

$$h^1(X, \mathscr{L}) = h^0(X, \mathscr{L}^{-1} \otimes \omega_X) = 0,$$

because any global section of a nontrivial torsion line bundle on a projective variety is 0. On the other hand, since Z is a nontrivial $\alpha_{\mathscr{L}}$-torsor, it corresponds to a nonzero class of the cohomology group $H^1(X, \alpha_{\mathscr{L}})$. The long exact sequence in cohomology attached to the short exact sequence of group schemes

$$0 \to \alpha_{\mathscr{L}} \to \mathscr{L} \to \mathscr{L}^{\otimes 2} \to 0,$$

along with the vanishing $H^0(X, \mathscr{L}^{\otimes 2}) = 0$, proves that there is an injection $H^1(X, \alpha_{\mathscr{L}}) \subseteq H^1(X, \mathscr{L})$. This latter group is 0, yet must have a nonzero class that corresponds to the nontrivial $\alpha_{\mathscr{L}}$-torsor Z, demonstrating the absurdity of our assumption. Therefore $\mathscr{L}^{-1} \otimes \omega_X \cong \mathcal{O}_X$. ∎

Chapter 4

The construction of regular del Pezzo surfaces with irregularity

In this chapter we construct examples of regular del Pezzo surfaces X with $h^1(X, \mathcal{O}_X) = 1$. By Corollary 2.2.8, these surfaces can only exist in characteristic 2 and must have anti-canonical degree $K_X^2 \in \{1, 2\}$. We construct such surfaces by applying Proposition 3.0.1 to an explicit regular del Pezzo surface Z and foliations $\mathscr{F}$ satisfying $\mathscr{F}^{\otimes 2} \cong \omega_Z$. Once constructed, it follows from Corollary 3.2.2 that $Z \to X$ is an α_{ω_X}-torsor.

4.1 The set-up

Let $\mathbb{F}_2$ be any perfect field of characteristic 2. Let $\mathcal{Z} \subseteq \mathbb{P}^3_{\mathbb{F}_2} \times \mathbb{A}^4_{\mathbb{F}_2}$ be the family of quasi-linear quadrics given by the vanishing of the form $Q := \alpha_0 X_0^2 + \alpha_1 X_1^2 + \alpha_2 X_2^2 + \alpha_3 X_3^2$, where the coordinates $[X_0 : X_1 : X_2 : X_3]$ are projective and $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ are affine. As a simplification, we sometimes omit the symbol $\mathbb{F}_2$ from our notation (e.g. we write $\Omega_{\mathbb{P}^3}$ instead of the more cluttered $\Omega_{\mathbb{P}^3_{\mathbb{F}_2}/\mathbb{F}_2}$). Let $\mathscr{I}_{\mathcal{Z}} \subseteq \mathcal{O}_{\mathbb{P}^3 \times \mathbb{A}^4}$ denote the ideal sheaf, generated by Q, that defines $\mathcal{Z}$ as a subscheme of $\mathbb{P}^3_{\mathbb{F}_2} \times \mathbb{A}^4_{\mathbb{F}_2}$.

The sequence

$$0 \to \mathcal{O}_{\mathcal{Z}}(-2) \xrightarrow{dQ} \mathcal{O}_{\mathcal{Z}} \otimes (\Omega^1_{\mathbb{P}^3} \oplus \Omega^1_{\mathbb{A}^4}) \to \Omega^1_{\mathcal{Z}/\mathbb{F}_2} \to 0 \tag{4.1.1}$$

is exact, and since $dQ = \sum X_i^2 d\alpha_i$ is nowhere vanishing, the cokernel $\Omega_{\mathcal{Z}/\mathbb{F}_2}$ is a rank 6 vector bundle on $\mathcal{Z}$. Hence, $\mathcal{Z}$ is a smooth $\mathbb{F}_2$-variety. Let $\mathcal{Z}_U$ denote the restriction of the family $\mathcal{Z}$ to the open subscheme $U \subseteq \mathbb{A}^4_{\mathbb{F}_2}$ that complements the 15 hyperplanes of the form $\sum_{i=0}^3 \varepsilon_i\alpha_i = 0$ for $\varepsilon_i \in \{0,1\}$. Let Z be the generic fibre of $\mathcal{Z}_U$ over U,

$$Z := (\sum \alpha_i X_i^2 = 0) \subseteq \mathbb{P}^3_{\mathbb{F}_2(\alpha_0,\dots,\alpha_3)}.$$

The adjunction formula implies $\omega_Z \cong \mathcal{O}_Z(-2)$, and hence Z is a regular del Pezzo surface with $K_Z^2 = 8$ and, being a hypersurface in $\mathbb{P}^3_{\mathbb{F}_2(\alpha_0,\dots,\alpha_3)}$, with $h^1(Z, \mathcal{O}_Z) = 0$.

To satisfy the hypothesis of Proposition 3.0.1, we shall construct, for specified subfields $k \subseteq \mathbb{F}_2(\alpha_0, \dots, \alpha_3)$, rank 1 foliations $\mathcal{O}_Z(-1) \cong \mathscr{F} \subseteq T_{Z/k} \subseteq T_{Z/\mathbb{F}_2}$ over the extension $k/\mathbb{F}_2$. To construct such $\mathscr{F}$, we find subsheaves $\mathscr{F}_{\mathcal{Z}} \subseteq T_{\mathcal{Z}/\mathbb{F}_2}$ that restrict to foliations on $\mathcal{Z}_U$ over the perfect field $\mathbb{F}_2$. We then take $\mathscr{F}$ to be the restriction of this foliation to Z, that is, $\mathscr{F} := (\mathscr{F}_{\mathcal{Z}})|_Z$.

4.2 An example of degree one

Define $\Theta_{\mathbb{P}} : \mathcal{O}_{\mathbb{P}^3}(-1) \to T_{\mathbb{P}^3}$ as the composition

$$\Theta_{\mathbb{P}} : \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\sum X_i^2 \partial_{X_i}} \sum_{i=0}^{3} \mathcal{O}_{\mathbb{P}^3}(1)\partial_{X_i} \xrightarrow{\phi} T_{\mathbb{P}^3},$$

where ϕ is the morphism coming from the Euler sequence,

$$0 \to \mathcal{O}_{\mathbb{P}^3_{\mathbb{F}_2}} \xrightarrow{\sum X_i \partial_{X_i}} \sum_{i=0}^{3} \mathcal{O}_{\mathbb{P}^3_{\mathbb{F}_2}}(1)\partial_{X_i} \xrightarrow{\phi} T_{\mathbb{P}^3} \to 0. \tag{4.2.1}$$

Let $\mathscr{F}_{\mathcal{Z}}$ denote the image of

$$\Theta_{\mathbb{P}} \oplus 0 : \mathcal{O}_{\mathcal{Z}} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathcal{Z}} \otimes (T_{\mathbb{P}^3} \oplus T_{\mathbb{A}^4}) \cong T_{\mathbb{P}^3 \times \mathbb{A}^4}|_{\mathcal{Z}}.$$

Notice that, as derivations in $\mathscr{F}_{\mathcal{Z}}$ preserve the ideal sheaf $\mathscr{I}_{\mathcal{Z}}$, as well as kill functions coming from $\mathcal{O}_{\mathbb{A}^4}$, the sheaf $\mathscr{F}_{\mathcal{Z}}$ is contained within the subsheaf $T_{\mathcal{Z}/\mathbb{A}^4} \subseteq T_{\mathbb{P}^3 \times \mathbb{A}^4}|_{\mathcal{Z}}$.

We now proceed to demonstrate that $\mathscr{F}_{\mathcal{Z}} \subseteq T_{\mathcal{Z}/\mathbb{F}_2}$ is foliation over $\mathbb{F}_2$ when restricted to $\mathcal{Z}_U$. First we prove that $\Theta_{\mathbb{P}} \oplus 0$ is injective on all fibres over $\mathcal{Z}_U$. It suffices to prove this injectivity after composing with the projection $T_{\mathbb{P}^3\times\mathbb{A}^4}|_{\mathcal{Z}} \to \mathcal{O}_{\mathcal{Z}}\otimes T_{\mathbb{P}^3}$. In view of (4.2.1), this composition fails to be injective precisely over the points where $\sum X_i^2\partial_{X_i}$ and $\sum X_i\partial_{X_i}$ fail to span a 2-dimensional subspace of the fibre of $\sum_{i=0}^3 \mathcal{O}_{\mathbb{P}^3_{\mathbb{F}_2}}(1)\partial_{X_i}$, which exactly constitutes the vanishing of all 2×2 minors of the matrix:

$$\begin{bmatrix} X_0^2 & X_1^2 & X_2^2 & X_3^2 \\ X_0 & X_1 & X_2 & X_3 \end{bmatrix}.$$

Such minors are of the form $X_iX_j(X_i+X_j)$, for $i\neq j$, and one quickly checks that they cannot simultaneously vanish on $\mathcal{Z}_U$.

As $\mathscr{F}_{\mathcal{Z}}$ is rank 1, it is preserved under the Lie bracket, and the only remaining criterion $\mathscr{F}_{\mathcal{Z}}$ must satisfy is closure under pth powers. It suffices to verify this condition on a local generator of $\mathscr{F}_{\mathcal{Z}}$. On the chart $(X_{i_0}\neq 0)$, the sheaf $\mathscr{F}_{\mathcal{Z}}$ is generated by the differential operator

$$\theta_{\mathbb{P}} := \Theta_{\mathbb{P}}(\frac{1}{X_{i_0}}) = \frac{1}{X_{i_0}}\sum X_i^2\partial_{X_i}.$$

If $x_i := \frac{X_i}{X_{i_0}}$ are the local affine coordinates, then

$$\theta_{\mathbb{P}} = \sum_{i\neq i_0}(x_i^2+x_i)\partial_{x_i},$$

because $X_i\partial_{X_i} = x_i\partial_{x_i}$, for $i\neq i_0$, and $X_{i_0}\partial_{X_{i_0}} = \sum_{i\neq i_0} x_i\partial_{x_i}$, as can be checked by evaluation on the functions $x_j = \frac{X_j}{X_{i_0}}$. We now expand $\theta_{\mathbb{P}}^2$, taking note that all higher-order operators in the expansion are either 0 (e.g. $\partial_{x_i}^2 = 0$) or are nonzero (e.g. $\partial_{x_i}\partial_{x_j}$, $i\neq j$) but occur with even, hence 0, coefficient:

$$\begin{aligned}\theta_{\mathbb{P}}^2 &= \sum_{i\neq i_0}(x_i^2+x_i)\partial_{x_i}\circ\sum_{j\neq i_0}(x_j^2+x_j)\partial_{x_j}\\ &= \sum_{i\neq i_0}(x_i^2+x_i)\sum_{j\neq i_0}\delta_{ij}\cdot\partial_{x_j}\\ &= \theta_{\mathbb{P}}.\end{aligned}$$

Thus, $\mathscr{F}_{\mathcal{Z}}\subseteq T_{\mathcal{Z}/\mathbb{A}^4}$ is a foliation on $\mathcal{Z}_U$, and the restriction $\mathscr{F} := (\mathscr{F}_{\mathcal{Z}})|_Z$ is therefore a foliation over the extension $\mathbb{F}_2(\alpha_0,\ldots,\alpha_3)/\mathbb{F}_2$. Let $X_1 := Z/\mathscr{F}$ be the resulting quotient.

Theorem 4.2.2. *The variety X_1 constructed above is a regular del Pezzo surface over the field $H^0(X_1, \mathcal{O}_{X_1}) = \mathbb{F}_2(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ with irregularity $h^1(X_1, \mathcal{O}_{X_1}) = 1$ and degree $K_{X_1}^2 = 1$.*

Proof. The variety Z defined above is a regular del Pezzo surface with $d_Z = 8$, $\chi(\mathcal{O}_Z) = 1$. Because $H^0(Z, \mathcal{O}_Z) = \mathbb{F}_2(\alpha_0, \dots, \alpha_3)$ and X_1 is an $\mathbb{F}_2(\alpha_0, \dots, \alpha_3)$-variety, $H^0(X_1, \mathcal{O}_{X_1}) = \mathbb{F}_2(\alpha_0, \dots, \alpha_3)$ as well. Proposition 3.0.1 therefore applies with $[k_Z : k_{X_1}] = 1$, proving the theorem. ■

Remark 4.2.3. *Actually, there exists a regular del Pezzo surface X_1' of degree and irregularity 1 defined over the subfield $\mathbb{F}_2(\alpha_0, \alpha_1, \alpha_2) \subseteq \mathbb{F}_2(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$. Indeed, the closed subscheme $\mathcal{Z}_{U \cap \mathbb{A}^3} \subseteq \mathcal{Z}_U$ sitting over the inclusion $U \cap \mathbb{A}^3 \subseteq U$ given by $\alpha_3 = 1$ is smooth. The foliation $\mathscr{F}_{\mathcal{Z}}|_{\mathcal{Z}_U}$ restricts to a foliation on $\mathcal{Z}_{U \cap \mathbb{A}^3}$, and the quotient of the generic fibre by this foliation is the desired surface X_1'. Any subvariety $B \subseteq U$ of dimension strictly less than 3 gives rise to a singular closed subscheme $\mathcal{Z}_{U \cap B} \subseteq \mathcal{Z}_U$, and here our method breaks down.*

4.3 An example of degree two

Let $\mathcal{Z}_U \to U$ be the family defined in §4.1. We again choose $\mathscr{F}_{\mathcal{Z}} \cong \mathcal{O}_{\mathcal{Z}}(-1)$, but this time to be the subsheaf of $T_{\mathcal{Z}/k}$ defined by the image of

$$\Theta_{\mathbb{P}} \oplus \Theta_{\mathbb{A}} : \mathcal{O}_{\mathcal{Z}}(-1) \to \mathcal{O}_{\mathcal{Z}} \otimes (T_{\mathbb{P}^3} \oplus T_{\mathbb{A}^4}) \cong T_{\mathbb{P}^3 \times \mathbb{A}^4}|_{\mathcal{Z}},$$

for $\Theta_{\mathbb{P}} = \sum X_i^2 \partial_{X_i}$ as before, and $\Theta_{\mathbb{A}} := (\sum X_j) \sum_k \alpha_k \partial_{\alpha_k}$. Again, we work locally on the chart $(X_{i_0} \neq 0)$, with affine coordinates $x_i := \frac{X_i}{X_{i_0}}$. A local generator of $\mathscr{F}_{\mathcal{Z}}$ is given by $\theta = \theta_{\mathbb{P}} + \theta_{\mathbb{A}}$, for $\theta_{\mathbb{P}} = \sum_{i \neq i_0} (x_i + x_i^2) \partial_{x_i}$ and $\theta_{\mathbb{A}} = (1 + \sum_{i \neq i_0} x_i) \sum \alpha_j \partial_{\alpha_j}$. We saw above that the image of $\Theta_{\mathbb{P}}$ preserves the ideal sheaf $\mathscr{I}_{\mathcal{Z}}$, and therefore is contained within $T_{\mathcal{Z}/\mathbb{F}_2}$. We now check that the image of $\Theta_{\mathbb{A}}$ does as well. On the chart $(X_{i_0} \neq 0)$, the ideal $\mathscr{I}_{\mathcal{Z}}$ is generated by

$$q := \frac{1}{X_{i_0}^2} \cdot Q = \alpha_{i_0} + \sum_{i \neq i_0} \alpha_i x_i^2.$$

As $\theta_{\mathbb{A}}(q) = (1 + \sum_{i \neq i_0} x_i)q = 0$, the image of $\Theta_{\mathbb{A}}$ preserves the ideal sheaf, and therefore the image of $\Theta_{\mathbb{P}} + \Theta_{\mathbb{A}}$ is contained in $T_{\mathcal{Z}/\mathbb{F}_2}$, that is, $\mathscr{F}_{\mathcal{Z}} \subseteq T_{\mathcal{Z}/\mathbb{F}_2}$.

We next begin to show that the subsheaf $\mathscr{F}_{\mathcal{Z}} \subseteq T_{\mathcal{Z}/\mathbb{F}_2}$ is a foliation over $\mathbb{F}_2$ on $\mathcal{Z}$. In the previous section, we showed that $\Theta_{\mathbb{P}}$ is injective on fibres over $\mathcal{Z}_U$, and it immediately follows that the same is true of the sum $\Theta_{\mathbb{P}} \oplus \Theta_{\mathbb{A}}$. Hence $\mathscr{F}_{\mathcal{Z}}$ is a subbundle of $T_{\mathcal{Z}/\mathbb{F}_2}$. The Lie bracket preserves $\mathscr{F}_{\mathcal{Z}}$ simply because $\mathscr{F}_{\mathcal{Z}}$ is rank 1, and as before, our final verification is whether $\mathscr{F}_{\mathcal{Z}}$ is closed under squaring. The following local calculation shows just that:

$$\begin{aligned}
\theta^2 &= (\theta_{\mathbb{P}} + \theta_{\mathbb{A}})^2 \\
&= \theta_{\mathbb{P}}^2 + \theta_{\mathbb{P}} \circ \theta_{\mathbb{A}} + \theta_{\mathbb{A}} \circ \theta_{\mathbb{P}} + \theta_{\mathbb{A}}^2 \\
&= \theta_{\mathbb{P}} + (\sum_{i \neq i_0} x_i + x_i^2)(\sum \alpha_j \partial_{\alpha_j}) + 0 + (1 + \sum_{i \neq i_0} x_i)^2 (\sum \alpha_j \partial_{\alpha_j}) \\
&= \theta_{\mathbb{P}} + \theta_{\mathbb{A}} = \theta.
\end{aligned}$$

This proves that $\mathscr{F}_{\mathcal{Z}}$ is a foliation on $\mathcal{Z}_U$ over $\mathbb{F}_2$. Let $\mathscr{F} := (\mathscr{F}_{\mathcal{Z}})|_Z$ be the restriction of this foliation to Z. If $k := \mathbb{F}_2(\alpha_i\alpha_j : 0 \leq i, j \leq 3) \subseteq \mathbb{F}_2(\alpha_0, \ldots, \alpha_3)$, then $\mathscr{F} \subseteq T_{Z/k}$, since the image of both $\Theta_{\mathbb{P}}$ and $\Theta_{\mathbb{A}}$ kills all elements of k. Let $X_2 := Z/\mathscr{F}$ be the resulting quotient k-variety.

Theorem 4.3.1. *The variety X_2 constructed above is a regular del Pezzo surface over the field $H^0(X_2, \mathcal{O}_{X_2}) = \mathbb{F}_2(\alpha_i\alpha_j : 0 \leq i, j \leq 3) \subseteq \mathbb{F}_2(\alpha_0, \ldots, \alpha_3)$ with irregularity $h^1(X_2, \mathcal{O}_{X_2}) = 1$ and degree $K_{X_2}^2 = 2$.*

Proof. We reiterate that Z is a regular del Pezzo surface with $K_Z^2 = 8, \chi(\mathcal{O}_Z) = 1$, and $H^0(Z, \mathcal{O}_Z) = \mathbb{F}_2(\alpha_0, \ldots, \alpha_3)$. The variety X_2 is defined over the field $k = \mathbb{F}_2(\alpha_i\alpha_j : 0 \leq i, j \leq 3)$, and therefore $k \subseteq H^0(X_2, \mathcal{O}_{X_2}) \subseteq H^0(Z, \mathcal{O}_Z)$. The foliation $\mathscr{F}$ does not kill all of $H^0(Z, \mathcal{O}_Z)$, since $\theta(\alpha_0) = \alpha_0(1 + \sum_{i \neq i_0} x_i) \neq 0$. Hence, α_0 is not contained in $H^0(X_2, \mathcal{O}_{X_2})$, which is therefore a proper subfield of $H^0(Z, \mathcal{O}_Z)$ containing k. As k is of index 2 in $H^0(Z, \mathcal{O}_Z)$, the fields $H^0(X_2, \mathcal{O}_{X_2})$ and k must coincide. We conclude by applying Proposition 3.0.1 with $[k_Z : k_{X_2}] = 2$. ∎

4.4 Geometric reducedness

We conclude this chapter by proving that, of our examples constructed above, the surface of degree 1 is geometrically reduced while the surface of degree 2 is geometrically non-reduced.

Proposition 4.4.1. *The regular del Pezzo surface X_1 is geometrically reduced, but the regular del Pezzo surface X_2 is geometrically non-reduced.*

Proof. Let $k_i := H^0(X_i, \mathcal{O}_{X_i})$ denote the field of global function on X_i, for $i \in \{1, 2\}$. We will begin with the case of $i = 1$, and we will use the notation established in §4.2. Since X_1 is Cohen-Macaulay, it is geometrically reduced if and only if it is so generically, and thus suffices to prove X_1 is geometrically reduced on the affine chart over which

$$\mathcal{O}_{Z_{\bar{k}_1}} = \bar{k}_1[x_1, x_2, x_3]/\ell^2, \quad \text{with } \ell := \sqrt{\alpha_0} + \sqrt{\alpha_1} \cdot x_1 + \sqrt{\alpha_2} \cdot x_2 + \sqrt{\alpha_3} \cdot x_3.$$

The ring $R := \mathcal{O}_{(X_1)_{\bar{k}_1}}$ is the subring of $\mathcal{O}_{Z_{\bar{k}_1}}$ on which the differential $\theta_{\mathbb{P}} := \sum_{i \neq 0}(x_i + x_i^2)\partial_{x_i}$ vanishes.

For the purpose of proving that R is reduced, assume $f \in \bar{k}_1[x_1, x_2, x_3]$ lifts a nilpotent element of R. This means that, in the ring $\bar{k}_1[x_1, x_2, x_3]$, the polynomial $\theta_{\mathbb{P}}(f)$ is divisible by ℓ, and secondly, for some $n > 0$, the quadratic form ℓ^2 divides f^n, which by unique factorization implies that $f = \ell \cdot g$ for some polynomial g. Consequently, ℓ divides the product $\theta_{\mathbb{P}}(\ell) \cdot g$ due to the Leibnitz rule:

$$\theta_{\mathbb{P}}(f) = \theta_{\mathbb{P}}(\ell) \cdot g + \ell \cdot \theta_{\mathbb{P}}(g).$$

We can compute $\theta_{\mathbb{P}}(\ell)$ explicitly as

$$\begin{aligned} \theta_{\mathbb{P}}(\ell) &= \sum_{i=1}^{3}(x_i + x_i^2)\frac{\partial f}{\partial x_i} \\ &= \ell + (\sqrt{\alpha_0} + \sum_{i=1}^{3} \sqrt{\alpha_i} \cdot x_i^2). \end{aligned}$$

Consider the morphism $\bar{k}_1[x_1, x_2, x_3]/\ell \twoheadrightarrow \bar{k}_1$ defined by $x_1, x_2 \mapsto 0$, and $x_3 \mapsto \sqrt{\alpha_0/\alpha_3}$. This morphism sends $\theta_{\mathbb{P}}(\ell) \mapsto \sqrt{\alpha_0} + \alpha_0/\sqrt{\alpha_3} \neq 0$, and so ℓ does not divide $\theta_{\mathbb{P}}(\ell)$. Therefore,

ℓ must divide g, and hence ℓ^2 divides f, which implies the image of f in R was 0 to begin with. Thus R is reduced.

Now, consider $f : Z \to X_2$, as in §4.3. Let $k_2 := H^0(X_2, \mathcal{O}_{X_2})$ and $k_2' := H^0(Z, \mathcal{O}_Z)$. As the degree of the field extension is $[k_2' : k_2] = 2$, and Z is geometrically a first-order neighborhood of a plane, the variety $Z \times_{k_2} \bar{k}_2$ has generic point $\bar{\xi}_Z$ whose local ring $k_2'(Z) \otimes_{k_2} \bar{k}_2$ is Artinian of length 4. If X_2 were geometrically reduced, then $k_2(X_2) \otimes_{k_2} \bar{k}_2$ would be a field, and $k_2'(Z) \otimes_{k_2} \bar{k}_2$ a 2-dimensional vector space over this field, with length at most 2, yielding a contradiction. ■

Chapter 5

A geometric description of the surface of degree one

In this chapter we study, through explicit computation, the regular del Pezzo surface X_1 over the field $\mathbb{F}_2(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ constructed in §4.2. Although Remark 4.2.3 asserts that there exists an analogous regular del Pezzo surface X_1' defined over the subfield $\mathbb{F}_2(\alpha_0, \alpha_1, \alpha_2)$, for the sake of symmetry in our calculations, we will restrict our attention to the surface X_1, which for convenience we will henceforth denote by X.

The surface X is geometrically integral and is of anti-canonical degree and irregularity one: $K_X^2 = 1$, $h^1(X, \mathcal{O}_X) = 1$. By Reid's classification of non-normal del Pezzo surfaces [35], the normalization of the geometric base change $X_{\bar{k}}$ is isomorphic to the projective plane, $X_{\bar{k}}^{\nu} \cong \mathbb{P}^2_{\bar{k}}$, and the normalization morphism consists of the collapse of a double line onto a cuspidal curve $C \subseteq X_{\bar{k}}$ of arithmetic genus $h^1(C, \mathcal{O}_C) = 1$. The upshot of our calculations is a concrete realization of this description of $X_{\bar{k}}$ in terms of our construction of X as the quotient by a foliation:

Proposition 5.0.1. *Let $k := \mathbb{F}_2(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ and $Z \to X$ denote the quotient morphism from the regular variety $Z := (\sum \alpha_i X_i^2 = 0) \subseteq \mathbb{P}^3_k$, defined by the foliation described in §4.2.*

1. *The reduced scheme $Z_{\bar{k}}^{red}$ is the hyperplane $(\sum \sqrt{\alpha_i} X_i = 0) \subseteq \mathbb{P}^3_{\bar{k}}$, and the induced morphism $Z_{\bar{k}}^{red} \to X_{\bar{k}}$ is the normalization of the variety $X_{\bar{k}}$.*

2. The singular locus of $X_{\bar{k}}$ is a rational cuspidal curve C of arithmetic genus one.

3. The inverse image of C in $Z^{red}_{\bar{k}}$ is the double line D described by the equation

$$(\sum \sqrt[4]{\alpha_i} X_i)^2 = 0.$$

4. The cusp of C sits below the unique point on D satisfying the additional equation

$$\sum \sqrt[8]{\alpha_i} X_i = 0.$$

This is proven in stages throughout the following sections.

5.1 Normalization of geometric base change

We recall the notation established in §4.2. The variety $Z := (\sum_{i=0}^{3} \alpha_i X_i^2 = 0) \subseteq \mathbb{P}^3_k$ is a regular del Pezzo surface over the field $k := \mathbb{F}_2(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, and $\mathscr{F} = \mathrm{Im}(\Theta_{\mathbb{P}}) \subseteq T_{Z/k}$ is the foliation on Z over the extension $k/\mathbb{F}_2$ defined by $\Theta_{\mathbb{P}} := \sum_{i=0}^{3} X_i^2 \partial_{X_i}$. Recall that X was defined as the quotient $X = Z/\mathscr{F}$ and as before $f : Z \to X$ will denote the quotient morphism.

Proposition 5.1.1. *The relative Frobenius morphism $\mathbf{F}_{Z/k}$ factors as*

$$\begin{tikzcd}
 & Z \arrow[d, "f"] \arrow[dl, "\bar{\mathbf{F}}_{Z/k}"'] \arrow[dd, bend left, "\mathbf{F}_{Z/k}"] \\
 & X \arrow[d, "g"] \arrow[dl, "\bar{g}"'] \\
(Z \times_{k,\mathbf{F}_k} k)^{red} \arrow[r] & Z \times_{k,\mathbf{F}_k} k,
\end{tikzcd} \tag{5.1.2}$$

with morphisms $\bar{\mathbf{F}}_{Z/k}$, $\bar{g}$, and f flat and finite with respective degrees $8, 4,$ and 2. The geometric base change of the top triangle admits a further factorization,

$$\begin{tikzcd}
Z^{red}_{\bar{k}} \arrow[r] \arrow[d, "h"'] \arrow[dr, "\bar{f}_{\bar{k}}"] & Z_{\bar{k}} \arrow[d, "f_{\bar{k}}"] \\
(Z \times_{k,\mathbf{F}_k} \bar{k})^{red} & X_{\bar{k}}, \arrow[l, "\bar{g}_{\bar{k}}"']
\end{tikzcd} \tag{5.1.3}$$

where the morphism $\bar{f}_{\bar{k}} : Z^{red}_{\bar{k}} \to X_{\bar{k}}$ identifies $Z^{red}_{\bar{k}} \cong \mathbb{P}^2_{\bar{k}}$ with the normalization of the variety $X_{\bar{k}}$.

Proof. Diagram (5.1.2) clearly exists and commutes since both Z and X are regular varieties and hence reduced schemes. By Proposition 3.1.4, the morphism $f : Z \to X$ is flat and finite of degree 2.

We next make computations on the affine chart $U = (X_0 \neq 0)$, and by symmetry, analogous assertions are true over any chart of the form $(X_i \neq 0)$. Restricted to U, the top triangle of (5.1.2) is dual to the following triangle of k-algebra morphisms:

$$\begin{tikzcd} & M := k[x_1, x_2, x_3]/(\alpha_0 + \sum \alpha_i x_i^2) \\ S := k[u_1, u_2, u_3]/(\alpha_0 + \sum \alpha_i u_i) \arrow[ur, "\bar{\mathbf{F}}^{\sharp}_{Z/k}"] \arrow[r, "\bar{g}^{\sharp}"] & R := \mathcal{O}_X|_U, \arrow[u, "f^{\sharp}"'] \end{tikzcd} \tag{5.1.4}$$

where $\bar{\mathbf{F}}^{\sharp}_{Z/k}$ is given by $u_i \mapsto x_i^2$. It is easy to check that $\bar{\mathbf{F}}_{Z/k}$ is flat and finite of rank 8 because M is a rank 8 free S-module with basis $\langle 1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3\rangle$. Also, since f is flat and surjective, it is faithfully flat. Thus, $\bar{g}$ is flat, and hence finite of degree $8/2 = 4$.

To finish the proof, consider the geometric base change diagram (5.1.3). The morphism h, given explicitly as a morphism between the hyperplanes $Z_{\bar{k}}^{\mathrm{red}} \cong (\sum_{i=0}^{3} \sqrt{\alpha_i} X_i = 0)$ and $(Z \times_{k,\mathbf{F}_k} \bar{k})^{\mathrm{red}} \cong (\sum_{i=0}^{3} \alpha_i U_i = 0)$, is defined by the rule $[X_0 : X_1 : X_2 : X_3] \mapsto [X_0^2 : X_1^2 : X_2^2 : X_3^2]$. This is easily seen to be a finite dominant morphism of degree 4. The morphism $\bar{g}_{\bar{k}}$ is also a dominant morphism of degree 4 that factors h. This implies that $\bar{f}_{\bar{k}}$ is finite of degree 1, and hence a birational morphism. Since $Z_{\bar{k}}^{\mathrm{red}}$ is a hyperplane in $\mathbb{P}^3_{\bar{k}}$, it is isomorphic to $\mathbb{P}^2_{\bar{k}}$, and thus $\bar{f}_{\bar{k}}$ is a normalization morphism. ∎

5.2 Local ring of functions

We compute $\mathcal{O}_X$ on an affine chart $(X_{i_0} \neq 0) \subseteq Z$, but for simplicity we assume $i_0 = 0$, as the computations on other charts are analogous by symmetry.

Proposition 5.2.1. *On the open $(X_0 \neq 0) \subseteq Z$, the ring of functions has presentation*

$$\mathcal{O}_X|_{(X_0 \neq 0)} = k[u_1, u_2, u_3, t_1, t_2, t_3]/(r_0, \ldots, r_6),$$

with the relations r_i defined as:

$$\begin{aligned}
r_0 &:= \alpha_0 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 & r_4 &:= t_2 t_3 + u_1 u_2 u_3 + (u_1 + u_1^2) t_1 + u_1 u_2 t_2 + u_1 u_3 t_3 \\
r_1 &:= t_1^2 + u_2 u_3 + u_2 u_3^2 + u_2^2 u_3 & r_5 &:= t_1 t_3 + u_1 u_2 u_3 + u_1 u_2 t_1 + (u_2 + u_2^2) t_2 + u_2 u_3 t_3 \\
r_2 &:= t_2^2 + u_1 u_3 + u_1^2 u_3 + u_1 u_3^2 & r_6 &:= t_1 t_2 + u_1 u_2 u_3 + u_1 u_3 t_1 + u_2 u_3 t_2 + (u_3 + u_3^2) t_3. \\
r_3 &:= t_3^2 + u_1 u_2 + u_1^2 u_2 + u_1 u_2^2 & &
\end{aligned}$$

Moreover, the inclusion of algebras $\mathcal{O}_X \subseteq \mathcal{O}_Z$ dual to the morphism $f : Z \to X$ is given by

$$k[u_1, u_2, u_3, t_1, t_2, t_3]/(r_0, \ldots, r_6) \to k[x_1, x_2, x_3]/(\sum \alpha_i x_i^2),$$

via $u_i \mapsto x_i^2$ and $t_i \mapsto x_j x_k(1 + x_j + x_k)$, for each assignment of indices $\{i, j, k\} = \{1, 2, 3\}$.

Proof. Recall the diagram (5.1.4), and the notation established there. The S-algebra $R = \mathcal{O}_X|_{(X_0 \neq 0)}$ is flat and hence projective as an S-module. As S is isomorphic to a polynomial ring in two variables, over which all projective modules are free, R is actually a free S-submodule of rank 4 of the free S-module M of rank 8 with basis $\langle 1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3 \rangle$.

The derivation $\theta_{\mathbb{P}} := \Theta_{\mathbb{P}}|_{(X_0 \neq 0)} = \sum_{i \neq 0}(x_i + x_i^2)\partial_{x_i}$ is S-linear because $S \subseteq R = \ker(\theta_{\mathbb{P}})$. Therefore, we may compute its matrix as an S-module morphism $M \xrightarrow{\theta_{\mathbb{P}}} M$:

$$\begin{array}{r c}
 & \begin{array}{cccccccc} 1 & x_1 & x_2 & x_3 & x_1x_2 & x_1x_3 & x_2x_3 & x_1x_2x_3 \end{array} \\
\begin{array}{r} 1 \\ x_1 \\ x_2 \\ x_3 \\ x_1x_2 \\ x_1x_3 \\ x_2x_3 \\ x_1x_2x_3 \end{array} &
\left(\begin{array}{c|ccc|ccc|c}
 & x_1^2 & x_2^2 & x_3^2 & & & & \\ \hline
 & 1 & & & x_2^2 & x_3^2 & & \\
 & & 1 & & x_1^2 & & x_3^2 & \\
 & & & 1 & & x_1^2 & x_2^2 & \\ \hline
 & & & & & & & x_3^2 \\
 & & & & & & & x_2^2 \\
 & & & & & & & x_1^2 \\ \hline
 & & & & & & & 1
\end{array}\right)
\end{array}$$

This is a block matrix, which makes computing its kernel easy:

$$\begin{pmatrix} 0 & A & 0 & 0 \\ 0 & 1 & B & 0 \\ 0 & 0 & 0 & C \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} Av_2 \\ v_2 + Bv_3 \\ Cv_4 \\ v_4 \end{pmatrix},$$

and this vector equals zero if and only if $v_4 = 0, v_2 = Bv_3$ and $ABv_3 = 0$. In our situation, the matrix $AB = 0$, and so we see that the kernel of M defined by $v_4 = 0, v_2 = Bv_3$. Thus, a basis of the kernel is given by the four elements

$$\begin{aligned} \phi(t_0) :=& \quad 1 \\ \phi(t_1) :=& \quad x_2x_3 + x_2^2x_3 + x_2x_3^2, \\ \phi(t_2) :=& \quad x_1x_3 + x_1^2x_3 + x_1x_3^2, \\ \phi(t_3) :=& \quad x_1x_2 + x_1^2x_2 + x_1x_2^2, \end{aligned}$$

and so $R = k[x_1^2, x_2^2, x_3^2, t_1, t_2, t_3] \subseteq k[x_1, x_2, x_3]/(\sum \alpha_i x_i^2)$.

Clearly, there is a surjective morphism ϕ from the polynomial algebra $k[u_1, u_2, u_3, t_1, t_2, t_3]$ onto R, defined by the rules $u_i \mapsto x_i^2$ and $t_i \mapsto \phi(t_i)$. The relations $r_0, \ldots, r_6$ listed above may be verified to be in $\ker \phi$ simply by writing the multiplication rules for the S-basis $\langle \phi(t_0), \phi(t_1), \phi(t_2), \phi(t_3) \rangle$. As a result, there is an induced surjective map of S-algebras,

$$\bar{\phi} : k[u_1, u_2, u_3, t_1, t_2, t_3]/(r_0, \ldots, r_6) \twoheadrightarrow R.$$

The domain is a free S-module with basis $\langle 1, t_1, t_2, t_3 \rangle$, since all monomials in the t_i can be written as S-linear combinations of these elements modulo the relations r_i. Therefore, $\bar{\phi}$ is an isomorphism. ■

5.3 An equation defining the singular locus

We apply the Jacobian criterion to the presentation of $R = \mathcal{O}_X|_{(X_0 \neq 0)}$ given in Proposition 5.2.1 to find the set of non-smooth points of X. It turns out that these points can be described set-theoretically as the vanishing locus of a single equation.

Proposition 5.3.1. *The non-smooth locus X^{sing} of X is set-theoretically equal to the codimension-1 locus defined by the single equation $\alpha_0+\alpha_1 u_1^2+\alpha_2 u_2^2+\alpha_3 u_3^2 = 0$. In particular, X is not geometrically normal.*

Proof. The Jacobian matrix is as follows:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \partial_{u_1} & \partial_{u_2} & \partial_{u_3} & \partial_{t_1} & \partial_{t_2} & \partial_{t_3} \end{array} \\
\begin{array}{c} r_0 \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{array} &
\left(\begin{array}{ccc|ccc}
\alpha_1 & \alpha_2 & \alpha_3 & & & \\
 & u_3+u_3^2 & u_2+u_2^2 & & & \\
u_3+u_3^2 & & u_1+u_1^2 & & & \\
u_2+u_2^2 & u_1+u_1^2 & & & & \\
\hline
 & & & u_1+u_1^2 & t_3+u_1u_2 & t_2+u_1u_3 \\
 & (*) & & t_3+u_1u_2 & u_2+u_2^2 & t_1+u_2u_3 \\
 & & & t_2+u_1u_3 & t_1+u_2u_3 & u_3+u_3^2
\end{array}\right)
\end{array}
$$

As R is a surface described in a 6-dimensional affine space, the singular locus is described by the ideal generated by the 4×4-minors of this matrix. As the Jacobian matrix comprises blocks in the form

$$
\begin{pmatrix} A & 0 \\ * & B \end{pmatrix},
$$

with $A = A^{3\times 4}$ and $B = B^{3\times 3}$, its 4×4 minors are either the product of two 2×2 minors of A and B, the product of an entry of A by the determinant of B, or the product of a 3×3 minor of A by an entry of B. Initially, the task of computing this ideal may appear daunting, but the following observation reduces the work dramatically.

Lemma 5.3.2. *Let B be the 3×3 matrix defined above.*

1. *The 2×2 minors of B are 0 in R.*

2. *The diagonal entries of B generate the unit ideal in R.*

Proof. Up to cyclic permutations of the indices $\{1,2,3\}$, there are only two types of 2×2-minors of B. A minor of the first type is $B_{3,3}$:

$$\begin{aligned} B_{3,3} &= (t_2+u_1u_3)(u_2+u_2^2)+(t_3+u_1u_2)(t_1+u_2u_3) \\ &= u_1u_2u_3+u_1u_2^2u_3+(u_2+u_2^2)t_2+t_1t_3+u_2u_3t_3+u_1u_2t_1+u_1u_2^2u_3 \\ &= r_5 = 0. \end{aligned}$$

A minor of the second type is $B_{1,3}$:

$$\begin{aligned} B_{1,3} &= (t_1+u_2u_3)^2+(u_3+u_3^2)(u_2+u_2^2) \\ &= t_1^2+u_2^2u_3^2+u_2u_3+u_2^2u_3+u_2u_3^2+u_2^2u_3^2 \\ &= r_1 = 0. \end{aligned}$$

This proves (1).

For (2), assume otherwise, and let $\mathfrak{m}$ be a maximal ideal containing the ideal generated by the entries of B. In the residue field $\kappa := R/\mathfrak{m}$, the image of the entry $u_i+u_i^2$ is 0, forcing $u_i=\varepsilon_i\in\kappa$, for $\varepsilon_i\in\{0,1\}$. The relation $r_0=0$ implies $\alpha_0+\varepsilon_1\alpha_1+\varepsilon_2\alpha_2+\varepsilon_3\alpha_3=0$ in $k\subseteq\kappa$, which contradicts the algebraic independence of the α_i's. ∎

From this lemma, it follows that the ideal generated by 4×4 minors of M is generated by the 3×3 minors of A. Denoting $h:=\alpha_0+\alpha_1u_1^2+\alpha_2u_2^2+\alpha_3u_3^2$, these minors of A are $A_0=0, A_1=(u_1+u_1^2)h, A_2=(u_2+u_2^2)h, A_3=(u_3+u_3^2)h$. Lemma 5.3.2(2) shows these minors generate the principal ideal (h). ∎

5.4 The geometry of the singular locus

Let C be the reduced subscheme corresponding to the non-smooth locus $X_{\bar{k}}^{\mathrm{sing}}\subseteq X_{\bar{k}}$. By Proposition 5.3.1, the curve C is set-theoretically cut out by the equation $\alpha_0+\alpha_1u_1^2+\alpha_2u_2^2+\alpha_3u_3^2=0$, which is simply the square of the equation $\sqrt{h}=0$ for

$$\sqrt{h} := \sqrt{\alpha_0}+\sqrt{\alpha_1}u_1+\sqrt{\alpha_2}u_2+\sqrt{\alpha_3}u_3.$$

We expect this equation to be insufficient to describe C scheme-theoretically, because X is not smooth along this locus, so the maximum ideal of the local ring $\mathcal{O}_{X,C}$ requires more than one generator. This is indeed the case, and the structure of C is as follows:

Proposition 5.4.1. *The curve C is isomorphic to a rational cuspidal curve with $h^1(C,\mathcal{O}_C) = 1$. The singular point of the curve sits below the point in $Z_{\bar{k}} \subseteq \mathbb{P}^3_{\bar{k}}$ described by the intersection of the three planes $(\sum \alpha_i^{1/2^j} X_i = 0)$, for $j = 1, 2, 3$.*

Proof of 5.4.1. Again we work over the chart $(X_0 \neq 0)$, and by symmetry our results will carry over to other opens $(X_i \neq 0)$. We must compute the quotient of the ring $R_{\bar{k}}/(\sqrt{h})$ by its nilradical ideal.

$$\bar{k}[u_1, u_2, u_3, t_1, t_2, t_3]/(\sqrt{h}, r_0, r_1, \ldots, r_6).$$

The first two relations $r_0 = \alpha_0 + \sum_{i=1}^3 \alpha_i u_i$ and $\sqrt{h} = \sqrt{\alpha}_0 + \sum_{i=1}^3 \sqrt{\alpha_i} u_i$ are $\bar{k}$-linearly independent relations. Therefore, in the ring $R_{\bar{k}}/(\sqrt{h})$, we can solve for u_2 and u_3 in terms of u_1, and rewrite

$$R_{\bar{k}}/(\sqrt{h}) \cong \bar{k}[u, t_1, t_2, t_3]/(r_1, \ldots, r_6),$$

where the variable u_1 is replaced by u and the variables u_2 and u_3 are replaced by the following expressions in u:

$$u_2(u) = \tfrac{(\alpha_0 + \sqrt{\alpha_0\alpha_3}) + (\alpha_1 + \sqrt{\alpha_1\alpha_3})u}{\alpha_2 + \sqrt{\alpha_2\alpha_3}}, \quad u_3(u) = \tfrac{\alpha_0 + \alpha_1 u + \alpha_2 u_2(u)}{\alpha_3}.$$

Hence the relations r_1, r_2, and r_3 read

$$\begin{aligned} r_1 &= \; t_1^2 + c_{10} &&+ c_{11}u + c_{12}u^2 + c_{13}u^3 \\ r_2 &= \; t_2^2 + 0 &&+ c_{21}u + c_{22}u^2 + c_{23}u^3 \\ r_3 &= \; t_3^2 + 0 &&+ c_{31}u + c_{32}u^2 + c_{33}u^3, \end{aligned}$$

for explicitly determined coefficients $c_{ij} \in \bar{k}$ whose concrete description, for the sake of exposition, will be omitted but made available in an auxiliary file on the author's homepage. When written explicitly, it is straight-forward to check that these coefficients, for any pair $i, j \in \{1, 2, 3\}$, satisfy the following relation:

$$c_{i1}c_{j3} + c_{j1}c_{i3} = 0. \tag{5.4.2}$$

Make the following change of variables

$$\begin{aligned} s_1 &:= t_1 + \sqrt{c_{10}} + \sqrt{c_{12}}u \\ s_2 &:= t_2 + \sqrt{c_{22}}u \\ s_3 &:= t_3 + \sqrt{c_{32}}u, \end{aligned}$$

so that the relations r_1, r_2, r_3 become

$$\begin{aligned} r_1 = &\quad s_1^2 + c_{11}u + c_{13}u^3 \\ r_2 = &\quad s_2^2 + c_{21}u + c_{23}u^3 \\ r_3 = &\quad s_3^2 + c_{31}u + c_{33}u^3. \end{aligned}$$

The relations (5.4.2) imply $s_2^2 = \frac{c_{21}}{c_{11}}s_1^2$ and $s_3^2 = \frac{c_{31}}{c_{11}}s_1^2$, so the nilradical of $R_{\bar{k}}/(\sqrt{h})$ must contain the relations $r_2' := s_2 + \frac{\sqrt{c_{21}}}{\sqrt{c_{11}}}s_1$ and $r_3' := s_3 + \frac{\sqrt{c_{31}}}{\sqrt{c_{11}}}s_1$. By setting $s := s_1$, we obtain an isomorphism

$$R_{\bar{k}}/(\sqrt{h}, r_2', r_3') \cong \bar{k}[u, s]/(s^2 + u(c_{11} + c_{13}u^2), r_4, r_5, r_6).$$

Since $\bar{k}[u, s]/(s^2 + u(c_{11} + c_{13})u^2)$ is an integral domain of dimension 1, the relations r_4, r_5, r_6 are already 0 in this ring. Thus,

$$\mathcal{O}_C|_{(X_0 \neq 0)} = \bar{k}[u, \sqrt{u}(u + \sqrt{c_{11}/c_{13}})] \subseteq \bar{k}[\sqrt{u}].$$

From this description, is is clear that the only singular point of C is an ordinary cuspidal singularity of (wild) order 2 occurring at

$$X_1^2/X_0^2 = u = \sqrt{c_{11}/c_{13}}.$$

Moreover, one can verify that $\sqrt[4]{c_{11}/c_{13}} = \det(A_1)/\det(A_0)$ where the matrix A_1 is defined by replacing the first column of the following matrix A by the vector b:

$$A := \begin{pmatrix} \sqrt{\alpha_1} & \sqrt{\alpha_2} & \sqrt{\alpha_3} \\ \sqrt[4]{\alpha_1} & \sqrt[4]{\alpha_2} & \sqrt[4]{\alpha_3} \\ \sqrt[8]{\alpha_1} & \sqrt[8]{\alpha_2} & \sqrt[8]{\alpha_3} \end{pmatrix}, \qquad b := \begin{pmatrix} \sqrt{\alpha_0} \\ \sqrt[4]{\alpha_0} \\ \sqrt[8]{\alpha_0} \end{pmatrix}.$$

Cramer's rule, implies that the cusp of C sits below the intersection of the 3 planes

$$(\sum \sqrt[2^j]{\alpha_i}X_i = 0) \subseteq \mathbb{P}^3_{\bar{k}}, \text{ for } j = 1, 2, 3.$$

By symmetry, this is the only singular point of C. ■

Chapter 6

Future research directions

Question 6.1. *Are there regular del Pezzo surfaces with positive irregularity in higher characteristic, that is, for $p \geq 3$?*

The inequality $q \geq \frac{d(p^2-1)}{6}$ of (1.2.2) relating the degree and irregularity becomes stronger as the characteristic grows, but it does not rule out the existence of such surfaces in any given characteristic. However, the author would find it surprising if examples exist in characteristic $p \geq 5$.

Question 6.2. *Are there regular del Pezzo surfaces X with positive irregularity over fields $k_X = H^0(X, \mathcal{O}_X)$ of inseparable degree $[k_X : k_X^p] \leq p^2$?*

As pointed out in Remark 4.2.3, the geometrically integral example X_1 may be constructed in characteristic 2 over a field of inseparable degree 2^3, and the geometrically non-reduced example was constructed over a field of inseparable degree 2^4. The case $[k_X : k_X^p] = p$ directly addresses a question of Kollár concerning 3-fold contractions [25, Rem. 1.2].

Question 6.3. *What is the geometry of the reduced structure on the geometric base change of the example X_2 constructed in §4.3?*

Presumably, one could explicitly compute local presentations of the ring of regular functions on $(X_2)^{\mathrm{red}}_{\bar{k}}$, as we did for the example X_1 in Chapter 5. This is left as an open exercise.

Part II

Intersection numbers on quotients in geometric invariant theory

Chapter 7

Introduction

7.1 A brief history

The cohomology of quotients arising from geometric invariant theory (GIT) has been the object of extensive study. In 1984, Kirwan [23] integrated the previous works of Hesselink, [17], Kempf [22], and Kempf and Ness [21] to explore the structure of GIT quotients from both the algebraic and symplectic perspectives, ultimately finding formulas to compute Hodge numbers. Five years later, Ellingsrud and Strømme [11] began to study the relationship between the Chow rings of the two GIT quotients $\mathbb{P}^n_{\bar{k}} /\!\!/ G$ and $\mathbb{P}^n_{\bar{k}} /\!\!/ T$, for a reductive group G over an algebraically closed field $\bar{k}$ with maximal torus $T \subseteq G$ acting on $\mathbb{P}^n_{\bar{k}}$ so that all semi-stable points have trivial stabilizers; their main result was a presentation of the Chow ring $A^*(\mathbb{P}^n /\!\!/ G)_{\mathbb{Q}}$ in terms of explicit generators and relations. Brion [3] then expanded this relationship to arbitrary linear actions of connected reductive groups G on smooth, projective varieties X over the complex numbers, proving that the G-equivariant cohomology of the locus of G-semi-stable points is isomorphic to the subgroup of Weyl anti-invariant classes of the T-equivariant cohomology group of the larger locus of T-semi-stable points:

$$\phi : H^*_G(X^{ss}_G; \mathbb{Q}) \xrightarrow{\cong} H^*_T(X^{ss}_T; \mathbb{Q})^a.$$

Later Brion and Joshua [5] extended these results further to the case of singular X, but with equivariant *intersection* cohomology used as a suitable replacement for the standard theory.

Brion's construction of the isomorphism ϕ is as follows (cf. [3]). As X_G^{ss} is a G-variety, the inclusion $T \subseteq G$ induces a homomorphism $\pi^* : H_G^*(X_G^{ss};\mathbb{Q}) \to H_T^*(X_G^{ss};\mathbb{Q})$. Because T is a maximal torus, π^* yields an isomorphism onto the submodule of W-invariant elements:

$$\pi : H_G^*(X_G^{ss};\mathbb{Q}) \xrightarrow{\cong} H_T^*(X_G^{ss};\mathbb{Q})^W.$$

Moreover, there is a W-equivariant isomorphism

$$H_T^*(X_G^{ss};\mathbb{Q}) \cong S \otimes_{S^W} H_G^*(X_G^{ss};\mathbb{Q}),$$

where $S := H_T^*(\operatorname{Spec}\mathbb{C};\mathbb{Q})$ is the T-equivariant cohomology of the point $\operatorname{Spec}\mathbb{C}$, and under this identification π^* becomes $1 \otimes \mathrm{id}$. The W-anti-invariant elements $S^a \subseteq S$ form a free module of rank 1 over the subring S^W of Weyl-invariant elements, and a generator is given by

$$\Delta := c_{\mathrm{top}}(\mathfrak{g}/\mathfrak{b}),$$

the top equivariant Chern class of the adjoint representation on $\mathfrak{g}/\mathfrak{b}$, where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{b}$ is the Lie algebra of a Borel subgroup containing T. Therefore, $\Delta \smile \pi^*(-)$ gives an isomorphism from $H_G^*(X_G^{ss};\mathbb{Q})$ onto the submodule of W-anti-invariant elements of $H_T^*(X_G^{ss};\mathbb{Q})$,

$$\Delta \smile \pi^*(-) : H_G^*(X_G^{ss};\mathbb{Q}) \xrightarrow{\cong} H_T^*(X_G^{ss};\mathbb{Q})^a.$$

The open inclusion $i : X_G^{ss} \hookrightarrow X_T^{ss}$ induces a homomorphism $i^* : H_T^*(X_T^{ss};\mathbb{Q}) \to H_T^*(X_G^{ss};\mathbb{Q})$, and Brion's key observation is that i^* is an isomorphism on the W-anti-invariant submodules:

$$i^* : H_T^*(X_T^{ss};\mathbb{Q})^a \xrightarrow{\cong} H_T^*(X_G^{ss};\mathbb{Q})^a.$$

The composition $\phi := (i^*)^{-1} \circ (\Delta \smile \pi^*)$ yields the desired isomorphism. Explicitly, if $\tilde{\sigma} \in H_T^*(X_T^{ss};\mathbb{Q})^W$ denotes some W-invariant lift of the class $\sigma \in H_G^*(X_G^{ss};\mathbb{Q})$, that is if $i^*\tilde{\sigma} = \pi^*\sigma$, then ϕ can be described as

$$\phi : \sigma \mapsto \Delta \smile \tilde{\sigma}.$$

7.2 The main goal

This thesis addresses the question of how the isomorphism ϕ interacts with the intersection pairings on the GIT quotients $X/\!\!/G$ and $X/\!\!/T$. When $X^{ss}_G = X^s_G$, there is a natural identification between the equivariant cohomology groups of the semi-stable locus and the ordinary cohomology groups of the GIT quotient,

$$H^*_G(X^{ss}_G;\mathbb{Q}) \cong H^*(X/\!\!/G;\mathbb{Q}),$$

(and similarly with T in place of G). For any $\sigma_1, \sigma_2 \in H^*(X/\!\!/G;\mathbb{Q})$, one can then, in such a case, compare the integrals

$$\int_{X/\!\!/G} \sigma_1 \smile \sigma_2 \quad \overset{?}{\leftrightarrow} \quad \int_{X/\!\!/T} \phi(\sigma_1) \smile \phi(\sigma_2).$$

Because $\phi(\sigma_1) \smile \phi(\sigma_2) = (\Delta \smile \Delta) \smile (\tilde{\sigma}_1 \smile \tilde{\sigma}_2)$ and $i^*(\tilde{\sigma}_1 \smile \tilde{\sigma}_2) = \pi^*(\sigma_1 \smile \sigma_2)$, we may simplify the expression by defining $\sigma := \sigma_1 \smile \sigma_2$. Moreover, it is more natural to consider the class $c_{\mathrm{top}}(\mathfrak{g}/\mathfrak{t})$, where $\mathfrak{t}$ is the Lie algebra of T, instead of $\Delta \smile \Delta$, which just differs from the former by the sign $(-1)^{|\Phi|/2}$. After these substitutions, the question becomes the comparison of the integrals $\int_{X/\!\!/G} \sigma$ and $\int_{X/\!\!/T} c_{\mathrm{top}}(\mathfrak{g}/\mathfrak{t}) \smile \tilde{\sigma}$ for $\sigma \in H^*(X/\!\!/G;\mathbb{Q})$.

Within an unpublished manuscript, Martin [28] provided an answer to the symplecto-geometric analogue of this question. There he proved the following formula for Hamiltonian actions of connected compact Lie groups G on symplectic manifolds X for which the moment map is proper and has 0 as a regular value:

$$\int_{X/\!\!/G} \sigma = \frac{1}{|W|} \int_{X/\!\!/T} c_{\mathrm{top}}(\mathfrak{g}/\mathfrak{t}) \smile \tilde{\sigma},$$

with $X/\!\!/G$ and $X/\!\!/T$ here denoting the symplectic reductions. Martin's formula may be deduced from the Jeffrey-Kirwan-Witten non-abelian localization theorem (cf. [20; 43]), although the proof Martin gave is much simpler than the proof of the general theorem (cf. [20, Thm. 8.1]). In addition to Martin's work [28], there has been a large body of literature devoted to understanding non-abelian localization; alternative approaches to the theorem may be found in the works of Guillemin and Kalkman [15], Paradan [33], and Vergne [41]. We note that the methods used in these works are analytic, and hence bound to characteristic 0, while the methods used in the works of Brion, Ellingsrud-Strømme, and Brion-Joshua referenced above are algebraic.

7.3 New results

This article generalizes Martin's result to the algebraic setting of varieties X over an arbitrary field k. Let G be a reductive group over k, with a maximal torus $T \subseteq G$ and Weyl group W, and let X be any projective, G-linearized (and possibly singular) variety over k for which $X^s_T = X^{ss}_T \neq \emptyset$. For any Chow 0-cycle $\sigma \in A_0(X/\!\!/\mathbb{G})_{\mathbb{Q}}$ with non-zero degree, we are led to define the *GIT integration ratio*,

$$r^{X,\sigma}_{G,T} := \frac{\int_{X/\!\!/T} c_{\mathrm{top}}(\mathfrak{g}/\mathfrak{t}) \frown \tilde{\sigma}}{\int_{X/\!\!/G} \sigma} \in \mathbb{Q},$$

where the Chow class $\tilde{\sigma} \in A_*(X/\!\!/T)_{\mathbb{Q}}$ denotes an arbitrary lift of the class σ (cf. Defn. 8.2.2). Understanding the invariance properties of this ratio will guide us to the proper generalization of Martin's theorem. The ratio $r^{X,\sigma}_{G,T}$ may depend on the choice of lift $\tilde{\sigma}$ or — as seems *a priori* more likely — the variety X, but neither is the case. This is our first main result (proved in Chapter 10):

Theorem 7.3.1. *Let G be a reductive group over an arbitrary field and $T \subseteq G$ a maximal torus. If X is a projective G-linearized variety satisfying $X^{ss}_T = X^s_T$ and $\sigma \in A_0(X/\!\!/G)$ is a Chow 0-cycle satisfying $\int_{X/\!\!/G} \sigma \neq 0$, then the GIT integration ratio $r^{X,\sigma}_{G,T}$ defined as above depends not on the choice of σ, T, or X. That is, $r_G := r^{X,\sigma}_{G,T}$ is an invariant of the group G.*

The GIT integration ratio is multiplicative under the group operation of direct product and invariant under central extension. This is the content of our second main result (proved in Chapter 11):

Theorem 7.3.2. *If G is a reductive group over a field k that, up to central extension, is the product of simple groups $G_1 \times \cdots \times G_n$, then*

$$r_G = \prod_{i=1}^{n} r_{G_i}.$$

As a result of these theorems, the determination of the value r_G for connected reductive groups G is reduced to the computation of r_G on a single example for each simple group. We do this explicitly in Chapter 12 for the simple group $G = PGL(n)$, where we verify $r_G = n! = |W|$. Thus, we obtain a strictly algebraic proof of the following corollary:

Corollary 7.3.3. *Let G be a connected reductive group over a field k and $T \subseteq G$ a maximal torus. If the root system of G decomposes into irreducible root systems of type $\mathbf{A}_n$, for various $n \in \mathbb{N}$, then for any G-linearized projective k-variety X for which $X^s_T = X^{ss}_T$ and any Chow class $\sigma \in A_0(X/\!\!/G)_{\mathbb{Q}}$ with lift $\tilde{\sigma} \in A_*(X/\!\!/T)_{\mathbb{Q}}$,*

$$\int_{X/\!\!/G} \sigma = \frac{1}{|W|} \int_{X/\!\!/T} c_{top}(\mathfrak{g}/\mathfrak{t}) \frown \tilde{\sigma}.$$

We can do even better than Corollary 7.3.3 if we do not restrict ourselves to purely algebraic arguments. Indeed, we may remove the restriction on root systems, proving that the GIT integration ratio is equal to the order of the Weyl group for any connected reductive group:

Amplification 7.3.4. *Let G be a connected reductive group over a field k and $T \subseteq G$ a maximal torus. For any G-linearized projective k-variety X for which $X^s_T = X^{ss}_T$ and any Chow class $\sigma \in A_0(X/\!\!/G)_{\mathbb{Q}}$ with lift $\tilde{\sigma} \in A_*(X/\!\!/T)_{\mathbb{Q}}$,*

$$\int_{X/\!\!/G} \sigma = \frac{1}{|W|} \int_{X/\!\!/T} c_{top}(\mathfrak{g}/\mathfrak{t}) \frown \tilde{\sigma}.$$

The proof of this result utilizes the theories of relative GIT (cf. [38]) and specialization (cf. [12, §20.3]), along with Theorem 7.3.1, to reduce the proof to the case of connected reductive groups over the complex numbers, where Martin's result [28, Thm. B] applies. This argument is found near the end of this part in Chapter 13.

Chapter 8

Lifting classes on smooth varieties

The goal of this chapter is to prove if X is a smooth G-linearized variety over an arbitrary field k for which $X_T^{ss} = X_T^s$, then any Chow class $\sigma \in A_*(X/\!\!/G)_{\mathbb{Q}}$ gives rise to a well-defined class $\Delta \frown \tilde{\sigma} \in A_*(X/\!\!/T)_{\mathbb{Q}}$ that is independent of the choice of lift $\tilde{\sigma}$, as defined in §8.2. The class $\Delta := c_{\text{top}}(\mathfrak{g}/\mathfrak{b}) \in A^*(BT)$ is defined to be the top Chern class of the equivariant vector bundle induced by the adjoint representation on $\mathfrak{g}/\mathfrak{b}$, where $\mathfrak{g}$ and $\mathfrak{b}$ are the Lie algebras of G and B, a Borel subgroup. As a consequence, the independence of the Chow class $c_{\text{top}}(\mathfrak{g}/\mathfrak{t}) \frown \tilde{\sigma}$ follows immediately, as

$$c_{\text{top}}(\mathfrak{g}/\mathfrak{t}) = (-1)^{|\Phi|/2} \Delta \frown \Delta.$$

It is this independence that is needed to make the GIT integration ratio $r_{G,T}^{X,\sigma}$ (cf. Ch. 10) well-defined. The case of singular X is treated separately in Chapter 9.

8.1 Geometric invariant theory

We begin by briefly reviewing geometric invariant theory, mainly to set notational conventions.

Definition 8.1.1. Let X be a variety over k with a linearized action of a reductive group G, i.e. an ample line bundle $\pi : \mathscr{L} \to X$ along with G-actions on $\mathscr{L}$ and X for which π is an equivariant map on whose fibres G acts linearly. Such an $\mathscr{L}$ is called a *G-linearization* of X.

· The *semi-stable locus* is the open subscheme defined by

$$X_G^{ss} := \{x \in X : \exists\; n > 0 \text{ and some } \phi \in \Gamma(X, \mathscr{L}^{\otimes n})^G \text{ satisfying } \phi(x) \neq 0\}.$$

· The *stable*[1] *locus* is the open subscheme defined by

$$X_G^{s} := \{x \in X_G^{ss} : x \cdot G \subseteq X_G^{ss} \text{ is a closed subscheme and } |\operatorname{stab}_G x| < \infty\}.$$

· The *unstable locus* is defined to be $X_G^{un} := X \setminus X_G^{ss}$.

· The *strictly semi-stable locus* is defined to be $X_G^{sss} := X_G^{ss} \setminus X_G^{s}$.

The following theorem justifies the making of the above definitions:

Theorem 8.1.2 (Mumford)**.** *The semi-stable locus of a G-linearized projective variety X admits a uniform categorical quotient $\pi : X_G^{ss} \to X/\!\!/G$ called the* GIT quotient of X by G*. Moreover, some positive tensor power of the G-linearization descends to an ample line bundle on the projective variety $X/\!\!/G$, and the restriction of π to the stable locus is a geometric quotient.*

Proof. See [32, Thm. 1.10]. ■

Next we describe how to compute the stable and semi-stable loci in practice. As these loci are compatible with field extensions (cf. [32, Prop. 1.14]), we may assume that we are working over the algebraically closed field $\bar{k}$. Let $T \subseteq G$ be a maximal torus with character group $\Lambda^*(T)$. Equivariantly embed X into $\mathbb{P}(V)$ for some G-representation V so that $\mathcal{O}_{\mathbb{P}(V)}(1)$ pulls-back to some positive tensor power of the G-linearization $\mathscr{L}$. The G-representation structure on V endows $\mathbb{P}(V)$ with a G-action, naturally linearized by the induced action on $\mathcal{O}(1)$, and for which both $\mathbb{P}(V)_G^{ss} \cap X = X_G^{ss}$ and $\mathbb{P}(V)_G^{s} \cap X = X_G^{s}$. Since $\bar{k}$ is algebraically closed, T is diagonalizable and hence V decomposes as the direct sum of weight spaces $V = \oplus_{\chi \in \Lambda^*(T)} V_\chi$.

[1] Sometimes in the literature this is referred to as "properly stable".

Definition 8.1.3. For any $x \in X \subseteq \mathbb{P}(V)$ as above, the *state of* x is defined to be

$$\Xi(x) := \{\chi \in \Lambda^*(T) : \exists\ v \in V_\chi \text{ such that } v(x) \neq 0\}.$$

We now state in the above notation the following well-known numerical criterion for stability:

Theorem 8.1.4 (Hilbert-Mumford criterion)**.** *A point $x \in X$ is semi-stable for the induced linearized T-action if and only if 0 is in the convex hull of $\Xi(x)$ in $\Lambda^*(T) \otimes \mathbb{Q}$. Moreover, $x \in X$ is stable for the T-action if and only if 0 is in the interior of the convex hull of $\Xi(x)$. Furthermore,*

$$X_G^{ss} = \bigcap_{g \in G} X_T^{ss} \cdot g, \quad \text{and} \quad X_G^{s} = \bigcap_{g \in G} X_T^{s} \cdot g.$$

Proof. See [32, Thm. 2.1]. ∎

Remark 8.1.5. *A corollary of the Hilbert-Mumford criterion is that the condition in T-stability that all semi-stable points are stable, $X_T^{ss} = X_T^s$, automatically implies the analogous condition in G-stability, $X_G^{ss} = X_G^s$.*

We conclude this review by presenting a stratification that describes the structure of the unstable locus. The stratification is due to Kirwan, but relies on the previous work of Hesselink, Kempf, and Ness (cf. [17; 22; 21]).

Theorem 8.1.6 (Kirwan)**.** *Let X be a projective variety with a linearized right G-action over an algebraically closed field $\bar{k}$. The unstable locus X_G^{un} admits a G-equivariant stratification,*

$$X_G^{un} = \bigcup_{\beta \in \mathbf{B}} S_\beta,$$

indexed by a finite partially ordered set $\mathbf{B}$, with the following properties:

1. *$S_\beta \subseteq X_G^{un}$ is a locally closed G-equivariant subscheme.*

2. *$S_\beta \cap S_{\beta'} = \emptyset$ for $\beta \neq \beta'$.*

3. *$\overline{S_\beta} \subseteq \bigcup_{\beta' \geq \beta} S_{\beta'}$.*

4. *There exist parabolic subgroups $T \subseteq P_\beta \subseteq G$, locally closed P_β-closed subschemes $Y_\beta \subseteq S_\beta \cap X_T^{un}$, and a surjective G-equivariant morphism,*

$$\phi : Y_\beta \times_{P_\beta} G \to S_\beta,$$

induced by the multiplication morphism $Y_\beta \times G \to S_\beta$.

If moreover X is smooth, then each S_β is smooth and the morphism ϕ in (4) *is an isomorphism.*

Proof. See [23, §12-§13]. ∎

8.2 Lifts

We now define precisely what it means to lift a Chow class $\sigma \in A_*(X/\!\!/G)_{\mathbb{Q}}$ to a class $\tilde{\sigma} \in A_*(X/\!\!/T)_{\mathbb{Q}}$. We make use of the notion of Chow groups of quotient stacks, a review of which may be found in the appendix to this thesis. The main result we will need is that when $X_G^{ss} = X_G^s$, the quotient $[X_G^{ss}/G]$ is a proper Deligne-Mumford stack with coarse moduli space morphism $\phi^G : [X_G^{ss}/G] \to X/\!\!/G$ that induces an isomorphism on Chow groups (cf. Thm. 1.1.1):

$$\phi_*^G : A_*([X_G^{ss}/G])_{\mathbb{Q}} \cong A_*(X/\!\!/G)_{\mathbb{Q}}. \tag{8.2.1}$$

Via the identification ϕ_*^G, we may think of a Chow class $\sigma \in A_*([X_G^{ss}/G])_{\mathbb{Q}}$ equivalently as $\phi_*^G(\sigma) \in A_*(X/\!\!/G)_{\mathbb{Q}}$, and we will henceforth denote both classes by the symbol σ.

Definition 8.2.2. If X is a G-linearized variety for which $X_T^{ss} = X_T^s$, then for any class $\sigma \in A_*(X/\!\!/G)_{\mathbb{Q}}$, a class $\tilde{\sigma} \in A_{*+g-t}(X/\!\!/T)_{\mathbb{Q}}$ is called a *lift of* σ provided that $i^*(\tilde{\sigma}) = f^*(\sigma)$, where

- $g := \dim G$, $t := \dim T$,
- $i : [X_G^{ss}/T] \hookrightarrow [X_T^{ss}/T]$ is the open immersion, and
- $f : [X_G^{ss}/T] \to [X_G^{ss}/G]$ is the flat fibration with fibre G/T.

Remark 8.2.3. *By the right exact sequence of Chow groups*

$$A_*([X_G^{un} \cap X_T^{ss}/T])_{\mathbb{Q}} \to A_*([X_T^{ss}/T])_{\mathbb{Q}} \xrightarrow{i^*} A_*(X_G^{ss}/T])_{\mathbb{Q}} \to 0,$$

any two lifts of σ differ by the push-forward of an element of $A_*([X_G^{un} \cap X_T^{ss}/T])_{\mathbb{Q}}$.

8.3 Vanishing on Kirwan strata

Throughout this section, we assume that X is a smooth, G-linearized projective variety over an algebraically closed field $\bar{k}$. In light of Remark 8.2.3, to show that $\Delta \frown \tilde{\sigma}$ is independent of the choice of lift, it suffices to show that Δ kills all elements in the image of $A_*([X_G^{un} \cap X_T^{ss}/T])_{\mathbb{Q}}$. For now we just prove that Δ vanishes on the T-semi-stable locus of each stratum, $S_\beta \cap X_T^{ss}$. Since X is smooth, the stratum S_β is fibred over the flag variety $P_\beta \backslash G$ of right P_β-cosets, and so we begin our study here.

We require some notation related to the Weyl group $W := T\backslash N_T$. For a parabolic subgroup $P \subseteq G$ containing the maximal torus T, denote by W_P the subgroup $T\backslash(N_T \cap P) \subseteq W$. Let the symbol $\dot{w}$ denote a choice of representative in N_T of a Weyl class $w \in W$, and let the symbol $\bar{w}$ denote the image of w in $W_P\backslash W$.

Lemma 8.3.1. *Let $P \subseteq G$ be a parabolic subgroup, containing a maximal torus T and with Lie algebra $\mathfrak{p}$. The inclusion $i : W_P\backslash W \to P\backslash G$, defined by $\bar{w} \mapsto P\dot{w}$, is the inclusion of the T-fixed points of $P\backslash G$, and the Gysin pull-back of the T-equivariant class $[P] \in A_*^T(P\backslash G)$ is given by*

$$i^*([P]) = c_{top}(\mathfrak{g}/\mathfrak{p}) \cdot [\bar{e}] \in A_*^T(W_P\backslash W),$$

where the T-action on $\mathfrak{g}/\mathfrak{p}$ is via the adjoint representation.

Proof. It is well-known that the T-invariant points of $P\backslash G$ are precisely $W_P\backslash W$. The Chow group $A_*^T(W_P\backslash W)$ is a free $A^*(BT)$-module with basis given by the elements of $W_P\backslash W$. The element $P \in P\backslash G$ is an isolated, nonsingular fixed point, disjoint from all other fixed points $wP \neq P$. Hence, $i^*([P])$ equals the product of $[\bar{e}]$ with the T-equivariant top Chern class of the normal bundle of P at P, which is just $c_{\text{top}}(\mathfrak{g}/\mathfrak{p})$. ∎

Lemma 8.3.2. *Let $P \subseteq G$ be a parabolic subgroup containing a maximal torus T, and let $U \subseteq P\backslash G$ denote the open complement of the finite set $W_P\backslash W \hookrightarrow P\backslash G$. As an element of the T-equivariant operational Chow group of U,*

$$\Delta = 0 \in A_T^*(U).$$

Proof. The variety U is smooth, so by Poincaré duality (Theorem 1.2.4) it suffices to prove that $\Delta \frown [U] = 0 \in A_*^T(U)$. Let $X := P\backslash G$ denote the flag variety, and let $i : X^T = W_P\backslash W \to X$ denote the inclusion of the T-fixed points. By the right-exact sequence of Chow groups

$$A_*^T(X^T) \xrightarrow{i_*} A_*^T(X) \to A_*^T(U) \to 0,$$

it suffices to show that $\Delta \frown [X]$ is in the image of i_*. Since X is a smooth projective variety, by the localization theorem (Theorem 1.1.3), $i^* : A_T^*(X) \to A_T^*(X^T)$ is an injective morphism of $A^*(BT)$-algebras. Thus, it suffices to prove that $i^*(\Delta \frown [X])$ is in the image of $i^* \circ i_*$.

The ring $A^*(X^T)$ is a free $A^*(BT)$-module with basis given by $\{[\bar{w}] : \bar{w} \in W_P\backslash W\}$. In terms of this basis,

$$i^*(\Delta \frown [X]) = \sum_{\bar{w} \in W_P\backslash W} \Delta \cdot [\bar{w}].$$

By Lemma 8.3.1, $i^* \circ i_*([\bar{e}]) = c_{\text{top}}(\mathfrak{g}/\mathfrak{p}) \cdot [\bar{e}]$. Since i is a W-equivariant inclusion, $i^* \circ i_*$ is compatible with the W-action. Therefore for any $w \in W$,

$$i^* \circ i_*([\bar{w}]) = (\prod_{\alpha \in \Phi(\mathfrak{g}/\mathfrak{p})} \alpha w) \cdot [\bar{w}],$$

where $\Phi(\mathfrak{g}/\mathfrak{p}) \subseteq \Phi$ denotes the subset of roots that appear in the diagonalization of the adjoint action on $\mathfrak{g}/\mathfrak{p}$. Notice that $\Phi(\mathfrak{g}/\mathfrak{p})$ is a subset of the negative roots $\Phi^- := \Phi(\mathfrak{g}/\mathfrak{b})$, for any choice of Borel subgroup $B \subseteq P$. Hence, for the Chow class $\beta_w := \prod_{\alpha \in \Phi^- \backslash \Phi(\mathfrak{g}/\mathfrak{p})} \alpha w \in A^*(BT)$,

$$i^* \circ i_*(\beta_w \cdot [\bar{w}]) = (\prod_{\alpha \in \Phi^-} \alpha w) \cdot [\bar{w}].$$

Since $\Delta = \prod_{\alpha\in\Phi^-}\alpha$, it follows that $\prod_{\alpha\in\Phi^-}\alpha w = \det(w)\cdot\Delta$. Therefore, $\Delta \frown [X]$ is in the image of $i^*\circ i_*$:

$$\sum_{\bar{w}\in W_P\backslash W}\Delta\cdot[\bar{w}] = i^*\circ i_*\left(\sum \det(w)\cdot\beta_w\cdot[\bar{w}]\right).$$

■

From this vanishing result on flag varieties, we now derive that Δ is zero on the T-semistable locus of each stratum $S_\beta\cap X_T^{ss}$.

Lemma 8.3.3. *Let X be a smooth, projective G-linearized variety over an algebraically closed field $\bar{k}$. As an element of the T-equivariant operational Chow group of $S_\beta\cap X_T^{ss}$,*

$$\Delta = 0\in A_T^*(S_\beta\cap X_T^{ss}).$$

Proof. By Theorem 8.1.6, there is a G-equivariant morphism $\pi: S_\beta\to P_\beta\backslash G$ with $\pi^{-1}(P_\beta) = Y_\beta\subseteq X_T^{un}$. By the Hilbert-Mumford criterion (Thm. 8.1.4), the normalizer N_T preserves the unstable locus X_T^{un}, so $Y_\beta\cdot N_T\subseteq X_T^{un}$. Hence, the image of π restricted to $S_\beta\cap X_T^{ss}$ is contained in U, the open complement in $P_\beta\backslash G$ of the finite set $P_\beta\backslash P_\beta W$. We finish by noting that the operational Chow class $\Delta\in A_T^*(S_\beta\cap X_T^{ss})$ is the pull-back of $\Delta\in A_T^*(U)$, which is 0 by Lemma 8.3.2. ■

Remark 8.3.4. *The arguments above in §8.3 are directly analogous to those used by Brion in [3] for equivariant cohomology, but the arguments to follow in §8.4 and §8.5 are original and yield results marginally stronger than what can be found in the literature for equivariant cohomology.*

8.4 Vanishing over an algebraically closed field

We continue to assume that X is smooth over an algebraically closed field $\bar{k}$, and we extend the vanishing of Δ on a stratum to vanishing over the entire locus $X_G^{un}\cap X_T^{ss}$, proving that Δ acts as 0 on $A_*^T(X_G^{un}\cap X_T^{ss})_{\mathbb{Z}}$.

Proposition 8.4.1. *Let X be a smooth G-linearized projective variety over $\bar{k}$. The operational Chow class Δ annihilates every class in $A_*^T(X_G^{un}\cap X_T^{ss})_{\mathbb{Z}}$.*

Proof. Label $\mathbf{B} = \{\beta_1, \ldots, \beta_n\}$ so that $\mathcal{S}_k := \bigcup_{i\leq k} S_{\beta_i}$ is a closed subvariety of X_G^{un}, for each $1 \leq k \leq n$; this is possible by Theorem 8.1.6 (3). We proceed to inductively prove that Δ annihilates every class in $A_*^T(\mathcal{S}_k \cap X_T^{ss})_{\mathbb{Z}}$ for all $k = 1, \ldots, n$, with $k = n$ being the desired result. By Lemma 8.3.3 the assertion is true when $k = 1$ and $\mathcal{S}_1 = S_{\beta 1}$ is an individual stratum. Assume the result holds for $1 \leq k < n$. By Proposition 1.1.2, it suffices to show that $\Delta \frown [Y] = 0 \in A_*^T(\mathcal{S}_{k+1} \cap X_T^{ss})$ for any T-invariant subvariety $Y \hookrightarrow \mathcal{S}_{k+1} \cap X_T^{ss}$. Since $\mathcal{S}_k \hookrightarrow \mathcal{S}_{k+1}$ is a closed immersion, the inductive hypothesis will imply $\Delta \frown [Y] = 0$ whenever Y is completely contained in $\mathcal{S}_k$. As $\mathcal{S}_{k+1} = S_{\beta_{k+1}} \cup \mathcal{S}_k$, the only remaining case is when Y intersects $S_{\beta_{k+1}}$ nontrivially.

For the sake of clarity, denote $\beta := \beta_{k+1}$. Consider the birational map $\overline{S_\beta} \dashrightarrow P_\beta\backslash G$ defined on S_β by $S_\beta \cong Y_\beta \times_{P_\beta} G \to P_\beta\backslash G$ (cf. Thm. 8.1.6 (4)). Our strategy will be to partially resolve the locus of indeterminacy in the following manner:

$$\begin{tikzcd} \overline{Y_\beta} \times_{P_\beta} G \arrow[d, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ \overline{S_\beta} \arrow[r, dashed, "f"] & P_\beta\backslash G. \end{tikzcd}$$

The morphism π is proper, since $P_\beta\backslash G$ is so, and is an isomorphism when restricted to the dense open $S_\beta \subseteq \overline{S_\beta}$. Moreover, $\tilde{f}^{-1}(P_\beta) = \overline{Y_\beta}$, and by G-invariance,

$$\tilde{f}^{-1}(P_\beta W) = \overline{Y_\beta} \times_{P_\beta} P_\beta \cdot W.$$

Since N_T preserves X_T^{un} and $\overline{Y_\beta} \subseteq X_T^{un}$, it follows that $\pi(\tilde{f}^{-1}(P_\beta W)) \subseteq X_T^{un}$, showing that the above diagram restricts as follows:

$$\begin{tikzcd} \pi^{-1}(\overline{S_\beta} \cap X_T^{ss}) \arrow[d, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ \overline{S_\beta} \cap X_T^{ss} \arrow[r, dashed, "f"] & U, \end{tikzcd}$$

where U is the open complement of the fixed-point locus, $U := P_\beta\backslash G - P_\beta W \subseteq P_\beta\backslash G$.

Let $\tilde{Y}$ denotes the strict transform of Y under the birational morphism π. As a Chow class, $\Delta \frown [\tilde{Y}] = 0$ since Δ is the pull-back via $\tilde{f}$ of $\Delta \in A_T^*(U)$, which is 0 by Lemma

8.3.2. At the same time, $\pi_*(\Delta \frown [\tilde{Y}]) = \Delta \frown [Y]$ by the projection formula, so $\Delta \frown [Y] = 0 \in A_*^T(\overline{S_\beta} \cap X_T^{ss})$. By the projection formula for the closed immersion $\overline{S_\beta} \hookrightarrow \mathcal{S}_{k+1}$, we conclude that $\Delta \frown [Y] = 0 \in A_*^T(\mathcal{S}_{k+1} \cap X_T^{ss})$. ■

8.5 Torsion over an arbitrary field

We now relax our previous assumption, henceforth allowing an arbitrary base field k. The stratification of Theorem 8.1.6 is constructed over an algebraically closed field, so our previous arguments do not immediately apply. However, if we weaken our statements by ignoring torsion, considering only Chow groups with rational coefficients, our previous results quickly extend to this more general setting. We outline the proof of the following well-established lemma (cf. [1, Lem. 1A.3]) for the reader's convenience.

Lemma 8.5.1. *If X is a variety over a field k, then any field extension K/k induces an injective morphism between Chow groups with rational coefficients: $A_*(X)_\mathbb{Q} \hookrightarrow A_*(X_K)_\mathbb{Q}$.*

Proof. If a field extension E/k is the union of a directed system of sub-extensions E_i/k, then $A_*(X_E) = \varinjlim A_*(X_{E_i})$. We may apply this result to the extension K/k to reduce the proof to the two cases: K/k is finite; or $K = k(x)$ for a transcendental element x. Let $\phi : X_K \to X$ denote the base change morphism.

If K/k is finite, then ϕ is flat and proper, and the composition $\phi_* \circ \phi^*$ is simply multiplication by $[K : k]$. This is an isomorphism since coefficients are rational, and hence ϕ^* is injective.

If $K = k(x)$, then X_K is the generic fibre of the projection $\pi : X \times \mathbb{P}^1_k \to \mathbb{P}^1_k$. If $\psi : X \times \mathbb{P}^1_k \to X$ denotes the other projection and $j : X_K \to X \times \mathbb{P}^1_k$ is the inclusion of the generic fibre, then $\phi = \psi \circ j$. There is an isomorphism of Chow groups,

$$A_{i+1}(X \times \mathbb{P}^1_k)_\mathbb{Q} \cong A_i(X)_\mathbb{Q} \oplus A_{i+1}(X)_\mathbb{Q} \cdot t,$$

where t is the class associated to a fibre of π. The pull-back $\psi^* : A_i(X)_\mathbb{Q} \to A_{i+1}(X \times \mathbb{P}^1_k)_\mathbb{Q}$ is identified with $\mathrm{id} \oplus 0 : A_i(X)_\mathbb{Q} \to A_i(X)_\mathbb{Q} \oplus A_{i+1}(X)_\mathbb{Q} \cdot t$ (cf. [12, Thm. 3.3]). As schemes,

$$X_K = \varinjlim_{\emptyset \neq U \subseteq \mathbb{P}^1} X \times U,$$

and there is an induced isomorphism on the level of Chow groups:

$$A_i(X_K)_{\mathbb{Q}} \cong \varinjlim_{\emptyset \neq U \subseteq \mathbb{P}^1} A_{i+1}(X \times U)_{\mathbb{Q}} \cong A_i(X)_{\mathbb{Q}} \oplus 0.$$

Under this identification, the pull-back $j^* : A_{i+1}(X \times \mathbb{P}^1_k) \to A_i(X_K)$ is the projection $A_i(X)_{\mathbb{Q}} \oplus A_i(X)_{\mathbb{Q}} \cdot t \to A_i(X)_{\mathbb{Q}}$. From this description, it is clear that $\phi^* = j^* \circ \psi^*$ is an isomorphism, ergo injective. ■

Proposition 8.5.2. *Let X be a smooth projective G-linearized variety over an arbitrary field k. The operational Chow class Δ annihilates every class in $A_*^T(X_G^{un} \cap X_T^{ss})_{\mathbb{Q}}$.*

Proof. Lemma 8.5.1 reduces the proof to the case of an algebraically closed base field, which is the content of Proposition 8.4.1. ■

We state an immediate corollary that we will later extend to the case singular varieties (cf. Prop. 9.3.1).

Corollary 8.5.3. *Let X be a smooth projective variety over a field k, with a G-linearized action for which $X_T^{ss} = X_T^s$. If $\tilde{\sigma}_0, \tilde{\sigma}_1 \in A_*(X/\!\!/T)_{\mathbb{Q}}$ are two lifts of the same class $\sigma \in A_*(X/\!\!/G)_{\mathbb{Q}}$, then for any operation Chow class $\Delta' \in A_*(X/\!\!/T)_{\mathbb{Q}}$,*

$$\int_{X/\!\!/T} \Delta' \frown (\Delta \frown \tilde{\sigma}_0) = \int_{X/\!\!/T} \Delta' \frown (\Delta \frown \tilde{\sigma}_1).$$

Chapter 9

Singular varieties and strictly semi-stable points

We wish to extend the results of Chapter 8 to the case of singular X by studying a compatible closed immersion $j : X \hookrightarrow \mathbb{P}(V)$, but several obstacles must be overcome. The first of which is the regrettable fact that the push-forward map $j_* : A_*([X^{ss}_T/T])_{\mathbb{Q}} \to A_*([\mathbb{P}(V)^{ss}_T/T])_{\mathbb{Q}}$ is not injective in general, thus preventing an easy reduction of the general proof to the smooth case (cf. Prop. 8.5.2). Suppose we attempt to circumvent this problem by limiting our ambitions to showing that any Chow class $\sigma \in A_*(X/\!\!/G)_{\mathbb{Q}}$ gives rise to a well-defined *numerical* equivalence class $\Delta \frown \tilde{\sigma}$ independent of the choice of lift $\tilde{\sigma}$. Ideally the closed immersion j would induce an injective map between the groups of algebraic cycles modulo numerical equivalence, and we would reduce our proof to the smooth case. Unfortunately, there is the possibility that $\mathbb{P}(V)^{ss}_T \neq \mathbb{P}(V)^s_T$, and in this case there is no notion of numerical equivalence on the Artin stack $[(\mathbb{P}(V))^{ss}_T/T]$ (cf. [7]). Once again, the naïve argument breaks down.

Our solution is to build an auxiliary smooth G-linearized variety Y for which $Y^{ss}_T = Y^s_T$ and then to relate integration on $X/\!\!/G$ and $X/\!\!/T$ to integration on $Y/\!\!/G$ and $Y/\!\!/T$. If one is guaranteed G-equivariant resolutions of singularities (e.g. if $\operatorname{char} k = 0$), then a result of Reichstein [34] generalizes the partial desingularizations of Kirwan [24] and produces such an auxiliary variety Y. Since resolution of singularities is still an open problem in positive characteristic, we provide an independent construction. As an application of our method, we prove in §9.3 an extension of Corollary 8.5.3 to singular varieties.

9.1 Construction

The auxiliary variety Y will be defined as an iterated product of the flag variety of right B-cosets, for a Borel subgroup B containing the maximal torus T. The only delicate point is the choice of a suitable linearization, which will rely upon an understanding of equivariant line bundles over flag varieties.

Lemma 9.1.1. *For any weight $\chi \in \Lambda^*(T)$ in the interior of the positive Weyl chamber, the line bundle $\mathscr{L}(\chi) := \mathbb{A}^1 \times_{\chi,B} G$ is a very ample G-linearization of $B\backslash G$. For any nontrivial subtorus $T' \subseteq T$ stabilizing some point $Bg \in B\backslash G$, the induced action on the fibre of $\mathscr{L}(\chi)$ over Bg is by the restriction to T' of the weight $(\chi \cdot w)$, where $w \in W$ corresponds to the Bruhat cell containing Bg.*

Proof. That $\mathscr{L}(\chi)$ is a very ample line bundle on $B\backslash G$ is simply a rephrasing of the standard construction in representation theory of highest weight modules for dominant weights. The Bruhat decomposition states that $B\backslash G = \coprod_{w\in W} BwU$, for $U \subseteq B$ the maximal normal unipotent subgroup. Furthermore, denoting by $\dot{w} \in N_T$ a lift of the element $w \in W = T\backslash N_T$, there is an isomorphism

$$\phi : B \times U_w \to B\dot{w}U \subseteq G$$

sending $(b, u) \mapsto b\dot{w}u$, where $U_w := U \cap (U^-)^w$ is the intersection of U with the $\dot{w}$-conjugate of the opposite unipotent subgroup, $(U^-)^w := \dot{w}^{-1}U^-\dot{w}$ (cf. [2, Thm. 14.12]). Note that U_w is normalized by T because both U and $(U^-)^w$ are; the latter being so because

$$\begin{aligned} t\dot{w}^{-1}U^-\dot{w}t &= \dot{w}^{-1}\left(t^{\dot{w}^{-1}}U^-(t^{\dot{w}^{-1}})^{-1}\right)\dot{w} \\ &= \dot{w}^{-1}U^-\dot{w}. \end{aligned}$$

We write $Bg = B\dot{w}u$ for some $w \in W$ and $u \in U_w$, and we assume Bg is stabilized by the right action of $t \in T'$. Observe

$$\dot{w}ut = t^{\dot{w}^{-1}} \cdot \dot{w} \cdot u^t,$$

where $u^t := t^{-1}ut \in U_w$ and $t^{\dot{w}^{-1}} := \dot{w}t\dot{w}^{-1} \in T \subseteq B$. Since t stabilizes Bg, there exists some $b \in B$ such that $b\dot{w}u = t^{\dot{w}^{-1}} \cdot \dot{w} \cdot u^t$. Since ϕ is an isomorphism, it must be that $u = u^t$, $b = t^{\dot{w}^{-1}}$, and $\dot{w}ut = t^{w^{-1}} \cdot \dot{w}u$. From this equalilty, it is clear that t acts on the fibre of $\mathscr{L}(\chi)$ via multiplication by $\chi(t^{\dot{w}^{-1}}) = (\chi \cdot w)(t)$. ∎

The following lemma will be the inductive step in the argument showing that for any G-linearized variety Z, the variety $Y := Z \times (B\backslash G)^{\operatorname{rank} G}$ admits a linearization such that $Y^{ss}_T = Y^s_T$ and all points in Y projecting to a stable point of Z are stable in Y.

Lemma 9.1.2. *Let $\mathscr{L}_Z$ be a G-linearization of a projective variety Z, and $T \subseteq G$ a fixed maximal torus. If $r > 0$ is the maximum rank of a subtorus of T that stabilizes a strictly semi-stable point in Z^{sss}_T, then there is a character $\chi \in \Lambda^*(T)$ and an integer $n \gg 0$ so that the induced right G-linearization $\mathscr{L}_Z^{\otimes n} \otimes \mathscr{L}(\chi)$ on $Z \times B\backslash G$ has the following properties:*

1. *The subtori of T that stabilize points in $(Z \times B\backslash G)^{sss}_T$ are at most rank $r - 1$.*

2. *A point $p \in Z \times B\backslash G$ is stable (resp. unstable) whenever $\pi(p)$ is so, where $\pi : Z \times B\backslash G \to Z$ is projection onto the first factor and the notion of stability is taken with respect to either the G- or the T-action.*

Proof. Let $T_1, \ldots, T_n$ denote the positive-dimensional subtori of T occurring as the connected components of stabilizers of points $z \in Z^{sss}_T$. Let $H_i := \Lambda^*(T/T_i) \subset \Lambda^*(T)$ denote the subgroup of T-characters vanishing on T_i. Since T_i is positive dimensional, the inclusion $H_i \subsetneq \Lambda^*(T)$ is strict. Choose an integral weight χ in the interior of the positive Weyl chamber of $\Lambda^*(T)$ avoiding the W-orbit of any $H_1, \ldots, H_n$. By Lemma 9.1.1, $\mathscr{L}(\chi)$ gives a G-linearization of $B\backslash G$.

By choosing n large enough, one can derive from the Hilbert-Mumford criterion (Thm. 8.1.4), without too much effort, that the linearization $\mathscr{L}_Z^{\otimes n} \otimes \mathscr{L}(\chi)$ satisfies:

· $p \in Z \times B\backslash G$ is stable if $\pi(p)$ is stable,

· $p \in Z \times B\backslash G$ is unstable if $\pi(p)$ is unstable, for stability with respect to either the G- or the T-action. This also follows immediately from a more general theorem of Reichstein (cf. [34, Thm. 2.1]). As a consequence, any strictly semi-stable point $p \in (Z \times B\backslash G)^{ss}_T$ must sit above a strictly semi-stable point $\pi(p) \in Z^{ss}_T$.

If T' is a torus stabilizing some point $(z, Bg) \in (Z \times B\backslash G)^{sss}_T$, then $z \in Z^{sss}_T$. Therefore $T' \subseteq T_i$ for some $1 \leq i \leq n$, and the weight of the action of T' on the fibre of $\mathscr{L}_Z$ over z is 0. Since T' stabilizes Bg, Lemma 9.1.1 implies the weight of the action of T' on the fibre of $\mathscr{L}(\chi)$ over Bg is $\chi \cdot w$ for some $w \in W$. Therefore, the weight of the action of T' on the fibre of $\mathscr{L}_Z^{\otimes n} \otimes \mathscr{L}(\chi)$ over (z, Bg) is $0 + (\chi \cdot w)|_{T'}$. Since (z, Bg) is strictly semi-stable, this weight must be 0 and so $T' \subseteq \ker((\chi \cdot w)|_{T_i})$. Our choice of χ was such that $(\chi \cdot w)|_{T_i} \neq 0 \in \Lambda^*(T_i)$ for any $w \in W$, and thus the rank of the rank of T' is at most the rank of $\ker((\chi \cdot w)|_{T_i})$, which is $r - 1$. ■

Proposition 9.1.3. *If Z is a G-linearized projective variety then for $r := \operatorname{rank} T$, the variety $Y := Z \times (B\backslash G)^r$ admits a G-linearization for which*

(i) $Y^{ss} = Y^s$, *and*

(ii) $Z^s \times (B\backslash G)^r \subseteq Y^{ss} \subseteq Z^{ss} \times (B\backslash G)^r$,

for both T- and G- (semi-)stability.

Proof. We prove the results for T-stability, and then the result for G-stability will follow (cf. Thm. 8.1.4). Recursively applying Lemma 9.1.2, we obtain a G-linearization of $Z \times (B\backslash G)^r$ for which no T-strictly semi-stable points have positive dimensional T-stabilizers. Hence, all T-semi-stable points are T-stable, proving (i). Lemma 9.1.2(2) guarantees (ii). ■

9.2 Integration

Here we prove that the integration of Chow classes and their lifts on GIT quotients of a singular variety which equivariantly embeds into a nonsingular variety can be related to the integration of Chow classes and lifts on an auxiliary variety constructed as in the previous section. We begin with an easy lemma describing how lifting Chow classes commutes with proper pushes-forward.

Lemma 9.2.1. *If $\pi : X \to Y$ is a G-equivariant proper morphism of varieties satisfying $X^{ss}_T = X^s_T$, $Y^{ss}_T = Y^s_T$, and $\pi^{-1}(Y^{ss}_T) = X^{ss}_T$, then for any Chow class $\sigma \in A_*(X/\!\!/G)_{\mathbb{Q}}$ and any lift $\tilde{\sigma} \in A_*(X/\!\!/T)_{\mathbb{Q}}$, the push-forward $\pi_*\tilde{\sigma} \in A_*(Y/\!\!/T)_{\mathbb{Q}}$ is a lift of the Chow class $\pi_*\sigma \in A_*(Y/\!\!/G)_{\mathbb{Q}}$.*

Proof. The Hilbert-Mumford criterion (Thm. 8.1.4) implies that $\pi^{-1}(Y^{ss}_G) = X^{ss}_G$, yielding the following fibre square,

$$\begin{tikzcd} X^{ss}_G \arrow[r, hook, "i_X"] \arrow[d, "\pi"'] & X^{ss}_T \arrow[d, "\pi"] \\ Y^{ss}_G \arrow[r, hook, "i_Y"] & Y^{ss}_T, \end{tikzcd}$$

where the horizontal arrows are open immersions, and the vertical arrows are proper morphisms. By Definition 8.2.2, the Chow classes $\sigma, \tilde{\sigma}$ correspond to classes $\sigma \in A_*([X^{ss}_G/G])_{\mathbb{Q}}$ and $\tilde{\sigma} \in A_*([X^{ss}_T/T])_{\mathbb{Q}}$ satisfying $i^*_X\tilde{\sigma} = f^*_X\sigma$, where $f_X : [X^{ss}_G/G] \to [X^{ss}_G/T]$ is induced by the inclusion $T \subseteq G$.

All that must be verified is that

$$i^*_Y(\pi_*\tilde{\sigma}) \stackrel{?}{=} f^*_Y(\pi_*\sigma),$$

for the morphism $f_Y : [Y^{ss}_G/G] \to [Y^{ss}_G/T]$ induced by the inclusion $T \subseteq G$. Since the above diagram is a fibre square, there is the commutativity $i^*_Y \circ \pi_* = \pi_* \circ i^*_X$ (cf. [12, Prop. 1.7]). Therefore,

$$\begin{aligned} i^*_Y(\pi_*\tilde{\sigma}) &= \pi_*(i^*_X\tilde{\sigma}) \\ &= \pi_*(f^*_X\sigma), \end{aligned}$$

since $\tilde{\sigma}$ is a lift of σ. It remains to show that $\pi_* \circ f^*_X = f^*_Y \circ \pi_*$, which follows from the fact that the commutative diagram of Deligne-Mumford stacks,

$$\begin{tikzcd} {[X^{ss}_G/T]} \arrow[r, "f_X"] \arrow[d, "\pi_T"'] & {[X^{ss}_G/G]} \arrow[d, "\pi_G"] \\ {[Y^{ss}_G/T]} \arrow[r, "f_Y"] & {[Y^{ss}_G/G]}, \end{tikzcd}$$

is a fibre square, as demonstrated by the following lemma (taking $S = \{e\}$). ■

Lemma 9.2.2. *Let $\tilde{G}$ be a reductive group, $\tilde{T}$ a maximal torus, and $S < \tilde{G}$ a central torus. Let G denote the reductive group $\tilde{G}/S$ which has a maximal torus $T := \tilde{T}/S$. Assume that $\pi : X \to Y$ is a $\tilde{G}$-equivariant morphism of varieties and that S acts trivially on Y. The following diagram forms a fibre square in the 2-category of Artin stacks:*

$$\begin{tikzcd} {[X/\tilde{T}]} \arrow[r, "f_X"] \arrow[d, "\pi_{\tilde{T}}"'] & {[X/\tilde{G}]} \arrow[d, "\pi_{\tilde{G}}"'] \\ {[Y/T]} \arrow[r, "f_Y"] & {[Y/G]}. \end{tikzcd} \tag{9.2.3}$$

Proof. We point the reader to [39, 04UV] for a definition of the quotient stack $[Z/H]$, of a scheme Z by a group H. For an arbitrary scheme B, we denote the B-points of $[Z/H]$, which consist of H-torsors E over B equipped with an equivariant morphism to Z, by

$$(B \leftarrow E \to Z).$$

It is left to the reader to easily verify the 2-commutativity of diagram (9.2.3), where the involved morphisms are defined by the following rules on B-points:

- $f_X : (B \leftarrow E \to X) \mapsto (B \leftarrow E \times_{\tilde{T}} \tilde{G} \to X)$.
- $f_Y : (B \leftarrow E \to Y) \mapsto (B \leftarrow E \times_T G \to Y)$.
- $\pi_{\tilde{G}} : (B \leftarrow E \to X) \mapsto (B \leftarrow E/S \to Y)$, with $E/S \to Y$ equal to the composition $E/S \to X/S \xrightarrow{\pi} Y$.
- $\pi_{\tilde{T}}$ is given by the same rule as $\pi_{\tilde{G}}$, but for $\tilde{T}$-torsors.

We conclude that there is a functor from $[X/\tilde{T}]$ to the fibre product stack $[Y/T] \times_{[Y/G]} [X/G]$. All that remains is to construct an inverse functor that yields an equivalence of categories. We do so, explicitly describing the inverse functor on B-points, but omitting the details on morphisms.

A B-point of the fibre product stack comprises a $\tilde{G}$-torsor admitting an equivariant map to X, $(B \leftarrow \tilde{E} \to X)$, a T-torsor admitting an equivariant map to Y, $(B \leftarrow E \to Y)$, and an isomorphism of the G-torsors over B, $\tilde{E}/S \cong E \times_T G$ respecting the equivariant maps to Y. From this data, we must construct a $\tilde{T}$-torsor over B along with an equivariant morphism to X. This can be accomplished by forming the fibre product $E \times_{\tilde{E}/S} \tilde{E}$, where one of the structure morphisms is the composition $E = E \times \{e\} \to E \times_T G \cong \tilde{E}/S$ and the other is the quotient $\tilde{E} \to \tilde{E}/S$. The $\tilde{T}$-equivariant morphism $E \times_{\tilde{E}/S} \tilde{E} \to X$ is the composition of the projection onto $\tilde{E}$ with the $\tilde{G}$-equivariant morphism to X. It is routine to verify that this is an inverse functor. ■

We are now prepared to prove the main result of this section.

Proposition 9.2.4. *Let $j : X \hookrightarrow Z$ be a G-equivariant embedding of varieties, and let $\mathscr{L}$ be a G-linearization of Z. There is a G-linearization of $Y := Z \times (B\backslash G)^r$, where $r := \operatorname{rank} T$, for which $Y_T^{ss} = Y_T^s$ and, furthermore, if $X_T^{ss} = X_T^s$ for the induced linearization $j^*\mathscr{L}$ of X, then for any $\sigma \in A_*(X/\!\!/G)_{\mathbb{Q}}$ with lift $\tilde{\sigma} \in A_*(X/\!\!/T)_{\mathbb{Q}}$, there exists a class $\tau \in A_*(Y/\!\!/G)_{\mathbb{Q}}$ with lift $\tilde{\tau} \in A_*(Y/\!\!/T)_{\mathbb{Q}}$ so that*

1. $\int_{X/\!\!/G} \sigma = \int_{Y/\!\!/G} \tau$, *and*
2. $\int_{X/\!\!/T} \Delta' \frown (\Delta \frown \tilde{\sigma}) = \int_{Y/\!\!/T} \Delta' \frown (\Delta \frown \tilde{\tau})$,

for any operational Chow class $\Delta' \in A_*(X/\!\!/T)_{\mathbb{Q}}$.

Proof. Let $\mathscr{M}$ denotes the linearization of the G-action on Y prescribed in Proposition 9.1.3. Define the variety $\hat{X} := X \times (B\backslash G)^r$ and endow it with the G-linearization $(j \times \mathrm{id})^*\mathscr{M}$ induced from the embedding $j \times \mathrm{id} : X \times (B\backslash G)^r \hookrightarrow Z \times (B\backslash G)^r$. Proposition 9.1.3 guarantees that $Y_T^{ss} = Y_T^s$, and hence $\hat{X}_T^{ss} = \hat{X}_T^s$. Moreover, it also shows that $(j \times \mathrm{id})^{-1}(Y_T^{ss}) = \hat{X}_T^{ss}$ and $\pi^{-1}(X_T^{ss}) = \hat{X}_T^{ss}$, for $\pi : \hat{X} \to X$ the projection morphism. Since $\pi|_{X_G^{ss}}$ is surjective, there exists a class $\rho \in A_*(\hat{X}/\!\!/G)_{\mathbb{Q}}$ such that $\pi_*\rho = \sigma$. Let $\tilde{\rho} \in A_*(\hat{X}/\!\!/T)_{\mathbb{Q}}$ be a lift of the Chow class ρ for which $\pi_*(\tilde{\rho}) = \tilde{\sigma}$, which is possible because $\pi|_{X_T^{ss}}$ is surjective. By Lemma 9.2.1, $\tilde{\tau} := (j \times \mathrm{id})_*\tilde{\rho}$ is a lift of the class $\tau := (j \times \mathrm{id})_*\rho$.

By the functoriality of the Chow groups under proper push-forward, the commutative diagram,

$$\begin{tikzcd} \hat{X}/\!\!/G \arrow[r, "j\times\mathrm{id}"] \arrow[d, "\pi"'] & Y/\!\!/G \arrow[d] \\ X/\!\!/G \arrow[r] & \operatorname{Spec} k, \end{tikzcd}$$

shows that $\int_{X/\!\!/G} \sigma = \int_{Y/\!\!/G} \tau$, proving (1). The equality in (2) follows in the same way from an analogous diagram involving T-quotients, recalling that the projection-formula guarantees the compatibility of the operational Chow class $\Delta' \frown \Delta$ under pushes-forward.

■

9.3 Application

Proposition 9.3.1. *Let X be a projective variety over a field k, with a G-linearized action for which $X_T^{ss} = X_T^s$. If $\tilde{\sigma}_0, \tilde{\sigma}_1 \in A_*(X/\!\!/T)_{\mathbb{Q}}$ are two lifts of the same class $\sigma \in A_*(X/\!\!/G)_{\mathbb{Q}}$, then for any operation Chow class $\Delta' \in A_*(X/\!\!/T)_{\mathbb{Q}}$,*

$$\int_{X/\!\!/T} \Delta' \frown (\Delta \frown \tilde{\sigma}_0) = \int_{X/\!\!/T} \Delta' \frown (\Delta \frown \tilde{\sigma}_1).$$

Proof. Embed the singular X into the smooth variety $\mathbb{P}(V)$ via some high tensor power of the given G-linearization. Construct the smooth G-linearized Y sitting over $\mathbb{P}(V)$. By Proposition 9.2.4, any two lifts $\tilde{\sigma}_0$ and $\tilde{\sigma}_1$ of a class $\sigma \in A_*(X/\!\!/G)_{\mathbb{Q}}$ have analogues $\tilde{\tau}_0$ and $\tilde{\tau}_1$ both lifting a class $\tau \in A_*(Y/\!\!/G)_{\mathbb{Q}}$ and satisfying

$$\int_{X/\!\!/T} \Delta' \frown (\Delta \frown \tilde{\sigma}_i) = \int_{Y/\!\!/T} \Delta' \frown (\Delta \frown \tilde{\tau}_i),$$

for $i = 0, 1$. The result immediately follows from the smooth case (cf. Cor. 8.5.3). ■

Chapter 10

Invariance of the GIT integration ratio

The goal of this chapter is to prove Theorem 7.3.1, that for a G-linearized projective variety X over a field k with no strictly T-semi-stable points, the following ratio is an invariant of the group G:

Definition 10.0.1. Assume X is a G-linearized projective variety for which $X_T^{ss} = X_T^s$, and $\sigma \in A_0(X/\!\!/G)$ is a 0-cycle satisfying $\int_{X/\!\!/G} \sigma \neq 0$. We define the *GIT integration ratio* to be

$$r_{G,T}^{X,\sigma} := \frac{\int_{X/\!\!/T} c_{\mathrm{top}}(\mathfrak{g}/\mathfrak{t}) \frown \tilde{\sigma}}{\int_{X/\!\!/G} \sigma},$$

where $\tilde{\sigma}$ is some lift of the class σ.

Note that the GIT integration ratio is well-defined, as seen by taking $\Delta' = (-1)^{|\Phi|/2}\Delta$ and applying Proposition 9.3.1. We prove the invariance of $r_{G,T}^{X,\sigma}$ in stages, showing that it is independent of the Chow class $\sigma \in A_0(X/\!\!/G)_{\mathbb{Q}}$, the choice of maximal torus T, and then the G-linearized variety X.

10.1 Chow class

Lemma 10.1.1. *The GIT integration ratio* $r^X_{G,T} := r^{X,\sigma}_{G,T}$ *is independent of the choice of Chow class* $\sigma \in A_0(X/\!\!/G)_{\mathbb{Q}}$.

Proof. The definition of $r^{X,\sigma}_{G,T}$ is independent of the algebraic equivalence class of σ, since numerical equivalence is coarser than algebraic equivalence. Let $B_*(-)$ denote the quotient of the Chow group $A_*(-)$ by the relation of algebraic equivalence (cf. [12, §10.3]). All connected projective schemes are algebraically connected, i.e. there is a connected chain of (possibly singular) curves connecting any two closed points. Therefore, $B_0(X/\!\!/G)_{\mathbb{Q}} = \mathbb{Q}$, and the result follows since $r^{X,\sigma}_{G,T}$ is invariant under the scaling of σ. ∎

10.2 Maximal torus

We begin by reducing to the case where $k = \bar{k}$ is an algebraically closed field.

Lemma 10.2.1. *Let* X *be a projective variety over a field* k *equipped with a linearized right-action of a reductive group* G*, with maximal torus* $T \subseteq G$*, such that* $X^{ss}_T = X^s_T$*. For any field extension* K/k*, the GIT integration ratios are equal,*

$$r^X_{G,T} = r^{X_K}_{G_K,T_K},$$

where X_K*,* G_K*, and* T_K *denote the base change to* K *of* X*,* G*, and* T*, respectively.*

Proof. By [32, Prop. 1.14], the induced G_K-linearized action on X_K satisfies $X^{ss}_G \times K = (X_K)^{ss}_{G_K}$ (and similarly for T replacing G). Since field extensions are flat, taking the uniform categorical quotient commutes with extending the field, so $(X/\!\!/G)_K \cong X_K/\!\!/G_K$ and $(X/\!\!/T)_K \cong X_K/\!\!/T_K$. The result can then be deduced from the fact that the degree of a Chow class is invariant under field extension (cf. [12, Ex. 6.2.9]). ∎

Lemma 10.2.2. *The GIT integration ratio* $r^X_G := r^X_{G,T}$ *does not depend on the choice of maximal torus* T.

Proof. By Lemma 10.2.1, we may assume that $k = \bar{k}$. For any two maximal tori $T, T' \subseteq G$, there exists some $g \in G(\bar{k})$ such that $T' = gTg^{-1}$. By assumption, G acts linearly on the projective variety X. Consider the map $\psi : T \to T'$ given by $t \mapsto gtg^{-1}$, and the map $\Psi : X \to X$ given by $x \mapsto x \cdot g^{-1}$. The pair of maps (ψ, Ψ) show that the linearized actions $\sigma : X \times T \to X$ and $\sigma' : X \times T' \to X$ are isomorphic: $\Psi(x) \cdot \psi(t) = \Psi(x \cdot t)$. By the Hilbert-Mumford numerical criterion (Thm. 8.1.4), $X^{ss}_{T'} = \Psi(X^{ss}_T)$, and hence the following square is commutative,

$$\begin{tikzcd} X^{ss}_G \arrow[r, "\Psi"] \arrow[d, hook, "i_T"'] & X^{ss}_G \arrow[d, hook, "i_{T'}"'] \\ X^{ss}_T \arrow[r, "\Psi"] & X^{ss}_{T'}, \end{tikzcd}$$

where the vertical arrows are open immersions and the horizontal arrows are isomorphisms. From this diagram, it is clear that for any lift $\tilde{\sigma} \in A_*(X/\!\!/T)_{\mathbb{Q}}$ of a Chow class $\sigma \in A_0(X/\!\!/G)_{\mathbb{Q}}$, the push-forward $\Psi_*\tilde{\sigma} \in A_*(X/\!\!/T')_{\mathbb{Q}}$ is a lift of $\Psi_*\sigma = \sigma$. We use the pairs $(\sigma, \tilde{\sigma})$ and $(\sigma, \Psi_*\tilde{\sigma})$ to compute the GIT integration ratios. Since Ψ is an isomorphism, the classes $c_{\text{top}}(\mathfrak{g}/\mathfrak{t}) \frown \tilde{\sigma}$ and $\Psi_*(c_{\text{top}}(\mathfrak{g}/\mathfrak{t}) \frown \tilde{\sigma}) = c_{\text{top}}(\mathfrak{g}/\mathfrak{t}') \frown \Psi_*\tilde{\sigma}$ have the same degree, where $\mathfrak{t}'$ denotes the Lie algebra of T'. Therefore, the ratios $r^X_{G,T}$ and $r^X_{G,T'}$ are equal. ■

10.3 Linearized variety

Lemma 10.3.1. *The GIT integration ratio $r_G := r^X_G$ does not depend on the choice of G-linearized variety X.*

Proof. Let X_i for $i = 1, 2$ be two G-linearized projective varieties for which $(X_i)^{ss}_T = (X_i)^s_T$. Some high tensor powers of these linearizations define G-equivariant embeddings $j_i : X_i \hookrightarrow \mathbb{P}(V_i)$ for G-representations V_i, $i = 1, 2$. For $i = 1, 2$ there are equivariant embeddings $X_i \hookrightarrow \mathbb{P}(V_1 \oplus V_2)$, defined from the embeddings j_i by setting the extraneous coordinates to 0. Moreover, these embeddings are compatible with the G-linearization on $\mathbb{P}(V_1 \oplus V_2)$ given by $\mathcal{O}(1)$ with the linearized action induced from the direct sum representation of G on $V_1 \oplus V_2$. By Proposition 9.2.4, for any classes $\sigma_i \in A_0(X_i/\!\!/G)_{\mathbb{Q}}$, there are classes $\tau_i \in A_0(Y/\!\!/G)_{\mathbb{Q}}$, for some smooth G-linearized Y over $\mathbb{P}(V)$ satisfying $Y^{ss}_T = Y^s_T \neq \emptyset$, such that $r^{Y,\tau_i}_G = r^{X_i,\sigma_i}_G$. By Lemma 10.1.1, $r^{Y,\tau_1}_G = r^{Y,\tau_2}_G$ is an invariant of the G-linearized space Y, and therefore $r^{X_1,\sigma_1}_G = r^{X_2,\sigma_2}_G$. ■

The above lemmas combine to demonstrate Theorem 7.3.1:

Theorem 7.3.1. *Let G be a reductive group over an arbitrary field and $T \subseteq G$ a maximal torus. If X is a projective G-linearized variety satisfying $X_T^{ss} = X_T^s$ and $\sigma \in A_0(X/\!\!/G)$ is a Chow 0-cycle satisfying $\int_{X/\!\!/G} \sigma \neq 0$, then the GIT integration ratio depends not on the choice of σ, T, or X. That is, $r_G := r_{G,T}^{X,\sigma}$ is an invariant of the group G.*

Chapter 11

Direct products and central extensions

In this chapter, we prove that the GIT integration ratio behaves well with respect to the group operations of direct product and central extension.

11.1 Direct products

Lemma 11.1.1. *If G_1, G_2 are two reductive groups over a field k, then $r_{G_1 \times G_2} = r_{G_1} \cdot r_{G_2}$.*

Proof. For each $i = 1, 2$, choose a projective X_i on which G_i acts linearly, and let $T_i \subseteq G_i$ denote a maximal torus. Clearly $G_1 \times G_2$ acts on $X_1 \times X_2$ linearly, and the stability loci are just the products of the corresponding loci from the factors. Let $\sigma_i \in A_0^{G_i}((X_i)^{ss}_{G_i})$, and consider $\sigma := \sigma_1 \times \sigma_2 \in A_0^{G_1 \times G_2}((X_1)^{ss}_{G_1} \times (X_2)^{ss}_{G_2})$. Also, take $\tilde{\sigma} := \tilde{\sigma}_1 \times \tilde{\sigma}_2$ to where each σ_i is be the lift of σ_i to the T_i-semi-stable locus of X_i. We may calculate the GIT integration ratio for $G_1 \times G_2$ using these classes, since the ratio is independent of such choices (cf. Thm. 7.3.1). The degree of a product of two classes is the product of the degrees, and so the result follows since the isomorphism $[\operatorname{Spec} k/T] \cong [\operatorname{Spec} k/T_1] \times_k [\operatorname{Spec} k/T_2]$ identifies $c_{\mathrm{top}}(\mathfrak{g}_1/\mathfrak{t}_1) \times c_{\mathrm{top}}(\mathfrak{g}_2/\mathfrak{t}_2)$ with $c_{\mathrm{top}}(\mathfrak{g}/\mathfrak{t})$. ∎

11.2 Central extensions

In this section we prove that r_G is invariant under central extensions, completing the proof of Theorem 7.3.2. The discussion is rather involved, and in a first reading, the reader is advised to skip straight to Proposition 11.2.6.

Lemma 11.2.1. *Let $\mathscr{L}$ be a G-linearization of a variety X for which the GIT quotient $\pi : X^{ss}_G \to X/\!\!/G$ is nonempty. There exists some $n > 0$ such that $\mathscr{L}^{\otimes n}$ descends to a line bundle $\mathscr{M}$ on $X/\!\!/G$, and moreover, $\mathscr{M}$ is the uniform categorical quotient of the induced action $G \times \mathscr{L}^{\otimes n} \to \mathscr{L}^{\otimes n}$.*

Proof. From [32, Thm. 1.10], we see that $X^{ss}_G \to X/\!\!/G$ is a uniform categorical quotient and that some power $\mathscr{L}^{\otimes n}$ descends to $\mathscr{M}$. Since $\mathscr{L} \to X/\!\!/G$ is flat, the base change morphism $\mathscr{L}^{\otimes n} \to \mathscr{M}$ is also a uniform categorical quotient. ∎

Lemma 11.2.2. *Let $1 \to S \to \tilde{G} \to G \to 1$ be a central extension of reductive groups and X a scheme on which $\tilde{G}$ acts. If $\pi_S : X \to X/S$ and $\pi_{\tilde{G}} : X \to X/\tilde{G}$ are uniform categorical quotients of X by S and $\tilde{G}$ respectively, then the induced morphism $\pi_G : X/S \to X/\tilde{G}$ is a uniform categorical quotient by the induced action of G on X/S.*

Proof. Consider the action $\sigma : X \times \tilde{G} \to X$. Compose with π_S to obtain the $S \times S$-invariant morphism $X \times \tilde{G} \to X/S$, which therefore descends to an action $X/S \times G \to X/S$. All that remains to be shown is that $\pi_G : X/S \to X/\tilde{G}$ is the uniform categorical quotient for this action.

It suffices to show for each open affine $U \subseteq X/\tilde{G}$ that the restriction $\pi_G : \pi_G^{-1}(U) \to U$ is a uniform categorical quotient (cf. [32, Rem. 0.2(5)] for the case of *universal* categorical quotients). This is true because of the easy fact that for an affine ring R on which $\tilde{G}$ acts, the rings of invariants satisfy $R^{\tilde{G}} = (R^S)^G$. ∎

Lemma 11.2.3. *Let $1 \to S \to \tilde{G} \to G \to 1$ be a central extension of reductive groups. Let $\tilde{G}$ act linearly on a variety X, and $\pi : X^{ss}_S \to X/\!\!/S$ denote the induced GIT quotient. There is an induced G-linearized action on $X/\!\!/S$, and the semi-stable loci satisfy*

$$X^{ss}_{\tilde{G}} = \pi^{-1}\left((X/\!\!/S)^{ss}_G\right). \tag{11.2.4}$$

Moreover, this yields a canonical isomorphism between GIT quotients $(X/\!\!/S)/\!\!/G \cong X/\!\!/\tilde{G}$.

Proof. Choose $n \in \mathbb{N}$ as in Lemma 11.2.1 so that $\mathscr{L}^{\otimes n}$ descends to $\mathscr{M}$, the ample line bundle on $X/\!\!/S$ that is the uniform categorical quotient of $\mathscr{L}^{\otimes n}$ by S. Lemma 11.2.2 results in compatible G-actions on $X/\!\!/S$ and $\mathscr{L}^{\otimes n}$, i.e. a G-linearized action on $X/\!\!/S$. Therefore, we can take the GIT quotient $(X/\!\!/S)/\!\!/G$, which by Lemma 11.2.2 is a uniform categorical $\tilde{G}$-quotient of $\pi^{-1}((X/\!\!/S)^{ss}_G)$.

By the Hilbert-Mumford criterion (Thm. 8.1.4), $X^{ss}_S \subseteq X^{ss}_{\tilde{G}}$. Since $\mathscr{M}$ pulls-back to the tensor power $\mathscr{L}^{\otimes n}$, G-equivariant sections of $\mathscr{M}^{\otimes m}$ pull-back to $\tilde{G}$-equivariant sections of $\mathscr{L}^{\otimes mn}$, and hence $\pi^{-1}(X/\!\!/S)^{ss}_G \subseteq X^{ss}_{\tilde{G}}$. Conversely, if $\sigma \in \Gamma(X, \mathscr{L}^{\otimes n})^{\tilde{G}}$ is a $\tilde{G}$-equivariant section, then due to its S-equivariance, it descends to a G-equivariant section $\bar{\sigma} \in \Gamma(X/\!\!/S, \mathscr{M})^G$ since $\mathscr{M}$ is a quotient of $\mathscr{L}^{\otimes n}$ by S. Therefore, the inclusion is full: $\pi^{-1}((X/\!\!/S)^{ss}_G) = X^{ss}_{\tilde{G}}$. Combining this with Lemma 11.2.2, we obtain isomorphisms

$$X/\!\!/\tilde{G} \cong (X^{ss}_{\tilde{G}}/\!\!/S)/\!\!/G \cong (X/\!\!/S)^{ss}_G/\!\!/G = (X/\!\!/S)/\!\!/G.$$

■

Lemma 11.2.5. *Let* $1 \to S \to \tilde{G} \twoheadrightarrow G \to 1$ *be a central extension of reductive groups. If* $\tilde{G}$ *acts linearly on a variety* X *so that* $X^{ss}_{\tilde{G}} = X^{s}_{\tilde{G}} \neq \emptyset$, *then there is a canonical morphism of stacks* $\phi : [X^{ss}_{\tilde{G}}/\tilde{G}] \to [(X^{ss}_{\tilde{G}}/\!\!/S)/G]$. *Moreover, the morphism* ϕ *is proper and makes the following diagram commute:*

$$\begin{tikzcd}
{[X^{ss}_{\tilde{G}}/\tilde{G}]} \arrow[r, "\phi"] \arrow[d] & {[(X^{ss}_{\tilde{G}}/\!\!/S)/G]} \arrow[d] \\
{[\operatorname{Spec} k/\tilde{G}]} \arrow[r] & {[\operatorname{Spec} k/G],}
\end{tikzcd}$$

where the vertical arrows are induced by the structure morphisms and the lower arrow by the surjection of groups.

Proof. We describe the functor ϕ on objects, omitting the description of the functor on morphisms. Let $\underline{E} := (B \xleftarrow{\pi} E \xrightarrow{f} X^{ss}_{\tilde{G}})$ be an object of $[X^{ss}_{\tilde{G}}/\tilde{G}]$, i.e. $\pi : E \to B$ is a $\tilde{G}$-torsor in the étale topology, and $f : E \to X^{ss}_{\tilde{G}}$ is a $\tilde{G}$-equivariant morphism. Consider the induced G-torsor, $E \times_{\tilde{G}} G$. It is clear that $E \times_{\tilde{G}} G \cong E/S$, the uniform categorical quotient of E by S, and therefore admits a morphism to $X^{ss}_{\tilde{G}} /\!\!/ S$. The result is the object $\underline{E} \times_{\tilde{G}} G := (B \xleftarrow{\pi} E \times_{\tilde{G}} G \xrightarrow{f} X^{ss}_{\tilde{G}}/S)$ of the quotient stack $[(X^{ss}_{\tilde{G}} /\!\!/ S)/G]$. The construction $\underline{E} \mapsto \underline{E} \times_{\tilde{G}} G$ is clearly functorial and thus defines ϕ as a morphism of stacks.

The diagram clearly commutes, and it is left to see that ϕ is proper. By Lemma 11.2.3, $(X^{ss}_{\tilde{G}}) /\!\!/ G \cong X /\!\!/ \tilde{G}$, and so the morphism $f : [(X^{ss}_{\tilde{G}} /\!\!/ S)/G] \to X /\!\!/ \tilde{G}$, in addition to the morphism $g : [X^{ss}_{\tilde{G}}/\tilde{G}] \to X /\!\!/ \tilde{G}$, is a coarse moduli space morphism. As $g = f \circ \phi$ and, being coarse moduli morphisms, g is proper and f is separated, consequently ϕ is proper. ∎

Proposition 11.2.6. *If $\tilde{G} \twoheadrightarrow G$ is a central extension of reductive groups, then $r_{\tilde{G}} = r_G$.*

Proof. The case of a finite central extension is trivial, as we may use the same G-variety X on which to calculate both ratios r_G and $r_{\tilde{G}}$. This allows us to then reduce to the case where the kernel S is a torus. Since S centralizes the maximal torus $\tilde{T} \subseteq \tilde{G}$, it must be contained within $\tilde{T}$; hence there is an analogous exact sequence, $1 \to S \to \tilde{T} \to T \to 1$. Also, notice that the class $c_{\mathrm{top}}(\mathfrak{g}/\mathfrak{t}) \in A^*(BT)$ pulls-back via $B\tilde{T} \to BT$ to $c_{\mathrm{top}}(\tilde{\mathfrak{g}}/\tilde{\mathfrak{t}}) \in A^*(B\tilde{T})$, since the map $\tilde{G} \twoheadrightarrow G$ induces an isomorphism of root systems. Let X be a projective $\tilde{G}$-linearized variety such that $X^{ss}_{\tilde{T}} = X^{s}_{\tilde{T}}$ and $X^{ss}_{\tilde{G}} \neq \emptyset$. By Lemma 11.2.3 the projective scheme $X /\!\!/ S$ has an induced G-linearization. For this linearization, $(X /\!\!/ S)^{ss}_T = (X /\!\!/ S)^{s}_T$ and $(X /\!\!/ S)^{ss}_G \neq \emptyset$, because of the analogous properties of the $\tilde{G}$-linearization on X. We will use X to compute $r_{\tilde{G}}$ and $X /\!\!/ S$ to compute r_G.

There is a commutative diagram:

$$\begin{tikzcd}
{[X^{ss}_{\tilde{T}}/\tilde{T}]} \arrow[d, "\phi_1"] & {[X^{ss}_{\tilde{G}}/\tilde{T}]} \arrow[l, hook', "\tilde{i}"'] \arrow[r, "\tilde{\pi}"] \arrow[d, "\phi_2"] & {[X^{ss}_{\tilde{G}}/\tilde{G}]} \arrow[d, "\phi_3"] \\
{[(X /\!\!/ S)^{ss}_T/T]} & {[(X /\!\!/ S)^{ss}_G/T]} \arrow[l, hook', "i"'] \arrow[r, "\pi"] & {[(X /\!\!/ S)^{ss}_G/G].}
\end{tikzcd}$$

Lemma 11.2.3 combined with Lemma 11.2.5 implies that each morphism ϕ_j induces an isomorphism $(\phi_j)_*$ on rational Chow groups. Moreover, the commutative diagram of Lemma 11.2.5 (with $\tilde{G}$ of the diagram equal to our $\tilde{T}$) together with the projection formula imply that for any Chow class $\sigma \in A_*((X/\!\!/\tilde{G})_{\mathbb{Q}}$ with lift $\tilde{\sigma} \in A_*(X/\!\!/\tilde{T})_{\mathbb{Q}}$,

$$(\phi_1)_*(c_{\mathrm{top}}(\tilde{\mathfrak{g}}/\tilde{\mathfrak{t}}) \frown \tilde{\sigma}) = c_{\mathrm{top}}(\mathfrak{g}/\mathfrak{t}) \frown (\phi_1)_*\tilde{\sigma}.$$

The theorem follows once we check that $(\phi_1)_*(\tilde{\sigma})$ is a lift of the class $(\phi_3)_*\sigma$, for then the equality

$$\frac{\int_{X/\!\!/\tilde{T}} c_{\mathrm{top}}(\tilde{\mathfrak{g}}/\tilde{\mathfrak{t}}) \frown \tilde{\sigma}}{\int_{X/\!\!/\tilde{G}} \sigma} = \frac{\int_{(X/\!\!/S)/\!\!/T} c_{\mathrm{top}}(\mathfrak{g}/\mathfrak{t}) \frown (\phi_1)_*\tilde{\sigma}}{\int_{(X/\!\!/S)/\!\!/G} (\phi_3)_*\sigma}$$

follows immediately from the functoriality of Chow groups under proper pushes-forward. The Chow class $(\phi_1)_*\tilde{\sigma}$ is indeed a lift of $(\phi_3)_*\sigma$:

$$(\phi_2)_* \circ \tilde{\pi}^* = \pi^* \circ (\phi_3)_*,$$

by [12, Prop. 1.7] because the right square is a fibre square by Lemma 9.2.2, and then

$$\begin{aligned} i^*(\phi_1)_*(\tilde{\sigma}) &= (\phi_2)_*(\tilde{i}^*\tilde{\sigma}) \\ &= (\phi_2)_*(\tilde{\pi}^*\sigma) \\ &= \pi^*(\phi_3)_*\sigma, \end{aligned}$$

where the first equality follows from [12, Prop. 1.7] applied now to the left fibre square of the diagram. ■

The results of this chapter combine to prove Theorem 7.3.2.

Theorem 7.3.2. *If G is a reductive group over a field k that, up to central extensions, is the product of simple groups $G_1 \times \cdots \times G_n$, then*

$$r_G = \prod_{i=1}^{n} r_{G_i}.$$

Chapter 12

A computation for groups of type $\mathbf{A}_n$

We compute the GIT integration ratio r_G for $G = PGL(n)$, and use this to prove Corollary 7.3.3.

Proposition 12.0.1. *The GIT integration ratio for the group $G = PGL(n)$ defined over any field is*

$$r_G = |W| = n!.$$

Proof. Let $SL(n)$ act on $\mathbf{M}_n$, the vector space of $n \times n$ matrices with k-valued entries, via right multiplication of matrices. This induces a dual right-action of $SL(n)$ on $\mathbf{M}_n^*$ and hence right-actions of $SL(n)$ and $PGL(n)$ on $\mathbb{P}(\mathbf{M}_n)$, the projective space of lines in $\mathbf{M}_n^*$. Choose the $PGL(n)$-linearization on $\mathcal{O}_{\mathbb{P}(\mathbf{M}_n)}(n)$ induced from these representations of $SL(n)$. Let $T \subseteq PGL(n)$ and $\tilde{T} \subseteq SL(n)$ denote the diagonal maximal tori. In this case, the $PGL(n)$-stability loci (resp. T-stability loci) are equal to the analogous $SL(n)$-stability loci (resp. $\tilde{T}$-stability loci), which we proceed to describe.

A basis of $\mathbf{M}_n$ is given by the matrices e_{ij}, each defined by its unique nonzero entry of 1 in the (i, j)th position. Moreover, e_{ij} is a weight vector of weight $\chi_j \in \Lambda^*(\tilde{T})$, with χ_j

defined by the rule

$$\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} \mapsto t_j.$$

Notice that $\chi_1, \ldots, \chi_{n-1}$ is a basis of the character group $\Lambda^*(\tilde{T})$ and $\chi_n = -\sum_{i=1}^{n-1} \chi_i$, so that the characters $\chi_1, \ldots, \chi_n$ form the vertices of a simplex centered at the origin in $\Lambda^*(\tilde{T})_{\mathbb{Q}}$. From the Hilbert-Mumford criterion (Thm. 8.1.4), one quickly concludes that the unstable locus $\mathbb{P}(\mathbf{M}_n)^{un}_{\tilde{T}}$ is the set of all points $x \in \mathbb{P}(\mathbf{M}_n)$ such that the matrix $e_{ij}(x)$ has a column with all entries 0; all other points are $\tilde{T}$-stable. Such stable points have trivial T-stabilizers, and so the T-quotient is easily seen to be

$$[\mathbb{P}(\mathbf{M}_n)^{ss}_T/T] \cong \mathbb{P}(\mathbf{M}_n)/\!\!/T \cong (\mathbb{P}^{n-1})^n.$$

As there are no strictly semi-stable points for the $\tilde{T}$-action, neither are there any for the $SL(n)$-action. The $SL(n)$-stable locus then comprises the set of $x \in \mathbb{P}(\mathbf{M}_n)$ such that the matrix $e_{ij}(x)$ is of full-rank; this comprises a dense $PGL(n)$ orbit. The stabilizer of this orbit is trivial in $PGL(n)$, and hence

$$[\mathbb{P}(\mathbf{M}_n)^{ss}_{PGL(n)}/PGL(n)] \cong \mathbb{P}(\mathbf{M}_n)/\!\!/PGL(n) \cong \operatorname{Spec} k.$$

The rational Chow ring of the T-quotient is

$$A^*(\mathbb{P}(\mathbf{M}_n)/\!\!/T)_{\mathbb{Q}} \cong \mathbb{Q}[t_1, \ldots, t_n]/(t_1^n, \ldots, t_n^n).$$

In this ring, the class of a point is clearly $\prod_{i=1}^n t_i^{n-1}$. The class $c_{\mathrm{top}}(\mathfrak{g}/\mathfrak{t}) \in A^*(BT)$ is the product of all the roots, which are of the form $\alpha_{ij} := \chi_i - \chi_j \in \operatorname{Sym}^* \Lambda^*(T) \cong A^*(BT)$, for $1 \leq i \neq j \leq n$. One can check that the pull-back of $\chi_i - \chi_j$ to $A^*(\mathbb{P}(\mathbf{M}_n)/\!\!/T)_{\mathbb{Q}}$ is $t_i - t_j$, and therefore $c_{\mathrm{top}}(\mathfrak{g}/\mathfrak{t}) = \prod_{i \neq j}(t_i - t_j)$. Let $\sigma \in A_0(\mathbb{P}(\mathbf{M}_n)/\!\!/G)_{\mathbb{Q}} \cong \mathbb{Q}$ denote the fundamental class; i.e. $\int_{\mathbb{P}(\mathbf{M}_n)/\!\!/PGL(n)} \sigma = 1$. Therefore, the GIT integration ratio r_G is just $\int_{\mathbb{P}(\mathbf{M}_n)/\!\!/T} c_{\mathrm{top}}(\mathfrak{g}/\mathfrak{t})$, which equals the coefficient of the monomial $\prod_{i=1}^n t_i^{n-1}$ in the expansion of $\prod_{i \neq j}(t_i - t_j)$, since the other monomials of degree $n^2 - n$ vanish in the ring $\mathbb{Q}[t_1, \ldots, t_n]/(t_1^n, \ldots, t_n^n)$.

We can compute this coefficient as follows. The class $c_{\text{top}}(\mathfrak{g}/\mathfrak{t})$ may be alternatively expressed as

$$\prod_{i\neq j}(t_i - t_j) = (-1)^{n(n-1)/2}(\det M_V)^2,$$

where $\det M_V$ is the determinant of the Vandermonde matrix

$$M_V := \begin{pmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{pmatrix}.$$

By definition, $\det M_V = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n t_i^{\sigma(i)-1}$. In the ring $A^*((\mathbb{P}^{n-1})^n)_{\mathbb{Q}}$, the products of monomials of the form $m_\sigma := \prod_{i=1}^n t_i^{\sigma(i)-1}$ for $\sigma \in S_n$ are defined by the rule:

$$m_\sigma \cdot m_{\sigma'} = \begin{cases} \prod_{i=1}^n t_i^{n-1} & : \sigma(j) + \sigma'(j) = n+1;\ \forall\, 1 \leq j \leq n; \\ 0 & : \text{ otherwise.} \end{cases}$$

If $w_0 := (1\ n)(2\ n-1)\cdots(\lceil n/2\rceil\ \lceil (n+1)/2\rceil) \in S_n$ denote the longest element of the Weyl group W, then for each $\sigma \in S_n$, the permutation σ' defined as the composition $\sigma' := w_0 \circ \sigma$ is the unique element of S_n for which $m_\sigma \cdot m_{\sigma'} \neq 0$. For such pairs (σ, σ'), the product of the signs satisfies $\operatorname{sgn}(\sigma)\cdot\operatorname{sgn}(\sigma') = \operatorname{sgn}(w_0) = (-1)^{(n^2-n)/2}$. Therefore,

$$\begin{aligned} c_{\text{top}}(\mathfrak{g}/\mathfrak{t}) &= (-1)^{(n^2-n)/2} \cdot \sum_{\sigma\in S_n} (-1)^{(n^2-n)/2} \prod_{i=1}^n t_i^{n-1} \\ &= n! \cdot \prod_{i=1}^n t_i^{n-1}. \end{aligned}$$

Thus, $r_G = n! = |W|$. ■

The proof of Corollary 7.3.3 is now anticlimactic:

Corollary 7.3.3. *Let G be a connected reductive group over a field k and $T \subseteq G$ a maximal torus. If the root system of G decomposes into irreducible root systems of type $\mathbf{A}_n$, for various $n \in \mathbb{N}$, then for any G-linearized projective k-variety X for which $X_T^s = X_T^{ss}$ and any Chow class $\sigma \in A_0(X/\!\!/G)_{\mathbb{Q}}$ with lift $\tilde{\sigma} \in A_*(X/\!\!/T)_{\mathbb{Q}}$,*

$$\int_{X/\!\!/G} \sigma = \frac{1}{|W|}\int_{X/\!\!/T} c_{top}(\mathfrak{g}/\mathfrak{t}) \frown \tilde{\sigma}.$$

Proof. Combine Theorems 7.3.1 and 7.3.2, and Proposition 12.0.1. ■

Chapter 13

Generalization

We conclude with a discussion of how to generalize Corollary 7.3.3 to arbitrary connected reductive groups, that is, how to prove Amplification 7.3.4. The argument will rely on Martin's original result [28, Thm. B], and the discovery of an independent proof is left as an open question.

Amplification 7.3.4. *Let G be a connected reductive group over a field k and $T \subseteq G$ a maximal torus. For any G-linearized projective k-variety X for which $X^s_T = X^{ss}_T$ and any Chow class $\sigma \in A_0(X/\!\!/G)_{\mathbb{Q}}$ with lift $\tilde{\sigma} \in A_*(X/\!\!/T)_{\mathbb{Q}}$,*

$$\int_{X/\!\!/G} \sigma = \frac{1}{|W|} \int_{X/\!\!/T} c_{top}(\mathfrak{g}/\mathfrak{t}) \frown \tilde{\sigma}.$$

Proof. We begin by pointing the reader to [38] as a reference on geometric invariant theory relative to a base. The base we will use is the spectrum of the ring $\mathbb{Z}_{(p)}$, the localization of $\mathbb{Z}$ at the prime p equal to the characteristic of the base field k.

By Theorems 7.3.1 and 7.3.2, it suffices to verify $r^X_G = |W|$ on a single G-linearized projective variety X for each simple Chevalley group G. Each Chevalley group G admits a model $G_{\mathbb{Z}}$ over the integers, with a split maximal torus $T_{\mathbb{Z}} \subseteq G_{\mathbb{Z}}$. We assert that there is a smooth projective $\mathbb{Z}_{(p)}$-scheme $X_{(p)}$ on which $G_{(p)} := G_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{Z}_{(p)}$ acts linearly and for which all $G_{(p)}$- (resp. $T_{(p)}$-) semi-stable points are stable and comprise an open locus that nontrivially intersects the closed fibre over $\mathbb{F}_p$. We justify this assertion briefly: Proposition 9.1.3 reduces the problem to finding some $\mathbb{Z}_{(p)}$-scheme for which there exist $G_{(p)}$-stable points in the closed fibre over $\mathbb{F}_p$; with the aid of the Hilbert-Mumford criterion, one discovers that many such schemes exist (e.g. take $\mathbb{P}(V_{\mathbb{Z}_{(p)}}^{\oplus n})$ with $V_{\mathbb{Z}_{(p)}}^{\oplus n}$ a large multiple of a general irreducible $G_{(p)}$-representation).

Having chosen such an $X_{(p)}$, the technique of specialization (cf. [12, §20.3]) implies that the integral of relative 0-cycles on $X_{(p)}/\!\!/G_{(p)}$ and $X_{(p)}/\!\!/T_{(p)}$ restricted to the generic fibre over $\mathbb{Q}$ is equal to the integral restricted to any closed fibre over $\mathbb{F}_p$. The ratio r_G is independent under field extension by Lemma 10.2.1, and so this reduces the calculation of r_G over the field k to the computation of $r_{G_\mathbb{C}}$, where $G_\mathbb{C} := G_\mathbb{Z} \times_\mathbb{Z} \mathbb{C}$. The Kirwan-Kempf-Ness theorem (cf. [23, §8] or [32, §8.2]) shows that in the analytic topology, the GIT quotient $X_\mathbb{C}/\!\!/G_\mathbb{C}$ is homeomorphic to the symplectic reduction of $X_\mathbb{C}$ by a maximal compact subgroup, and so Martin's theorem [28, Thm. B] implies $r_{G_\mathbb{C}} = |W|$. ■

Question 13.0.1. *What is a purely algebraic proof that $r_G = |W|$ for a general connected reductive group G?*

In light of Theorems 7.3.1 and 7.3.2, to answer Question 13.0.1 it suffices to verify $r_G = |W|$ for all simple groups G. Such a verification was done in Chapter 12 for simple groups of type $\mathbf{A}_n$. Can r_G be calculated (algebraically) for any other simple groups G?

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Appendix

Appendix A

Chow groups and quotient stacks

Here we recall the basic properties of Chow groups for schemes and quotient stacks.

A.1 Chow groups

For a scheme X defined over a field k, let $A_i(X)$ denote the $\mathbb{Z}$-module generated by i-dimensional subvarieties over k modulo rational equivalence (cf. [12]). We call $A_*(X) := \oplus_i A_i(X)$ the *Chow group* of X. To indicate rational coefficients, we write $A_*(X)_{\mathbb{Q}} := A_*(X) \otimes \mathbb{Q}$.

For a scheme X over a field k and an algebraic group G acting on X, the Chow group of the quotient stack $[X/G]$ is defined by Edidin and Graham in [8] to be the limit of Chow groups using Totaro's finite approximation construction (cf. [40]):

$$A_i([X/G]) := A_{i-g+\ell}(X \times_G U),$$

where U is an open subset of an ℓ-dimensional G-representation V on which G acts freely and whose complement $V \setminus U$ has sufficiently large codimension. It is a result of Edidin and Graham that this group is well-defined, independent of the presentation of the stack $[X/G]$ as a quotient, and recovers Gillet's original definition (cf. [13]) of Chow groups on Deligne-Mumford stacks (see [8, §5]). When convienient, we may also think of $A_i[X/G]$ as the G-equivariant Chow group of X to highlight the functorality with respect to group homomorphisms, and we make use of the notation $A_*^G(X) := A_*([X/G])$. One benefit of the stack interpretation, which we use repeatedly in this chapter, arises when X admits a geometric quotient $X/\!\!/G$. In this case, there is a coarse moduli morphism from the quotient stack to the geometric quotient that induces an isomorphism of Chow groups with rational coefficients:

Theorem 1.1.1 (Gillet, Vistoli)**.** *If X is a variety that admits a geometric quotient $X/\!\!/G$ by a smooth, affine algebraic group G, then the induced morphism $\phi : [X/G] \to X/\!\!/G$ is a coarse moduli space morphism whose push-forward homomorphism,*

$$\phi_* : A_*([X/G])_{\mathbb{Q}} \to A_*(X/\!\!/G)_{\mathbb{Q}},$$

is an isomorphism of Chow groups with rational coefficients.

Proof. See [42, Prop. 2.11] for the proof that ϕ is a coarse moduli space morphism, and see [13, Thm. 6.8] for the proof that a coarse moduli space morphism induces an isomorphism of Chow groups with rational coefficients. ■

The Chow groups of quotient stacks are functorial with respect to the usual operations (e.g. flat pull-back, proper push-forward), and when X is a smooth n-dimensional variety, there is an intersection product that endows these groups with the structure of a commutative ring with identity, graded by codimension and denoted by $A^*([X/G]) := A_{n-g-*}([X/G])$. Hence the Chow group of the stack $A_*([X/G])$ is naturally a module over the ring $A^*(BG)$ where $BG = [\operatorname{Spec} k/G]$ is the trivial quotient. In the case $T = \mathbb{G}_m^r$ is a split torus of rank r,

$$A^*(BT) \cong \operatorname{Sym} \Lambda^*(T) \cong \mathbb{Z}[\chi_1, \ldots, \chi_r],$$

where $\chi_1, \ldots, \chi_r$ is some $\mathbb{Z}$-basis of $\Lambda^*(T)$. A character $\chi \in \Lambda^*(T)$ is equivalent to a line bundle $\mathscr{L}_\chi$ over BT whose Chern class $c_1(\mathscr{L}_\chi) \in A^*(BT)$ corresponds to χ under the above isomorphism.

As usual, the structure of the Chow group $A_*([X/T])$ of the stacky quotient of X by a torus T is especially well-understood (see [4]).

Proposition 1.1.2 (Brion)**.** *Let X be a scheme with the action of a torus T over an algebraically closed field $\bar{k}$. The T-equivariant Chow group $A_*^T(X)$ is generated as an $A^*(BT)$-module by the classes $[Y]$ associated to T-invariant closed subschemes $Y \hookrightarrow X$.*

Proof. See [4, Thm. 2.1]. ∎

Moreover, there is a localization theorem useful for making calculations in T-equivariant Chow groups. The following version of the localization theorem will suffice for our purposes:

Theorem 1.1.3 (Localization)**.** *Let X be a smooth projective scheme with the action of a torus T over an algebraically closed field $\bar{k}$, and let $i : X^T \to X$ denote the inclusion of the scheme of T-fixed points. Then the morphism*

$$i^* : A_*^T(X)_{\mathbb{Q}} \to A_*^T(X^T)_{\mathbb{Q}}$$

is an injective $A^*(BT)$-algebra morphism. Furthermore, if X^T consists of finitely many points, then the morphism

$$i^* : A_*^T(X) \to A_*^T(X^T)$$

of Chow groups with integer coefficients is injective as well.

Proof. See [4, Cor. 3.2.1]. ∎

A.2 Operational Chow groups

The ith operational Chow group $A^i(X)$ is defined to be the group of "operations" c that comprise a system of group homomorphisms $c_f : A_*(Y) \to A_{*-i}(Y)$ associated to morphisms of schemes $f : Y \to X$ and compatible with proper push-forward, flat pull-back, and the refined Gysin map (cf. [12, §17]). Similarly, Edidin and Graham define equivariant operational Chow groups $A^i_G(X)$ via systems of group homomorphisms $c_f^G : A_*^G(Y) \to A_{*-i}^G(Y)$ compatible with the G-equivariant analogues of the above maps (cf. [8, §2.6]).

The clearest examples of equivariant operational Chow classes are equivariant Chern classes $c_i(\mathscr{E})$ of G-linearized vector bundles $\mathscr{E}$ (i.e. Chern classes of vector bundles on $[X/G]$). Moreover, $A_G^*(X)$ equipped with composition forms an associative, graded ring with identity. When X is smooth, there is a Poincaré duality between the equivariant operational Chow group and the usual equivariant Chow group. For any operational Chow class $c = \{c_f^G\} \in A_G^i(X)$ and Chow class $\sigma \in A_*^G(Y)$, we introduce the following "cap product" notation:

$$c \frown \sigma := c_f^G(\sigma) \in A_{*-i}^G(Y).$$

Theorem 1.2.4 (Poincaré duality)**.** *If X is a smooth n-dimensional variety, then the map $A_G^i(X) \to A_{n-i}^G(X)$ defined by $c \mapsto c \frown [X]$ is an isomorphism.*

Proof. See [8, Prop. 4]. ∎

Remark 1.2.5. *When X is a smooth n-dimensional variety, this allows us to write $A_G^k(X)$ to denote the codimension k Chow group $A_{n-k}^G(X)$, without any ambiguity in notation. Furthermore, this identification forms an isomorphism of rings $A_G^*(X) \cong A_{n-*}^G(X)$, with the multiplication structure on $A_{n-*}^G(X)$ given by the intersection product.*