Columbia University

Department of Economics
Discussion Paper Series

Democracy Undone: Systematic Minority Advantage in Competitive Vote Markets

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Discussion Paper No.: 1213-10

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New York, NY 10027

September 2012

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Systematic Minority Advantage in Competitive Vote Markets[1]

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Novermber 2012

[1]We thank Aniol Llorente-Saguer, Tom Palfrey, and participants to the Leitner Seminar at Yale University and the Straus Fellows seminar at the NYU Law School for their comments; Antoine Arnoud for his help on Theorem 3, and Georgy Egorov for an illuminating discussion of Lemma 1. Casella thanks the National Science Foundation for its support (SES-0617934), Paris-Jourdan Sciences Economiques for its hospitality, and the Straus Institute at the NYU Law School for its support and hospitality.

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Abstract

We study the competitive equilibrium of a market for votes where voters can trade votes for a numeraire before making a decision via majority rule. The choice is binary and the number of supporters of either alternative is known. We identify a sufficient condition guaranteeing the existence of an ex ante equilibrium. In equilibrium, only the most intense voter on each side demands votes and each demand enough votes to alone control a majority. The probability of a minority victory is independent of the size of the minority and converges to one half, for *any* minority size, when the electorate is arbitrarily large. In a large electorate, the numerical advantage of the majority becomes irrelevant: democracy is undone by the market.

JEL Classification: C62, C72, D70, D72, P16

Keywords: Voting, Majority Voting, Markets for Votes, Vote Trading, Competitive Equilibrium

1 Introduction

Consider a group making a single binary decision via majority voting. We know that majority voting treats all voters equally and both options symmetrically, and that it responds positively to changes in the preferences of the electorate (May, 1972), inducing voters to vote sincerely. We also know, however, that it ignores the intensity of voters' preferences and can lead to decisions that do not maximize utilitarian welfare. Can this difficulty be resolved?

It is a classic question in political economy and has led to a classic line of inquiry: Could a market for votes be the answer? Markets are designed to allocate goods to the individuals who value them the most; why wouldn't the same logic apply to a market for votes? A large literature developed in the Sixties and Seventies, whether focusing on a market for votes or on vote-trading when more than one decision is at stake: Buchanan and Tullock (1962), Coleman (1966, 1967), Park (1967), Wilson (1969), Tullock (1970), Haefele (1971), Kadane (1972), Riker and Brams (1973), Mueller (1973), Bernholtz (1973, 1974). The question has a long history, but providing an answer has been difficult. The problem is a fundamental non-convexity associated with vote trading: votes are intrinsically worthless; their value depends on the influence they provide, and therefore on the holdings of votes by all other individuals. Thus, demands are interdependent, and payoffs discontinuous at the point at which a voter becomes pivotal. Both in a market for votes and in log-rolling games, traditional equilibrium concepts such as competitive equilibrium or the core typically fail to exist. Writing in 1974, Ferejohn concluded: "[W]e really know very little theoretically about vote trading. We cannot be sure about when it will occur, or how often, or what sort of bargains will be made. We don't know if it has any desirable normative or efficiency properties" (p. 25). Ferejohn's early observation was echoed in later works (Schwartz (1977, 1981), Shubik and van der Heyden (1978), Weiss (1988), Philipson and Snyder (1996)), and with very few exceptions (Piketty (1994), Kultti and Salonen (2005)), the theoretical interest in voters trading votes among themselves effectively came to an end. This paper is part of a larger research project aimed at reopening the debate on competitive vote markets. We focus on a competitive market because it is both the first tool of an economist and the paradigm of the efficient market, and it is within this paradigm that we want to evaluate the early normative recommendations for markets for votes.

We study the *ex ante* competitive equilibrium of a market for votes where voters can trade votes for a numeraire before taking a single decision via majority rule. The choice is binary and the number of supporters of each alternative is known. The concept of *ex ante* competitive equilibrium was introduced in Casella, Llorente-Saguer and Palfrey (2012): it requires the market to clear in expectation, as opposed to clearing ex post, and allows traders to express probabilistic demands. These two features reestablish the existence of an equilibrium, while preserving the discipline imposed by market clearing: any mismatch between demand and supply must be unsystematic and unexpected.[1]

Casella, Llorente-Saguer and Palfrey show that an equilibrium exists when each voter has equal probability of favoring either alternative: without vote trading, the expected outcome of the vote is a tie. In most circumstances, however, one of the two alternatives is expected to win, and in many cases, lacking vote-trading, the sizes of the two opposing groups are known. Consider for example a decision with a clear partisan bias fought over by two opposing political parties; or taken by a committee where repeated interactions have led to a clear understanding of members' positions. What is often less clear is the intensity with which preferences are held. In a general election, for example, it is this uncertainty that translates into uncertainty about participation and explains the empirical importance of getting-out-the-vote campaigns.[2]

It is this case, when the size of the majority is known but the intensity of preferences is private information, that we study in this paper. Note that the existence of an ex ante equilibrium cannot be taken for granted: in general, we expect that the non-convexity problems associated with votes will be made worse by more precise information.[3]

We obtain two main results. First, we construct an ex ante equilibrium with vote trading that exists for arbitrary electorate size and majority/minority partition, as long as a condition on the intensities of preferences is satisfied. The condition rules out the possibility that multiple members of one group all have preferences that are much more intense (in a precise sense) than any member of the opposite group. The ex ante likelihood that the condition is satisfied depends on the distribution from which voters' intensities are drawn, on the size

[1] Kultti and Salonen (2005) also propose a Walrasian approach to vote markets based on probabilistic demands, but do not impose any market clearing condition.

[2] In U.S. Presidential elections, even voters who classify themselves as "undecided" also overwhelmingly identify themselves as partisan. See for example Bartels and Vavreck, "Meet the Undecided", on *Campaign Stops*, July 30, 2012 (http://campaignstops.blogs.nytimes.com/2012/07/30/meet-the-undecided/)

[3] See for example the discussion in Piketty (1994). Casella, Palfrey and Turban (2012) show that an ex ante equilibrium with trade exists in the case of five voters, divided into two groups of sizes 3 and 2, under a condition on the realized ranking of intensities. This finding allows them to run laboratory experiments comparing a market for votes to centralized bargaining by group leaders. It opens, however, the question of how robust the existence result may be in a more general model with arbitrary, known majority size.

of the electorate, and on the sizes of the two groups. At small electorate sizes, we find such likelihood to be high for standard intensity distributions–for example, if the minority is a third of the electorate and the distribution of intensities is uniform, the equilibrium exists with probability larger than 98 percent with nine voters, and larger than 99.9 percent with 21 voters. The stronger conclusion, however, concerns large electorates, where an ex ante equilibrium exists with probability arbitrarily close to 1, for any intensity distribution.

Second, the equilibrium we characterize has strong properties that translate into a systematic bias in favor of the minority. In equilibrium, only the highest intensity member of each group demands votes with positive probability; all other individuals offer their vote for sale. But the two voters who are willing to buy do not demand one or a few votes, rather they demand enough votes to single-handedly control a majority. The competition for votes becomes a competition for dictatorship between the highest-intensity member of the majority and the highest-intensity member of the minority, independently of the size of the minority, and of the intensities of all other voters. In a sense, the market functions as we should have expected: votes per se are worthless; what is traded is decision power, and the market allocates it to the individuals who want it most. Still, this also means that the frequency of minority victories reflects the relative intensity of the most intense minority member, without taking into account the smaller size of the minority and the aggregate group values. As a result, if the minority is very small or if the distribution of intensities assigns low probability to large outliers, the minority wins too often, relative to utilitarian efficiency. The bias can be strong enough that ex ante welfare is lower with a vote market than in the absence of trade. Again, this is particularly clear in large electorates. In such settings, if the minority is a non-negligible share of the total electorate, the highest intensities of majority and minority voters converge to the upper bound of the support of the intensities distribution, and thus are arbitrarily close. As a result, the minority is always expected to win half of the time, for *any* distribution of intensities, and *regardless of its share of the electorate.* The outcome is inefficient and inferior to simple majority voting with no vote market. As we summarize in the title of this paper: in a large market, democracy–the power of majority rule–is undone: the numerical superiority of the majority becomes irrelevant.

The equilibrium we construct echoes the equilibrium in Casella, Llorente-Saguer and Palfrey (2012): a vote market leads individuals to either demand bundles of votes–in fact a majority–or sell. It is an interesting conclusion, surprisingly robust to different assumptions about the expected size of the opposing groups and the structure of information. But when the two groups' sizes are known, the model delivers a number of additional predictions. First, because in both groups most individuals are offering their vote for sale, demand for additional votes is just as likely to arise from the majority as from the minority. Second, in equilibrium, intra-group trade and super-majorities always arise with high probability, even though votes command a positive price and the majority size is known. The intuition is clear: high intensity individuals need to preempt sales to the opposite group by their own weak allies. We believe that the predictions are empirically very plausible, but intra-group trades are absent from all vote-buying models we are familiar with.[4]

The goals of our project can be clarified by briefly addressing three fundamental objections to vote markets. First, there is the philosophical objection to the very idea of a vote market: votes just should not be traded because individual political rights and obligations in a democracy cannot be sold or bought (for recent voices, see, for example, Tobin, 1970, Marshall, 1977, Waltzer, 1989, Anderson, 1993, Sandels, 2012, Satz, 2012). We do not disagree, even as we note that vote trading and vote buying most likely do occur, both in popular elections and in committees.[5] And yet, we may still want to ask what the results of a vote market would be if one were allowed. Knowing how the institution would behave can only help in evaluating the moral objection. Second, it is also argued, again from a normative view, that vote markets are unacceptable because differences in income and wealth would translate into different incentives to trade, reducing the political voice of poorer individuals. Again, we agree in principle. But understanding the role that inequality would play is impossible if we do not understand how a market for votes would work in an ideal (abstract) world *without* inequality. This is the question we ask. Third, trades of votes create externalities on individuals who are not part of the trades. Failing a full Coasian bargain, a market for

[4] Groseclose and Snyder's (1996) conclusion that vote-buying leads to supermajorities has the same flavor. Their paper studies vote-buying in a legislature by two competing outside buyers, as opposed to vote trading among voters, and their result is due to the buyers taking turns in proposing a deal to the legislators, as opposed to the one-shot market studied here.

[5] There is a large empirical literature on vote-buying by candidates and parties in general elections. Studies of vote-buying among voters concentrate on committee votes. Building on political science's traditional focus on urban politics (for example, Goodman (1975)), Philipson and Snyder (1998) discuss indirect evidence of vote-buying in municipal decisions. A recent literature attempts to quantify vote-buying in international organizations. See, for example, Kuzienko and Werker (2006), Eldar (2008), Drexel et al. (2009), Dippel (2010), and Carter and Stone (2011). There is some anecdotal evidence of vote-buying among voters in popular elections: for example, http://www.slate.com/articles/news_and_politics/net_election/2000/10/want_to_sell_your_vote_not_so_fast.html, or http://www.fbi.gov/louisville/press-releases/2011/salyersville-man-convicted-of-buying-votes-in-2010-general-election.

votes, even in its more favorable design, will not be fully efficient. Externalities, however, do not imply that a market for votes should be expected to be inferior to simple majority voting with no trading, since the latter is likely to be inefficient as well. The comparison between the two then remains unsettled and interesting.

Finally, two other strands of literature should be mentioned. First, there is the important but different literature on vote markets where candidates or lobbies buy voters' or legislators' votes: for example, Myerson (1993), Groseclose and Snyder (1996), Dal Bò (2007), Dekel, Jackson and Wolinsky (2008) and (2009). These papers differ from the problem we study because in our case vote trading happens *within* the committee (or the electorate). The individuals buying votes are members, not external traders, groups or parties. Second, vote markets are not the only remedy advocated for majority rule's failure to recognize intensity of preferences in binary decisions. The mechanism design literature has proposed mechanisms with side payments, building on Groves-Clarke taxes (e.g., d'Apremont and Gerard-Varet 1979). However, these mechanisms have problems with bankruptcy, budget balance, and collusion (Green and Laffont 1980, Mailath and Postlewaite 1990). A recent literature suggests combining insights from mechanism design into the design of voting rules. Goeree and Zhang (2012) and Weyl (2012) propose allowing voters to purchase votes from a central agency at a price equal to the square of the number of votes purchased, a scheme with strongly desirable asymptotic properties. Casella (2005, 2012), Jackson and Sonnenschein (2007) and Hortala-Vallve (2012) propose mechanisms without transfer that allow agents to express their relative intensity of preferences by linking decisions across issues. Casella, Gelman and Palfrey (2006), Engelmann and Grimm (2012), and Hortala-Vallve and Llorente-Saguer (2010) test the performance of these mechanisms experimentally and find that efficiency levels are very close to theoretical equilibrium predictions, even in the presence of some deviations from theoretical equilibrium strategies.

The rest of the paper is organized as follows. Section 2 presents the model; section 3 characterizes the ex ante equilibrium whose properties we discuss in the rest of the paper; section 4 studies the expected frequency of minority victories and expected welfare, and compares these measures to the equivalent measures in the absence of a vote market and in the utilitarian first best. Section 5 discusses the robustness of the results to more general assumptions on the stochastic process generating intensities and to an alternative rationing rule. Section 6 concludes. The Appendix collects the proofs.

2 The Model

A committee of size n (odd) must decide between two alternatives, A and B. The committee is divided in two publicly known groups with opposite preferences: M individuals prefer alternative A, and m prefer alternative B, with $m = n - M < M$. We will use M and m to indicate not only the size of the two groups, but also the groups' names. While the direction of each individual's preference is known, the intensity of such preference is private information. Intensity is summarized by a value v_i representing the utility that individual i attaches to obtaining his preferred alternative, relative to the competing one: thus the utility experienced by i as a result of the committee's decision is v_i if i's preferred alternative is chosen, and 0 if it is not. We will use *intensity* and *value* interchangeably. Individual values are independent draws from a common and commonly known distribution $F(v)$ with support $[0, 1]$. We call $\mathbf{v}$ the vector of realized values.

Each individual is endowed with one indivisible vote. The group decision is taken through majority voting. Prior to voting, however, individuals can purchase or sell votes among themselves in exchange for a numeraire. The trade of a vote is an actual transfer of the vote and of all rights to its use. We normalize each voter's endowment of the numeraire to zero and allow borrowing at no cost. The important point is that no voter is budget constrained and all are treated equally. Individual utility u_i is given by:

$$u_i = v_i I + t_i \tag{1}$$

where I equals 1 if i's preferred decision is chosen and 0 otherwise, and t_i is i's net monetary transfer, positive if i is a net seller of votes, or negative if he is a net buyer.

With two alternatives and a single voting decision, voting sincerely is always a weakly dominant strategy, and we restrict our attention to sincere voting equilibria: after trading, each individual casts all votes in his possession, if any, in support of the alternative he prefers. Our focus is on the vote trading mechanism. We are interested in a competitive spot market for votes.

We allow for probabilistic (mixed) demands for votes. Let $S = \{s \in \mathbb{Z} \geq -1\}$ be the set of possible pure demands for each agent, where $\mathbb{Z}$ is the set of integers, and a negative demand corresponds to supply: agent i can offer to sell his vote, do nothing, or demand any positive integer number of votes. The set of strategies for each voter is the set of probability measures on S, ΔS, denoted by Σ. Elements of Σ are of the form $\sigma : S \to \mathbb{R}$ where, for each voter, $\sum_{s\in S}\sigma(s)=1$ and $\sigma(s)\geq 0$ for all $s\in S$.

If individuals adopt mixed strategies, the aggregate amounts of votes demanded and of votes offered need not coincide ex post. A rationing rule R maps the profile of voters' demands to a feasible allocation of votes. We denote the set of feasible vote allocations by $X=\left\{x\in\mathbb{Z}_+^n \middle| \sum x_i=n\right\}$. The rule R is a function from realized demand profiles to the set of probability distributions over vote allocations: $R:S^n\rightarrow\Delta X$. We require: $R(s)(x)=0$ if: $\exists i\in\{1,..,n\}|x_i>1+s_i$ and $s_i\geq 0$; or $\exists i\in\{1,..,n\}|x_i=0$ and $s_i\geq 0$; or $\exists i\in\{1,..,n\}|x_i\notin\{0,1\}$ and $s_i=-1$; we require: $R(s)(x)=1$ if $\sum s_i=0$ and $x=1+s$. In words, no voter with positive demand can be required either to buy more votes than he demanded, or to sell his vote; no voter who offered his vote for sale can be required to buy votes, and all demands must be respected if they are all jointly feasible.

The particular mixed strategy profile, $\sigma\in\Sigma$, and the rationing rule, R, imply a probability distribution over the set of final vote allocations that we denote as $r_{\sigma,R}(x)$. For every possible allocation, we denote by $\varphi_{i,x}$ the probability that the committee decision coincides with voter i's favorite alternative. Thus, given some strategy profile σ, the rationing rule R, a vote price p, and equation (1), the ex ante expected utility of voter i is given by:

$$U_i(\sigma,R,p)=\sum_{x\in X}r_{\sigma,R}(x)\left[\varphi_{i,x}v_i-(x_i-1)\,p\right] \tag{2}$$

Each individual makes his trading and voting choices so as to maximize (2).

3 The Equilibrium

We borrow the concept of *ex ante competitive equilibrium* from Casella, Llorente-Saguer and Palfrey (2012), (CLP from now onward). To allow for the existence of mixed strategies, the traditional requirement of market balance is substituted with the weaker condition that market demand and supply coincide *in expectation*. The discipline imposed by market equilibrium is softened to the requirement that deviations from market balance be unsystematic and unpredictable.

With two opposing groups of known and different sizes, best response strategies will generally differ between the two groups. As a result, even when demands are anonymous, if the equilibrium exists, it will convey information about the direction of preferences associated to each demand, and individual strategies will take that information into account. In the spirit of rational expectations models, we call an equilibrium *fully revealing* if either: (1) the equilibrium price, together with the set of others' equilibrium strategies and market equilibrium, fully convey to voter i the direction of preferences associated to each demand; or (2) the information conveyed is partial but voter i has a unique best response, identical to his best response under full information. Thus in a fully revealing equilibrium the price and individual strategies are identical to what they would be with full information. Define $\sigma_i^*(\mathbf{v})$ as individual i's equilibrium strategy when all preferences are known, where $\mathbf{v}$ stands for the vector of realized intensity values. Then:

Definition 1 *The vector of strategies σ^* and the price p^* constitute a fully revealing ex ante competitive equilibrium relative to rationing rule R if the following conditions are satisfied:*

1. *For each agent i, σ_i^* satisfies*

$$\sigma_i^* \in \arg\underset{\sigma_i \in \Sigma}{Max} U_i\left(\sigma_i, \sigma_{-i}^*, R, p^*\right)$$

2. *In expectation, the market clears, i.e.,*

$$\sum_{i=1}^{n} \sum_{s \in S} \sigma_i^*(s) \cdot s = 0$$

3. *Given $\{\sigma_{-i}^*,\, p^*\}$ and the knowledge that the equilibrium is fully revealing,*

$$\sigma_i^* = \sigma_i^*(\mathbf{v}) \text{ for all } i.$$

In equilibrium, individuals select strategies that maximize their expected utility, given the strategies used by others and the price. Demands are interdependent and best-respond to others' demands. In a market for votes, such interdependence is inevitable because the value of a vote depends on the full profile of votes allocation.[6] In competitive equilibrium theory, it is found in analyses of contributions to public goods (for example, Arrow and Hahn. 1971, pp.132-6). In the present setting, with two opposite groups of different sizes, the interdependence of demands plays a second important role. Together with the price, it supports the information revelation that occurs in equilibrium. Surveying the literature

[6] As a transparent example, all remaining votes have zero value if one voter holds a majority on his own.

on the existence of rational expectations equilibria, Allen and Jordan (1998) identify the "competitive message"–the price and the set of others' demands–as the smallest possible information message that supports a fully revealing equilibrium.

In our competitive market, demands are known but anonymous. What is crucial is not the full knowledge of realized values but an individual's ability to associate a demand σ_j with a direction of preferences for voter j. In a fully revealing equilibrium, others' strategies and the price are sufficient to convey such information and thus to identify uniquely one's own best response strategy. An important corollary is that if a fully revealing equilibrium exists, then it is also an equilibrium of the complete information game. We have assumed above that individual intensities of preferences are private information. But everything we say below will apply identically if all preferences are publicly known.[7]

In general, the existence and the characterization of the equilibrium will depend on the rationing rule. The selection of any particular rule is necessarily debatable, but we concentrate on a rule that seems well-suited to a market for votes. First, in line with the spirit of competitive markets, it is anonymous: every demand is treated equally, independently of the identity and preferences of the trader, or of the amount of the demand. Second, the rule does not require voters to accept and pay for partially filled orders: any individual demand is either satisfied in full or not at all. Being held responsible for partial orders is an expensive proposition in a market for votes where a single vote may make the difference between control and irrelevance. As in CLP, we call such a rule $R1$ or Rationing-by-Voter: any outstanding positive order for votes is equally likely to be selected; it is then either satisfied in full, if there exists sufficient outstanding supply; or not at all, in which case the voter exits the market; a second positive order is then randomly selected from those remaining, and the process continues until either all orders are satisfied or the only orders left outstanding are all unfeasible. Formally: call $\pi_i(s)$ the probability of being recognized when s are realized demands, $n^{R1}_{s>0}$ the number of voters with outstanding positive demands, and n_{-1} the number of voters with outstanding offers to sell. Then $\pi_i(s) = 1/n^{R1}_{s>0}$ if $s_i > 0$; $\pi_i(s) = 1/n_{-1}$ if $s_i = -1$, and $R(s)(x) = 0$ if $\exists i \in \{1, .., n\} | x_i \notin \{1, 1 + s_i\}$, where the last condition just states that no allocation of votes has positive probability unless all individuals either fulfill their full demand or do not trade at all. At the end of the paper, we discuss the conditions

[7]Note that the reverse does not hold: an equilibrium of the full information game need not be a fully revealing equilibrium of the incomplete information game, because it may be impossible for an agent to extract all relevant information.

under which our results are robust to an alternative rationing rule where each vote offered for sale is randomly allocated to any voter with positive demand, and orders can be filled only partially. Up to that point, all our results are to be read as *relative to rationing rule* $R1$.[8]

An equilibrium with no trade always exists–if no-one else is trading, an individual is rationed with probability one–and is, trivially, fully revealing–strategies are identical to what they would be with full information. Our interest is in equilibria with trade.

If an equilibrium existed in pure strategies, market balance would hold not only ex ante but ex post, and no rationing would occur. We need to allow for mixed strategies and ex ante equilibrium because in a market for votes with two opposing groups of known sizes, no fully revealing competitive equilibrium with trade exists in pure strategies. This result is well-known but we reproduce it here because it is the point of departure of our analysis.

Remark[9]**.** *For all n odd, m, F, and $\mathbf{v}$, there is no price p^* and vector of strategies $s^*(\mathbf{v}, p^*)$ such that $s_i^*(\mathbf{v}, p^*) = \arg\underset{s_i \in S}{Max} U_i\left(s_i, s_{-i}^*, p^*\right)$ for all i and $\sum_i s_i^*(\mathbf{v}, p^*) = 0$, unless $s_i^*(\mathbf{v}, p^*) = 0$ for all i.*

The logic is simple. If there is trade, for all $p > 0$, $\sum_{i \in m} s_i^*(\mathbf{v}, p) \in \{-m, (M-m+1)/2\}$: if the aggregate demand of minority voters is positive, it must equal the minimum number of votes required to win; alternatively, at any positive price all losing votes must be offered for sale. But $\sum_{i \in M} s_i^*(\mathbf{v}, p) \leq 0$: in equilibrium, the aggregate demand by majority voters cannot be positive. In addition, $\sum_{i \in M} s_i^*(\mathbf{v}, p) \neq -(M-m+1)/2$: if $(M-m+1)/2$ votes were traded, the remaining $(M+m-1)/2$ votes collectively held by M voters would be worthless and thus offered for sale too. Thus for all $p > 0$, $\sum_{i \in m} s_i^*(\mathbf{v}, p) + \sum_{i \in M} s_i^*(\mathbf{v}, p) \neq 0$. If $p = 0$, $\sum_{i \in m} s_i^*(\mathbf{v}, p) \geq (M-m+1)/2$[10], but $\sum_{i \in M} s_i^*(\mathbf{v}, p) \geq -(M-m-1)/2$, because the only supply can come from M voters whose vote is not pivotal. Thus for $p = 0$, $\sum_{i \in m} s_i^*(\mathbf{v}, p) + \sum_{i \in M} s_i^*(\mathbf{v}, p) > 0.\square$

The question this paper addresses then is whether a fully revealing *ex ante* competitive equilibrium with trade exists, given the knowledge of m and M.

[8] $R1$ resembles All-or-Nothing (AON) orders used in securities trading: the order is executed at the specified price only if it can be executed in full. See for example the description of AON orders by the New York Stock Exchange http://www.nyse.com/futuresoptions/nysearcaoptions/

[9] Ferejohn (1974), Philipson and Snyder (1996), Piketty (1994), Kultti and Salonen (2005), Casella, Palfrey, Turban (2012)

[10] We are assuming that at $p = 0$, voters on the losing side demand rather than sell votes. This is equivalent to the standard assumption that goods are in excess demand at 0 price.

3.1 An example

It is useful to begin by discussing a simple example. Call $\overline{v}_m$ the highest realized value in the minority group, group m, and $\overline{v}_M$ the highest realized value in group M, and call $v_{(2)m}$ and $v_{(2)M}$ the second highest realized values in each of the two groups.

Example 1. *Suppose* $\mathbf{v}$ *is such that* $\overline{v}_m \geq \overline{v}_M \geq v_{(2)m}$. *Then for all* n *odd,* m*, and* F*, there exists a fully revealing ex ante equilibrium with trade where:* $p = (2\overline{v}_M)/(n+1)$*;* $\overline{v}_m$ *demands* $(n-1)/2$ *votes;* $\overline{v}_M$ *randomizes between demanding* $(n-1)/2$ *votes (with probability* $(n-1)/(n+1)$*) and selling his vote; and all other individuals offer their vote for sale.*

The ranking $\overline{v}_m \geq \overline{v}_M \geq v_{(2)m}$ means that the (weakly) highest intensity voter belongs to the minority and that there exist two voters with intensities not lower than anyone else who disagree. The result is implied by Theorem 1 below and proved there. It states that in this example there exists an equilibrium where the *only* two voters with some probability of positive demand are the highest-value individuals in the two groups, and the only amount demanded is such that, if demand is satisfied, the individual will hold a majority of votes. The equilibrium does not depend on F, or, strikingly, on m, the size of the minority. Nor does the equilibrium depend on the value ranking of the other minority and majority members, besides requiring $\overline{v}_m \geq \overline{v}_M \geq v_{(2)m}$. For clarity, recall that values are private information: the group membership of the two highest-value voters, the values' ranking, and $\overline{v}_M$ are revealed in equilibrium.

The market equilibrium recalls an auction for dictatorship: the equilibrium price is pinned down by the condition that $\overline{v}_M$, the individual with second-highest value if the inequalities are strict, is indifferent between selling his vote and demanding dictatorship. Equilibrium strategies are identical to those characterized in CLP, a surprising result because the information about preferences is very different in the two models: in CLP each voter is expected to favor either option with equal probability; here the size of the group favoring each alternative is known, and the two groups may be very unequal.[11] The conclusion that the vote market does not allocate votes somewhat smoothly among higher value individuals seems counterintuitive, but the robustness of the result to the different information assumptions in the two models suggests a central aspect of markets for votes. Votes have no value in them-

[11] The different assumption explains the difference in the equilibrium price. In Example 1, the second-highest value voter is sure to lose the election if rationed; in CLP, if rationed, he loses only with probability 1/2, the probability that his competitor for votes disagrees with him. Hence the price he is willing to pay in the first case is double the price he is willing to pay in the second.

selves, and in this equilibrium a well-functioning market for votes approximates a market for decision power. In the absence of income constraints, the market allocates decision power to one of the two individuals with the highest incentive to compete for it.

An implication of this result is that the equilibrium allocation of decision power is *independent* of the relative size of the minority, and of the values of all voters but the two highest. The equilibrium holds whether $m = 1$ or $m = M - 1$, and for any m it holds whether large differences in values are possible, or all voters' values are equal, as long as $\overline{v}_m \geq \overline{v}_M \geq v_{(2)m}$.[12]

Four implications follow immediately. They are particularly transparent in this example, but will continue to be valid in the more general case. First, if we define efficiency in utilitarian terms–as the allocation of decision power to the group with higher aggregate value–then there can be no presumption that the market for votes will be efficient. More strongly, there can be no presumption that the market for votes is more efficient than no trading, or simple majority voting. For example, if all values are equal, it is clearly superior to let the majority win, an outcome that in the equilibrium characterized here the market delivers only if the highest value majority voter demands votes and is not rationed, or with probability $(n-1)/[2(n+1)]$, smaller than 50 percent.

Second, note that in such a case the only realized purchases of votes are by a majority member. The result is less paradoxical than it seems: all other majority members are offering their votes for sale, and the highest value majority member buys to prevent the transfer of votes to the minority. Preemptive purchases by the majority are very plausible–any sponsor of a bill needs to worry about the support of his weakest allies. But to our knowledge they have no role in usual formalizations of vote trading.

Third, and related, the equilibrium predicts intra-group trading with probability that for all m and M is positive and high. Again, most voters are offering their vote for sale, and high value individuals need to preempt sales to the opposite group by their own weak allies.

Finally, unless *all* of one's group votes are purchased, the winning majority will be larger than the minimal winning coalition. Thus in general the equilibrium predicts super-majority, a counter-intuitive result in a market for votes where votes command a positive price. Note that the result holds although the sizes of the two groups are known, and thus the number

[12] Even if all values are equal, the strategies of the two voters with positive demands cannot be interchanged. If $v_i = v_j = v$, there is no equilibrium where $v_i \in M$ demands $(n-1)/2$ votes, $v_j \in m$ randomizes between selling and demanding $(n-1)/2$ votes, and all other sell. See Theorem 1. The reason is the main theme of this paper: in the market, minority equilibrium strategies are systematically more aggressive.

of additional votes the minority needs to win is common knowledge.

3.2 Two theorems

In this section we extend the example studied above to less restrictive configurations of realized values. Theorem 1 characterizes an ex ante equilibrium for a large range of value realizations. Theorem 2 shows that although the range is not exhaustive, with large electorates realized value configurations must fall under Theorem 1 with probability arbitrarily close to 1.

Given realized values $\mathbf{v}$, we denote by $v_{(1)}$ the highest realized value; by $G \in \{m, M\}$ the group such that $v_{(1)} \in G$–the group to which the highest intensity individual belongs–, and by g the opposite group. We call $\overline{v}_G$ ($\overline{v}_g$) the highest realized value in G (g) (thus by definition $\overline{v}_G = v_{(1)}$). Finally, we denote by $v_{(2)G}$ the second highest value in G: $v_{(2)G} = \max\{v_i \in G, v_i \neq \overline{v}_G\}$.[13]

Theorem 1. *For all n odd, m, and F, there exists a threshold $\mu(n) \in (0,1)$ such that if $\overline{v}_g \geq \mu(n)v_{(2)G}$, there exists a fully revealing ex ante equilibrium with trade where $\overline{v}_G$ and $\overline{v}_g$ randomize between demanding $(n-1)/2$ votes (with probabilities $q_{\overline{G}}$ and $q_{\overline{g}}$ respectively) and selling their vote, and all other individuals sell. The randomization probabilities $q_{\overline{G}}$ and $q_{\overline{g}}$ and the price p depend on $\overline{v}_g$ and $\overline{v}_G$, but for all $\overline{v}_g \geq \mu(n)v_{(2)G}$, and $\overline{v}_G$, $q_{\overline{G}} \in [\frac{n-1}{n+1}, 1]$ and $q_{\overline{g}} \in [\frac{n-1}{n+1}, 1]$. The threshold $\mu(n)$ is given by:*

$$\mu(n) = \begin{cases} \frac{2}{3} & \text{if } n = 3 \\ \max\left\{\frac{(n-2)(n-1)}{2(n^2+n-5)}, \frac{(n-2)(n-1)(n+1)}{2(n^3+3n^2-19n+21)}\right\} & \text{if } n > 3 \end{cases}$$

The theorem is proved in the Appendix, where we also report the explicit solutions for $q_{\overline{G}}$, $q_{\overline{g}}$ and p. The expressions are simple but not very enlightening because of the need to consider different cases, depending on the realizations of $\overline{v}_g$ and $\overline{v}_G$. An important observation is that $\mu(n) < 1$ for all n, and $\mu(n) < 1/2$ for all $n > 3$.[14] The condition $\overline{v}_g \geq \mu(n)v_{(2)G}$ is necessary and sufficient for the existence of the equilibrium characterized in the theorem, and is thus

[13] Throughout the paper, we use v_i to denote the value of i but also occasionally, with abuse of notation, the name of voter i. We use the notation $v_{(1)}$ to indicate the highest draw, as opposed to the more standard $v_{(n)}$, for consistency with $v_{(2)G}$.

[14] For all $n > 3$, $\mu(n)$ is increasing in n, and approaches $1/2$ asymptotically for n arbitrarily large.

sufficient for the existence of a fully revealing ex ante equilibrium with trade.[15]

Since $\mu(n) < 1$ for all n, the example discussed in the previous subsection ($G = m$, $g = M$, and $\overline{v}_g \geq v_{(2)G}$) satisfies the condition of the theorem, and the equilibrium characterized here applies. The condition $\overline{v}_g \geq \mu(n)v_{(2)G}$ is weaker than $\overline{v}_g \geq v_{(2)G}$ and is compatible with multiple highest-ranked value realizations all belonging to members of the same group, but the qualitative properties of Example 1 extend to the more general case. In particular: the market for votes amounts to a competition for dictatorship between $\overline{v}_M$ and $\overline{v}_m$, and after rationing there always is one dictator; the price is always such that $\min(\overline{v}_M, \overline{v}_m)$ is indifferent between selling his vote and demanding dictatorship; finally, the market results in intra-group trade and in a super-majority with strictly positive probability. Given n, both the existence and the properties of the equilibrium depend exclusively on the realizations of three values, $\overline{v}_g$, $\overline{v}_G$, and $v_{(2)G}$. If the equilibrium exists, both the strategies and the price are fully independent of all other realized values, and, surprisingly, of the size of the two groups.

Figure 1 represents graphically the areas over which the equilibrium described in Theorem 1 exists and uses different colors to describe the equilibrium mixing probabilities. In all panels, the vertical axis measures $\overline{v}_g/\overline{v}_G$ and the horizontal axis $v_{(2)G}/\overline{v}_G$, and thus both axes range between 0 and 1. The panels on the left are drawn for the case $G = m$–the highest value realization belongs to a minority voter–and the panels on the right for $G = M$–the highest value realization belongs to a majority voter. The upper panels correspond to $n = 9$, and the lower panels, in both columns, to $n = 21$**.** Because the existence and characterization of the equilibrium do not depend on the size of the minority, the figure applies for *any* $m < M$**,** as long as $v_{(2)m}$ exists and thus $m \geq 2$.[16]

In all panels, the equilibrium exists above the diagonal line $\overline{v}_g = \mu(n)v_{(2)G}$. Blue areas correspond to an equilibrium where the highest intensity minority voter, $\overline{v}_m$, demands $(n-1)/2$ votes with probability 1; the highest intensity majority voter, $\overline{v}_M$, demands $(n-1)/2$ votes with probability $(n-1)/(n+1)$ and sells his vote otherwise, and all other voters sell. Note that such an equilibrium exists not only when the highest value belongs to the minority (the panels on the left) but also when the highest value belongs to the majority (the panels on the right) as long as $\overline{v}_m$ is high enough, relative to $\overline{v}_M$–higher than a value

[15] Theorem 1 does not state that no fully revealing equilibrium with trade exists if $\overline{v}_g < \mu(n)v_{(2)G}$, and in a specific example ($M = 3$, $m = 2$), we have constructed an equilibrium when the condition is violated (Casella, Palfrey and Turban, 2012).

[16] If $m = 1$, the panel on the right ($G = M$) is unchanged; the panel on the left ($G = m$) has no white area in the lower right corner because the condition $\overline{v}_M > \mu(n)v_{(2)m}$ is trivially satisfied.

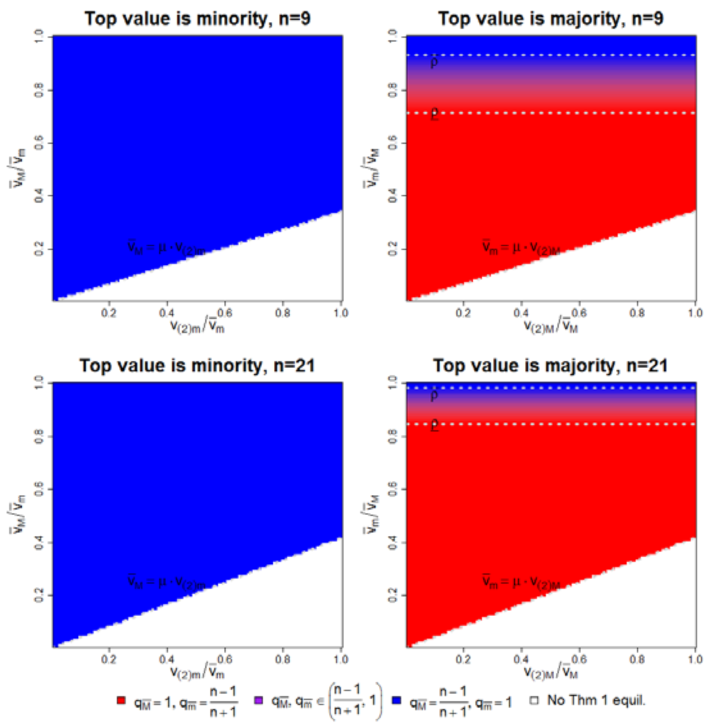

Figure 1: Equilibrium strategies in Theorem 1, as function of $\overline{v}_G$, $\overline{v}_g$, and $\overline{v}_{(2)G}$.

$\overline{\rho}(n)\overline{v}_M < \overline{v}_M$ that appears as the upper horizontal line in the panels on the right. The red area corresponds to an equilibrium where $\overline{v}_M$ demands $(n-1)/2$ votes with probability 1; $\overline{v}_m$ demands $(n-1)/2$ votes with probability $(n-1)/(n+1)$ and sells his vote otherwise, and all other voters sell. Such an equilibrium exists when the highest value belongs to the majority and $\overline{v}_m$ is low enough, relative to $\overline{v}_M$–lower than a value $\underline{\rho}(n)\overline{v}_M < \overline{\rho}(n)\overline{v}_M$ that appears as the lower horizontal line in the panels on the right. Both $\underline{\rho}(n)$ and $\overline{\rho}(n)$ are defined precisely in the Appendix; for all n they satisfy $1/2 \leq \underline{\rho}(n) < \overline{\rho}(n) < 1$, and both converge to 1 at large n. Finally, in the purple area, for $\overline{v}_m \in (\underline{\rho}(n)\overline{v}_M < \overline{\rho}(n)\overline{v}_M)$, both $\overline{v}_m$ and $\overline{v}_M$ randomize between demanding $(n-1)/2$ votes, with probabilities $q_{\overline{m}}$ and $q_{\overline{M}}$ strictly between $(n-1)/(n+1)$ and 1, and selling their vote, and all others sell. The values of $\mu(n)$, $\underline{\rho}(n)$, and $\overline{\rho}(n)$, and thus the exact borders between the different areas, depend on n, but qualitatively the figure is unchanged for all n.

Figure 1 represents vividly the equilibrium strategies in Theorem 1, but could be misleading. It is important to note that the relative size of an area in the figure is *not* informative about the probability with which values in the area are realized. The figure's axes report relative values of order statistics whose realizations depend on F, n, and the size of the two groups, m and M. Figure 2 reports the same panels drawn in Figure 1, now using shading to represents probability mass: darker areas correspond to value realizations with higher probability. The probabilities were obtained from one hundred million simulations of random independent draws from a uniform distribution, fixing $m = (1/3)n$. As in Figure 1, the upper panels report results for $n = 9$, and the lower panels for $n = 21$; the left panels are drawn for the case $G = m$ and the right panels for $G = M$. Because the minority is by definition small, realizations in the right panels always have higher probability than realizations in the left panels, as reflected in the slightly darker shades. Note that this does not imply that a majority victory is necessarily more probable than a minority victory. As described in Figure 1, at equal relative values, in equilibrium $\overline{v}_m$ demands votes more aggressively than $\overline{v}_M$.

Figure 2 shows two main regularities: first, in each panel, the probability mass is concentrated in the upper right corner; second, the concentration is stronger at higher n.[17] The figure gives a clear visual representation, but both results can be obtained analytically. As shown in Figure 1, the realizations of $\overline{v}_g$, $\overline{v}_G$, and $v_{(2)G}$ that support the equilibrium of Theorem 1 can be divided into three areas, corresponding to the different mixing probabilities: blue (B), where $q_{\overline{m}} = 1$, red (R), where $q_{\overline{M}} = 1$, and purple (P), where both $q_{\overline{m}}$ and $q_{\overline{M}} \in (\frac{n-1}{n+1}, 1)$. Call $\Pr(B)$ the probability of value realizations in B, and similarly for R and P. Thus:

$$\begin{aligned}
\Pr(B) &= \Pr(\overline{v}_m \geq \overline{\rho}\overline{v}_M, \overline{v}_M \geq \mu v_{(2)m}) \\
\Pr(P) &= \Pr(\underline{\rho}\overline{v}_M < \overline{v}_m < \overline{\rho}\overline{v}_M) \\
\Pr(R) &= \Pr(\overline{v}_m \leq \underline{\rho}\overline{v}_M, \overline{v}_m \geq \mu v_{(2)M})
\end{aligned}$$

Given F, the different probabilities can be calculated. Suppose, for example, that F is

[17] The different patterns in the left and right panels reflect the different sizes of the two groups. Because $M > m$, $\overline{v}_M$ is likely to be higher than $v_{(2)m}$ (and thus the probability mass in the left panels concentrates around the upper horizontal boundary), and $v_{(2)M}$ is likely to be higher than $\overline{v}_m$ (and thus the probability mass in the right panels concentrates around the upper vertical boundary).

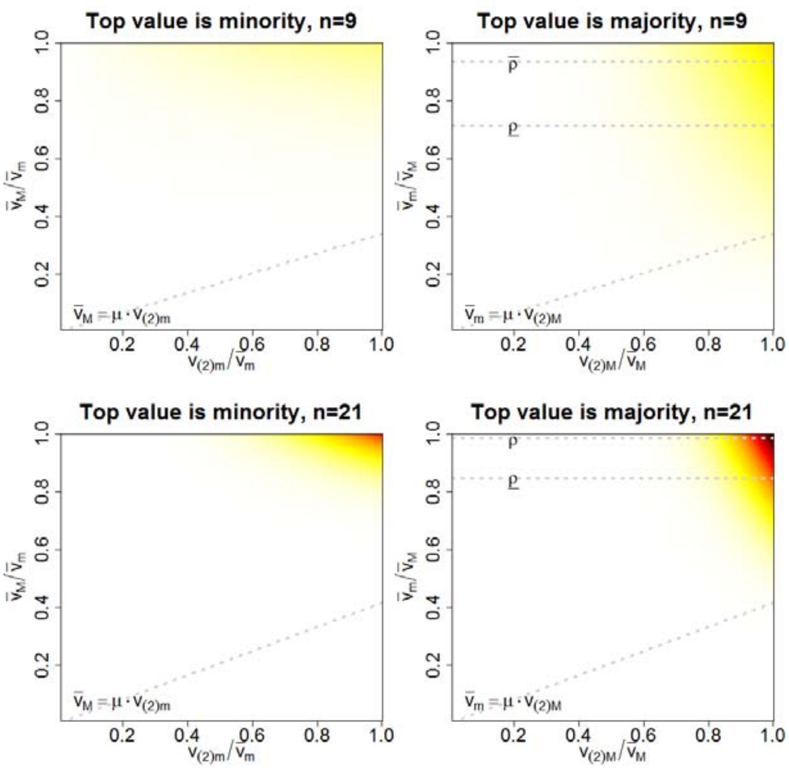

Figure 2: Probability of ordered value realizations; $F(v)$ uniform. A darker shade indicates higher probability.

uniform. Then:[18]

$$
\begin{aligned}
\Pr(B) &= 1-\frac{m(m-1)}{n(n-1)}\mu^M-\frac{M}{n}\overline{\rho}^m \\
\Pr(P) &= \frac{M}{n}(\overline{\rho}^m-\underline{\rho}^m) \\
\Pr(R) &= \underline{\rho}^m\frac{M}{n}-\frac{M(M-1)}{n(n-1)}\mu^m
\end{aligned} \tag{3}
$$

Specific values of n and m will then yield precise numerical values. For example, if $n = 9$ and $m = 3$, as in the upper panels of Figure 2, the probability of falling in the blue area is 47.8 percent, in the red area is 22.6 percent, and in the purple area 29.9 percent. Thus the probability of value realizations for which the equilibrium of Theorem 1 does not exist is 1.7 percent. At $n = 21$ and $m = 7$, as in the lower panels of Figure 2, the numbers become: $\Pr(B) = 0.401$, $\Pr(P) = 0.392$, and $\Pr(R) = 0.206$; the probability of value realizations that do not support the equilibrium of Theorem 1 is less than 1 in $1,000$.

As we increase n, both the concentration of probability mass in the upper right corner of each panel and the sharply decreased likelihood of realizations outside the equilibrium area are clear both from the figure and from the numbers. They arise from the increase

[18] Using our notation, call $x_{(1)}$ and $x_{(2)}$ the two highest order statistics out of n independent draws, where each variable is distributed according to the cumulative distribution function G_x, with density g_x. Then the joint density of $x_{(1)}$ and $x_{(2)}$, $g_{x_{(1)},x_{(2)}}$ is given by:

$$g_{x_{(1)},x_{(2)}}(x,y)=n(n-1)[G_x(x)]^{n-2}g_{x_{(1)}}(y)g_{x_{(2)}}(x)$$

where, calling $x_{(r)}$ the rth highest order statistics:

$$g_{x_{(r)}}(x)=\frac{n!}{(n-r)!(r-1)!}[G_x(x)]^{n-r}[1-G_x(x)]^{r-1}g_x(x)$$

See Gibbons and Chakraborty, (2003).

The expressions in (3) are obtained from solving the integrals in:

$$
\begin{aligned}
\Pr(B) &= \int_{\overline{v}_M=0}^{1}\int_{v_{(2)m}=0}^{\min\left(\frac{\overline{v}_M}{\mu},1\right)}\int_{\overline{v}_m=\max\left(v_{(2)m},\overline{\rho}\overline{v}_M\right)}^{1} m(m-1)(v_{(2)m})^{m-2}M(\overline{v}_M)^{M-1}d\overline{v}_m dv_{(2)m}d\overline{v}_M \\
\Pr(P) &= \frac{M}{M+m}(\overline{\rho}^m-\underline{\rho}^m) \\
\Pr(R) &= \int_{\overline{v}_m=0}^{\underline{\rho}}\int_{v_{(2)M}=0}^{\min\left(\frac{\overline{v}_m}{\mu},1\right)}\int_{\overline{v}_M=\max\left(v_{(2)M},\frac{\overline{v}_m}{\underline{\rho}}\right)}^{1} M(M-1)(v_{(2)M})^{M-2}m(\overline{v}_m)^{m-1}d\overline{v}_M dv_{(2)M}d\overline{v}_m
\end{aligned}
$$

in n and are independent, qualitatively, of the uniform distribution assumption used in these examples. If the minority is a non-vanishing fraction of the electorate[19], then with independent draws from any common distribution F, at large n, both $\overline{v}_g/\overline{v}_G$ and $v_{(2)G}/\overline{v}_G$ must approach the upper boundary of the distribution's support. At large n, the probability mass must concentrate towards the upper right corner of the panel. It then follows that when the electorate is large, the restriction on realized values required for the existence of the equilibrium described in Theorem 1 is almost certainly satisfied. Indeed this is our second result. Suppose $m = \lfloor \alpha n \rfloor$, for all n, where $\lfloor \alpha n \rfloor$ is the largest integer not greater than αn, and α is a constant in $(0, 1/2)$. We can state:

Theorem 2. *Consider a sequence of vote markets. For all n, $\alpha \in (0, 1/2)$, and F, for any $\delta > 0$ there exists a finite $n'(F, \delta)$ such that if $n > n'$, $\Pr_n(\overline{v}_g \geq \mu(n) v_{(2)G}) > (1 - \delta)$.*

The proof of the theorem is immediate. Given $\mu(n) < 1/2$, the theorem follows if $\lim_{n\to\infty} \Pr_n(\overline{v}_g > 1/2) = 1$. But $\lim_{n\to\infty} \Pr_n(\overline{v}_g > 1/2) = \lim_{n\to\infty} 1 - [F(1/2)]^{\lfloor \alpha n \rfloor} = 1$, and the result is established.□

The uniform distribution provides a transparent example. From (3):

$$\begin{aligned}
&\lim_{n\to\infty} \Pr{}_n(B) = \alpha \\
&\lim_{n\to\infty} \Pr{}_n(P) = (1-\alpha)(1 - e^{-4\alpha}) \\
&\lim_{n\to\infty} \Pr{}_n(R) = (1-\alpha) e^{-4\alpha}.
\end{aligned} \tag{4}$$

As predicted, $\lim_{n\to\infty}(\Pr(B) + \Pr(P) + \Pr(R)) = 1$. The explicit solution allows us to see how the probability of realizations in the three different areas depends on the relative size of the minority, α. As expected, the probability of $\overline{v}_m/\overline{v}_M$ realizations high enough to support the more aggressive minority strategy (the Blue area) increases monotonically with α; conversely, the probability of the more aggressive majority strategy (the Red area, or low enough $\overline{v}_m/\overline{v}_M$ realizations) falls monotonically with α; the intermediate case (the Purple area) is not monotonic in α. Figure 3 depicts system (4) graphically. The horizontal axis measures α; each line is drawn in the color of the area whose probability it represents.[20]

The uniform distribution provides a clean example, but Theorem 2 holds generally. It

[19] I.e. $\frac{m}{n}$ does not converge to 0 as $n \longrightarrow \infty$.

[20] The figure requires $\alpha < 1/2$. The equilibrium strategies are derived taking into account that the minority loses with probability 1 if no trade takes place. The discountinuity at $\alpha = 1/2$, where the red and blue curves must overlap, reflects the discontinuity of payoffs at the point of pivotality.

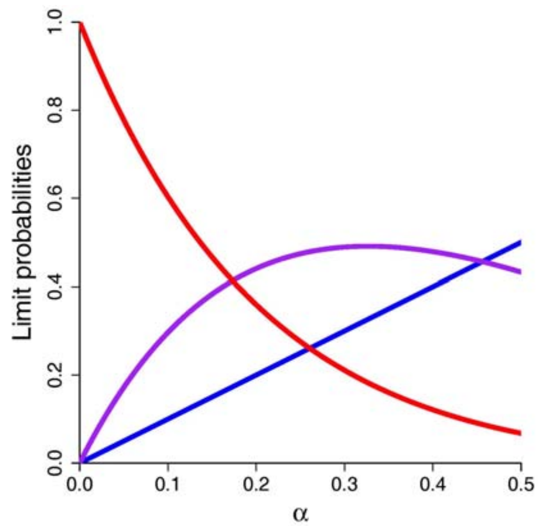

Figure 3: Asymptotic probability of the red area (in red), of the purple area (in purple) and of the blue area (in blue) as functions of the minority share α. $F(v)$ uniform.

implies that for large n the equilibrium described in Theorem 1 exists with probability that approaches 1. In addition, because in such an equilibrium the probabilities with which $\overline{v}_G$ *and* $\overline{v}_g$ demand $(n-1)/2$ votes are bound below by $(n-1)/(n+1)$, at large n both probabilities must also approach 1. Theorem 2 thus implies the following Corollary:

Corollary to Theorem 2. *For any* $\alpha \in (0, 1/2)$ *and* F, *for any* $\delta > 0$ *there exists a finite* $n''(F, \delta)$ *such that for all* $n > n''$ $q_{\overline{m},n} > (1-\delta)$, and $q_{\overline{M},n} > (1-\delta)$.[21]

4 Frequency of minority victories and welfare

The most unexpected feature of Theorem 1 is that when the equilibrium exists the market outcome depends on the size of the minority only indirectly. It is easy to see why: only two market participants demand votes with positive probability–the highest-value minority voter and the highest-value majority voter–and the only quantity demanded is such that, if demand is satisfied, the voter will hold a strict majority of the votes. Given these strategies, the relative size of the two groups is irrelevant to the outcome, i.e. to which group exits the market controlling a majority of votes. Group size matters indirectly because it affects the

[21] Because n' in Theorem 2 depends on F, we allow for the possibility that $n" > n'$.

probability of different value rankings, and thus the existence of the trading equilibrium and the equilibrium mixing probabilities of the two high-value voters. Given the single highest realized value in each group, however, if the equilibrium exists the expected outcome is the same whether there is a single minority voter or the minority comprises almost half of the electorate. This result suggests a systematic vote market bias in favor of the minority group: a higher frequency of minority victories than efficiency dictates.

To evaluate this conjecture, we need to construct an equilibrium that exists for all value draws, and define an efficiency benchmark. Since an equilibrium with no trade exists trivially for all value realizations, we can construct an equilibrium such that if $\overline{v}_g \geq \mu(n)v_{(2)G}$, then trade occurs and the equilibrium of Theorem 1 is selected; if $\overline{v}_g < \mu(n)v_{(2)G}$, then no vote-trading takes place and the majority wins with probability 1. Our equilibrium construction thus minimizes the frequency of minority victories when the condition is not met.[22] We call ω_m the ex ante expected frequency of minority victories in such an equilibrium, before values are drawn: $\omega_m \equiv \Pr_F(\sum_{i\in m} x_i(\mathbf{v}) > \sum_{j\in M} x_j(\mathbf{v}))$

In line with the anonymity of the competitive market and of majority voting, we measure efficiency by ex ante efficiency, treating each voter identically–expected utility before the voter knows the group he belongs to and before values are drawn. Ex ante efficiency is equivalent to the utilitarian criterion: it is maximized when, for each realization of values, the group with higher aggregate value prevails. We call W the ex ante expected utility in the equilibrium we have constructed; W_0 the ex ante expected utility in the absence of vote trading (i.e. with simple majority voting), and W^* the ex ante expected utility under full efficiency. Finally, we call ω_m^* the expected frequency of majority victories under our efficiency benchmark: $\omega_m^* \equiv \Pr_F(\sum_{i\in m} v_i > \sum_{j\in M} v_j)$.

We begin by establishing a preliminary result. Because it can be of some general interest, we report it here as a separate lemma.

Lemma 1. *If all* v_i, $i \in m$ and $i \in M$, *are i.i.d. according to some* $F(v)$*, then for all* F*,* n*, and* m *,* $\omega_m^* \leq m/n$*.*

The lemma is proved in the Appendix. It states that if values are i.i.d., then for any distribution F the expected share of value configurations such that the aggregate minority value is larger than the aggregate majority value, and thus a minority victory is efficient, cannot be larger than the share of the minority in the electorate. The statement is intuitive

[22] As noted earlier, equilibria with trade may exist when $\overline{v}_g < \mu(n)v_{(2)G}$, in which case the expected fraction of minority victories must be weakly higher than in our equilibrium construction.

and it is useful here because it establishes an upper bound for ω_m^* that holds for all F, n, and m and can be compared to ω_m, the equilibrium fraction of expected minority victories.

Conditional on value realizations, ω_m is either characterized precisely by the strategies in Theorem 1, or equals 0, by our equilibrium construction, if the condition in Theorem 1 is not satisfied. In particular, because under Theorem 1 the final votes' allocation depends only on the probability with which $v_{\overline{m}}$ and $v_{\overline{M}}$ demand votes, we can write:

$$\omega_m = \begin{cases} q_{\overline{m}}(1-q_{\overline{M}}) + (1/2)q_{\overline{m}}q_{\overline{M}} & \text{if } \overline{v}_g \geq \mu(n)v_{(2)G} \\ 0 & \text{if } \overline{v}_g < \mu(n)v_{(2)G} \end{cases}$$

where the equilibrium values of $q_{\overline{m}}$ and $q_{\overline{M}}$ depend on the realized values. It is convenient to refer to the regions of the value space according to their color in Figure 1: recall that Blue (B) corresponds to value realizations such that $\overline{v}_M \in \left[\mu(n)v_{(2)m}, \overline{v}_m/\overline{\rho}(n)\right]$; Red ($R$) corresponds to $\overline{v}_m \in \left[\mu(n)v_{(2)M}, \underline{\rho}(n)\overline{v}_M\right]$, and Purple ($P$) to $\overline{v}_m \in [\underline{\rho}(n)\overline{v}_M, \overline{\rho}(n)\overline{v}_M]$. Then:

$$q_{\overline{m}} = \begin{cases} 1 \text{ if } B \\ \frac{n-1}{n+1} \text{ if } R \\ q'_{\overline{m}} \in (\frac{n-1}{n+1}, 1) > \frac{n-1}{n+1} \text{ if } P \end{cases} \qquad q_{\overline{M}} = \begin{cases} \frac{n-1}{n+1} \text{ if } B \\ 1 \text{ if } R \\ q'_{\overline{M}} \in (\frac{n-1}{n+1}, 1) < 1 \text{ if } P \end{cases}$$

Hence:

$$\omega_m \geq \left[\left(1-\frac{n-1}{n+1}\right)+\frac{1}{2}\left(\frac{n-1}{n+1}\right)\right]\Pr(B) + \left[\frac{1}{2}\left(\frac{n-1}{n+1}\right)\right]\Pr(R) + \left[\frac{1}{2}\left(\frac{n-1}{n+1}\right)\right]\Pr(P)$$

with strong inequality if $\Pr(P) > 0$. Or:

$$\omega_m \geq \left(\frac{n+3}{2(n+1)}\right)\Pr(B) + \left(\frac{n-1}{2(n+1)}\right)[\Pr(R)+\Pr(P)] \equiv \underline{\omega_m} \tag{5}$$

The probability of realizations in the different regions of the value space depends on F, and thus so does ω_m. Yes, as we prove in the Appendix, some claims can be made without specifying F:

Proposition 1. (a) *There exists a value $m' > 0$ such that if $m < m'$, $\omega_m > \omega_m^*$ for all n and F.* (b) *There exist distributions $\mathbf{F}'$ such that if $F \in \mathbf{F}'$, $\omega_m > \omega_m^*$ for all n and m.*

As the proposition states, for any value distribution and electorate size, it is always possible to find a minority size for which the equilibrium we are studying predicts excessive

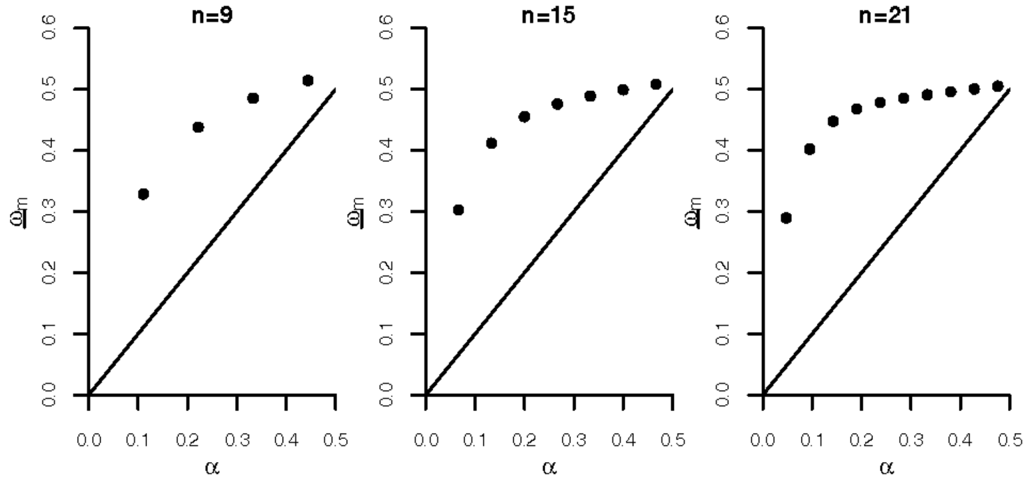

Figure 4: Lower bound on the probability of minority victories, as function of $\alpha = m/n$. $F(v)$ uniform.

minority victories. The proof shows that this must always be the case if $m = 1$. It relies on sufficient conditions that can be weakened substantially if we make specific assumptions on the shape of the value distribution. In particular, we show in the Appendix that if $F = v^b$, then for any $b \geq 1$, $\omega_m > \omega_m^*$ for any electorate size and for any minority size, supporting the second claim in the proposition.

Consider the example of F uniform ($b = 1$). Substituting (3) in (5), we obtain an explicit expression for $\underline{\omega_m}$, as function of n and m. Figure 4 plots $\underline{\omega_m}$, on the vertical axis, against $m/n \equiv \alpha$ on the horizontal axis, with $m = 1, .., (n-1)/2$. The different panels correspond to different values of n: $n = 9$, 15, and 21. In each panel, the 45^0 line thus equals $m/n = \alpha$, and by Lemma 1, since $\omega_m^* \leq m/n$, if $\underline{\omega_m} > m/n$, it follows that $\omega_m > \omega_m^*$. Thus in all cases in all three panels, $\omega_m > \omega_m^*$.

The figure shows that $\underline{\omega_m}$ can be surprisingly large. In all three panels, $\underline{\omega_m} > 1/2$ if $m = (n-1)/2 = M - 1$ and assumes unexpectedly high values even at low m/n. For example, if $m = 1$, $\underline{\omega_m}$ is 33 percent at $n = 9$ (when m is 11 percent of the voters) and remains almost 29 percent at $n = 21$ (when m is just below 5 percent of the voters).

Analyzing $\underline{\omega_m}$ when $F = v^b$, for arbitrary $b > 0$, provides a simple intuition for the role played by F. The higher is b, the larger the probability mass at high value realizations, the smaller the ratio $Ev_{(1)}/Ev$–the ratio of the expected highest order statistics to the mean–

and the smaller the probability that an unusually high value realization can compensate for the minority's smaller size. Hence the higher is b the lower is the probability that the aggregate minority value is higher than the aggregate majority value. Conversely, the lower is b, the larger the probability mass at low value realizations, the larger the ratio $Ev_{(1)}/Ev$, and the less important the relative size of the two groups in determining which group has higher aggregate value. Hence the lower is b, the less costly is the high frequency of minority victories built into the vote market.[23] Thus, as stated, if $b \geq 1$, $F = v^b \in \mathbf{F}'$ in Proposition 1 for all n and m, but there exists a $\underline{b} \in (0,1)$ such that for $b \leq \underline{b}$, $F = v^b \notin \mathbf{F}'$.

Proposition 1 establishes that there always are parameter values for which the first best solution has a lower probability of minority victory than the market delivers. The lack of full efficiency is not too surprising, given the externalities inherent in trading votes. But is the market inferior to majority voting with no trading? Are the excessive minority victories so costly that no minority victory at all, as delivered by simple majority voting, is in fact preferable? If n is small, the answer depends on the shape of the value distribution. We find:

Proposition 2. *There exist distributions $\mathbf{F}''$ such that if $F \in \mathbf{F}''$, then $W < W_0$ for all n and m.*

In principle, the set $\mathbf{F}''$ could be larger or smaller than the set $\mathbf{F}'$. The Appendix shows that $F = v^b$, with $b \geq 1$, belongs to both sets, i.e. it is such that for all n and m, not only $\omega_m > \omega_m^*$, but $W < W_0$. The intuition remains as discussed above: when the value distribution is such that the expected frequency of efficient minority victories is sufficiently low, ex ante welfare is higher when the minority always loses than with the pro-minority bias implied by the market.[24]

The complications tied to the specific shape of F disappear when the market is large. Both Propositions 1 and 2 become simpler and stronger. The point of departure is the Corollary to Theorem 2 in the previous section: if n is large, with probability approaching 1, realized values satisfy the condition in Theorem 1, and again with probability approaching 1, voters $\overline{v}_m$ and $\overline{v}_M$ both demand $(n-1)/2$ votes, while all other voters offer their votes for sale. An immediate and unexpected result then follows: the final outcome depends exclusively on

[23] To be clear: by Lemma 1, $\omega_m^* \leq m/n$ always, but the shape of F determines how close ω_m^* is to m/n. If $F = v^b$, $b > 0$, the higher is b, the closer ω_m^* is to 0; the lower is b, the closer ω_m^* is to m/n.

[24] The same logic implies, correctly, that for other value distributions the market can be welfare superior. An example is $F = v^b$ with $b = 0.1$ and $n = 7$. As discussed below, however, the market can be welfare improving only for small n.

which one of $\overline{v}_m$ and $\overline{v}_M$ has his order filled, and since both have identical chances, both win with equal probability. Theorem 2 and its Corollary directly imply:

Proposition 3. *Consider a sequence of vote markets. For all n, $m = \lfloor \alpha n \rfloor$, with $\alpha \in (0, 1/2)$. Then for any α, F, and $\delta > 0$ there exists a finite $n'''(F, \delta)$ such that for all $n > n'''$, $|\omega_n - 1/2| < \delta$.*[25]

At sufficiently large market size, the minority is expected to win with probability arbitrarily close to 1/2, for *any* minority share and for *any* distribution from which values are drawn. Given the previous results, the intuition is straightforward, but the result remains surprising. Whether the minority is 40 percent of the total electorate, 25 percent, or 10 percent, as long as it is not negligible, in a sufficiently large vote market there is an equilibrium such that the minority wins with probability 1/2 for any shape of the value distribution. After trade, the minority and the majority group are *equally likely* to control a majority of the votes. The market nullifies majority voting: following the will of the electorate becomes identical to flipping a coin.

The welfare implications are equally immediate. Call $V_m = \sum_{i \in m} v_i$ the aggregate minority value, $V_M = \sum_{j \in M} v_j$ the aggregate majority value, and Ev the expected value draw: $Ev = \int_0^1 v dF(v)$. With i.i.d. value draws, by the law of large numbers, with probability arbitrarily close to 1 both V_m/m and V_M/M converge to Ev. Thus, since $\alpha < 1/2$ and, ignoring integer constraints, $M/m = (1-\alpha)/\alpha > 1$, with probability arbitrarily close to 1, $V_M > V_m$: in a large electorate with i.i.d. value draws any minority victory is inefficient. The welfare loss implied by vote trading is easily quantified. Indexing variables to make their dependence on the size of the market clear, we can write:

$$\begin{aligned}
W_n &= \frac{M_n}{n} E_n\left(\frac{V_{M_n}}{M_n}\right)\left(\frac{1}{2}\right) + \frac{m_n}{n} E_n\left(\frac{V_{m_n}}{m_n}\right)\left(\frac{1}{2}\right) \\
W_{0n} &= \frac{M_n}{n} E_n\left(\frac{V_{M_n}}{M_n}\right) \\
W_n^* &= \frac{M_n}{n} E_n\left(\frac{V_{Mn}}{M_n} | V_{M_n} > V_{m_n}\right) \Pr(V_{M_n} > V_{m_n}) + \frac{m_n}{n} E_n\left(\frac{V_{m_n}}{m_n} | V_{m_n} > V_{M_n}\right) \Pr(V_{m_n} > V_{M_n})
\end{aligned}$$

[25] Denote G^n the CDF of the joint distribution of n values.

$$\lim_{n \to \infty} \omega_n = \lim_{n \to \infty} \int_{R \cup P \cup B} \left[q_{\overline{m}}(\mathbf{v}) q_{\overline{M}}(\mathbf{v}) \frac{1}{2} + (1 - q_{\overline{m}}(\mathbf{v})) q_{\overline{M}}(\mathbf{v}) \right] dG^n$$

But by Theorem 2 and its Corollary, as $n \longrightarrow \infty$, $q_{\overline{m}} \longrightarrow 1$, and $q_{\overline{M}} \longrightarrow 1$ for all $\mathbf{v} \in R \cup P \cup B$, and $\int_{R \cup P \cup B} dG^n \longrightarrow 1$. Hence: $\omega_n \longrightarrow \frac{1}{2}$.

Thus:

$$\begin{aligned}
\lim_{n \longrightarrow \infty} W_n &= \left(\frac{1}{2}\right) Ev \\
\lim_{n \longrightarrow \infty} W_{0n} &= (1-\alpha) Ev \\
\lim_{n \longrightarrow \infty} W_n^* &= (1-\alpha) Ev
\end{aligned}$$

Hence:

$$\lim_{n->\infty} \left(\frac{W_n}{W_{0n}}\right) = \frac{1}{2(1-\alpha)} < 1$$

Note that the limit is independent of the distribution of valuations. The following proposition summarizes the result.

Proposition 4. *Consider a sequence of vote markets. For any* $\alpha \in (0, 1/2)$, F, *and* $\delta > 0$, *there exists a finite* $\widetilde{n}(F, \delta)$ *such that for all* $n > \widetilde{n}$, $W_n < W_{0n}$, and $\left|\frac{W_n}{W_{0n}} - \frac{1}{2(1-\alpha)}\right| < \delta$.

For any minority size and for any distribution of values, with a sufficiently large electorate vote-trading lowers welfare. Note the contribution of the proposition. The assumption of i.i.d. value draws implies that majority voting without trade must be asymptotically efficient[26], but a priori a market for votes need not imply sizable minority victories when the electorate is very large. If the price becomes negligible (as the probability that a single vote be pivotal becomes negligible), a market for votes could in principle support an equilibrium with negligible minority victories, and negligible efficiency losses. The proposition makes clear that this is not the case, at least in the equilibrium we are considering. Because the minority wins with 50 percent probability, the efficiency loss is both significant and precisely quantifiable.

5 Robustness of the equilibrium

5.1 Correlated and not identically distributed values

We have assumed so far that values are independent both across groups and within groups, and identically distributed according to some distribution F. The assumption has allowed us to look at simple examples and provide closed form solutions, but the logic of the arguments shows that neither independence nor a common distribution are necessary for our more

[26] A special case of the result in Ledyard and Palfrey (2002).

substantive results. Theorem 1 states a sufficient condition for a trading equilibrium that depends only on the existence of a sufficient wedge between $\overline{v}_g$ and $\overline{v}_{(2)G}$, the *realized* highest values in the two groups. Nor does the equilibrium depend on F: given m, M, R, p, and others' strategies, a voter's best response is fully identified. The probability that the condition in Theorem 1 is satisfied does depend on F, but the asymptotic result in Theorem 2 is robust to significant generalization.

Particularly relevant to our voting environment is the possibility of correlation in values. Consider then the following standard model, where the assumption of independence is weakened to conditional independence:

$$\begin{aligned} v_i &= v_m + \varepsilon_i \text{ for all } i \in m \\ v_j &= v_M + u_j \text{ for all } j \in M \end{aligned}$$

where v_m (v_M) is a common value shared by all m (M) voters, and ε_i and u_j are idiosyncratic components, independently drawn from distribution $G_m(\varepsilon)$, with full support $[0, \overline{\varepsilon}]$, and $G_M(u)$, with full support $[0, \overline{u}]$. For all fixed $\alpha \in (0, 1/2)$, as $n \longrightarrow \infty$, $\overline{v}_m \longrightarrow v_m + \overline{\varepsilon}$, and $\overline{v}_M \longrightarrow v_M + \overline{u}$. Thus for all $2(v_M + \overline{u}) \geq (v_m + \overline{\varepsilon}) \geq \frac{v_M + \overline{u}}{2}$ the equilibrium of Theorem 1 exists with probability approaching 1 asymptotically.[27] And if the equilibrium exists, Proposition 3 follows: asymptotically, the minority is expected to win with probability 1/2.

Relative to our previous results, there are then two qualifications. First, to ensure that the equilibrium always exists asymptotically, we need additional conditions on the distributions of values, here on v_m, $\overline{v}_M$, $\overline{\varepsilon}$, and $\overline{u}$. Second, the welfare results need to be reevaluated and again in general will depend on the distributions. In this example, if $v_m + E_{G_m}(\varepsilon)$ is sufficiently larger than $v_M + E_{G_M}(u)$, then, depending on α, the vote market could be asymptotically superior to simple majority voting. Notice however that both qualifications stem from the assumption of different distributions of values for the two groups, not from relaxing independence. We see no convincing reason to assume systematically different intensities between the majority and the minority. Much stronger arguments can be made for allowing for richer forms of dependence in intensities, but the logic here is so simple that it makes us confident that the model can be extended with little change. Our asymptotic results require that the extremum statistic of the value draws in each group should converge to the upper bound of the support. The condition is violated if all values are perfectly correlated, but can

[27] We are using $\lim_{n \longrightarrow \infty} \mu(n) = 1/2$.

accommodate high degrees of dependence.[28]

5.2 An alternative rationing rule

The equilibrium strategies we have characterized have an extreme flavor: individuals either demand a majority of votes or sell. Intuitively, the behavior seems in line with the unusual nature of the goods being traded: because votes per se are worthless, the market allocates not votes but decision power. Yet, could the extreme strategies instead be the result of the all-or-nothing rationing rule (either an order is fully filled or it is passed over)? We show in this section that the result is robust to a different rationing rule that allocates offered votes with equal probability to any individual with unfilled demand. Under this alternative rule, a fully revealing ex ante competitive equilibrium with trade is guaranteed to exist under a condition that recalls the condition characterized in Theorem 1. The equilibrium we have constructed mimics the equilibrium in Theorem 1: $\overline{v}_G$ and $\overline{v}_g$ randomize between demanding a majority of votes and selling their vote, while all other voters sell.[29]

Consider the following rule, which we call $R2$, or rationing-by-vote: if voters' orders result in excess demand, any vote supplied is randomly allocated to one of the individuals with outstanding purchasing orders, with equal probability. An order remains outstanding until it has been completely filled. When all supply is allocated, each individual who put in an order must purchase all units that have been directed to him, even if the order is only partially filled. If there is excess supply, the votes to be sold are chosen randomly from each seller, with equal probability. Formally, $\pi_i(s) = 1/n_{s>0}^{R2}$ if $s_i > 0$; $\pi_i(s) = 1/n_{-1}$ if $s_i = -1$, and $R2\,(s)\,(x) = 1$ if, for all i, $x_i \in \{0, 1, 2, .., 1 + s_i\}$ and $\sum x_i = n$, where, as earlier, $\pi_i(s)$ is the probability of being recognized, n_{-1} the number of voters with outstanding offers to sell, and $n_{s>0}^{R2}$ the number of voters with outstanding positive demands. Under $R2$, $n_{s>0}^{R2}$ is the number of voters whose demand has not been fully filled, whether or not they have been recognized in the past.

Like $R1$, $R2$ is anonymous, in line with competitive analysis in this paper; contrary to

[28] For example, statisticians working on limit distributions for maxima have proposed the concept of *m-dependence*. When values are drawn in a natural sequence (think of floods over time), m-dependence applies when there exists a finite m such that draws that are more than m steps apart are independent (Hoeffding and Robbins, 1948). In our application, the concept could be relevant for geographically or ideologically concentrated subgroups of voters. Theorem 2 and Proposition 3 continue to hold in this case, under minor regularity assumptions.

[29] A similar result, on the robustness of the equilibrium to this alternative rationing rule, holds for the model in CLP.

$R1$, it guarantees that only one side of the market is ever rationed. However, $R2$ requires voters to accept and pay for partially filled orders, a scenario that can be very costly in a market for votes, where the value of votes hinges on pivotality, and thus on the exact number of votes transacted.

At $n = 3$, $R2$ and $R1$ are identical and Theorem 1 applies. Suppose then $n > 3$:

Theorem 3. *Suppose $R2$ is the rationing rule. For all $n > 3$ odd, m, and F, there exists a threshold $\mu_{R2}(n) > 0$ such that if $\overline{v}_g \geq \mu_{R2}(n) Max[v_{(2)G}, v_{(2)g}]$, there exists a fully revealing ex ante equilibrium with trade where $\overline{v}_G$ and $\overline{v}_g$ randomize between demanding $(n-1)/2$ votes (with probabilities $q'_{\overline{G}}$ and $q'_{\overline{g}}$ respectively) and selling their vote, and all other individuals sell. The randomization probabilities $q'_{\overline{G}}$ and $q'_{\overline{g}}$ and the price p' depend on the realized values $\overline{v}_g$ and $\overline{v}_G$, but for all $\overline{v}_G$ and $\overline{v}_g \geq \mu_{R2}(n) Max[v_{(2)G}, v_{(2)g}]$, $q'_{\overline{G}} \in [\frac{n-1}{n+1}, 1]$ and $q'_{\overline{g}} \in [\frac{n-1}{n+1}, 1]$. The threshold $\mu_{R2}(n)$ is given by:*

$$\mu_{R2}(n) = \frac{(n-1)^2}{2^{n-2} n} \binom{n-3}{\frac{n-3}{2}}$$

The theorem is proved in the Appendix. Its similarity to Theorem 1 is apparent. There are two main differences: first, the thresholds in the two theorems differ, and $\mu_{R2}(n) > \mu(n)$, implying that the equilibrium exists under $R2$ under more restrictive conditions than under $R1$. In particular, $\lim_{n \longrightarrow \infty} \mu_{R2}(n) = \infty$: whereas under $R1$ the probability that the equilibrium exists in a very large market converges to 1, the probability converges to 0 under $R2$. Second, as can be verified in the Appendix, when the equilibrium exists, the equilibrium price p' is consistently lower than p, the equilibrium price under $R1$. The intuition is clear: when both $\overline{v}_G$ and $\overline{v}_g$ submit demands for $(n-1)/2$ votes, one of the two will receive and be charged for $(n-3)/2$ votes, useless votes, since the opponent will hold a majority. To compensate for this risk, the equilibrium price must be lower[30].

The choice of rationing rule poses a number of interesting but challenging questions. We know that in general the equilibrium must depend on the exact rule, and we can debate whether the rationing rule is better thought of as part of the institution, controlled by the market designer, or as part of the equilibrium, and interpreted as reduced form for the

[30] There is a third difference as well. As the proof in the Appendix makes clear, the condition $\overline{v}_g \geq \mu_{R2}(n) Max[v_{(2)G}, v_{(2)g}]$ is sufficient for the existence of the equilibrium in Theorem 3–there are value realizations for which weaker conditions are necessary–whereas under $R1$ the condition in Theorem 1 is necessary and sufficient for the equilibrium characterized there.

complex, decentralized system of search that underlies the trades.[31] Our goal here is not to address these broad questions but to make a narrower point: Theorem 3 shows that the equilibrium discussed in this paper is not the artifact of one specific rationing rule, and in particular of the all-or-nothing nature of $R1$. We now revert to $R1$ for our concluding section.

6 Conclusions

We have shown in this paper that an ex ante competitive equilibrium exists in a market for votes in which the precise numerical advantage enjoyed by the majority is known, while the intensity of preferences is private information. The assumption seems faithful to many voting situations, but the knowledge of the precise number of votes on which the decision hangs defines pivotality sharply and makes the existence of an equilibrium particularly problematic. We have characterized a sufficient condition for the existence of an ex ante equilibrium with trade for any electorate size, any majority advantage, and any distribution of intensities.

It is well-known that the equilibrium of a competitive market for votes must involve randomization. The concept of ex ante competitive equilibrium is designed to accommodate probabilistic demands and weakens the requirement of market-clearing to market-clearing in expectation. The equilibrium we have constructed is such that only two voters, the highest intensity voters on each side, demand votes with positive probabilities; all others offer their votes for sale. The two voters who randomize assign positive probability to only two actions: either selling, or demanding enough votes to alone control a majority of all votes. The equilibrium exists unless multiple members of one of the two groups, whether the majority or the minority, have much higher intensities than all members of the opposite group. Although at first sight counterintuitive, the equilibrium reflects the unusual characteristics of votes: per se, votes are worthless; what matters is decision power. As in Casella, Llorente-Saguer and Palfrey (2012), in equilibrium the market comes to resemble an auction for decision power among those who value it most.

Because the probability of either group's victory depends only on the action of its most intense member and gives no direct weight to the size of the group, the equilibrium yields a systematic minority bias. If the minority is sufficiently small, then, for any number of voters and any distribution of intensities, the markets leads it to win more frequently than efficiency dictates. And for any minority size, there are well-behaved distributions of intensities such

[31] Green (1980) proposes the second interpretation.

that not only the minority wins too frequently relative to efficiency, but the market decreases expected welfare relative to majority voting alone. The results are particularly clean in a large electorate. There, the equilibrium always exists. Strikingly, for any distribution of intensities, the minority always wins as frequently as the majority does, for any minority sizes, and expected welfare is diminished by the market for votes.

The results are surprisingly clear-cut for such a long-debated problem, but depend on the specific equilibrium we have studied. It would be good to know to what extent the minority bias we uncovered is a general property of competitive markets for votes. The experimental results in Casella, Palfrey and Turban (2012) support the conjecture: in every experimental session, in fact in every committee of voters, the frequency of minority victories is higher than efficiency dictates. The experiment, however, concerned a specific case: a committee of five voters, with a minority of size two. Can the theory tell us more?

This is difficult question because it addresses the possible multiplicity of equilibria, an issue we are unable to resolve satisfactorily. The equilibrium we have discussed exists for a large range of value realizations (and with probability one asymptotically), and we have not identified any other equilibrium with trade when the condition in Theorem 1 is satisfied. We know however that other equilibria can be supported in special cases, if we simplify the model and focus on a degenerate distribution of intensities. Suppose that all voters in the same group share the same value, and define *quasi-symmetrical* an equilibrium where all voters in the same group adopt an identical strategy. Then, for example:

Example 2. *Suppose $n = 5$, $m = 2$ and $M = 3$; $v_{im} = v_m$ for all $i \in m$, and $v_{jM} = v_M$ for all $j \in M$. Then, if $v_M/v_m \in [0.57, 1.55]$, there exists a quasi-symmetrical ex ante equilibrium with trade where all voters randomize over demanding one vote, staying out of the market, and offering their vote for sale. The mixing probabilities differ across the two groups; together with the price p, the probabilities depend on v_M/v_m.*[32]

The quasi-symmetry of the strategies makes the equilibrium trivially fully revealing. The example may be quite special: we do not know whether the equilibrium exists for other n's, or for other m's, or whether there is a similar equilibrium when values are allowed to differ within each group. It is an interesting equilibrium because its strategies are such that voters do not demand bundles of votes, contrary to the equilibrium characterized in Theorem 1.[33]

[32] We have identified the equilibrium in the example with numerical methods in R and in Mathematica. The programs are available upon request.

[33] Note that the asymmetrical equilibrium of Theorem 1 continues to exist in this example: it requires

And yet, our numerical results show that equilibrium strategies again induce a bias in favor of the minority. Suppose for example $v_M/v_m = 1$. To evaluate whether a bias exists, consider three different benchmarks: the efficient probability of a minority victory in this case is 0; a fair-division perspective suggests a probability of 2/5, or 40 percent; finally, if all voters adopted the same strategy, given the three actions over which the voters randomize and expected market balance, the minority would win at most with 28 percent probability.[34] In equilibrium, however, the minority wins with 57 percent probability, i.e. more than half of the times. With fixed values within each group, the efficient frequency of minority victories is a corner solution (1 for $v_M/v_m < 2/3$ and 0 otherwise), and thus there are low v_M/v_m values for which the minority wins less in equilibrium than efficiency demands, but for 90 percent of all v_M/v_m realizations that support the equilibrium–for all $v_M/v_m > 2/3$–the minority wins more than it should. This is not because there is a positive but small probability of a minority victory: for all v_M/v_m such that the equilibrium exists, the minority wins with probability higher than 40 percent.

Looking at the equilibrium strategies in more detail, it is not difficult to see the source of the bias. In equilibrium, the minority consistently adopts more aggressive strategies. If we call γ_m (σ_m) the equilibrium probability of demanding one vote (selling) for each member of the minority group, and analogously for the majority group, then:

Example 2 continued. *For all v_M/v_m such that the equilibrium exists, $\gamma_m > \gamma_M \geq 0$, and $0 \leq \sigma_m < \sigma_M$. In addition, $\gamma_m > \sigma_m \geq 0$, and $\sigma_M > \gamma_M \geq 0$.*[35]

Nor is it difficult to understand the source of the difference in strategies: the minority is smaller and suffers from a weaker free-rider problem. This is what we would expect, once the externalities present in the market are recognized. It is this observation that leads us to conjecture that the pro-minority bias may a general feature of a market for votes, if an equilibrium exists.

$v_M/v_m \in [\mu(5), 1/\mu(5)] = [2/7, 7/2]$, and thus exists over a larger range of value realizations.

[34] With symmetrical strategies, ex ante equilibrium requires that the probability of selling be equal to the probability of demanding one vote in each group–call it γ. The probability of a minority victory goes from 0 when $\gamma = 0$ to a maximum of 0.28 when γ is maximal, at 0.5.

[35] The equilibrium strategies are the following: (1) For $v_M/v_m \in [0.57, 1.05]$, $\gamma_M = 0$, and over this range of v_M/v_m values, σ_M rises from 0.25 to 0.36, γ_m falls from 0.56 to 0.54, and σ_m falls from 0.19 to 0. (2) For $v_M/v_m \in [1.03, 1.56]$, $\sigma_m = 0$, γ_m falls from 0.62 to 0.57, σ_M rises from 0.41 to 0.49, and γ_M increases from 0 to 0.11. All probability changes are monotonic in v_M/v_m.

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7 Appendix

7.1 Proof of Theorem 1.

Theorem 1. *For all n odd, m, and F, there exists a threshold $\mu(n) \in (0,1)$ such that if $\overline{v}_g \geq \mu(n) v_{(2)G}$, there exists a fully revealing ex ante equilibrium with trade where $\overline{v}_G$ and $\overline{v}_g$ randomize between demanding $(n-1)/2$ votes (with probabilities $q_{\overline{G}}$ and $q_{\overline{g}}$ respectively) and selling their vote, and all other individuals sell. The randomization probabilities $q_{\overline{G}}$ and $q_{\overline{g}}$ and the price p depend on $\overline{v}_g$ and $\overline{v}_G$, but for all $\overline{v}_G$ and $\overline{v}_g \geq \mu(n) v_{(2)G}$, $q_{\overline{G}} \in [\frac{n-1}{n+1}, 1]$ and $q_{\overline{g}} \in [\frac{n-1}{n+1}, 1]$. The threshold $\mu(n)$ is given by:*

$$\mu(n) = \begin{cases} \frac{2}{3} & \text{if } n = 3 \\ \max\left\{ \frac{(n-2)(n-1)}{2(n^2+n-5)}, \frac{(n-2)(n-1)(n+1)}{2(n^3+3n^2-19n+21)} \right\} & \text{if } n > 3 \end{cases} \tag{6}$$

Proof. The theorem is implied by the following two lemmas. Lemma A1 characterizes the case $G = M$ and Lemma A2 the case $G = m$.

Lemma A1.

Suppose $G = M$ (or $\overline{v}_G = \overline{v}_M$ and $\overline{v}_g = \overline{v}_{m,}$). Then if $\overline{v}_m \in \left[\mu(n) v_{(2)M}, \overline{v}_M\right]$, the strategies described in the theorem are a fully revealing ex ante competitive equilibrium for all n odd, m, and F. The mixing probabilities $q_{\overline{M}}$ and $q_{\overline{m}}$ and the price p depend on the realizations of $\overline{v}_m$ and $\overline{v}_M$. There exist two thresholds $1/2 \leq \underline{\rho}(n) < \overline{\rho}(n) < 1$ such that:

(a) $n > 3$.

1. *If $\overline{v}_m \in \left[\mu(n) v_{(2)M}, \underline{\rho}(n) \overline{v}_M\right]$, $q_{\overline{M}}$, $q_{\overline{m}}$, and p satisfy:*

$$\begin{aligned} q_{\overline{M}} &= 1 \\ q_{\overline{m}} &= \frac{n-1}{n+1} \\ p &= 2\frac{\overline{v}_m}{n+1} \end{aligned} \tag{7}$$

2. *If* $\overline{v}_m \in [\underline{\rho}(n)\overline{v}_M, \overline{\rho}(n)\overline{v}_M]$*,* $q_{\overline{M}}$*,* $q_{\overline{m}}$*, and* p *satisfy:*

$$\begin{aligned} q_{\overline{m}} + q_{\overline{M}} &= \frac{2n}{n+1} \\ p &= \frac{2q_{\overline{m}}\overline{v}_M}{(n-3)(1-q_{\overline{m}})+n+1} \\ p &= \frac{2(2-q_{\overline{M}})\overline{v}_m}{(n-3)(1-q_{\overline{M}})+n+1}. \end{aligned} \tag{8}$$

3. *If* $\overline{v}_m \in [\overline{\rho}(n)\overline{v}_M, \overline{v}_M]$*,* $q_{\overline{M}}$*,* $q_{\overline{m}}$*, and* p *satisfy:*

$$\begin{aligned} q_{\overline{m}} &= 1 \\ q_{\overline{M}} &= \frac{n-1}{(n+1)} \\ p &= 2\frac{\overline{v}_M}{n+1} \end{aligned} \tag{9}$$

The two thresholds $\underline{\rho}(n)$ and $\overline{\rho}(n)$ *are given by:*

$$\begin{aligned} \underline{\rho}(n) &= \frac{n+1}{n+5} \\ \overline{\rho}(n) &= \frac{(n-1)(n+5)}{(n+3)(n+1)} \end{aligned} \tag{10}$$

(b) $n = 3$*.*

If $v_{(2)M} \leq (3/4)\overline{v}_M$*, then* $\mu(3)v_{(2)M} \leq \underline{\rho}(3)\overline{v}_M$*, and the characterization in part (a) above applies unchanged. If* $v_{(2)M} > (3/4)\overline{v}_M$*, then:*

If $\overline{v}_m \in [\mu(3)v_{(2)M}, \overline{\rho}(3)\overline{v}_M]$*,* $q_{\overline{M}}$*,* $q_{\overline{m}}$*, and* p *satisfy (8); if* $\overline{v}_m \in [\overline{\rho}(3)\overline{v}_M, \overline{v}_M]$*,* $q_{\overline{M}}$*,* $q_{\overline{m}}$*, and* p *satisfy (9).*

Lemma A2. *Suppose* $G = m$ *(or* $\overline{v}_G = \overline{v}_m$ *and* $\overline{v}_g = \overline{v}_M$*). Then if* $\overline{v}_M \in \left[\mu(n)v_{(2)m}, \overline{v}_m\right]$*, where* $\mu(n)$ *is given by (6) above,the strategies described in the theorem, together with the price and mixing probabilities given by (9) are a fully revealing ex ante competitive equilibrium for all* n *odd,* m*, and* F*.*

The proof is organized in two stages. First, we show that if the direction of preferences associated with each demand is commonly known, the strategies and price described above are an equilibrium. Second, we show that when preferences are private information the equilibrium is fully revealing: given others' strategies and the market price, each individual's best response is identical to what it would be under full information. Others' strategies and the market price, together with the notion that the market is in equilibrium, fully reveal others' direction of preferences.

7.1.1 Ex ante equilibrium with full information.

Suppose first that preferences are publicly known. We show here that the three systems (7), (8), and (9) characterize an ex ante equilibrium for each corresponding range of realized valutions.

1. Consider a candidate equilibrium with $q_{\overline{M}} \in (0,1)$, $q_{\overline{m}} \in (0,1)$. Expected market balance requires $(q_{\overline{M}} + q_{\overline{m}})(n-1)/2 = (n-2) + (1-q_{\overline{M}}) + (1-q_{\overline{m}})$, or:

$$q_{\overline{M}} + q_{\overline{m}} = \frac{2n}{n+1} \tag{11}$$

Denote by $U_M(s)$ the expected utility to voter $\overline{v}_M$ from demand s. Then:

$$\begin{aligned} U_M\left(\frac{n-1}{2}\right) &= q_{\overline{m}}\left(\frac{q\overline{v}_M}{2} - \frac{n-1}{4}p\right) + (1-q_{\overline{m}})\overline{v}_M - \frac{n-1}{2} \\ U_M(-1) &= q_{\overline{m}}\left(\frac{p}{2}\right) + (1-q_{\overline{m}})\left(\overline{v}_M\right) \end{aligned}$$

where we are assuming that voter $\overline{v}_M$ is informed that the other voter randomizing with probability $q_{\overline{m}}$ belongs to the minority. Voter $\overline{v}_M$ is indifferent between the two pure demands if and only if:

$$p = \frac{2q_{\overline{m}}\overline{v}_M}{n+1+(n-3)(1-q_{\overline{m}})} \tag{12}$$

Similarly, denoting by $U_m(s)$ the expected utility from demand s to voter $\overline{v}_m$:

$$\begin{aligned} U_m\left(\frac{n-1}{2}\right) &= q_{\overline{M}}\left(\frac{\overline{v}_m}{2} - \frac{n-1}{4}p\right) + (1-q_{\overline{M}})\left(\overline{v}_m - \frac{n-1}{2}\right) \\ U_m(-1) &= q_{\overline{M}}\left(\frac{p}{2}\right) + (1-q_{\overline{M}})(0), \end{aligned} \tag{13}$$

again assuming full information. Indifference requires:

$$p = \frac{2(2 - q_{\overline{M}})\overline{v}_m}{n + 1 + (n - 3)q_{\overline{M}}} \tag{14}$$

System (11), (12) and (14) corresponds to system (7) in Lemma 2. The existence of a solution is not guaranteed. There is a solution if and only if there exists $q_{\overline{M}} \in [0, 1]$ and $q_{\overline{m}} \in [0, 1]$ with $q_{\overline{M}} + q_{\overline{m}} = (2n)/(n + 1)$ such that (12)=(14). Such conditions are satisfied if and only if:

$$\overline{v}_m \in [\underline{\rho}(n)\overline{v}_M, \bar{\rho}(n)\overline{v}_M]$$

where:

$$\begin{aligned} \underline{\rho}(n) &= \frac{n+1}{n+5} \\ \bar{\rho}(n) &= \frac{(n-1)(n+5)}{(n+3)(n+1)}, \end{aligned}$$

conditions (10) in Lemma A2. Note that $1/2 \leq \underline{\rho}(n) < \bar{\rho}(n) < 1$ for all $n \geq 3$.

To verify that this is indeed an equilibrium, we need to rule out profitable deviations. Note that for any voter any demand $s_i > n-1$ is always fully rationed, and thus is equivalent to $s_i = 0$.

(i) Consider first voter $\overline{v}_M$. For any $s_M \in (\frac{n-1}{2}, n-1]$, $U_M(s_M) < U_M(\frac{n-1}{2})$: demanding more votes than required to achieve a strict majority does not affect the probability of rationing and is strictly costly. For any $s_M \in [0, \frac{n-1}{2})$, $U_M(s_M) < U_M(-1)$: demanding less than $\frac{n-1}{2}$ votes is dominated by selling. To see this, note that when $s_m = \frac{n-1}{2}$, any $s_M < \frac{n-1}{2}$ guarantees that $\overline{v}_m$ will not be rationed and will win (because all other voters are selling). Thus, whether $s_M \in (0, \frac{n-1}{2})$ and the action is strictly costly, or $s_M = 0$ and voter $\overline{v}_M$ stays out of the market, when $s_m = \frac{n-1}{2}$, any $s_M \in [0, \frac{n-1}{2})$ is strictly dominated by selling. When $s_m = -1$, any $s_M \in (0, \frac{n-1}{2}]$ is dominated by $s_M \in \{-1, 0\}$ and these two actions are equivalent because both $s_M = -1$ and $s_M = 0$ induce no trade and guarantee a majority victory. Therefore, when facing the strategy profile defined in the candidate equilibrium, $\overline{v}_M$'s best response can only be either $s_M = -1$ or $s_M = \frac{n-1}{2}$. System (7) guarantees that $\overline{v}_M$ is indifferent between the two demands.

(ii) Consider now voter $\overline{v}_m$. As above, for any $s_m \in (\frac{n-1}{2}, n-1]$, $U_m(s_m) < U_m(\frac{n-1}{2})$: demanding more votes than required to achieve a strict majority does not affect the probability of rationing and is strictly costly. It is also clear that $U_m(0) < U_m(-1)$: the two demands are equivalent if $s_M = -1$ and selling is strictly superior to staying out of the market if $s_M = \frac{n-1}{2}$. The question is whether $\overline{v}_m$ could gain by demanding less than $\frac{n-1}{2}$ votes: although such a strategy is dominated by selling when $s_M = \frac{n-1}{2}$, it could in principle be superior when $s_M = -1$. Consider the relevant expected utilities:

$$\begin{aligned} U_m\left(\frac{n-1}{2}\right) &= (1-q_{\bar{M}})\left(\bar{v}_m - \frac{n-1}{2}p\right) + q_{\bar{M}}\left(\frac{\bar{v}_m}{2} - \frac{n-1}{4}p\right) \\ U_m(-1) &= (1-q_{\bar{M}})\cdot 0 + q_{\bar{M}}\left(\frac{p}{2}\right) \\ U_m(x) &= (1-q_{\bar{M}})\left(P(x)\bar{v}_m - xp\right) + q_{\bar{M}}(-xp) \end{aligned} \tag{15}$$

where $P(x)$ is the probability of a minority victory when $\overline{v}_m$ demands $x \in (0, \frac{n-1}{2})$ votes and $\overline{v}_M$ offers his vote for sale. Since $P(x) < 1$ for all $x \in (0, \frac{n-1}{2})$, and $U_m(x)$ is increasing in $P(x)$ and decreasing in x, it follows that $U_m(x) < (1-q_{\bar{M}})(\overline{v}_m - p) + q_{\bar{M}}(-p)$. Hence $U_m\left(\frac{n-1}{2}\right) > (1-q_{\bar{M}})(\overline{v}_m - p) + q_{\bar{M}}(-p)$ is sufficient to rule out a profitable deviation to $x \in (0, \frac{n-1}{2})$. The condition is equivalent to:

$$\frac{q_{\bar{M}}}{2}\bar{v}_m \geq \frac{2(1-q_{\bar{M}})(n-1) + q_{\bar{M}}(n-1) - 4}{4}p$$

Substituting p from (14) and simplifying, the condition amounts to:

$$(2-n)q_{\bar{M}}^2 + (3n-5)q_{\bar{M}} - 2n + 6 \geq 0$$

This function is increasing in $q_{\bar{M}}$ for all $n \geq 3$. By (11) $q_{\bar{M}} \geq \frac{n-1}{n+1}$. Hence, we can evaluate the condition at $q_{\bar{M}} = \frac{n-1}{n+1}$. If it is positive, the deviation is not profitable. Substituting, we obtain:

$$n^2 + 2n + 13 \geq 0$$

which is trivially satisfied for all n. Hence for any $s_m \in [1, \frac{n-1}{2})$, $U_m(s_m) < U_m(\frac{n-1}{2})$. We can conclude that when facing the strategy profile defined in the candidate equilibrium, $\overline{v}_m$'s best response can only be either $s_M = -1$ or $s_M = \frac{n-1}{2}$. System (7) guarantees that $\overline{v}_m$ is indifferent between them.

(iii) Consider $v_i \in M$, $v_i \neq \overline{v}_M$. We show here that, given others' specified strategies, v_i's best response is selling: $s_i = -1$. First notice that, as argued above and for the same reasons, $U_i(s_i) < U_i(\frac{n-1}{2})$ for any $s_i \in (\frac{n-1}{2}, n-1]$. We need to treat the cases $n \geq 5$ and $n = 3$ separately.

(iii.a) Suppose first $n > 3$. In this case, for the same reasons descrobed above $U_i(0) < U_i(-1)$. If a deviation from $s_i = -1$ is profitable, it must be to some $s_i \in (0, \frac{n-1}{2}]$. Suppose first $s_M = -1$. Then in the candidate equilibrium the profile of others' strategies faced by v_i is identical to the profile faced by $\overline{v}_M$. In particular, $U_i(-1) = U_M(-1) = U_M(\frac{n-1}{2}) > U_M(s)$ for all $s \in [0, \frac{n-1}{2})$. But $U_i(s)$ is increasing in v_i for all $s \in (0, \frac{n-1}{2}]$; hence for all s in this interval $U_i(s) < U_M(s)$, and thus $U_i(-1) > U_i(s)$ for all $s \in (0, \frac{n-1}{2}]$. Thus if $s_M = -1$, $s_i = -1$ is v_i's best response. Suppose then $s_M = \frac{n-1}{2}$. For all $s_i \in [0, \frac{n-3}{2})$, v_i is never rationed, but there is always another voter, either $\overline{v}_M$ or $\overline{v}_m$, who exits the market holding a majority of the votes. Hence the strategy is costly for v_i and never increases the probability of his side winning. It is dominated by $s_i = -1$. Consider then the two remaining strategies $s_i = \frac{n-1}{2}$, and $s_i = \frac{n-3}{2}$. Conditional on $s_M = \frac{n-1}{2}$, the relevant expected utilities are:

$$\begin{aligned}
U_{i\in M}\left(\frac{n-1}{2}\right)\Bigg|_{s_M=\frac{n-1}{2}} &= (1-q_{\overline{m}})\left(v_i - \frac{n-1}{4}p\right) + q_{\overline{m}}\left(\frac{2v_i}{3} - \frac{n-1}{6}p\right) \\
U_{i\in M}\left(\frac{n-3}{2}\right)\Bigg|_{s_M=\frac{n-1}{2}} &= (1-q_{\overline{m}})\left(v_i - \frac{n-3}{2}p\right) + q_{\overline{m}}\left(\frac{2v_i}{3} - \frac{n-3}{6}p\right) \qquad (16) \\
U_{i\in M}(-1)\Bigg|_{s_M=\frac{n-1}{2}} &= (1-q_{\overline{m}})\left(v_i + \frac{p}{2}\right) + q_{\overline{m}}\left(\frac{v_i}{2} + \frac{n-1}{2(n-2)}p\right)
\end{aligned}$$

Taking into account $q_{\overline{m}} \in \left[\frac{n-1}{n+1}, 1\right]$, (12), and $v_i \leq \overline{v}_M$, it is then straightforward to show that, conditional on $s_M = \frac{n-1}{2}$, $U_{i\in M}(-1) > U_{i\in M}\left(\frac{n-1}{2}\right)$, and $U_{i\in M}(-1) > U_{i\in M}\left(\frac{n-3}{2}\right)$. But if $s_i = -1$ is v_i's best response both when $s_M = -1$ and when $s_M = \frac{n-1}{2}$, than it is v_i's best response when $\overline{v}_M$ randomizes between $s_M = -1$ and $s_M = \frac{n-1}{2}$. No profitable deviation exists.

(iii.b) Suppose now $n = 3$. There are two M voters; hence $v_i \in M$, $v_i \leq \overline{v}_M$, is $v_{(2)M}$, the M voter with second highest value. This case must be considered separately because if $n = 3$, and only if $n = 3$, $v_{(2)M}$ can induce no trade with probability $q_{\overline{m}}q_{\overline{M}}$ by unilaterally deviating and staying out of the market. Conditional on $s_M = \frac{n-1}{2} = 1$, the relevant expected utilities are:

$$\begin{aligned} U_{(2)M}(1)\Big|_{s_M=1} &= (1-q_{\overline{m}})\left(v_i - \frac{n-1}{4}p\right) + q_{\overline{m}} v_i \\ U_{(2)M}(0)\Big|_{s_M=1} &= v_i \\ U_{(2)M}(-1)\Big|_{s_M=1} &= (1-q_{\overline{m}})\left(v_i + \frac{p}{2}\right) + q_{\overline{m}}\left(\frac{v_i}{2} + p\right) \end{aligned} \tag{17}$$

It is immediately clear that $U_{(2)M}(0) > U_{(2)M}(1)$. Given (14) and (11), $U_{(2)M}(-1) > U_{(2)M}(0)$ for all $\overline{v}_m \in [\underline{\rho}(3)\overline{v}_M, \bar{\rho}(3)\overline{v}_M] \Longleftrightarrow \overline{v}_m > (2/3)v_{(2)M}$. Thus $s_i = -1$ is indeed a best response for $v_{(2)M}$ as long as $\overline{v}_m \in [\max\{(2/3)v_{(2)M}, \underline{\rho}(3)\overline{v}_M\}, \bar{\rho}(3)\overline{v}_M]$.

(iv) Finally, consider $v_i \in m$, $v_i \neq \overline{v}_m$. Note that such a voter only exists for $n > 3$. Again, we show here that, given others' specified strategies, v_i's best response is selling: $s_i = -1$. The proof proceeds as above. First notice that, as above, $U_i(s_i) < U_i(\frac{n-1}{2})$ for any $s_i \in (\frac{n-1}{2}, n-1]$, and $U_i(0) < U_i(-1)$. If a deviation from $s_i = -1$ is profitable, it must be to some $s_i \in (0, \frac{n-1}{2}]$. Suppose first $s_m = -1$. Then in the candidate equilibrium the profile of others' strategies faced by v_i is identical to the profile faced by $\overline{v}_m$. In particular, $U_i(-1) = U_m(-1) = U_m(\frac{n-1}{2}) > U_m(s)$ for all $s \in [0, \frac{n-1}{2})$. But $U_i(s)$ is increasing in v_i for all $s \in (0, \frac{n-1}{2}]$; hence for all s in this interval $U_i(s) < U_m(s)$, and thus $U_i(-1) > U_i(s)$ for all $s \in (0, \frac{n-1}{2}]$. Thus if $s_m = -1$, $s_i = -1$ is v_i's best response.

Suppose then $s_m = \frac{n-1}{2}$. Eaxctly as argued above, if $s_i \in [0, \frac{n-3}{2})$, v_i is never rationed, but there is always another voter, either $\overline{v}_M$ or $\overline{v}_m$, who exits the market holding a majority of the votes. Hence the strategy is costly for v_i and never increases the probability of his side winning. It is dominated by $s_i = -1$. Consider then the two remaining strategies $s_i = \frac{n-1}{2}$, and $s_i = \frac{n-3}{2}$. Conditional on $s_m = \frac{n-1}{2}$, the relevant expected utilities are:

$$\begin{aligned} U_{i\in m}\left(\frac{n-1}{2}\right)\Big|_{s_m=\frac{n-1}{2}} &= (1-q_{\bar{M}})\left(v_i - \frac{n-1}{4}p\right) + q_{\bar{M}}\left(\frac{2v_i}{3} - \frac{n-1}{6}p\right) \\ U_{i\in m}\left(\frac{n-3}{2}\right)\Big|_{s_m=\frac{n-1}{2}} &= (1-q_{\bar{M}})\left(v_i - \frac{n-3}{2}p\right) + q_{\bar{M}}\left(\frac{2v_i}{3} - \frac{n-3}{6}p\right) \\ U_{i\in m}(-1)\Big|_{s_m=\frac{n-1}{2}} &= (1-q_{\bar{M}})\left(v_i + \frac{p}{2}\right) + q_{\bar{M}}\left(\frac{v_i}{2} + \frac{n-1}{2(n-2)}p\right) \end{aligned}$$

Taking into account $q_{\overline{M}} \in \left[\frac{n-1}{n+1}, 1\right]$, (14), and $v_i \leq \overline{v}_m$, it is then straightforward to show that, conditional on $s_m = \frac{n-1}{2}$, $U_{i\in m}(-1) > U_{i\in m}\left(\frac{n-1}{2}\right)$, and $U_{i\in m}(-1) > U_{i\in m}\left(\frac{n-3}{2}\right)$. But if $s_i = -1$ is v_i's best response both when $s_m = -1$ and when $s_m = \frac{n-1}{2}$, than it is v_i's best response when $\overline{v}_m$ randomizes between $s_m = -1$ and $s_m = \frac{n-1}{2}$. No profitable deviation exists.

We can conclude that if $\overline{v}_m \in [\max\{\mu(n)v_{(2)M}, \underline{\rho}(n)\overline{v}_M\}, \bar{\rho}(n)\overline{v}_M]$, where $\mu(n)$ is given by (6), and$\underline{\rho}(n)$ and $\bar{\rho}(n)$ are given by (10), the strategies described in the theorem, together with the price and the mixing probabilities characterized in system (8), are indeed an ex ante equilibrium of the full information game. Note that $\underline{\rho}(n)\overline{v}_M > \mu(n)v_{(2)M}$ for all $n > 3$; if $n = 3$, $\underline{\rho}(3)\overline{v}_M > (2/3)v_{(2)M} \iff v_{(2)M} < (3/4)\overline{v}_M$.

2. Consider now $\overline{v}_m \in [\mu(n)v_{(2)M}, \underline{\rho}(n)\overline{v}_M]$, where $\mu(n)$ is given by (6). Note that this case is relevant if $\underline{\rho}(n)\overline{v}_M > \mu(n)v_{(2)M}$, and thus for all $n > 3$, or for $v_{(2)M} < (3/4)\overline{v}_M$ if $n = 3$. Suppose all voters adopt the strategies described in the theorem, and $q_{\overline{M}} = 1$. Expected market clearing (equation (11)) implies $q_{\overline{m}} = \frac{n-1}{n+1}$, and $U_m(-1) = U_m(\frac{n-1}{2})$ (or equation (14)) implies $p = \frac{2\overline{v}_m}{n+1}$. Thus suppose system (8) holds. We show here that such strategies and price are an ex ante equilibrium of the full information game. As above, we rule out any profitable deviation for each voter in turn. Again, note that for any voter any demand $s_i > n-1$ is always fully rationed, and thus is equivalent to $s_i = 0$.

(i) Consider first voter $\overline{v}_M$. In the candidate equilibrium, $s_M = \frac{n-1}{2}$. As argued earlier, it remains true that for any $s_M \in (\frac{n-1}{2}, n-1]$, $U_M(s_M) < U_M(\frac{n-1}{2})$: demanding more votes than required to achieve a strict majority does not affect the probability of rationing and is strictly costly. Similarly, it remains true that for any $s_M \in [0, \frac{n-1}{2})$, $U_M(s_M) < U_M(-1)$: demanding less than $\frac{n-1}{2}$ votes is dominated by selling. The argument is identical to what described earlier. Thus the only deviation we need to consider is to $s_M = -1$. The relevant expected utilities are:

$$\begin{aligned} U_M\left(\frac{n-1}{2}\right) &= q_{\bar{m}}\left(\frac{\overline{v}_M}{2} - \frac{n-1}{4}p\right) + (1-q_{\bar{m}})\left(\overline{v}_M - \frac{n-1}{2}\right) \\ U_M(-1) &= q_{\bar{m}}\left(\frac{p}{2}\right) + (1-q_{\bar{m}})(\overline{v}_M) \end{aligned}$$

Substituting $q_{\overline{m}} = \frac{n-1}{n+1}$, we obtain:

$$U_M\left(\frac{n-1}{2}\right) \geq U_M(-1) \Leftrightarrow \frac{\overline{v}_M}{p} > \frac{n+5}{2}$$

Given $p = \frac{2\overline{v}_m}{n+1}$, the condition amounts to:

$$U_M\left(\frac{n-1}{2}\right) \geq U_M(-1) \Leftrightarrow \overline{v}_M \geq \frac{n+5}{n+1}\overline{v}_m = \frac{1}{\underline{\rho}(n)}\overline{v}_m$$

The requirement established the upper bound of the range of $\overline{v}_m$ values considered here: $\overline{v}_m \in [\mu(n)v_{(2)M}, \underline{\rho}(n)\overline{v}_M]$.

(ii) Consider voter $\overline{v}_m$. The arguments discussed under point 1.(ii) apply. With $s_M = \frac{n-1}{2}$ and all other voters selling, $s_m = \frac{n-1}{2}$ and $s_m = -1$ dominate all other $v_{\overline{m}}$'s strategies. With $p = \frac{2\overline{v}_m}{n+1}$, $\overline{v}_m$ is indifferent between them and has no profitable deviation.

(iii) Consider now $v_i \in M$, $v_i \neq \overline{v}_M$. We show here that, given others' specified strategies, v_i's best response is selling: $s_i = -1$. By the arguments under point 1.(iii) above, the only deviations we need to consider are $s_i = \frac{n-1}{2}$ and $s_i = \frac{n-3}{2}$. The relevant expected utilities are given by (16) for $n > 3$, and (17) for $n = 3$. Substituting $p = \frac{2\overline{v}_m}{n+1}$, and $q_{\overline{m}} = \frac{n-1}{n+1}$, we derive the following conditions. If $n > 3$:

$$U_{i\in M}\left(\frac{n-1}{2}\right) \leq U_{i\in M}(-1) \Leftrightarrow v_i\frac{(n-2)(n-1)}{2(n^2+n-5)} \leq \overline{v}_m$$

and

$$U_{i\in M}\left(\frac{n-3}{2}\right) \leq U_{i\in M}(-1) \Leftrightarrow v_i\frac{(n-2)(n-1)(n+1)}{2(n^3+3n^2-19n+21)} \leq \overline{v}_m$$

The two conditions are satisfied if and only if $\mu(n)v_i \leq \overline{v}_m$. Thus they are satisfied for all $v_i \in M$, $v_i \leq \overline{v}_M$ if they are satisfied for $v_i = v_{(2)M}$. If $n = 3$:

$$U_{(2)M}(1) \leq U_{(2)M}(-1) \Leftrightarrow \frac{v_{(2)M}}{2} \leq \overline{v}_m$$

and:

$$U_{(2)M}(0) \leq U_{(2)M}(-1) \Leftrightarrow \frac{2}{3}v_{(2)M} \leq \overline{v}_m$$

This latter condition is stricter and again is satisfied if and only if $\mu(3)v_{(2)M} \leq \overline{v}_m$. For all n, we have established the lower bound of the range of $\overline{v}_m$ values considered here: $\overline{v}_m \in [\mu(n)v_{(2)M}, \underline{\rho}(n)\overline{v}_m]$. Recall that$\underline{\rho}(n)\overline{v}_M > \mu(n)v_{(2)M}$ for all $n > 3$, but if $n = 3$, $\underline{\rho}(3)\overline{v}_M > \mu(3)v_{(2)M} \Longleftrightarrow v_{(2)M} < (3/4)\overline{v}_M$ if $n = 3$.

(iv) Finally, consider $v_i \in m$, $v_i \neq \overline{v}_m$. Again, this voter only exists if $n > 3$. The arguments in 1.(iv) above can be applied identically here and establish that $s_i = -1$ is v_i's unique best response. In particular, if $s_m = -1$, the profile of others' strategies faced by v_i is identical to the profile faced by $\overline{v}_m$. Given others' specified strategies, the differential utility from selling, relative to any other action, is decreasing in v_i; hence if $s_m = -1$ is $\overline{v}_m$'s best response, then it must be a best response for $v_i \leq \overline{v}_m$. If $s_m = \frac{n-1}{2}$, the identical proof detailed in 1.(iv) is relevant. The proof made use of the constraint $q_{\bar{M}} \in \left[\frac{n-1}{n+1}, 1\right]$, which is still satisfied here.

We conclude that for all $\overline{v}_m \in [\mu(n)v_{(2)M}, \underline{\rho}(n)\overline{v}_M]$, where $\mu(n)$ is given by (6), the strategies described in the theorem, together with the price and the mixing probabilities characterized in system (7), are indeed an ex ante equilibrium of the full information game. If $n = 3$, this case is only relevant if $v_{(2)M} < (3/4)\overline{v}_M$.

3. Consider now $\overline{v}_m > \bar{\rho}(n)\overline{v}_M$, where $\bar{\rho}(n)$ is defined in (10). Suppose all voters adopt the strategies described in the theorem, and $q_{\overline{m}} = 1$. Expected market clearing (equation (11)) implies $q_{\overline{M}} = \frac{n-1}{n+1}$, and $U_M(-1) = U_M(\frac{n-1}{2})$ (or equation (12)) implies $p = \frac{2\overline{v}_M}{n+1}$. Thus suppose system (9) holds. We show here that such strategies and price are an ex ante equilibrium of the full information game. As above, we rule out any profitable deviation for each voter in turn. The proofs follow immediately from the arguments used earlier. In particular:

(i) Consider first voter $\overline{v}_M$. The arguments discussed under point 1.(i) apply. With $s_m = \frac{n-1}{2}$ and all other voters selling, $s_M = \frac{n-1}{2}$ and $s_M = -1$ dominate all other $\overline{v}_M$'s strategies. With $p = \frac{2\overline{v}_M}{n+1}$, $\overline{v}_M$ is indifferent between them and has no profitable deviation.

(ii) Consider then voter $\overline{v}_m$. Recall that when $\overline{v}_M$ randomizes between $s_M = \frac{n-1}{2}$ and $s_M = -1$ and all others sell, $s_m = \frac{n-1}{2}$ and $s_m = -1$ dominate all other $\overline{v}_m$'s strategies. The relevant expected utilities are given by (13). Hence, substituting $q_{\overline{M}} = \frac{n-1}{n+1}$:

$$U_m\left(\frac{n-1}{2}\right) \geq U_m(-1) \Leftrightarrow \frac{\overline{v}_m}{p} \geq \frac{(n-1)(n+5)}{2(n+3)}$$

With $p = \frac{2\overline{v}_M}{n+1}$, therefore:

$$U_m\left(\frac{n-1}{2}\right) \geq U_m(-1) \Leftrightarrow \overline{v}_m \geq \frac{(n-1)(n+5)}{(n+1)(n+3)}\overline{v}_M = \bar{\rho}(n)\overline{v}_M$$

The condition establishes the lower bound of the range of $\overline{v}_m$ values considered under this case.

(iii) Consider $v_i \in M$, $v_i \neq \overline{v}_M$. If $n > 3$, the arguments in 1.(iii.a) above can be applied identically here and establish that $s_i = -1$ is v_i's unique best response. In particular, if $s_M = -1$, the profile of others' strategies faced by v_i is identical to the profile faced by $\overline{v}_M$. Hence if $s_M = -1$ is $\overline{v}_M$'s best response, then it must be a best response for $v_i \leq \overline{v}_M$. If $s_M = \frac{n-1}{2}$, the identical proof detailed in 1.(iii) is relevant. The proof made use of the constraint $q_m \in \left[\frac{n-1}{n+1}, 1\right]$, which is still satisfied here. If $n = 3$, $v_i \equiv v_{(2)M}$ and:

$$\begin{aligned}
U_{(2)M}(1)\Big|_{s_m=1} &= q_{\overline{M}} v_{(2)M} + (1 - q_{\overline{M}}) \left(\frac{v_{(2)M}}{2} + \frac{p}{2}\right) \\
U_{(2)M}(0)\Big|_{s_m=1} &= q_{\overline{M}} v_{(2)M} \\
U_{(2)M}(-1)\Big|_{s_m=1} &= q_{\overline{M}} \left(\frac{v_{(2)M}}{2} + p\right) + (1 - q_{\overline{M}}) \left(\frac{p}{2}\right)
\end{aligned}$$

With $p = \frac{2\overline{v}_M}{n+1}$ and $q_{\overline{M}} = 1/2$ by (11), it is trivial to verify that $U_{(2)M}(-1) > U_{(2)M}(1)$ and $U_{(2)M}(-1) > U_{(2)M}(0)$.

(iv) Finally, when $n > 3$, consider $v_i \in m$, $v_i \neq \overline{v}_m$. The problem faced here by $v_i \in m$ is identical to the problem faced by $v_i \in M$, $v_i \neq \overline{v}_M$ in case 2.(iii) above, when $q_{\overline{M}} = 1$, $q_{\overline{m}} = \frac{n-1}{n+1}$. Taking into account $p = \frac{2\overline{v}_M}{n+1}$, all profitable deviations can be ruled out if and only if $v_i \max\left\{\frac{(n-2)(n-1)}{2(n^2+n-5)}, \frac{(n-2)(n-1)(n+1)}{2(n^3+3n^2-19n+21)}\right\} \leq \overline{v}_M$, or $v_i \mu(n) \leq \overline{v}_M$.

Because $\mu(n) < 1$, two observations follow immediately. First, if $\overline{v}_M \geq \overline{v}_m$, the condition $v_i \mu(n) \leq \overline{v}_M$ for all $v_i \in m$, $v_i \neq \overline{v}_m$ is always satisfied. Thus the strategies described in the theorem, together with the price and mixing probabilities characterized in system (9) are indeed an ex ante equilibrium of the full information game for all $\overline{v}_m \in (\bar{\rho}(n)\overline{v}_M, \overline{v}_M]$. Second, the condition $\overline{v}_M \geq \overline{v}_m$ has not been imposed anywhere in the proof of the equilibrium of case 3. The equilibrium requires $\overline{v}_m > \bar{\rho}(n)\overline{v}_M$, where $\bar{\rho}(n) < 1$, and, for $n > 3$, $v_i \mu(n) \leq \overline{v}_M$ $\forall v_i \in m$, $v_i \neq \overline{v}_m$. Thus it is compatible with $\overline{v}_m > \overline{v}_M$, as long as $\overline{v}_M \geq \mu(n) v_{(2)m}$ if $n > 5$, and with no additional constraint if $n = 3$. Hence Lemma A2 follows immediately.

We now show that when preferences are private information, the strategies and price identified above constitute a fully revealing ex ante equilibrium.

7.1.2 Fully revealing equilibrium.

We conjecture an equilibrium identical to the full information equilibrium characterized above and show that given others' strategies, the equilibrium price and the knowledge that the market is in a fully revealing equilibrium, each voter's best response when preferences are private information is uniquely identified and equals the voter's best response with full information. Thus the equilibrium exists when preferences are private information and is indeed fully revealing.

(i) Consider first the perspective of voter $\overline{v}_M$, in equilibrium. In any of the scenarios identified above, expected market equilibrium requires $\overline{v}_M$ to demand a positive number of votes with positive probability. It then follows that the other voter who demands a positive number of votes with positive probability must belong to the minority. If not, $\overline{v}_M$'s best response would be to sell, violating expected market equilibrium. Thus $\overline{v}_M$ also knows that $M-1$ majority members and $m-1$ minority members are offering their vote for sale; he cannot identify them individually, but that is irrelevant. Given that the other net demand for votes comes from a minority voter, $\overline{v}_M$'s best response is identified uniquely and is identical to his best response under full information.

(ii) Consider then the perspective of voter $\overline{v}_m$. If $n = 3$, he is the only minority voter and the problem is trivial. Suppose $n > 3$. Suppose first that $\overline{v}_m \in \left[\mu(n)v_{(2)M}, \underline{\rho_M}(n)\overline{v}_M\right]$, and hence $s_M = \frac{n-1}{2}$ with probability 1. Expected market balance requires $\overline{v}_m$ to demand a positive number of votes with positive probability. But that can only be a best response if the voter who demands $\frac{n-1}{2}$ votes belongs to the majority; if not, $\overline{v}_m$'s best response would be to sell. Again, $\overline{v}_m$ also knows that $M-1$ majority members and $m-1$ minority members are offering their vote for sale; he cannot identify them individually, but that is irrelevant.

Suppose now $\overline{v}_m \in [\underline{\rho}(n)\overline{v}_M, \bar{\rho}(n)\overline{v}_M]$. Expected market balance rules out that $\overline{v}_m$ could sell with probability 1 (because over this range of valuations the minimal expected demand of votes by $\overline{v}_m$ required for expected market balance is $\min(q_{\overline{m}})(\frac{n-1}{2}) + (1- \min(q_{\overline{m}})(-1) = \left(\frac{n-1}{n+1}\right)\left(\frac{n-1}{2}\right) + \left(1 - \frac{n-1}{n+1}\right)(-1) = \frac{n-5}{2(n+1)} > -1$ for all $n \geq 3$). Given the profile of strategies faced by $\overline{v}_m$, staying out of the market ($s_m = 0$) is always dominated by selling. Thus $\overline{v}_m$'s best response in equilibrium must include demanding a positive number of votes with positive probability. As in all previous cases, demanding more than $\frac{n-1}{2}$ votes is always dominated by demanding $\frac{n-1}{2}$ votes. Thus the actions over which $\overline{v}_m$ can randomize with positive probability are $s_m = \frac{n-1}{2}$, $s_m = x$, with $0 \leq x < \frac{n-1}{2}$, and $s_m = -1$. Suppose that the voter demanding $\frac{n-1}{2}$ with probability $q_{\overline{M}}$ (with $q_{\overline{M}}$ identified in (7)), and selling otherwise, belonged to the minority. Then:

$$\begin{aligned}
U_m\left(\frac{n-1}{2}\right)\Bigg|_{(\overline{v}_M\in m)_e} &= (1-q_{\bar{M}})\left(\overline{v}_m-\frac{n-1}{2}p\right)+q_{\bar{M}}\left(\overline{v}_m-\frac{n-1}{4}p\right)\\
U_m(-1)\Bigg|_{(\overline{v}_M\in m)_e} &= (1-q_{\bar{M}})\cdot 0+q_{\bar{M}}\left(\overline{v}_m+\frac{p}{2}\right)\\
U_m(x)\Bigg|_{(\overline{v}_M\in m)_e} &= (1-q_{\bar{M}})\left(P(x)\overline{v}_m-xp\right)+q_{\bar{M}}\left(\overline{v}_m-xp\right)
\end{aligned} \tag{18}$$

where the index $(\overline{v}_m \in m)_e$ indicates the belief that the other voter with positive expected demand belongs to the minority. System (18) is similar to system (15). In particular: (1) The differential utility from selling relative to demanding $x \in [0, \frac{n-1}{2})$ votes, $U_m(-1) - U_m(x)$, is identical. We saw earlier that such term must be positive for all $q_{\bar{M}} \in \left[\frac{n-1}{n+1}, 1\right]$, a result that thus applies immediately here. (2) For all $\overline{v}_m > 0$, the differential utility from selling relative to demanding $\frac{n-1}{2}$ votes, $U_m(-1) - U_m\left(\frac{n-1}{2}\right)$, is strictly higher than in system (15), where, at equilibrium $q_{\bar{M}}$, it equalled 0. Hence at equilibrium $q_{\bar{M}}$ it must be positive here. It follows that if the voter demanding $\frac{n-1}{2}$ with probability $q_{\overline{M}}$ belonged to the minority, $\overline{v}_m$'s best response would be to sell. But that would violate expected market balance. Hence the voter demanding $\frac{n-1}{2}$ with probability $q_{\overline{M}}$ must belong to the majority. Of all remaining voters offering their votes for sale, $M-1$ belongs to the majority, and $m-1$ to the minority. They cannot be distinguished but that has no impact on $\overline{v}_m$'s unique best response.

Finally, suppose either $\overline{v}_m \in (\bar{\rho}(n)\overline{v}_M, \overline{v}_M]$, or $\overline{v}_M \in \left[\mu(n)v_{(2)m}, \overline{v}_m\right]$. Expected market balance requires $s_m = \frac{n-1}{2}$ with probability 1. But then the other voter demanding $\frac{n-1}{2}$ votes with positive probability cannot belong to the minority (because in a fully revealing equilibrium, if $s_m = \frac{n-1}{2}$ with probability 1, all other minority voters would prefer to sell). Hence again the other voter with positive demand for votes must be a majority voter. All remaining voters are sellers; identifying the group each of them belongs to is not possible but has no impact on $\overline{v}_m$'s unique best response.

(iii) Consider now the perspective of all voters who in the full information equilibrium offer their vote for sale with probability 1: $v_i \in M$, $v_i \neq \overline{v}_M$, or $v_i \in m$, $v_i \neq \overline{v}_m$. By the arguments above, each of them knows that in a fully revealing equilibrium the two voters with positive expected demand must belong to the two different parties. Which one belongs to the majority and which one to the minority cannot be distinguished, but is irrelevant: since in the full information case v_i's best response is $s_i = -1$ with probability 1 whether $v_i \in M$, or $v_i \in m$, it follows that identifying which of the two voters with positive expected demand belongs to which group is irrelevant to v_i's best response. Equally irrelevant is identifying which of the sellers belongs to which group. Although the direction of preferences associated with each individual voter cannot be identified, v_i's best response is unique and identical to his best response with full information.

We can conclude that the equilibrium strategies and price identified by Lemmas A1 and A2 are indeed a fully revealing ex ante equilibrium with private information.$\square$

7.2 Proof of Lemma 1.

Lemma 1. *If all v_i, $i \in m$ and $i \in M$, are i.i.d. according to some $F(v)$, then for all F, n, and α , $\omega^*_m \leq m/n$.*

Proof. Call a realization of n values a *profile* Π, and call a *partition* $\mathcal{P}(\Pi)$ a corresponding minority profile $\mathfrak{m}$ and majority profile $\mathfrak{M}$: $\mathcal{P}(\Pi)=\{\mathfrak{m},\mathfrak{M}\}$.[36] The probability of a profile Π depends on the distribution F, but note that because values are i.i.d., given Π any partition $\mathcal{P}(\Pi)$ is equally likely. Call V_m the sum of realized minority values ($V_m = \sum_{i\in m} v_i$), and similarly for V_M ($V_M = \sum_{j\in M} v_j$). Consider any $\mathcal{P}(\Pi)=\{\mathfrak{m},\mathfrak{M}\}$ such that $V_m > V_M$, supposing that at least one such profile Π and partition $\mathcal{P}(\Pi)$ exist. Now, keeping Π fixed, consider an alternative partition $\mathcal{P}'(\Pi)$ such that the values in the minority profile $\mathfrak{m}$ are reassigned to majority voters. By construction, $V_M > V_m$. The values assigned to the remaining $M - m$ majority voters are chosen freely among all realized values in the original majority profile $\mathfrak{M}$. Thus for any $\mathfrak{m}$, there are $\binom{n-m}{M-m} = \binom{M}{M-m} = \binom{M}{m}$ equally likely partitions $\mathcal{P}'(\Pi)$ such that $V_M > V_m$. But then:

$$\Pr(V_M > V_m|\Pi) \geq \binom{M}{m} \Pr(V_m > V_M|\Pi),$$

with inequality because for given Π we are ignoring partitions $\mathcal{P}''(\Pi)$ such that some of $\mathfrak{m}$ values are associated with minority and some with majority voters and $V_M > V_m$.[37].

[36] For clarity: for any Π, there are $\binom{n}{m}$ possible partitions $\mathcal{P}(\Pi)$, and for any partition $\mathcal{P}(\Pi)$ there are $m!M!$ possible permutations of values among the different voters, all keeping $\mathcal{P}(\Pi) = \{\mathfrak{m},\mathfrak{M}\}$ constant.

[37] We are not ignoring those such that $V_m > V_M$ because they are taken into account as different initial partitions $\widetilde{\mathcal{P}}(\Pi)$.

Now:

$$\Pr(V_m > V_M) = \int_\Pi \Pr(V_m > V_M|\Pi)dG$$

$$\Pr(V_M > V_m) = \int_\Pi \Pr(V_M > V_m|\Pi)dG \geq \binom{M}{m} \int_\Pi \Pr(V_m > V_M|\Pi)dG = \binom{M}{m} \Pr(V_m > V_M)$$

where $G = F^n$ is the joint density of a profile Π. But $\Pr(V_m > V_M) = 1 - \Pr(V_M > V_m)$. Hence:

$$\Pr(V_m > V_M) \leq \frac{1}{1 + \binom{M}{m}}.$$

The Lemma then follows if:

$$\frac{1}{1 + \binom{M}{m}} \leq \frac{m}{m + M}. \tag{19}$$

Condition (19) is equivalent to:

$$\frac{m!(M-m)!}{m!(M-m)! + M!} \leq \frac{m}{m+M}$$

or, after some manipulations:

$$(m-1)!(M-m)! \leq (M-1)!$$

which is equivalent to:

$$\binom{M-1}{m-1} \geq 1,$$

an inequality that holds for all $m \geq 1$.□

7.3 Proof of Proposition 1.

Proposition 1. (a) *There exists a value $m' > 0$ such that if $m < m'$, $\omega_m > \omega_m^*$ for all n and F.* (b) *There exist distributions $\mathbf{F}'$ such that if $F \in \mathbf{F}'$, $\omega_m > \omega_m^*$ for all n and m.*

Proof of part (a). We know that if $\overline{v}_g > v_{(2)G}$, the equilibrium in Theorem 1 always applies. If $G = m$ (i.e. $v_n \in m$), m wins with probability $\frac{n+3}{2(n+1)}$; if $G = M$ (i.e. $v_n \in M$), m wins with probability $\frac{n-1}{2(n+1)}$ if $\overline{v}_m < \underline{\rho}\overline{v}_M$, and with some probability $\in (\frac{n-1}{2(n+1)}, \frac{n+3}{2(n+1)})$ otherwise. Hence:

$$\omega_m > \frac{n+3}{2(n+1)} \Pr(G = m \cap \overline{v}_M > v_{(2)m}) + \frac{n-1}{2(n+1)} \Pr(G = M \cap \overline{v}_m > v_{(2)M}) \tag{20}$$

The inequality is strict both because (20) sets to $\frac{n-1}{2(n+1)}$ the probability of minority victories whenever $\overline{v}_g > \overline{v}_{(2)G}$ and $G = M$, and because it ignores value realizations such that $\overline{v}_g \in (\mu(n)v_{(2)G}, v_{(2)G})$–the condition in Theorem 1 is satisfied, and the minority wins with positive probability.[38]

With i.i.d. value draws:

$$\Pr(G = m \cap \overline{v}_M > \overline{v}_{(2)m}) = \Pr(G = M \cap \overline{v}_m > \overline{v}_{(2)M}) = \frac{mM}{n(n-1)}$$

Thus:

$$\omega_m > \frac{n+3}{2(n+1)} \frac{mM}{n(n-1)} + \frac{n-1}{2(n+1)} \frac{mM}{n(n-1)} = \frac{mM}{n(n-1)}$$

Given Lemma 1, the proposition follows if there exists a m' such that for all $m < m'$, $mM/[n(n-1)] \geq 1/n$. This last condition holds with equality at $m = 1$, and thus the proposition always holds at $m' = 2$. □

Proof of part (b) Assume that $F(v) = v^b$. Then we can derive:

$$P(B) = 1 - \frac{m(m-1)}{n(n-1)} \mu^{bM} - \frac{M}{n} \bar{\rho}^{bm} \tag{21}$$

$$P(P) = \frac{M}{n} \left(\bar{\rho}^{bm} - \underline{\rho}^{bm} \right) \tag{22}$$

$$P(R) = \underline{\rho}^{bm} \frac{M}{n} - \frac{M(M-1)}{n(n-1)} \mu^{bm} \tag{23}$$

Ignoring integer constraints, we can rewrite equations (21)-(23) above as:

$$P(B) = 1 - \frac{\alpha(\alpha n - 1)}{n-1} \mu^{b(1-\alpha)n} - (1-\alpha) \bar{\rho}^{b\alpha n}$$

$$P(P) = (1-\alpha) \left(\bar{\rho}^{b\alpha n} - \underline{\rho}^{b\alpha n} \right)$$

$$P(R) = \underline{\rho}^{b\alpha n} (1-\alpha) - \frac{(1-\alpha)((1-\alpha)n - 1)}{n-1} \mu^{b\alpha n}.$$

[38] Note that such realizations have positive probability for all F with full support.

Therefore:

$$\begin{aligned}\omega_m &\geq \frac{n+3}{2(n+1)}\left[1-\frac{\alpha(\alpha n-1)}{n-1}\mu^{b(1-\alpha)n}-(1-\alpha)\bar{\rho}^{b\alpha n}\right] \\ &+ \frac{n-1}{2(n+1)}\left[(1-\alpha)\bar{\rho}^{b\alpha n}-\frac{(1-\alpha)((1-\alpha)n-1)}{n-1}\mu^{b\alpha n}\right]\end{aligned}$$

The condition that $\omega_m \geq m/n$ is equivalent to

$$\begin{aligned}(n+3)(n-1)-4(n-1)(1-\alpha)\bar{\rho}^{b\alpha n}-(n-1)(1-\alpha)((1-\alpha)n-1)\mu^{b\alpha n}- \\ -(n+3)\alpha(\alpha n-1)\mu^{b(1-\alpha)n}-2\alpha(n+1)(n-1)\geq 0 \quad (24)\end{aligned}$$

If $n = 3$, then $\alpha = 1/3$, $\bar{\rho}(3) = \mu(3) = 2/3$. Substituting these values, we can verify immediately that the condition is satisfied for all $b > 0$. Suppose then $n > 3$, and consider some loose bounds on the various parameters. We know that $\bar{\rho}^{b\alpha n} < 1$ and $\mu^{b(1-\alpha)n} < \mu^{b\alpha n} < \mu < \frac{1}{2}$ provided $b\alpha n = bm \geq 1$ and, for the last inequality, $n > 3$. Hence, a sufficient condition for (24) is:

$$(n+3)(n-1)-4(n-1)(1-\alpha)-\frac{n-1}{2}(1-\alpha)((1-\alpha)n-1)-\frac{n+3}{2}\alpha(\alpha n-1)-2\alpha(n+1)(n-1)\geq 0$$

This is equivalent to:

$$a(\alpha)n^2+c(\alpha)n+\frac{1}{2}\geq 0 \quad (25)$$

where:

$$\begin{aligned}a(\alpha) &= \frac{1}{2}-\alpha-\alpha^2 \\ c(\alpha) &= -(\alpha^2+1-3\alpha)\end{aligned}$$

If $f(\alpha) \equiv \alpha^3 - 6\alpha^2 + 13\alpha - 4 \leq 0$, the discriminant of the left handside of inequality (25) is negative. Provided $a(\alpha) \geq 0$, inequality (25) would then hold. Note that $f(\alpha)$ is increasing in α and that there exists an $\alpha^* > 0$ such that $f(\alpha^*) = 0$. It can be easily verified that $a(\alpha) \geq 0$ for all $\alpha \leq \frac{\sqrt{3}-1}{2}$, and $\alpha^* < \frac{\sqrt{3}-1}{2}$. It follows that (25) is satisfied, and thus (24) is satisfied and $\omega_m \geq \alpha$ for all $n > 3$ for all $\alpha \in (1/n, \alpha^*)$. To conclude the proof, we need to show that (24) is also satisfied when $\alpha \in \left[\alpha^*, \frac{n-1}{2n}\right]$[39].

[39] The upper bound comes from the fact that $m = \alpha \cdot n$, n is odd. and $m < M$ so that $m \leq \frac{n-1}{2}$

For any $b \geq 1$ and $n > 3$, $\mu^{b(1-\alpha)n} < \mu^{b\alpha n} < \mu^{\alpha n} < \frac{1}{2^{\alpha^* \cdot n}}, \forall \alpha \in \left[\alpha^*, \frac{n-1}{2n}\right]$. Thus a sufficient condition for (24) is:

$$n^2 \left[1 - 2\alpha - \frac{1 - 2\alpha + 2\alpha^2}{2^{\alpha^* \cdot n}}\right] + n \left[4\alpha - 2 + \frac{1 - \alpha - \alpha^2}{2^{\alpha^* \cdot n - 1}}\right] + 1 - 2\alpha + \frac{4\alpha - 1}{2^{\alpha^* \cdot n}} \geq 0$$

Rearrange the terms as :

$$(1 - 2\alpha)(n-1)^2 + \frac{1}{2^{\alpha^* \cdot n}} \left[-2(n^2 + n)\alpha^2 + 2(n^2 - n + 2)\alpha - (n-1)^2\right] \geq 0 \tag{26}$$

One can show that $-2(n^2+n)\alpha^2 + 2(n^2 - n + 2)\alpha \geq \alpha(1-\alpha)(n-1)^2$ for $\alpha \in \left[\alpha^*, \frac{n-1}{2n}\right]$. A sufficient condition for inequality (26) is then:

$$\forall \alpha \in \left[\alpha^*, \frac{n-1}{2n}\right], 1 - 2\alpha - \frac{1}{2^{\alpha^* \cdot n}} + \frac{\alpha(1-\alpha)}{2^{\alpha^* \cdot n}} \geq 0$$

The left handside of this equation is decreasing in α, and thus the inequality is satisfied if it is satisfied at $\alpha = \frac{n-1}{2n}$. The condition becomes:

$$\frac{2^{\alpha^* \cdot n}}{n} \geq 1 - \frac{n^2 - 1}{4n^2} = \frac{3n^2 + 1}{4n^2}$$

The difference of the two terms is increasing in n for $n \geq 5$. Moreover, it is satisfied at $n = 7$[40], so is satisfied for all $n \geq 7$. Thus for $n \geq 7$, we can conclude that (24) is satisfied for all $b \geq 1$ and for all α. To complete the proof, we need to consider the last remaining case: $n = 5$. But at $n = 5, \alpha \in \{\frac{1}{5}, \frac{2}{5}\}, \bar{\rho} = \frac{5}{6}, \mu = \frac{2}{7}$ and $b = 1$, (24) is satisfied. Hence at $n = 5$, it is also satisfied for $b > 1$. The proposition is proven. $\square$

7.4 Proof of Proposition 2.

Proposition 2. *There exist distributions* $\mathbf{F}''$ *such that if* $F \in \mathbf{F}''$, *then* $W < W_0$ *for all* n *and* m.

Proof. Recall that V_m denotes the sum of realized minority values ($V_m = \sum_{i \in m} v_i$), and V_M the sum of realized majority values ($V_M = \sum_{j \in M} v_j$). Suppose $F(v) = v^b$, $b > 0$. We show here that $W < W_0$ if $b \geq 1$, for all n, m.

[40] To see that, one can use the fact that $\alpha^* > 0.36$

For value realizations such that the condition in Theorem 1 is not satisfied, the equilibrium construction selects the majority voting outcome, and thus $(W|\overline{v}_g < \mu v_{(2)G}) = (W_0|\overline{v}_g < \mu v_{(2)G})$. When the value realizations are in areas R ($\underline{\rho}\overline{v}_M > \overline{v}_m > \mu v_{(2)M}$) and P ($\overline{\rho v}_M > \overline{v}_m > \underline{\rho}\overline{v}_M$), $\overline{v}_M > \overline{v}_m$, and given $m < M$ and i.i.d. values, it follows that $E[V_M|R,P] > E[V_m|R,P]$. Thus $(W|\overline{\rho v}_M > \overline{v}_m > \mu v_{(2)M}) < (W_0|\overline{\rho v}_M > \overline{v}_m > \mu v_{(2)M})$. Hence, for all n and m, a sufficient condition for $W < W_0$ is $E[V_M|B] > E[V_m|B]$, where B is the area of value realizations such that $\overline{v}_m > \overline{\rho v}_M, \overline{v}_M > \mu v_{(2)m}$. The proposition is an immediate result of the following Lemma.

Lemma A.1. *If* $F(v) = v^b$, *then:*

$$\Pr(B)E(V_m|B) = \frac{bm}{b+1} - \frac{b^2Mm}{(bn+1)(b+1)}\rho^{bm+1} - \\ - \mu^{bM}\frac{b^2m(m-1)}{bn+1}\left[\frac{1}{b(n-1)} + \frac{(b(m-1)+1)}{(b+1)(b(n-1)+1)}\right]$$

$$\Pr(B)E(V_M|B) = \frac{bM}{b+1} - \frac{bM(bM+1)}{(b+1)(bn+1)}\bar{\rho}^{bm} - \frac{b^3Mm(m-1)\mu^{bM+1}}{(b+1)(b(n-1)+1)(bn+1)}$$

Proof of Lemma A1. Recall that $\overline{v}_m > \overline{\rho v}_M, \overline{v}_M > \mu v_{(2)m}$. If we call $x = \bar{v}_m, y = v_{(2)m}$, and $z = \bar{v}_M$, then:

$$\Pr(B)E[V_M|B] = \int_{x=0}^{1}\int_{y=0}^{x}\int_{z=\mu y}^{min\left(\frac{x}{\rho},1\right)}\left[\frac{bM+1}{b+1}z\right]b^2m(m-1)y^{b(m-1)-1}x^{b-1}Mbz^{bM-1}$$
$$= \frac{bM}{b+1} - \frac{bM(bM+1)}{(b+1)(bn+1)}\bar{\rho}^{bm} - \frac{b^3Mm(m-1)\mu^{bM+1}}{(b+1)(b(n-1)+1)(bn+1)}$$

$$\Pr(B)E[V_m|B] = \int_{x=0}^{1}\int_{y=0}^{x}\int_{z=\mu y}^{min\left(\frac{x}{\rho},1\right)}\left[x + \frac{b(m-1)+1}{b+1}y\right]b^2m(m-1)y^{b(m-1)-1}x^{b-1}Mbz^{bM-1}$$
$$= \frac{bm}{b+1} - \frac{b^2Mm}{(bn+1)(b+1)}\rho^{bm+1} - \mu^{bM}\frac{b^2m(m-1)}{bn+1}\left[\frac{1}{b(n-1)} + \frac{(b(m-1)+1)}{(b+1)(b(n-1)+1)}\right]$$

□

The proof of Proposition 2 proceeds in two stages. First, we show that if $W < W_0$ for

$b=1$, the uniform case, then $W < W_0$ for $b > 1$. Second, we show that $W < W_0$ for $b = 1$.

Given Lemma A.1, for any b, a sufficient condition for $W < W_0$ is:

$$\begin{aligned} W < W_0 \;\; &\Leftarrow \;\; 2\frac{b}{b+1}\left[\frac{M-m}{2} - \frac{M(bM+1-bm\bar{\rho})}{2(bn+1)}\bar{\rho}^{bm}\right] \\ &- \;\; \frac{b^2 m(m-1)}{(b+1)(b(n-1)+1)(bn+1)}\left[bM\mu - \frac{b^2 m(n-1)+b(n-1)+bn+1}{b(n-1)}\right]\mu^{bM} > 0 \\ &\Leftrightarrow \;\; 2\frac{b}{b+1}\left[\frac{M-m}{2} - \frac{M(bM+1-bm\bar{\rho})}{2(bn+1)}\bar{\rho}^{bm}\right] \\ &- \;\; 2\frac{b}{b+1}\left[\frac{bm(m-1)}{b(n-1)}\frac{1}{2(b(n-1)+1)(bn+1)}\left[b^2(n-1)(M\mu-m-1)-bn-1\right]\mu^{bM}\right] > 0 \end{aligned}$$

Note first that:

$$\begin{aligned} b^2(n-1)(M\mu - m) - bn - 1 \;\; &< \;\; b^2(n-1)(n\mu-1) - b(n-1) \\ &< \;\; b^2(n-1)(n-1) - b(n-1) \end{aligned}$$

Hence:

$$\begin{aligned} \frac{b^2(n-1)(M\mu-m-1)-bn-1}{2(b(n-1)+1)(bn+1)} \;\; &< \;\; \frac{b(n-1)}{2(bn+1)} \\ &\leq \;\; \frac{1}{2} \end{aligned}$$

Thus $W < W_0$ if:

$$\frac{M-m}{2} - \frac{M(bM+1-bm\bar{\rho})}{2(bn+1)}\bar{\rho}^{bm} - \frac{m(m-1)}{(n-1)}\frac{\mu^{bM}}{2} \geq 0 \tag{27}$$

Straightforward manipulations show that $\frac{M(bM+1-bm\bar{\rho})}{2(bn+1)}\bar{\rho}^{bm}$ is decreasing in b. The term in μ^{bM} is obviously decreasing in b. Hence, if condition (27) is satisfied at $b=1$, then it is satisfied for all $b > 1$.

(b) We can thus focus on the case $b=1$. First, we consider the case $n=3$, which is slightly different from the general case. If $n=3$, then $m=1$, $M=2$, $\bar{\rho}=2/3$, and $\mu = 2/3$. Condition (27) becomes: $1/2 - 7/18 > 0$ and is satisfied.

Suppose then $n > 3$. Substituting $b=1$ and $M = n-m$, condition (27) becomes:

$$\frac{n-2m}{2}-\frac{(n-m)(n-m+1-m\bar{\rho})}{2(n+1)}\bar{\rho}^m-\frac{m(m-1)}{2(n-1)}\mu^{n-m}\geq 0$$

Suppose first $n \geq (2m+3)$, or $m \leq (n-3)/2$. Given $n > 3$ and $\bar{\rho} = 1 - \frac{8}{(n+1)(n+3)}$, if $n \geq (2m+3)$:

$$\frac{n-2m}{2}-\frac{(n-m)(n-m+1-m\bar{\rho})}{2(n+1)}\bar{\rho}^m \geq \frac{37m}{2(n+1)^2(n+3)}$$

Hence the condition becomes:

$$\frac{37m}{(n+1)^2(n+3)}-\frac{m(m-1)}{(n-1)}\left(\frac{1}{2}\right)^{n-m}\geq 0$$

But $37m(n-1)-m(m-1)(n+1)^2(n+3)\left(\frac{1}{2}\right)^{n-m} \geq 37m(n-1)-m(m-1)(n+1)^2(n+3)\left(\frac{1}{2}\right)^{\frac{n+1}{2}} \geq 37(n-1)-(m-1)(n+1)^2(n+3)\left(\frac{1}{2}\right)^{\frac{n+1}{2}}$. Finally, notice that $37(n-1)-(m-1)(n+1)^2(n+3)\left(\frac{1}{2}\right)^{\frac{n+1}{2}}$ evaluated at $m=\frac{n-1}{2}$ is always positive for any $n>3$, and thus must be positive for all $m \leq (n-3)/2$. Therefore, condition (27) is always satisfied for $n > 3$ and $n \geq (2m+3)$

The condition $n \geq (2m+3)$ excludes the only case $m=\frac{n-1}{2}$. Suppose then $m=\frac{n-1}{2}$. Inthis case, $M\mu - m < 0$ and the term in μ^M in condition 27 is positive. A sufficient condition for $W < W_0$ is then:

$$2\frac{b}{b+1}\left[\frac{M-m}{2}-\frac{M(bM+1-bm\bar{\rho})}{2(bn+1)}\bar{\rho}^{bm}\right]>0$$

or, with $m=\frac{n-1}{2}$:

$$\frac{1}{2}-\frac{\frac{n+3}{2}-\frac{n-1}{2}\bar{\rho}}{4}\bar{\rho}^m=\frac{1}{2}-\frac{2+\frac{8}{(n+1)(n+3)}}{4}\bar{\rho}^m>0$$

Or:

$$\left[1+\frac{4}{(n+1)(n+3)}\right]exp\left(\frac{n-1}{2}ln\left(1-\frac{8}{(n+1)(n+3)}\right)\right)<1$$

Denote $x = \frac{4}{(n+1)(n+3)}$. Note that:

$$\begin{aligned} exp\left(\frac{n-1}{2} ln\left(1 - \frac{8}{(n+1)(n+3)}\right)\right) &= exp\left(\frac{n-1}{2} ln\left(1-2x\right)\right) \\ &< exp\left(-(n-1)x\right) \end{aligned}$$

But $f(x) = (1+x)exp(-(n-1)x)$ is decreasing in x and is equal to 1 at $x = 0$. Hence, the inequality is satisfied, for any n. This concludes the proof. □

7.5 Proof of Theorem 3

Theorem 3. *Suppose R2 is the rationing rule. For all $n > 3$ odd, m, and F, there exists a threshold $\mu_{R2}(n) > 0$ such that if $\overline{v}_g \geq \mu_{R2}(n) Max[v_{(2)G}, v_{(2)g}]$, there exists a fully revealing ex ante equilibrium with trade where $\overline{v}_G$ and $\overline{v}_g$ randomize between demanding $(n-1)/2$ votes (with probabilities $q'_{\overline{G}}$ and $q'_{\overline{g}}$ respectively) and selling their vote, and all other individuals sell. The randomization probabilities $q'_{\overline{G}}$ and $q'_{\overline{g}}$ and the price p' depend on $\overline{v}_g$ and $\overline{v}_G$, but for all $\overline{v}_G$ and $\overline{v}_g \geq \mu_{R2}(n) Max[v_{(2)G}, v_{(2)g}]$, $q'_{\overline{G}} \in [\frac{n-1}{n+1}, 1]$ and $q'_{\overline{g}} \in [\frac{n-1}{n+1}, 1]$. The threshold $\mu_{R2}(n)$ is given by:*

$$\mu_{R2}(n) = \frac{(n-1)^2}{2^{n-2} n} \binom{n-3}{\frac{n-3}{2}}$$

Proof. The theorem is implied by the following three lemmas.

Lemma A4. *Suppose $\overline{v}_M / \overline{v}_m \geq (n+1)/(n-1)$. Then for all $n > 3$ odd, m, and F, if $\overline{v}_m \geq \mu_{R2}(n) Max[v_{(2)M}, v_{(2)m}]$, there exists a fully revealing ex ante equilibrium with trade where $\overline{v}_M$ demands $(n-1)/2$ votes with probability 1, $\overline{v}_m$ randomizes between demanding $(n-1)/2$ votes (with probability $q'_{\overline{m}} = (n-1)/(n+1)$ and selling, and all others sell. The equilibrium price p' equals $\overline{v}_m/(n-1)$.*

Lemma A5. *Suppose $\overline{v}_M / \overline{v}_m \leq (n+3)/(n+1)$. Then for all $n > 3$ odd, m, and F, if $\overline{v}_M \geq \mu_{R2}(n) Max[v_{(2)M}, v_{(2)m}]$, there exists a fully revealing ex ante equilibrium with trade where $\overline{v}_m$ demands $(n-1)/2$ votes with probability 1, $\overline{v}_M$ randomizes between demanding $(n-1)/2$ votes (with probability $q'_{\overline{M}} = (n-1)/(n+1)$ and selling, and all others sell. The equilibrium price p' equals $\overline{v}_M/(n-1)$.*

Lemma A6. *Suppose $\overline{v}_M / \overline{v}_m \in ((n+3)/(n+1), (n+1)/(n-1))$. Then for all $n > 3$*

odd, m, and F, if :

$$\overline{v}_m \geq \mu_{R2}(n)\frac{2(nx-n-1)}{(n-1)(x-1)x}Max[v_{(2)M}, v_{(2)m}]$$

where $x \equiv \overline{v}_M/\overline{v}_m$, *there exists a fully revealing ex ante equilibrium with trade where $\overline{v}_M$ and $\overline{v}_m$ randomize between demanding $(n-1)/2$ votes (with probabilities $q'_{\overline{M}}$ and $q'_{\overline{m}}$ respectively) and selling their vote, and all other individuals sell. The randomization probabilities $q'_{\overline{M}}$ and $q'_{\overline{m}}$ and the price p'* solve:

$$\begin{aligned} q'_{\overline{M}} + q'_{\overline{m}} &= \frac{2n}{n+1} \\ p' &= \left(\frac{2-q'_{\overline{M}}}{n-1}\right)\overline{v}_m \\ p' &= \left(\frac{q'_{\overline{m}}}{n-1}\right)\overline{v}_M. \end{aligned}$$

Note that in lemmas A4 and A6, $\overline{v}_m < \overline{v}_M$, or $\overline{v}_m \equiv \overline{v}_g$, and the condition thus applies to $\overline{v}_g$, as stated in the theorem. In Lemma A5 the condition is stated in terms of $\overline{v}_M$, and $\overline{v}_M \lesseqgtr \overline{v}_m$, but if the condition is satisfied for $\overline{v}_g = \min[\overline{v}_M, \overline{v}_m]$, then it is always satisfied for $\overline{v}_M$ (i.e. the condition stated in the theorem is sufficient for the condition stated in the lemma). Finally, in Lemma A6, the condition depends on $x \equiv \overline{v}_M/\overline{v}_m$. Over the interval $x \in (\frac{n+3}{n+1}, \frac{n+1}{n-1})$, the expression $\frac{2(nx-n-1)}{(n-1)(x-1)x}$ is increasing in x, and maximal at $x = \frac{n+1}{n-1}$ where $\frac{2(nx-n-1)}{(n-1)(x-1)x} = 1$ for all n. Hence again the condition stated in the theorem is sufficient for the condition stated in the lemma.

As in the case of Theorem 1, the proof is organized in two stages. First, we show that the strategies and price described in the lemmas are an equilibrium if the direction of preferences associated with each demand is commonly known. Second, we show that when preferences are private information the equilibrium is fully revealing.

7.5.1 Ex ante equilibrium with full information.

Suppose first that the direction of preferences associated with each demand is commonly known. Expected market balance requires $(q'_{\overline{M}}+q'_{\overline{m}})(n-1)/2 = (n-2)+(1-q'_{\overline{M}})+(1-q'_{\overline{m}})$, or:

$$q'_{\overline{M}} + q'_{\overline{m}} = \frac{2n}{n+1}$$

We begin by proving Lemma A4.

Proof of Lemma A4. Recall that we denote by $U_m(s)$ the expected utility to voter $\overline{v}_m$ from demand s (and similarly for $U_M(s)$). Then, in the candidate equilibrium:

$$\begin{aligned} U_m(-1) &= \frac{p'}{2} \\ U_m\left(\frac{n-1}{2}\right) &= \frac{\overline{v}_m}{2} - \frac{n-2}{2}p' \end{aligned}$$

Indifference between the two actions requires:

$$p' = \frac{\overline{v}_m}{n-1}$$

By expected market balance, if $q'_{\overline{M}} = 1$, then:

$$q'_{\overline{m}} = \frac{n-1}{n+1}.$$

To verify that this is indeed an equilibrium, we need to rule out profitable deviations.

(i) Consider first voter $\overline{v}_M$. For any $s_M \in (\frac{n-1}{2}, n-1]$, $U_M(s_M) < U_M(\frac{n-1}{2})$: demanding more votes than required to achieve a strict majority is strictly costly and does not affect the probability of rationing $\overline{v}_m$ (because $s_M > \frac{n-1}{2}$ becomes relevant only once $s_M = \frac{n-1}{2}$ is satisfied, at which point $\overline{v}_m$ is already rationed and $\overline{v}_M$ holds a majority of votes). For any $s_M \in [0, \frac{n-1}{2})$, $U_M(s_M) < U_M(-1)$: demanding less than $\frac{n-1}{2}$ votes is dominated by selling because demanding any positive number of votes less than $\frac{n-1}{2}$ would be costly and not affect the outcome, whether $\overline{v}_m$ is selling or demanding $\frac{n-1}{2}$. Therefore, the majority leader is optimizing if and only if the deviation to selling is not profitable. In the candidate equilibrium:

$$\begin{aligned} U_M(-1) &= q'_{\overline{m}}(\frac{1}{2}p') + (1 - q'_{\overline{m}})(\bar{v}_M) \\ U_M\left(\frac{n-1}{2}\right) &= q'_{\overline{M}}(\frac{\overline{v}_m}{2} - \frac{n-2}{2}p') + (1 - q'_{\overline{M}})(\overline{v}_m - \frac{n-1}{2}p) \end{aligned}$$

The deviation is not desirable if and only if $\overline{v}_M/\overline{v}_m \geq (n+1)/(n-1)$.

(ii) Consider voter $\overline{v}_m$. Given $s_M = \frac{n-1}{2}$, $U_m(s_m) < U_m(\frac{n-1}{2})$ for all $s_m > 0 \neq \frac{n-1}{2}$, and $U_m(0) < U_m(-1)$. Hence no deviation dominates randomizing over selling or demanding $\frac{n-1}{2}$.

(iii) Consider now $v_i \in M$, $v_i \neq \overline{v}_M$. Here the rationing rule makes an important difference. With $R2$, any incremental demand has a positive incremental impact on the probability that $\overline{v}_m$ and/or $\overline{v}_M$ will be rationed. We need to consider and exclude deviation to any $s_i \in [0, \frac{n-1}{2}]$. We show here, however, that for all $v_i \in M$, $v_i \neq \overline{v}_M$, $U_i(-1) \geq U_i(0)$ is sufficient to guarantee $U_i(-1) \geq U_i(s_i)$ for all $s_i \in [0, \frac{n-1}{2}]$. Hence only one possible deviation, to $s_i = 0$, needs to be ruled out. It is this step that makes the proof possible.

Consider the utilities from demanding $s+1$ votes and demanding s. The probability of receiving 0 to $s-1$ votes is identical when demanding s or $s+1$ votes. The probability of receiving s votes when demanding s votes is equal to the probability of receiving s or $s+1$ votes when demanding $s+1$ votes. Therefore, calling x the number of votes received after rationing, for all $s \in [0, \frac{n-5}{2}]$:

$$\begin{aligned} U_i(s+1) - U_i(s) &= (1-q'_{\overline{m}})(-p') + q'_{\overline{m}}[P(x_i = s+1|s+1)] \cdot \\ &\cdot \left[(P(x_{\overline{m}} = \frac{n-1}{2}|s_i = s) - P(x_{\overline{m}} = \frac{n-1}{2}|s_i = s+1))v - p' \right] \end{aligned}$$

Calling $[(P(x_{\overline{m}} = \frac{n-1}{2}|s_i = s) - P(x_{\overline{m}} = \frac{n-1}{2}|s_i = s+1))v - p'] \equiv \Delta(s)$, we can rewrite the expression more concisely as:

$$U_i(s+1) - U_i(s) = q'_{\overline{m}}[P(x_i = s+1|s+1)\Delta(s)] - (1-q'_{\overline{m}})p' \tag{28}$$

and thus, for $s \in [0, \frac{n-5}{2}]$:

$$U_i(s) - U_i(s-1) = q'_{\overline{m}}(P(x_i = s|s)\Delta(s-1) - (1-q'_{\overline{m}})p' \tag{29}$$

where, as argued above, $P(x_i = s|s) > P(x_i = s+1|s+1)$.

Given:

$$P(x_{\overline{m}} = \frac{n-1}{2}|s_i = s) = \sum_{z=\frac{n-1}{2}}^{n-3-s} \binom{n-3-s}{z} \left(\frac{1}{2}\right)^{n-3-s} \quad \forall s \in \left[0, \frac{n-5}{2}\right],$$

and hence:

$$\Delta(s) = \sum_{z=\frac{n-1}{2}}^{n-3-s} \left(\frac{1}{2}\right)^{n-4-s} \left[\binom{n-4-s}{z-1}(1 - \frac{n-3-s}{2z})\right] v - p',$$

it is possible to show that $\Delta(s) \leq 0$ implies $\Delta(s+1) \leq 0$ for all $s \in \left[0, \frac{n-5}{2}\right]$[41]. It follows that if 0 is preferred to 1, then 0 dominates all strategies up to buying $\frac{n-3}{2}$ votes.

From (28):

$$U_i(1) - U_i(0) = q'_{\overline{m}}(P(x_i = 1|1)\Delta(0) - (1 - q'_{\overline{m}})p'$$

and since

$$\Delta(0) = \left[\frac{\binom{n-4}{\frac{n-7}{2}}}{2^{n-4}} - \frac{\binom{n-3}{\frac{n-3}{2}}}{2^{n-2}}\right] v_i - p'$$

it follows that $U_i(1) < U_i(0)$ if $\Delta(0) \leq 0$ or, given $p' = \frac{\overline{v}_m}{n-1}$, $U_i(1) < U_i(0)$ for all $v_i \in M$, $v_i \neq \overline{v}_M$, if $\overline{v}_m \geq \frac{(n-1)\left[4\binom{n-4}{\frac{n-7}{2}} - \binom{n-3}{\frac{n-3}{2}}\right]}{2^{n-2}} v_{(2)M}$. But:

$$\begin{aligned} U_i(0) &= q'_{\overline{m}}\left[\frac{1}{2} + \frac{\binom{n-3}{\frac{n-3}{2}}}{2^{n-2}}\right] v_i + (1 - q'_{\overline{m}})v_i \\ U_i(-1) &= q'_{\overline{m}}\left(\frac{v_i}{2} + p'\right) + (1 - q'_{\overline{m}})\left(v_i + \frac{p'}{2}\right) \end{aligned}$$

and thus:

$$U_i(0) < U_i(-1) \text{ for all } v_i \in M, v_i \neq \overline{v}_M \text{ if } \frac{\binom{n-3}{\frac{n-3}{2}}}{2^{n-2}} v_{(2)M} \leq \frac{v_{\bar{m}}}{n-1}$$

or:

$$\overline{v}_m \geq \frac{(n-1)\binom{n-3}{\frac{n-3}{2}}}{2^{n-2}} v_{(2)M}. \tag{30}$$

Note that $4\binom{n-4}{\frac{n-7}{2}} - \binom{n-3}{\frac{n-3}{2}} \leq \binom{n-3}{\frac{n-3}{2}}$. Hence, the last condition is sufficient for $\Delta(0) \leq 0$. It is the condition in the lemma, and it is sufficient to establish both that $s_i = -1$ dominates $s_i = 0$, and that $s_i = 0$, and hence $s_i = -1$, dominate all $s_i \in [1, \frac{n-3}{2}]$.

The last step in proof is verifying that a deviation to $\frac{n-1}{2}$is not profitable. Note that $P(x_{\overline{m}} = \frac{n-1}{2}|s_i = \frac{n-1}{2}) = P(x_{\overline{m}} = \frac{n-1}{2}|s_i = \frac{n-3}{2})$: demanding $\frac{n-1}{2}$ does not change the probability that $\overline{v}_m$ receive $\frac{n-1}{2}$ votes, relative to demanding $\frac{n-3}{2}$. It may however lead to a

[41] The proof requires some work. Details are posted at: columbia.edu/~st2511/demundone/theorem3_supp.pdf.

higher number of votes paid. Thus $s_i = \frac{n-1}{2}$ is dominated by $s_i = \frac{n-3}{2}$ which, as we have seen, is dominated by $s_i = 0$. Ruling out a profitable deviation to 0 is thus sufficient to rule out all other deviations. It follows that no deviation is profitable if (30) is satisfied.

(iv). Finally, consider $v_i \in m$, $v_i \neq \overline{v}_m$. With probability $q'_{\overline{m}}$, $\overline{v}_m$ demands $\frac{n-1}{2}$ votes, as does $\overline{v}_M$. In this case, a demand of votes by v_i is justified if it increases the probability that $\overline{v}_M$ is rationed. This is exactly the reasoning we considered in point (iii) above, for $v_i \in M$. We established there that if $s_i = -1$ dominates $s_i = 0$, then it dominates all $s_i \in [0, \frac{n-3}{2}]$. With probability $(1 - q'_{\overline{m}})$, however, $\overline{v}_m$ sells his vote. Since $\overline{v}_M$ demands $\frac{n-1}{2}$ votes with probability 1, in this case $s_i = 0$ is dominated by $s_i = -1$ and any $s_i \in [1, \frac{n-3}{2}]$ is dominated by $s_i = \frac{n-1}{2}$ (because for any $s_i \in [1, \frac{n-3}{2}]$, neither $\overline{v}_M$ nor v_i are rationed, $\overline{v}_M$ wins, and v_i pays $s_i p'$). We conclude the only deviations from $s_i = -1$ that cannot be excluded are to $s_i = 0$, and $s_i = \frac{n-1}{2}$. The condition $U_i(0) < U_i(-1)$ leads to a condition parallel to (30):

$$\overline{v}_m \geq \frac{(n-1)\binom{n-3}{\frac{n-3}{2}}}{2^{n-2}} v_{(2)m}. \tag{31}$$

Consider now $U_i\left(\frac{n-1}{2}\right)$. If $\overline{v}_m$ demands $\frac{n-1}{2}$ votes, v_i can expect to receive $\dfrac{n-3}{3}$ votes. The minority wins unless $\overline{v}_M$ receives $\dfrac{n-1}{2}$ votes. If $\overline{v}_m$ sells his vote, v_i receives $\dfrac{n-1}{2}$ votes with probability 1/2 (and wins), and $\dfrac{n-3}{2}$ votes with probability 1/2 (and loses). Hence:

$$\begin{aligned} U_i\left(\frac{n-1}{2}\right) &= q'_{\overline{m}}[(1 - P(x_{\overline{M}} = \frac{n-1}{2}|s_m = \frac{n-1}{2}, s_i = \frac{n-1}{2}))v_i - \frac{n-3}{3}p'] + \\ &+(1 - q'_{\overline{m}})(\frac{1}{2}v_i - \frac{n-2}{2}p') \end{aligned}$$

Call $P(x_{\overline{M}} = \frac{n-1}{2}|s_m = \frac{n-1}{2}, s_i = \frac{n-1}{2}) = \sum_{z=\frac{n-1}{2}}^{n-3} \sum_{y=0}^{n-3-z} \binom{n-3}{n-3-z-y, z, y} \left(\frac{1}{3}\right)^{n-3} \equiv \delta$. For all $v_i \in m$, $v_i \neq \overline{v}_m$, the deviation to buying $\frac{n-1}{2}$ is not desirable if:

$$\overline{v}_m \geq \frac{n(3 - 6\delta) + 3 + 6\delta}{2n + 6} v_{(2)m}.$$

This constraint is not binding if $n = 5$ (when the ratio equals $\frac{24}{23} > 1$) and when $n = 7$ (when the ratio equals 1), and it is less stringent than (31) for all $n \geq 9$.

We conclude that the equilibrium exists if $\overline{v}_M/\overline{v}_m \geq (n+1)/(n-1)$, and

$$\overline{v}_m \geq \frac{(n-1)\binom{n-3}{\frac{n-3}{2}}}{2^{n-2}} Max[v_{(2)M}, v_{(2)m}],$$

as stated in the lemma.□

Proof of Lemma A5. In the candidate equilibrium:

$$\begin{aligned} U_M(-1) &= \frac{p'}{2} \\ U_M\left(\frac{n-1}{2}\right) &= \frac{\overline{v}_M}{2} - \frac{n-2}{2}p' \end{aligned}$$

Indifference between the two actions requires:

$$p' = \frac{\overline{v}_M}{n-1}$$

By expected market balance, if $q'_{\overline{m}} = 1$, then:

$$q'_{\overline{M}} = \frac{n-1}{n+1}.$$

To verify that this is indeed an equilibrium, we need to rule out profitable deviations.

(i) Consider first voter $\overline{v}_M$. Given $s_m = \frac{n-1}{2}$, $U_M(s_M) < U_M(\frac{n-1}{2})$ for all $s_M > 0 \neq \frac{n-1}{2}$, and $U_M(0) < U_M(-1)$. Hence no deviation dominates randomizing over selling or demanding $\frac{n-1}{2}$.

(ii) Consider voter $\overline{v}_m$. Call $P(k)$ the probability of a minority victory when $\overline{v}_m$ demands k votes, for $k < \frac{n-1}{2}$. Then:

$$\begin{aligned} U_m(-1) &= q'_{\overline{M}}(\frac{1}{2}p') \\ U_m\left(\frac{n-1}{2}\right) &= q'_{\overline{M}}(\frac{\overline{v}_m}{2} - \frac{n-2}{2}p') + (1-q'_{\overline{M}})(\overline{v}_m - \frac{n-1}{2}p) \\ U_m(k) &= q'_{\overline{M}}(-kp') + (1-q'_{\overline{M}})(P(k)\overline{v}_m - kp') \end{aligned}$$

where

$$P(k|n,m) = \frac{\sum_{i=\frac{n+1}{2}-m}^{k} \binom{n-m}{i}\binom{m-1}{k-i}}{\binom{n-1}{k}}$$

Note that $P(k) = 0$ if $k < \frac{n+1}{2} - m$. Moreover, with n fixed, $P(k)$ is increasing in m for $k \in \left[\frac{n+1}{2} - m, \frac{n-3}{2}\right]$[42] Thus if $U_m\left(\frac{n-1}{2}\right) > U_m(k)$ when $m = M - 1$, then $U_m\left(\frac{n-1}{2}\right) > U_m(k)$ for all $m < M$. Suppose then $m = M - 1$. In this case:

$$P(k) = 1 - \frac{\binom{m-1}{k}}{\binom{n-1}{k}}$$

and, for $0 \leq k \leq \frac{n-5}{2}$:

$$\begin{aligned} U(k+1) - U(k) &= -p + (1 - q'_{\overline{M}})(P(k+1) - P(k))\overline{v}_m \\ &= -p + \frac{\binom{m-1}{k}}{\binom{n-1}{k}} \frac{n-m}{n-1-k} \overline{v}_m \end{aligned}$$

The difference is decreasing in k: if one defines $h(k) = \frac{\binom{m-1}{k}}{\binom{n-1}{k}} \frac{n-m}{n-1-k}$,

$$\frac{h(k-1)}{h(k)} = \frac{n-1-k}{m-k} > 1$$

It follows that $U_m\left(\frac{n-1}{2}\right) \geq U_m(k)$ for all $k \in [0, \frac{n-3}{2}]$ if $U_m\left(\frac{n-1}{2}\right) \geq U_m(0)$. But note that $U_m(-1) \geq U_m(0) = 0$. Hence $s_m = -1$ is the only possibly profitable deviation for $\overline{v}_m$. $U_m\left(\frac{n-1}{2}\right) \geq U_m(-1)$ yields the condition: $\overline{v}_M / \overline{v}_m \leq (n+3)/(n+1)$.

(iii) Consider now $v_i \in M$, $v_i \neq \overline{v}_M$. The incentives are identical to (iv) in the proof of Lemma A4: with $\overline{v}_m$ demanding $\frac{n-1}{2}$ with probability 1, the only possibly profitable deviation for $v_i \in M$ are either $s_i = 0$ or $s_i = \frac{n-1}{2}$. For all $v_i \in M$, $v_i \leq v_{(2)M}$, $U_i(-1) \geq U_i(0)$ if:

$$\overline{v}_m \geq \frac{(n-1)\binom{n-3}{\frac{n-3}{2}}}{2^{n-2}} v_{(2)M}, \tag{32}$$

and $U_i(-1) \geq U_i(\frac{n-1}{2})$ if:

$$\overline{v}_m \geq \frac{n(3-6\delta) + 3 + 6\delta}{2n+6} v_{(2)M}.$$

where $\delta \equiv P(x_{\overline{m}} = \frac{n-1}{2} | s_M = \frac{n-1}{2}, s_i = \frac{n-1}{2}) =.\sum_{z=\frac{n-1}{2}}^{n-3} \sum_{y=0}^{n-3-z} \binom{n-3}{n-3-z-y, z, y} \left(\frac{1}{3}\right)^{n-3}$. This latter condition is not binding for $n = \{5, 7\}$ and is less stringent than (32) for all $n \geq 9$.

[42] See columbia.edu/~st2511/demundone/theorem3_supp.pdf.

Thus (32) is sufficient ot guarantee that no $v_i \in M$, $v_i \neq \overline{v}_M$ has an incentive to deviate.

(iv) Finally, consider $v_i \in m$, $v_i \neq \overline{v}_m$. The incentives are identical to (iii) in the proof of Lemma A4: with $\overline{v}_m$ demanding $\frac{n-1}{2}$ with probability 1, the only possibly profitable deviation for $v_i \in m$ is to stay out of the market. For all $v_i \in m$, $v_i \leq v_{(2)m}$, $U_i(-1) \geq U_i(0)$ if:

$$\overline{v}_M \geq \frac{(n-1)\binom{n-3}{\frac{n-3}{2}}}{2^{n-2}} v_{(2)m}.$$

We conclude that the equilibrium exists if $\overline{v}_M/\overline{v}_m \leq (n+3)/(n+1)$, and

$$\overline{v}_M \geq \frac{(n-1)\binom{n-3}{\frac{n-3}{2}}}{2^{n-2}} Max[v_{(2)M}, v_{(2)m}],$$

as stated in the lemma.□

Proof of Lemma A6. In the candidate equilibrium:

$$\begin{aligned} U_m(-1) &= q'_{\overline{M}}(\frac{1}{2}p') \\ U_m\left(\frac{n-1}{2}\right) &= q'_{\overline{M}}(\frac{\overline{v}_m}{2} - \frac{n-2}{2}p') + (1-q'_{\overline{M}})(\overline{v}_m - \frac{n-1}{2}p') \end{aligned}$$

and

$$\begin{aligned} U_M(-1) &= q'_{\overline{m}}(\frac{1}{2}p') + (1-q'_{\overline{m}})(\overline{v}_M) \\ U_M\left(\frac{n-1}{2}\right) &= q'_{\overline{m}}(\frac{\overline{v}_M}{2} - \frac{n-2}{2}p') + (1-q'_{\overline{m}})(\overline{v}_M - \frac{n-1}{2}p') \end{aligned}$$

The two equivalence conditions yield:

$$\begin{aligned} p' &= \left(\frac{2-q'_{\overline{M}}}{n-1}\right)\overline{v}_m \\ p' &= \left(\frac{q'_{\overline{m}}}{n-1}\right)\overline{v}_M \end{aligned}$$

This system has a solution if and only if:

$$\frac{n+3}{n+1} \leq \frac{\overline{v}_M}{\overline{v}_m} \leq \frac{n+1}{n-1}$$

Given the expected market clearing constraint $q'_{\overline{M}} + q'_{\overline{m}} = \frac{2n}{n+1}$, we obtain:

$$
\begin{aligned}
q'_{\overline{M}} &= 2\frac{nx-n-1}{(n+1)(x-1)} \\
q'_{\overline{m}} &= \frac{2}{(n+1)(x-1)}
\end{aligned}
$$

with $x = \frac{\overline{v}_M}{\overline{v}_m}$.

Consider now the scope for deviations.

(i) Consider first voter $\overline{v}_M$. As in the proof of Lemma A5, $U_M(s_M) < U_M(\frac{n-1}{2})$ for all $s_M > 0 \neq \frac{n-1}{2}$, and $U_M(0) < U_M(-1)$. Hence no deviation dominates randomizing over $s_M = -1$ and $s_M = \frac{n-1}{2}$.

(ii) Consider voter $\overline{v}_m$. Again, exactly as in the proof of Lemma A5, the only two possible best responses are $s_m = -1$ and $s_m = \frac{n-1}{2}$. Hence no profitable deviation exists when $\overline{v}_m$ randomizes over the two actions.

(iii) Consider now $v_i \in M$, $v_i \neq \overline{v}_M$. We have established above that if $s_M = \frac{n-1}{2}$ and $s_m = \frac{n-1}{2}$, v_i's best response can put positive probability on only two actions, either $s_i = -1$ or $s_i = 0$. If $s_M = \frac{n-1}{2}$ and $s_m = -1$, v_i's best response is $s_i = -1$. If $s_M = -1$ and $s_m = -1$, v_i's best response is either $s_i = -1$ or $s_i = 0$, which in this case are equivalent. Finally, if $s_M = -1$ and $s_m = \frac{n-1}{2}$, v_i's best response can put positive probability on only two actions, either $s_i = -1$ or $s_i = \frac{n-1}{2}$. It follows that all demands $s_i \in [1, \frac{n-3}{2}]$ are strictly dominated. Only $s_i = 0$ and $s_i = \frac{n-1}{2}$ are possible alternatives to $s_i = -1$: no profitable deviation exists if $U_i(-1) \geq U_i(0)$, and $U_i(-1) \geq U_i(\frac{n-1}{2})$.

We have:

$$
U_i(-1) = q'_{\overline{M}} q'_{\overline{m}} (p' + \frac{1}{2} v_i) + q'_{\overline{M}} (1 - q'_{\overline{m}})(v_i + \frac{1}{2} p') + (1 - q'_{\overline{M}}) q'_{\overline{m}} \frac{1}{2} p' + (1 - q'_{\overline{M}})(1 - q'_{\overline{m}}) v_i;
$$

$$
U_i(0) = q'_{\overline{M}} q'_{\overline{m}} \left(\frac{1 + \binom{n-3}{(n-3)/2} (1/2)^{n-3}}{2} v_i \right) + q'_{\overline{M}} (1 - q'_{\overline{m}}) v_i + (1 - q'_{\overline{M}})(1 - q'_{\overline{m}}) v_i,
$$

where $\left(\dfrac{1+\binom{n-3}{(n-3)/2}(1/2)^{n-3}}{2}\right) = P(x_{\overline{M}} = \frac{n-1}{2}|s_m = \frac{n-1}{2}, s_i = 0)$, and:

$$\begin{aligned} U\left(\frac{n-1}{2}\right) &= q'_{\overline{m}}\left[q'_{\overline{M}}\left((1-\delta)v_i - \frac{n-3}{3}p'\right) + (1-q'_{\overline{M}})\left(\frac{v_i}{2} - \frac{n-2}{2}p'\right)\right] + \\ &\quad +(1-q'_{\overline{m}})\left[v_i - \left(q'_{\overline{M}}\cdot\frac{n-2}{2} + (1-q'_{\overline{M}})\cdot\frac{n-1}{2}\right)p'\right]. \end{aligned}$$

Hence, for all $v_i \in M$, $v_i \leq v_{(2)M}$, $U_i(-1) \geq U_i(0)$ if:

$$\overline{v}_M \geq \frac{\binom{n-3}{\frac{n-3}{2}}(n-1)}{2^{n-2}n}\frac{2(nx-n-1)}{(x-1)}v_{(2)M}. \tag{33}$$

and $U_i(-1) \geq U_i(\frac{n-1}{2})$ if:

$$\overline{v}_M \geq \frac{3(n^2-1)(x-1)[(1+n)(1-x-4\delta)+4\delta nx)]}{15+11n-7n^2-3n^3-6x-18nx+10n^2x+6n^3x+3x^2+3nx^2-3n^2x^2-3n^3x^2}v_{(2)M}. \tag{34}$$

where $x \equiv \frac{\overline{v}_M}{\overline{v}_m}$. For $n = 5, 7$, the right-hand side of (34) is above 1 for any $x \in [\dfrac{n+3}{n+1}, \dfrac{n+1}{n-1}]$ and therefore the constraint is not binding. For $n \geq 9$, (34) is less stringent than (33).[43] Hence (33) is sufficient to guarantee that all $v_i \in M$, $v_i \leq v_{(2)M}$, have no profitable deviation. By dividing both sides of (33) by x, we obtain the condition in the lemma.

(iii) Finally, consider $v_i \in m$, $v_i \neq \overline{v}_m$. Exactly as described in point (iii) above, the analysis so far has established that only $s_i = 0$ and $s_i = \frac{n-1}{2}$ are possible alternatives to $s_i = -1$: no profitable deviation exists if $U_i(-1) \geq U_i(0)$, and $U_i(-1) \geq U_i(\frac{n-1}{2})$.

We have:

$$U_i(-1) = q'_{\overline{M}}q'_{\overline{m}}(p' + \frac{1}{2}v_i) + q'_{\overline{M}}(1-q'_{\overline{m}})\frac{1}{2}p' + (1-q'_{\overline{M}})q'_{\overline{m}}(v_i + \frac{1}{2}p');$$

$$U_i(0) = q'_{\overline{M}}q'_{\overline{m}}(\frac{1+\binom{n-3}{(n-3)/2}(1/2)^{n-3}}{2}v_i) + (1-q'_{\overline{M}})q'_{\overline{m}}v_i,$$

[43] The details are available from the authors.

and:

$$U_i\left(\frac{n-1}{2}\right) = q'_M q'_m\left[(1-\delta)v_i - \frac{n-3}{3}p'\right] + q'_M(1-q'_m)(\frac{1}{2}v_i - \frac{n-2}{2}p') + \\ +(1-q'_M)q'_m(v_i - \frac{n-2}{2}p') + (1-q'_M)(1-q'_m)(v_i - \frac{n-1}{2}p').$$

Thus, for all $v_i \in m$, $v_i \leq v_{(2)m}$, $U_i(-1) \geq U_i(0)$ if:

$$\overline{v}_m \geq \frac{\binom{n-3}{\frac{n-3}{2}}(n-1)}{2^{n-2}n}\frac{2(nx-n-1)}{(x-1)x}v_{(2)m} \tag{35}$$

and $U_i(-1) \geq U_i(\frac{n-1}{2})$ if:

$$\overline{v}_m \geq \frac{3(n^2-1)(x-1)[(1+n)(1+3x+x^2-4\delta)+4\delta nx)]}{x(15+11n-7n^2-3n^3-6x-18nx+10n^2x+6n^3x+3x^2+3nx^2-3n^2x^2-3n^3x^2)}v_{(2)m}. \tag{36}$$

As under point (iii) above, it is possible to show that (35) is a more stringent condition than (36).[44] It is then the sufficient condition, guaranteeing that no profitable deviation exists for all $v_i \in m$, $v_i \neq \overline{v}_m$. $\square$

We now show that when preferences are private information, the strategies and price identified above constitute a fully revealing ex ante equilibrium.

7.5.2 Fully revealing equilibrium.

We proceed as for Theorem 1. We conjecture an equilibrium identical to the full information equilibrium characterized above and show that given others' strategies, the equilibrium price and the knowledge that the market is in a fully revealing equilibrium, each voter's best response when preferences are private information is uniquely identified and equals the voter's best response with full information. Thus the equilibrium exists when preferences are private information and is indeed fully revealing.

(i) Consider first the perspective of voter $\overline{v}_M$, in equilibrium. When the equilibrium exists, expected market balance requires $\overline{v}_M$ to demand a positive number of votes with positive probability. It then follows that the other voter who demands a positive number of votes with positive probability must belong to the minority. If not, $\overline{v}_M$'s best response

[44]See columbia.edu/~st2511/demundone/theorem3_supp.pdf

would be to sell, violating expected market equilibrium. Thus $\overline{v}_M$ also knows that $M-1$ majority members and $m-1$ minority members are offering their vote for sale; he cannot identify them individually, but that is irrelevant. Given that the other net demand for votes comes from a minority voter, $\overline{v}_M$'s best response is identified uniquely and is identical to his best response under full information.

(ii) Consider then the perspective of voter $\overline{v}_m$. Suppose first that $\overline{v}_M/\overline{v}_m \geq \frac{n+1}{n-1}$, and hence $s_M = \frac{n-1}{2}$ with probability 1. Expected market balance requires $\overline{v}_m$ to demand a positive number of votes with positive probability. But that can only be a best response if the voter who demands $\frac{n-1}{2}$ votes belongs to the majority. Again, $\overline{v}_m$ also knows that $M-1$ majority members and $m-1$ minority members are offering their vote for sale; he cannot identify them individually, but that is irrelevant.

Suppose now $\overline{v}_M/\overline{v}_m \in (\frac{n+3}{n+1}, \frac{n+1}{n-1})$. By market balance, the minimal demand on which $\overline{v}_m$ must put positive probability is $\frac{n-3}{2}$ (because $\frac{n-3}{2} = \left(\frac{n-1}{n+1}\right)\left(\frac{n-1}{2}\right) - (1-.\frac{n-1}{n+1}))$. Suppose that the voter demanding $\frac{n-1}{2}$ votes with probability $q'_{\overline{M}}$ were in fact a member of group m. Then, given that all others offer to sell:

$$\begin{aligned}
U_{m,m}(-1) &= q'_{\overline{M}}(\overline{v}_m + \frac{p'}{2}) \\
U_{m,m}(\frac{n-3}{2}) &= q'_{\overline{M}}(\overline{v}_m - \frac{n-3}{2}p') + (1-q'_{\overline{M}})(P(\frac{n-3}{2})\overline{v}_m - \frac{n-3}{2}p') \\
&\leq \overline{v}_m - \frac{n-3}{2}p'
\end{aligned}$$

where $P(\frac{n-3}{2}) < 1$ is, as earlier, the probability that the minority wins when $\overline{v}_m$ is the only buyer in the market and purchases $\frac{n-3}{2}$ votes. The index m,m indicates $\overline{v}_m$'s expected utility if the voter demanding $\frac{n-1}{2}$ votes with probability $q'_{\overline{M}}$ is a member of group m. Given $p' = \overline{v}_m(2-q'_{\overline{M}})/(n-1)$, it is easy to verify that $U_{m,m}(-1) > U_{m,m}(\frac{n-3}{2})$ for all $q'_{\overline{M}} \in (\frac{n-1}{n+1}, 1)$ if $U_{m,m}(-1) > U_{m,m}(\frac{n-3}{2})$ at $q'_{\overline{M}} = \frac{n-1}{n+1}$, a condition satisfied for all $n \geq 5$. Thus, any strategy for $\overline{v}_m$ that satisfies expected market balance cannot be his best response, if the voter demanding $\frac{n-1}{2}$ votes with probability $q'_{\overline{M}}$ belongs to group m. Hence such a voter must belong to group M. Of all remaining voters offering their votes for sale, $M-1$ belongs to the majority, and $m-1$ to the minority. They cannot be distinguished but that has no impact on $\overline{v}_m$'s unique best response.

Finally, suppose either $\overline{v}_M/\overline{v}_m \leq \frac{n+3}{n+1}$. Expected market balance requires $s_m = \frac{n-1}{2}$ with probability 1. But then the other voter demanding $\frac{n-1}{2}$ votes with positive probability cannot belong to the minority (because in a fully revealing equilibrium, if $s_m = \frac{n-1}{2}$ with probability 1, all other minority voters would prefer to sell). Hence again the other voter with positive demand for votes must be a majority voter. All remaining voters are sellers; identifying the group each of them belongs to is not possible but has no impact on $\overline{v}_m$'s unique best response.

(iii) Consider now the perspective of all voters who in the full information equilibrium offer their vote for sale with probability 1: $v_i \in M$, $v_i \neq \overline{v}_M$, or $v_i \in m$, $v_i \neq \overline{v}_m$. By the arguments above, each of them knows that in a fully revealing equilibrium the two voters with positive expected demand must belong to the two different parties. Which one belongs to the majority and which one to the minority cannot be distinguished, but is irrelevant: since in the full information case v_i's best response is $s_i = -1$ with probability 1 whether $v_i \in M$, or $v_i \in m$, it follows that identifying which of the two voters with positive expected demand belongs to which group is irrelevant to v_i's best response. Equally irrelevant is identifying which of the sellers belongs to which group. Although the direction of preferences associated with each individual voter cannot be identified, v_i's best response is unique and identical to his best response with full information.

We can conclude that the equilibrium strategies and price identified by Lemmas A4, A5, and A6 are indeed a fully revealing ex ante equilibrium with private information.□