Purity of the stratification by Newton polygons and Frobenius-periodic vector bundles

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ABSTRACT

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This thesis includes two parts. In the first part, we show a purity theorem for stratifications by Newton polygons coming from crystalline cohomology, which says that the family of Newton polygons over a noetherian scheme have a common break point if this is true outside a subscheme of codimension bigger than 1. The proof is similar to the proof of [dJO99, Theorem 4.1].

In the second part, we prove that for every ordinary genus-2 curve X over a finite field κ of characteristic 2 with automorphism group $\mathbb{Z}/2\mathbb{Z} \times S_3$, there exist $\mathrm{SL}(2, \kappa[\![s]\!])$-representations of $\pi_1(X)$ such that the image of $\pi_1(\bar{X})$ is infinite. This result produces a family of examples similar to Laszlo's counterexample [Las01] to a question regarding the finiteness of the geometric monodromy of representations of the fundamental group [dJ01].

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To My Parents and My Husband

Chapter 1

Introduction

This thesis includes two topics. The first topic is purity of the stratification by Newton polygons from crystalline cohomology; the second topic is Frobenius-periodic vector bundles and the induced representations over $\mathbb{F}_q[\![s]\!]$. On the one hand, both topics are closely related to Artin-Schreier coverings; on the other hand, both topics are independent from each other. Therefore, in the following we will focus on Frobenius structures that are trivializable by Artin-Schreier coverings; for each topic, we will give a separate introduction at the beginning of each part.

Artin-Schreier theory is a branch of Galois theory, specifically in positive characteristic. A typical Artin-Schreier polynomial is as follows:

$$f_\alpha(x) = x^p - x + \alpha \text{ for } \alpha \in k, \tag{1.0.1}$$

where k is a field of characteristic p. Artin-Schreier coverings are étale covers defined by equations of the following form:

$$X^{(q)} = X + A, \tag{1.0.2}$$

where A is a given matrix with entries to be regular functions, q is some power of p, $X = (x_{ij})$ is a matrix of variables and $X^{(q)} = (x_{ij}^q)$.

The key reason for Frobenius structures inducing representations is that Frobenius structures can be trivialized under Artin-Schreier coverings. As we will see, two types of representations arise from this point, i.e. representations over $W(\mathbb{F}_q)$ and over $\mathbb{F}_q[\![s]\!]$. Both types come up to be an inverse limit of a sequence of representations ρ_n over $W(\mathbb{F}_q)/(p^{n+1})$ (resp. $\mathbb{F}_q[\![s]\!]/(s^{n+1})$). When $n > 0$, the reason for ρ_{n-1} liftable to ρ_n is exactly because some Artin-Schreier covering trivializes the given Frobenius structures after modulo p^{n+1} (resp. modulo s^{n+1}). The initial term ρ_0 comes to existence because the given Frobenius structures after modulo p (resp. modulo s) is trivializable by an étale covering defined by equations of the following form:

$$AX^{(q)} = X, \tag{1.0.3}$$

where A is an invertible matrix with entries to be regular functions.

Consider ρ_0 over a one-point space $\operatorname{Spec} k$. Assume that the field $k \supset \mathbb{F}_q$ and $\sigma : k \to k$ is the q^{th}-power Frobenius morphism. The Frobenius structure over $\operatorname{Spec} k$ modulo p or s is none other than a σ-linear endomorphism $\psi : V \to V$ of a k-vector space V defined by an invertible matrix, i.e. over a basis $\{e_1, \cdots, e_r\}$ of V, ψ is defined by $\psi\{e_1, \cdots, e_r\} = \{e_1, \cdots, e_r\}A$ with $A \in \mathrm{GL}(r, k)$. Clearly Equation (1.0.3) has a nontrivial solution X in a separable closure k^{sep} of k. Take a basis of $V \otimes k^{\mathrm{sep}}$ to be $\{\bar{e}_1, \cdots, \bar{e}_r\} = \{e_1, \cdots, e_r\} \otimes X$. Obviously $\{\bar{e}_1, \cdots, \bar{e}_r\}$ is fixed by $\psi \otimes k^{\mathrm{sep}}$. This is exactly the proof in [Kat73b, Proposition 1.2] of the fact that every finte-dimensional vector space over a separably closed field of characteristic p with a Frobenius-linear endomorphism whose linearization is an isomorphism has a basis fixed by the endomorphism. The associated representation is defined as

$$\rho_0 : \mathrm{Gal}(k^{\mathrm{sep}}/k) \to \mathrm{GL}(r, \mathbb{F}_q), \quad g \mapsto X^{-1}g(X). \tag{1.0.4}$$

Clearly the key point in the definition of ρ_0 is that the Frobenius structure $\psi : V \to V$ becomes trivialized after base change to an étale cover of $\operatorname{Spec} k$. This idea has been applied extensively to many kinds of Frobenius structures on locally free sheaves. We first list the most basic ones in each type.

We start with the $W(\mathbb{F}_q)$-case. Let k be a perfect field containing $\mathbb{F}_q$ and X/k be a connected smooth scheme. Let $\mathcal{X}/W(k)$ be a formally smooth lifting of X/k with a

q^{th}-power Frobenius lifting $\sigma : \mathcal{X} \to \mathcal{X}$. Then every locally free sheaf $\mathcal{E}$ over $\mathcal{X}/W(k)$ satisfying that $\sigma^*\mathcal{E} \simeq \mathcal{E}$ induces a representation

$$\rho : \pi_1(X) \to \mathrm{GL}(r, W(\mathbb{F}_q)), \text{ where } r = \mathrm{rk}\, \mathcal{E}. \tag{1.0.5}$$

In the $\mathbb{F}_q[\![s]\!]$-case, let X be a connected smooth scheme over $\operatorname{Spec} \mathbb{F}_q$. Let $\sigma : X \times \operatorname{Spf} \mathbb{F}_q[\![s]\!] \to X \times \operatorname{Spf} \mathbb{F}_q[\![s]\!]$ be a q^{th}-power Frobenius lifting such that $\sigma^*(s) = s$. Then every locally free sheaf $\mathcal{E}$ over $X \times \operatorname{Spf} \mathbb{F}_q[\![s]\!]$ satisfying that $\sigma^*\mathcal{E} \simeq \mathcal{E}$ induces a representation

$$\rho : \pi_1(X) \to \mathrm{GL}(r, \mathbb{F}_q[\![s]\!]), \text{ where } r = \mathrm{rk}\, \mathcal{E}. \tag{1.0.6}$$

In general, we and many others are interested in representations coming from unit-root F-(iso)crystals, overconvergent unit-root F-isocrystals, Frobenius-periodic vector bundles and so on.

In the $W(\mathbb{F}_q)$-case, the sheaf $\mathcal{E}$ that induces ρ in (1.0.5) is called a unit-root F-lattice. It has been proved in [Kat73a, Prop 4.1.1] (also refer to [Cre87, Theorem 2.2]) that there is a natural equivalence of categories

$$\mathrm{Rep}_{W(\mathbb{F}_q)}(\pi_1(X)) \Leftrightarrow (\text{Unit-root } F\text{-lattices on } \mathcal{X}/W(k)). \tag{1.0.7}$$

Based on the above equivalence, [Cre87, Theorem 2.1] has set up another equivalence

$$\mathrm{Rep}_{K^{\tilde{\sigma}}}(\pi_1(X)) \Leftrightarrow (\text{Unit-root } F\text{-isocrystals on } X/K). \tag{1.0.8}$$

For discussions on overconvergent unit-root F-isocrystals, we refer to [Cre87] and [Cre92].

Besides unit-root F-(iso)crystals, isoclinic F-crystals (i.e. all Newton slopes equal) can also be associated to representations. Particularly, we investigate rank-one representations decided by break points of Newton polygons of F-crystals, see Definition 6.1.2. A key fact in proving Theorem 6.3.4 is Proposition 6.2.2, which claims an equivalent relation between the unramified property of representations and the coincidence of break points.

In the $\mathbb{F}_q[\![s]\!]$-case, the sheaf $\mathcal{E}$ that induces ρ in (1.0.6) can be viewed as a family of Frobenius-periodic vector bundles on X. In the case of a single vector bundle, it is known from [LS77] and [BD07] that Frobenius-periodic vector bundles correspond bijectively to representations of algebraic fundamental groups over finite fields. A similar idea of [LS77, Prop 1.2] can be applied to the family case and set up an equivalence, see Propostion 8.1.8. [Las01, Lemma 3.3] makes use of torsors to obtain representations and that is equivalent to the notion of the lisse $\mathbb{F}_q[\![s]\!]$-sheaves in (8.1.2).

When restricted to projective smooth varieties over a finite field, we obtain in Proposition 8.2.7 a stronger equivalence between representations with an infinite geometric monodromy and a non-constant family of slope-stable Frobenius-periodic vector bundles. Therefore, the search for representations with an infinite geometric monodromy is converted to looking for Frobenius-periodic vector bundles. Specifically, by showing the existence of a family of Frobenius-periodic vector bundles over a family of genus-2 curves in characteristic 2, we obtain examples of representations with an infinite geometric monodromy in Theorem 9.3.1. Particularly, the example in [Las01] is re-discovered.

Part I

Purity in crystalline cohomology

Chapter 2

Introduction

In this part, we show a purity theorem for stratifications by Newton polygons coming from crystalline cohomology, which is analogous to [Yan11, Theorem 1.1], a purity theorem for stratifications by Newton polygons coming from an F-crystal. Both theorems follow from a similar proof to [dJO99, Theorem 4.1] and state a common property of stratifications by Newton polygons, which says that the family of Newton polygons over a scheme have a common break point if this is true outside a subset of codimension bigger than 1.

First we introduce the stratification by Newton polygons coming from crystalline cohomology. Given a proper smooth morphism $f : X \to S$, every point $e \in S$ is related to an F-module $(H^i_{cris}(X_e/W(k_e^{\mathrm{pf}})), F)$ and thus associated to a Newton polygon (see Definition 5.1.1). In this way, we obtain a stratification of S. We are interested to investigate the properties of this type of stratifications. What is already known is an analogue of Grothendieck's specialization theorem [Kat79, Theorem 2.3.1], proved in [Cre86, Theorem 2.7] via a specialization theorem for convergent F-isocrystals. We would like to know whether other results for F-crystals, such as the purity properties in [dJO99, Theorem 4.1], [Yan11, Theorem 1.1] and [Vas06, Main Theorem B] and so on, have an analogue for crystalline cohomology. The topic of this part is to prove an analogous purity theorem of crystalline cohomology to [Yan11, Theorem 1.1].

One would ask whether properties regarding stratifications by Newton polygons in the case of crystalline cohomology can directly and naturally follow from that in the case of F-crystals by establishing some relation between crystalline cohomology groups of the fibers of f and the higher direct image of crystalline sheaves $R^if_{cris*}\mathcal{O}_{X/\mathbb{Z}_p}$, just like what we do with Zariski cohomology. Or more directly one would ask whether $R^if_{cris*}\mathcal{O}_{X/\mathbb{Z}_p}$ is an F-crystal. The answer to the above questions is negative. The key reason is that the most that the base change theorems in crystalline cohomology can guarantee is that $R^*f_{cris*}\mathcal{O}_{X/\mathbb{Z}_p}$ is a "crystal in the derived category" (see [BO78, Cor 7.11]), therefore, in general each relative cohomology sheaf $R^if_{cris*}\mathcal{O}_{X/\mathbb{Z}_p}$ is not even a crystal on the crystalline site $\mathrm{Cris}(S/\mathbb{Z}_p)$.

Our solution is to investigate the Zariski sheaf $R^*f_{X/\hat{S}}\mathcal{O}_{X/\hat{S}}$ rather than the crystalline sheaf $R^if_{cris*}\mathcal{O}_{X/\mathbb{Z}_p}$. As we work on affine bases shown in Situations 4.1.0.1 and 4.1.0.2, it is more convenient to formulate in terms of modules. We focus on the crystalline cohomology group $H^i_{\mathrm{cris}}(X/\widehat{S})$, a finitely generated module over $\Gamma(\widehat{S},\mathcal{O}_{\widehat{S}})$. We first see that $H^i_{\mathrm{cris}}(X/\widehat{S})$ is an (F,∇)-module in Proposition 4.1.2; then we show that over an open subscheme, the Newton polygons induced from $H^i_{\mathrm{cris}}(X/\widehat{S})$ are identified with those of F-modules $(H^i_{cris}(X_e/W(k_e^{\mathrm{pf}})),F)$, see Corollary 5.2.2; moreover, over bases in Situation 4.1.0.2, we see that $H^i_{\mathrm{cris}}(X/\widehat{S})$ not only completely records the Newton polygons of $(H^i_{cris}(X_e/W(k_e^{\mathrm{pf}})),F)$ at every point e, but also is isogenous to an F-crystal $\mathcal{P}_i$, as shown in Lemma 5.4.5 and Proposition 5.4.6. The above results make it possible to apply the known results for F-crystals to prove both specialization and purity theorems of crystalline cohomology. A key fact in proving [Yan11, Theorem 1.1] is an equivalence between unramified representations and coincidence of Newton polygons in [Yan11, Prop 4.4]. A similar fact in the coholomology case is shown in Proposition 6.2.2 via the F-crystal $\mathcal{P}_i$, which carries the same Newton polygons and representation as the cohomology does (see Corollary 6.1.9). As a byproduct, we reprove the specialization theorem in Theorem 5.5.2.

This part is organized as follows. In Chapter 3, we recall basics about crystalline cohomology. In Chapter 4, we show that $H^i_{\mathrm{cris}}(X/\widehat{S})$ is an (F, ∇)-module. In Chapter 5, we study the Newton polygons from $H^i_{\mathrm{cris}}(X/\widehat{S})$ and relate them to those of F-modules $(H^i_{cris}(X_e/W(k_e^{\mathrm{pf}})), F)$. In Chapter 6, we study representations coming from $(H^i_{cris}(X_e/W(k_e^{\mathrm{pf}})), F)$ and prove the main result – Theorem 6.3.4.

Chapter 3

Preliminaries on crystalline cohomology

We recall some basic concepts and statements about crystalline cohomology.

3.1 Basic definitions and statements

In this section, all schemes will be assumed to be of a finite characteristic, unless other specified. Assume that $(S, \mathcal{I}, \gamma)$ is a PD scheme and $p^m \mathcal{O}_S = 0$ for some p^m.

3.1.1 Crystalline sites

We first recall the definitions of the crystalline site and topos.

Definition 3.1.1. Let X be an S-scheme to which γ extends. The crystalline site $\mathrm{Cris}(X/S)$ of X relative to S is defined as follows:

1. Its objects are pairs $(U \hookrightarrow T, \delta)$, where U is a Zariski open subscheme of X, $U \hookrightarrow T$ is a closed S-immersion defined by an ideal $\mathcal{J}$, and δ is a PD structure on $\mathcal{J}$ compatible with γ. We often abuse notation by writing (U, T, δ) or even T for $(U \hookrightarrow T, \delta)$. T is called an S-PD thickening of U.

2. A morphism $T \xrightarrow{u} T'$ in Cris(X/S) is a commutative square:

$$u: \begin{array}{ccc} U & \hookrightarrow & T \\ \downarrow & & \downarrow \\ U' & \hookrightarrow & T' \end{array} \tag{3.1.1}$$

such that $U \to U'$ is an inclusion in the Zariski topology of X and $T \to T'$ is an S-PD morphism.

3. A covering family is a collection of morphisms $\{u_i : T_i \to T\}$ such that each $T_i \to T$ is an open immersion and $T = \cup T_i$.

The crystalline topos of sheaves on Cris(X/S), denoted by $(X/S)_{\text{cris}}$, is the full subcategory of the category of presheaves Cris$(X/S) \to$ ((sets)) whose objects satisfy the sheaf axiom. The structure sheaf $\mathcal{O}_{X/S}$ is defined to be the cofunctor $(U, T, \delta) \mapsto \Gamma(T, \mathcal{O}_T)$.

We would consider $((X/S)_{cris}, \mathcal{O}_{X/S})$ systematically as a ringed topos. Note that the category of $\mathcal{O}_{X/S}$-modules on Cris(X/S) has enough injectives. We are now in a good position to introduce cohomology.

Definition 3.1.2. Let $e \in (X/S)_{cris}$ be the final object, the functor of taking global sections is defined as $\Gamma(\cdot) : (X/S)_{cris} \to ((\text{sets})), \mathcal{F} \to Hom(e, \mathcal{F})$. Then we define $H^i(X/S, \cdot)$ to be the i^{th} derived functor of $\Gamma(\cdot)$.

Given a commutative diagram of PD-schemes as follows:

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S \end{array} \tag{3.1.2}$$

A morphism of topoi $g_{\text{cris}} : (X'/S')_{\text{cris}} \to (X/S)_{\text{cris}}$ is given in [BO78, Proposition 5.8].

Moreover, a natural morphism of topoi: $u_{X/S} : (X/S)_{\text{cris}} \to X_{\text{zar}}$ is defined in [BO78, Proposition 5.18], as follows:

1. For $\mathcal{F} \in (X/S)_{\text{cris}}$ and $j : U \to X$ an open immersion,

$$(u_{X/S*}\mathcal{F})(U) = \Gamma((U/S)_{\text{cris}}, j^*_{\text{cris}}\mathcal{F});$$

2. For $\mathcal{E} \in X_{\text{zar}}$ and $(U, T, \delta) \in \text{Cris}(X/S)$,

$$(u^*_{X/S}\mathcal{E})(U, T, \delta) = \mathcal{E}(U).$$

3.1.2 Crystals

Definition 3.1.3. A crystal on $\text{Cris}(X/S)$ is a sheaf $\mathcal{F}$ of $\mathcal{O}_{X/S}$-modules such that for any morphism $u : (U', T', \delta') \to (U, T, \delta)$ in $\text{Cris}(X/S)$, the map $u^*\mathcal{F}_{(U,T,\delta)} \to \mathcal{F}_{(U',T',\delta')}$ is an isomorphism.

There are two alternative ways to describe crystals as follows:

Proposition 3.1.4. *([Ber74, IV Proposition 1.6.3] or [BO78, Theorem 4.12, 6.6]) Let X/S be smooth and $p^m\mathcal{O}_S = 0$. Then the following categories are equivalent:*

1. *The category of crystals of $\mathcal{O}_{X/S}$-modules on Cris(X/S).*

2. *The category of $\mathcal{O}_X$-modules with an HPD-stratification.*

3. *The category of $\mathcal{O}_X$-modules with an integrable, quasi-nilpotent connections.*

Remark 3.1.5. Proposition 3.1.4 still holds for the case $\text{Spec}\, A[\![x_1, \cdots, x_d]\!] \to \text{Spec}\, A$ if the definition of HPD-stratification is adjusted to incorporate the topology of the structure ring.

As the definition of HPD stratifications will be applied later, we now recall it from [BO78, Definition 4.3H]. We first fix some notations.

Suppose that X is a separated S-scheme and γ extends to S. Let $(X/S)^{\nu+1}$ be the $\nu + 1$-fold Cartesian product of X with itself over S and $\triangle : X \to (X/S)^{(\nu+1)}$ be the diagonal immersion. Denote by $D_{X/S}(\nu)$ the divided power envelope of X as a closed subscheme of $(X/S)^{(\nu+1)}$ under $\triangle$ and by $\mathcal{D}_{X/S}(\nu)$ the structure sheaf of $D_{X/S}(\nu)$. We also have a natural PD algebra homomorphism ([BO78, Remark 4.2])

$$\delta : \mathcal{D}_{X/S}(1) \to \mathcal{D}_{X/S}(1) \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}(1),\ \xi^{[k]} \mapsto \sum_{i+j=k} \xi^{[i]} \otimes \xi^{[j]}, \tag{3.1.3}$$

where $\xi = 1 \otimes x - x \otimes 1$ for any section x of $\mathcal{O}_X$.

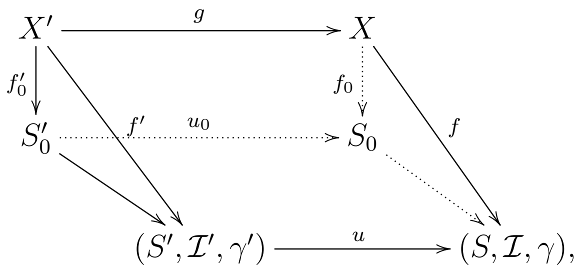
Definition 3.1.6. With the above notations. An HPD stratification on an $\mathcal{O}_X$-module $\mathcal{E}$ is a $\mathcal{D}_{X/S}(1)$-linear isomorphism $\epsilon : \mathcal{D}_{X/S}(1) \otimes \mathcal{E} \to \mathcal{E} \otimes \mathcal{D}_{X/S}(1)$ such that: (1) ϵ reduces to the identity mod $\bar{\mathcal{J}}$, where $\bar{\mathcal{J}}$ is the PD ideal sheaf of $\mathcal{D}_{X/S}(1)$ generated by the ideal sheaf of the diagonal immersion $\triangle : X \to X \times_S X$; (2) the following cocycle condition holds:

$$\begin{array}{ccc} \mathcal{D}_{X/S}(1) \otimes \mathcal{D}_{X/S}(1) \otimes \mathcal{E} & \xrightarrow{\delta^*(\epsilon)} & \mathcal{E} \otimes \mathcal{D}_{X/S}(1) \otimes \mathcal{D}_{X/S}(1) \\ & {\scriptstyle \mathrm{id}\otimes\epsilon} \searrow \quad \swarrow {\scriptstyle \epsilon\otimes\mathrm{id}} & \\ & \mathcal{D}_{X/S}(1) \otimes \mathcal{E} \otimes \mathcal{D}_{X/S}(1) & \end{array} \tag{3.1.4}$$

Remark 3.1.7. For the definition of divided power envelope, see [BO78, Theorem 3.19]. If X/S is smooth with local coordinates $\{x_1, \cdots, x_d\}$, then $\mathcal{D}_{X/S}(1)$ is isomorphic to the PD polynomial algebra $\mathcal{O}_X < \xi_1, \cdots, \xi_d >$, where $\xi = 1 \otimes x_i - x_i \otimes 1$ and it is a flat $\mathcal{O}_X$-module for both projections $p_i^* : \mathcal{O}_X \to \mathcal{D}_{X/S}(1)$, see [Ber74, I Corollary 4.5.3]

3.1.3 Base change theorem and finiteness theorem

Theorem 3.1.8 (Base change theorem)**.** *(see [Ber74, V Proposition 3.5.2], [BO78, Theorem 7.8]) Suppose that the morphisms f' and f fit into the following commutative* *diagram:*



S_0 is quasi-compact, f_0 is smooth, quasi-compact and quasi-separated, u is a PD morphism and $S_0 \subset S$ and $S_0' \subset S'$ are defined by sub PD ideals of $\mathcal{I}$ and $\mathcal{I}'$ respectively. Let $f_{X/S}$ and $f'_{X'/S'}$ be the following morphisms of topos:

$$f_{X/S} : (X/S)_{cris} \overset{u_{X/S}}{\to} X_{zar} \to S_{zar}, \tag{3.1.5}$$

$$f'_{X'/S'} : (X'/S')_{cris} \overset{u_{X'/S'}}{\to} X'_{zar} \to S'_{zar}. \tag{3.1.6}$$

Then for all quasicoherent crystals $\mathcal{E}$ with finite tor-dimension on X, there exists a canonical morphism in the derived category $\mathcal{D}^-(S'_{zar}, \mathcal{O}_{S'})$

$$Lu^*(Rf_{X/S*}\mathcal{E}) \overset{\Phi_u}{\to} Rf'_{X'/S'*}(Lg^*_{cris}\mathcal{E}). \tag{3.1.7}$$

Moreover, if $X' = X \times_S S'$ and $\mathcal{E}$ is a flat crystal, then the morphism in (3.1.7) is an isomorphism.

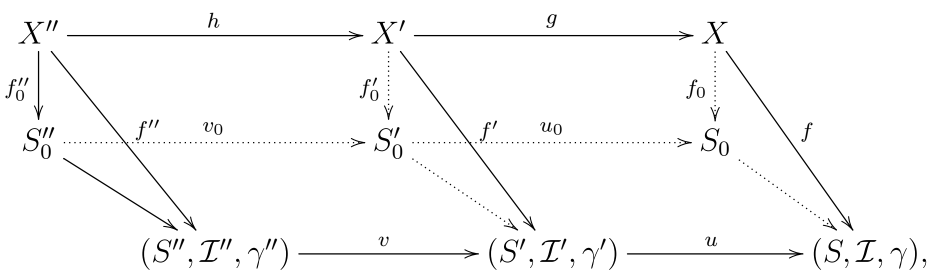
Corollary 3.1.9. *([Ber74, V Corollary 3.5.7]) Under the hypotheses of Theorem 3.1.8, $X' = X \times_S S'$ and $\mathcal{E}$ is a flat crystal. Assume that $S = \operatorname{Spec} A$ and $S' = \operatorname{Spec} A'$. Then the canonical morphism*

$$R\Gamma(X/S, \mathcal{E}) \otimes^L_A A' \xrightarrow{\Phi_u} R\Gamma(X'/S', g^*_{cris}\mathcal{E}) \tag{3.1.8}$$

is an isomorphism.

The following corollary has been applied extensively and are very useful to deal with composite morphims.

Corollary 3.1.10. *Consider the following commutative diagram:*



Assume that S_0 and S_0' are quasi-compact, f_0 and f_0' are smooth, quasi-compact and quasi-separated, u and v are PD morphisms, $S_0 \subset S$, $S_0' \subset S'$ and $S_0'' \subset S''$ are defined by sub PD ideals of $\mathcal{I}$, $\mathcal{I}'$ and $\mathcal{I}''$ respectively. Then in the derived category $\mathcal{D}^-(S''_{zar}, \mathcal{O}_{S''})$, the canonical morphism $\Phi_{u\circ v}$ for $g \circ h : X'' \to X$ over $u \circ v : S'' \to S$ given in Theorem 3.1.8

$$L(u\circ v)^*(Rf_{X/S*}\mathcal{O}_{X/S}) \xrightarrow{\Phi_{u\circ v}} Rf''_{X''/S''*}(\mathcal{O}_{X''/S''}) \tag{3.1.9}$$

is the composite of canonical morphisms

$$L(u\circ v)^*(Rf_{X/S*}\mathcal{O}_{X/S}) \overset{Lv^*(\Phi_u)}{\to} Lv^*(Rf'_{X'/S'*}\mathcal{O}_{X'/S'}) \overset{\Phi_v}{\to} Rf''_{X''/S''*}(\mathcal{O}_{X''/S''}). \tag{3.1.10}$$

Moreover, if S, S' and S'' are all affine, assume that $S = \operatorname{Spec} A$, $S' = \operatorname{Spec} A'$ and $S'' = \operatorname{Spec} A''$, then the canonical morphism

$$R\Gamma(X/S, \mathcal{O}_{X/S}) \otimes^L_A A'' \xrightarrow{\Phi_{v\circ u}} R\Gamma(X''/S'', \mathcal{O}_{X''/S''}) \tag{3.1.11}$$

is the composite of canonical morphisms

$$R\Gamma(X/S, \mathcal{O}) \otimes^L_A A'' \overset{\Phi_u \otimes^L_{A'} A''}{\longrightarrow} R\Gamma(X'/S', \mathcal{O}) \otimes^L_{A'} A'' \overset{\Phi_v}{\to} R\Gamma(X''/S'', \mathcal{O}). \tag{3.1.12}$$

Theorem 3.1.11 (Finiteness theorem)**.** *([Ber74, VII Theorem 1.1.1] or [BO78, Theorem 7.16]) Suppose $f : X \to S_0$ is a smooth proper map, $S_0 \hookrightarrow S$ is defined by a sub PD ideal of $\mathcal{I}$, and S is noetherian. If $\mathcal{E}$ is a crystal of locally free $\mathcal{O}_{X/S}$-modules of finite rank, then $Rf_{X/S*}\mathcal{E}$ is a perfect complex of $\mathcal{O}_S$-modules, i.e. locally on S, $Rf_{X/S*}\mathcal{E}$ is quasi-isomorphic to a strict perfect complex.*

Corollary 3.1.12. *[Ber74, VII Corollary 1.1.2] Under the hypotheses of Theorem 3.1.11, then for all i, $R^i f_{X/S*}\mathcal{E}$ is a coherent $\mathcal{O}_S$-module. Moreover if S is affine, assume that $A = \Gamma(S, \mathcal{O}_S)$, then for all i, $H^i(X/S, \mathcal{E})$ is a finitely generated A-module.*

3.2 Crystalline cohomology over a p-adic base

3.2.1 Crystalline cohomology revisited

Situation 3.2.1.1. Let $(\widehat{A}, I, \gamma)$ be a noetherian PD-ring together with a sub PD ideal I_0 of I such that I_0 contains some prime number p, and $\widehat{A}$ is I_0-adically separated and complete. For each n, we let $A_n = \widehat{A}/I_0^{n+1}$, $S_n = \operatorname{Spec} A_n$ and $\widehat{S} = \operatorname{Spf} \widehat{A}$ for the I_0-adic topology. And $f : X \to S_0$ is a proper smooth morphism.

The crystalline site $\operatorname{Cris}(X/\widehat{S})$ of X relative to $\widehat{S}$ include all objects of $\operatorname{Cris}(X/S_n)$ for all n and can be viewed as a direct limit of the sites $\operatorname{Cris}(X/S_n)$ with the obvious inclusions. Denote by $(X/\widehat{S})_{cris}$ the topos of sheaves on $\operatorname{Cris}(X/\widehat{S})$. For all n, there exists a canonical morphism of topoi $i_n : (X/S_n)_{cris} \to (X/\widehat{S})_{cris}$.

Given a crystal $\mathcal{E}$ of locally free $\mathcal{O}_{X/\widehat{S}}$-modules of finite rank, then $i_n^*\mathcal{E}$ is a crystal over $\operatorname{Cris}(X/S_n)$. By (3.1.8), for every n, there exists a canonical isomorphism

$$R\Gamma(X/S_{n+1}, i_{n+1}^*\mathcal{E}) \otimes^L_{A_{n+1}} A_n \longrightarrow R\Gamma(X/S_n, i_n^*\mathcal{E}) \tag{3.2.1}$$

When varying n, we obtain a projective system in the sense of derived category.

Definition 3.2.1. Under the assumptions in Situation 3.2.1.1, the i^{th} crystalline cohomology of X relative to (A, I_0, γ) is defined to be $\varprojlim H^i(X/S_n, \mathcal{O}_{X/S_n})$, denoted by $H^i_{\text{cris}}(X/\widehat{S})$. For a crystal $\mathcal{E}$ of locally free $\mathcal{O}_{X/\widehat{S}}$-modules of finite rank, we set $H^i(X/\widehat{S}, \mathcal{E}) = \varprojlim H^i(X/S_n, i_n^*\mathcal{E})$.

Remark 3.2.2. In [BO78], another appoach to define $H^i(X/\widehat{S}, \mathcal{O})$ via taking the derived functors of Γ is applied to deal with the nonproper case. It is proved in [BO78, Theorem 7.24.3] that the two approaches are equivalent under the assumption in Situation 3.2.1.1

Theorem 3.2.3 (Finiteness theorem)**.** *[Ber74, VII Proposition 1.1.5, Corollary 1.1.6] Under the assumptions in Situation 3.2.1.1. Let $\mathcal{E}$ be a crystal of locally free $\mathcal{O}_{X/\widehat{S}}$-modules of finite rank. Then there exists a strictly perfect complex, i.e. a bounded complex of projective finite $\widehat{A}$-modules denoted by $R\Gamma(X/\widehat{S},\mathcal{E})$, such that:*

1. *For all n, there exists an isomorphism*

$$R\Gamma(X/\widehat{S},\mathcal{E})\otimes_{\widehat{A}} A_n \xrightarrow{\ \sim\ } R\Gamma(X/S_n, i_n^*\mathcal{E}) \tag{3.2.2}$$

in the derived category $D^b(A_m)$, which is compatible with (3.2.1)

2. *For all i, the natural homomorphism*

$$H^i(R\Gamma(X/\widehat{S},\mathcal{E})) \longrightarrow \varprojlim H^i(X/S_n, i_n^*\mathcal{E}) \tag{3.2.3}$$

is an isomorphism. In particular, $H^i(X/\widehat{S},\mathcal{E})$ is a finite $\widehat{A}$-module.

Proposition 3.2.4 (Flat base change)**.** *[Ber74, Proposition 1.1.8] Under the assumptions in Situation 3.2.1.1. Let $(A,I,\gamma)\to(A',I',\gamma')$ be a PD morphism. A' is a noetherian flat A-algebra PD-ring with a sub PD ideal I_0' of I' such that I_0' contains I_0A', and A' is I_0'-adically separated and complete. For each n, set $A_n'=A'/(I_0')^{n+1}$, $S_n'=\operatorname{Spec}A_n'$, $\widehat{S}'=\operatorname{Spf}A'$ for the I_0-adic topology and $X'=X\times_{S_0}S_0'$. Let $\mathcal{E}$ be a crystal of locally free $\mathcal{O}_{X/\widehat{S}}$-modules of finite rank and $\mathcal{E}'$ be its the inverse image on Cris$(X'/\widehat{S}')$. Then there exists an canonical isomorphism*

$$H^i(X/\widehat{S},\mathcal{E})\otimes_A A' \longrightarrow H^i(X'/\widehat{S}',\mathcal{E}'). \tag{3.2.4}$$

An analogue of Theorem 3.1.8 over a p-adic base could be true, yet it does not seem to exist in literature. We are going to show a result for base change to fibers.

Proposition 3.2.5. *Under the assumptions in Situation 3.2.1.1. Take a point e : $\operatorname{Spec}k\to S_0$, where k is a perfect field. Let $X_e=X\times_e\operatorname{Spec}k$. Assume that $\hat{e}$ is a*

PD morphism fitting into the following commutative diagram:

$$\begin{array}{ccc} X_e & \overset{g}{\hookrightarrow} & X \\ \downarrow f & & \downarrow f \\ \operatorname{Spec} k & \xrightarrow{e} & S_0 \end{array} \qquad \begin{array}{ccc} \operatorname{Spec} k & \xrightarrow{e} & S_0 \\ \downarrow & & \downarrow \\ \operatorname{Spf} W(k) & \xrightarrow{\hat{e}} & \widehat{S} \end{array} \tag{3.2.5}$$

Then there exists a canonical isomorphism of finite $W(k)$-modules

$$H^i(X_e/W(k), g^*_{cris}\mathcal{E}) \xrightarrow{\sim} H^i(R\Gamma(X/\widehat{S}, \mathcal{E}) \otimes_{\hat{e}} W(k)) \tag{3.2.6}$$

Proof. For all $n \geq 0$, let $W_n(k) = W(k)/(p^{n+1})$ and $e_n : \operatorname{Spec} W_n(k) \to S_n$ be the reduction of $\hat{e}$. By Corollary 3.1.9,

$$R\Gamma(X/S_n, i_n^*\mathcal{E}) \otimes^L_{e_n} W_n(k) \xrightarrow{\sim} R\Gamma(X_e/W_n(k), i_n^* g^*_{\text{cris}}\mathcal{E}).$$

By (3.2.2) and as $R\Gamma(X/\widehat{S}, \mathcal{E})$ is projective, we have

$$R\Gamma(X/\widehat{S}, \mathcal{E}) \otimes_{\hat{e}} W(k) \otimes_{W(k)} W_n(k) \xrightarrow{\sim} R\Gamma(X_e/W_n(k), i_n^* g^*_{\text{cris}}\mathcal{E}). \tag{3.2.7}$$

As $H^i(X_e/W(k), g^*_{\text{cris}}\mathcal{E}) = \varprojlim H^i(X_e/W_n(k), i_n^* g^*_{\text{cris}}\mathcal{E})$, thus

$$H^i(X_e/W(k), g^*_{\text{cris}}\mathcal{E}) \xrightarrow{\sim} \varprojlim H^i(R\Gamma(X/\widehat{S}, \mathcal{E}) \otimes_{\hat{e}} W(k) \otimes_{W(k)} W_n(k)) \tag{3.2.8}$$

(3.2.6) is shown by applying Lemma 3.2.6 to the complex $R\Gamma(X/\widehat{S}, \mathcal{E}) \otimes_{\hat{e}} W(k)$. □

Lemma 3.2.6. *Let $\mathcal{C}$ be a complex of free $W(k)$-modules of finite rank. Then*

$$H^i(\mathcal{C}) = \varprojlim H^i(\mathcal{C} \otimes W_n(k)). \tag{3.2.9}$$

Proof. After a suitable choice of bases, any morphism between two free $W(k)$-modules of finite rank correspond to a matrix of the form $\left(\begin{smallmatrix} A_{n\times n} & 0_{n\times m} \\ 0_{l\times n} & 0_{l\times m} \end{smallmatrix}\right)$, where $A_{n\times n}$ is a diagonal matrix whose nonzero entries are powers of p. With this, it is easy to verify Equation (3.2.9). □

3.2.2 Crystals revisited

First we recall a p-adic analogue of Proposition 3.1.4 to describe crystals in terms of modules with connections. Such an analogue exists in a very general setting, see [dJ95, Proposition 2.2.2] or [dJso, Proposition 40.22.4]. For our purpose of discussion, we state the following analogy of Proposition 3.1.4.

Proposition 3.2.7. *([BM90, Proposition 1.3.3]) Given $A_0, \widehat{A}, A_n, S_0$ same as in Situations 4.1.0.1 or 4.1.0.2. Then the category of crystals of quasi-coherent $\mathcal{O}_{S_0/\mathbb{Z}_p}$-modules on Cris$(S_0/\mathbb{Z}_p)$ is equivalent to the category of separated, complete $\widehat{A}$-modules with an integrable and topologically quasi-nilpotent connection.*

When $S_0 = \operatorname{Spec} k$ for a perfect k field, then a crystal of coherent $\mathcal{O}_{S_0/\mathbb{Z}_p}$-modules is a finite $W(k)$-module, as the condition on connection automatically holds.

Recall that a connection on an $\widehat{A}$-module is an additive morphism $\nabla : M \to M \otimes \Omega_{\widehat{S}/W(k)}$ such that $\nabla(fm) = df \otimes m + f\nabla(m)$, where $\Omega_{\widehat{S}/W(k)} = \varprojlim \Omega_{S_n/W_n}$. We say that ∇ is topologically quasi-nilpotent with respect to a local coordinate $\{x_i\}$ if for any section e, there are only finitely many pairs (i,k) such that $\nabla_i^k(e) \notin p\mathcal{E}$, where ∇_i is the $\frac{\partial}{\partial x_i}$-derivative. For an integrable connection, being topologically quasi-nilpotent is independent of the choice of local coordinates.

Now we turn to Frobenius structures on crystals. Given a morphism $f : X \to S_0$ in characteristic p, we denote by F_X its absolute Frobenius endomorphism. Consider the following commutative diagrams, where $X' = X \times_{F_{S_0}} S_0$.

$$\begin{array}{ccccc} X & \xrightarrow{F_{X/S_0}} & X' & \xrightarrow{W_{X/S_0}} & X \\ & \searrow^{f} & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ & & S_0 & \xrightarrow{F_{S_0}} & S_0 \end{array}, \qquad \begin{array}{ccc} S_0 & \xrightarrow{F_{S_0}} & S_0 \\ \downarrow & & \downarrow \\ T & \xrightarrow{F_T} & T \end{array},$$

where T is a PD thickening of S_0 and F_T is a lifting of F_{S_0}. T is p-adically complete and of positive characteristic or characteristic 0.

By abuse of notations, we respectively denote by $F_{X/T} : (X, S_0, T) \to (X', S_0, T)$

and $F_T : (X', S_0, T) \to (X, S_0, T)$ the following two diagrams:

$$\begin{CD} X @>{f}>> S_0 @>>> T \\ @V{F_{X/S_0}}VV @| @| \\ X' @>{f'}>> S_0 @>>> T \end{CD} \,, \qquad \begin{CD} X' @>{f'}>> S_0 @>>> T \\ @V{W_{X/S_0}}VV @V{F_{S_0}}VV @V{F_T}VV \\ X @>{f}>> S_0 @>>> T \end{CD} \,.$$

Denote by $F_{X,T} : (X, S_0, T) \to (X, S_0, T)$ the composite $F_T \circ F_{X/T}$. Clearly $F_{X,T}^*$ pulls back crystals of $\mathcal{O}_{X/T}$-modules to crystals of $\mathcal{O}_{X/T}$-modules.

Recall that an F-crystal $\mathcal{E}$ over $\mathrm{Cris}(X/\mathbb{Z}_p)$ is defined to be a crystal of locally free $\mathcal{O}_{X/\mathbb{Z}_p}$-modules of finite rank with a morphism $\Psi : F_{X,\mathbb{Z}_p}^* \mathcal{E} \to \mathcal{E}$ such that $\Psi \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an isomorphism. We generalize the concept of F-crystal as below:

Definition 3.2.8. Under the assumptions in Situations 4.1.0.1 or 4.1.0.2. An (F, ∇)-module over $\widehat{A}$ is defined to be a triple (M, F, ∇) consisting of a finite $\widehat{A}$-module M, a σ-linear morphism $F : M \to M$ and an integrable and topologically quasi-nilpotent connection ∇ on M such that the linearization Ψ of F becomes an isomorphism after tensoring with $\mathbb{Q}_p$ and Ψ is a horizontal map when assuming $M \otimes_\sigma \widehat{A}$ is endowed with the induced connection from (M, ∇). Denoted by (M, F, ∇).

Remark 3.2.9. By Proposition 3.2.7, an (F, ∇)-module is the same as a crystal $\mathcal{E}$ of quasi-coherent, finitely generated $\mathcal{O}_{X/\mathbb{Z}_p}$-modules with a morphism $\Psi : F_{X,\mathbb{Z}_p}^* \mathcal{E} \to \mathcal{E}$ such that $\Psi \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an isomorphism. In particular, an F-crystal is an (F, ∇)-module. In Situation 4.1.0.2, an (F, ∇)-module is usually called as an (F, θ)-module, see Definition 5.4.1.

For later use, we now give a description of the horizontal map $\Psi : M \otimes_\sigma \widehat{A} \to M$ in Definition 3.2.8.

Lemma 3.2.10. *Under the assumptions in Situations 4.1.0.1 or 4.1.0.2. The morphism $\Psi : M \otimes_\sigma \widehat{A} \to M$ in Definition 3.2.8 is horizontal if and only if the diagram* *as below is commutative:*

$$\begin{array}{ccc} M & \xrightarrow{\nabla} & M \otimes_{\widehat{A}} \Omega_{\widehat{A}/W(k)} \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F\otimes F_{\Omega}} \\ M & \xrightarrow{\nabla} & M \otimes_{\widehat{A}} \Omega_{\widehat{A}/W(k)} \ , \end{array} \tag{3.2.10}$$

where $\Omega_{\widehat{A}/W(k)} = \varprojlim \Omega_{A_n/W_n}$*,* $F_{\Omega} = \varprojlim F_{\Omega,n}$ *and* $F_{\Omega,n} : \Omega_{A_n/W_n} \to \Omega_{A_n/W_n}$ *is a* σ*-linear map induced from* $\sigma : A_n \to A_n$*.*

Proof. The induced connection on $M \otimes_{\sigma} \widehat{A}$ is determined by a commutative diagram as below:

$$\begin{array}{ccc} M & \xrightarrow{\quad\nabla\quad} & M \otimes_{\widehat{A}} \Omega_{\widehat{A}/W(k)} \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_1\otimes F_{\Omega}} \\ M \otimes_{\sigma} \widehat{A} & \xrightarrow{\nabla^f} & M \otimes_{\sigma} \widehat{A} \otimes_{\widehat{A}} \Omega_{\widehat{A}/W(k)} \ , \end{array} \tag{3.2.11}$$

where q_1 is defined by $m \mapsto m \otimes 1$. Ψ is a horizontal map if and only if it fits into a commutative diagram as follows:

$$\begin{array}{ccc} M \otimes_{\sigma} \widehat{A} & \xrightarrow{\nabla^f} & M \otimes_{\sigma} \widehat{A} \otimes_{\widehat{A}} \Omega_{\widehat{S}/W(k)} \\ \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi\otimes 1_{\Omega}} \\ M & \xrightarrow{\quad\nabla\quad} & M \otimes_{\widehat{A}} \Omega_{\widehat{S}/W(k)} \ . \end{array} \tag{3.2.12}$$

By combining (3.2.10) and (3.2.12), we prove the lemma. □

Chapter 4

$H^i_{\mathbf{cris}}(X/\widehat{S})$ is an (F,∇)-module

In this chapter, we are going to show that in some cases, $H^i_{\text{cris}}(X/\widehat{S})$ is an (F,∇)-module. In other words, $H^i_{\text{cris}}(X/\widehat{S})$ is similar to an F-crystal, except that it is not locally free.

4.1 Preliminaries

We first list some facts and then propose two situations to be investigated.

Facts 4.1.1. *([Kat79, 2.4]) Let A_0 be a smooth finitely generated algebra over a perfect field k of characteristic $p > 0$. Then there exists a p-adically complete, flat $W(k)$-algebra $\widehat{A}$ and a homomorphism $\sigma : \widehat{A} \to \widehat{A}$ such that $\widehat{A}/p\widehat{A} \simeq A_0$, $A_n = \widehat{A}/(p^{n+1})$ is formally smooth over $W_n = W(k)/(p^{n+1})$ and σ is a (not unique) Frobenius lifting of the absolute Frobenius morphism $\sigma : A_0 \to A_0$. Moreover, there exists a unique embedding $i_\sigma : \widehat{A} \to W(A_0^{pf})$ compatible with Frobenius liftings.*

Situation 4.1.0.1. Given $A_0, \widehat{A}$ and $\sigma : \widehat{A} \to \widehat{A}$ same as in Fact 4.1.1. Let $S_0 = \operatorname{Spec} A_0$, $S_n = \operatorname{Spec} A_n$ and $\widehat{S} = \operatorname{Spf} \widehat{A}$. Let $F_{S_n}; S_n \to S_n, F_{\widehat{S}} : \widehat{S} \to \widehat{S}$ be morphisms implied by σ.

Situation 4.1.0.2. Given a perfect field k of characteristic $p > 0$. Let $A_0 = k[[t]]$,

$\widehat{A} = W(k)[\![t]\!]$, $A_n = \widehat{A}/(p^{n+1}) = W_n[\![t]\!]$ and fix $\sigma : W(k)[[t]] \to W(k)[[t]], t \mapsto t^p$. Let $S_n, \widehat{S}, F_{S_n}, F_{\widehat{S}}$ be similarly defined as in Situation 4.1.0.1.

Our purpose in this chapter is to prove the following statement:

Proposition 4.1.2. *Given $S_0, \widehat{S}$ as in Situations 4.1.0.1 or 4.1.0.2. Let $f : X \to S_0$ be a proper smooth morphism. Then for all i, $H^i_{cris}(X/\widehat{S})$ is an (F,∇)-module.*

Proof. The above statement is a summary of Propositions 4.2.1, 4.3.1 and 4.4.1. □

4.2 A σ-linear map F on $H^i_{\mathbf{cris}}(X/\widehat{S})$

We may refer to Section 3.2.2 for some notations. The purpose of this section is to prove

Proposition 4.2.1. *With the assumptions in Proposition 4.1.2. Then there exists a σ-linear map $F : H^i_{cris}(X/\widehat{S}) \to H^i_{cris}(X/\widehat{S})$ such that its linearization $\Psi : H^i_{cris}(X/\widehat{S}) \otimes_\sigma \widehat{A} \to H^i_{cris}(X/\widehat{S})$ fitting into the following commutative diagram:*

$$\begin{array}{ccc} H^i_{cris}(X/\widehat{S}) \otimes_\sigma \widehat{A} & \xrightarrow{\quad\Psi\quad} & H^i_{cris}(X/\widehat{S}) \\ & {\searrow}^{\simeq} \qquad\qquad {\nearrow}_{R^iF_{X/\widehat{S}}} & \\ & H^i_{cris}(X'/\widehat{S}) & \end{array} \tag{4.2.1}$$

Moreover, $\Psi \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an isomorphism.

Proof. As $H^i_{\mathrm{cris}}(X/\widehat{S}) = \varprojlim H^i(X/S_n, \mathcal{O}_{X/S_n})$, to obtain a σ-linear map on $H^i_{\mathrm{cris}}(X/\widehat{S})$, it suffices to find a compatible system of σ-linear maps on $H^i(X/S_n, \mathcal{O}), \forall n$. Consider the following diagram of morphisms:

$$\begin{array}{ccccc} (X, S_0, S_n) & \xrightarrow{F_{X/S_n}} & (X', S_0, S_n) & \xrightarrow{F_{S_n}} & (X, S_0, S_n) \\ \downarrow{\scriptstyle i_{n,n+1}} & & \downarrow{\scriptstyle i_{n,n+1}} & & \downarrow{\scriptstyle i_{n,n+1}} \\ (X, S_0, S_{n+1}) & \xrightarrow{F_{X/S_{n+1}}} & (X', S_0, S_{n+1}) & \xrightarrow{F_{S_{n+1}}} & (X, S_0, S_{n+1}) \end{array},$$

where $i_{n,n+1} : (X, S_0, S_n) \to (X, S_0, S_{n+1})$ consists of the identity map of $f : X \to S_0$ and the closed immersion $S_n \to S_{n+1}$. By applying Corollaries 3.1.9 and 3.1.10 to $\mathcal{O}_{X/S_{n+1}}$, we have a commutative diagram in $D^b(A_n)$ as follows:

$$\begin{array}{ccccc} R\Gamma(X/S_n) & \longleftarrow & R\Gamma(X'/S_n) & \xleftarrow{\simeq} & R\Gamma(X/S_n)\otimes^L_\sigma A_n \\ \uparrow & & \uparrow & & \uparrow \\ R\Gamma(X/S_{n+1})\otimes^L A_n & \longleftarrow & R\Gamma(X'/S_{n+1})\otimes^L A_n & \xleftarrow{\simeq} & R\Gamma(X/S_{n+1})\otimes^L_\sigma A_{n+1}\otimes^L A_n \end{array}$$

where $R\Gamma(X/S_n)$ is $R\Gamma(X/S_n, \mathcal{O}_X/S_n)$ for short. Note that $\sigma : A_n \to A_n$ is flat for all n. After taking cohomology, we have a commutative diagram of A_{n+1}-modules as follows:

$$\begin{array}{ccccc} H^i(X/S_n,\mathcal{O}) & \longleftarrow & H^i(X'/S_n,\mathcal{O}) & \xleftarrow{\simeq} & H^i(X/S_n,\mathcal{O})\otimes_\sigma A_n \\ \uparrow & & \uparrow & & \uparrow \\ H^i(X/S_{n+1},\mathcal{O}) & \longleftarrow & H^i(X'/S_{n+1},\mathcal{O}) & \xleftarrow{\simeq} & H^i(X/S_{n+1},\mathcal{O})\otimes_\sigma A_{n+1} \end{array} \tag{4.2.2}$$

By taking inverse limit of (4.2.2), we get

$$\Psi : H^i(X/\widehat{S},\mathcal{O}_{X/\widehat{S}})\otimes_\sigma \widehat{A} \xrightarrow{\simeq} H^i(X'/\widehat{S},\mathcal{O}_{X'/\widehat{S}}) \xrightarrow{R^iF_{X/\widehat{S}}} H^i(X/\widehat{S},\mathcal{O}_{X/\widehat{S}}). \tag{4.2.3}$$

Moreover, by [BO83, Theorem 1.3], there exists an isomorphism

$$F^*_{X/\widehat{S}} : R^if'_{X'/\widehat{S}*}\mathcal{O}_{X'/\widehat{S}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \longrightarrow R^if_{X/\widehat{S}*}(\mathcal{O}_{X/\widehat{S}})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \tag{4.2.4}$$

induced by $F_{X/\widehat{S}}$, where the functor $f_{X/\widehat{S}}$ is defined as follows:

$$f_{X/\widehat{S}} : (X/\widehat{S})_{cris} \xrightarrow{u_{X/\widehat{S}}} X_{zar} \xrightarrow{f_{\mathrm{zar}}} (\widehat{S})_{\mathrm{zar}}. \tag{4.2.5}$$

Clearly $\Gamma(\widehat{S}, R^if_{X/\widehat{S}*}(\mathcal{O}_{X/\widehat{S}})) = H^i_{\mathrm{cris}}(X/\widehat{S})$, thus $R^iF_{X/\widehat{S}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is an isomorphism, so is $\Psi\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. □

By applying Proposition 4.2.1 to a fiber X_e of $f : X \to S_0$ at the point $e : \operatorname{Spec} k' \to S_0$, we obtain a σ-linear map on $H^i_{\mathrm{cris}}(X_e/W(k))$ as follows:

$$\Psi_e : H^i_{\mathrm{cris}}(X_e/W(k))\otimes_\sigma W(k') \to H^i_{\mathrm{cris}}(X_e/W(k)). \tag{4.2.6}$$

Now we are going to verify that the compatibility of Ψ with Ψ_e.

Corollary 4.2.2. *With the same assumptions as Proposition 4.1.2. Let $e : \operatorname{Spec} k' \to S_0$ be a point of S_0 and k' is a perfect field. Let $X_e = X \times_e \operatorname{Spec} k$. Let $\hat{e} : \operatorname{Spf} W(k') \to \widehat{S}$ be a canonical lifting of e such that $\hat{e}^* : \widehat{A} \to W(k')$ is the composite map of i_σ in Fact 4.1.1 with the natural homomorphism $W(A_0^{pf}) \to W(k')$. Then Ψ on $H^i_{cris}(X/\widehat{S})$ is compatible with Ψ_e on $H^i_{cris}(X_e/W(k))$ in the following sense:*

$$\begin{array}{ccc} H^i_{cris}(X/\widehat{S}) \otimes_\sigma \widehat{A} \otimes_{\hat{e}} W(k') & \xrightarrow{\Psi\otimes id} & H^i_{cris}(X/\widehat{S}) \otimes_{\hat{e}} W(k') \\ \downarrow & & \downarrow \\ H^i_{cris}(X_e/W(k')) \otimes_\sigma W(k') & \xrightarrow{\Psi_e} & H^i_{cris}(X_e/W(k')) \end{array} , \tag{4.2.7}$$

where the vertical maps are induced from (3.2.6).

Proof. Let $W_n = W(k)/(p^{n+1})$ and denote by $e_n : (X_e, k', W_n) \to (X, S_0, S_n)$ the diagram:

$$\begin{array}{ccccc} X_e & \xrightarrow{f} & \operatorname{Spec} k' & \hookrightarrow & \operatorname{Spec} W_n \\ \downarrow & & \downarrow e & & \downarrow e_n \\ X & \xrightarrow{f} & S_0 & \hookrightarrow & S_n \end{array} . \tag{4.2.8}$$

Consider the following diagram of morphisms:

$$\begin{array}{ccccc} (X_e, k', W_n) & \xrightarrow{F_{X_e/W_n}} & (X'_e, k', W_n) & \xrightarrow{F_{W_n}} & (X_e, k', W_n) \\ \downarrow e_n & & \downarrow e_n & & \downarrow e_n \\ (X, S_0, S_n) & \xrightarrow{F_{X/S_n}} & (X', S_0, S_n) & \xrightarrow{F_{S_n}} & (X, S_0, S_n) \end{array} ,$$

By applying Corollaries 3.1.9 and 3.1.10 to $\mathcal{O}_{X/S_n}$, we have a commutative diagram in $\mathcal{D}^-(W_n)$ as follows:

$$\begin{array}{ccccc} R\Gamma(X_e/W_n) & \longleftarrow & R\Gamma(X'_e/W_n) & \xleftarrow{\simeq} & R\Gamma(X_e/W_n) \otimes^L_\sigma W_n \\ \uparrow & & \uparrow & & \uparrow \\ R\Gamma(X/S_n) \otimes^L_{e_n} W_n & \longleftarrow & R\Gamma(X'/S_n) \otimes^L_{e_n} W_n & \xleftarrow{\simeq} & R\Gamma(X/S_n) \otimes^L_\sigma A_n \otimes^L_{e_n} W_n \end{array}$$

After taking cohomology, we have

$$\begin{array}{ccccc} H^i(X_e/W_n,\mathcal{O}) & \longleftarrow & H^i(X'_e/W_n,\mathcal{O}) & \xleftarrow{\simeq} & H^i(X_e/W_n,\mathcal{O})\otimes_\sigma W_n \\ \uparrow & & \uparrow & & \uparrow \\ H^i(X/S_n,\mathcal{O})\otimes_{e_n} W_n & \longleftarrow & H^i(X'/S_n,\mathcal{O})\otimes_{e_n} W_n & \xleftarrow{\simeq} & H^i(X/S_n,\mathcal{O})\otimes_\sigma A_n \otimes_{e_n} W_n \ . \end{array} \tag{4.2.9}$$

It is easy to see that (4.2.9) forms a diagram of inverse systems when varying n. By taking inverse limit, we obtain (4.2.7). □

By Proposition 3.2.4, $H^i_{\mathrm{cris}}(X_e/W(k))$ behaves properly under field extension, as a consequence of Corollary 4.2.2, we see that the σ-linear maps are compatible under base change.

Corollary 4.2.3. *Let X be a proper smooth scheme over a perfect field k of characteristic $p>0$. Assume that $k'\supset k$ is a perfect field extension and $X_{k'}=X\times_k k'$. Then we have a commutative diagram as below:*

$$\begin{array}{ccc} H^i_{cris}(X/W(k))\otimes_\sigma W(k)\otimes_{W(k)} W(k') & \longrightarrow & H^i_{cris}(X/W(k))\otimes_{W(k)} W(k') \\ \downarrow\simeq & & \downarrow\simeq \\ H^i_{cris}(X_{k'}/W(k'))\otimes_\sigma W(k') & \longrightarrow & H^i_{cris}(X_{k'}/W(k')) \ , \end{array} \tag{4.2.10}$$

where the the vertical isomorphisms are given by (3.2.4) and the horizontal maps are isomorphic after tensoring with $\mathbb{Q}_p$.

4.3 A connection ∇ on $H^i_{\mathbf{cris}}(X/\widehat{S})$

In this section, we are going to introduce an integrable and topologically quasi-nilpotent connection ∇ on $H^i_{\mathrm{cris}}(X/\widehat{S})$.

Proposition 4.3.1. *With the same assumptions as Proposition 4.1.2. Then there exists an integrable, topologically quasi-nilpotent $W(k)$-connection*

$$\nabla : H^i_{cris}(X/\widehat{S}) \to H^i_{cris}(X/\widehat{S})\otimes_{\widehat{A}} \Omega_{\widehat{A}/W(k)}. \tag{4.3.1}$$

Proof. As $H^i_{\text{cris}}(X/\widehat{S}) = \varprojlim H^i(X/S_n, \mathcal{O}_{X/S_n})$, we first introduce an integrable, quasi-nilpotent connection on $H^i(X/S_n, \mathcal{O}_{X/S_n})$ for every n. By Proposition 3.1.4, the data of an integrable, quasi-nilpotent connection is equivalent to the data of an HPD stratification. Thus it remains to check the existence of HPD stratifications on $H^i(X/S_n, \mathcal{O}_{X/S_n})$. This will be done by definition, similar to the proof of [Ber74, V Corollary 3.6.3].

Denote by $D_n(\nu)$ the divided power envelope of S_n as a closed subscheme of $(S_n/W_n)^{(\nu+1)}$ under the diagonal immersion $\triangle : S_n \to D_n(\nu+1)$. Let $p_k : D_n(1) \to S_n$ for $k = 1, 2$ be the two natural projections. Let $D_{A_n/W_n}(\nu) = \Gamma(D_n(\nu), \mathcal{O}_{D_n(\nu)})$.

First, to define a compatible system of $D_{A_n/W_n}(1)$-linear isomorphisms

$$\epsilon_n : D_{A_n/W_n}(1) \otimes_{A_n} H^i(X/S_n, \mathcal{O}_{X/S_n}) \to H^i(X/S_n, \mathcal{O}_{X/S_n}) \otimes_{A_n} D_{A_n/W_n}(1), \quad (4.3.2)$$

we consider a commutative diagram as follows

$$\begin{array}{ccccc} (X, S_0, S_n) & \xleftarrow{p_1} & (X, S_0, D_n(1)) & \xrightarrow{p_2} & (X, S_0, S_n) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ (X, S_0, S_{n+1}) & \xleftarrow{p_1} & (X, S_0, D_{n+1}(1)) & \xrightarrow{p_2} & (X, S_0, S_{n+1}) \end{array} \quad . \qquad (4.3.3)$$

By applying Corollaries 3.1.9 and 3.1.10 to , we obtain a commutative diagram as follows:

$$\begin{array}{ccccc} R\Gamma(X/S_{n+1}) \otimes^L_{p_1} D_{A_{n+1}/W_{n+1}}(1) & \xrightarrow{\simeq} & R\Gamma(X/D_{n+1}(1)) & \xleftarrow{\simeq} & D_{A_{n+1}/W_{n+1}}(1) \otimes^L_{p_2} R\Gamma(X/S_{n+1}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ R\Gamma(X/S_n) \otimes^L_{p_1} D_{A_n/W_n}(1) & \xrightarrow{\simeq} & R\Gamma(X/D_n(1)) & \xleftarrow{\simeq} & D_{A_n/W_n}(1) \otimes^L_{p_2} R\Gamma(X/S_n) \end{array}$$

As p_1, p_2 are flat by Remark 3.1.7, after taking cohomology, we obtain the following commutative diagram with $\mathcal{O}_{X/S_n}$ removed from $H^i(X/S_n, \mathcal{O}_{X/S_n})$ for short:

$$\begin{array}{ccccc} H^i(X/S_{n+1}) \otimes_{p_1} D_{A_{n+1}/W_{n+1}}(1) & \xrightarrow{\simeq} & H^i(X/D_{n+1}) & \xleftarrow{\simeq} & D_{A_{n+1}/W_{n+1}}(1) \otimes_{p_2} H^i(X/S_{n+1}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ H^i(X/S_n) \otimes_{p_1} D_{A_n/W_n}(1) & \xrightarrow{\simeq} & H^i(X/D_n) & \xleftarrow{\simeq} & D_{A_n/W_n}(1) \otimes_{p_2} H^i(X/S_n). \end{array}$$

(4.3.4)

Thus we obtain a compatible system of isomorphisms $\{\epsilon_n\}$. It is routine to verify that $\{\epsilon_n\}$ is a compatible system of HPD stratifications, and we omit the lengthy details. Note that the connection $\nabla_n : H^i(X/S_n, \mathcal{O}_{X/S_n}) \to H^i(X/S_n, \mathcal{O}_{X/S_n}) \otimes \Omega_{A_n/W_n}$ induced from ϵ_n is integrable and quasi-nilpotent by Proposition 3.1.4, therefore, $\nabla = \varprojlim \nabla_n$ is integrable and topologically quasi-nilpotent. □

Remark 4.3.2. Recall from Proposition 3.1.4, ∇_n induced from ϵ_n is defined as below:

$$H^i(X/S_n) \xrightarrow{\epsilon_n \circ q_2 - q_1} H^i(X/S_n) \otimes_{A_n} D_{A_n/W_n}(1) \xrightarrow{\text{mod } \bar{J}_n^{[2]}} H^i(X/S_n) \otimes \Omega^1_{A_n/W_n}, \quad (4.3.5)$$

where q_2 and q_1 are the natural projection morphisms, $\bar{J}_n$ is the PD ideal of $\triangle^* : D_{A_n/W_n}(1) \to A_n$ and $\bar{J}_n^{[2]}$ is the PD ideal generated by $\bar{J}_n^2$, see definition in [BO78, Definition 3.24]

4.4 The compatibility of F with ∇

Proposition 4.4.1. *With the same assumptions as Proposition 4.1.2. The σ-linear map $F : H^i_{cris}(X/\widehat{S}) \to H^i_{cris}(X/\widehat{S})$ given in Proposition 4.2.1 is a horizontal map with respect to the connection ∇ given in 4.3.1, i.e. (F,∇) fits into (3.2.10).*

Proof. As $H^i_{\text{cris}}(X/\widehat{S}) = \varprojlim H^i(X/S_n, \mathcal{O}_{X/S_n})$, it suffice to show that for every n, the diagram as below is commutative:

$$\begin{array}{ccc} H^i(X/S_n, \mathcal{O}_{X/S_n}) & \xrightarrow{\nabla_n} & H^i(X/S_n, \mathcal{O}_{X/S_n}) \otimes_{A_n} \Omega_{A_n/W_n} \\ \downarrow F_n & & \downarrow F_n \otimes F_{\Omega,n} \\ H^i(X/S_n, \mathcal{O}_{X/S_n}) & \xrightarrow{\nabla_n} & H^i(X/S_n, \mathcal{O}_{X/S_n}) \otimes_{A_n} \Omega_{A_n/W_n} \end{array}, \quad (4.4.1)$$

where F_n is the σ-linear map induced from Ψ_n given in Proposition 4.2.1. As ∇_n is deduced from an HPD stratification ϵ_n by (4.3.5), it suffices to show that ϵ_n is compatible with F_n. Consider a commutative diagram as below:

$$\begin{array}{ccccc} (X, S_0, S_n) & \xleftarrow{p_1} & (X, S_0, D_n(1)) & \xrightarrow{p_2} & (X, S_0, S_n) \\ \downarrow F_{X,S_n} & & \downarrow F_{X,D_n} & & \downarrow F_{X,S_n} \\ (X, S_0, S_n) & \xleftarrow{p_1} & (X, S_0, D_n(1)) & \xrightarrow{p_2} & (X, S_0, S_n) \end{array}. \quad (4.4.2)$$

Then apply Corollaries 3.1.9 and 3.1.10 to $\mathcal{O}_{X/S_n}$, we have a commutative diagram:

$$\begin{array}{ccccc}
R\Gamma(X/S_n)\otimes^L_{p_1} D_{A_n/W_n}(1) & \xrightarrow{\simeq} & R\Gamma(X/D_n(1)) & \xleftarrow{\simeq} & D_{A_n/W_n}(1)\otimes^L_{p_2} R\Gamma(X/S_n) \\
\Big\downarrow{\scriptstyle F_n\otimes F_{D_n}} & & \Big\downarrow & & \Big\downarrow{\scriptstyle F_{D_n}\otimes F_n} \\
R\Gamma(X/S_n)\otimes^L_{p_1} D_{A_n/W_n}(1) & \xrightarrow{\simeq} & R\Gamma(X/D_n(1)) & \xleftarrow{\simeq} & D_{A_n/W_n}(1)\otimes^L_{p_2} R\Gamma(X/S_n)
\end{array}\ , \tag{4.4.3}$$

where $F_{D_n}: D_{A_n/W_n}(1)\to D_{A_n/W_n}(1)$ is a σ-linear map induced from $\sigma: A_n\to A_n$, the vertical map in the middle is F_{D_n}-linear. As p_1, p_2 are flat, by taking cohomology, we obtain

$$\begin{array}{ccccc}
H^i(X/S_n)\otimes_{p_1} D_{A_n/W_n}(1) & \xrightarrow{\simeq} & H^i(X/D_n(1)) & \xleftarrow{\simeq} & D_{A_n/W_n}(1)\otimes_{p_2} H^i(X/S_n) \\
\Big\downarrow{\scriptstyle F_n\otimes F_{D_n}} & & \Big\downarrow & & \Big\downarrow{\scriptstyle F_{D_n}\otimes F_n} \\
H^i(X/S_n)\otimes_{p_1} D_{A_n/W_n}(1) & \xrightarrow{\simeq} & H^i(X/D_n) & \xleftarrow{\simeq} & D_{A_n/W_n}(1)\otimes_{p_2} H^i(X/S_n)
\end{array}\ . \tag{4.4.4}$$

This completes the proof of the proposition. □

Now we turn to a morphism $h: X'\to X$ of proper smooth S_0-schemes: Then there is a canonical morphism given by (3.1.7):

$$H^i_{\mathrm{cris}}(X/\widehat{S}) \xrightarrow{\Phi_h} H^i_{\mathrm{cris}}(X'/\widehat{S}). \tag{4.4.5}$$

With a similar proof as Proposition 4.4.1, we see that Φ_h is a morphism (F,θ)-modules, i.e. Φ_h commutes with the σ-linear maps and is a horizontal map with respect to the connections.

Proposition 4.4.2. *Given $S_0, \widehat{S}$ as in Situations 4.1.0.1 or 4.1.0.2. Given a morphism $h: X'\to X$ of proper smooth S_0-schemes. Then for all i, the canonical morphism Φ_h is a morphism of (F,θ)-modules.*

Chapter 5

Newton polygons and specialization theorem

Given a smooth proper morphism $X \to S_0$, we will introduce two types of Newton polygons associated to every point e of S_0, one from the crystalline cohomology group $H^i_{\text{cris}}(X_e/W(k'), \mathcal{O}_{X_e/W(k')})$ and the other from the base change of $H^i_{\text{cris}}(X/\widehat{S})$ to e. We will see that on a nonempty open subscheme $U \subset S_0$ the above two types of Newton polygons coincide; moreover, in Situation 4.1.0.2, we will show that the above two types of Newton polygons coincide everywhere and there exists an F-crystal with the same Newton polygons and isogenous to the (F, ∇)-module $H^i_{\text{cris}}(X/\widehat{S})$. This result makes it possible to apply the known results for F-crystals to prove both specialization and purity theorems of crystalline cohomology.

5.1 Basics about Newton polygons

In this section, we first recall the definition of Newton polygons associated to F-crystals over a perfect field k, and then introduce two types of Newton polygons.

Recall that the notion of an F-crystal over a perfect field k of characteristic $p > 0$ is equivalent to a pair (M, F) consisting of a finite free $W(k)$-module M together with a σ-linear endomorphism $F : M \to M$ such that $F \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an automorphism, where $\sigma : W(k) \to W(k)$ is the absolute Frobenius. Newton slopes of (M, F), roughly speaking, are the p-adic values of "eigenvalues" of F. Precisely, they can be defined as follows: choose an integer $N \geq 1$ divisible by $r!$, $r = \text{rank } M$, and consider the discrete valuation ring

$$R = W(\bar{k})[X](X^N - p) = W(\bar{k})[p^{1/N}], \tag{5.1.1}$$

where $\bar{k}$ is an algebraic closure of k. Extend σ to R by requiring that $\sigma(X) = X$ and extend F to a σ-linear map of $M \otimes_{W(k)} K$, where K is the fraction field of R. By Dieudonné's theory ([Man63]), $M \otimes_{W(k)} K$ admits a K-basis $\{e_1, \cdots, e_r\}$ such that

$$F(e_i) = p^{\lambda_i} e_i, \lambda_i \in \mathbb{Q}^{\geq 0}. \tag{5.1.2}$$

The rational numbers $\{\lambda_1, \cdots, \lambda_r\}$ are called Newton slopes of (M, F).

We may assume that $\lambda_1 \leq \cdots \leq \lambda_r$. The Newton polygon of (M, F), denoted by $\text{NP}(M, F)$ is the graph of the Newton function on $[0, r]$ defined on integers by

$$\text{Newton}_F(i) = \lambda_1 + \cdots + \lambda_i \text{ for } 1 \leq i \leq r \text{ and } 0 \text{ for } i = 0. \tag{5.1.3}$$

Let $\text{mult}(\lambda)$ be the number of times that λ occurs among $(\lambda_1, \ldots, \lambda_r)$. Then by Dieudonné's theory,

$$\Sigma_{\lambda \in \mathbb{Q}} \text{mult}(\lambda) = r, \quad \lambda \cdot \text{mult}(\lambda) \in \mathbb{Z}$$

In other words, break points of Newton polygons where Newton slopes jump are integral.

Similarly we can define Newton polygon of an (F, ∇)-module over $W(k)$, which is usually called as an F-module over $W(k)$, since the connection is insignificant. It is easy to see that Newton polygon is invariant under isogeny, i.e. $\text{NP}(M_1, F_1) = \text{NP}(M_1, F_1)$ if there exists an isogeny $\phi : (M_1, F_1) \to (M_2, F_2)$ of F-modules, which is defined to be a morphism $\phi : M_1 \to M_2$ such that $F_2 \circ \phi = \phi \circ F_1$ and $\phi \otimes \mathbb{Q}_p$ is isomorphic

Now we are ready to define two types of Newton polygons.

Definition 5.1.1. Let $f : X \to S_0$ be a proper smooth morphism of schemes in characteristic $p > 0$. For every point $e \in S_0$, choose a perfect field k' such that e is the image of $e : \operatorname{Spec} k' \to S_0$. Recall the notations $X_e, \hat{e}$ and Ψ_e from Corollary 4.2.2. We define a Newton polygon associated to (e, i, f) to be the Newton polygon of the F-module $(H^i_{\text{cris}}(X_e/W(k')), \Psi_e)$, denote by $\text{NP}^1(e, i)$. Note that $\text{NP}^1(e, i)$ is independent of the choice of k' by Corollary 4.2.3.

To define the second type of Newton polygon, we require the assumptions of Proposition 4.1.2. With the notations $e, X_e, \hat{e}$ same as above. We turn to the (F, ∇)-module $H^i_{\text{cris}}(X/\widehat{S})$. Let $M_{i,e} = H^i_{\text{cris}}(X/\widehat{S}) \otimes_{\hat{e}^*} W(k')$. Note that $M_{i,e}$ naturally becomes an F-module with a σ-linear map induced from that on $H^i_{\text{cris}}(X/\widehat{S})$, i.e.

$$\begin{array}{ccc} H^i_{\text{cris}}(X/\widehat{S}) & \xrightarrow{F} & H^i_{\text{cris}}(X/\widehat{S}) \\ \downarrow & & \downarrow \\ H^i_{\text{cris}}(X/\widehat{S}) \otimes_{\hat{e}^*} W(k')(\simeq M_{i,e}) & \xrightarrow{F_{i,e}} & H^i_{\text{cris}}(X/\widehat{S}) \otimes_{\hat{e}^*} W(k')(\simeq M_{i,e}) \end{array} \quad (5.1.4)$$

Definition 5.1.2. Under the assumptions of Proposition 4.1.2, the second type of Newton polygon associated to e is the Newton polygon of $(M_{i,e}, F_{i,e})$, denoted by $\text{NP}^2(e, i)$. Note that $\text{NP}^2(e, i)$ is independent of the choice of k'.

By (4.2.7), we have a commutative diagram as below:

$$\begin{array}{ccc} M_{i,e} \otimes_\sigma W(k') & \xrightarrow{\Psi_{i,e}} & M_{i,e} \\ \downarrow{\scriptstyle \chi_e \otimes \text{id}} & & \downarrow{\scriptstyle \chi_e} \\ H^i_{\text{cris}}(X_e/W(k')) \otimes_\sigma W(k') & \xrightarrow{\Psi_e} & H^i_{\text{cris}}(X_e/W(k')) \end{array} , \quad (5.1.5)$$

we will show that for points of an open subscheme, $\chi_e \otimes \mathbb{Q}_p$ is isomorphic, i.e. $M_{i,e}$ and $H^i_{\text{cris}}(X_e/W(k'))$ are isogenous and thus $\text{NP}^1(e, i) = \text{NP}^2(e, i)$.

5.2 Comparison of Newton polygons

To compare $\text{NP}^1(e, i)$ with $\text{NP}^2(e, i)$, we will need a lemma as follows:

Lemma 5.2.1. *Let A be a noetherian integral domain and $\mathcal{C}$ be a complex of finite A-modules. For a homomorphism $\psi : A \to k$ to a field k, we denote by $h^i(\psi)$ the natural map $H^i(\mathcal{C}) \otimes_\psi k \to H^i(\mathcal{C} \otimes_\psi k)$ and identify ψ with the point defined by* $\operatorname{Spec} k \to \operatorname{Spec} A$. *Then for each integer i, there exists a nonempty open subscheme $U \subset \operatorname{Spec} A$ such that for all points $\psi : A \to k$ of U, $h^i(\psi)$ is an isomorphism.*

Proof. Note that for a finite module over a noetherian ring, it is locally free if and only if it is projective. Let $\mathcal{C} = (\cdots \to C^{i-1} \stackrel{d^{i-1}}{\to} C^i \stackrel{d^i}{\to} C^{i+1} \to \cdots)$. First, shrink $\operatorname{Spec} A$ to an affine open subscheme U_1 such that C^{i-1}, C^i and C^{i+1} are locally free on U_1.

Second, shrink U_1 to an affine open subscheme U_2 such that $C^{i+1}/\operatorname{Im}(d^i)$ restricted to U_2 is locally free. Then there is a split exact sequence of $\Gamma(U_2, \mathcal{O}_X)$-modules as follows:

$$0 \longrightarrow \operatorname{Im}(d^i) \longrightarrow C^{i+1} \longrightarrow C^{i+1}/\operatorname{Im}(d^i) \longrightarrow 0. \tag{5.2.1}$$

Clearly $\operatorname{Im}(d^i)$ is a projective $\Gamma(U_2, \mathcal{O}_X)$-module and we have another split exact sequence:

$$0 \longrightarrow \operatorname{Ker}(d^i) \longrightarrow C^i \xrightarrow{d^i} \operatorname{Im}(d^i) \longrightarrow 0. \tag{5.2.2}$$

Thus $\operatorname{Ker}(d^i)$ is a projective module as it is a direct summand of a projective module.

Last, shrink U_2 to an affine open subscheme U_3 such that $\operatorname{Ker}(d^i)/\operatorname{Im}(d^{i-1})$ restricted to U_3 is locally free. Then there is a split exact sequence of $\Gamma(U_3, \mathcal{O}_X)$-modules as follows:

$$0 \longrightarrow \operatorname{Im}(d^{i-1}) \longrightarrow \operatorname{Ker}(d^i) \longrightarrow \operatorname{Ker}(d^i)/\operatorname{Im}(d^{i-1}) \longrightarrow 0. \tag{5.2.3}$$

Thus $\operatorname{Im}(d^{i-1})$ is a projective $\Gamma(U_3, \mathcal{O}_X)$-module and we have another split exact sequence:

$$0 \longrightarrow \operatorname{Ker}(d^{i-1}) \longrightarrow C^{i-1} \xrightarrow{d^{i-1}} \operatorname{Im}(d^{i-1}) \longrightarrow 0. \tag{5.2.4}$$

With (5.2.1), (5.2.2), (5.2.3) and (5.2.4), we have a commutative diagram as follows:

$$\begin{array}{ccccc} C^{i-1} & \xrightarrow{d^{i-1}} & C^i & \xrightarrow{d^i} & C^{i+1} \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Ker}(d^{i-1}) \oplus \mathrm{Im}(d^{i-1}) & \xrightarrow{\left(\begin{smallmatrix} 0 & \mathrm{id} \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)} & \mathrm{Im}(d^{i-1}) \oplus \frac{\mathrm{Ker}(d^i)}{\mathrm{Im}(d^{i-1})} \oplus \mathrm{Im}(d^i) & \xrightarrow{\left(\begin{smallmatrix} 0 & 0 & \mathrm{id} \\ 0 & 0 & 0 \end{smallmatrix}\right)} & \mathrm{Im}(d^i) \oplus \frac{C^{i+1}}{\mathrm{Im}(d^i)}. \end{array}$$

Clearly for every point $\psi : A \to k$ of U_3, the natural map $h^i(\psi)$ is an isomorphism. □

Now we show that $\mathrm{NP}^1(e,i) = \mathrm{NP}^2(e,i)$ for points of an open subscheme.

Corollary 5.2.2. *With the assumptions of Proposition 4.1.2. There exists a nonempty open subscheme $U \subset S_0$ such that for all points $e : \operatorname{Spec} k' \to U$, the morphism $\chi_e : M_{i,e} \to H^i_{cris}(X_e/W(k'))$ in (5.1.5) is an isogeny of F-modules and $NP^1(e,i) = NP^2(e,i)$.*

Proof. We may assume that S_0 is irreducible. By Theorem 3.2.3, there exists of a strictly perfect complex $R\Gamma(X/\widehat{S}, \mathcal{O}_{X/\widehat{S}})$ such that $H^i_{\mathrm{cris}}(X/\widehat{S}) = H^i(R\Gamma(X/\widehat{S}, \mathcal{O}_{X/\widehat{S}}))$ and $H^i_{\mathrm{cris}}(X_e/W(k')) \xrightarrow{\sim} H^i(R\Gamma(X/\widehat{S}, \mathcal{O}_{X/\widehat{S}}) \otimes_{\hat{e}} W(k'))$ by (3.2.6).

By applying Lemma 5.2.1 to the complex $R\Gamma(X/\widehat{S}, \mathcal{O}_{X/\widehat{S}})$, we may assume that there exists an open affine subscheme $\operatorname{Spec} \widehat{A}[\frac{1}{\alpha}] \subset \operatorname{Spec} \widehat{A}$ such that for every point ψ of $\operatorname{Spec} \widehat{A}[\frac{1}{\alpha}]$, $h^i(\psi)$ is isomorphic. Assume that $\alpha = p^n\beta$ with $\beta \in \widehat{A} \backslash p\widehat{A}$. Let $\alpha_0 = (\beta \bmod p) \in A_0$ and $U = \operatorname{Spec} A_0[\frac{1}{\alpha_0}]$. Consider the canonical lifting $\hat{e} : \operatorname{Spf} W(k') \to \widehat{S}$ for every point $e : \operatorname{Spec} k' \to U$, note that the homomorphism

$$\hat{e}^* \otimes \mathbb{Q}_p : \widehat{A} \to W(k') \hookrightarrow W(k') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

factors through $\widehat{A}[\frac{1}{\alpha}]$ and can be viewed as a point of $\operatorname{Spec} \widehat{A}[\frac{1}{\alpha}]$. Therefore, for every point $e : \operatorname{Spec} k' \to U$, $h^i(\hat{e}^* \otimes \mathbb{Q}_p)$ is an isomorphism. Moreover, it is easy to see that $h^i(\hat{e}^* \otimes \mathbb{Q}_p)$ is exactly $\chi_e \otimes \mathbb{Q}_p$ in (5.1.5). This completes the proof. □

5.3 Specialization theorem for F-prelattices

As the proof of specialization theorem for F-crystals [Kat79, Theorem 2.3.1] only involves the Frobenius structure, it can be employed to prove similar property for objects endowed with only the Frobenius structure, for example, F-prelattices.

Definition 5.3.1. In Situation 4.1.0.1, an F-prelattice over $\widehat{A}$ is a pair (M, F) consisting of a finite $\widehat{A}$-module M and a σ-linear map $F : M \to M$ such that the linearization $\Psi : M \otimes_\sigma \widehat{A} \to M$ becomes an isomorphism after tensoring with $\mathbb{Q}_p$. Denoted by (M, Ψ).

Clearly the (F, ∇)-module $H^i_{\mathrm{cris}}(X/\widehat{S})$ is an F-prelattice over $\widehat{A}$. Given an F-prelattice, same as defining $\mathrm{NP}^2(e, i)$ in Definition 5.1.2, we can associate a Newton polygon to every point e of S_0, denoted by $\mathrm{NP}(e, M)$.

Lemma 5.3.2 (Specialization theorem for F-prelattice)**.** *In Situation 4.1.0.1, let (M, F) be an F-prelattice over $\widehat{A}$. Then there exists an open subscheme $U \subset S_0$ such that the family of Newton polygons $\{NP(e, M) \mid e \in U\}$ is constant.*

Proof. Without loss of generality, we may assume that S_0 is irreducible. First there exists an open affine subscheme $\mathrm{Spec}\, \widehat{A}[\frac{1}{\alpha}]$ such that $M \otimes_{\widehat{A}} \widehat{A}[\frac{1}{\alpha}]$ is a free $\widehat{A}[\frac{1}{\alpha}]$-module of finite rank. Assume that $\alpha = p^n \beta$ with $\beta \in \widehat{A} \backslash p\widehat{A}$. Let $\alpha_0 = (\beta \bmod p) \in A_0$ and $U = \mathrm{Spec}\, A_0[\frac{1}{\alpha_0}]$. It is easy to see that the composite map $\widehat{A} \xrightarrow{i_\sigma} W(A_0^{\mathrm{pf}}) \longrightarrow W(A_0^{\mathrm{pf}}[\frac{1}{\alpha_0}])[\frac{1}{p}]$ factors through $\widehat{A}[\frac{1}{\alpha}]$. Let $R = W(A_0^{\mathrm{pf}}[\frac{1}{\alpha_0}])$. Clearly $M \otimes_{\widehat{A}} R[\frac{1}{p}]$ is a free $R[\frac{1}{p}]$-module of finite rank and the σ-linear map $F : M \otimes_{\widehat{A}} R[\frac{1}{p}] \to M \otimes_{\widehat{A}} R[\frac{1}{p}]$ can be expressed as a matrix with entries in $R[\frac{1}{p}]$. We may replace F by $p^{n_0} F$ so that all entries of the matrix of F are in R. Let $\lambda_1 \leq \ldots \leq \lambda_r$ be the Newton slopes of $\mathrm{NP}(\eta, M)$ at the generic point of S_0. By a similar proof to that of ([Kat79, Theorem 2.3.1]), we see that for every $1 \leq i \leq r$, the set of points in $Spec A_0[\frac{1}{\alpha_0}]$ at which all Newton slopes of $(\wedge^i M, \wedge^i F)$ are $= \sum_{k=1}^i \lambda_k$ is a Zariski open subset. This proves the lemma. □

Remark 5.3.3. A key step in the proof of [Kat79, Theorem 2.3.1] is to apply basic slope estimate [Kat79, 1.4.3] to convert the condition on Newton slopes to conditions on Hodge slopes, which can be described in terms of entries of the matrix of the σ-linear map.

Corollary 5.3.4. *With the assumptions of Proposition 4.1.2. There exists a nonempty open subscheme $U \subset S_0$ such that the family of Newton polygons $\{NP^1(e,i) \mid e \in U\}$ is constant.*

Proof. May assume that S_0 is irreducible. First, by Corollary 5.2.2, there exists a nonempty open subscheme $U_1 \subset S_0$ such that for all points $e \in U_1$, $\mathrm{NP}^1(e,i) = \mathrm{NP}^2(e,i)$. Then apply Lemma 5.3.2 to $H^i_{\mathrm{cris}}(X/\widehat{S})$, we obtain an open subscheme $U_2 \subset S_0$ such that the family of Newton polygons $\{\mathrm{NP}^2(e,i) \mid e \in U_2\}$ is constant. Thus $U = U_1 \cap U_2$ is as required. □

5.4 F-Crystals isogenous to $H^i_{\mathbf{cris}}(X/\widehat{S})$

In this section, we restrict to Situation 4.1.0.2 when $S_0 = \operatorname{Spec} k[\![t]\!]$ and show that there exists an F-crystal over $\mathrm{Cris}(S_0/W(k))$ isogenous to the (F,∇)-module $H^i_{\mathrm{cris}}(X/\widehat{S})$. In this case, an (F,∇)-module is usually called as an (F,θ)-module defined as below.

Definition 5.4.1. [dJ98, Definition 4.9] In Situation 4.1.0.2, an (F,θ)-module over $\widehat{A} = W(k)[\![t]\!]$ is a triple (M,F,θ) consisting of a finitely generated $\widehat{A}$-module, a σ-linear map $F : M \to M$ and an additive map $\theta : M \to M$ such that

1. The $\widehat{A}$-linear morphism $M \otimes_\sigma \widehat{A} \to M$ becomes isomorphic after tensoring with $\mathbb{Q}_p$;
2. $\theta(fm) = f\theta(m) + \frac{d}{dt}(f)m$, where $\frac{d}{dt} : \widehat{A} \to \widehat{A}$ is defined by $\Sigma a_n t^n \mapsto \Sigma n a_n t^{n-1}$;
3. $\theta(F(m)) = pt^{p-1}F(\theta(m))$.

An isogeny of (F,θ)-modules is a morphism $\phi : (M,F,\theta) \to (M',F',\theta')$ such that $\phi \otimes_{\widehat{A}} K_W[\![t]\!]$ become an isomorphism, where $K_W = W(k)[\frac{1}{p}]$.

Note that θ is the same as an connection, which is automatically integrable.

Definition 5.4.2. We say that an (F,θ)-module (M,F,θ) has a topologically quasi-nilpotent connection if for every $m \in M$, there exists some $k > 0$ such that $\theta^k(m) \in pM$.

Given an (F,θ)-module, similarly as Definition 5.1.2, we can associate a Newton polygon to both points of S_0. Clearly, Newton polygons are invariant under isogeny.

We will first see some properties of (F,θ)-modules.

Lemma 5.4.3. *Let (M,F,θ) be an (F,θ)-module over $\widehat{A} = W(k)[\![t]\!]$ and K_W be the fraction field of $W(k)$. Then $M \otimes_{\widehat{A}} K_W[\![t]\!]$ is a free $K_W[\![t]\!]$-module.*

Proof. Let $M_K = M \otimes_{\widehat{A}} K_W[\![t]\!]$ and $M_K^\tau \subset M_K$ be the maximal submodule of torsions.

To see $M_K^\tau = 0$, we first prove by induction on n that if $m \in M_K^\tau$ satisfies that $t^n m = 0$, then $m \in tM_K^\tau$. Note that θ extends to M_K. The initial case: $n = 1$, i.e. $tm = 0$, then $\theta(tm) = t\theta(m) + m = 0$ and $m \in tM_K^\tau$. The induction step: if $t^n m = 0$, then $\theta(t^n m) = t^n\theta(m) + nt^{n-1}m = 0$. Since $t^{n-1}(t\theta(m) + nm) = 0$, by assumption, $t\theta(m) + nm \in tM_K^\tau$, thus $m \in tM_K^\tau$. By induction, $M_K^\tau \subset tM_K^\tau$. Then by Nakayama's Lemma, $M_K^\tau = 0$. □

Lemma 5.4.4. *([dJ98, Lemma 6.1]) Let (M,F,θ) be an (F,θ)-module over $\widehat{A} = W(k)[\![t]\!]$. Then there exists an isogeny $M \to M'$ to a free (F,θ)-module over $\widehat{A}$. If (M,F',θ') has a topologically quasi-nilpotent connection, so does (M',F,θ).*

Proof. See [dJ98, Lemma 6.1]. As $\widehat{A}$ is regular local of dimension 2, the dual of a finitely generated $\widehat{A}$-module is finite free. We may take M' as below:

$$M' = Hom_{\widehat{A}}(Hom_{\widehat{A}}(M,\widehat{A}),\widehat{A}). \tag{5.4.1}$$

The evaluation map $\mathrm{ev} : M \to M'$ has a finite length cokernel. The fact that $\mathrm{ev} \otimes_{\widehat{A}} K_W[\![t]\!]$ is an isomorphism follows from Lemma 5.4.3.

Let $K_{\widehat{A}}$ is the fraction field of $\widehat{A}$. M' can also be viewed as a submodule of $M \otimes_{\widehat{A}} K_{\widehat{A}}$ consisting of $m \in M \otimes_{\widehat{A}} K_{\widehat{A}}$ such that $m_1 = p^{n_1}m, m_2 = t^{n_2}m \in M$ for some $n_1, n_2 \in \mathbb{N}$ and $t^{n_2} m_1 = p^{n_1} m_2$ in M. From this point of view, it is easy to check that the (F, θ)-actions on M extends to (F', θ')-actions on M'. Moreover, if θ on M is topologically quasi-nilpotent, then for every $m \in M$ and every $N > 0$, there exists some $k > 0$ such that $\theta^k(m) \in p^{N+1}M$. Thus for every $m' \in M'$ of the form $\frac{m}{p^N}$, we have $(\theta')^k(m') \in pM'$. Therefore, θ' on M' is topologically quasi-nilpotent. □

Before replacing $H^i_{\mathrm{cris}}(X/\widehat{S})$ by an isogenous F-crystal, we would like to verify that $\mathrm{NP}^1(e,i) = \mathrm{NP}^2(e,i)$ for both points $e \in \operatorname{Spec} k[\![t]\!]$.

Lemma 5.4.5. *Given $S_0, \widehat{S}$ as in Situations 4.1.0.2. Let s and η respectively be the special and generic point of $S_0 = \operatorname{Spec} k[\![t]\!]$. Let $f : X \to S_0$ be a proper smooth morphism. Then $NP^1(e,i) = NP^2(e,i)$ for both points of S_0; $NP^1(s,i)$ and $NP^1(\eta,i)$ have the same endpoints.*

Proof. $\mathrm{NP}^1(\eta,i) = \mathrm{NP}^2(\eta,i)$ follows from Corollary 5.2.2. If $e = s$, we need to review the map χ_e in (5.1.5). Note that $\widehat{A} = W(k)[\![t]\!]$ and $\hat{s}^* : \widehat{A} \to W(k)$ is defined by $t \mapsto 0$. By (3.2.6), χ_s is given as follows (write $R\Gamma(X/\widehat{S}, \mathcal{O}_{X/\widehat{S}})$ as $R\Gamma(X/\widehat{S})$ for short):

$$H^i(R\Gamma(X/\widehat{S})) \otimes_{\hat{s}^*} W(k) \xrightarrow{\chi_s} H^i(R\Gamma(X/\widehat{S}) \otimes_{\hat{s}^*} W(k)). \tag{5.4.2}$$

To show that $\mathrm{NP}^1(s,i) = \mathrm{NP}^2(s,i)$, it suffices to show $\chi_s \otimes_{W(k)} K_W$ is an isomorphism. Let K_W be the fraction field of $W(k)$. Still denote by $\hat{s}^*$ the map $K_W[\![t]\!] \to K_W$ given by $t \mapsto 0$. Let $\mathcal{C}$ be the complex $R\Gamma(X/\widehat{S}, \mathcal{O}_{X/\widehat{S}}) \otimes_{\widehat{A}} K_W[\![t]\!]$. As the natural map $\widehat{A} = W(k)[\![t]\!] \to K_W[\![t]\!]$ is flat, we have a commutative diagram as below:

$$\begin{array}{ccc} H^i(R\Gamma(X/\widehat{S})) \otimes_{\hat{s}^*} W(k) \otimes K_W & \xrightarrow{\chi_s \otimes K_W} & H^i(R\Gamma(X/\widehat{S}) \otimes_{\hat{s}^*} W(k)) \otimes K_W \\ \simeq \downarrow & & \simeq \downarrow \\ H^i(\mathcal{C}) \otimes_{\hat{s}^*} K_W & \xrightarrow{\chi'_e} & H^i(\mathcal{C} \otimes_{\hat{s}^*} K_W) \end{array}, \tag{5.4.3}$$

Then apply the universal coefficient theorem to the complex $\mathcal{C}$ of finite free $K_W[\![t]\!]$-modules and $\hat{s}^* : K_W[\![t]\!] \to K_W$, we obtain an exact sequence

$$0 \to H^i(\mathcal{C}) \otimes_{\hat{s}^*} K_W \to H^i(\mathcal{C} \otimes_{\hat{s}^*} K_W) \to \mathrm{Tor}_1(K_W[\![t]\!]/(t), H^{i+1}(\mathcal{C})) \to 0, \tag{5.4.4}$$

where $\mathrm{Tor}_1(K_W[\![t]\!]/(t), H^{i+1}(\mathcal{C})) = \{m \in H^{i+1}(\mathcal{C}) \mid tm = 0\}$. Since $H^{i+1}(\mathcal{C}) \simeq H^{i+1}_{\mathrm{cris}}(X/\widehat{S}) \otimes_{\widehat{A}} K_W[\![t]\!]$ and $H^{i+1}_{\mathrm{cris}}(X/\widehat{S})$ is an (F,θ)-module over $\widehat{A}$, by Lemma 5.4.3, $H^{i+1}_{\mathrm{cris}}(X/\widehat{S}) \otimes_{\widehat{A}} K_W[\![t]\!]$ is free, so is $H^{i+1}(\mathcal{C})$. Therefore, χ'_s is an isomorphism, so is $\chi_s \otimes_{W(k)} K_W$. Thus $\mathrm{NP}^1(s,i) = \mathrm{NP}^2(s,i)$.

To see that $\mathrm{NP}^1(s,i)$ and $\mathrm{NP}^1(\eta,i)$ have the same endpoints, it suffices to show that $\mathrm{NP}^2(s,i)$ and $\mathrm{NP}^2(\eta,i)$ have the same endpoints. Since $NP^2(e,i)$ is based on the (F,θ)-module $H^i(\mathcal{C})$ and $H^i(\mathcal{C}) \otimes_{\widehat{A}} K_W[\![t]\!]$ is a free $K_W[\![t]\!]$-module by Lemma 5.4.3, it is not hard to see that $\mathrm{NP}^2(s,i)$ and $\mathrm{NP}^2(\eta,i)$ have the same endpoints (r, n_r), where $r =$ rank $H^i(\mathcal{C}) \otimes_{\widehat{A}} K_W[\![t]\!]$ and n_r is the p-adic value of the determinant of the σ-linear self-map of $H^i(\mathcal{C})$. □

Now we are ready to claim that $H^i_{\mathrm{cris}}(X/\widehat{S})$ is isogenous to an F-crystal.

Proposition 5.4.6. *With the assumptions in Lemma 5.4.5. Then there exists an isogeny $\xi : H^i_{cris}(X/\widehat{S}) \to P_i$ of (F,θ)-modules to a free (F,θ)-module (P_i, F, θ) over $\widehat{A}$ with a topologically quasi-nilpotent connection; $NP^1(e,i) = NP(e, P_i)$ for both points of S_0. Denote by $\mathcal{P}_i$ the F-crystal over Cris$(\mathrm{Spec}\, k[\![t]\!]/\mathbb{Z}_p)$ corresponding to (P_i, F, θ).*

Proof. It follows from Lemmas 5.4.4 and 5.4.5. □

Corollary 5.4.7. *With the notations in Lemma 5.4.5. $NP^1(s,i)$ lies on or over $NP^1(\eta,i)$.*

Proof. It follows from applying Grothendieck's specialization theorem [Kat79, Theorem 2.3.1] to the F-crystal $\mathcal{P}_i$ in Proposition 5.4.6. □

Remark 5.4.8. The above proof also implies that $\mathrm{NP}^1(s,i)$ and $\mathrm{NP}^1(\eta,i)$ have the same endpoints.

5.5 Specialization theorem of crystalline cohomology

In this section, we reprove the specialization theorem of crystalline cohomology. First we generalize Corollary 5.3.4 to the case when S_0 is not necessarily smooth.

Lemma 5.5.1. *Consider a proper smooth morphism $f : X \to S_0$ over an affine and irreducible $\mathbb{F}_p$-scheme S_0 of finite type. Let $\eta \in S_0$ be the generic point. Then there exists an open subscheme $U \subset S_0$ such that $NP^1(e, i) = NP^1(\eta, i)$ for all $e \in U$.*

Proof. Let $\widetilde{S}_0$ be the normalization of S_0 in its function field. By [Mat80, 31.H], $h : \widetilde{S}_0 \to S_0$ is a finite and surjective morphism. As $\widetilde{S}_0$ is normal, then it has a smooth open affine subscheme U_0. As U_0 is affine, smooth and of finite type over $\mathbb{F}_p$ and $f' : X \times_{S_0} U_0 \to U_0$ is proper smooth, the assumptions of Corollary 5.3.4 are satisfied, thus there exists an open subscheme $U_1 \subset U_0$ such that the family of Newton polygons $\{\mathrm{NP}^1(e, i) | e \in U_1\}$ is constant. As h is an open map, then $h(U_1) \subset S_0$ is an open subscheme as required. □

Theorem 5.5.2 (Speciallization)**.** *Let S be an $\mathbb{F}_p$-scheme and $f : X \to S$ be a proper smooth morphism. Then there exists a locally finite stratification $S = \coprod_\alpha U_\alpha$ consisting of locally closed subsets of S such that for every α, the family of Newton polygons $\{NP^1(e, i) | e \in U_\alpha\}$ is constant. Moreover, if η is a generization of s, then $NP^1(s, i)$ lies on or over $NP^1(\eta, i)$ and they have the same endpoints.*

Proof. First, because of the local nature of the theorem, we may assume that $S = \operatorname{Spec} A$ is affine. Second, by EGA IV 8.9.1, 8.10.5 and 17.7.8, there exists an affine $\mathbb{F}_p$-scheme S_0 of finite type, a proper smooth morphism $f_0 : X_0 \to S_0$ and a cartesian diagram:

$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} \\ S & \xrightarrow{\pi} & S_0 \end{array} \quad . \tag{5.5.1}$$

As $\pi^{-1}\{\text{closed set}\} = \{\text{closed set}\}$ and $\pi^{-1}\{\text{open set}\} = \{\text{open set}\}$, it suffices to prove the theorem for f_0. May assume that S is an affine and irreducible $\mathbb{F}_p$-scheme of finite type.

By Lemma 5.5.1 and EGA $0_{III}9.2.6$, we conclude that the stratification by Newton polygons consists of a finite disjoint union of locally closed subsets.

To see that Newton polygons rise under specialization from η to s, we first find a morphism $\operatorname{Spec} k[\![t]\!] \to S$ such that it maps the special and generic point of $\operatorname{Spec} k[\![t]\!]$ to s and η respectively, where k is a perfect field. Then the proof is done by applying Lemma 5.4.5 and Corollary 5.4.7 to the base change of f under $\operatorname{Spec} k[\![t]\!] \to S$. $\square$

Chapter 6

Representations and purity theorem

We are going to prove a purity theorem of crystalline cohomology. The proof is similar as that of [Yan11, Theorem 1.1]. A key point in the proof is to employ an equivalence between the unramified property of representations and the coincidence of Newton polygons.

6.1 Rank-1 representations associated to F-modules

In this section, k will denote a perfect field of characteristic $p > 0$, unless otherwise specified. Set $\mathrm{Gal}_k = \mathrm{Aut}(k^{\mathrm{sep}}/k)$. Note that for a perfect field k, the separable closure k^{sep} is also the algebraic closure $\bar{k}$. The following fact will be very useful to make transitions between a field and its perfect closure.

Facts 6.1.1. *([Rom06, Theorems 3.6.1, 3.6.4]) Let k be a field of characteristic $p > 0$ and $\bar{k}$ be its algebraic closure. Let k^{sep} and k^{pf} be the separable and perfect closure of k inside $\bar{k}$ respectively. Denote by Gal_k (resp. $Gal_{k^{pf}}$) the automorphism group of*

k^{sep} (resp. $\bar{k}$) over k (resp. k^{pf}). Then there exists a canonical isomorphism

$$Gal_k \simeq Aut(\bar{k}/k) \simeq Gal_{k^{pf}}. \tag{6.1.1}$$

We first define a representation for a rank-one F-crystal.

Definition 6.1.2. Let (M, F) be a rank-one F-crystal over a perfect field k and $\{e\}$ be a basis. Assume that $F(e) = p^m \mu e$ for some unit $\mu \in W(k)$. Extend (M, F) to an F-crystal over $\bar{k}$. By Dieudonné's theory ([Man63]), there exists a unit $\alpha \in W(\bar{k})$ such that $F(\alpha e) = p^m \alpha e$, i.e. $\mu \cdot \sigma(\alpha) = \alpha$. With the Galois action Gal_k on $\bar{k}$ canonically lifted to an action on $W(\bar{k})$, we can define a representation as follows:

$$\rho_M : \mathrm{Gal}_k \to \mathbb{Z}_p^*, \quad g \mapsto g(\alpha) \cdot \alpha^{-1}. \tag{6.1.2}$$

We would like to define representations depending on break points of Newton polygons. To do this, we first manage to find an F-crystal of rank 1 as follows:

Proposition 6.1.3. *([Yan11, Proposition 2.1]) Let (M, F) be an F-crystal over a perfect field k of characteristic $p > 0$. If the first break point of $NP(M)$ is $(1, N)$ for $N \in \mathbb{N}$, then (M, F) has a unique subcrystal $M_1 \subset M$ of rank 1 and slope N such that any subcrystal of slope N is a subcrystal of M_1.*

Proof. See [Yan11, Proposition 2.1]. □

Start with a break point (r, N) of the Newton polygon of an F-crystal (M, F), we consider the exterior power $(\wedge^r M, \wedge^r F)$, which is an F-crystal with $(1, N)$ as its first break point. By applying Proposition 6.1.3 to $(\wedge^r M, \wedge^r F)$, we find an F-crystal M_1 of rank 1 and then obtain a representation from M_1 by applying Definition 6.1.2.

Definition 6.1.4. Let (M, F) be an F-module over $W(k)$ and β be a break point of $\mathrm{NP}(M)$. Take the maximal free submodule $M' \subset M$. Note that M' naturally becomes an F-crystal and $\mathrm{NP}(M') = \mathrm{NP}(M)$. We define the representation associated to (M, β) to be the one induced from (M', β) shown as above, denoted by $\rho_{(M,\beta)} : \mathrm{Gal}_k \to \mathbb{Z}_p^*$.

Facts 6.1.5. *Given an isogeny $(M, F) \to (M', F')$ of F-modules over $W(k)$ and a break point β of $NP(M) = NP(M')$. Then the representations $\rho_{(M,\xi)}$ and $\rho_{(M',\xi)}$ are the same.*

Now we define a representation induced from an (F, θ)-module.

Definition 6.1.6. Let (M, F, θ) be an (F, θ)-module over $\widehat{A} = W(k)[\![t]\!]$. Fix a morphism $\eta : \operatorname{Spec} k(\!(t)\!)^{\mathrm{pf}} \to \operatorname{Spec} k[\![t]\!]$ to the generic point $\eta \in \operatorname{Spec} k[\![t]\!]$ and take the canonical lifting $\hat{\eta}^* : W(k)[\![t]\!] \to W(k(\!(t)\!)^{\mathrm{pf}})$. With a break point β of $\mathrm{NP}^1(\eta, M)$ of the F-module $M \otimes_{\hat{\eta}^*} W(k(\!(t)\!)^{\mathrm{pf}})$, by Definition 6.1.4, we obtain a map $\rho : \mathrm{Gal}_{k(\!(t)\!)^{\mathrm{pf}}} \to \mathbb{Z}_p^*$. We define the representation associated to (η, M, β) to be the composite map $\mathrm{Gal}_{k(\!(t)\!)} \simeq \mathrm{Gal}_{k(\!(t)\!)^{\mathrm{pf}}} \xrightarrow{\rho} \mathbb{Z}_p^*$, denoted by $\rho_{(\eta,M,\beta)}$. Note that $\rho_{(\eta,M,\beta)}$ is invariant under isogeny of (F, θ)-modules.

We are interested in representations induced from crystalline cohomology groups.

Definition 6.1.7. Let $f : X \to S_0$ be a proper smooth morphism of schemes in characteristic $p > 0$. For every point $e \in S_0$, let k_e be the residue field at e and $X_{k_e^{\mathrm{pf}}}$ be the base change of f to k_e^{pf}, a perfect closure of k_e. With a break point β of $\mathrm{NP}^1(e, i)$ of the F-module $H^i_{\mathrm{cris}}(X_e/W(k_e^{\mathrm{pf}}))$, by Definition 6.1.4, we obtain a map $\rho : \mathrm{Gal}_{k_e^{\mathrm{pf}}} \to \mathbb{Z}_p^*$. With the canonical isomophism $\mathrm{Gal}_{k_e} \simeq \mathrm{Gal}_{k_e^{\mathrm{pf}}}$ in Facts 6.1.1, we define the representation associated to $(e, H^i_{\mathrm{cris}}, \beta)$ to be the composite map $\mathrm{Gal}_{k_e} \simeq \mathrm{Gal}_{k_e^{\mathrm{pf}}} \xrightarrow{\rho} \mathbb{Z}_p^*$, denoted by $\rho_{(e,i,\beta)}$.

We would like to see how representations are related under base change.

Lemma 6.1.8. *With the assumptions in Definition 6.1.7. Let $k' \supset k_e^{pf}$ be a perfect field and $e' : \operatorname{Spec} k' \to S_0$. Then $\rho_{(e,i,\beta),k'}$ is isomorphic to the representation associated to $(e', H^i_{cris}(X_{k'}/W(k')), \beta)$, where*

$$\rho_{(e,i,\beta),k'} : Gal_{k'} \to Gal_{k_e} \xrightarrow{\rho_{(e,i,\beta)}} \mathbb{Z}_p^*. \tag{6.1.3}$$

Proof. It follows from an isomorphism of F-modules in Corollary 4.2.3. □

Corollary 6.1.9. *With the assumptions in Proposition 5.4.6. Let η be the generic point of $S_0 = \operatorname{Spec} k[\![t]\!]$ and β be a break point of $NP^1(\eta, i) = NP(\eta, P_i)$. Then $\rho_{(\eta,i,\beta)}$ and $\rho_{(\eta,P_i,\beta)}$ are the same representation $Gal_{k((t))} \to \mathbb{Z}_p^*$.*

Proof. By Proposition 5.4.6, $H^i_{\text{cris}}(X/\widehat{S})$ and P_i are isogenous, thus $\rho_{(\eta,H^i_{\text{cris}}(X/\widehat{S}),\beta)} = \rho_{(\eta,P_i,\beta)}$ by Definition 6.1.6. Apply Corollary 5.2.2 to the generic point $\eta : \operatorname{Spec}(k((t)))^{\text{pf}} \to S_0$, we obtain an isogeny of F-modules; by Facts 6.1.5, $\rho_{(\eta,H^i_{\text{cris}}(X/\widehat{S}),\beta)} = \rho_{(\eta,i,\beta)}$. □

6.2 Criteria for unramified representations

For F-crystals over $\operatorname{Cris}(\operatorname{Spec} k[\![t]\!]/\mathbb{Z}_p)$, we have an equivalence between unramified representations and coincidence of break points; we show an analogous equivalence for the cohomology case. First we recall the equivalence for F-crystals.

Proposition 6.2.1. *([Yan11, Proposition 4.4]) Let R be a discrete valuation ring of characteristic $p > 0$ with fraction field K and residue field k. Let $\mathcal{E}$ be an F-crystal over Spec R. Let η and s be the generic and closed point of Spec R. Assume that the first break point of $NP(\mathcal{E})_\eta$ is $(1, m)$. Then the following two conditions are equivalent:*

1. *the Galois representation associated to $\mathcal{E}$ is unramified, i.e., it factors through*

$$\phi : Gal_K \to \pi_1(\text{Spec } R).$$

2. *the first break point of $NP(\mathcal{E})_s$ is $(1, m)$.*

Proof. See [Yan11, Proposition 4.4] or below for a sketch of the proof for the case when $R = k[\![t]\!]$ with k algebraically closed. The general case can be reduced to this case following similar discussion as the proof of Proposition 6.2.2. In this case, $\pi_1(\operatorname{Spec} R)$ is trivial, thus it turns to show that (1) the associated representation to $\mathcal{E}$ is trivial if and only if (2) $(1, m)$ is also the first break point of $NP(\mathcal{E})_s$.

(1)⇒(2): By [Yan11, Lemma 4.3], $\mathcal{E}_{\text{Spec } K}$ has a trivial subcrystal of rank 1 and slope m. Then we get an injection $\Phi : \mathcal{L}_{\text{Spec}K} \to \mathcal{E}_{\text{Spec } K}$, where $\mathcal{L}$ is a trivial F-crystal of rank 1 and slope m over Spec R. Apply [dJ98, Theorem 1.1] to $\mathcal{E}, \mathcal{L}$ and Φ. We obtain a nontrivial map $\mathcal{L} \to \mathcal{E}$. Restricting to s,we see that $\mathcal{E}_s$ contains a subcrystal of rank 1 and slope m. On the other hand, by Grothendieck's specialization theorem [Kat79, 2.3.1], $NP(\mathcal{E})_s$ lies on or above $NP(\mathcal{E})_\eta$. Hence $(1, m)$ is the first break point of $NP(\mathcal{E})_s$.

Condition (2)⇒ (1):by [Kat79, Corollary 2.6.2], $\mathcal{E}$ is isogenous to an F-crystal $\mathcal{E}'$ which is divisible by p^m, which contains a subcrystal $\mathcal{E}'_1$ of rank 1 and slope m. By [Kat79, Theorem 2.7.4], $\mathcal{E}'_1$ becomes isogenous to a constant F-crystal over $k((t))^{pf}$, and therefore the associated representation to $\mathcal{E}'_1$ is trivial. As representations associated to F-crystals are preserved under isogeny, thus we obtain (1). □

Now we show a similar equivalence for the cohomology case.

Proposition 6.2.2. *Let R be a discrete valuation ring of characteristic $p > 0$ with fraction field K and residue field k and $f : X \to \operatorname{Spec} R$ be a proper smooth morphism. Let η and s respectively be the generic and special point of* Spec R. *Let β be a break point of $NP^1(\eta, i)$. Then β is a break point of $NP^1(s, i)$ if and only if the representation $\rho_{(\eta,i,\beta)} : Gal_K \longrightarrow \mathbb{Z}_p^*$ is unramified, i.e., it factors through the canonical map $Gal_K \xrightarrow{\phi} \pi_1(\operatorname{Spec} R)$.*

For the proof, we recall a lemma regarding the kernel of $\operatorname{Gal}_K \xrightarrow{\phi} \pi_1(\operatorname{Spec} R)$.

Lemma 6.2.3. *([Yan11, Corollary 3.4]) Let R be a discrete valuation ring of characteristic $p > 0$ with fraction field K and residue field k. Let $\widehat{R}$ be the completion of R and fix an isomorphism $\widehat{R} = k[[t]]$. Let $\bar{k}$ be the algebraic closure of k. Then the kernel of the canonical homomorphism $Gal_K \xrightarrow{\phi} \pi_1(\operatorname{Spec} R)$ is the normal subgroup of Gal_K generated by the image of the composition $Gal_{\bar{k}((t))} \longrightarrow Gal_{k((t))} \longrightarrow Gal_K$.*

Proof. See [Yan11, Corollary 3.4]. □

Proof of Proposition 6.2.2. We first prove the case when $R = k[\![t]\!]$ for a perfect field k. May assume that β is the first break point, as other cases are reduced to this one by taking exterior powers. By Proposition 5.4.6 and Corollary 6.1.9, there exists an F-crystal $\mathcal{P}_i$ over $S_0 = \operatorname{Spec} k[\![t]\!]$ such that $\mathrm{NP}^1(e,i) = \mathrm{NP}(e,P_i)$ for both points of S_0 and $\rho_{(\eta,P_i,\beta)} = \rho_{(\eta,i,\beta)}$. Thus the proposition in this case follows directly from Proposition 6.2.1.

In the general case, let $\widehat{R}$ be the completion of R and fix an isomorphism $\widehat{R} = k[[t]]$. Consider the base change of f to $\operatorname{Spec} k^{\mathrm{pf}}[\![t]\!]$ and let $K_1 = k^{\mathrm{pf}}(\!(t)\!)$. On the one hand, by Corollary 4.2.3, $\mathrm{NP}^1(\eta,i)$ is invariant under base change. On the other hand, by Lemma 6.1.8, the representation associated to $(\eta, H^i_{\mathrm{cris}}(X_{K_1^{\mathrm{pf}}}/W(K_1^{\mathrm{pf}})),\beta)$ is the composite map $\rho_{(\eta,i,\beta),K_1} : \mathrm{Gal}_{K_1} \longrightarrow \mathrm{Gal}_K \xrightarrow{\rho_{(\eta,i,\beta)}} \mathbb{Z}_p^*$. Clearly

$$
\begin{aligned}
& \rho_{(\eta,i,\beta),K_1} \text{ is unramified.} \\
\Longleftrightarrow\ & \mathrm{Gal}_{\bar{k}(\!(t)\!)} \subset \mathrm{Gal}_{K_1} \text{ is in the kernel of } \rho_{(\eta,i,\beta),K_1}. \\
\Longleftrightarrow\ & \text{The composite map } \mathrm{Gal}_{\bar{k}(\!(t)\!)} \longrightarrow \mathrm{Gal}_{k(\!(t)\!)} \longrightarrow \mathrm{Gal}_K \xrightarrow{\rho_{(\eta,i,\beta)}} \mathbb{Z}_p^* \text{ is trivial.} \\
\Longleftrightarrow\ & \rho_{(\eta,i,\beta)} \text{ is unramified. (By Lemma 6.2.3)}
\end{aligned}
$$

Thus the general case follows from the case for $R = k^{\mathrm{pf}}[\![t]\!]$. □

6.3 Purity theorem of crystalline cohomology

The proof of purity theorem (Theorem 6.3.4) applies the same method as [dJO99, Theorem 4.1] and [Yan11, Theorem 1.1]. We first prove purity theorem in a special case (Theorem 6.3.3), then complete the proof of Theorem 6.3.4 by reducing the general case to the case in Theorem 6.3.3. Now we list some facts to be referred to in the proof.

Facts 6.3.1. *([Neu99, II Propostion 5.5]) Let $T_p = 1 + p^2\mathbb{Z}_p$ be a subgroup of $(\mathbb{Z}_p^*, *)$. There exists an isomorphism $(T_p, *) \overset{log}{\to} (p^2\mathbb{Z}_p, +), 1 + x \mapsto \log(1+x) =$*

$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}$. *We fix an isomorphism* $(T_p,*)\simeq(\mathbb{Z}_p,+)$ *to be* log *composite with* $(p^2\mathbb{Z}_p,+)\simeq(\mathbb{Z}_p,+)$.

Lemma 6.3.2. *Let* $f: X\to S_0$ *be a proper smooth morphism of schemes in characteristic* $p>0$. *Consider the cartesian diagram as below:*

$$\begin{array}{ccccc} X & \xrightarrow{f} & S_0 \ni g(e') & \qquad & \operatorname{Spec} k_{g(e')} \\ \uparrow & & \uparrow g & & \uparrow \\ X' & \xrightarrow{f'} & S_0' \ni e' & & \operatorname{Spec} k_{e'} \end{array} \tag{6.3.1}$$

Let $k_{e'}$ *(resp.* $k_{g(e')}$*) be the residue field at* $e'\in S_0'$ *(resp.* $g(e')\in S_0'$*). Then we have*

1. $NP^1(e',i)=NP^1(g(e'),i)$.

2. *For any break point* β *of* $NP^1(e',i)$*, the representation* $\rho_{(e',i,\beta)}$ *is the composite map of the canonical map* $Gal_{k_{e'}}\to Gal_{k_{g(e')}}$ *with* $\rho_{(g(e'),i,\beta)}$.

Proof. (1) follows from Corollary 4.2.3 and invariance of Newton polygons under isogeny; (2) follows from Lemma 6.1.8. They will respectively be referred to as invariance of Newton polygons and compatibility of representations under base change. □

Theorem 6.3.3. *Let* $(A,\mathfrak{m},k)$ *be a Noetherian complete local normal integral domain of dimension 2 and its residue field* k *be algebraically closed. Assume that* $S_0=\operatorname{Spec} A$ *and* $f: X\to S_0$ *is a proper smooth morphism. Let* $e_0\in S_0$ *be the closed point and* $U=S_0\backslash\{e_0\}$ *be the open subscheme. If the family of Newton polygons* $\{NP^1(e,i)\mid e\in U\}$ *have a common break point* β*, then* β *is also a break point of* $NP^1(e_0,i)$.

Proof. Let η be the generic point of S_0 and K be the fraction field of A. Consider $\rho_{(\eta,i,\beta)}:\mathrm{Gal}_K\to\mathbb{Z}_p^*$. First, we may assume that $\rho_{(\eta,i,\beta)}(\mathrm{Gal}_K)\subset T_p$, the latter introduced in Facts 6.3.1. Otherwise we take a finite Galois extension L of K such that $\rho_{(\eta,i,\beta)}(\mathrm{Gal}_L)\subset T_p$, replace A by the integral closure A' of A inside L and f by its base change to $S_0'=\operatorname{Spec} A'$. Let e_0' be the closed point of S_0' and $U'=S_0'\backslash\{e_0'\}$. Note that A' have the same properties as A, the natural map $S_0'\to S_0$ maps e_0' to e_0

and U' to U. By invariance of Newton polygons under base change, the general case reduces to the case when $\rho_{(\eta,i,\beta)}(\mathrm{Gal}_K) \subset T_p$.

Second, we show that $\rho_{(\eta,i,\beta)}$ factors through the canonical map $\mathrm{Gal}_K \to \pi_1(U)$. By [Yan11, Prop 3.2], it suffices to show that for every $e \in U$, $\rho_{(\eta,i,\beta)}$ factors through $\mathrm{Gal}_K \to \pi_1(\mathrm{Spec}\,\mathcal{O}_{U,e})$, where $\mathcal{O}_{U,e}$ is the local ring at e. Consider the base change of f to $\mathrm{Spec}\,\mathcal{O}_{U,e}$, denoted by f_e. By Lemma 6.3.2, $\rho_{(\eta,i,\beta)}$ can be viewed as coming from f_e; and the Newton polygon associated to both points of $\mathrm{Spec}\,\mathcal{O}_{U,e}$ have the same break points. By applying Proposition 6.2.2 to f_e, we obtain that $\rho_{(\eta,i,\beta)}$ factors through $\mathrm{Gal}_K \to \pi_1(\mathrm{Spec}\,\mathcal{O}_{U,e})$. Therefore, $\rho_{(\eta,i,\beta)}$ factors through $\mathrm{Gal}_K \to \pi_1(U)$. Consider the composite map

$$\begin{tikzcd} \mathrm{Gal}_K \arrow[r] & \pi_1(U) \arrow[r] \arrow[rr, bend left, "\iota"] & T_p \arrow[r, "\simeq"] & \mathbb{Z}_p. \end{tikzcd} \tag{6.3.2}$$

Next, take a resolution of singularities $\widetilde{S}_0 \to S_0$; if A happens to be regular, let $\widetilde{S}_0$ be the blowup of S_0 with respect to e_0. By [dJO99, Theorem 3.2], $H^1_{et}(\widetilde{S}_0, \mathbb{Q}_p) \simeq H^1_{et}(U, \mathbb{Q}_p)$, thus ι defined in (6.3.2) viewed as an element of $H^1_{et}(U, \mathbb{Q}_p)$ is extended to an element of $H^1_{et}(\widetilde{S}_0, \mathbb{Q}_p)$, i.e. ι factors through $\pi_1(U) \to \pi_1(\widetilde{S}_0)$. Let η_0 be the generic point of some connected component of the exceptional fiber and $\mathcal{O}_{\widetilde{S}_0,\eta_0}$ be the local ring at η_0. We have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Gal}_K \arrow[r] \arrow[d] \arrow[rr, bend left, "\rho_{(\eta,i,\beta)}"] & \pi_1(U) \arrow[r] \arrow[d] \arrow[rd, "\iota"] & T_p \arrow[d, leftrightarrow, "\simeq"] \\ \pi_1(\mathrm{Spec}\,\mathcal{O}_{\widetilde{S}_0,\eta_0}) \arrow[r] & \pi_1(\widetilde{S}_0) \arrow[r] & \mathbb{Z}_p \end{tikzcd} \qquad \begin{tikzcd} X \times_{S_0} \mathrm{Spec}\,\mathcal{O}_{\widetilde{S}_0,\eta_0} \arrow[d, "f_{\eta_0}"] \\ \mathrm{Spec}\,\mathcal{O}_{\widetilde{S}_0,\eta_0}. \end{tikzcd} \tag{6.3.3}$$

Note that $\rho_{(\eta,i,\beta)}$ viewed as the representation associated to the generic point of $\mathrm{Spec}\,\mathcal{O}_{\widetilde{S}_0,\eta_0}$, is unramified as shown above. By applying Proposition 6.2.2 to f_{η_0}, we see that β is a break point of $\mathrm{NP}^1(\eta_0, i) = \mathrm{NP}^1(e_0, i)$. □

Theorem 6.3.4 (Purity)**.** *Let S_0 be a locally Noetherian scheme of characteristic $p > 0$ and $f : X \to S_0$ be a proper smooth morphism. Take a point $e_0 \in S$. If the family of Newton polygons $\{NP^1(e, i) \mid e \text{ is a generization of } e_0\}$ have a common* *break point β, then either the codimension of the Zariski closure of e_0 in S_0 is ≤ 1, or β is a break point of $NP^1(e_0, i)$.*

Proof. Let $\{e_0\}^-$ be the Zariski closure of e_0 in S_0. May assume that $\operatorname{codim}(\{e_0\}^-, S_0) \geq 2$, then it suffices to show that β is a break point of $\mathrm{NP}^1(e_0, i)$. First, consider the base change f_{e_0} of f to $\operatorname{Spec} \mathcal{O}_{S_0,e_0}$, where $\mathcal{O}_{S_0,e_0}$ is the local ring at e_0. By invariance of Newton polygons under base change, it reduces to prove the theorem for f_{e_0}. We may assume that $S_0 = \operatorname{Spec} R$ of dimension ≥ 2, where R is a local Noetherian ring and e_0 is the closed point.

Note that if there exists a morphism of local rings $\psi : R \to R'$ such that the induced morphism of schemes $g : S_0' \to S_0$ maps all points other than the close point to the open subscheme $S_0 \backslash \{e_0\}$, and R' is a Noetherian complete local normal integral domain of dimension 2 with an algebraically closed residue field, then consider the base change f' of f to S_0', by invariance of Newton polygons under base change, it reduces to prove the theorem for f', which is proved in Theorem 6.3.3.

Therefore, it suffices to find a morphism $\psi : R \to R'$ satisfying the above assumptions. First take the completion $\widehat{R}$ of R. By Cohen structure theorem, we may assume that $\widehat{R} = k[\![x_1, \ldots, x_d]\!]/I$ for some ideal I, where k is the residue field. Second, consider $R_{\bar{k}} = \bar{k}[\![x_1, \ldots, x_d]\!]/I\bar{k}[\![x_1, \ldots, x_d]\!]$, where $\bar{k}$ is the algebraic closure of k. Next, choose a prime ideal $\mathfrak{p} \subset R_{\bar{k}}$ such that $A = R_{\bar{k}}/\mathfrak{p}$ is of dimension 2. The existence of such a prime ideal is shown in [Mat80, Proposition 12.K]. Last, take the integral closure $\widetilde{A}$ of A in its fraction field. So far, we have obtained a chain of morphisms as below:

$$\begin{array}{ccccccccc}
R & \longrightarrow & \widehat{R} & \longrightarrow & R_{\bar{k}} & \longrightarrow & A & \longrightarrow & \widetilde{A} \\
 & & \| & & \| & & \| & & \\
 & & k[\![x_1, \ldots, x_d]\!]/I & \longrightarrow & \bar{k}[\![x_1, \ldots, x_d]\!]/I\bar{k}[\![x_1, \ldots, x_d]\!] & \longrightarrow & R_{\bar{k}}/\mathfrak{p} & &
\end{array}
\qquad (\psi : R \dashrightarrow \widetilde{A}) \tag{6.3.4}$$

We claim that $\psi : R \to \widetilde{A}$ is as required. First, $\widetilde{A}$ is a finite A-module by [Mat80, Corollary(Nagata)]; as a finite algebra over a henselian ring and also as an integral domain, $\widetilde{A}$ is a local ring by [Mil80, Theorem 4.2]. It is easy to check that ψ satisfies all other assumptions. This completes the proof. □

Part II

Representations and Frobenius-periodic vector bundles

Chapter 7

Introduction

It was conjectured by de Jong in [dJ01, Conjecture 2.3] that given a finite field $\mathbb{F}$ of characteristic l and a normal variety Y over a finite field κ of characteristic $p \neq l$, every representation $\rho : \pi_1(Y) \to \mathrm{GL}(r, \mathbb{F}(\!(s)\!))$ has a finite geometric monodromy. This conjecture was proved by de Jong in the GL_2-case [dJ01], by Böckle–Khare in the GL_n-case under some mild condition [BK06] and by Gaitsgory modulo the theory of $\mathbb{F}(\!(s)\!)$-sheaves [Gai07]. Then a natural question comes up: if the hypothesis $l \neq p$ is dropped and moreover Y is proper over κ, does the conjecture remain true? Note that when Y/κ is not proper, a counterexample has already been given in [dJ01]

In [Las01], Laszlo gave a negative answer to the above question. He showed that there exists a non-trivial family of rank-2 bundles fixed by the square of Frobenius over a specific genus-2 curve $C_0/\mathbb{F}_2$. From this he deduced the existence of the desired representations of $\pi_1(C_0 \otimes \mathbb{F}_{2^d})$. Recently, Esnault and Langer [EL11] have employed Laszlo's example to improve the statement of a p-curvature conjecture in characteristic p.

It is suspected by de Jong that the representations with an infinite geometric monodromy are rare. Thus one would like to understand the underlying mechanics of Laszlo's example and to obtain such representations in other characteristics.

In this part, we give a geometric interpretation of Laszlo's example based on the study of the action of the automorphism group of the curve; this interpretation allows us to produce a family of similar examples. Meanwhile, our method also provides some indication in characteristics 3 and 5, though it does not directly provide examples.

Now we give a brief summary of our results. In [Las01], Laszlo deduced representations from a non-trivial family of bundles. We show that the converse also holds, see Theorem 8.3.1. The equivalence between a non-constant family of stable Frobenius-periodic vector bundles and a representation with an infinite geometric monodromy is of course well-known to the experts.

Because of the equivalence in Theorem 8.3.1, the question of looking for representations with an infinite geometric monodromy is converted to studying the fixed point locus of the rational map $V : \mathcal{M}_Y(r,0) \dashrightarrow \mathcal{M}_Y(r,0)$ defined by taking pullback of vector bundles under the geometric Frobenius of Y/κ. In [Las01], the expression of V_{C_0} for the specific curve C_0 was applied to locate a projective line $\triangle$ in $\mathcal{M}_{C_0}(2,0)$ such that $(V_{C_0}^2)|_\triangle$ is the identity map. Here our observation is that $\triangle$ is the fixed point locus of the G-action on $\mathcal{M}_{C_0}(2,0)$, where $G = \mathrm{Aut}(C_0 \otimes \mathbb{F}_{2^2}/\mathbb{F}_{2^2}) = \mathbb{Z}/2\mathbb{Z} \times S_3$. Indeed, this property is common to all genus-2 ordinary curves X in characteristic 2 with a G-action. As Frobenius morphisms commute with the G-actions, it is natural to consider the restriction of V to the fixed point locus $\mathcal{M}_X(2,0)^G$ of the G-action on $\mathcal{M}_X(2,0)$. With some trick, we manage to show that some power of V restricted to $\mathcal{M}_X(2,0)^G$ is the identity map and thus obtain a family of curves that carry a non-constant family of stable Frobenius-periodic vector bundles, see Theorem 9.3.1. Moreover, we give an explicit construction of the family of vector bundle, see Lemma 10.1.1 and Corollary 10.3.3.

Combining with Theorem 8.3.1, for every curve in Theorem 9.3.1, there exist representations of the fundamental group with an infinite geometric monodromy.

A large part of the proof of Theorem 9.3.1 can be applied to other characteristics, particularly the application of a group action in locating a sublocus in the moduli space. However, when considering whether the restriction of the Verschiebung to the sublocus is reduced to a linear map, the condition regarding the existence of a single base point on the sublocus is sufficient only in charateristic 2. In other characteristics, more is required to ensure that the restriction of the Verschiebung is the identity. Examples of representations with an infinite geometric monodromy in charateristic other than 2 remain to be discovered.

Chapter 8

Frobenius-periodic bundles and Representations

8.1 Definitions and basic properties

In this section, we establish equivalences among the categories of Frobenius-periodic vector bundles, smooth étale sheaves and representations.

Notations: κ is a finite field of order $q = p^d$, $S = \operatorname{Spec} \kappa[\![s]\!]$, $\mathcal{S} = \operatorname{Spf} \kappa[\![s]\!]$ and $S_n = \operatorname{Spec} \kappa[\![s]\!]/(s^n)$ for $n \geq 0$; F_Y is the absolute Frobenius of a noetherian κ-scheme Y.

8.1.1 Smooth étale $\kappa[\![s]\!]$-sheaves

Our definition of smooth étale $\kappa[\![s]\!]$-sheaves is similar to that of a lisse l-adic sheaf in [Mil80, Chap.V, §1]. When Y is connected, there is an equivalence between the category of locally free smooth $\kappa[\![s]\!]$-sheaves over Y_{et} and the category of continuous $\pi_1(Y)$-modules that are free $\kappa[\![s]\!]$-modules of finite rank, denoted by $\mathcal{C}_{1,Y_{et}} \rightleftharpoons \mathcal{C}_{2,\pi_1(Y)}$.

Definition 8.1.1. Let Y be a noetherian scheme. An *étale $\kappa[\![s]\!]$-sheaf* (or sheaf of $\kappa[\![s]\!]$-modules) on Y_{et} is a projective system $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ of sheaves on Y_{et} such that the given map $\mathcal{F}_{n+1} \to \mathcal{F}_n$ is isomorphic to the canonical map $\mathcal{F}_{n+1} \to \mathcal{F}_{n+1} \otimes_{\kappa[\![s]\!]} \kappa[\![s]\!]/(s^n)$ for every n. Implicitly, $\mathcal{F}_0 = 0$ and $\mathcal{F}_n$ is a $\kappa[\![s]\!]/(s^n)$-module. A $\kappa[\![s]\!]$-sheaf is *locally constant* or *constructible* if each $\mathcal{F}_n$ has the corresponding property. A locally constant constructible $\kappa[\![s]\!]$-sheaf is also called *smooth*. A $\kappa[\![s]\!]$-sheaf $\mathcal{F}$ is *locally free of rank* r if each $\mathcal{F}_n$ is locally free of rank r over $\kappa[\![s]\!]/(s^n)$.

Similarly, the stalk functor $\mathcal{F} \hookrightarrow \mathcal{F}_{\bar{y}} = \varprojlim (\mathcal{F}_n)_{\bar{y}}$ defines an equivalence between the category of constructible, locally constant $\kappa[\![s]\!]$-sheaves over Y_{et} and the category of continuous $\pi_1(Y, \bar{y})$-modules that are finitely generated.

Facts 8.1.2. *Assume that Y is a connected noetherian scheme with geometric point $\bar{y}$. Then there is an equivalence between the category $\mathcal{C}_{Y_{et}}$ of locally free smooth $\kappa[\![s]\!]$-sheaves over Y_{et} and the category $\mathcal{C}_{\pi_1(Y)}$ of continuous $\pi_1(Y, \bar{y})$-modules that are free $\kappa[\![s]\!]$-modules of finite rank, given by $\mathcal{F} \hookrightarrow \mathcal{F}_{\bar{y}} = \varprojlim (\mathcal{F}_n)_{\bar{y}}$. Denoted by $\mathcal{C}_{Y_{et}} \rightleftharpoons \mathcal{C}_{\pi_1(Y)}$.*

8.1.2 Frobenius-periodic bundles

Now we turn to Frobenius-periodic bundles.

Definition 8.1.3. Let Y and T be noetherian κ-schemes. A vector bundle $\mathcal{F}$ over $Y \times_\kappa T$ is said to be *Frobenius-periodic* if $\exists$ an isomorphism $\xi : \mathcal{F} \to (F_Y^d \times \mathrm{id}_T)^* \mathcal{F}$ (resp. $\xi : \mathcal{F} \to (F_Y^d \times \mathrm{id}_{S_n})^* \mathcal{F}$), denoted by $(\mathcal{F}, \xi)$. A *Frobenius-periodic vector bundle* $(\mathcal{F}, \xi)$ over $Y \times_\kappa \mathcal{S}$ is a projective system $((\mathcal{F}_n, \xi_n))_{n \in \mathbb{Z}^+}$ such that for each n, $(\mathcal{F}_n, \xi_n)$ is a Frobenius-periodic vector bundle over $Y \times_\kappa S_n$, the given map $\mathcal{F}_{n+1} \to \mathcal{F}_n$ is isomorphic to the natural map $\mathcal{F}_{n+1} \to \mathcal{F}_{n+1} \otimes_{\kappa[\![s]\!]} \kappa[\![s]\!]/(s^n)$ and compatible with ξ_n's.

For any morphism $Z \xrightarrow{f} Y$, the pull back of a Frobenius-periodic vector bundle $(\mathcal{F}, \xi)$ over $Y \times_\kappa T$ to $Z \times_\kappa T$ can be viewed as a Frobenius-periodic vector bundle over $Z \times_\kappa T$, denoted by $f^*(\mathcal{F}, \xi_n)$ or $(f^*\mathcal{F}, f^*\xi)$.

Definition 8.1.4. Given a Frobenius-periodic vector bundle $(\mathcal{F}, \xi)$ over $Y \times_\kappa T$. A section $s \in \Gamma(Y \times_\kappa T, \mathcal{F})$ is said to be *fixed by* ξ if $\xi(s) = (F_Y^d \times \mathrm{id}_T)^* s$; $(\mathcal{F}, \xi)$ is said to be *trivializable* if $\mathcal{F}$ has a global basis fixed by ξ; it is said to be *étale trivializable* if there exists a finite étale map $Z \xrightarrow{f} Y$ such that $f^*(\mathcal{F}, \xi)$ is trivializable. In this case we also say that Z/Y trivializes $(\mathcal{F}, \xi)$.

Given a Frobenius-periodic vector bundle $(\mathcal{F}_n, \xi_n)$ over $Y \times_\kappa S_n$, we define an étale sheaf $(\mathcal{F}_n)^{\xi_n}$ as follows:

$$(U \xrightarrow{f} Y) \in \mathrm{Et}(Y) \longmapsto \{s \in \Gamma(U, f^*\mathcal{F}) | f^*(\xi)(s) = 1 \otimes s\}. \tag{8.1.1}$$

We are going to see that $(\mathcal{F}_n)^{\xi_n}$ is a locally free smooth $\kappa[\![s]\!]/(s^n)$-sheaf. In the following, by a covering space of Y, we mean a finite étale morphism $f : Z \to Y$ and it is Galois if $\#\mathrm{Aut}(Z/Y) = \deg(f)$.

Lemma 8.1.5. *Given a Frobenius-periodic vector bundle* $(\mathcal{F}, \xi) = ((\mathcal{F}_n, \xi_n))_{n \in \mathbb{Z}^+}$ *over* $Y \times_\kappa \mathcal{S}$. *Then there exists a family of covering spaces* $Y \leftarrow Y_1 \leftarrow Y_2 \leftarrow \cdots \leftarrow Y_n \leftarrow \cdots$ *such that for all* n, Y_n/Y *trivializes* $(\mathcal{F}_n, \xi_n)$.

Proof. According to [Mil80, V Proposition 1.1], a locally constant sheaf with finite stalks over Y_{et} is represented by a finite étale group scheme over Y. Thus, for every n, to show that $(\mathcal{F}_n, \xi_n)$ is étale trivializable, it suffices to show that the étale sheaf $(\mathcal{F}_n)^{\xi_n}$ associated to $(\mathcal{F}_n, \xi_n)$ is locally constant and has finite stalks.

Prove by induction on n. Case $n = 1$ is exactly [LS77, Proposition 1.2]. First there exists an open covering $\{U_\alpha\}$ of Y such that $\mathcal{F}_1|_{U_\alpha \times_\kappa S_1}$ is free for all α. We are going to see that for every U_α, there exists a finite étale morphism $U'_\alpha \to U_\alpha$ that trivializes $(\mathcal{F}_1, \xi_1)|_{U_\alpha \times_\kappa S_1}$. let $\{v_1, ..., v_r\}$ be a basis of $\mathcal{F}_1|_{U_\alpha \times_\kappa S_1}$. Assume that

$$\xi_1\{v_1, ..., v_r\} = \{(F_Y^d \times \mathrm{id}_{S_1})^* v_1, ..., (F_Y^d \times \mathrm{id}_{S_1})^* v_r\} N_0,$$

for some $N_0 \in \mathrm{GL}(r, \mathcal{O}_Y(U_\alpha))$. To find a basis $\{v'_1, ..., v'_r\}$ of the form $\{v_1, ..., v_r\} M_1$ that is fixed by ξ_1, it is equivalent to find $M_1 = (m_{ij})_{i,j=1}^r$ such that

$$N_0 M_1 \equiv M_1^{(q)}, \quad \text{where} \quad M_1^{(q)} = (m_{ij}^q), \quad q = p^d.$$

Thus we define $U_{\alpha,1} = \operatorname{Spec} \mathcal{O}_Y(U_\alpha)[m_{11}, ..., m_{rr}, 1/\det\ M_1]/(N_0M_1 - M_1^{(q)})$. It is easy to see that $U'_\alpha \to U_\alpha$ is finite étale and trivializes $(\mathcal{F}_1, \xi_1)|_{U_\alpha \times_\kappa S_1}$. Therefore, the étale sheaf $(\mathcal{F}_1)^{\xi_1}$ is locally constant and has finite stalk $\kappa^{\oplus r}$.

Induction step: assume that there is a covering space $Y_n \to Y$ that factors through Y_{n-1} and trivializes $(\mathcal{F}_n, \xi_n)$. Take an open covering $\{U_\alpha\}$ of Y_n such that $\mathcal{F}_{n+1}|_{U_\alpha \times_\kappa S_{n+1}}$ is free for all α. We are going to see that for every U_α, there exists a finite étale morphism $U'_\alpha \to U_\alpha$ that trivializes $(\mathcal{F}_{n+1}, \xi_{n+1})|_{U_\alpha \times_\kappa S_{n+1}}$. Let $\{v_1, ..., v_r\}$ be a basis of $\mathcal{F}_{n+1}|_{U_\alpha \times_\kappa S_{n+1}}$ such that it extends to a global basis of $\mathcal{F}_n|_{Y_n \times_\kappa S_n}$ fixed by ξ_n, i.e.

$$\xi_{n+1}\{v_1, \cdots, v_r\} = \{(F^d_{Y_n} \times \mathrm{id}_{S_{n+1}})^* v_1, \cdots, (F^d_{Y_n} \times \mathrm{id}_{S_{n+1}})^* v_r\}(I_{(r)} + s^n D_n),$$

for some $D_n \in \mathrm{Mat}(r \times r, \mathcal{O}_{Y_n}(U_\alpha))$. To find a basis $\{v'_1, ..., v'_r\}$ that is fixed by ξ_{n+1} and of the form $\{v_1, ..., v_r\}(I_{(r)} + s^n \triangle_{n+1})$, it is equivalent to find $\triangle_{n+1} = (m_{ij})$ such that

$$D_n + \triangle_{n+1} = \triangle_{n+1}^{(q)},$$

where $\triangle_{n+1}^{(q)} = (m_{ij}^q)$. Then define $U'_\alpha = \operatorname{Spec} \mathcal{O}_{Y_n}(U_\alpha)[m_{11}, ..., m_{rr}]/(D_n + \triangle_{n+1} - \triangle_{n+1}^{(q)})$. Clearly $U'_\alpha \to U_\alpha$ is finite étale and trivializes $(\mathcal{F}_{n+1}, \xi_{n+1})|_{U_\alpha \times_\kappa S_{n+1}}$. Therefore, the étale sheaf $(\mathcal{F}_{n+1})^{\xi_{n+1}}$ associated to $(\mathcal{F}_{n+1}, \xi_{n+1})$ is is locally constant and has finite stalks. □

Remark 8.1.6. The family of finite étale maps $\{U'_\alpha \to U_\alpha\}$ over an open covering $\{U_\alpha\}$ of Y_n can naturally be glued to a global finite étale map of Y_n.

The trivial line bundle with a non-trivializable Frobenius structure can be trivialized under base field extension and may induce representations with an infinite monodromy yet a trivial geometric monodrony. To avoid such cases, we give the following definition.

Definition 8.1.7. A Frobenius-periodic vector bundle $(\mathcal{F}, \xi)$ over $Y \times_\kappa T$ is said to be *strictly Frobenius-periodic* if $(\det(\mathcal{F}), \det(\xi))$ is trivializable, denoted by $(\mathcal{F}, \xi, \det =$

1); A Frobenius-periodic vector bundle $(\mathcal{F},\xi)$ over $Y\times_\kappa \mathcal{S}$ is said to be *strictly Frobenius-periodic* if every $(\mathcal{F}_n,\xi_n)$ is.

Note that the determinant bundle of a strictly Frobenius-periodic vector bundle $(\mathcal{F},\xi)$ is trivial.

Proposition 8.1.8. *(1) Let $\mathcal{C}_{Y_{et}}$ be the category of locally free smooth $\kappa[\![s]\!]$-sheaves over Y_{et} and $\mathcal{C}_{Y_{zar}}$ be the category of Frobenius-periodic vector bundles over $Y\times_\kappa \mathcal{S}$. Then there is an equivalence $\mathcal{C}_{Y_{et}} \rightleftharpoons \mathcal{C}_{Y_{zar}}$.*
(2) Asssume that Y is connected. Let $\mathcal{C}^{sl}_{\pi_1(Y)}$ be the full subcategory of $\mathcal{C}_{\pi_1(Y)}$ whose objects are SL-representations of $\pi_1(Y)$ and $\mathcal{C}^{str}_{Y_{zar}}$ be the full subcategory of $\mathcal{C}_{Y_{zar}}$ whose objects are strictly Frobenius-periodic vector bundles over $Y\times_\kappa \mathcal{S}$. Then there is an equivalence

$$\mathcal{C}^{sl}_{\pi_1(Y)} \rightleftharpoons \mathcal{C}^{str}_{Y_{zar}}, \quad \rho \leftrightarrow (\mathcal{F}_\rho,\xi_\rho, det=1) \quad or \quad \rho_{(\mathcal{F},\xi)} \leftrightarrow (\mathcal{F},\xi, det=1).$$

Proof. (1)May assume that Y is connected. The functor $\mathcal{C}_{Y_{zar}} \to \mathcal{C}_{Y_{et}}$ is given by

$$\{(\mathcal{F}_n,\xi_n)\}_{n\in\mathbb{Z}^+} \hookrightarrow \{(\mathcal{F}_n)^{\xi_n}\}_{n\in\mathbb{Z}^+} \tag{8.1.2}$$

By the proof of Lemma 8.1.5, $(\mathcal{F}_n)^{\xi_n}$ is a locally constant and locally free $\kappa[\![s]\!]/(s^n)$-sheaf.

The functor $\mathcal{C}_{Y_{et}} \to \mathcal{C}_{Y_{zar}}$ is defined as below: let $\{\mathcal{F}_n\}_{n\in\mathbb{Z}^+} \in \mathrm{Ob}(\mathcal{C}_{Y_{et}})$. For every n, $\mathcal{F}_n$ over Y_{et} is equivalent to a finite and free $\kappa[\![s]\!]/(s^n)$-sheaf $\mathcal{F}'_n$ over a Galois covering Y_n/Y with an $\mathrm{Aut}(Y_n/Y)$-action; the latter can be viewed as the étale sheaf in 8.1.1 associated to a trivializable Frobenius-periodic vector bundle $\mathcal{E}_n$ over $Y_n\times S_n$ with an equivariant $\mathrm{Aut}(Y_n/Y)$-action; by descent theory (e.g. [Mum74, §12, Thereom 1]), $\mathcal{E}_n$ is the pullback of a Frobenius-periodic vector bundle $\mathcal{E}'_n$ over Y. The functor $\mathcal{C}_{Y_{et}} \to \mathcal{C}_{Y_{zar}}$ is defined by $\{(\mathcal{F}_n,\xi_n)\} \hookrightarrow \{\mathcal{E}'_n\}$ and clearly we have an equivalence of categories.

(2) We only need to show that $(\mathcal{F}_n,\xi_n,\det=1)$ induces a SL-representation. Let $Y_n \xrightarrow{f_n} Y$ be a Galois covering space that trivializes $(\mathcal{F}_n,\xi_n)$ by Lemma 8.1.5. Then the

induced representation is the composition $\pi_1(Y,\overline{y}) \to \mathrm{Gal}(Y_n/Y) \to \mathrm{GL}(r,\kappa[\![s]\!]/(s^n))$; with a basis $\{e_1^n,\cdots,e_r^n\}$ of $f_n^*\mathcal{F}_n$ preserved by $f_n^*\xi_n$, the latter is defined by $g \mapsto M_{g^{-1}}$, where $(g^{-1}\times \mathrm{id}_{S_n})^*\{e_1^n,\cdots,e_r^n\} = \{e_1^n,\cdots,e_r^n\}M_{g^{-1}}$. Thus $M_{g^{-1}} \in \mathrm{SL}(r,\kappa[\![s]\!]/(s^n))$. □

8.2 Frobenius-periodic bundles over a projective smooth base

Now we turn to the case when Y is a projective smooth geometrically connected scheme over κ. Let $\bar{\kappa}$ be the algebraic closure of κ and $\bar{Y} = Y \otimes_\kappa \bar{\kappa}$. Note that by Grothendieck's existence theorem, the category of vector bundles over $Y \times_\kappa \mathcal{S}$ is equivalent to the category of vector bundles over $Y \times_\kappa S$. There exists an exact sequence of profinite groups:

$$1 \longrightarrow \pi_1(\bar{Y}) \longrightarrow \pi_1(Y) \longrightarrow \mathrm{Gal}(\bar{\kappa}/\kappa) \longrightarrow 1. \tag{8.2.1}$$

We now recall some definitions and facts.

Definition 8.2.1. We call a Frobenius-periodic vector bundle $(\mathcal{F},\xi)$ over $Y \times_\kappa T$ to be constant if it is isomorphic to the pullback of a Frobenius-periodic vector bundle under the projection $Y \times_\kappa T \to Y$.

Definition 8.2.2. [HL10] A vector bundle $\mathcal{F}$ over Y is said to be *geometrically slope-stable* if $\mathcal{F} \otimes_\kappa \bar{\kappa}$ is slope-stable over $\bar{Y}$.

Facts 8.2.3. *[Isa76] Let K be a field. A continuous representation $\rho : \pi_1(Y) \to SL(r,K)$ is said to be (absolutely) irreducible if and only if ρ is (absolutely) irreducible as a representation of the abstract group $\pi_1(Y)$. Note that a representation of an abstract group $\rho : H \to SL(r,K)$ is absolutely irreducible $\Leftrightarrow$ $\rho \otimes_K L$ is irreducible for every field extension $K \subset L$ $\Leftrightarrow$ $\rho \otimes_K L$ is irreducible for every finite field extension $K \subset L$.*

Lemma 8.2.4. *[dJ01, Lemma 3.15] Let $\rho : H \to GL(r, K[\![s]\!])$ be a representation of a finite group H, where K is a field. If $\rho_0 = \rho$ mod s is absolutely irreducible, then $\rho \simeq \rho_0 \otimes_K K[\![s]\!]$.*

Lemma 8.2.5. *[dJ01, Lemma 2.7] Let $1 \to \Gamma \to H \to \hat{Z} \to 0$ be an exact sequence of profinite groups. Suppose that $\rho : H \to SL(V)$ is a continuous representation such that $\rho|_\Gamma$ is absolutely irreducible. Then $\#\rho(\Gamma) < \infty \Leftrightarrow \#\rho(H) < \infty$.*

Lemma 8.2.6. *Let $(\mathcal{F}, \xi, det = 1) \leftrightarrow \rho$ be a pair under the equivalence in Proposition 8.1.8. If $(\mathcal{F}, \xi)$ is constant, then $\#\rho(\pi_1(Y)) < \infty$. If ρ mod s is absolutely irreducible, then $(\mathcal{F}, \xi)$ is constant $\Leftrightarrow \#\rho(\pi_1(Y)) < \infty$.*

Proof. As a Frobenius-periodic vector bundle over Y is étale trivializable by Lemma 8.1.5, thus if $(\mathcal{F}, \xi)$ is constant, then $\#\rho(\pi_1(Y)) < \infty$.

If ρ mod s is absolutely irreducible and $\Leftrightarrow \#\rho(\pi_1(Y)) < \infty$, by Lemma 8.2.4, $\rho \simeq \rho_0 \otimes_K K[\![s]\!]$. By descent theory, $(\mathcal{F}, \xi)$ is constant. □

Proposition 8.2.7. *Let $(\mathcal{F}, \xi, det = 1) \leftrightarrow \rho$ be a pair under the equivalence in Proposition 8.1.8. Let $\mathcal{F}_0$ be the pullback of $\mathcal{F}$ under the closed immersion $Y \hookrightarrow Y \times_\kappa S$ and ρ_0 be the composite map $\pi_1(Y) \xrightarrow{\rho} SL(r, \kappa[\![s]\!]) \xrightarrow{mod\ s} SL(r, \kappa)$. Then the followings are equivalent:*

$$\mathcal{F}_0 \text{ is geometrically slope-stable (g.s.s.)} \tag{8.2.2}$$

$$\Leftrightarrow (\rho_0)|_{\pi_1(\bar{Y})} \text{ is absolutely irreducible (a.i.).} \tag{8.2.3}$$

If one of the above equivalent conditions hold, then $\mathcal{F}$ is non-constant $\Leftrightarrow \#\rho(\pi_1(\bar{Y})) = \infty$.

Proof. (g.s.s.) $\implies$ (a.i.): The reducibility of $\rho_0|_{\pi_1(\bar{Y})} \otimes \bar{\kappa}$ implies the existence of a proper subbundle of $\mathcal{F}_0 \otimes \bar{\kappa}$ with slope 0.

(a.i.) $\implies$ (g.s.s.): As $\mathcal{F}_0$ is étale trivialized, it is geometrically slope-semistable. Since a slope-zero subbundle of a trivial bundle is trivial, then the existence of a proper slope-zero subbundle of $\mathcal{F}_0 \otimes \bar{\kappa}$ over $\bar{Y}$ implies the reducibility of $\rho_0|_{\pi_1(\bar{Y})} \otimes \bar{\kappa}$. Since the absolute irreducibility of $(\rho_0)|_{\pi_1(\bar{Y})}$ implies the same property for $\rho|_{\pi_1(\bar{Y})}$ and ρ_0, then the second equivalence follows from Lemmas 8.2.6 and 8.2.5. □

8.3 An equivalence between vector bundles and representations

Recall from [HL10] that the geometric Frobenius map of a scheme Y over a finite field κ of characteristic p is defined to be F_Y^d, where $d = [\kappa : \mathbb{F}_p]$.

Theorem 8.3.1. *Let Y be a projective smooth geometrically connected curve over a finite field κ and $\mathcal{M}_Y(r,0)$ be the coarse moduli space of rank-r semistable bundles over $\bar{Y} = Y \otimes_\kappa \bar{\kappa}$ with trivialized determinant. Denote by $V : \mathcal{M}_Y(r,0) \dashrightarrow \mathcal{M}_Y(r,0)$ the rational map defined by $[E] \mapsto [F^*E]$ with respect to the geometric Frobenius map F of Y over κ. Then the following are equivalent:*
(1)There exists a finite extension $\widetilde{\kappa}$ of κ and a representation $\rho : \pi_1(Y \otimes_\kappa \widetilde{\kappa}) \to SL(r, \widetilde{\kappa}[\![s]\!])$ such that $(\rho \bmod s)|_{\pi_1(\bar{Y})}$ is absolutely irreducible and $\#\rho(\pi_1(\bar{Y})) = \infty$.
(2) There exists some positive integer N such that the Zariski closure of the fixed point locus $Fix(V^N)$ is of positive dimension and contains a stable point in a connected component, where $Fix(V^N)$ consists of closed points $x \in \mathcal{M}_Y(r,0)$ such that $V^N(x) = x$.

Proof. (1) $\Longrightarrow$ (2): By Proposition 8.1.8, there exists a strictly Frobenius-periodic rank-r vector bundle $(\mathcal{F}, \xi, \det = 1)$ over $(Y \otimes_\kappa \widetilde{\kappa}) \times_{\widetilde{\kappa}} \mathrm{Spec}\, \widetilde{\kappa}[\![s]\!]$. Locally, $(\mathcal{F}, \xi, \det = 1)$ is defined by transition matrices and linear maps. Note that there exists a finitely generated $\widetilde{\kappa}$-algebra $A \subset \widetilde{\kappa}[\![s]\!]$ and a a strictly Frobenius-periodic bundle $(\mathcal{F}', \xi', \det = 1)$ over $(Y \otimes_\kappa \widetilde{\kappa}) \times_{\widetilde{\kappa}} \mathrm{Spec}\, A$ such that its pullback to $Y \times_\kappa \mathrm{Spec}\, \widetilde{\kappa}[\![s]\!]$ is exactly $(\mathcal{F}, \xi, \det = 1)$. As $\mathcal{F}'$ can be viewed as a family of bundles over $\bar{Y}$ fixed by the geometric Frobenius map of $Y \otimes_\kappa \widetilde{\kappa}$ over $\widetilde{\kappa}$, i.e. the N^{th}-power of the geometric Frobenius map of Y over

κ, where $N = [\widetilde{\kappa} : \kappa]$. Thus the image of the modular morphism $\operatorname{Spec} A \to \mathcal{M}_Y(r, 0)$ is in $\operatorname{Fix}(V^N)$. By Proposition 8.2.7, $\mathcal{F}'$ is a non-constant family and consists mostly of stable bundles, thus $\operatorname{Fix}(V^N)$ has the required properties.

(2) $\Longrightarrow$ (1): Let $Z \subset \mathcal{M}_{X_t}(2, 0)$ be the connected component of the Zariski closure of $\operatorname{Fix}(V^N)$ that contains a stable point. Using the construction of $\mathcal{M}_{X_t}(2, 0)$ as a GIT quotient, then there exists a quasi-projective scheme T over some finite extension κ' of κ and a strictly Frobenius-periodic vector bundle $(\mathcal{F}, \xi)$ over $(Y_{\kappa'}) \times_{\kappa'} T$ such that the image of the modular morphism $T \to \mathcal{M}_{X_t}(2, 0)$ is dense in Z. Then pick up a stable point $t \in T$. Let $\widetilde{\kappa}$ be the residue field at t and fix an injection $\mathcal{O}_{T,t} \to \widetilde{\kappa}[\![s]\!]$. The pullback of $(\mathcal{F}, \xi)$ under $(Y_{\kappa'}) \times_{\kappa'} T \to (Y_{\widetilde{\kappa}}) \times_{\widetilde{\kappa}} \operatorname{Spec} \widetilde{\kappa}[\![s]\!]$ is a strictly Frobenius-periodic vector bundle over $(Y_{\widetilde{\kappa}}) \times_{\widetilde{\kappa}} \operatorname{Spec} \widetilde{\kappa}[\![s]\!]$. By Propositions 8.1.8 and 8.2.7, we obtain a representation $\rho : \pi_1(Y_{\widetilde{\kappa}}) \to \operatorname{SL}(r, \widetilde{\kappa}[\![s]\!])$ as required. □

From now on, in order to obtain representations with an infinite geometric monodromy, we turn to study the fixed point locus of the Verschiebung.

Chapter 9

The Verschiebung map on $\mathcal{M}_X(2,0)$

Notations: For a scheme X over a field k of characteristic $p > 0$, let $\sigma : k \to k$ be the absolute Frobenius map; denote by $X(n)$ the scheme deduced from X by the extension of scalars $k \xrightarrow{\sigma^n} k$: let $F_n : X(n) \to X(n+1)$ be the relative Frobenius.

9.1 General facts on the Verschiebung map

Let X be a smooth algebraic curve of genus 2 over a field k of characteristic $p = 2$ and $\bar{k}$ be the algebraic closure of k. Let $\mathcal{M}_X(2,0)$ denote the moduli space of rank 2 semi-stable vector bundles over $X_{\bar{k}}$ with trivialized determinant. Let J^1_X be the space of line bundles of degree 1 over $X_{\bar{k}}$ and denote by Θ the divisor denoted by $X_{\bar{k}} \subset J^1_X$.

We first recall from [Ray82] the theta characteristic B of X. Consider $F_{-1} : X(-1) \to X$, we have an exact sequence as below:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow (F_{-1})_*\mathcal{O}_{X(-1)} \longrightarrow B \longrightarrow 0 \qquad (9.1.1)$$

Clearly B is a line bundle of degree 1 and it is shown in [Ray82] that $B^2 \simeq \Omega_{X/k}$.

As in the complex case [NR69], there also exists a regular morphism in characteristic $p = 2$ by [Ray82]:

$$D : \mathcal{M}_X(2,0) \longrightarrow |2\Theta|, \qquad (9.1.2)$$

which maps the class of the semi-stable bundle E to the divisor $D(E)$ with

$$\mathrm{supp} D(E) = \{L \in J_X | H^0(E \otimes B \otimes L) \neq 0\}.$$

A proof of that D is an isomorphism is given by M.S. Narasimhan along the lines of the original paper [NR69]. It is also proved in [LP02, Proposition 5.1].

Proposition 9.1.1. *[LP02, Proposition 5.1] $D : \mathcal{M}_X(2,0) \longrightarrow \mathbb{P}^3 \simeq |2\Theta|$ is an isomorphism.*

Consider the Kummer morphism $\phi : J_X \to \mathcal{M}_X(2,0), L \mapsto [L \oplus L^{-1}]$.

Proposition 9.1.2. *[LP02, Proposition 4.1] In a canonical Theta basis of $H^0(J_X, 2\Theta)$, the Kummer surface $Km_X = \phi(J_X^1) \subset |2\Theta|$ is defined by a quartic equation.*

Now we turn to the Verschiebung map. Consider the relative Frobenius $F_0 : X \to X(1)$. The Verschiebung map is defined as below:

$$V_{X(1)} : \mathcal{M}_{X(1)}(2,0) \longrightarrow \mathcal{M}_X(2,0), \qquad [E] \hookrightarrow [F_0^* E]. \tag{9.1.3}$$

The Verschiebung map $V_{X(1)}$ can be identified with a rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ via D.

Proposition 9.1.3. *[LP02, Proposition 6.1] Let X be an ordinary genus-2 curve in characteristic 2. The Verschiebung map $V_{X(1)} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is defined by quadratic polynomials.*

1. *$V_{X(1)}$ maps the Kummer surface of $X(1)$ to the Kummer surface of X.*

2. *There exists a unique stable bundle E^0 destabilized by the Frobenius map, i.e., $F_0^* E^0$ is not semi-stable. We have $E^0 = (F_0)_* B^{-1}$.*

Proof. See [LP02, Proposition 6.1]. As $\deg(B) = 1$, $E^0 = (F_0)_* B^{-1}$ is destabilized by the Frobenius map following from the exact sequence:

$$0 \longrightarrow B \longrightarrow (F_0)^*(F_0)_* B^{-1} \longrightarrow B^{-1} \longrightarrow 0.$$

□

9.2 $\mathcal{M}_X(2,0)$ over a genus-2 curve in characteristic 2

In this section, $G = \mathbb{Z}/2\mathbb{Z} \times S_3$ and X is a projective smooth ordinary curve of genus 2 over a field κ of characteristic 2 with $\mathrm{Aut}(X/\kappa) = G$. Except in Theorem 9.3.1, κ can be infinite. Consider the relative Frobenius $F_0 : X \to X(1)$. Note that the G-action on X deduces a G-action on $X(1)$ such that F_0 commutes with the G-actions.

9.2.1 A group action

In this subsection, we study the fixed point locus of the G-action on the Kummer surface Km_X of X and the coarse moduli space $\mathcal{M}_X(2,0)$.

Let $\pi_X : X \to |K_X| = \mathbb{P}^1$ be the canonical morphism of X. As X is ordinary, the double covering π_X has three ramification points according to Fact 9.2.1.

Facts 9.2.1. *Let Y be a projective smooth curve of genus 2 over an algebraically closed field of characteristic $p > 0$. Assume that $\mathcal{L} \in Pic^0(Y)$. Then $\mathcal{L}$ is of the form $\mathcal{O}_X(P - Q)$, where P, Q are closed points of Y. Moreover, if $\mathcal{L}^2 = \mathcal{O}_Y$, then $\mathcal{L}$ is of the form $\mathcal{O}_Y(R_1 - R_2)$, where $R_1, R_2 \in Y$ are ramification points of the canonical morphism $\pi_Y : Y \to |K_Y| = \mathbb{P}^1$.*

We assume that the image of the ramification points of π_X are $\{0, 1, \infty\}$. The $\mathbb{Z}/2\mathbb{Z}$-action on X is generated by the hyperelliptic involution of π_X, denoted by ι; the S_3-action on X induces an action on the canonical linear system $|K_X|$ and hence can be identified as the permutation group of the branch points $\{0, 1, \infty\}$. Fix such an identification and denote by τ_{01} (resp. σ) the automorphism of X corresponding to $(01)(\infty)$ (resp. (01∞)).

The G-action on X induces a G-action on the Jacobian J_X and thus a G-action on the Kummer surface Km_X of X. We first study the fixed points of G on Km_X.

Lemma 9.2.2. *Let* $Q \in \bar{X}$ *is a fixed point of* σ*. Then The set of the fixed points* $(Km_X)^G$ *of the G-action on* Km_X *consists of three points:* $\mathcal{O}_X^{\oplus 2}$*,* $E_{1,X} = \mathcal{O}_X(Q - \tau_{01}(Q)) \oplus \mathcal{O}_X(\tau_{01}(Q) - Q)$*,* $E_{2,X} = \mathcal{O}_X(Q - \iota \circ \tau_{01}(Q)) \oplus \mathcal{O}_X(\iota \circ \tau_{01}(Q) - Q)$*.*

Proof. It suffices to find all line bundles $\mathcal{L} \in \mathrm{Pic}^0(\bar{X})$ such that $g^*\mathcal{L} \simeq \mathcal{L}$ or $\mathcal{L}^{-1}$ for $g = \tau_{01}, \tau_{0\infty}$ and σ. By Fact 9.2.1, $\mathcal{L} \simeq \mathcal{O}_X(Q_1 - Q_2)$ for $Q_1, Q_2 \in \bar{X}$. The lemma is proved by a case-by-case analysis according to the three types of points: (I) the three ramification points; (II) the four fixed points of σ; (III) all the others. □

Note that σ fixes four points on $\bar{X}$, yet the definition of $E_{1,X}$ and $E_{2,X}$ is independent of the choice of Q. Take a fixed point Q_1 of σ on $\bar{X}(1)$, we similarly define $E_{1,X(1)}$ and $E_{2,X(1)}$. We have

Lemma 9.2.3. *For* $j = 1, 2$*,* $F_0^* E_{j,X(1)} = E_{j,X}$*.*

Proof. Take a fixed point Q of σ on $\bar{X}$, then $Q_1 = F_0(Q)$ is a fixed point of σ on $\bar{X}(1)$. Since $F_0^*[\mathcal{O}_{X(1)}(Q_1 - \tau_{01}(Q_1))] = \mathcal{O}_X(2Q - 2\tau_{01}(Q))$ and $F_0^*[\mathcal{O}_{X(1)}(Q_1 - \iota \circ \tau_{01}(Q_1))] = \mathcal{O}_X(2Q - 2\iota \circ \tau_{01}(Q))$, it suffices to prove that $3Q \sim 3\tau_{01}(Q) \sim 3\iota \circ \tau_{01}(Q)$. This follows from that $Q, \tau_{01}(Q), \iota(Q), \iota \circ \tau_{01}(Q)$ are the ramification points of the finitemap $X \to X/\langle \sigma \rangle \simeq \mathbb{P}^1$ with ramification index 3. □

The following lemma will be useful to find $(\mathcal{M}_X(2,0))^G$.

Lemma 9.2.4. *Let* H *be a subgroup of* $Aut(\mathbb{P}^n_k/k)$*, where* k *is a field of characteristic* p*. Assume that* H *is generated by elements with order of the form* p^r*. Let* $P_1, P_2 \in \mathbb{P}^n_k$ *be fixed by* H*, then the projective line* $\bar{P_1 P_2}$ *is fixed by* H*.*

Proof. Identify points P_1, P_2 with vectors $v_1, v_2 \in k^{n+1}$. Let $h \in H$ have order p^r and $\widetilde{h} \in \mathrm{GL}(n+1, k)$ be a preimage of h. Then $\widetilde{h}^{p^r} = \mu I_{n+1}$. By assumption, $\widetilde{h}(v_1) = \mu_1 v_1$ and $\widetilde{h}(v_2) = \mu_2 v_2$. Thus $\mu_1^{p^r} = \mu_2^{p^r} \implies \mu_1 = \mu_2$. Therefore, h fixes the line $\bar{P_1 P_2}$. □

By ([LP02]), $\mathcal{M}_X(2,0)$ is isomorphic to $|2\Theta| \simeq \mathbb{P}^3_\kappa$ and the Kummer surface Km_X is a quartic hypersurface. To find the fixed point locus $(\mathcal{M}_X(2,0))^G$, we need the following.

Proposition 9.2.5. *The fixed point locus of the* G*-action on* $\mathcal{M}_X(2,0)$ *is a projective line, denoted by* $\triangle_X$*, which intersects with the Kummer surface of* X *at* 3 *points.*

Proof. By Proposition 9.1.1, $\mathcal{M}_X(2,0) \simeq \mathbb{P}^3$; as G is generated by 2-torsion elements, by Lemma 9.2.4, the fixed point locus $(\mathcal{M}_X(2,0))^G$ of G on $\mathcal{M}_X(2,0)$ is a projective subspace. Since $(\mathcal{M}_X(2,0))^G$ intersects with the Kummer surface Km_X at three points by Lemma 9.2.2, thus $\dim(\mathcal{M}_X(2,0))^G = 1$ and it is a projective line. □

9.2.2 The Verschiebung map

Recall that $X(n)$ is the scheme deduced from X by the extension of scalars $\kappa \xrightarrow{\sigma^n} \kappa$. As X is a projective smooth ordinary curve of genus 2 over a field κ of characteristic 2 with $\mathrm{Aut}(X/\kappa) = G$, so is $X(n)$. For all n, the G-action on $X(n)$ is deduced from the G-action on X such that the relative Frobenius $F_n : X(n) \to X(n+1)$ commutes with the G-actions.

Denoted by V_n the Verschiebung map

$$V_n = V_{X(n+1)} : \mathcal{M}_{X(n+1)}(2,0) \dashrightarrow \mathcal{M}_{X(n)}(2,0)\ , [E] \mapsto [F_n^* E].$$

Clearly, the previous results for $(X, V_{X(1)})$ also holds for $(X(n), V_n)$. Therefore, the fixed point locus of G on $\mathcal{M}_{X(n)}(2,0)$ is a projective line by Proposition 9.2.5, denoted by $\triangle_{X(n)}$.

As $F_n : X(n) \to X(n+1)$ commutes with the G-actions, the pullback of a G-bundle is a G-bundle, hence $V_n(\triangle_{X(n+1)}) \subset \triangle_{X(n)}$.

Lemma 9.2.6. *The restriction of* $V_n|_{\triangle_{X(n+1)}} : \triangle_{X(n+1)} \dashrightarrow \triangle_{X(n)}$ *can be identified with a linear map* $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$.

Proof. As $\triangle_{X(n)}$ is a projective line of $\mathcal{M}_{X(n)}(2,0)$ for all n, then $V_n|_{\triangle_{X(n+1)}}$ is identified with a rational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^1$. As V_n is defined by quadratic polynomials by Propositon 9.1.3, thus $V_n|_{\triangle_{X(n+1)}}$ is given by two quadratic polynomials $\{h_1, h_2\}$ in two variables. Note that V_n has a base point $(F_n)_*B_n^{-1}$, where B_n is the theta characteristic of $X(n)$ defined by a similar exact sequence as (9.1.2). Clearly $(F_n)_*B_n^{-1}$ is a G-bundle, i.e. $[(F_n)_*B_n^{-1}] \in \triangle_{X(n+1)}$. Thus h_1 and h_2 have a common linear factor and $V_n|_{\triangle_{X(n+1)}}$ is reduced to a linear map. □

9.3 A non-trivial family of vector bundles

Theorem 9.3.1. *Let X be a projective smooth ordinary curve of genus 2 over a finite field κ of characteristic 2 with $Aut(X/\kappa) = \mathbb{Z}/2\mathbb{Z} \times S_3 \doteq G$. Consider the rational map defined by taking pullback of bundles with respect to the geometric Frobenius map of X over κ, denoted by $V : \mathcal{M}_X(2,0) \dashrightarrow \mathcal{M}_X(2,0)$. Then the fixed point locus of the G-action on $\mathcal{M}_X(2,0)$ is a projective line, denoted by $\triangle_X$, which intersects with the Kummer surface of X at 3 points and $V|_{\triangle_X} = id_{\triangle_X}$.*

Proof. Assume that $\#\kappa = 2^d$. Note that $X(d) = X$. Note that the rational map V is a composite map $V_0 \circ V_1 \circ \cdots \circ V_{d-1}$. As $V_n|_{\triangle_{X(n+1)}} : \triangle_{X(n+1)} \dashrightarrow \triangle_{X(n)}$ is linear for $0 \leq n \leq d-1$ by Lemma 9.2.6, thus $V|_{\triangle_X}$ is linear. Moreover, Lemma 9.2.3 holds for every $X(n)$, i.e. there are semistable bundles $E_{1,X(n)}, E_{2,X(n)}$ such that $V_n([E_{j,X(n+1)}]) = [E_{j,X(n)}]$ for $j = 1, 2$. Thus $V|_{\triangle_X}$ has three distinct fixed points, i.e. $[\mathcal{O}_X^{\oplus 2}]$, $[E_{1,X}]$ and $[E_{2,X}]$. Therefore, $V|_{\triangle_X}$ is the identity map. □

Next chapter we give an explicit construction of the vector bundles represented by $\triangle_X$.

Chapter 10

An invariant construction of Frobenius-periodic vector bundles

In this chapter, X_t is a projective smooth ordinary curve of genus 2 over a field κ of characteristic 2 with $\mathrm{Aut}(X_t/\kappa) = G \doteq \mathbb{Z}/2\mathbb{Z} \times S_3$. We give an explicit construction of the bundles over X_t parametrized by the projective line $\triangle_{X_t}$ in Proposition 9.2.5. Let $\bar{\kappa}$ be the algebraic closure of κ and $\bar{X}_t = X_t \otimes_\kappa \bar{\kappa}$.

Recall from [Anc43] that an ordinary curve of genus 2 in characteristic 2 is determined by the equation

$$y^2 + (x^2 + x)y + \lambda x^5 + \mu x^3 + \nu x = 0. \tag{10.0.1}$$

Among the three-parameter family of curves, there is a one-parameter family of curves with a G-action defined by

$$X_t: \quad y^2 + (x^2 + x)y + (t^2 + t)(x^5 + x) + t^2 x^3 = 0, \quad t \in \kappa, \quad t \neq 0, 1. \tag{10.0.2}$$

Let $(x_0 : x_1)$ be the homogeneous coordinates of $\mathbb{P}^1$ and identify $x = \frac{x_1}{x_0}$. Then the projection $(x, y) \mapsto x$ extends to a double covering $\pi_t : X_t \to \mathbb{P}^1$. Denote by $P_{0,t}, P_{1,t}, P_{\infty,t} \in X_t$ the ramified points of π_t over $0, 1, \infty \in \mathbb{P}^1$ respectively. Near ∞, let $x_- = x^{-1}, y_- = yx^{-3}$, then X_t is given by

$$X_t : y_-^2 + (x_-^2 + x_-)y_- + (t^2 + t)(x_-^5 + x_-) + t^2 x_-^3 = 0. \tag{10.0.3}$$

It is not hard to see that $\mathrm{Aut}(X_t/\kappa) = \mathbb{Z}/2\mathbb{Z} \times S_3$.

$$\begin{array}{ccc} X_t & \xrightarrow{\iota} & X_t \\ & {\scriptstyle \pi_t} \searrow \quad \swarrow {\scriptstyle \pi_t} & \\ & \mathbb{P}^1 & \end{array} \qquad \begin{array}{l} \iota^*(x) = x, \\ \\ \iota^*(y) = y + x^2 + x. \end{array}$$

The $\mathbb{Z}/2\mathbb{Z}$-action on X_t is generated by the hyperelliptic involution of π_t, denoted by ι; the S_3-action on X_t can be identified as the permutation group of the branch points $\{0, 1, \infty\}$ of π_t. Fix such an identification and denote by τ_{01} (resp. $\tau_{0\infty}$) the automorphism of X_t corresponding to $(01)(\infty)$ (resp. $(0\infty)(1)$). Explicitly, the S_3-action on X_t and $\mathbb{P}^1$ are defined as below (in terms of structure sheaves):

$$\tau_{01}^* : \quad x \mapsto x + 1, \quad y \mapsto y + t(x^2 + x + 1); \quad x_0 \mapsto x_0, \quad x_1 \mapsto x_0 + x_1; \tag{10.0.4}$$

$$\tau_{0\infty}^* : \quad x \mapsto x_-, \quad y \mapsto y_-; \quad x_0 \mapsto x_1, \quad x_1 \mapsto x_0. \tag{10.0.5}$$

The S_3-action on $\mathbb{P}^3$ induces a canonical S_3-action ϕ on $\mathcal{O}_{\mathbb{P}^1}(n)$ as follows:

$$\phi_{\tau_{01}} : \tau_{01}^* \mathcal{O}_{\mathbb{P}^1}(n) \to \mathcal{O}_{\mathbb{P}^1}(n), \quad \tau_{01}^*([x_0^n]) \mapsto [x_0^n], \quad \tau_{01}^*([x_1^n]) \mapsto [(x_0 + x_1)^n]; \tag{10.0.6}$$

$$\phi_{\tau_{0\infty}} : \tau_{0\infty}^* \mathcal{O}_{\mathbb{P}^1}(n) \to \mathcal{O}_{\mathbb{P}^1}(n), \quad \tau_{0\infty}^*([x_0^n]) \mapsto [x_1^n], \quad \tau_{0\infty}^*([x_1^n]) \mapsto [x_0^n]. \tag{10.0.7}$$

where $[x_0^n], [x_1^n]$ denote global sections of $\mathcal{O}_{\mathbb{P}^1}(n)$.

10.1 A family of G-bundles $\mathcal{E}_t$

In this section, we define a family of G-bundles over X_t parametrized by $\Lambda_t = \mathrm{Spec}\, \kappa[\lambda]$. By a G-bundle, we mean a vector bundle with an equivariant G-action. Recall that an equivariant G-action ϕ on a vector bundle $\mathcal{E}$ is a set of isomorphisms $\{\phi_g : g^* \mathcal{E} \xrightarrow{\sim} \mathcal{E} | g \in G\}$ with some compatible conditions.

Denote by ω_t the canonical sheaf of X_t/κ. Note that $\omega_t \simeq \pi_t^*\mathcal{O}_{\mathbb{P}^1}(1)$. Consider the exact sequence as follows:

$$0 \longrightarrow \omega_t^{-1} \oplus \omega_t^{-2} \xrightarrow{(i_1,i_2)} \omega_t^2 \oplus \omega_t \xrightarrow{(q_1,q_2)} \mathcal{Q}_1 \oplus \mathcal{Q}_2 \longrightarrow 0, \tag{10.1.1}$$

where i_1, i_2 are defined by tensoring with the section $\pi_t^*x_0 \otimes \pi_t^*(x_0 + x_1) \otimes \pi_t^*x_1 \in \Gamma(X_t, \omega_t^3)$ and the quotient sheaves $\mathcal{Q}_1 = \mathcal{Q}_2$, which is the structure sheaf of $2P_{0,t} + 2P_{1,t} + 2P_{\infty,t}$ as a closed subscheme of X_t.

Let $\Lambda_t = \operatorname{Spec} \kappa[\lambda]$ and $p_1 : X_t \times_\kappa \Lambda_t \to X_t$ be the natural projection to X_t. As p_1 is flat, the pullback of (10.1.1) under p_1 is still exact:

$$0 \longrightarrow p_1^*\omega_t^{-1} \oplus p_1^*\omega_t^{-2} \xrightarrow{(i_1,i_2)} p_1^*\omega_t^2 \oplus p_1^*\omega_t \xrightarrow{(q_1,q_2)} p_1^*\mathcal{Q}_1 \oplus p_1^*\mathcal{Q}_2 \longrightarrow 0, \tag{10.1.2}$$

Note that $\operatorname{Aut}(X_t \times_\kappa \Lambda_t/\Lambda_t) \simeq \operatorname{Aut}(X_t/\kappa) = G$ and $p_1^*\omega_t^n = p_1^*\pi_t^*\mathcal{O}_{\mathbb{P}^1}(n)$ has an equivariant G-action that is compatible with the action ϕ on $\mathcal{O}_{\mathbb{P}^1}(n)$ in (10.0.6) and (10.0.7).

Now we define a family of G-bundles as below.

Lemma 10.1.1. *There exists a unique rank-2 bundle $\mathcal{E}_t$ over $X_t \times_\kappa \Lambda_t$ with an equivariant G-action ϕ such that*

1. *The diagram as below is commutative:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p_1^*\omega_t^{-1} \oplus p_1^*\omega_t^{-2} & \xrightarrow{\zeta_1} & \mathcal{E}_t & \xrightarrow{q_3} & p_1^*\mathcal{Q}_2 & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \zeta_2} & & \downarrow{\scriptstyle s\mapsto(0,s)} & & \\ 0 & \longrightarrow & p_1^*\omega_t^{-1} \oplus p_1^*\omega_t^{-2} & \xrightarrow{(i_1,i_2)} & p_1^*\omega_t^2 \oplus p_1^*\omega_t & \xrightarrow{(q_1,q_2)} & p_1^*\mathcal{Q}_1 \oplus p_1^*\mathcal{Q}_2 & \longrightarrow & 0. \end{array} \tag{10.1.3}$$

2. *ζ_1 and ζ_2 commute with the G-actions.*

3. *The two rows in (10.1.3) are exact.*

4. *The sequence $0 \longrightarrow p_1^*\omega_t^{-1} \xrightarrow{\zeta_1} \mathcal{E}_{t,\lambda} \xrightarrow{\zeta_2} p_1^*\omega_t \longrightarrow 0$ is exact.*

5. *Locally on the open neighborhood* $U = p_1^{-1}(X_t \backslash \{P_{\infty,t}\})$ *of* $p_1^{-1}(P_{0,t})$, $\mathcal{E}_t$ *identified with a sub-sheaf of* $p_1^*\omega_t^2 \oplus p_1^*\omega_t$ *is generated by*

$$\mathcal{E}_t|_U = \langle x(x+1)[x_0^2], (y+\lambda)[x_0^2] + [x_0] \rangle, \tag{10.1.4}$$

where $[x_0^n]$ *is identified with* $p_1^*\pi_t^*([x_0^n])$ *and viewed as a global section of* $p_1^*\omega_t^n$.

Proof. Consider the open subscheme $U' = p_1^{-1}(X_t \backslash \{P_{0,t}, P_{1,t}, P_{\infty,t}\})$. Because the rows in (10.1.3) are exact, $\mathcal{E}_t|_{U'} \simeq (p_1^*\omega_t^2 \oplus p_1^*\omega_t)|_{U'}$. To determine $\mathcal{E}_t$, it suffices to define $\mathcal{E}_t$ around $p_1^{-1}(P_{0,t})$,$p_1^{-1}(P_{1,t})$ and $p_1^{-1}(P_{\infty,t})$. To endow $\mathcal{E}_t$ with a G-action such that ζ_2 commutes with the G-actions, the definition of $\mathcal{E}_t$ around $p_1^{-1}(P_{0,t})$ as a subsheaf of $p_1^*\omega_t^2 \oplus p_1^*\omega_t$ determines the definition of $\mathcal{E}_t$ around $p_1^{-1}(P_{1,t})$ and $p_1^{-1}(P_{\infty,t})$ as a subsheaf of $p_1^*\omega_t^2 \oplus p_1^*\omega_t$. Thus $\mathcal{E}_t$ is uniquely determined by the above assumptions. It suffices to verify that there exists a G-action on $\mathcal{E}_t$ as required. This is routine and implicit in (10.0.4)∼(10.0.7), thus omitted. □

10.2 Properties of $\mathcal{E}_t$

We will show that $\mathcal{E}_t$ in Lemma 10.1.1 is a non-constant family of stable bundles over $\bar{X}_t$.

Lemma 10.2.1. *let* $\mathcal{L}$ *be a line bundle on* X_t *with an isomorphism* $\phi : \iota^*\mathcal{L} \to \mathcal{L}$ *s.t.* $\phi \circ \iota^*\phi = 1$. *Consider the exact sequence* $0 \to \pi_t^*(\pi_{t*}\mathcal{L})^\iota \to \mathcal{L} \to \mathcal{Q}_\mathcal{L} \to 0$, *then Supp*$(\mathcal{Q}_\mathcal{L}) \subset \{P_{0,t}, P_{1,t}, P_{\infty,t}\}$ *and* $\mathcal{Q}_\mathcal{L}$ *has length* 1 *at each point in its support.*

Proof. First $\pi_t^*(\pi_{t*}\mathcal{L})^\iota|_U \simeq \mathcal{L}|_U$ for $U = X_t \backslash \{P_{0,t}, P_{1,t}, P_{\infty,t}\}$. Take $P_{0,t}$ for example. Assume that $\mathcal{L}_{P_{0,t}} = \mathcal{O}_{X_t,P_{0,t}}\langle e \rangle$ and $\phi(\iota^*e) = \mu e$ with $\mu \in \mathcal{O}^*_{X_t,P_{0,t}}$, then $\mu(\iota^*\mu) = 1$. By Hilbert's Satz 90, $\mu = \frac{\nu}{\iota^*\nu}$ for ν in the fraction field of $\mathcal{O}_{X_t,P_{0,t}}$; as ν can be chosen to have order 0 or 1. Thus $\pi_t^*(\pi_{t*}\mathcal{L})^\iota|_{P_{0,t}} = \mathcal{O}_{X_t,P_{0,t}}\langle \nu e \rangle$ and $\mathcal{Q}_{\mathcal{L},P_{0,t}}$ is of length ≤ 1. □

Lemma 10.2.2. *Let Y be a projective smooth scheme over an algebraically closed field K of characteristic $p > 0$. Let $H \subset Aut(Y/K)$ be generated by elements with order of the form p^n. Let $\mathcal{F}$ be a slope-stable vector bundle over Y. Then there is at most one way to lift the action of H on Y to an equivariant action on $\mathcal{F}$.*

Proof. Suppose that there are two equivariant H-actions on $\mathcal{F}$, denoted by ϕ_1 and ϕ_2. It suffices to show that $\phi_{1,h} = \phi_{2,h}$ for $h \in H$ with $\operatorname{ord}(h) = p^n$ for some n. Let $\delta = \phi_{2,h} \circ \phi_{1,h}^{-1}$. Since $\mathcal{F}$ is slope-stable, then $\delta \in K^*$. Consider the following diagram:

$$\begin{array}{ccccccccccc}
\mathcal{F} & \xleftarrow{\phi_{1,h}} & h^*\mathcal{F} & \xleftarrow{h^*\phi_{1,h}} & h^{2*}\mathcal{F} & \xleftarrow{h^{2*}\phi_{1,h}} & h^{3*}\mathcal{F} & \longleftarrow & \cdots & \longleftarrow & h^{p^n*}\mathcal{F} = \mathcal{F} \\
\downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & & & \downarrow \mathrm{id} \\
\mathcal{F} & \xleftarrow{\phi_{2,h}} & h^*\mathcal{F} & \xleftarrow{h^*\phi_{2,h}} & h^{2*}\mathcal{F} & \xleftarrow{h^{2*}\phi_{2,h}} & h^{3*}\mathcal{F} & \longleftarrow & \cdots & \longleftarrow & h^{p^n*}\mathcal{F} = \mathcal{F}
\end{array}$$

Note that the largest square is commutative and it is split into p^n small squares, the commutator of which are all δ. Therefore, $\delta^{p^n} = 1$ and hence $\delta = 1$. □

Proposition 10.2.3. *Let $\mathcal{E}_t$ be the vector bundle in Lemma 10.1.1. For every $\lambda \in \bar{\kappa}$, denote by $\mathcal{E}_{t,\lambda}$ the pullback of $\mathcal{E}_t$ under $\bar{X}_t \hookrightarrow X_t \times_\kappa \Lambda_t$ defined by λ. Then*

1. *$\mathcal{E}_{t,\lambda}$ is a semistable vector bundle over $\bar{X}_t$ and $\mathcal{E}_{t,\lambda} \neq \mathcal{O}_{\bar{X}_t}^{\oplus 2}$ for all $\lambda \in \bar{\kappa}$.*
2. *If $\lambda \neq t, t+1$, $\mathcal{E}_{t,\lambda}$ is stable.*
3. *For $\lambda_1, \lambda_2 \neq t, t+1$, then $\mathcal{E}_{t,\lambda_1} \simeq \mathcal{E}_{t,\lambda_2}$ if and only if $\lambda_1 = \lambda_2$.*
4. *As points of $\mathcal{M}_{X_t}(2,0)$, $[\mathcal{E}_{t,t+1}] = [E_{1,X_t}]$; $[\mathcal{E}_{t,t}] = [E_{2,X_t}]$.*

Proof. (1) Suppose that $\mathcal{E}_{t,\lambda}$ is unstable and let $\mathcal{L}$ be its maximal line subbundle. Because of the uniqueness of Harder-Narasimhan filtration, $\phi_\iota(\iota^*\mathcal{L}) = \mathcal{L}$. By Lemma 10.2.1, $\pi_t^*(\pi_{t*}\mathcal{L})^\iota (\subset \mathcal{L})$ is of degree > -3, thus $\pi_t^*(\pi_{t*}\mathcal{L})^\iota \subset \omega_t^{-1}$. As ω_t^{-1} is saturated and of negative degree, contradiction with that $\mathcal{L}$ is of positive degree. Similarly, $\mathcal{E}_{t,\lambda} \neq \mathcal{O}_{X_t}^{\oplus 2}$, because whatever G-action is given on $\mathcal{O}_{X_t}^{\oplus 2}$, the ι-invariant part always contains a trivial bundle as its subbundle.

(2) Because of Lemma 9.2.2, it suffices to show that when $\lambda \neq t, t+1$, $\mathcal{E}_{t,\lambda}$ is not isomorphic to E_{1,X_t} or E_{2,X_t}. To do this, we consider the action of ϕ_ι and $\phi_{\tau_{01}}$ on the stalk at $P_{1,\infty}$. By (10.1.4),

$$(\mathcal{E}_{t,\lambda})_{P_{1,t}} = \langle x(x+1)[x_0^2], (y + t(x^2+x+1)+\lambda)[x_0^2] + [x_0]\rangle \doteq \langle \vec{e}_1, \vec{e}_2\rangle.$$

By computation, $\phi_\iota\{\iota^*\vec{e}_1, \iota^*\vec{e}_2\} = \{\vec{e}_1, \vec{e}_2\}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and

$$\phi_{\tau_{0\infty}}\{\tau_{0\infty}^*\vec{e}_1, \tau_{0\infty}^*\vec{e}_2\} \equiv \{\vec{e}_1, \vec{e}_2\}\left(\begin{smallmatrix} 1 & t+\lambda \\ 0 & 1 \end{smallmatrix}\right) \bmod m_{X_t, P_{1,t}}. \tag{10.2.1}$$

Let $Q \in X_t$ be a fixed point of $\sigma = \tau_{0\infty} \circ \tau_{01}$ and clearly $\tau_{01}(Q) = \tau_{0\infty}(Q)$. As

$$\begin{aligned}
&\iota^*(\mathcal{O}(Q - \tau_{01}(Q))) = \mathcal{O}(\tau_{01}(Q) - Q), \quad \tau_{0\infty}^*(\mathcal{O}(Q - \tau_{01}(Q))) = \mathcal{O}(\tau_{01}(Q) - Q);\\
&\iota^*(\mathcal{O}(Q - \iota \circ \tau_{01}(Q))) = \mathcal{O}(\iota \circ \tau_{01}(Q) - Q),\\
&\tau_{0\infty}^*(\mathcal{O}(Q - \iota \circ \tau_{01}(Q))) = \mathcal{O}(Q - \iota \circ \tau_{01}(Q)).
\end{aligned}$$

Therefore, whatever G-action ϕ is given on $E_{1,X_t} = \mathcal{O}(Q - \tau_{01}(Q)) \oplus \mathcal{O}(\tau_{01}(Q) - Q)$ or $E_{2,X_t} = \mathcal{O}(Q - \iota \circ \tau_{01}(Q)) \oplus \mathcal{O}(\iota \circ \tau_{01}(Q) - Q)$, there exists a local basis such that

$$\text{on } (E_{1,X_t})_{P_{1,t}} \otimes \bar{\kappa}, \quad \phi_\iota = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \quad \phi_{\tau_{0,\infty}} \equiv \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \quad \bmod m_{X_t, P_{1,t}}; \tag{10.2.2}$$

$$\text{on } (E_{2,X_t})_{P_{1,t}} \otimes \bar{\kappa}, \quad \phi_\iota = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \quad \phi_{\tau_{0,\infty}} \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \quad \bmod m_{X_t, P_{1,t}}. \tag{10.2.3}$$

Then (2) is clear by comparing the matrices of $\phi_{\tau_{0\infty}}$.

(3) It follows from Lemma 10.2.2 and the matrices of $\phi_{\tau_{0\infty}}$ in (10.2.1).

(4)Let $\triangle_{X_t} \subset \mathcal{M}_{X_t}(2,0)$ be the projective line in Proposition 9.2.5. Consider the modular morphism $i_t : \Lambda_t \longrightarrow \triangle_{X_t}\backslash\{[\mathcal{O}_{X_t}^{\oplus 2}]\}$ defined by $\mathcal{E}_t$. Because of the above, i_t restricted to $\Lambda_t\backslash\{t, t+1\}$ is an isomorphism, so is i_t. Thus $\mathcal{E}_{t,t}$ and $\mathcal{E}_{t,t+1}$ are in the S-isomorphism class of E_{1,X_t} or E_{2,X_t}. Then it is done by comparing the matrices of $\phi_{\tau_{0\infty}}$. □

Corollary 10.2.4. *The modular morphism given by $\mathcal{E}_t$ in Lemma 10.1.1 is an isomorphism $i_t : \Lambda_t \xrightarrow{\sim} \triangle_{X_t}\backslash\{[\mathcal{O}_{X_t}^{\oplus 2}]\}$.*

10.3 Frobenius pullback of $\mathcal{E}_t$

We will show that $\mathcal{E}_t$ is Frobenius-periodic over $X_t \times_\kappa \Lambda_t$ if κ is finite. We first consider the Verschiebung map V_{X_t}. Note that the scheme deduced from X_t by the extension of scalars $\kappa \xrightarrow{\sigma} \kappa$ is X_{t^2}. Then the relative Frobenius of X_t is a morphism $F_{X_t/\kappa} : X_t \to X_{t^2}$ and the Verschiebung map is $V_{X_t} : \mathcal{M}_{X_{t^2}}(2,0) \dashrightarrow \mathcal{M}_{X_t}(2,0)$.

By Lemma 9.2.6, the map $V_{X_t}|_{\triangle_{X_{t^2}}} : \triangle_{X_{t^2}} \dashrightarrow \triangle_{X_t}$ is linear. With the isomorphisms i_t and i_{t^2} in Corollary 10.2.4, we consider the restriction of $V_{X_t}|_{\triangle_{X_{t^2}}}$ to Λ_{t^2}, denoted by $f_t : \Lambda_{t^2} \to \Lambda_t$. Clearly f_t is linear. Note that for almost all $\lambda \in \bar{\kappa}$, $F^*_{X_t/\kappa}\mathcal{E}_{t^2,\lambda}$ is stable; moreover, as $F^*_{X_t/\kappa}\mathcal{E}_{t^2,\lambda}$ has a G-action deduced from the G-action on $\mathcal{E}_{t^2,\lambda}$, $F^*_{X_t/\kappa}\mathcal{E}_{t^2,\lambda}$ is isomorphic to some $\mathcal{E}_{t,\lambda'}$. By Lemma 10.2.2 and the matrices of $\phi_{\tau_{0\infty}}$ in (10.2.1), $\lambda' = \lambda + t + t^2$. Therefore, we arrive at the following result.

Theorem 10.3.1. *Let κ be a field in characteristic 2. For all $t \in \kappa$ and $t \neq 0, 1$, let X_t be an ordinary curve defined by (10.0.2) , $\mathcal{E}_t$ be the bundle over $X_t \times_\kappa \Lambda_t$ defined in Lemma 10.1.1, $F_{X_t/\kappa} : X_t \to X_{t^2}$ be the relative Frobenius of X_t and $g_t : \Lambda_{t^2} \to \Lambda_t$ be a linear map defined by $\lambda \mapsto \lambda + t + t^2$. Then there exists a morphism $\chi_t : \mathcal{E}_t \to (F_{X_t/\kappa} \times g_t)^*\mathcal{E}_{t^2}$ that becomes an isomorphism on an open subscheme $X_t \times_\kappa U_t$, where $\Lambda_t \backslash U_t$ has a unique point corresponding to the unique stable bundle destablized under Frobenius pullback.*

Remark 10.3.2. Theorem 10.3.1 can be proved directly by finding the explicit expressions of χ_t. When $t \in \mathbb{F}_{2^2} \backslash \mathbb{F}_2$, we recover Laszlo's example.

Now we turn to the case when κ is finite . Assume that $[\kappa : \mathbb{F}_2] = d$, then the geometric Frobenius of X_t is $F^d_{X_t/\kappa}$; the composite map $g_{t^{2^{d-1}}} \circ \cdots \circ g_{t^2} \circ g_t$ is the identity map of Λ_t and we obtain a composite morphism $\chi_t^{(d)} : \mathcal{E}_t \to (F_t^{(d)} \times \mathrm{id}_{\Lambda_t})^*\mathcal{E}_t$.

Corollary 10.3.3. *With the assumptions in Theorem 10.3.1. Moreover, assume that $[\kappa : \mathbb{F}_2] = d$. Let $\chi_t^{(d)} : \mathcal{E}_t \to (F_t^{(d)} \times id_{\Lambda_t})^*\mathcal{E}_t$ be the morphism defined as above. Then $\chi_t^{(d)}$ is an isomorphism on an open subscheme $X_t \times_\kappa U'_t$, where U'_t is obtained by removing d closed points from Λ_t.*

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