



Columbia University

Department of Economics *Discussion Paper Series*

On Human Capital and Team Stability

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Discussion Paper No.: 1213 09

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New York, NY 10027

November 2012, updated July 2016

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February 3, 2016

¹We thank Yeon-Koo Che, Navin Kartik, and especially Arnaud Dupuy, Fuhito Kojima and Phil Reny for useful comments. This paper builds on material from an unpublished manuscript circulated under the name “The Roommate Problem Is More Stable Than You Think,” which is now obsolete.

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Abstract

In many economic contexts, agents from a same population team up to better exploit their human capital. In such contexts (often called “roommate matching problems”), stable matchings may fail to exist even when utility is transferable. We show that when each individual has a close substitute, a stable matching can be implemented with minimal policy intervention. Our results shed light on the stability of partnerships on the labor market. Moreover, they imply that the tools crafted in empirical studies of the marriage problem can easily be adapted to many roommate problems.

1 Introduction

The search for a more efficient use of human capital often leads several individuals to team up. Lawyers gather in law firms, doctors associate in medical practices, architects congregate in architectural firms, . . . Such partnerships are typical of the professions; but they extend to other services firms such as consultancies. The mechanisms through which agents match to form these partnerships, and sometimes leave or dissolve them, vary in interesting ways. Yet they have not attracted the attention they deserve in the matching literature.

This statement may seem surprising, given the huge body of work that has followed the seminal works of Gale and Shapley (1962), Koopmans and Beckmann (1957), Becker (1973) and Shapley and Shubik (1971) among others. Yet these matching models mostly consider bi- or multipartite matching games, in which each “team” consists of partners that come from prespecified, separate subpopulations. This feature is especially prominent in models with transferable utility (hereafter TU): analyses of the marriage market have traditionally considered man/woman matches (see Chiappori, Salanié, and Weiss (2015), Galichon and Salanié (2015), or Dupuy and Galichon (2014)), labor market models have employers and employees (Gabaix and Landier (2008), Tervio (2008)), and so on. The partnerships we have mentioned, however, have a very specific feature that distinguishes them from the bipartite literature: the agents who team up to jointly exploit their human capital very often belong to the *same population*. Lawyers who join forces in a law firm may each have their own specialty, but in larger firms several lawyers share the same specialty. Dentists rarely match with oncologists, and consultancies congregate experts who often come from the same pool of individuals.

It has been known for some time that this apparently minor difference in settings may generate largely divergent results. Take, for instance, the specific case in which teams consist of exactly two people, both coming from the same population—what is classically called the *roommate matching problem*. In the No Transferable Utility (NTU) context, Gale and Shapley (1962) showed that stable matchings may not exist. In the TU framework, any potential couple generates a surplus that is (endogenously) shared by its members. The resulting matching must satisfy a stability property, reflecting robustness to unilateral and bilateral deviations. Properties of stable matchings in bipartite matching models under TU are by now well-known. A stable matching exists under mild continuity and compactness conditions; it maximizes aggregate surplus, and the associated individual surpluses solve

the dual imputation problem. Yet as we will see, stable roommate matchings may also fail to exist when transfers are allowed.

Since Gale and Shapley (1962), a few papers have studied the property of NTU stable roommate matchings when they do exist. Gusfield and Irving (1989) showed that the set of singles is the same in all stable matchings; Klaus and Klijn (2010) study whether any of them can be “fair”. Efficient algorithms have also been available since Irving (1985). Necessary and sufficient existence conditions under strict preferences have been found by Tan (1991) for complete stable matchings and by Sotomayor (2005) for stable matchings. Chung (2000) shows that a condition he calls “no odd rings” is sufficient for stable matchings to exist under weak preferences. Rodrigues-Neto (2007) introduces “symmetric utilities” and Gudmunsson (2014) uses “weak cycles.”

The TU case has been less studied in the theoretical literature, in spite of its relevance in empirical applications. All examples at the beginning of this introduction, and arguably the vast majority of relevant economic applications, involve transfers. The profit generated by a law firm is split between partners, and the rule governing that division is (at least partly) endogenous to the matching game¹. Chung (2000) shows that when the division of surplus obeys an exogenous rule, odd rings are ruled out and the roommate problem has a stable matching; but that is clearly not an appealing assumption. Karlander and Eriksson (2001) provide a graph-theoretic characterization of stable outcomes when they exist; and Klaus and Nichifor (2010) studies their properties. Talman and Yang (2011) give a characterization in terms of integer programming.

In this paper, we provide an extensive analysis of the two-partner roommate matching problem under TU. We first argue that the existence problem is still present. However, its economic implications may be much less damaging than one would expect. Specifically, we consider a model in which agents belong to various “types”, where each type consists of individuals of indistinguishable characteristics and tastes. In this context, we show two main results. First, a stable matching always exists when the number of individuals in each type is *even*. Second, when the number of individuals of any given type is large enough, there always exist “quasi-stable” matchings: even if a stable matching does not exist, existence can be restored with minimal policy intervention. To do this, one only needs to convince *one* individual to leave the game in each type with an odd number of individuals.

¹If only because it is usually negotiated as part of the matching agreement, particularly when a new partner joins in.

If this requires a compensation to be paid, this can be done at a per capita cost that goes to zero when the population of each type goes to infinity.

The results of this paper are related to those of Azevedo, Weyl, and White (2013), who show the existence of a Walrasian equilibrium in an economy with indivisible goods, a continuum of agents and quasilinear utility. Unlike their main results, ours apply in markets with finite numbers of agents. Our methods are also original. As is well-known, in bipartite problems all feasible matchings that maximize social surplus are stable. This is not true in roommate problems; but we show how any roommate problem can be “cloned” in order to construct an associated bipartite problem. We then exploit this insight to prove existence of stable matchings in roommate problems with even numbers of agents within each type.

To the best of our knowledge, the connection between the unipartite and bipartite problems stressed in this paper is new. Importantly, this implies that the empirical tools devised for the bipartite matching setting should carry over directly to the roommate context when the populations under consideration are large. Some of the results of the present paper are applied in this direction in Ciscato, Galichon, and Goussé (2015).

Our results have clear implications for the partnerships enumerated at the beginning of this introduction. To put things a bit loosely, such a partnership is more likely to be stable when any member has a close substitute within the partnership. “Close substitute” here has a precise meaning: i is a close substitute to j if the joint surplus a partnership that includes j can generate is very similar to what could be generated by replacing j with i ². A partner is less likely to have a close substitute when it has a rare, valuable skill; a doctor may for instance be the only expert available on one particular disease. In professional partnerships (or in academia!), management skills may also be very unevenly distributed; and our analysis suggests that partnerships that depend on rare leadership skills are more susceptible to break up. This would also true of firms that rely on a very charismatic individual for inspiration. The early (1969–84) trajectory of Apple under Steve Jobs may be a case in point. To take the polar opposite, medical practices formed by a largish number of doctors with similar specialties should be rather stable. Consultancies are an intermediate case: while junior consultants may be relatively interchangeable, leadership matters in finding clients and conserving them. Firm-specific capital is also likely to stabilize a partnership.

When partnerships are least likely to be stable, regulation may play a

²Note that it does not imply that j (or i) obtain no surplus in equilibrium, since the condition also applies to all other members of the partnership.

useful role. While we do not pursue this here, one can imagine cases when non-compete or “no poaching” clauses that make mobility more costly could actually be welfare-improving, if the courts allow them. Becker (1991, p. 330) already cited the ability of homosexual unions to “dissolve without judiciary proceedings, alimony, or child support payments” as one reason why they are less stable than heterosexual unions. This is an interesting topic for further research.

2 A Simple Example

We start by giving the intuition of our main results on an illustrative example.

2.1 Unstable Matchings

It has been known since Gale and Shapley that a stable matching may not exist for the roommate problem under non-transferable utility. As it turns out, it is almost equally easy to construct an example of non-existence of a stable matching with transferable utility. Here a *matching* defines who is matched to whom *and* how the corresponding surplus is divided between the partners. Stability requires that

- no partner would be better off by leaving the partnership
- no group of individuals could break off their current match, rematch together, and generate a higher joint surplus than the sum of their current individual utilities.

Consider the following example, in which only two-member matches are possible:

Example 1 *The population has three individuals. Any unmatched individual has zero utility. The joint surplus created by the matching of any two of them is given by the off-diagonal terms of the matrix*

$$\Phi = \begin{pmatrix} - & 6 & 8 \\ 6 & - & 5 \\ 8 & 5 & - \end{pmatrix} \quad (1)$$

so that individuals 1 and 2 create, if they match, a surplus of 6; 1 and 3 create a surplus of 8, etc.

Assume that there exists a stable matching. A matching in which all individuals remain single is obviously not stable; any stable matching must be such that one person remains single and the other two are matched together. Let (u_x) be the utility that individual of type $x = 1, 2, 3$ gets out of this game; stability imposes $u_x + u_y \geq \Phi_{xy}$ for all potential matches, with equality if x and y are actually matched—and $u_x \geq 0$ with equality if x is single. One can readily check, however, that no set of numbers (u_1, u_2, u_3) satisfying these relationships for all x and y exists: whichever the married pair is, one of the matched partners would increase her utility by matching with the single person. Indeed, if the matched pair is $\{1, 2\}$, then

$$u_1 + u_2 = 6, u_3 = 0, u_2 \geq 0$$

contradicts $u_1 + u_3 \geq 8$: agent 3, being single, is willing to give up any amount smaller than 8 to be matched with 1, while the match between 1 and 2 cannot provide 1 with more than 6. Similarly, if the married pair is $\{2, 3\}$, then

$$u_2 + u_3 = 5, u_1 = 0, u_2 \geq 0, u_3 \geq 0$$

contradicts both $u_1 + u_2 \geq 6$ and $u_1 + u_3 \geq 8$ (so that 1 is willing to give more than 5 and less than 6 to agent 2 to match with her, and more than 5 and less than 8 to 3.) Finally, if the married pair is $\{1, 3\}$, then

$$u_1 + u_3 = 8, u_2 = 0, u_1 \geq 0, u_3 \geq 0$$

is incompatible with $u_1 + u_3 \geq 11$, which follows from combining $u_1 + u_2 \geq 6$ and $u_2 + u_3 \geq 5$ with $u_2 = 0$ (since agent 2 is single 1 could match with her and capture almost 6, while 3 could match with her and capture almost 5; these outside options are more attractive than anything 1 and 3 can achieve together.) We conclude that no stable matching exists.

Note that there is nothing pathological in Example 1. The surpluses can easily be (locally) modified without changing the result. Also, the conclusion does not require an odd number of agents; one can readily introduce a fourth individual, who generates a small enough surplus with any roommate, without changing the non-existence finding.

2.2 Cloning

However, there exists a simple modification that restores existence in Example 1. Let us now *duplicate* the economy by “cloning” each agent; technically,

we now have three *types* $x = 1, 2, 3$ of agents, with two (identical) individuals of each type. The joint surplus created by a matching between two individuals of different types $x \neq y$ is as in Example 1; but we now also need to define the surplus generated by the matching of two clones (two individuals of the same type.) Take it to be 2 for every type—more on this later. We then have the matrix:

$$\Phi' = \begin{pmatrix} 2 & 6 & 8 \\ 6 & 2 & 5 \\ 8 & 5 & 2 \end{pmatrix} \quad (2)$$

Consider the following matching μ^* : there is one match between a type 1 and a type 2 individuals, one between type 1 and type 3, and one between type 2 and type 3. Assume individuals share the surplus so that each individual of type 1 gets 4.5, each individual of type 2 gets 1.5, and each individual of type 3 gets 3.5. This is clearly feasible; and it is easy to verify that it is a stable matching.

Less obvious but still true is the fact (proved later on) that existence would still obtain for *any* values chosen for the diagonal of the matrix, although the stable matching pattern that would emerge may be different³. In other words, our cloning operation always restores the existence of a stable match, irrespective of the values of the joint surpluses created by matches between clones.

2.3 Surplus Maximization

Our main result is better understood when related to another, closely linked problem: finding a feasible matching that maximizes total surplus. Total surplus is simply the sum of the joint surpluses of every match (keeping to a normalized utility of zero for singles). In the standard, bipartite framework, the adjective “feasible” refers to the fact that each individual can only be matched to one partner or stay single. Roommate matching, however, introduces an additional feasibility constraint. For any two types $x \neq y$, denote μ_{xy} the number of matches between an individual of type x and an individual of type y ; since a roommate matching for which μ_{xy} and μ_{yx} differ would clearly not be feasible, it must be the case that $\mu_{xy} = \mu_{yx}$. This additional symmetry constraint is absent from the bipartite model, where these two individuals would belong to two separate subpopulations and the number of marriages between say, a college-educated man and a woman

³For instance, if the diagonal elements are large enough, the stable matching matches each individual with her clone.

who is a high-school graduate may well differ (and typically does) from the number of marriages between a college-educated woman and a man who is a high-school graduate.

This symmetry constraint is the source of the difficulty in finding stable roommate matchings; and our cloning operation addresses it. To see this on our Example 1, first go back to roommate matching with one individual of each type $x = 1, 2, 3$, and neglect the symmetry constraint. Since there is only one individual of each type x , she cannot match with herself: $\mu_{xx} \equiv 0$; and neglecting symmetry, the only other feasibility constraints are

$$\text{for every } x, \sum_{y \neq x} \mu_{xy} \leq 1$$

and

$$\text{for every } y, \sum_{x \neq y} \mu_{xy} \leq 1.$$

The two matchings

$$\mu^1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \mu^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

are feasible in this limited sense; and they both achieve the highest possible surplus when the symmetry conditions are disregarded. The existence of two solutions is not surprising: given the symmetric nature of the surplus matrix Φ , if a matrix μ maximizes total surplus, so does its transpose μ^t . Unfortunately, neither is symmetric, and therefore neither makes any sense in the roommate problem. For instance, μ^1 has agent 1 matched both with agent 3 (in the first row) and with agent 2 (in the first column). Also, note that a third solution to this relaxed problem is the unweighted mean of μ^1 and μ^2 ,

$$\mu^m = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

However, while this matrix is indeed symmetric, its coefficients are not integer and thus it is not a feasible matching either; moreover, and quite interestingly, it cannot be interpreted as the outcome of randomization since it is *not* a convex combination of feasible roommate matching matrices⁴.

⁴For any stable roommate matching matrix, the sum of coefficients equals 2, reflecting the fact that one agent must remain single. This property is preserved by convex combination; however, the sum of coefficients of μ^m equals 3.

Let us now reintroduce the symmetry constraint. The (now fully) feasible matching that maximizes total surplus can only have one matched pair and one single; and the pair that should be matched clearly consists of individuals 1 and 3:

$$\bar{\mu} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Obviously, $\bar{\mu}$ is *not* a solution to the maximization problem without symmetry constraint; in other words, the symmetry constraint is binding in this example. As we shall see below, this is characteristic of situations in which the roommate matching problem with transferable utility does not have a stable matching. Indeed, we prove in the next section that *a stable matching exists if and only if the symmetry constraint does not bind*.

Now take the “cloned” version of Example 1, in which each type x has two individuals. It is easy to see that the solution to the relaxed problem which neglects the symmetry constraint is the μ^* of section 2.2, which is symmetric; therefore the symmetry constraint does not bind, and a stable matching exists. This is a general result: we shall see below that in any cloned roommate matching setup, at least one solution to the relaxed problem is symmetric—which implies the existence of a stable match.

2.4 A Bipartite Interpretation

The relaxed problem, in turn, has a natural interpretation in terms of bipartite matching. Start from the three-agent Example 1, and define an *associated bipartite matching problem* as follows: clone the population again, but this time assign a label (such as “man” or “woman”) to each of the two subpopulations. Then consider the bipartite matching problem between these subpopulations of “men” and “women”, with the joint surplus matrix given by Φ' in (2).

By standard results, there always exists a stable matching in this associated bipartite matching problem; and it maximizes the associated total surplus. In our example, μ^1 and μ^2 are the two stable matchings. Any convex combination such as μ^m can be interpreted as a randomization between these two matchings; it is natural to focus on μ^m since it is the only symmetric one and feasible roommate matchings must be symmetric. As remarked above, in the original roommate problem μ^m cannot be stable, since it has non-integer element.

Now if the roommate matching problem is cloned we can proceed as in the above paragraph, except that with twice the number of individuals we

should work with $2\mu^m$. As an integer symmetric matrix, reinterpreted in the cloned roommate matching setup, it defines a feasible roommate matching which is stable—in fact it is the stable matching μ^* of section 2.2. This construction is general: we shall see below that any roommate matching problem in which the number of individuals in each type is even has a symmetric stable match.

We now provide a formal derivation of these results.

3 The Formal Setting

We consider a population of individuals who belong to a finite set of types \mathcal{X} . Individuals of the same type are indistinguishable. We denote n_x the number of individuals of type $x \in \mathcal{X}$, and

$$N = \sum_{x \in \mathcal{X}} n_x$$

the total size of the population.

Without loss of generality, we normalize the utilities of singles to be zero throughout.

3.1 Roommate Matching

A match consists of two partners of types x and y . An individual of any type can be matched with any individual of the same or any other type, or remain single. In particular, there is no restriction that matches only involve two partners of different “genders.”

Let a match $\{x, y\}$ generate a surplus Φ_{xy} . In principle the two partners could play different roles. In sections 3 and 4 we will assume that they are in fact symmetric within a match, so that Φ_{xy} is assumed to be a symmetric function of (x, y) :

Assumption 1 *The surplus Φ_{xy} is symmetric in (x, y) .*

We show in section 5 that, surprising as it may seem, there is in fact no loss of generality in making this assumption. The intuition is simple: if Φ_{xy} fails to be symmetric in (x, y) , so that the partners’ roles are not exchangeable, then they should choose their roles so to maximize output. This boils down to replacing Φ_{xy} with the symmetric $\max(\Phi_{xy}, \Phi_{yx})$. Thus our results extend easily when we do not impose Assumption 1; but it is easier to start from the symmetric case.

A matching can be described by a matrix of numbers (μ_{xy}) indexed by $x, y \in \mathcal{X}$, such that

- μ_{x0} is the number of singles of type x
- when $y \neq 0$, μ_{xy} is the number of matches between types x and y .

The numbers μ_{xy} should be integers; given Assumption 1, they should be symmetric in (x, y) ; and they should satisfy the scarcity constraints. More precisely, the number of individuals of type x must equal the number μ_{x0} of singles of type x , plus the number of pairs in which only one partner has type x , plus twice the number of pairs in which the two partners are of type x —since such a same-type pair has two individuals of type x .

Finally, the set of *feasible roommate matchings* is

$$\mathcal{P}(n) = \left\{ \mu = (\mu_{xy}) : \begin{pmatrix} 2\mu_{xx} + \sum_{y \neq x} \mu_{xy} \leq n_x \\ \mu_{xy} = \mu_{yx} \\ \mu_{xy} \in \mathbb{N} \end{pmatrix} \right\} \quad (3)$$

3.2 TU stability and optimality

We define an *outcome* (μ, u) as the specification of a feasible roommate matching μ and an associated vector of payoffs u_x to each individual of type x . These payoffs have to be feasible: that is, the sum of payoffs across the population has to be equal to the total output under the matching μ . Now in a roommate matching μ , the total surplus created is⁵

$$S_R(\mu; \Phi) = \sum_x \mu_{xx} \Phi_{xx} + \sum_{x \neq y} \mu_{xy} \frac{\Phi_{xy}}{2}. \quad (4)$$

This leads to the following definition of a feasible outcome: an outcome (μ, u) is *feasible* if μ is a feasible roommate matching and

$$\sum_{x \in \mathcal{X}} n_x u_x = S_R(\mu; \Phi). \quad (5)$$

We define stability as in Gale and Shapley (1962): an outcome (μ, u) is stable if it cannot be blocked by an individual or by a pair of individuals.

⁵Note that in the second sum operator the pair $\{x, y\}$ appears twice, one time as (x, y) and another time as (y, x) ; but the joint surplus Φ_{xy} it creates must only be counted once, hence the division by 2.

More precisely, an outcome (μ, u) is *stable* if it is feasible, and if for any two types $x, y \in \mathcal{X}$, (i) $u_x \geq 0$, and (ii) $u_x + u_y \geq \Phi_{xy}$. By extension, a matching μ is called stable if there exists a payoff vector (u_x) such that the outcome (μ, u) is stable.

In bipartite matching the problem of stability is equivalent to the problem of *optimality*: stable matchings maximize total surplus. Things are obviously more complicated in roommate matchings—there always exist surplus-maximizing matchings, but they may not be stable. The maximum of the aggregate surplus over the set of feasible roommate matchings $\mathcal{P}(n)$ is

$$\begin{aligned} \mathcal{W}_{\mathcal{P}}(n, \Phi) &= \max S_R(\mu; \Phi) \\ \text{s.t.} \quad & 2\mu_{xx} + \sum_{y \neq x} \mu_{xy} \leq n_x \\ & \mu_{xy} = \mu_{yx} \\ & \mu_{xy} \in \mathbb{N}. \end{aligned} \tag{6}$$

While no stable matching may actually achieve this value, it plays an important role in our argument.

3.3 The Associated Bipartite Matching Problem

We shall now see that to every roommate matching problem we can associate a bipartite matching problem which generates almost the same level of aggregate surplus. More precisely, we will prove that for every vector of populations of types $n = (n_x)$ and every symmetric surplus function $\Phi = (\Phi_{xy})$, the highest possible surplus in the roommate matching problem is “close to” that achieved in a bipartite problem with mirror populations of men and women and half the surplus function:

$$\mathcal{W}_{\mathcal{P}}(n, \Phi) \simeq \mathcal{W}_{\mathcal{B}}(n, n, \Phi/2).$$

where $\mathcal{W}_{\mathcal{B}}(n, n, \Phi/2)$ is defined as the maximal surplus of the bipartite matching problem:

$$\mathcal{W}_{\mathcal{B}}(n, n, \Phi/2) = \max_{\nu \in \mathcal{B}(n, n)} S_B(\nu; \Phi) \tag{7}$$

where $S_B(\nu; \Phi) = \sum_{x, y \in \mathcal{X}} \nu_{xy} \frac{\Phi_{xy}}{2}$ and $\mathcal{B}(n, n)$ is the set of feasible matchings in the bipartite problem:

$$\mathcal{B}(n, n) = \left\{ \nu = (\nu_{xy}) : \begin{pmatrix} \sum_y \nu_{xy} \leq n_x \\ \sum_x \nu_{xy} \leq n_y \\ \nu_{xy} \in \mathbb{N} \end{pmatrix} \right\} \tag{8}$$

We also define stability for a feasible bipartite matching (ν_{xy}) in the usual way: there must exist payoffs (u_x, v_y) such that

$$\begin{aligned} S_B(\nu; \Phi) &= \sum_{x \in \mathcal{X}} n_x u_x + \sum_{y \in \mathcal{X}} n_y v_y & (9) \\ u_x + v_y &\geq \frac{\Phi_{xy}}{2} \\ u_x &\geq 0, \quad v_y \geq 0 \end{aligned}$$

By classical results of Shapley and Shubik (1971), there exist stable matchings ν , and they coincide with the solutions of (7). Moreover, the associated payoffs (u, v) solve the dual program; that is, they minimize $\sum_{x \in \mathcal{X}} n_x u_x + \sum_{y \in \mathcal{X}} n_y v_y$ over the feasible set of program (9). Finally, for any stable matching, $\mu_{xy} > 0$ implies $u_x + v_y = \Phi_{xy}/2$, and $\mu_{x0} > 0$ implies $u_x = 0$.

Remark 3.1 *The marriage problem obviously is a particular case of the roommate problem: if in a roommate matching problem $\Phi_{xy} = -\infty$ whenever x and y have the same gender, then any optimal or stable matching will be heterosexual.*

3.3.1 Links Between $\mathcal{W}_{\mathcal{P}}$ and $\mathcal{W}_{\mathcal{B}}$

It is not hard to see that $\mathcal{W}_{\mathcal{P}}(n, \Phi) \leq \mathcal{W}_{\mathcal{B}}(n, n, \Phi/2)$. In fact, we can bound the difference between these two values:

Theorem 1 *Under Assumption 1,*

$$\mathcal{W}_{\mathcal{P}}(n, \Phi) \leq \mathcal{W}_{\mathcal{B}}(n, n, \Phi/2) \leq \mathcal{W}_{\mathcal{P}}(n, \Phi) + |\mathcal{X}|^2 \bar{\Phi}$$

where

$$\bar{\Phi} = \sup_{x, y \in \mathcal{X}} \Phi_{xy}.$$

and $|\mathcal{X}|$ is the cardinal of the set \mathcal{X} , i.e. the number of types in the population.

Proof. See appendix. ■

In some cases, $\mathcal{W}_{\mathcal{P}}(n, \Phi)$ and $\mathcal{W}_{\mathcal{B}}(n, n, \Phi/2)$ actually coincide. For instance:

Proposition 2 *If n_x is even for each $x \in \mathcal{X}$, then under Assumption 1,*

$$\mathcal{W}_{\mathcal{P}}(n, \Phi) = \mathcal{W}_{\mathcal{B}}(n, n, \Phi/2).$$

Proof. See appendix. ■

3.3.2 Stable Roommate Matchings

The existence of stable roommate matchings is directly related to the divergence of $\mathcal{W}_{\mathcal{P}}(n, \Phi)$ and $\mathcal{W}_{\mathcal{B}}(n, n, \Phi/2)$. Indeed, one has:

Theorem 3 *Under Assumption 1,*

(i) *There exist stable roommate matchings if and only if*

$$\mathcal{W}_{\mathcal{P}}(n, \Phi) = \mathcal{W}_{\mathcal{B}}(n, n, \Phi/2).$$

(ii) *Whenever they exist, stable roommate matchings achieve the maximal aggregate surplus $\mathcal{W}_{\mathcal{P}}(n, \Phi)$ in (6).*

(iii) *Whenever a stable roommate matching exists, individual utilities at equilibrium (u_x) solve the following, dual program:*

$$\begin{aligned} \min_{u, A} \quad & \sum_x u_x n_x & (10) \\ \text{s.t.} \quad & u_x \geq 0 \\ & u_x + u_y \geq \Phi_{xy} + A_{xy} \\ & A_{xy} = -A_{yx} \end{aligned}$$

Proof. See appendix. ■

Note that while the characterization of the existence of a stable matching in terms of equality between an integer program and a linear program is a well-known problem in the literature on matching (see Talman and Yang (2011) for the roommate problem), the link with a bipartite matching problem is new.

Also note that in program (10), the antisymmetric matrix A has a natural interpretation: A_{xy} is the Lagrange multiplier of the symmetry constraints $\mu_{xy} = \mu_{yx}$ in the initial program (6). Our proof shows that if $\mu_{xy} > 0$ in a stable roommate matching, then the corresponding A_{xy} must be non-positive; but since $\mu_{yx} = \mu_{xy}$ the multiplier A_{yx} must also be non-positive, so that both must be zero. The lack of existence of a stable roommate matching is therefore intimately linked to a binding symmetry constraint.

Given Proposition 2, Theorem 3 has an immediate corollary: with an even number of individuals per type, there must exist a stable roommate matching. Formally:

Corollary 3.1 *If n_x is even for each $x \in \mathcal{X}$, then under Assumption 1, there exists a stable roommate matching.*

In particular, for any roommate matching problem, its “cloned” version, in which each agent has been replaced with a couple of clones, has a stable matching; and this holds irrespective of the surplus generated by the matching of two identical individuals. Of course, in general much less than full cloning is needed to restore existence; we give this statement a precise meaning in the next paragraph.

Our next result shows that one can restore the existence of a stable matching by removing at most one individual of each type from the population; if these individuals have to be compensated for leaving the game, this can be done at limited total cost:

Theorem 4 (Approximate stability) *Under Assumption 1, in a population of N individuals, there exists a subpopulation of at least $N - |\mathcal{X}|$ individuals among which there exist a stable matching, where $|\mathcal{X}|$ is the number of types. The total cost for the regulator to compensate the individuals left aside is bounded above by $|\mathcal{X}|\bar{\Phi}$.*

Proof. See appendix. ■

4 Matching in Large Numbers

We now consider the case of a “large” game, in which there are “many” agents of each type. Intuitively, even though an odd number of agents in any type may result in non existence of a stable roommate matching, the resulting game becomes “close” to one in which a stable matching exists. We now flesh out this intuition by providing a formal analysis.

We start with a formal definition of a large game. For that purpose, we consider a sequence of games with the same number of types and the same surplus matrix, but with increasing populations in each type. If n_x^k denotes the population of type x in game k and $N^k = \sum_x n_x^k$ is the total population of that game, then we consider situations in which, when $k \rightarrow \infty$:

$$N^k \rightarrow \infty \text{ and } n_x^k/N^k \rightarrow f_x$$

where f_x are constant numbers.

As the population gets larger, aggregate surplus increases proportionally; it is therefore natural to consider the *average surplus*, computed by dividing aggregate surplus by the size of the population. We also extend the definition of $\mathcal{W}_{\mathcal{B}}$ in program (7) to non-integers in the obvious way so as to define the

limit average bipartite problem $\mathcal{W}_{\mathcal{B}}(f, f, \Phi/2)$. Note that the linearity of the program implies

$$\mathcal{W}_{\mathcal{B}}(cn, cm, \Phi/2) = c\mathcal{W}_{\mathcal{B}}(n, m, \Phi/2)$$

for any $c > 0$.

Proposition 5 *In the large population limit, under Assumption 1, the average surplus in the roommate matching problem converges to the limit average surplus in the related bipartite matching problem. That is,*

$$\lim_{k \rightarrow \infty} \frac{\mathcal{W}_{\mathcal{P}}(n^k, \Phi)}{N^k} = \lim_{N^k \rightarrow \infty} \frac{\mathcal{W}_{\mathcal{B}}(n^k, n^k, \Phi/2)}{N^k} = \mathcal{W}_{\mathcal{B}}(f, f, \Phi/2).$$

Proof. See appendix. ■

Our approximation results crucially rely on the number of types becoming small relative to the total number of individuals. By definition, two individuals of the same type are indistinguishable in our formulation, both in their preferences and in the way potential partners evaluate them. This may seem rather strong; however, a closer look at the proof of Theorem 5 shows that our bound can easily be refined. In particular, we conjecture that with a continuum of types, Theorem 5 would hold exactly.

A related effect of the number of individuals becoming much larger than the number of types is that the costs of the policy to restore stability in Theorem 4 become negligible:

Proposition 6 *In the large population limit and under Assumption 1,*

- (i) *one may remove a subpopulation of asymptotically negligible size in order to restore the existence of stable matchings.*
- (ii) *the average cost per individual of restoring the existence of stable matchings tends to zero.*

Proof. See appendix. ■

In particular, in the case of a continuum of individuals (that is, when there is a finite number of types and an infinite number of individuals of each type), we recover the results of Azevedo, Weyl, and White (2013) (hereafter, AWW). To make the connection with this paper, the partner types in our

setting translates into goods in AWW's. The social welfare in our setting translates into the utility u of a single consumer in AWW. u is such that $u(C) = \Phi(\{x, y\})$ for $C = \{x, y\}$, $u(\{x\}) = 0$, and $u = -\infty$ elsewhere (or very negative). Then it can be shown without difficulty that the existence of a TU stable matching in our setting is equivalent to the existence of a Walrasian equilibrium in the AWW setting. Thus existence and TU stability in the case of a continuum of individuals follows from Theorem and Proposition in AWW.

5 The Nonexchangeable Roommate Problem⁶

We now investigate what happens when the surplus Φ_{xy} is not necessarily symmetric. This will arise when the roles played by the partners are not exchangeable. For instance, a pilot and a copilot on a commercial airplane have dissymmetric roles, but may be both chosen from the same population. Hence, in this section, we shall assume away Assumption 1, and we refer to the “nonexchangeable roommate problem”; it contains the exchangeable problem as a special case.

As it turns out, this can be very easily recast in the terms of an equivalent symmetric roommate problem. Indeed if $\Phi_{xy} > \Phi_{yx}$, then any match of an (ordered) 2-uple (y, x) will be dominated by a matching of a (x, y) 2-uple, and the partners may switch the roles they play and generate more surplus. Therefore, in any optimal (or stable) solution there cannot be such a (y, x) 2-uple. As a consequence, the nonexchangeable roommate problem is equivalent to an exchangeable problem where the surplus function is equal to the maximum joint surplus x and y may generate together, that is

$$\Phi'_{xy} = \max(\Phi_{xy}, \Phi_{yx});$$

and since this is symmetric our previous results apply almost directly. Denoting π_{xy} the number of (x, y) pairs (in that order), one has

$$\begin{aligned} \mu_{xy} &= \pi_{xy} + \pi_{yx}, \quad x \neq y \\ \mu_{xx} &= \pi_{xx} \end{aligned}$$

and obviously, π_{xy} need not equal π_{yx} . The population count equation is

$$n_x = \sum_{y \in \mathcal{X}} (\pi_{xy} + \pi_{yx}), \quad \forall x \in \mathcal{X}$$

⁶We are grateful to Arnaud Dupuy for correcting a mistake in a preliminary version of the paper.

and the social surplus from a matching π is

$$\sum_{x,y \in \mathcal{X}} \pi_{xy} \Phi_{xy}.$$

so that the optimal surplus in the nonexchangeable problem is

$$\begin{aligned} \mathcal{W}'_{\mathcal{P}}(n, \Phi) &= \max \sum_{x,y \in \mathcal{X}} \pi_{xy} \Phi_{xy} \\ \text{s.t. } n_x &= \sum_{y \in \mathcal{X}} (\pi_{xy} + \pi_{yx}), \quad \forall x \in \mathcal{X}. \end{aligned}$$

The following result extends our previous analysis to the nonexchangeable setting:

Theorem 7 *The nonexchangeable roommate matching problem is solved by considering the surplus function*

$$\Phi'_{xy} = \max(\Phi_{xy}, \Phi_{yx})$$

which satisfies Assumption 1. Call optimized symmetric problem the problem with surplus Φ'_{xy} and population count n_x . Then:

(i) *the optimal surplus in the nonexchangeable roommate problem coincides with the optimal surplus in the corresponding optimized symmetric problem, namely*

$$\mathcal{W}'_{\mathcal{P}}(n, \Phi) = \mathcal{W}_{\mathcal{P}}(n, \Phi')$$

(ii) *the nonexchangeable roommate problem has a stable matching if and only if the optimized symmetric problem has a stable matching.*

Given Theorem 7, all results in Sections 3 and 4 hold in the general (nonexchangeable) case. In particular:

- Theorem 1 extends to the general case: the social surplus in the roommate problem with asymmetric surplus Φ_{xy} is approximated by a bipartite problem with surplus function $\Phi'_{xy} = \max(\Phi_{xy}, \Phi_{yx})/2$, or more formally:

$$\mathcal{W}'_{\mathcal{P}}(n, \Phi) \leq \mathcal{W}_{\mathcal{B}}(n, n, \Phi'/2) \leq \mathcal{W}'_{\mathcal{P}}(n, \Phi) + |\mathcal{X}|^2 \bar{\Phi},$$

and as an extension of Proposition 2, equality holds in particular when the number of individuals in each types are all even.

- Theorem 3 extends as well: there is a stable matching in the roommate problem with asymmetric surplus Φ_{xy} if and only if there is equality in the first equality above, that is:

$$\mathcal{W}'_{\mathcal{P}}(n, \Phi) = \mathcal{W}_{\mathcal{B}}(n, n, \Phi'/2).$$

- All the asymptotic results in Section 4 hold true: in the asymmetric roommate problem, there is approximate stability and the optimal matching solves a linear programming problem.

6 Conclusion

Some roommate problems involve extensions to situations where more than two partners can form a match; but the two-partner case is a good place to start the analysis. Here, we have shown that when the population is large enough with respect to the number of observable types, the structure of the roommate problem is the same as the structure of the bipartite matching problem. Most empirical applications of matching models under TU use a framework as in this paper in order to understand, depending on the context, how the sorting on a given matching market depends on age, education or income, but also height, BMI, marital preferences, etc.⁷.

As a consequence of our results, the empirical tools developed in the bipartite setting, especially for the analysis of the marriage markets (see Choo and Siow (2006), Chiappori, Salanié, and Weiss (2015), Fox (2010), Galichon and Salanié (2015), to cite only a few⁸) can be extended to other contexts where the bipartite constraint is relaxed. These include law firms or doctor practices, but also team jobs such as pilot/copilot (and more generally team sports), as well as “tickets” in US presidential elections, marriage markets incorporating single-sex households, and many others. Also, we conjecture that the same “cloning” technique could be applied to matches involving more than two partners—the multipartite reference, in that case, being the “matching for teams” context studied by Carlier and Ekeland (2010). We leave this for future research.

While our analysis has been conducted in the discrete case, it would be interesting to extend our results to the case where there is an infinite number of agents with a continuum of types. We conjecture that this could

⁷See for instance Choo and Siow (2006), Chiappori and Oreffice (2008) Chiappori, Oreffice, and Quintana-Domeque (2012) among many others.

⁸Graham (2011) has a good discussion of this burgeoning literature.

be done, at some cost in terms of the mathematics required⁹. Stability would probably require that the distribution of agents' characteristics should not feature regions with very low density.

Finally, it seems natural to apply our “cloning technique” when utility is not transferable. One may think of assigning arbitrarily genders to both clones of each type, and considering a bipartite stable matching between the two genders. Such a matching will be stable in the roommate matching framework if the bipartite matching of the cloned populations is symmetric. However, such a symmetric stable bipartite matching of the cloned population may not exist. Therefore, the usefulness of cloning to restore stability in the non-transferable utility version of the roommate problem is an open question.

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⁹The relevant tools here come from the theory of optimal transportation, see Villani (2003) and McCann and Guillen (2013). For the precise connection between matching models and optimal transportation theory, see Ekeland (2010), Gretsky, Ostroy, and Zame (1999) and Chiappori, McCann, and Nesheim (2010). It is also worth mentioning the recent contribution of Ghoussoub and Moameni (2013), which uses the same type of mathematical structure for very different purposes.

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A Appendix: Proofs

Our proofs use an auxiliary object: the highest possible surplus for a *fractional* roommate matching, namely

$$\mathcal{W}_{\mathcal{F}}(n, \Phi) = \max_{\mu \in \mathcal{F}(n)} \left(\sum_x \mu_{xx} \Phi_{xx} + \sum_{x \neq y} \mu_{xy} \frac{\Phi_{xy}}{2} \right). \quad (11)$$

where $\mathcal{F}(n)$ is the set of *fractional (roommate) matchings*, which relaxes the integrality constraint on μ :

$$\mathcal{F}(n) = \left\{ (\mu_{xy}) : \begin{pmatrix} 2\mu_{xx} + \sum_{y \neq x} \mu_{xy} \leq n_x \\ \mu_{xy} = \mu_{yx} \\ \mu_{xy} \geq 0 \end{pmatrix} \right\}. \quad (12)$$

The program (11) has no immediate economic interpretation since fractional roommate matchings are infeasible in the real world; and while obviously $\mathcal{W}_{\mathcal{P}}(n, \Phi) \leq \mathcal{W}_{\mathcal{F}}(n, \Phi)$, the inequality in general is strict. We are going to show, however, that the difference between the two programs vanishes when the population becomes large. Moreover, we will establish a link between (11) and the surplus at the optimum of the associated bipartite matching problem.

We start by proving:

Lemma A.1

$$\mathcal{W}_{\mathcal{F}}(n, \Phi) = \mathcal{W}_{\mathcal{B}}(n, n, \Phi/2). \quad (13)$$

Moreover, problem (11) has a half-integral solution.

Proof of Lemma A.1. First consider some fractional roommate matching $\mu \in \mathcal{F}(n)$, and define

$$\begin{aligned} \nu_{xy} &= \mu_{xy} \text{ if } x \neq y \\ \nu_{xx} &= 2\mu_{xx}. \end{aligned}$$

As a (possibly fractional) bipartite matching, clearly $\nu \in \mathcal{B}(n, n)$; and

$$\sum_x \mu_{xx} \Phi_{xx} + \sum_{x \neq y} \mu_{xy} \frac{\Phi_{xy}}{2} = \frac{1}{2} \sum_{x, y \in \mathcal{X}} \nu_{xy} \Phi_{xy}.$$

Now the right-hand side is the aggregate surplus achieved by ν in the bipartite matching problem with margins (n, n) and surplus function $\Phi/2$. It follows that

$$\mathcal{W}_{\mathcal{F}}(n, \Phi) \leq \mathcal{W}_{\mathcal{B}}(n, n, \Phi/2). \quad (14)$$

Conversely, let (ν_{xy}) maximize aggregate surplus over $\mathcal{B}(n, n)$ with surplus $\Phi/2$. By symmetry of Φ , (ν_{yx}) also is a maximizer; and since (7) is a linear program, $\nu'_{xy} = \frac{\nu_{xy} + \nu_{yx}}{2}$ also maximizes it. Define

$$\begin{aligned}\mu'_{xy} &= \nu'_{xy} \text{ if } x \neq y \\ \mu'_{xx} &= \frac{\nu_{xx}}{2}.\end{aligned}$$

Then

$$\begin{aligned}2\mu'_{xx} + \sum_{y \neq x} \mu'_{xy} &= \nu_{xx} + \frac{1}{2} \sum_{y \neq x} (\nu_{xy} + \nu_{yx}) \\ &= \frac{1}{2} (\nu_{xx} + \sum_{y \neq x} \nu_{xy}) \\ &\quad + \frac{1}{2} (\nu_{xx} + \sum_{y \neq x} \nu_{yx}).\end{aligned}$$

Now $\nu_{xx} + \sum_{y \neq x} \nu_{xy} \leq n_x$ by the scarcity constraint of “men” of type x , and $\nu_{xx} + \sum_{y \neq x} \nu_{yx} \leq n_x$ by the scarcity constraint of “women” of type x . It follows that $\mu' \in \mathcal{F}(n)$, and

$$\sum_x \mu'_{xx} \Phi_{xx} + \sum_{x \neq y} \mu'_{xy} \frac{\Phi_{xy}}{2} = \frac{1}{2} \sum_{x, y \in \mathcal{X}} \nu_{xy} \Phi_{xy}.$$

Therefore the values of the two programs coincide.

Half-integrality follows from the Birkhoff-von Neumann theorem: there always exists an integral solution ν of the associated bipartite matching problem, and the construction of μ' makes it half-integral¹⁰. ■

Given Lemma A.1, we can now prove Theorem 1.

Proof of Theorem 1. The first inequality simply follows from the fact that $\mathcal{P}(n) \subset \mathcal{F}(n)$. Let us now show the second inequality. Lemma A.1 proved that $\mathcal{W}_{\mathcal{F}}(n, \Phi) = \mathcal{W}_{\mathcal{B}}(n, n, \Phi/2)$. Let μ achieve the maximum in $\mathcal{W}_{\mathcal{F}}(n, \Phi)$, so that

$$\mathcal{W}_{\mathcal{F}}(n, \Phi) = \sum_x \mu_{xx} \Phi_{xx} + \sum_{x \neq y} \mu_{xy} \frac{\Phi_{xy}}{2}.$$

¹⁰The half-integrality of the solution of problem (11) also follows from a general theorem of Balinski (1970); but the proof presented here is self-contained.

Let $\lfloor x \rfloor$ denote the floor rounding of x ; by definition, $x < \lfloor x \rfloor + 1$, so that

$$\mathcal{W}_{\mathcal{F}}(n, \Phi) < \sum_x \lfloor \mu_{xx} \rfloor \Phi_{xx} + \sum_{x \neq y} \lfloor \mu_{xy} \rfloor \frac{\Phi_{xy}}{2} + \sum_x \Phi_{xx} + \sum_{x \neq y} \frac{\Phi_{xy}}{2}.$$

The right-hand side can also be rewritten as

$$\sum_{x,y} \lfloor \mu_{xy} \rfloor \Phi_{xy} + \sum_{x,y} \Phi_{xy}.$$

But $\lfloor \mu \rfloor$ is in $\mathcal{B}(n, n)$, and is integer by construction; therefore

$$\sum_{x,y \in \mathcal{X}} \lfloor \mu_{xy} \rfloor \Phi_{xy} \leq \mathcal{W}_{\mathcal{P}}(n, \Phi).$$

Finally,

$$\sum_{x,y \in \mathcal{X}} \Phi_{xy} \leq |\mathcal{X}|^2 \bar{\Phi}$$

so that

$$\mathcal{W}_{\mathcal{F}}(n, \Phi) \leq \mathcal{W}_{\mathcal{P}}(n, \Phi) + |\mathcal{X}|^2 \bar{\Phi}.$$

■

A.1 Proof of Proposition 2

Proof. Let $n'_x = \frac{n_x}{2}$. By Lemma A.1, problem $\mathcal{W}_{\mathcal{F}}(n', \Phi)$ has an half-integral solution μ' ; therefore problem $\mathcal{W}_{\mathcal{F}}(n, \Phi)$ has an integral solution $2\mu'$, which must also solve (7). It follows that

$$\mathcal{W}_{\mathcal{P}}(n, \Phi) = \mathcal{W}_{\mathcal{F}}(n, \Phi).$$

■

A.2 Proof of Theorem 3

Proof. By Theorem A.1, Problem (11) coincides with a bipartite matching problem between marginal (n_x) and itself. By well-known results on bipartite matching, there exist vectors (v_x) and (w_y) such that

$$\begin{aligned} v_x &\geq 0, & w_y &\geq 0 \\ v_x + w_y &\geq \Phi_{xy} \end{aligned}$$

and the latter inequality is an equality when $\mu_{xy} > 0$. Setting

$$u_x = \frac{v_x + w_x}{2}$$

the symmetry of Φ implies

$$\begin{aligned} u_x &\geq 0 \\ u_x + u_y &\geq \Phi_{xy} \end{aligned}$$

and

$$\sum_{x \in \mathcal{X}} n_x u_x = \sum_{x \in \mathcal{X}} \mu_{xx} \Phi_{xx} + \sum_{x \neq y} \mu_{xy} \frac{\Phi_{xy}}{2}$$

so that the outcome (μ, u) is stable.

Conversely, assume that μ is a stable roommate matching. Then by definition, there is a vector (u_x) such that

$$\begin{aligned} u_x &\geq 0 \\ u_x + u_y &\geq \Phi_{xy} \end{aligned}$$

and

$$\sum_{x \in \mathcal{X}} n_x u_x = \sum_{x \in \mathcal{X}} \mu_{xx} \Phi_{xx} + \sum_{x \neq y} \mu_{xy} \frac{\Phi_{xy}}{2}.$$

Therefore $(u, A = 0)$ are Lagrange multipliers for the linear programming problem (11), and μ is an optimal solution of (11); finally, μ is integral since it is a feasible roommate matching. QED.

(i), (ii) and (iii) follow, as there exist integral solutions of (11) if and only if

$$\mathcal{W}_{\mathcal{P}}(n, \Phi) = \mathcal{W}_{\mathcal{F}}(n, \Phi),$$

and $\mathcal{W}_{\mathcal{F}}(n, \Phi)$ coincides with $\mathcal{W}_{\mathcal{B}}(n, n, \Phi/2)$ from Lemma A.1. ■

A.3 Proof of Theorem 4

Proof. For each type x , remove one individual of type x to the population if n_x is odd. The resulting subpopulation differs from the previous one by at most $|\mathcal{X}|$ individuals, and there is an even number of individuals of each type; hence by Proposition 3.1 there exists a stable matching.

Each individual so picked can be compensated with his payoff u_x . Since $u_x \leq \bar{\Phi}$, the total cost of compensating at most one individual of each type is bounded from above by $|\mathcal{X}| \bar{\Phi}$. ■

A.4 Proof of Proposition 5

Proof. By Theorem 1, in the large population limit

$$\lim_{k \rightarrow \infty} \frac{\mathcal{W}_{\mathcal{P}}(n^k, \Phi)}{N^k} = \mathcal{W}_{\mathcal{F}}(f, \Phi)$$

and Lemma A.1 yields the conclusion. ■

A.5 Proof of Proposition 6

Proof. (i) The number of individuals to be removed is bounded from above by $|\mathcal{X}|$, hence its frequency tends to zero as $|\mathcal{X}|/N \rightarrow 0$. (ii) follows from the fact that

$$\frac{\mathcal{W}_{\mathcal{F}}(n, \Phi) - \mathcal{W}_{\mathcal{P}}(n, \Phi)}{N} \rightarrow 0.$$

■

A.6 Proof of Theorem 7

Proof. (i) Consider an optimal solution μ_{xy} to $\mathcal{W}_{\mathcal{P}}(n, \Phi')$. For any pair $x \neq y$ such that $\Phi_{xy} > \Phi_{yx}$, set $\pi_{xy} = \mu_{xy}$, and $\pi_{yx} = 0$ if $\Phi_{xy} < \Phi_{yx}$. If $\Phi_{xy} = \Phi_{yx}$, set π_{xy} and π_{yx} arbitrarily nonnegative integers such that $\pi_{xy} + \pi_{yx} = \mu_{xy}$; set $\pi_{xx} = \mu_{xx}$. Then π is feasible for the optimized symmetric problem, and one has

$$\sum_{x \in \mathcal{X}} \mu_{xx} \Phi'_{xx} + \sum_{x \neq y} \mu_{xy} \frac{\Phi'_{xy}}{2} = \sum_{x, y \in \mathcal{X}} \pi_{xy} \Phi_{xy}$$

so that

$$\mathcal{W}_{\mathcal{P}}(n, \Phi') \leq \mathcal{W}'_{\mathcal{P}}(n, \Phi).$$

Conversely, consider π_{xy} an optimal solution to $\mathcal{W}'_{\mathcal{P}}(n, \Phi)$. First observe that if $\Phi_{xy} < \Phi_{yx}$ then $\pi_{xy} = 0$; otherwise subtracting one from μ_{xy} and adding one to π_{yx} would lead to an improving feasible solution, contradicting the optimality of π . Set

$$\begin{aligned} \mu_{xy} &= \pi_{xy} + \pi_{yx}, \quad x \neq y \\ \mu_{xx} &= \pi_{xx} \end{aligned}$$

so that

$$\sum_{x \in \mathcal{X}} \mu_{xx} \Phi'_{xx} + \sum_{x \neq y} \mu_{xy} \frac{\Phi'_{xy}}{2} = \sum_{x, y \in \mathcal{X}} \pi_{xy} \Phi_{xy}$$

and hence

$$\mathcal{W}'_{\mathcal{P}}(n, \Phi) \leq \mathcal{W}_{\mathcal{P}}(n, \Phi').$$

(ii) Assume there is a stable matching π_{xy} in the nonexchangeable roommate problem. Then if there is a matched pair (x, y) in that order, one cannot have $\Phi_{yx} > \Phi_{xy}$; otherwise the coalition (y, x) would be blocking. Hence one can define

$$\begin{aligned}\mu_{xy} &= \pi_{xy} + \pi_{yx}, \quad x \neq y \\ \mu_{xx} &= \pi_{xx}\end{aligned}$$

and the matching μ is stable in the optimized symmetric problem. Conversely, assume that the matching μ is stable in the optimized symmetric problem. Then it is not hard to see that, defining π from μ as in the first part of (i) above, the matching π is stable in the nonexchangeable roommate problem. ■