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PREEMPTION, LEAPFROGGING AND COMPETITION IN PATENT RACES*

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This paper investigates when patent races will be characterized by vigorous competition and when they will degenerate into a monopoly. Under some conditions, a firm with an arbitrarily small headstart can preempt its rivals. Such 'ε-preemption' is shown to depend on whether a firm that is behind in the patent race, as measured by the expected time remaining until discovery, can 'leapfrog' the competition and become the new leader. An example of an R&D game with random discovery illustrates how ε-preemption can occur when leapfrogging is impossible. A multi-stage R&D process allows leapfrogging and thus permits competition. A similar conclusion emerges in a model of a deterministic patent race with imperfect monitoring of rival firms' R&D investment activities.

1. Introduction

In markets where the first entrant or first inventor earns a substantial premium, competition takes on the characteristics of a race. Patents offer a stylized example: the first to make a patentable discovery 'wins'. While the patent does not generally preclude imitation, typically the payoffs to the competitors will be discontinuous in the times of discovery.

This paper analyzes patent races as dynamic games. We focus on determining when such races will be characterized by vigorous competition and when they will degenerate into a monopoly with 'blockaded entry'. In particular, we examine the conditions that allow a firm with an arbitrarily small headstart in the race to preempt its rivals, which we call 'ε-preemption'. The key to these questions turns out to be the possibility that a firm which is 'behind' in the race, as measured by the expected time remaining until discovery if both firms incur the same costs, can 'leapfrog' the competitor and jump ahead. When a firm has a chance of pulling ahead it competes vigorously, but a follower should drop out of the patent race if the leader

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can ensure that it will retain the lead until the race is over. This is not to say that the followers compete whenever they have some chance of winning the patent. Rather, they must have some hope of becoming the *favorite* to remain active. Even if the follower's chance of winning is almost as good as the leader's, the follower will not compete unless it can jump ahead. We stress the importance of the role of the information structure of the game and the specification of the technology of R&D for determining whether 'leapfrogging' is possible and whether competition or monopoly will prevail.

Most previous studies of patent races have predicted that there will be vigorous competition until the patent is obtained. Loury (1979), Dasgupta–Stiglitz (1980), Lee–Wilde (1980), and Reinganum (1981, 1982a, b) restrict attention to R&D technologies in which a firm's past R&D has no effect on its current likelihood of discovery. With such technologies the patent game is not a 'race' because no firm can ever pull ahead. Thus, preemption is impossible and firms compete vigorously until a discovery is made.

Dasgupta–Stiglitz (1980) and Gilbert–Newbery (1982), on the other hand, consider deterministic models of a patent race in which an arbitrarily small advantage allows the lead firm to act as a monopolist and still preempt its rivals. We call such striking absence of competition 'ε-preemption'. Clearly, both ε-preemption and continuous competition until discovery are extreme cases.

We have two goals in this paper. First, we show that ε-preemption does not depend on a deterministic technology for R&D. Rather, as claimed above, it suffices that the lead firm can ensure that it remains the favorite at every stage of the race. Second, and more importantly, we present patent races whose equilibria lie 'between' competition and ε-preemption. In these equilibria, firms compete vigorously if they are nearly even, while if one firm pulls far enough ahead the others drop out. Thus there is preemption, but not ε-preemption.

We consider two quite different sources of the 'leapfrogging' which generate such 'intermediate' equilibria. The first is a model in which the R&D process must pass through a number of discrete steps, so that even though one firm may be favored in the current stage a follower may still jump ahead. The second reason for leapfrogging is that each firm may be unable to monitor the *current* R&D efforts of its rivals, observing their progress only with a lag. The information lag allows a follower to catch up before the leader can respond.

Throughout the paper we restrict attention to those Nash equilibria which are subgame-perfect. This requirement, due to Selten (1965), prevents firms from being fooled by empty threats. The perfectness constraint allows us to study the circumstances in which preemption is credible; without perfectness, preemption can be obtained so easily that followers can preempt leaders. Most previous work on patent races has studied 'open-loop' equilibria in which firms pre-commit to time paths of R&D expenditures. An exception is Reinganum (1981, 1982a). However, past expenditures do not influence the current success probability in Reinganum's models. Hence, pre-commitment is irrelevant and, as she has shown, the open-loop and perfect equilibria coincide.

The structure of the paper is a series of three examples which illustrate how ε-preemption can blockade entry into a patent race and how the possibility of leapfrogging can give rise to vigorous competition. Section 2 uses a continuous-time model in which the intensity of R&D is not variable to demonstrate that ε-preemption can occur even when the follower has a chance of winning the patent race. Section 3 extends the fixed-intensity model to patent races with a number of stages to show that once leapfrogging is possible ε-preemption need not occur. Section 4 presents a discrete-time model inspired by earlier work in Gilbert–Stiglitz (1979) in which the intensity of R&D is variable, but firms cannot instantaneously monitor the R&D activities of their rivals. We show that while information lags allow leapfrogging and thus competition, as the information lags shrink to zero, the equilibrium converges to ε-preemption.

2. Why compete if you can't catch up: ε-preemption in the absence of leapfrogging

This section considers an extreme example of preemptive competition in a patent race, in which the timing of entry into the race takes on particular importance. To emphasize the investment-like qualities of expenditures on R&D, we assume that a firm's current chances of making the discovery depend only on its stock of 'experience'. Current expenditures are of value solely as an increase in that stock. We further assume that the intensity of R&D is not variable, so that a firm's experience is just the total time that it has engaged in R&D. We then show that if both firms would make losses if they engaged in R&D until a discovery, then an arbitrarily small headstart preempts followers.

Let $\omega_i(t)$ be firm i's total experience at date t. Then

$$\omega_i(t)=\int_{t_i}^{t} e_i(\tau)\,\mathrm{d}\tau, \tag{1}$$

where t_i is the date at which firm i entered the patent race, and

$$\begin{aligned} e_i(t)&=1 \quad \text{if the firm is engaged in R\&D at date } t,\\ &=0 \quad \text{otherwise.}\end{aligned}$$

Let $\mu(w_i(t))$ be firm i's hazard rate at time t; that is, the probability that firm i discovers between t and $t+\mathrm{d}t$, conditional or not having discovered by t, is $\mu(w_i(t))\,\mathrm{d}t$. We assume that $\mu(\cdot)$ is continuous and monotonically increasing, and use the convention

$$\mu_i(t)=\mu(w_i(t)).$$

Thus firm i's discovery probability is an exponential waiting time process with the time-varying parameter $\mu_i(t)$.

Note that the fixed intensity of R&D effort implies that if

$$\mu_i(t_0)>\mu_j(t_0),$$

and if i engages in R&D from t_0 on, then

$$\mu_i(t)>\mu_j(t) \quad \text{for all} \quad t\geqq t_0.$$

In particular, if firm i enters the patent race before j and if both do research, then

$$\mu_i(t)>\mu_j(t) \quad \text{for all} \quad t,$$

beginning with the entry date of firm i.

These assumptions mean that a late entrant cannot undertake an accelerated R&D program and leapfrog an early entrant (as measured by the success probability, μ), provided that the early entrant continues to do research. However, the late entrant may still win the patent race because discovery is stochastic. The model in this section distinguishes between leapfrogging in the accumulation of research knowledge (a higher ω) and getting lucky by making the patent discovery. We apply the term leapfrogging to the accumulation of knowledge — the research input that determines the probability of success.

A main objective of this section is to demonstrate the importance of the timing of entry into a market when there is no leapfrogging. The term ε-preemption refers to the ability of a firm to prevent the entry of competitors by virtue of being a marginally early entrant into the patent race. The example shows that ε-preemption can occur, despite the stochastic nature of research. The key to this observation is the inability of rivals to leapfrog their competitors in the accumulation of research knowledge.

Consider two firms $i=1,2$ in a patent race. If V is the value of the patent, c is the cost of R&D per unit time, and both firms are expected profit-maximizers, the expected instantaneous profit for firm i at date t conditional on no firm having yet made the discovery is

$$\mu_i(t)V-c.$$

The probability that neither firm makes the discovery before t is

$$\exp\left[-\int_0^t(\mu_1(\tau)+\mu_2(\tau))\,\mathrm{d}\tau\right].$$

If r is the (common) discount rate and t_i is the entry date of firm i, the expected value of the patent race to firm i is

$$\Pi_i=\int_{t_i}^{\infty}\exp\left\{-\left[rt+\int_0^t(\mu_1(\tau)+\mu_2(\tau))\,\mathrm{d}\tau\right]\right\}\{\mu_i(t)V-c\}\,\mathrm{d}t. \tag{2}$$

Firm one enters the race at time $t_1=0$, and firm two enters at an exogenously given time $t_2>0$. Both firms begin with no experience $[\omega_1(0)=\omega_2(0)=0]$.

Note that in the profit eq. (2), firm i's experience affects the firm's profits directly in the instantaneous profit term, as well as in the probability of previous discovery. This is the key difference of this model from those of Reinganum (1981, 1982a), in which past expenditures do not influence that current hazard rate.

Assume that R&D is a viable activity for a monopolist, meaning

$$\int_0^{\infty}\exp\left\{-\left[rt+\int_0^t\mu(\tau)\,\mathrm{d}\tau\right]\right\}\{\mu(t)V-c\}\,\mathrm{d}t>0, \tag{3}$$

but it is not profitable for both firms always to engage in R&D (otherwise R&D is a dominant strategy and preemption is impossible). A sufficient condition for the latter assumption is

$$\int_0^{\infty}\exp\left\{-\left[rt+2\int_0^t\mu(\tau)\,\mathrm{d}\tau\right]\right\}\{\mu(t)V-c\}\,\mathrm{d}t<0. \tag{4}$$

The assumption that R&D is profitable for a monopolist requires

$$\mu(\omega)V-c>0 \quad \text{for} \quad \omega>\bar{\omega}, \quad \text{for some} \quad \bar{\omega}\geqq 0,$$

while the assumption that it is not profitable for both firms to always do R&D implies

$$\mu(0)V-c<0.$$

The R&D process is an investment in that the expected instantaneous payoff is negative at the start but positive for sufficiently large levels.

We assume that once a firm stops doing R&D it must drop out of the race. This might seem an unnecessary assumption, because if a firm stops for a while its competitive position can only deteriorate. In fact the assumption is not required in a discrete-time finite-horizon model; its role here is to rule out 'supergame' phenomena.

The state of the competition for the discovery is summarized by the experience levels $\tilde{\omega}=(\omega_1,\omega_2)$. The strategy of firm i specifies whether firm i drops out or not at state $\tilde{\omega}$. Each firm's decision at a given state must be part of an optimal sequential decision beginning with that state. In particular firms will not be able to precommit themselves to doing R&D 'no matter what'. Were such commitments possible, it would be an (open-loop) equilibrium for firm one to drop out of the patent race and for firm two to stay in even if firm two has less experience, because profits would be negative with two firms and any one firm could commit itself to a research program.

More interesting is the case where firms cannot make arbitrary commitments, but are restricted to actions that are equilibria of the patent game as it evolves over time [specifically, subgame-perfect equilibria, as defined by Selten (1965)]. The following proposition demonstrates that in the absence of commitments, a marginal lead in the patent race example of this section is sufficient to guarantee a monopoly position in R&D.

Proposition 1 (*ε-preemption*). *Assume*:

(i) *R&D is a viable activity for a monopolist [inequality (3)];*
(ii) *it is not profitable for both firms always to engage in R&D [inequality (4)];*
(iii) *firm one enters the patent race before firm two* $[t_2 > t_1 = 0]$.

With these assumptions, a unique perfect equilibrium exists and has the property that whatever t_2, firm one engages in R&D and firm two drops out of the patent race.

The proof of the proposition is given in the appendix, but the intuition behind the result can be explained with the help of fig. 1. This diagram shows the critical level of experience that yields firm j a zero expected payoff from the patent race if firm i has experience ω_i and both engage in R&D until the discovery is made. This critical level of experience is labelled $\Omega(\omega_i)$; the function is symmetric and continuous for both firms. For states $\tilde{\omega}$ above the curve $\Omega(\omega_1)$, firm two makes profits if both do R&D, and for states to the right of $\Omega(\omega_2)$, firm one makes profits if both do R&D.

If both firms stay in from the start, both make losses by assumption. Thus, if firm two does not drop out at the beginning with certainty, there must be

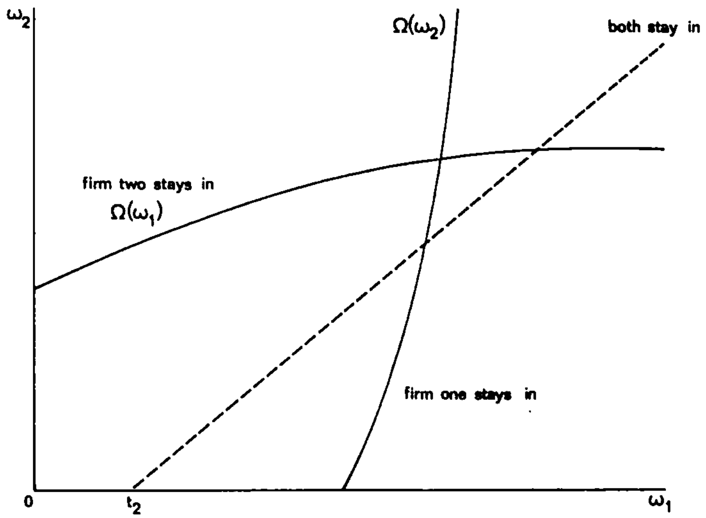

Fig. 1. Breakeven experience levels for the two firms. The dotted line is the path of R&D experience when both firms do R&D.

some probability that firm one drops out. But note that if both firms engage in R&D, the time path of their experience levels is a ray with slope 1 that cuts the ω_1 axis reflecting the headstart of firm one. Such a ray is shown in the figure. The crucial point is that firm one will reach the level of experience that guarantees a zero profit when both compete until the discovery is made, *before* firm two does. That is the ray in fig. 1 cuts $\Omega(\omega_2)$ before it cuts $\Omega(\omega_1)$. Thus, if that point is reached, firm one stays in, and since firm two would incur a loss if it continued, it must drop out. Then backwards induction precludes firm two from chasing firm one out, because both firms realize that later on firm two will have to drop out. The key to the result is that firm one can guarantee that it stays ahead, and thus that whenever firm one would make zero profits if both firms stayed in, firm two would have a loss. This is the same reasoning as in Dasgupta–Stiglitz (1980) and Gilbert-Newbery (1982), extended to a stochastic technology for R&D. As we shall see once firm two can 'leapfrog' firm one the above reasoning fails.

The key aspect of the ε-preemption argument is the impossibility of leapfrogging. While we presented the above model with a fixed intensity of R&D, this is not central to the ε-preemption result. In section 3, we show that in a discrete-time framework with variable intensity of research, the equilibrium converges to the ε-preemption equilibrium as the time periods shrink because the leader is able to perfectly monitor the activity of its rivals and hence avoid the possibility of leapfrogging.

3. Multi-stage patent races

In the fixed-intensity model of the preceding section, discovery is stochastic, but the probability of discovery is always greater for the firm with more experience. This section alters the fixed-intensity model by adding randomness in the link between R&D and the accumulation of pay-off relevant experience. This allows a firm to leapfrog another firm in terms of expected progress toward innovation. Specifically, we follow Roberts–Weitzman (1981) and assume that R&D takes place in a number of stages. To keep matters simple, assume there are only two. In the first stage a preliminary invention must be made, corresponding, as an example, to the conceptual definition of a research program. We will sometimes refer to completion of the first stage as the 'preliminary invention'. The second stage entails the progress toward a patentable design, with the rewards going to the winner of the patent race. Alternatively, the model in this section also would apply to a more realistic competition where the first stage is research and demonstration and the second is development, provided that the payoff disproportionately favors the first firm to complete the research, demonstration, and development cycle.

As in section 2, there are two firms ($i=1,2$) with the same research technology. Firm one begins the race an exogenously specified time t_2 ahead of firm two, and firm two makes an (expected) loss if both do (the two stages of) R&D until one of them obtains the patent. However, the firm that is behind in the first stage has a chance of being the first to make the preliminary invention and thus to induce the other firm to drop out.

The first phase of R&D costs c_1 per unit of time, and firm i's probability of making the preliminary discovery if it has not made it earlier is $\mu(\omega_i^1(t))$ per unit of time, where $\omega_i^1(t)$ is its first-stage experience at t. Similarly, the second phase costs c_2 and the probability of winning the patent is $\theta(\omega_i^2(t))$, where $\omega_i^2(t)$ is firm i's experience in the second phase. Both μ and θ are non-decreasing functions. Firms cannot accumulate second-stage experience without having made the preliminary discovery. We assume the fact that a firm has made the preliminary discovery becomes public knowledge at the date of discovery.

Leapfrogging can occur in the following way. Suppose both firms are in stage 1 and $\omega_1^1(t) > \omega_2^1(t)$, so that firm two lags firm one in expected value terms. With probability $\mu_2(t)$, firm two will make the preliminary discovery and its probability of success discontinuously jumps to $\theta(0)$. This will exceed firm one's success probability if firm one is still in the first stage, and the difference can be large enough to force firm one to drop out of the patent race. It is in this sense that one firm can leapfrog another.

Leapfrogging can be illustrated in a simple example where the discovery probabilities are constant in each stage. Both firms may compete in the first stage, and when one makes the preliminary discovery, the other will drop out if the race is a natural monopoly when both firms do R&D. Preemption occurs at the date of the preliminary discovery, but it is not ε-preemption in the sense that one firm drops out of the race if the other firm has greater experience, as occurred in the model of section 2. An example that illustrates both leapfrogging and ε-preemption is one where the probability of discovery is constant in the second stage (and equal for both firms), while in the first stage the probability of making the preliminary discovery increases with experience. This yields the following result:

Proposition 2. When the conditional probability of discovery increases in the first stage and is constant in the second, there exists a unique perfect equilibrium. The leading firm always does R&D unless the other firm does R&D and completes the first stage before a specified (possibly infinite) time $\bar{\omega}^1$. Depending on the values of the parameters, the follower either (i) *drops out at the start,* (ii) *does R&D until $\bar{\omega}^1$, or* (iii) *always does R&D, unless the leader passes the first stage before* $(\bar{\omega}^1+t_2)$.

Thus, depending on the parameter values, the outcome is either ε-preemption as in (i), or the two firms will compete despite the fact that both firms cannot earn positive profits if they continue R&D indefinitely. The firm that is behind in the first-stage race may continue R&D because it could make the preliminary discovery first and leapfrog its rival. Just how long the follower can stay in the race will of course depend on the progress of the other firm.

Proof. The structure of the proof of Proposition 2 is similar to that for the first proposition, although complicated by the two-stage nature of the patent race. Note that the payoff for a second-stage monopolist is

$$M^2=\int_0^{\infty} \exp\{-(r+\theta)t\}\{\theta V-c_2\}\,\mathrm{d}t$$

$$=(\theta V-c_2)/(r+\theta),$$

which is positive by assumption. If both firms pass the first stage, they both compete in the second stage and earn identical, positive expected profits,

$$W^2=(\theta V-c_2)/(r+2\theta).$$

Expected profits in the second stage are the same for both firms because the success probability, θ, does not depend on experience.

The expected profits are positive if both firms are in the second stage, but the decision points in the patent race occur when only one firm has made the preliminary discovery, or when both firms remain in the first stage. Let $\bar{\omega}^1$ denote the (possibly infinite) level of first-stage experience such that a firm with experience less than $\bar{\omega}^1$ (exceeding $\bar{\omega}^1$) drops out (stays in) when the other firm has completed the first stage. First note that $\bar{\omega}^1$ cannot be zero; otherwise a perpetual duopoly would be viable.[1] It is easy to show that if μ_1 is unbounded, $\bar{\omega}^1$ is finite and is defined by

$$\int_0^\infty \exp\left\{-(r+\theta)s-\int_0^s \mu(\bar{\omega}^1+\tau)\,\mathrm{d}\tau\right\}\{\mu(\bar{\omega}^1+s)W^2-c_1\}\,\mathrm{d}s=0. \tag{5}$$

If μ is bounded, say, by $\bar{\mu}$, then $\bar{\omega}^1$ is given by (5) if $\bar{\mu}W^2>c_1$, and is infinite if $\bar{\mu}W^2\leqq c_1$.

Assume that $\bar{\omega}^1$ is finite and consider the game when only one firm has completed the first stage. Firm one stays in if and only if $t\geqq\bar{\omega}^1$, when firm two has made the preliminary invention. In the reverse situation, firm two stays in if and only if $t-t_2\geqq\bar{\omega}^1$. Suppose that neither firm made the preliminary discovery at time $\bar{\omega}^1$. Two cases are possible. Either firm two makes an expected profit when both firms stay in 'forever' or it does not. In the first case, firm one, which has more experience, also has 'stay in' as a dominant strategy. Then both firms stay in. In the second case, the race from $t=\bar{\omega}^1$ on is a natural monopoly. It is straightforward to extend the proof of Proposition 1 to show that there is ε-preemption: Firm two drops out at $\bar{\omega}^1$.[2] Thus, if both firms are still in the first-stage R&D race at time $\bar{\omega}^1$, either firm one becomes a monopolist [case (ii) in Proposition 2] or both firms always do research, unless firm one passes the first stage before $(\bar{\omega}^1+t_2)$ [case (iii)].

Now consider the race before time $\bar{\omega}^1$. The value for either firm of making the preliminary discovery is M^2. Note also that the value at $\bar{\omega}^1$ — i.e., the value when both firms have done R&D until $\bar{\omega}^1$, but none has discovered — is higher for firm one than for firm two, whether firm two drops at $\bar{\omega}^1$ or not. Again a straight-forward extension of the argument in section 2 shows that either firm two drops at time 0 [case (i) in Proposition 2] or both firms do R&D until $\bar{\omega}^1$.

Let us now assume that $\bar{\omega}^1$ is infinite. Let $\bar{\mu}$ be the upper bound on the first-stage discovery probability. Note that a necessary condition for a monopoly to be viable is that $\bar{\mu}M^2>c_1$. But this condition does not imply

[1]Firm two loses money if both firms start with no first-stage experience and always do R&D. *A fortiori* it loses money if it starts with no experience and the other has already passed the first stage.

[2]In this race firm one indeed has a new advantage relative to section 2: Firm two will drop out if firm one makes the preliminary invention before $(\bar{\omega}^1+t_2)$.

that $\bar{\mu}W^2 > c_1$, and thus the case of an infinite $\bar{\omega}^1$ is possible. When $\bar{\omega}^1$ is infinite, the firm that is preempted at the preliminary discovery drops out immediately. The payoff for the firm that wins the first-stage race is M^2. This situation is equivalent to a simple competition for a patent of value M^2. There can be two cases: If this simple race is a natural monopoly, firm two drops out at time zero. If duopoly is viable,[3] then both firms do R&D until one passes the first stage and becomes a monopolist. Q.E.D.

A curious feature of the two-stage patent race is that the payoff to the winner of the first-stage competition may decrease with time. Suppose firm two begins the first-stage race with less experience, and consider its return from making the preliminary discovery before firm one. From Proposition 2, we know that firm one will drop out of the patent race if and only if firm two makes the preliminary discovery first at some time before $\bar{\omega}^1$. Thus firm two has an expected value from the preliminary discovery of M^2 if it makes the discovery before $\bar{\omega}^1$. After $\bar{\omega}^1$, firm one will continue R&D until one of the firms wins the patent race in the second stage. Since the probability that firm one will complete the first stage increases with time, the payoff to firm two decreases with time, provided that firm one remains in the first stage. The behavior of the return to firm two from the first-stage discovery is summarized in fig. 2.

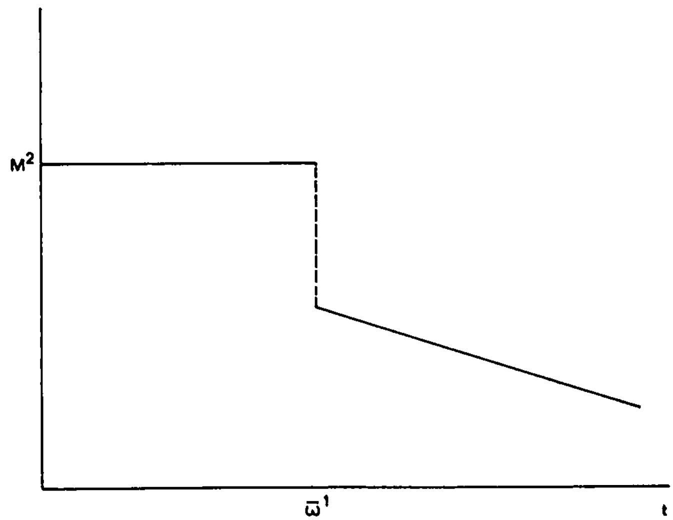

Fig. 2. Firm two's value of making the preliminary invention first.

[3]Again duopoly may be viable in the first stage even if it is not globally, i.e., when both firms never drop out before one of them obtains the patent.

The case in which the probability of discovery is constant in the first stage and increasing in the second is equally simple. In this case again firms compete even if the whole race has the natural monopoly property. The reader can check that both firms do R&D in the first stage, and that when one firm makes the preliminary invention, the other firm drops out either instantaneously or after some specified time if it has not succeeded in passing the first stage by that date.

The model in this section has the characteristic of a race for a patent whose value decreases over time. Imagine, for example, that in the one-stage model of section 2, a substitute for the good that can be produced under patent protection will be introduced T periods from the start of the patent race. Then the value of discovery decreases and potentially is nil at time T. The equilibrium for such a patent race very much resembles that of case (ii) in Proposition 2. The follower, if it does R&D at all, drops out after some given time. Of course an important difference in the multi-stage patent race is that the follower may make the discovery before $\bar{\omega}^1$ and leapfrog the leader, which may cause the leader to drop out of the patent race.

4. A patent race with information lags

Leapfrogging could occur in the preceding model because the probability of success was a stochastic function of R&D effort. Completion of the first stage was a random event, and by being lucky in the first stage, a firm with less experience could leapfrog a more experienced rival. Leapfrogging also may occur if firms have only imperfect information about the activities of their competitors. In this case a firm may leapfrog not because the success probability is stochastic, but because it can make progress toward invention without revealing its progress to a more experienced competitor. The impossibility of up-to-the-minute monitoring of a rival's R&D program allows firms that are only slightly behind to catch up before the leader can react. In this way information lags allow leapfrogging, and an arbitrarily small lead need not preempt rivals.

Information lags can be represented in a discrete-time formulation of the patent race. Each firm has a choice of R&D levels within a time period. At period t, firms are completely informed about the R&D activities of competitors up to period $t-1$. To keep the model simple, suppose invention occurs when the first firm accumulates a specified level of knowledge, as measured by total R&D (the extension to discovery as a stochastic function of R&D is straightforward). Firms may 'learn' at the rate of one or two units (steps) per period, with learning at the slower rate being more efficient. Information lags occur because within the current time period, each firm must choose its R&D effort without knowing what rival firms are doing. Thus, if a firm is one unit ahead of a rival, and it elects to proceed at the more economical learning speed, the follower may catch up with an effort level of two, and in the next period it may move ahead.

We show that the firms will proceed at the high effort level only if they are separated by no more than one unit of experience. If either firm lags by two or more units, it drops out of the race. The leader then proceeds at the more economical lower effort level. Allowing for random discovery where the probability of discovery depends on experience would not alter this basic result. In the limit of shorter information lags, the follower always drops out, so that both firms compete only if they are tied. This limit result shows that ε-preemption, while special, is not an artifact of the fixed-intensity specification of section 2. We also show that the equilibrium is unique if the race is not too long. An interesting feature of (every) equilibrium path when firms begin with equal experience is that there is a burst of R&D followed by the eventual emergence of a monopolist. The initial competition for the patent on average dissipates the monopoly profits.

There are two firms $i=1,2$. R&D competition takes place in discrete time, $t=0,1,\ldots$, and $\omega_i(t)$ denotes firm i's experience at time t. Discovery occurs with certainty when a firm's experience is equal to or exceeds a given number of 'units of experience' N. If both firms reach the critical level N in the same period, a firm receives the patent if its total experience exceeds that of its rival. If they tie, each firm wins with probability 0.5. The value of the patent for any firm is V.

Let $e_i(t)$ denote the effort level of firm i in period t, and define the cumulative effort as

$$\omega_i(t)=\sum_{\tau=0}^{t-1} e_i(\tau).$$

Each firm can choose from the alternatives shown in table 1 in any period, with associated costs.

Table 1
R&D effort levels and costs.

Effort	e_i	c_i
Zero-effort level	0	0
Low-effort level	1	c_1
High-effort level	2	c_2

Despite the temporal nature of the competition, explicitly introducing a positive rate of time preference would serve no purpose. We will assume that firms prefer to obtain a given reward sooner rather than later, which would follow from an arbitrarily small rate of time preference or a lexicographic ordering on money and time.

Let $\bar{k}$ be the largest even integer such that firms with an experience level $N-\bar{k}$ could both compete at the high-effort level and earn non-negative profits. That is, playing the high-effort level $\bar{k}/2$ times to get $V/2$ is worthwhile:[4]

$$V/2-(\bar{k}/2)c_2>0>V/2-(\bar{k}/2+1)c_2.$$

We make the following assumptions:

(A.1) $c_2>2c_1$,
(A.2) $V/2-c_2>0$,
(A.3) $N\leqq 2\bar{k}$.

(A.1) implies that a monopolist would always choose the low-effort level. (A.2) says that maintaining the high-effort level for one period in order to tie for the patent is worthwhile. (A.3) bounds the amount of effort required to discover in terms of c_2. Its only purpose is to shorten the analysis.

Let $k_i=N-\omega_i$ be the amount of effort remaining for firm i. A strategy for each firm i, $s_i^{\bullet}$, is a mapping from experience levels of both firms to the choice of effort,

$$s_i:(k_i,k_j)\to e_i.$$

We allow mixed strategies. Define $V^i(k_i,k_j)$ as the expected return to firm i when it has k_i steps to go, firm j has k_j steps, and firms follow their optimal strategies. [We will show that for $\min(k_i,k_j)<\bar{k}$ the equilibrium is symmetric so that $V(k_i,k_j)$ depends only on the state and not on the name of the firm.]

The main results are summarized in the following proposition:

Proposition 3. In any perfect equilibrium, if the follower is two or more steps behind, it drops out. If it is one step behind, and if the number of steps left is not too large, it randomizes between dropping out and incurring the high cost; the leader randomizes between the high- and the low-effort levels. If both firms are tied, they compete vigorously (i.e., incur the high cost) if the number of steps is less than $\bar{k}+1$*, and otherwise they randomize. The equilibrium is unique if the number of steps remaining for the leader is less than* $\bar{k}+1$.

Proof. Let k be the number of steps left for the leader. We treat only $k<\bar{k}$ in the text; the proof for $k\geqq\bar{k}$ is in the appendix. For $k<\bar{k}$ we show that the

[4]For simplicity we shall not consider the case in which there exists a $\bar{k}$ such that $V/2=(\bar{k}/2)c_2$.

valuations do not depend on the name of the firm and that the following postulates hold:

(i) $V(k,k+1)-V(k+1,k+1)\geqq V/2$,

(ii.a) $V(k+j,k)=0$ for $j\geqq 1$,

(ii.b) if $j>1$ the follower drops out,

(iii) $V(k,k)=V/2-n(k)c_2$, $\quad n(k)=k/2$ for k even, $\quad =(k+1)/2$ for k odd.

(i) says a firm which is tied in the patent race would like to have incurred the extra cost $V/2$ [and *a fortiori* (c_2-c_1) from assumption (A.2)] in order to be leading by one step. (ii) says that the valuation of the follower is always zero, and that he drops out when behind by two or more. (iii) gives the valuation 'on the diagonal', i.e., when firms are tied, where $n(k)$ is the number of periods until discovery when both firms compete vigorously. Note that $k\leqq\bar{k}$ implies $V/2>n(k)c_2$, and firms earn positive profits if they have equal initial experience. For $k>\bar{k}$, the appendix shows that firms will choose mixed R&D strategies and earn zero expected profits.

Assume that firm one is the leader $(k=k_1)$. Using (i), (ii) and (iii), we will show that the optimal actions for each firm are as given in table 2, where $[a,b]$ denotes a mixed strategy on actions a and b.

Table 2

Optimal-effort levels.

k_2-k_1	e_1	e_2
$\geqq 2$	1	0
1	[1,2] or 1	[0,2] or 2
0	2	2

Consider the optimal strategies for firms one and two when $k_1=1$, so that firm one has only one step remaining in the patent race. If the two firms are tied with $k_2=1$, the unique equilibrium is $e_1=e_2=2$. Although there is only one step to go, both firms compete at the high-effort level because the patent is awarded to the firm with the most experience when both reach the critical experience level, N, in the same period. Note that the resulting state valuation for $(k_1,k_2)=(1,1)$ is

$$V(1,1)=V/2-c_2,$$

and this is consistent with postulate (iii) above.

Next consider $k_2=2$ and $k_1=1$, so that firm two is behind by one step. Would firm two drop out of the patent race with probability 1? If so, the optimal strategy for firm one would be to proceed at the monopoly pace ($e_1=1$). But then the follower could catch up and earn $V/2$ by incurring cost c_2 with a positive net return. Hence, playing $e_2=0$ with probability 1 could not be an equilibrium strategy for $(k_1,k_2)=(1,2)$. It should be clear that the leader would never play $e_1=0$. The leader can guarantee itself $V-c_2$ by choosing the high-effort level, and this strictly exceeds the return from $e_1=0$ given any R&D choice by the follower (if the follower plays $e_2=0$, firm one gets the same payoff, but later). Thus, the leader can play only $e_1=1$ and/or $e_1=2$ when $(k_1,k_2)=(1,2)$.

Note that the leader must randomize between effort levels of 1 and 2, since the follower would drop out if $e_1=2$ (and the leader would then choose $e_1=1$), and the follower would choose $e_2=2$ if $e_1=1$ (in which case the leader would be better off with $e_1=2$). With the leader randomizing between 1 and 2, it is never optimal for the follower to choose $e_2=1$, because the follower cannot even tie for the patent. Also, the follower cannot choose the high-effort level with probability 1, because the leader then would do the same, and the follower would always lose. We conclude that firm two randomizes between 0 and 2, while firm one randomizes between 1 and 2.

Since the follower plays $e_2=0$ with some probability and this gives a zero return, the return from $e_2=2$ also must be zero, otherwise the firm would either drop out or proceed at the high level with probability 1. The follower must be indifferent between dropping out and trying to catch up. Thus, for the follower,

$$V(2,1)=0,$$

which is consistent with postulate (ii.a). The leader plays $e_1=2$ with some probability and this gives a guaranteed return of $V-c_2$, which must equal firm one's valuation. Thus,

$$V(1,2)=V-c_2.$$

Note that $V(2,2)=V(1,1)=V/2-c_2$, so that

$$V(1,2)-V(2,2)=V/2,$$

and thus, postulate (i) is satisfied for $k=1$.

Proceeding by induction, assume that (i) and (iii) are satisfied when the leader has at most k steps to go, and consider the case where $k_1=k+1$.

Case 1: $k_1 = k_2$. *The firms are tied.*

We will show that both firms playing 2 is the unique equilibrium. First note that it *is* an equilibrium: each firm receives $V/2 - n(k+1)c_2$, compared to $(-c_1)$ from playing 1, or zero from playing 0. Next note that it is not optimal for either firm to do no R&D in any period. If firm i chose $e_i = 0$, its valuation would be

$$V(k+1,k+1) = t_0 V(k+1,k+1) + t_1 \times 0 + t_2 \times 0,$$

(where t_0, t_1, t_2 denote the other firm's probabilities of playing 0, 1, 2). If $t_0 < 1$, this implies $V(k+1,k+1) = 0$. Yet playing $e_i = 2$ ensures a positive payoff. If $t_0 = 1$, then playing 1 dominates playing 0.

If the other firm never plays 0, it is not difficult to see that playing 2 dominates playing 1. If firm i plays 1, the return is

$$t_1 V(k,k) + t_2 \times 0 - c_1, \tag{6}$$

while the return from playing 2 is

$$t_1 V(k-1,k) + t_2 V(k-1,k-1) - c_2. \tag{7}$$

Subtracting (6) from (7), noting that $t_1 + t_2 = 1$, gives

$$t_1[V(k-1,k) - V(k,k) - c_2] + t_2[V(k-1,k-1) - c_2] + c_1. \tag{8}$$

Since the postulates (i)–(iii) are assumed to hold for $k_1 \leqq \bar{k}$, it follows that both terms in brackets in (8) are positive, and $e_i = 2$ is the unique equilibrium strategy. Thus,

$$V(k+1,k+1) = V/2 - n(k+1)c_2,$$

and this confirms (iii) by induction. Finally, $V(k,k+1)$ is greater than or equal to $V - n(k)c_2$, so that $V(k,k+1) - V(k+1,k+1) \geqq V/2$, which confirms (i).

Case 2: $k_2 - k_1 = 1$. *Firm one leads by one step.*

Let t_i be the probability that the follower plays $e_2 = i$ for $i = 0, 1, 2$. If the leader plays $e_1 = 0$ with positive probability, the return must be

$$V(k+1,k+2) = t_0 V(k+1,k+2) + t_1 V(k+1,k+1) + t_2 \times 0,$$

and, since $t_0 = 1 - (t_1 + t_2)$, this is equivalent to

$$V(k+1,k+2) = (t_1/(t_1 + t_2)) V(k+1,k+1). \tag{9}$$

Now from postulate (iii),

$$V(k+1,k+1)=V/2-n(k+1)c_2, \tag{10}$$

and clearly if the leader plays zero, making use of (10) and (9) shows that

$$V(k+1,k+2)\leqq V/2-n(k+1)c_2.$$

But note that, if the leader plays $e_1=2$ each period until discovery, he earns

$$V-n(k+1)c_2.$$

This strictly exceeds the return from $e_1=0$, so the leader does not play 0.

Could firm one proceed at the monopoly pace with certainty? In this case, the follower's payoff to catching up is

$$V(k,k)-c_2=V/2-[n(k)+1]c_2,$$

which is positive because, by assumption, $k+1<\bar{k}$. Thus the follower would want to play 2 with probability 1; but then the leader would prefer playing 2 by induction hypothesis (i). The follower cannot play 0 with certainty either. In this case the leader would proceed at the monopoly pace. Lastly the leader cannot play 2 with certainty: the follower could not catch up and would drop out of the race, inducing the leader to proceed at the monopoly pace.

We thus conclude that the leader randomizes between 1 and 2, and the follower between 0 and 2. Let p be the probability that the leader plays 1 and q the probability that the follower plays 0. Since the follower must be indifferent between dropping out with a value of zero and trying to catch up, the return from the high-effort level must be zero; that is,

$$pV(k,k)-c_2=0, \tag{11}$$

or

$$p=c_2/(V/2-n(k)c_2). \tag{12}$$

Since $k+1<\bar{k}$, the denominator is at least as large as c_2 and $0<p<1$. Also, q must be such that the leader is indifferent between R&D at the high- and low-cost levels, so that

$$q[V-c_2-(k-1)c_1]+[1-q][V(k-1,k)-c_2]$$
$$=q[V-(k+1)c_1]+[1-q][V(k,k)-c_1], \tag{13}$$

or

$$q=[V(k-1,k)-V(k,k)-(c_2-c_1)]/[V(k-1,k)-V(k,k)-c_1]. \tag{14}$$

From (i) and assumption (A.2), $0<q<1$.

Lastly note that dropping out is an optimal strategy for the follower. Thus, his valuation is zero and (ii) is satisfied:

$$V(k+2,k+1)=0.$$

Case 3: $k_2-k_1 \geqq 2$. *Firm 1 has a lead of two steps or more.*

Let us first show that playing zero is dominated by playing 1 for firm one. If firm two plays 0, playing 1 is optimal since it is the monopoly pace. If the follower plays 1, or if the lag is at least 3, playing 1 ensures, from induction hypothesis (ii), that the follower will drop out next period. If the follower lags by 2 and plays 2, playing 0 gives $V(k+1,k+1)$ to the leader and playing 1 yields $V(k,k+1)-c_1$. But from (i), $V(k,k+1)-c_1>V(k+1,k+1)$.

Suppose the follower does R&D and incurs c_1 or c_2. In the next period the follower will continue to lag the leader by at least one step. Thus the follower's valuation will be zero, and he is better off not doing any R&D. We thus conclude that the follower drops out and the leader proceeds at the monopoly pace. This proves that (ii) holds when the leader has $(k+1)$ steps to go and the lead is at least two. Q.E.D.

We have thus shown that for $k<\bar{k}$ there exists a unique perfect equilibrium, and that this equilibrium has the properties given in Proposition 3. The case where $k \geqq \bar{k}$ is treated in the appendix. Summarizing the results shown in the appendix, if $V/2-(\bar{k}/2)c_2-c_1<0$, the equilibrium is unique and the induction postulates (i) and (ii) hold; (iii) is replaced by $V(\bar{k}+j,\bar{k}+j)=0$. In the complementary case $V/2-(\bar{k}/2)c_2-c_1>0$, condition (ii) is still satisfied, but there may exist asymmetric equilibria in which $V(\bar{k}+j,\bar{k}+j)$ is positive for one firm.

Comparative statics of the patent race

The simple model of the patent race contains only a few parameters and it is straightforward to examine the competitive consequences of alternative assumptions.

(a) Suppose that *the value of the patent* is increased, for example by means of a longer patent life or a liberalization of antitrust laws against the exploitation of a patent monopoly. This will have no impact when the gap between firms is two or more effort units. If they are tied, it will increase the amount of competition, in the sense that firms can sustain a high level of R&D effort over a longer time interval.

Note that the smallest experience level, $(N-\bar{k})$, for which firms compete with pure strategies is defined by the largest even number $\bar{k}$ that satisfies $V/2>(\bar{k}/2)c_2$. Thus increasing V clearly increases $\bar{k}$. For $k<\bar{k}$, the firms will invest in R&D at the rapid pace and their activities will be redundant. Unless the gains from accelerated discovery are sufficient to compensate for the extra costs, the welfare consequences of increased competition resulting from the larger patent reward are unambiguously negative in this simple model.

A larger patent reward also increases the range of experience levels over which a firm will continue to compete if it is one period behind the leader. The follower with $(N-k)$ units of experience will compete provided the value of catching up exceeds the cost,

$$V(k-2,k-2)>c_2.$$

Since $V(k-2,k-2)$ increases with the size of the reward, for $k\leqq\bar{k}$, a larger reward increases the range of experience levels over which firms compete.

Consider the consequence of an increased reward on the leader's strategy. Let p be the probability that the leader chooses $e=1$. The condition for a mixed strategy by the follower requires [see eq. (12)]

$$p=c_2/V(k-2,k-2) \quad \text{for} \quad V(k-2,k-2)>c_2.$$

A higher reward lowers p and therefore increases the expected effort by the leader. It can also be shown that a larger reward increases the expected effort by the follower. Thus, a larger reward increases the likelihood of leapfrogging as well as the range of experience levels where the follower continues to compete.

(b) A crucial parameter in the patent race model is the *length of the period* over which firms cannot observe the actions of their rivals. This is the source of imperfect information in the model. Let T be the period length (or lag in information). Assume that this length is cut by half: $T/2$. To keep things comparable, we leave the (physical) number of units of experience N constant, but we assume that firms can in period t acquire $\frac{1}{2}$ unit of experience at cost $c_1/2$ or 1 unit of experience at cost $c_2/2$. From the definition of $\bar{k}$, we see that the number of periods over which there may be competition between a leader and a follower doubles, so the real length of time stays constant. But notice that now the follower competes only if the lag does not exceed $\frac{1}{2}$. More generally, we see that when the length of the period decreases, the lag in experience for which the follower competes gets smaller. Thus *the imperfect-information variable-intensity equilibrium converges to the ε-preemption equilibrium when the lag in information tends to zero.*

A striking feature of this model is that the follower always has valuation zero in equilibrium regardless of the length of the lead. While we think that there is an important point here, we do not want to insist that arbitrarily small disadvantages cause a zero valuation in the presence of information lags. The zero-valuation result will not hold if there is sufficient uncertainty about discovery (there was none in this section) and a large enough information lag. To see this consider the open-loop equilibrium described in Lee–Wilde (1980). Open-loop strategies are perfect when the information lag is infinite. Although there is no experience and no initial asymmetry between the firms in Lee–Wilde, it is easy to introduce these two features and still get the outcome that more than one firm can make positive profit in the patent race. Whether the follower has a positive valuation or not should depend on the degree of uncertainty, the length of the information lags and the specification of the hazard function.

5. Concluding remarks

The central issue addressed in this paper has been a characterization of the conditions that facilitate competition in patent races. Although we have not attempted a general analysis, the three examples illustrate the importance of leapfrogging in the accumulation of experience relevant to the probability of discovery. This is evident in a comparison of the first two models. When the relative ordering of firms' cumulative experience is immutable, as in the example of section 2, any lead in the patent race is sufficient to preempt a rival firm when the market is otherwise too small to accommodate both firms always engaged in R&D. This ε-preemption occurs even though discovery is stochastic, and each firm has a positive probability of winning the patent race. Allowing for the possibility of leapfrogging in relevant experience levels enables both firms to compete for the patent, at least over finite intervals, provided the firms do not begin too far apart.

The third model differs from the others, among other reasons, because success is not a stochastic function of experience: the first firm to reach a critical level of experience wins. But here again the main determinant of competition is the possibility that a firm which is behind in the patent race can change places with the current leader. The mechanism for this leapfrogging behavior is information lags, which essentially allow a firm to sneak up on and eventually pass the current leader in the patent race. It should be clear in the development of the patent race model with information lags that introducing uncertainty in the patent discovery would not alter the main results. In particular, ε-preemption would occur if the leader could perfectly monitor the R&D effort of the follower, and potential leapfrogging would remain the key to competition.

In section 2 we argued that a one-stage game with variable intensity may lead to a natural monopoly situation. If this is the case in the second stage of the two-stage game described in section 3, the first firm that passes stage one (at any time) becomes a monopoly. We can attribute a fictitious patent value to passing the first stage equal to the expected present discounted monopoly profit in stage two. Thus the game is 'like a one-stage game', and if rent can be dissipated by high enough intensity, we again obtain ε-preemption at the beginning of the first stage.[5] We thus see that the possibility of leapfrogging in the multi-stage (or random experience) game will depend on the degree of uncertainty, on the lags in information and on the R&D technology.

The behavior exhibited in our patent races differs from the implications of models studied by Loury (1979), Dasgupta–Stiglitz (1980), Reinganum (1981, 1982a, b) and others. Experience is not a factor in these models. We would expect that introducing experience and dynamic strategies into these models would result in market structures that become more concentrated over time as firms' probabilities of success become dispersed. In some specifications, a sufficient lead in payoff-relevant experience might cause the patent race to degenerate to a monopoly from the start.

We have assumed in this paper that all firms were equally efficient in the performance of R&D. Once firms' efficiencies may differ, it is no longer reasonable to assume that each firm knows the payoff functions of its rivals. That is, the information structure will typically be incomplete in Harsanyi's (1967) sense. This leads to an alternative to leapfrogging as an explanation of competition with selection (but without ε-preemption) in R&D. In a patent race time selects the most efficient or motivated firms, an idea developed in Fudenberg–Tirole (1983).

This paper is an exploration of the middle ground between patent races without preemption and those in which preemption is virtually automatic. We stressed that the extreme results in previous work were due to very special information structures and technologies, and we provided examples of patent races whose equilibria were, we argued, more plausible. As our purpose was primarily to exhibit such 'intermediate' patent races, we relied on quite simple models. We did not fully develop the comparative statics of the equilibria nor did we pose such questions as the optimal length of a patent. These issues are well worth addressing. While this paper endows us with a small headstart in understanding patent races, we encourage potential leapfroggers.

[5] Roger Guesnerie and Patrick Rey suggested this reasoning to us.

Appendix

A.1. Proof of Proposition 1

Let $\Omega(\omega)$ be the level of experience such that a firm with initial experience $\Omega(\omega)$ has a zero expected payoff when the other firm has experience ω, and both firms do R&D 'forever'. That is, $\Omega(\omega)$ is defined by

$$\int_0^\infty \exp\left\{-\left[rs+\int_0^s [\mu(\Omega+\tau)+\mu(\omega+\tau)]\,\mathrm{d}\tau\right]\right\}\{\mu(\Omega+s)V-c\}\,\mathrm{d}s=0, \quad \text{(A.1)}$$

where (Ω,ω) is the initial state and $[\mu(\Omega+s),\ \mu(\omega+s)]$ are the conditional success probabilities after a time s has elapsed from the initial date.

Assume that the probability μ is a continuous function of experience (this assumption can be relaxed with a somewhat more complicated proof). Then $\Omega(\omega)$ is continuous and it is easy to check that $\Omega(0)>0$ (when both firms with no experience always do R&D, they both lose money) and that $\Omega(\bar{\omega})<\bar{\omega}$ (a firm with experience $\bar{\omega}$ or more is willing to stay in, whatever the experience of its competitor). Note that there exists an experience level $\underset{\sim}{\omega}$ $(0<\underset{\sim}{\omega}<\bar{\omega})$ such that $\Omega(\underset{\sim}{\omega})=\underset{\sim}{\omega}$; when both firms start with experience $\underset{\sim}{\omega}$ and stay in, they just break even.

The equilibrium strategies on Proposition 1 have the property that a firm either does R&D until discovery or drops out; staying in for a while and then dropping out before discovery is never optimal. To see this, assume the initial state is $\tilde{\omega}=(\omega_1,\omega_2)$, both firms do R&D until date t, and firm 1 drops out of the race at t with zero expected profits. They by definition the state at t is

$$\tilde{\omega}(t)=(\Omega(\omega_2+t),\ \omega_2+t),$$

and

$$\Pi_1(\Omega(\omega_2+t),\omega_2+t)=\Pi_1(\tilde{\omega}(t))=0.$$

Firm 1's expected profit at the initial state $\tilde{\omega}$ can be written as

$$\Pi_1(\tilde{\omega})=\int_0^t \exp\left\{-\left[rs+\int_0^s [\mu(\omega_1+\tau)+\mu(\omega_2+\tau)]\,\mathrm{d}\tau\right]\right\}\{\mu(\omega_1+s)V-c\}\,\mathrm{d}s$$
$$+\Pi_1(\tilde{\omega}(t))\exp\left\{-\left[rt+\int_0^t [\mu(\omega_1+\tau)+\mu(\omega_2+\tau)]\,\mathrm{d}\tau\right]\right\}.$$

Since firm one's profits are zero at state $\omega(t)$, it must be the case that

$$\mu(\omega_1+s)V-c<0 \quad \text{for} \quad s\leqq t.$$

Therefore $\Pi_1(\tilde{\omega})<0$ and firm one would never join the patent race. For similar reasons, firm two would never begin an R&D program and then stop before discovery. Also, since monopoly profits are positive, if either firm drops out, the other should continue R&D.

Note that this argument proves that the graph of Ω has the concave curvature shown in fig. 1. More precisely,

$$\Omega(\omega+t)-\Omega\omega)<(\omega+t)-\omega.$$

Now consider the following strategies:

(a) If $\omega_1 \geqq \omega_2$, firm one stays in (until it discovers). Firm two drops out if and only if $\omega_2 \leqq \Omega(\omega_1)$.
(b) If $\omega_1 < \omega_2$, firm two stays in (until it discovers). Firm one drops out if and only if $\omega_1 \leqq \Omega(\omega_2)$.

It is easy to check that these strategies form a perfect equilibrium. For example, when firm one is ahead ($\omega_1 > \omega_2$), firm two can never induce firm one to stop, so that the latter will stay ahead. Thus, firm two can do no better than to stay in if and only if $\omega_2 > \Omega(\omega_1)$.

Next we show that in any equilibrium firm two drops out immediately, while firm one never drops out. Choose any equilibrium and assume to the contrary that there exists a time t such that, conditional on neither firm having dropped out at t, firm one drops out with some probability. Let $\bar{t}_1$ denote the supremum of such dates, and define $\bar{t}_2$ analogously for firm two (we will show below that $\bar{t}_2$ exists). We know that $\bar{t}_1 \leqq \bar{\omega}$ and $\bar{t}_2 \leqq \bar{\omega}+t_2$.

First we show that if $\bar{t}_1$ exists, then $\bar{t}_1 < \bar{t}_2$. Imagine that $\bar{t}_1 \geqq \bar{t}_2$. Then past $\bar{t}_1$, no firm will drop out. But firm one's probability of discovery at each instant strictly exceeds that of firm two. Thus, if firm one is willing to drop out at $\bar{t}_1$, firm two must strictly prefer to drop out, which contradicts $\bar{t}_2 \leqq \bar{t}_1$.

Now we show that firm two drops out with probability 1 at $\bar{t}_1$. Since firm one stays in from $\bar{t}_1$ on, firm two's best response is either to drop out or to stay forever. If firm two stays in at $\bar{t}_1$ then firm two's experience at date $\bar{t}_1$, $\omega_2(\bar{t}_1)=\bar{t}_1-t_2$, is no less than $\Omega(\bar{t}_1)$. But if

$$\bar{t}_1-t_2 \geqq \Omega(\bar{t}_1),$$

then $\bar{t}_1 > \Omega(\bar{t}_1-t_2)$. Thus firm one strictly prefers to stay in at $\bar{t}_1$, which contradicts the definition of $\bar{t}_1$. Thus firm two drops out with certainty at $\bar{t}_1$.

Lastly, because firm two is sure to drop out at $\bar{t}_1$ and a monopoly makes strictly positive profit, firm one would not drop out at $(\bar{t}_1-\varepsilon)$ for small ε, again contradicting the existence of $\bar{t}_1$. We thus conclude that firm one never drops out, which implies that firm two never does R&D.

To complete the proof we observe that the above arguments imply that $\bar{t}_2$ does indeed exist: because the race is a natural monopoly at least one of the firms must sometimes drop out. Thus if firm two never dropped out firm one would have to; and reasoning as above would yield a contradiction in the subgame starting at time $\bar{t}_1$. Q.E.D.

A.2. Proof of Proposition 3 when the leader has at least $\bar{k}$ steps remaining

There are two cases as $V/2-(\bar{k}/2)c_2-c_1 \gtrless 0$. In each case the follower has valuation zero and drops out when two or more stages behind. In the first case the follower may play a mixed strategy when one step behind, while in the second case the follower always drops out. Only in the second case is the equilibrium unique.

Case I: $V/2-(\bar{k}/2)c_2-c_1>0$

We will prove the following claims by induction on step $\bar{k}+j$ from $j=0$:

(i) The leader never plays $e_1=0$.

(ii) $V_i(\bar{k}+j+1,\bar{k}+j)=0$ for all j (this implies that if the follower trails by 2 or more, it drops out).

(iii) At $(\bar{k}+j,\bar{k}+j)$, either there exists a j such that $V_i(\bar{k}+j,\bar{k}+j)=0$ or for all j, $V_i(\bar{k}+j,\bar{k}+j)\leqq \max[0,c_2-jc_1]$. If $V_i(\bar{k}+j,\bar{k}+j)>0$, firm i plays 2 with positive probability.

To start the induction we consider states $(\bar{k}+1,\bar{k}+1)$, $(\bar{k},\bar{k}+1)$ and $(\bar{k}+1,\bar{k}+2)$ [the treatment of the case $(\bar{k},\bar{k})$ is as in case 1 in the text].

At $(\bar{k}+1,\bar{k}+1)$, if neither firm has a zero valuation, neither firm plays 0. Then both must sometimes play 1; otherwise, if one firm put no weight on 1, its rival could not have a positive valuation [as it could at best catch up, with a net return $V/2-n(\bar{k}+1)c_2<0$]. If both firms sometimes play 1, their valuation cannot exceed $V(\bar{k},\bar{k})-c_1<c_2-c_1$. Moreover, if neither firm plays 0, both must put some weight on 2: if either played 1 with certainty, playing 2 would yield $V(\bar{k}-1,\bar{k})-c_2>V(\bar{k},\bar{k})-c_1$. Finally, if one firm has a zero valuation, its opponent must put some weight on 2. Otherwise, the firm could obtain $V(\bar{k}-1,\bar{k})-c_2>0$ by playing 2. Thus (iii) is satisfied. It is easy to show that there exists asymmetric equilibria (for example, one firm randomizes between 1 and 2, and the other randomizes between 0 and 1).

Second, consider $(\bar{k},\bar{k}+1)$. First note that the leader can guarantee $V(\bar{k}-2,\bar{k}-1)-c_2>0$ by playing 2. If the leader plays 0, he gets at most $V(\bar{k},\bar{k})\leqq V(\bar{k}-1,\bar{k}-1)<V(\bar{k}-2,\bar{k}-1)-c_2$. Thus the leader does not play 0. Hence, the follower drops out: if he played 1, he would remain the follower and lose c_1; if he played 2, he would get at most $V(\bar{k}-1,\bar{k}-1)-c_2<0$. Thus the leader proceeds at the monopoly pace and the follower drops out.

Lastly, consider $(\bar{k}+1, \bar{k}+2)$. First note that the leader can guarantee $V(\bar{k}-1, \bar{k})-c_2>0$ by playing 2. Now we will confirm that the leader doesn't play 0 and the follower has value zero. There are three cases, corresponding to the three possible equilibria at $(\bar{k}+1, \bar{k}+1)$: either (a) the current leader's value, $V_{\mathrm{L}}(\bar{k}+1, \bar{k}+1)=0$, or (b) the current follower's value, $V_{\mathrm{F}}(\bar{k}+1, \bar{k}+1)=0$, or (c) $\max[V_{\mathrm{L}}(\bar{k}+1, \bar{k}+1), V_{\mathrm{F}}(\bar{k}+1, \bar{k}+1)]<c_2-c_1$.

In case (a), the leader's payoff to playing 0 is zero, so doing R&D dominates. Thus the follower drops out, as $V(k,k)-c_2<0$. In case (b), the follower will put no weight on 1, which yields only $-c_1$. Then the leader's payoff to 0 is again zero, so he will do R&D and the follower will drop out. Finally, in case (c), if the leader played 0, an optimal strategy at $(\bar{k}+1, \bar{k}+1)$, if his valuation is positive, is 'play 2' and yields at best $V(\bar{k}-1, \bar{k}+1)-c_2$ [notice that if the follower plays 2 at $(\bar{k}+1, \bar{k}+2)$ and the leader plays 0, the leader has a zero payoff]. But the leader could guarantee exactly this payoff *one period earlier* by playing 2 now, and so would do so from our 'impatience' assumption. Thus again the leader does not play 0, and the follower drops out.

Now we assume the induction hypotheses (i)–(iii) hold when the leader has $\bar{k}+j$ steps to go and extend them to $\bar{k}+j+1$.

Consider first $(\bar{k}+j+1, \bar{k}+j+1)$. We claim that either one firm has a zero valuation or both play 1 and 2. If neither firm has a zero valuation, neither plays 0; but then, if one firm never played 1, its rival's valuation could not be positive. If one firm played *only* 1, its rival could play level 2 until $\bar{k}$ and obtain at least $V-((j+1)/2)c_2-\bar{k}c_1>c_2-jc_1-c_1$, which is the maximum return to playing 1. [Remember that at $(\bar{k}, \bar{k}+1)$ and $(\bar{k}+1, \bar{k}+2)$ the follower drops out.] Thus if no firm has a zero value each must have value less than $c_2-(j+1)c_1$. Finally, if one firm has a zero valuation its rival must put some weight on 2: otherwise the firm could play 2 until $\bar{k}$ and gain a positive reward.

Next we consider $(\bar{k}+j+1, \bar{k}+j+2)$. The leader can guarantee a positive value by playing 2 until $\bar{k}$. Now we confirm that the leader does not play 0 and that the follower has value 0. There are three cases, corresponding to the three possible cases at $(\bar{k}+j+1, \bar{k}+j+1)$:

(a) $V_{\mathrm{L}}(\bar{k}+j+1, \bar{k}+j+1)=0$,

(b) $V_{\mathrm{F}}(\bar{k}+j+1, \bar{k}+j+1)=0$,

(c) $\max[V_{\mathrm{L}}(\bar{k}+j+1, \bar{k}+j+1), V_{\mathrm{F}}(\bar{k}+j+1, \bar{k}+j+1)]<c_2-(j+1)c_1$.

In case (a), the leader will not play 0. Therefore the follower will not play 1 from induction hypothesis (ii). Next we claim the follower must put some weight on 0. This is clear if $V_{\mathrm{F}}(\bar{k}+j, \bar{k}+j)<c_2$, otherwise $V_{\mathrm{L}}(\bar{k}+j, \bar{k}+j)=0$, and if the follower played 2 with certainty the leader would play 2 and the follower would drop out. So $V_{\mathrm{F}}(\bar{k}+j+1, \bar{k}+j+2)=0$.

In case (b), the follower will not play 1. If the follower played only 2, the leader would play 2 and the follower would drop out. So the follower must have value 0. Because the follower never plays 1, the leader won't play 0, which would yield 0 unless the follower played 0 with certainty in which case the leader would play 1.

In case (c), if the leader played 0 it would have at most $c_2(j+1)c_1$, which is less than what the leader can guarantee by playing 2 until $\bar{k}$. But then the follower drops out: playing 1 yields $(-c_1)$, and playing 2 yields at most $V_F(\bar{k}+j,\bar{k}+j)-c_2<-jc_1$. Q.E.D.

Case II: $V/2-(\bar{k}/2)c_2-c_1<0$

In this case there exists a unique perfect equilibrium. It is symmetric and satisfies:

(i) $V(\bar{k}+j,\bar{k}+j+1)=V-(\bar{k}+j)c_1$,

(ii) $V(\bar{k}+j+1,\bar{k}+j)=0$, and the follower drops out with probability 1 (a fortiori *the follower drops out when more than one step behind*),

(iii) $V(\bar{k}+j,\bar{k}+j)=0$.

To start the induction we consider state $(\bar{k}+1,\bar{k}+1)$. At this state we claim there is a unique equilibrium. In this (symmetric) equilibrium, both firms put positive weight on all three actions (and so in particular have value 0). First, at least one firm must play 0. If the other firm never plays 0, playing 1 yields at most $V/2-(\bar{k}/2)c_2-c_1<0$, yet if both play 2 both receive $V/2-(\bar{k}/2)c_2-c_2<0$. Let firm i be the firm which plays 0.

(a) Firm i does not play 0 with probability one, or j would play 1 and i would play 2.

(b) Firm i must put some weight on 2. Otherwise, j can guarantee $v-n(\bar{k}+1)c_2>0$ by playing 2, so j would not play 0, and i would get at most $V/2-(\bar{k}/2)c_2-c_1<0$.

(c) Firm j must play 2. Otherwise, i would prefer 2 to 0.

(d) Firm i must play 1. Otherwise, j would prefer 1 to 2: both acts yield the same contingent payoff if i plays 0, if i plays 2, j's gain from playing 1 instead of 2 is $-c_1-(V/2-(\bar{k}/2)c_2-c_2)>c_1-V/2+(\bar{k}/2)c_2>0$.

(e) Firm j must play 0, or i would not play 1.

(f) Firm j must play 1, or i would not play 2 [as in (d) above].

We conclude that the equilibrium must be totally mixed. As the payoffs are symmetric, the totally mixed strategies are unique.

Next we assume (i), (ii), and (iii) hold for $(j-1)$ and extend them to j.

(i) At $(\bar{k}+j,\bar{k}+j)$ there is again a unique equilibrium which is totally mixed. Again, some firm i must play 0 with some probability — if the other firm never plays 0, playing 1 yields $-c_1$, while if both play 2 they receive $\max(V/2-n(\bar{k}+j-2)c_2, 0)-c_2<0$.

(a) Firm i cannot play 0 with probability 1, or j plays 1 and i plays 2.
(b) Firm i must play 2. Otherwise, j guarantees itself $V-(\bar{k}+j-2)c_1-c_2>0$ by playing 2, and so j would not play 0, and i would not play 1.
(c) Firm j must play 2. Otherwise, i would prefer 2 to 0.
(d) Firm i must play 1. Otherwise, j does not play 2, which is dominated by 1. Playing 1 is preferable if i plays 0, while if i plays 2, playing 2 instead of 1 costs $c_2-c_1>0$.
(e) Firm j must play 0. Otherwise, i would not play 1.
(f) Firm j must play 1. Otherwise, i would not play 2 [as in (d)].

Again the equilibrium is totally mixed, and thus unique.

(ii) At $(\bar{k}+j, \bar{k}+j+1)$ the follower drops out with certainty. The leader can guarantee $V-c_2-(\bar{k}+j-2)c_1$ by playing 2, and thus won't play 0. Then the most the follower could hope to obtain by catching up is $-c_2+\max(0, V(\bar{k},\bar{k}))<0$. Thus the follower drops out and the leader plays one. Q.E.D.

Note that in this case the only bound on N needed is $V-(N-2)c_1-c_2>0$, so that it pays to proceed at level 2 for 1 period to become a monopolist.

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