Modeling Customer Behavior for Revenue Management

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ABSTRACT

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In this thesis, we model and analyze the impact of two behavioral aspects of customer decision-making upon the revenue maximization problem of a monopolist firm. First, We study the revenue maximization problem of a monopolist firm selling a homogeneous good to a market of risk-averse, strategic customers. Using a discrete (but arbitrary) valuation distribution, we show how the dynamic pricing problem with strategic customers can be formulated as a mechanism design problem, thereby making it more amenable to analysis. We characterize the optimal solution, and solve the problem for several special cases. We perform asymptotic analysis for the low risk-aversion case and show that it is asymptotically optimal to offer at most two products. Second, we consider a revenue-maximizing monopolist firm that serves a market of customers that are heterogeneous with respect to their valuations and desire for a quality attribute. Instead of optimizing the net utility that results from an appropriate combination of product price and quality, as in the traditional model of customer behavior, we consider a setting where customers purchase the cheapest product subject to its quality exceeding a customer specific quality threshold. We call such preferences threshold preferences. We solve the firm's product design problem in this setting, and contrast with the traditional model of customer choice behavior. We consider several scenarios where such preferences might arise, and identify the optimal solution in each case. In addition to these product design problems, we study the problem of identifying the optimal putting strategy for a golfer. We develop a model of golfer putting skill, and combine it with a putt trajectory and holeout model to identify a golfer's optimal putting strategy. The problem of identifying the optimal putting strategy is shown to be equivalent to a two-dimensional stochastic shortest path problem, with continuous state and control space, and solved using approximate dynamic programming. We calibrate the golfer model to professional and amateur player data, and use the calibrated model to answer sev-

eral interesting questions, e.g., how does green reading ability affect golfer performance, how do professional and amateur golfers differ in their strategy, how do uphill and downhill putts compare in difficulty, etc.

Contents

List of Figures

List of Tables

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To my family, friends and teachers

Chapter 1

Introduction

In this thesis, we consider three problems: (1) Dynamic pricing when customers strategize over time, (2) Product design with threshold preferences, and (3) Optimal putting strategies in golf. The first two problems belong to the broad area of revenue management, while the third is motivated by the game of golf. In what follows, we motivate the first two of these problems in Section 1.1, and the third in Section 1.2. The three problems are discussed in detail in Chapters 2, 3 and 4 of this thesis, respectively.

1.1 Modeling customer behavior in revenue management

In recent years, there has been a continued emphasis in the field of revenue management to develop realistic, yet tractable models of customer choice behavior. Modeling strategic customer behavior, in particular, has received a lot of attention. It has been observed that in certain settings such as retail and electronic marketplaces, customers have become increasingly sophisticated; they monitor the pricing and product differentiation strategies adopted by firms, and incorporate it in their purchasing decisions. This makes it imperative for the firms to take into account strategic customer behavior while developing their pricing and product differentiation strategies.

Several researchers have addressed the dynamic pricing problem with strategic customers, e.g., Liu and van Ryzin [39], Su [63], Cachon and Swinney [13], etc. A common feature of the past analysis, however, is the stylized nature of the problem with respect to customer heterogeneity, e.g., assuming two discrete customer valuations, uniformly distributed customer valuations, no price control, no risk-aversion, and/or focus on two product problems. Our goal in Chapter 2 is to suggest an approach that is intuitive and allows one to relax these restrictive market assumptions. In particular, we study the revenue maximization problem of a monopolist firm selling a homogeneous good to a market of risk-averse, strategic customers. Using a discrete (but arbitrary) valuation distribution, we show how the dynamic pricing problem with strategic customers can be formulated as a mechanism design problem, thereby making it more amenable to analysis. We characterize the optimal solution, and solve the problem for several special cases. We perform asymptotic analysis for the low risk-aversion case and show that it is asymptotically optimal to offer at most two products.

In Chapter 3, we model a different aspect of customer choice behavior. We consider a revenue-maximizing monopolist firm that serves a market of customers that are heterogeneous with respect to their valuations and desire for a quality attribute. Instead of optimizing the net utility that results from an appropriate combination of product price and quality, as in the traditional model of customer behavior, we consider a setting where customers purchase the cheapest product subject to its quality exceeding a customer specific quality threshold. We call such preferences threshold preferences. While not previously examined in the revenue management literature, this model of choice behavior is well-known in marketing and psychology literature, and dates back to Simon [59]. It can be thought of as a model of bounded rationality or as a depiction of how customers actually make decisions for certain application settings such as Voice over IP. We solve the firm's product design problem in this setting, and contrast with the traditional model of customer choice behavior. We consider several scenarios where such preferences might arise, and identify the optimal solution in each case.

1.2 Optimal putting strategies in golf

The third problem we study is related to putting in golf. Golf is popular sport and putting is an important aspect of the game, with around 35-45% of the shots in a typical 18-hole round being putts. While there are a number of instructional books, articles and books devoted to putting, almost all are focussed on putting technique, with relatively few focusing on putting strategy. By putting strategy, we mean a golfer's choice of target velocity and direction for a putt, that given the initial position, green conditions, and golfer skill, minimizes the expected number of putts required to achieve a holeout. In Chapter 4, we develop a model of golfer putting skill, and combine it with a putt trajectory and holeout model to identify a golfer's optimal putting strategy. The problem of identifying the optimal putting strategy is shown to be equivalent to a two-dimensional stochastic shortest path problem, with continuous state and control space, and solved using approximate dynamic programming. We calibrate the golfer model to professional and amateur player data, and use the calibrated model to answer several interesting questions, e.g., how does green reading ability affect golfer performance, how do professional and amateur golfers differ in their strategy, how do uphill and downhill putts compare in difficulty, etc.

Chapter 2

Dynamic pricing with strategic customers

We study the dynamic pricing problem of a monopolist firm in presence of strategic customers that differ in their valuations and risk-preferences. We show that this problem can be formulated as a static mechanism design problem, which is more amenable to analysis. We highlight structural properties of the optimal solution, and solve the problem for several special cases, including the case where the seller only uses two price points, and the case with risk-neutral customers. Focusing in settings with low risk-aversion, we show through an asymptotic analysis that the "two-price point" strategy is near-optimal, offering partial validation for its wide use in practice, but also highlighting when it is indeed suitable to adopt it.

2.1 Introduction

The wide adoption of promotional and markdown pricing by major retailers has "trained" many consumers to anticipate these events and accordingly time their purchases. Given this observation, a natural question that arises is the following: How should the retailer price and allocate its inventory over time in presence of customers that strategize their purchasing decisions? Most pricing and revenue management modeling frameworks and associated commercial systems do not explicitly incorporate this level of strategic consumer behavior. This chapter is part of a small but growing literature that tries to model this effect and study its impact on the firm's strategy and profitability.

In more detail, we consider a revenue-maximizing monopolist firm, the seller, that sells a homogeneous good over some time horizon to a market of heterogeneous strategic customers that differ in their valuations and risk-preferences. The firm seeks to discriminate customers by selling the product at different points in time at different prices by introducing rationing risk, i.e., the risk of not being able to procure the product because its availability is limited. This introduces an incentive for customers with higher valuations, or that are more risk-averse, to pay more for the product offered during periods with higher availability. Customers observe or anticipate correctly the price and associated product availability at different points in time and decide when, if at all, to attempt to purchase the product in a way that maximizes their expected net utility from their purchase that accounts for the rationing risk associated with each time period. The seller's problem is to choose its dynamic pricing and product availability strategies to maximize its profitability taking into account the strategic customer choice behavior.

Though some of these aspects have been studied elsewhere, a common feature of the past analysis is the stylized nature of the problem with respect to customer heterogeneity, e.g., assuming two discrete types of customers, uniformly distributed valuations, no price control, and/or focus on two product variants (i.e., product offered at only two price points). The goal of this chapter is to suggest an approach to this problem that is intuitive and allows one to relax these restrictive market assumptions.

Using a discrete (but arbitrary) customer valuation distribution, we show that the above dynamic pricing problem can be reformulated as a mechanism design problem. While rationing, a key quality attribute in our setting, affects capacity consumption and differentiates our problem from the standard product design problem, the "standard machinery" applied in mechanism design is still useful in our setting.

The structure of the solution is as follows. A) prices are monotonic in fill-rates, i.e., periods with higher prices are also characterized by higher availability rates. B) It suffices to offer as many products as there are discrete valuations (types). C) If the optimal number of offered products is strictly less than the number of customer types, then the offered products partition the customer types in accordance with their strategic choice behavior into contiguous classes (sets of customer types).

These properties allow us to reformulate the problem as a non-linear optimization problem in terms of the price decisions alone. While the resulting problem is not easy to solve in general, it does provide a formulation that is amenable to a brute force computational solution. In many special cases, the problem simplifies considerably. For example, the two product problem with multiple types and general valuation distribution is solved in closed-form. The problem with risk-neutral customers is reformulated as an LP, and hence can be easily solved. The revenue resulting from this LP is shown to be a lower bound of the revenue achievable in the problem with risk-averse customers. Its solution consists of offering either one or two products, irrespective of the size and composition of the market, with the two product solution being offered only if the capacity constraint is tight.

Focusing on application setting where risk-aversion is low, we show that the general problem can be addressed hierarchically through two LPs.

1. The first LP solves the seller's problem treating all consumer types as being risk-neutral.

2. The second LP solves for price and rationing risk perturbations around the risk-neutral solution, taking into account the risk-aversion of the various customer types.

The above decomposition is justified asymptotically as the risk-aversion coefficients of all market participants approach one (i.e., the risk-neutral case). The perturbation in item 2 above is such that the two-product nature of the risk-neutral solution is preserved even for the problem with low risk aversion. This lends credibility to a practical heuristic that would focus on identifying the optimal two product offering, which as mentioned earlier can be solved very efficiently even without this asymptotic decomposition. It also offers justification for the two price point strategy adopted widely in the literature, but highlighting that it is optimal to do so in the low risk-aversion case. Numerical results are used to benchmark this heuristic against the brute-force computational solution.

Our approach unifies and extends several previously established results under a common and intuitive framework. The above problem can also be extended to include, for example, the possibility for some of the customers to be myopic (and to always purchase in a given time period), customer utility to be time-dependent, and customers to differ in terms of their best outside opportunity which affects their no-purchase utility threshold, to name but a few.

The remainder of this chapter is organized as follows: this section concludes with a brief literature review. In section §2.2, we pose the firm's dynamic pricing problem, and develop its reformulation as a product design problem. Section §2.3 characterizes the structure of the optimal solution, and section §2.4 solves the two-product problem, the problem with risk-neutral customers, and the problem with low risk-aversion. Section §2.5 presents some numerical results.

Literature review: An early motivation for modeling strategic customer behavior is in the treatment of durable goods problem by Coase [15], who considers a monopolist selling durable good to a market of strategic customers. Other early work in this area includes Bulow [12], Stokey [62] and Harris and Raviv [24]. Liu and van Ryzin [39] study a problem of offering two product variants at predetermined prices to a market of risk-averse consumers with uniformly distributed valuations. Su [63] looks at a problem with high-value and low-value (i.e with a two-point mass valuation distribution) that are either strategic or myopic (purchase at their time of arrival, if at all). Cachon and Swinney [13] study a problem of offering two product variants to a market of myopic, strategic and bargain-hunting customers (that only purchase if the price is low). Zhang and Cooper [74] consider the two-product problem with strategic and myopic customers under a linear and multiplicative demand model.

Besanko and Winston [9] study the dynamic pricing problem of a monopolist facing rational risk-neutral customers using dynamic programming under the assumptions of uniformly distributed valuations and deterministic demand. Levin et. al [38] use dynamic programming to study the dynamic pricing problem of a monopolist firm when demand is stochastic. Aviv and Pazgal [5] consider a two product problem with strategic risk-neutral customers, where customer valuations are a decreasing function of time. Ozge et. al [20] consider a two-period problem with risk-neutral customers where prices are exogenously fixed and the firm's decision involves the amount of inventory to offer for sale. They show that if customers are strategic, it might be optimal to leave some inventory unsold rather than mark it down. Elmaghraby et. al [19] consider the dynamic pricing problem for a monopolist when the price schedule is preannounced and price decreases over time. Zhou et. al [75] analyze two threshold policies for strategic customers, one based on time to the end of horizon and the other based on price falling below a threshold. Strategic rationing as a way to differentiate customers is also discussed in Dana [17], where the primary motivation for rationing is demand uncertainty. Chen and Seshadri [14] study the product design problem of a seller serving information goods to a market of heterogeneous customers in presence of outside opportunities. Xu and Hopp [73] characterize the optimal prices to offer for a continuous model where price sensitivity is time dependent. Asvanunt and Kachani [4] consider the problem of identifying the optimal purchasing decision for a strategic customer in a single leg, single airline setting where the pricing policy is known. Ho et. al [25] study the shopping behavior of strategic customers faced with firms that differ in their price variability. Jerath et. al [31] consider a two-period problem with two firms and under the possibility that in the second period a firm may choose to sell to strategic customers via an opaque intermediary. Su and Zhang [64, 65] examine the effect of strategic customer behavior in the setting of a newsvendor seller and extend their analysis to other related settings. Wilson et. al [72] analyze the two-period problem of an airline facing high and low fare strategic customers. Recent work that addresses learning with strategic customers includes Levin et. al [37], and Liu and van Ryzin [40]. A thorough literature survey of work in this area can be found in Shen and Su [57].

Our use of the direct revelation principle (Myerson [48]) towards solving the product design reformulation of the dynamic pricing problem is very similar to the approach adopted in Harris and Raviv [24] and Moorthy [44]. Our notion of fill-rate corresponds to their notion of quality. However, while quality affects cost in [24], in our case fill-rates are tied together through the capacity constraint. This, in addition to the risk-averse behavior of customers, makes our problem different and more complicated.

2.2 Dynamic pricing with strategic customers

In this section, we formulate the firm's dynamic pricing problem, and show how it can be reformulated as a static mechanism design problem, thereby making it more amenable to analysis.

2.2.1 Problem formulation

Seller: A monopolist firm seeks to sell a homogeneous good to a market of strategic customers that differ in their valuations and risk-aversion. In order to optimally segment the market and maximize revenues, the monopolist sells the product over some time horizon different prices and fill-rates. Different (price, fill-rate) combinations can be interpreted as product variants. Time is discrete, and indexed by $t = 1, ..., T$, where T is the length of the sales horizon. The capacity, denoted by C, can be endogenous (an optimization variable) or exogenously given (fixed). The capacity cost is linear, and there is no inventory carrying cost. We denote by p_t and r_t the price and the fill-rate associated with the product offered by this monopolist in the t^{th} period. We also refer to it as the t^{th} product.

The seller's policy (p, r) is assumed to be known to the customers, either because it is announced to the market, or because customers have estimated it through repeated interactions with the firm. The seller's strategy (p, r) should be credible in the sense that the seller commits to it at the start of the selling horizon and cannot deviate from it at any point after that even if that would be optimal from that instant onwards; e.g., the seller cannot announce that the low price product variant will be offered with a significant rationing risk, and then once the high valuation customers buy the high price variant, decide to offer the lower priced variant at full availability so as to capture more revenue.

Customers: We allow the customer valuation distribution to be arbitrary but discrete. We assume that there are N distinct customer valuations, $v_1, v_2, ..., v_N$. Without loss of generality, we assume that $N \leq T$. The discrete valuations could be obtained as a result of some clustering analysis or by dividing the support of valuation distribution uniformly. Corresponding to each valuation v_i, there is an associated number π_i, denoting the size of customer segment with this valuation. The pair (v, π) defines an arbitrary discrete valuation distribution in a market with N types. We assume that the number of customers π_i with valuation v_i is deterministic. For notational convenience, we will assume $v_1 > v_2 > ... > v_N > 0$ and refer to the customer segment that has valuation v_i as "type i". Type i customers, apart from their valuation, are also characterized by a risk-aversion parameter, γ_i, assumed to be rational, and are endowed with the power-utility function. Specifically, the net expected utility for a type i customer from product t is given by $(v_i - p_t)^{\gamma_i} r_t$. We also assume that higher valuation types are at least as risk-averse as the low valuation types, i.e., $0 < \gamma_1 \leq \gamma_2 \leq ... \leq \gamma_N \leq 1$. Customers seek to purchase a product as long as their net expected utility is non-negative. If $v_i < p_t$, then we define the resulting utility to be 0^-. Customers choose the variant that maximizes their expected net utility according to:

$$\chi(i,p,r) = \begin{cases} \arg\max_{1\leq t\leq T} \ (v_i - p_t)^{\gamma_i} \ r_t, & \text{if}(v_i - p_t)^{\gamma_i} \ r_t \geq 0 \ \text{ for some t,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

In the following, we will often abbreviate $\chi(i,p,r)$ to $\chi(i)$. Note that given the discrete type space and the assumption that customers of each type are homogeneous, all customers of each particular type will make the same choice. We assume that each customer makes the decision to buy one of the offered products only once and buys only one unit of product. In particular, if a customer decides to enter the system in a particular period and does not obtain a unit of the product (because of being rationed out), then the customer leaves and does not contend to buy any other product offered by this firm. In a more general formulation, this customer could be expected to attempt to buy the product in a later period. However, we do not model this flexibility. We also assume that there is no strategic interaction amongst the customers (i.e., the firm operates in a large market).

Dynamic pricing problem formulation: Under the above assumptions, the revenue maximization problem faced by the monopolist is given by:

$$\max_{p,r} \ \ \Sigma_{t=1}^{T} \ (\Sigma_{i=1}^{N} 1_{\{\chi(i)=t\}}\pi_i) \ r_t \ p_t \tag{2.2}$$

$$s.t. \ \ \Sigma_{t=1}^{T} \ (\Sigma_{i=1}^{N} 1_{\{\chi(i)=t\}}\pi_i) \ r_t \leq C, \tag{2.3}$$

$$0 \leq r_t \leq 1, \quad 0 \leq p_t, \ t = 1,2,...,T. \tag{2.4}$$

The objective is the sum of the revenues from all product variants: $(\Sigma_{i=1}^{N} 1_{\{\chi(i)=t\}}\pi_i)$ is the number of customers that wish to purchase in period t, r_t is the fraction of customers that are served, and p_t is the price per unit sold. For each time period t, the price, fill-rate combination (p_t, r_t) can be interpreted as a "product" offered by the firm. The T products are sequenced in time, $t = 1, ..., T$, with $t = 1$ denoting the first and $t = T$ denoting the last product respectively. Capacity is consumed in this sequence as well, and hence defining $C_0 = C$, C_t to be the capacity at the end of period t, we observe that $C_t = C_{t-1} - (\Sigma_{i=1}^{N} 1_{\{\chi(i)=t\}} \pi_i)\ r_t$. Equation (2.3) enforces the constraint that the cumulative sales over the sales horizon cannot exceed the available capacity, and hence $C_t \geq 0,\ \forall\ t = 1, ..., T$. The optimization variables are the price and fill-rate to offer in each of these T periods, and prices are non-negative, fill-rates are between 0 and 1, that the total sales cannot exceed the available capacity.

2.2.2 Reformulation as a mechanism design problem

The strategic consumer choice in (2.2)-(2.4) complicates the head-on treatment of this problem and at times obfuscates the underlying intuition. This section develops a mechanism design formulation that is equivalent to the problem specified in (2.2)-(2.4).

Sufficiency of N products: As a starting point, we show that the firm needs to offer at most N distinct products, N being the number of customer types.

Lemma 1 *Let k^* be the optimal number of products for formulation (2.2)-(2.4). Then, $k^* \leq N$.*

The above result does not preclude the case where the firm may optimally choose less than N products, or even just one product. Hence the firm needs to segment the sales horizon of T periods into at most N intervals such that a distinct product (price, fill-rate) combination is offered in each interval. Since all customers are assumed to be fully strategic, the length (as long as it is non-zero) or the ordering of the intervals during which distinct products are offered does not matter.

Reformulation as a static, product design problem: Next note that we can rewrite the revenue in (2.2),

$$\Sigma_{t=1}^{T}\ (\Sigma_{i=1}^{N} 1_{\{\chi(i)=t\}} \pi_i)\ r_t\ p_t = \Sigma_{i=1}^{N}\ \pi_i\ p_{\chi(i)}\ r_{\chi(i)}, \tag{2.5}$$

where we define $p_0 := 0,\ r_0 := 0$. Hence, the T period optimization problem in (2.2)-(2.4) can be viewed as a single period problem, when we interpret the fill-rates associated with different time periods as quality attributes of the different product variants that the firm offers to this market of strategic customers. While customers choose the optimal time to enter the system and purchase a product (if at all), for the firm, time does not explicitly enter the problem. The firm needs to compute the optimal prices and fill-rates as if it were a single period problem and all the customers arrive, purchase (if at all), and depart in the same period. This mapping is illustrated in Figure 2.1.

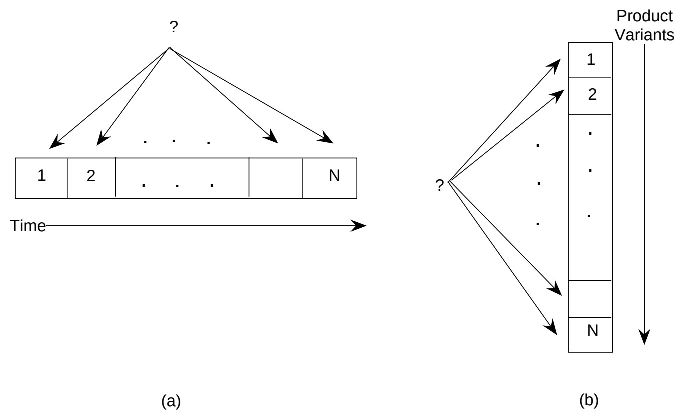


Figure 2.1: In model (a), customers strategize over the timing of their purchases. Model (b) interprets each time period as a product variant, and customers strategize over which variant to choose, if any. Also, a solution to model (a) can be mapped to a solution to model (b), and vice-versa.

The above observation allows us to reformulate the dynamic pricing problem as a static mechanism design problem. Customers arrive and observe the product menu offered by the firm, and make their choices accordingly. Each customer is characterized by its type designation, which is private information, i.e., not directly observed by the firm. The firm's problem is to design the optimal product menu so as to maximize its profitability.

To begin with, one can restrict the firm's optimization problem to so called "direct mechanisms", wherein the firm designs a payment and product allocation policy ("the mechanism"), under which the customers choose to truthfully self-report their type, as described in Myerson [47]. Essentially, in order to elicit this private type information, the mechanism is designed in a way such that the customer is allocated the product variant that she/he would have selected on her/his own. Following lemma 1, which ensures that we need to offer at most N distinct products, the resulting problem can be formulated as follows:

$$\max \quad \Sigma_{i=1}^{N} \ p_i \ \pi_i \ r_i \tag{2.6}$$

$$s.t. \quad (v_i - p_i)^{\gamma_i} \ r_i \geq (v_i - p_j)^{\gamma_i} \ r_j, \quad \forall j \neq i, \tag{2.7}$$

$$(v_i - p_i) \ r_i \geq 0, \ \ i = 1, 2, ..., N, \tag{2.8}$$

$$\Sigma_{i=1}^{N} \ \pi_i \ r_i \ \ \leq C, \tag{2.9}$$

$$0 \leq p_i, \quad 0 \leq r_i \leq 1, \ \ i = 1, 2, ..., N. \tag{2.10}$$

Equations (2.6) assumes, without loss of generality, that customer type i buys product i, i.e., $\chi(i) = i$. Equations (2.7) are the Incentive Compatibility (IC) conditions, enforcing that customer type i (at least weakly) prefers product i over all other products offered by the firm. Equations (2.8) are the Individual Rationality (IR) conditions, enforcing that customer type i buys from the firm only if the resulting consumer surplus is non-negative. Equation (2.9) enforces the capacity constraint. In the optimal solution, some products can be the same, thereby allowing less that N distinct products to be offered. In our solution, $r_i = 0$ implies that customer type i is not offered a product. The above discussion leads to the following theorem.

Theorem 1 *The problem (2.2)-(2.4) is equivalent to the product design problem (2.6)-(2.10) in the sense that both formulations lead to the same optimal solution.*

Translation of product design solution to a dynamic policy: Given a solution to (2.6)-(2.10), a solution to (2.2)-(2.4) can be obtained by assigning to each unique variant (price, fill-rate combination), an interval of time over which it will be sold. For example, if the solution (2.6)-(2.10) involves offering k distinct products, then one possible assignment is to offer the k variants in k disjoint intervals, each of length $\lfloor T/k \rfloor$. Since customers are fully-strategic, arrive at the beginning of the time horizon, and demand is deterministic, the order in which different variants are offered, or the duration of time for which they are offered, does not matter in our stylized model. In a richer model, it might be optimal to offer products in a certain order, e.g., in increasing order of prices, as in Su [63]. The mechanism design formulation can incorporate other model attributes, such as time-discounting, myopic behavior, etc. that would force the solution to "define" the sequencing of product variants over time.

The mechanism design framework that we propose is general, and can be extended to examine several other cases of interest, including myopic customers, customers with time-sensitive utility, and customers in markets with outside opportunities.

2.3 Analysis of the mechanism design problem

The mechanism design formulation (2.6) - (2.10) allows us to deduce several structural properties of the optimal solution. We consider these next. In what follows, without loss of generality, we will assume that $\chi(i) = i,\ i = 1, ..., N$, whenever we consider a N product solution. We first show that the optimal prices and fill-rates are monotonic, with higher prices offered at higher fill-rates to customer types with high valuations.

Lemma 2 ***(Monotonicity of prices and fill-rates)*** *At the optimal solution for (2.6)-(2.10), $p_1 \geq p_2 \geq ... \geq p_N$ and $r_1 \geq r_2 \geq ... \geq r_N$. Moreover, if for some $i \neq j$, $p_i > p_j$ then $r_i > r_j$, and if $p_i = p_j$, then $r_i = r_j$.*

Our next lemma shows that if suffices for type i customers (buying product i) to only check the IC constraints for products intended for types $i-1$ and $i+1$, if any.

Lemma 3 ***(Transitivity of IC conditions)*** *The IC conditions in (2.7) are equivalent to:*

$$(v_i - p_i)^{\gamma_i}\ r_i \geq (v_i - p_{i+1})^{\gamma_i}\ r_{i+1},\ \ i = 1, 2, ...N-1, \tag{2.11}$$

$$(v_i - p_i)^{\gamma_i}\ r_i \geq (v_i - p_{i-1})^{\gamma_i}\ r_{i-1},\ i = 2, ...N. \tag{2.12}$$

We will refer to (2.11) and (2.12) as the downstream and upstream IC constraints, respectively. Through (2.11)-(2.12) we have reduced the number of IC constraints from $N(N-1)$ to $2(N-1)$, and the firm's problem (2.6)-(2.10) can be reduced to (2.6), (2.8)-(2.12). Lemma 4 shows that products offered by the firm partition the customer types into contiguous sets, so that if customer types $i-1$ and $i+1$ buy the same product l, then customer type i also buys product l. This property will be exploited in subsequent computational algorithms.

Lemma 4 ***(Contiguous Partitioning)*** *Suppose the firm offers $k \leq N$ distinct products with $p_1 > p_2 > ... > p_k$ and $r_1 > r_2 > ... > r_k$, and such that each generates non-zero demand. Then, these products partition the customer types into contiguous sets $\{1, ..., i_1\}$, $\{i_1+1, ..., i_2\}$, ..., $\{i_{k-1}+1, ..., i_k\}$, buying product 1, 2, ..., k, respectively, $1 \leq i_1 < i_2 < ... < i_k \leq N$, and customer types $\{i_k+1, ..., N\}$, if any, not buying from the firm. In addition,*
a) $r_1 = \min\left(1, \frac{C}{\Sigma_{l=1}^{i_1}\pi_l}\right)$,
b) $p_j = v_{i_j}$, *where* $j = \max\{1 \leq l \leq k \mid r_l > 0\}$ *in the optimal solution.*

In what follows, we will assume that whenever $k \leq N$ products are offered, they partition the customer types as in lemma 4, i.e., $\chi(1) = ... = \chi(i_1) = 1$, $\chi(i_1+1) = ... = \chi(i_2) = 2$, ..., $\chi(i_{k-1}+1) = ... = \chi(i_k) = k$, and $\chi(i_k+1) = \chi(i_k+2) = ... = 0$, if any.

Using lemma 4, we can characterize the optimal one product solution.

Corollary 1 ***(One product solution)*** *The optimal one product solution is $p^* = p_i$, $r^* = \min\left(1, \frac{C}{\Sigma_{k=1}^{i}\pi_k}\right)$ where $i = \text{argmax} \min(C, \Sigma_{k=1}^{j}\pi_k)v_j$. Moreover, if $C \leq \pi_1$, then the globally optimal solution is to offer a single product to type 1 customers with $p_1 = v_1$ and $r_1 = \frac{C}{\pi_1}$.*

To avoid trivial solutions, hereafter we will assume that $C > \pi_1$. Our next result shows that the downstream IC constraints for types 1, ..., N-1 are tight and the upstream IC constraints for types

2, ..., N can be dropped.

Proposition 1 *Formulation (2.6), (2.8)-(2.12) has the same (optimal) solution as formulation (2.6), (2.8)-(2.11).*

The proof of proposition 1 also establishes that constraint (2.11) is tight in the optimal solution, and so we can also write it as an equality constraint and use it to express the optimal fill-rates in terms of the optimal prices. This is summarized in the following corollary.

Corollary 2 *a) A price vector p defines a partitioning of the customer types, specifically, $p_1 = p_2 = ... = p_{i_1} > p_{i_1+1} = ... = p_{i_2} > ... > p_{i_{k-1}+1} = ... = p_{i_k},\ p_{i_j} \leq v_{i_j},\ j = 1, ..., k-1,\ p_k = v_{i_k}$, partitions the customer types as described in lemma 4.*
b) Fixing the price vector as above, the optimal fill-rates for $j = 1, ..., k$, are given as follows:

$$r_j = \min\left(\max \left(\frac{C - \Sigma_{l=1}^{j-1} \ (\Sigma_{m=i_{l-1}+1}^{i_l} \pi_m) \ r_l}{\Sigma_{m=i_{j-1}+1}^{i_j} \pi_m},\ 0 \right),\ \Pi_{l=1}^{j-1} \left(\frac{v_{i_l} - p_l}{v_{i_l} - p_{l+1}} \right)^{\gamma_{i_l}} \right). \tag{2.13}$$

Clearly, the above observation holds at the optimal solution. Also, it follows that given an exogenously fixed price vector p satisfying the monotonicity condition in corollary 2, problem (2.6), (2.8)-(2.11) is solvable in closed form. Formulation (2.6), (2.8)-(2.11) also leads to the following corollary, which relates the optimal revenue with risk-neutral customers to optimal revenue with risk-averse customers.

Corollary 3 *Let $R(\gamma)$ be the optimal revenue achieved for (2.6), (2.8)-(2.11) with risk-aversion vector γ. Let $\mathbf{1}$ denote the N-vector of ones. Then $R(\gamma) \geq R(\mathbf{1})$, $\forall \gamma \leq \mathbf{1}$, where $R(\mathbf{1})$ denotes the optimal revenue achieved for (2.6), (2.8)-(2.11), for risk-neutral customers.*

2.4 Computations

In general, (2.6), (2.8)-(2.11) appears to be a hard problem, in part due to the bilinear objective, but mostly due to the non-convex nature of the constraint (2.11) for $\gamma < 1$. We next discuss two special cases where problem (2.6), (2.8)-(2.11) can be solved efficiently, and then relate them for our key computational and managerial insight regarding the near optimality of two-product strategies in low risk-aversion settings.

2.4.1 Risk-neutral case

When customers are risk-neutral, objective (2.6) and constraint (2.11) can be simplified through appropriate variable substitutions to lead into an equivalent LP formulation.

Proposition 2 *If $\gamma_i = 1,\ i = 1, 2, ..., N,$ then (2.6), (2.8)-(2.11) can be reformulated as the following LP: choose $x_i,\ y_i,\ i = 1, ..., N$ to*

$$\max\ \Sigma_{i=1}^{N}\ \pi_i\ x_i \tag{2.14}$$

$$s.t.\ \ x_i - x_{i+1} = v_i\ y_i,\quad i = 1, 2, ..., N-1, \tag{2.15}$$

$$\Sigma_{i=1}^{N}\ (\Sigma_{i=i}^{N}\ y_i)\ \pi_i \leq C, \tag{2.16}$$

$$v_i\ \Sigma_{i=i}^{N}\ y_i \geq x_i, \tag{2.17}$$

$$0 \leq x_i,\quad 0 \leq \Sigma_{i=i}^{N}\ y_i \leq 1, \tag{2.18}$$

where $x_i := p_i r_i$ and $y_i := r_i - r_{i+1},\ i = 1, ..., N,\ y_{N+1} := 0.$

This LP gives us x_i and y_i as solution. However, $y_N = r_N$, and $y_i = r_i - r_{i+1}$, so we can obtain $r_i,\ i = 1, ..., N$. Next the relation $x_i = p_i r_i$ gives us the value of $p_i,\ i = 1, ..., N$. Proposition 2 implies that the firm's revenue-maximization is easy to solve in the case of risk-neutral customers. Our next proposition shows that furthermore, there exists a solution to (2.14)-(2.18) that involves offering at most two distinct products, irrespective of the customer valuation distribution and the available capacity.

Proposition 3 *If $\gamma_i = 1,\ i = 1, ..., N$, then the optimal number of products to offer to risk-neutral customers, $\overline{k}$, is at most 2. In particular, $\overline{k} = 1$ if the capacity constraint is slack in the optimal solution, and $\overline{k} = 2$ if the capacity constraint is tight in the optimal solution.*

The proof of the above proposition also leads to the following corollary, which states that under our assumption that $C > \pi_1$, the highest fill-rate is always equal to 1 in the optimal risk-neutral solution irrespective of the available capacity.

Corollary 4 *For risk-neutral customers, under $\pi_1 \leq C$, it is never optimal to set $r_1 < 1$.*

2.4.2 Two product case ($k = 2$)

We now turn to solving the monopolist's problem when the number of products it seeks to offer to the market is small. This may be due to administrative reasons ("menu" costs associated with offering new products) or branding considerations (with more than two products being offered, a customer might be rationed out while the product is available in the next period), both of which are not captured of our model. As a special case, we consider the case where the monopolist can offer at most two products. In this case, the monopolist effectively partitions the N customer types into 3 segments. The first segment of customers from types 1 to i_1 buys product 1, the second segment of customers from types $i_1 + 1$ to i_2 buys product 2, and the remaining customer types, if any, do not buy from the firm. Algorithm 1 outlines how to solve this problem efficiently in $O(N^2)$ time (the proof is presented in Appendix B).

From lemma 1, we know that at most two distinct products need to be offered in a market with two customer types, and so it follows that we can solve the two customer type problem as a special case of the two product problem. The two product solution provides a lower bound to the optimal

Algorithm 1 To calculate two distinct product solution

$R^* = 0$
for $i_1 = 1$ to $N-1$ **do**
  for $i_2 = i_1 + 1$ to N **do**
    $R_{i_1,i_2} = 0$
    if $\gamma_{i_1} < 1$ **then**
      if $C \geq (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)(\frac{\Sigma_{l=i_1+1}^{i_2}\pi_l v_{i_2}\gamma_{i_1}}{(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2})})^{\frac{\gamma_{i_1}}{1-\gamma_{i_1}}}$ && $(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1} - v_{i_2}) > (\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}$ **then**
        $R_{i_1,i_2} = (\Sigma_{l=1}^{i_1}\pi_l)v_1 + (\frac{(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}}{(\Sigma_{l=1}^{i_1}\pi_l)^{\gamma_{i_1}}(v_{i_1}-v_{i_2})^{\gamma_{i_1}}})^{\frac{1}{1-\gamma_{i_1}}}(\frac{1}{\gamma_{i_1}} - 1)$
      else if $C < (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)(\frac{(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}}{(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2})})^{\frac{\gamma_{i_1}}{1-\gamma_{i_1}}}$ && $C < (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)$ **then**
        $R_{i_1,i_2} = (\Sigma_{l=1}^{i_1}\pi_l)v_{i_1} - (\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1} - v_{i_2})(\frac{C-\Sigma_{l=1}^{i_1}\pi_l}{\Sigma_{l=i_1+1}^{i_2}\pi_l})^{\frac{1}{\gamma_{i_1}}} + (\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}(\frac{C-\Sigma_{l=1}^{i_1}\pi_l}{\Sigma_{l=i_1+1}^{i_2}\pi_l})$
    else if $\gamma_{i_1} == 1$ **then**
      if $C \geq (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)$ && $(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1} - v_{i_2}) == (\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}$ **then**
        $R_{i_1,i_2} = (\Sigma_{l=1}^{i_1}\pi_l)v_1$
      else if $C < (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)$ && $(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1} - v_{i_2}) < (\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}$ **then**
        $R_{i_1,i_2} = (\Sigma_{l=1}^{i_1}\pi_l)v_{i_1} - (\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1} - v_{i_2})(\frac{C-\Sigma_{l=1}^{i_1}\pi_l}{\Sigma_{l=i_1+1}^{i_2}\pi_l}) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}(\frac{C-\Sigma_{l=1}^{i_1}\pi_l}{\Sigma_{l=i_1+1}^{i_2}\pi_l})$
    if $R^* < R_{i_1,i_2}$ **then**
      $R^* = R_{i_1,i_2}$
if $R^* == 0$ **then**
  Not optimal to offer two distinct products

revenues attainable in problem (2.6), (2.8)-(2.11). Since the general problem is hard to solve, this provides a heuristic solution to the problem. The next subsection show that it is asymptotically optimal in settings with low risk-aversion, and its overall effectiveness is evaluated numerically in section 2.5.

2.4.3 Low risk-aversion: offering two products is near-optimal

We now focus on the setting where customers have low risk-aversion, i. e., γ close to 1. To that end, we will rewrite $\gamma_i = 1 - x_i$, where $x_i := \frac{1-\gamma_i}{1-\gamma_1}$ and $x_1 = 1 \geq x_2 \geq ... \geq x_N \geq 0$. We assume that $\gamma_1 < 1$, else the problem involves risk-neutral customers only. We will consider a sequence of problems indexed by n, where the n^{th} problem is characterized by the risk-aversion parameter vector γ^n, given by $\gamma_i^n = 1 - \frac{x_i}{n}$ for $i = 1, ..., N$. When $n = \frac{1}{1-\gamma_1}$, we “recover” the original model parameters, or in other words, the element in the above sequence that corresponds to $n = \frac{1}{1-\gamma_1}$ is exactly the one we started with. We are interested in the case where $\gamma_1 \uparrow 1$. [1]

We will denote the optimal solution to the problem with γ^n as the risk-aversion parameter vector as (p^n, r^n), and the optimal solution to the risk-neutral problem as $(\overline{p}, \overline{r})$. Following corollary 2, given price-vector p^n, the corresponding fill-rate vector r^n is uniquely determined. So, in what follows, we will often abbreviate (p^n, r^n) to p^n and $(\overline{p}, \overline{r})$ to $\overline{p}$. We will also refer to the problem with γ^n as the risk-aversion parameter vector as $\mathcal{P}^n$ and the risk-neutral problem as $\overline{\mathcal{P}}$. We will denote the feasible set, given by equations (2.8)-(2.12), for $\mathcal{P}^n$ as $\mathcal{S}^n$ and the feasible set for $\overline{\mathcal{P}}$ as $\overline{\mathcal{S}}$.

Asymptotic optimality of risk-neutral solution: We will make the following assumption in the subsequent analysis.

Assumption 1: The risk-neutral solution $(\overline{p}, \overline{r})$ is unique.

Proposition 4 *Under assumption 1, for any convergent subsequence* $\{p^{n_k}\}$, $p^{n_k} \to \overline{p}$, *and* $R(\gamma^{n_k}) \to R(\mathbf{1})$, *as* $n_k \uparrow \infty$.

The proof of proposition 4 also shows that for n sufficiently large, the optimal risk-neutral solution $(\overline{p}, \overline{r})$ is feasible for the problem with risk-averse customers, $\mathcal{S}^n$, and that the optimal solution p^n is "close" to a feasible risk-neutral solution $p' \in \overline{\mathcal{S}}$.

Perturbations around $(\overline{p}, \overline{r})$: With this knowledge, we will look at the perturbed solution to $\overline{P}$ as a candidate optimal solution for P^n for n sufficiently large. Specifically, for $\mathcal{P}^n$, we will consider solutions of the form $(\overline{p} + \delta^n, \overline{r} + \rho^n)$ for the risk-averse problem, where $\delta^n := \frac{\delta}{n}$, and $\rho^n := \frac{\rho}{n}$, such that $\delta^n = p^n - \overline{p}$, and $\rho^n = r^n - \overline{r}$. Suppose the optimal risk-neutral solution partitions involves offering $k \leq N$ distinct products, where the products partition the customer types as in lemma 4. If $i_k < N$, define $j := i_k + 1$, and set $r_i = 0,\ p_i = v_i,\ i \geq j$. Then, the revenue-maximization

[1] It is also possible to consider the case where $\gamma^n \downarrow 0$, wherein the asymptotically optimal solution can be characterized as follows. Let $i^* = \min\{i \mid \Sigma_{l=1}^{i} \pi_l > C\}$. Then, $p_i^n \to v_i,\ i < i^*$, $p_i^n = v_i,\ i \geq i^*$, $r_i^n \to 1,\ i < i^*$, $r_{i^*}^n = \frac{C - \Sigma_{l=1}^{i^*-1} \pi_l}{\pi_{i^*}}$, $r_i^n = 0,\ i > i^*$, $r_1^n > r_2^n > ... > r_{i^*}^n > 0$, and the optimal revenue converges to revenue achievable with myopic customers. This case corresponds to extremely high risk-aversion, and we do not analyze it in detail.

problem $\mathcal{P}^n$ can be written as follows.

$$\max \quad \Sigma_{i=1}^N \pi_i(\overline{p}_i + \delta_i^n)(\overline{r}_i + \rho_i^n) \tag{2.19}$$

$$s.t. \quad \Sigma_{i=1}^N \pi_i(\overline{r} + \rho_i^n) \leq C, \tag{2.20}$$

$$(v_i - \overline{p}_i - \delta_i^n)^{\gamma_i}(\overline{r}_i + \rho_i^n) \geq (v_i - \overline{p}_{i+1} - \delta_{i+1}^n)^{\gamma_i}(\overline{r}_{i+1} + \rho_{i+1}^n), \quad i = 1, ..., N-1, \tag{2.21}$$

$$(v_i - \overline{p}_i - \delta_i^n)(\overline{r}_i + \rho_i^n) \geq 0, \quad i = 1, ..., N-1, \tag{2.22}$$

$$\delta_i^n \geq \delta_{i+1}^n, \ i \notin \{i_1, i_2, ..., i_k\}, \ i < i_k, \quad \rho_i^n \geq \rho_{i+1}^n, \quad i \notin \{i_1, i_2, ..., i_k\}, \tag{2.23}$$

$$\delta_{i_k}^n \leq 0, \quad \rho_i^n \leq 0, \quad i = 1, ..., i_1, \tag{2.24}$$

$$\delta_i^n \leq 0, \quad \rho_i^n \geq 0, \quad i \geq j. \tag{2.25}$$

Equations (2.19) and (2.20) represent the objective and the capacity constraint, respectively, for the problem $\mathcal{P}^n$. Equation (2.21) is the downstream IC constraint. Equation (2.22) ensures that the IR condition is satisfied. Equation (2.23) ensures that prices and fill-rates are monotonically non-increasing with respect to customer types. For n sufficiently large, we need to enforce this condition only for indices $i \notin \{i_1, i_2, ..., i_k\}$. Equation (2.24) ensures that $p_{i_k}^n$ cannot increase from the optimal price $\overline{p}_{i_k} = v_{i_k}$ for the risk-neutral case, and similarly that the fill-rate r_1^n cannot increase from the optimal fill-rate $\overline{r}_1 = 1$ for the risk-neutral case (Following corollary 4 and our assumption that $\pi_1 \leq C$, the optimal solution involves setting $\overline{r}_1 = 1$). Finally, equation (2.25) ensures that prices and fill-rates for types that were not being sold a product in the risk-neutral case (indices $i \geq j$), if any, are all non-negative.

Characterization of the optimal perturbation around $(\overline{p}, \overline{r})$: Since $\gamma^n = 1 - \frac{x}{n}$, in what follows, we will use Taylor expansion and focus on the second-order terms. This will lead to a LP in terms of x, δ and ρ. This would imply that the price and rationing risk "corrections" needed because customers are not risk-neutral are captured through a solution to a LP. We proceed to derive this LP as follows.

The objective in (2.19) can be re-written as $\Sigma_{i=1}^{N}(\pi_i\overline{p}_i\overline{r}_i + \pi_i\overline{p}_i\rho_i^n + \pi_i\delta_i^n\overline{r}_i + \pi_i\rho_i^n\delta_i^n)$, wherein we note that first term is objective is a constant, while the last term is $O(\frac{1}{n^2})^2$. Similarly, equation (2.20) can be re-written as $\Sigma_{i=1}^{N}(\pi_i\overline{r}_i + \pi_i\rho_i^n) \leq C$. If the capacity constraint for $\overline{\mathcal{P}}$ is slack at the optimal solution $(\overline{p}, \overline{r})$, then this constraint can be dropped (since as n grows large, the first-order term will dominate). If not, since the optimal risk-neutral solution was capacitated, it can be re-written as $\Sigma_{i=1}^{N}\pi_i\rho_i^n \leq 0$. Finally, using Taylor expansion, the IC constraint (2.21) can be written as

$$\begin{aligned}
&(v_i - \overline{p}_i)\rho_i^n - \delta_i^n\overline{r}_i - (v_i - \overline{p}_i)\overline{r}_i(1-\gamma_i^n)\log(v_i - \overline{p}_i) + O\left(1/n^2\right) \geq \\
&(v_i - \overline{p}_{i+1})\rho_{i+1}^n - \delta_{i+1}^n\overline{r}_{i+1} - (v_i - \overline{p}_{i+1})\overline{r}_{i+1}(1-\gamma_i^n)\log(v_i - \overline{p}_{i+1}) + O\left(1/n^2\right),
\end{aligned} \tag{2.26}$$

where we used that $(v_i - \overline{p}_{i+1})\overline{r}_{i+1} = (v_i - \overline{p}_i)\overline{r}_i$, which follows from the tightness of the downstream IC condition in the optimal solution to $\overline{\mathcal{P}}$, as shown in proposition 1. We will substitute this constraint by the following constraint.

$$\begin{aligned}
&(v_i - \overline{p}_i)\rho_i^n - \delta_i^n\overline{r}_i - (v_i - \overline{p}_i)\overline{r}_i(1-\gamma_i^n)\log(v_i - \overline{p}_i) + \geq \\
&(v_i - \overline{p}_{i+1})\rho_{i+1}^n - \delta_{i+1}^n\overline{r}_{i+1} - (v_i - \overline{p}_{i+1})\overline{r}_{i+1}(1-\gamma_i^n)\log(v_i - \overline{p}_{i+1}) + \epsilon_i,
\end{aligned} \tag{2.27}$$

where $\epsilon_i > 0$, if $i \in \{i_1, i_2, ..., i_{k-1}\}$, $\epsilon_i = 0$ otherwise. For n sufficiently large, feasibility of constraint (2.27) implies feasibility of constraint (2.26). In what follows we will use the following notation.

$$u_{i,i} = v_i - \overline{p}_i, \quad w_{i,i} = (v_i - \overline{p}_i)\overline{r}_i\log(v_i - \overline{p}_i), \tag{2.28}$$

$$u_{i,i+1} = v_i - \overline{p}_{i+1}, \quad w_{i,i+1} = (v_i - \overline{p}_i)\overline{r}_i\log(v_i - \overline{p}_{i+1}). \tag{2.29}$$

The above discussion leads to the following first-order optimization problem.

$$\max\ \Sigma_{i=1}^{N}(\pi_i\overline{p}_i\rho_i + \pi_i\overline{r}_i\delta_i) \tag{2.30}$$

[2] $g(x) = O(f(x))$ denotes that $\lim_{x\downarrow 0}\frac{g(x)}{f(x)} = c < \infty$

$$s.t.\ \Sigma_{i=1}^{N}\pi_i\rho_i \leq 0, \tag{2.31}$$

$$\rho_i \leq 0, \quad i = 1, ..., i_1, \tag{2.32}$$

$$\delta_{i_k} \leq 0, \tag{2.33}$$

$$\delta_i \geq \delta_{i+1}, \quad i \notin \{i_1, i_2, ..., i_k\}, \quad i < i_k, \quad \rho_i \geq \rho_{i+1}, \quad i \notin \{i_1, i_2, ..., i_k\}, \tag{2.34}$$

$$\delta_i \leq 0, \quad \rho_i \geq 0, \quad i \geq j, \tag{2.35}$$

$$u_{i,i}\rho_i + \delta_{i+1}\overline{r}_{i+1} + w_{i,i+1}x_i \geq u_{i,i+1}\rho_{i+1} + \delta_i\overline{r}_i + w_{i,i}x_i + \epsilon_i, \quad i = 1, ..., N-1. \tag{2.36}$$

Analyzing the dual of problem (2.30)-(2.36), verifying its feasibility and using strong duality for LPs leads to the following proposition.

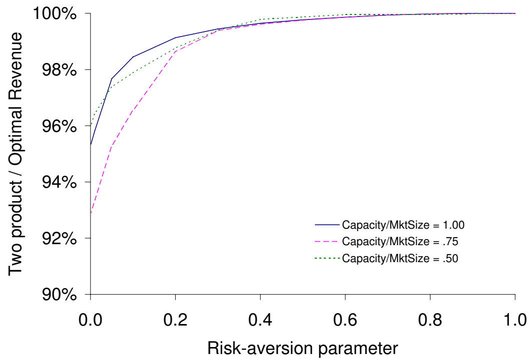
Proposition 5 *The problem (2.30)-(2.36) is feasible and has a finite solution. Let $\overline{k}$ denote the optimal number of products to offer for the risk-neutral problem $\overline{\mathcal{S}}$. Then,*
i) if $\overline{k} = 1$, $\rho_i = \delta_i = 0,\ i = 1, ..., N$,
ii) if $\overline{k} = 2$, $\rho_i = 0,\ i = 1, ..., N$, $\delta_1 = \delta_2 = ... = \delta_{i_1} = (w_{i_1,i_1+1} - w_{i_1,i_1})x_{i_1} - \epsilon_{i_1}$, $\delta_{i_1+1} = ... = \delta_N = 0$, for $\epsilon_{i_1} > 0$ sufficiently small.

Proposition 5 implies that as $\gamma \uparrow 1$, it becomes asymptotically optimal to offer at most two products. Following lemma 1 and algorithm 1, the optimal one product and the optimal two distinct product solution can be computed efficiently, and together with proposition 5, this implies that we can solve for the optimal prices and fill-rates for the low risk-aversion case. This section has methodically showed when and why is a two product solution, i. e., offering the product at two price points at different fill-rates, near optimal. This allows justification for this practical heuristic, and allows one to circumvent the intractability of the general formulation (2.6), (2.8)-(2.11). It also lends credibility to numerous papers that have restricted attention to two product models but without any theoretical justification, highlighting the conditions under which it is suitable to do so.

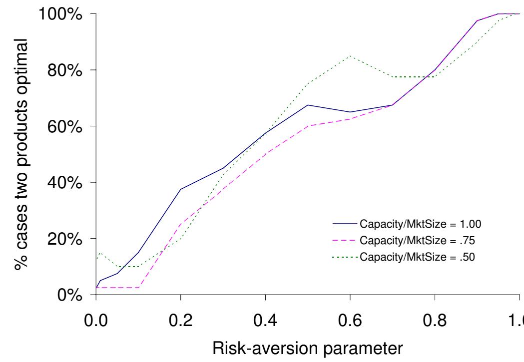
2.5 Numerical Results

We next present some numerical results to evaluate the performance of the two-product heuristic. We consider a market with seven customer types ($N = 7$), with uniform valuations $v_i = 8 - i,\ i = 1, ..., 7$. Type i population-size is sampled from a normal distribution with mean μ_i, and variance σ_i^2, where $\mu_1 = 1,\ \mu_2 = 3,\ \mu_3 = 2,\ \mu_4 = 1,\ \mu_5 = 3.5,\ \mu_6 = 5,\ \mu_7 = 3$, and $\sigma_i = 0.2\mu_i$. Correlation is assumed to be 0, though it can be easily added. For $\sigma_i = 0,\ i = 1, ..., 7$, this corresponds to a bimodal distribution of customer valuations. Other valuation distributions, e.g., uniform, geometric, lead to similar results and are therefore not included. Capacity is fixed at 1, while the capacity to market-size ratio $\frac{C}{\Sigma_{i=1}^{N} \pi_i}$ varies between $(0, 1]$ and is also a simulation input. The risk-aversion parameter varies between $(0, 1]$ and is a simulation input. $\gamma_2, ..., \gamma_N$ are assumed to be order-statistics of random samples drawn uniformly from the interval $[\gamma_1, 1]$, so that $\gamma_1 \leq \gamma_2 \leq ... \leq \gamma_N \leq 1$. Given γ_1 and a fixed capacity to market-size ratio, 50 scenarios are generated, wherein for each scenario, the risk-aversion parameters $\gamma_2, ..., \gamma_N$ are generated randomly as described above, and the customer population size π is sampled from a normal distribution with the parameters given above. Negative demand, if any, is truncated to zero, and customer population sizes are scaled proportionately to achieve the given capacity to market-size ratio. For each scenario, the optimal one-product, two-product and k-product solutions are computed. These are averaged over scenarios to obtain results corresponding to a data point in Figure 2.2.

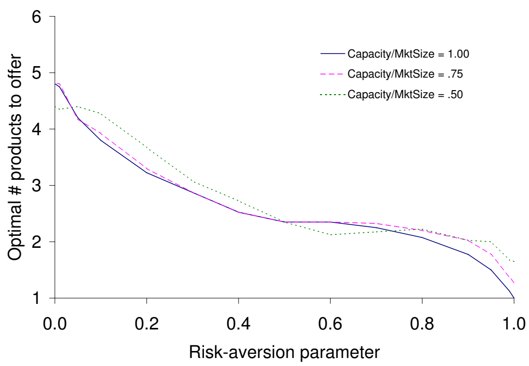
Figure 2.2 (a) shows how the two-product revenue compares with the k-product revenue for different capacity to market-size ratios as γ_1 varies between zero and one. We observe that as γ_1 approaches one, the two-product revenue approaches the k-product revenue. Even when γ_1 is small and not close to 1, the two-product revenue achieves within 8% of the k-product revenues on average, therefore serving as a useful approximation and lower bound. (As described in Footnote 1 in section 2.4.3, as $\gamma \downarrow 0$, the optimal number of products may grow large.) Figure 2.2 (b) shows how the fraction of scenarios where it suffices to offer at most two products varies with γ_1. We



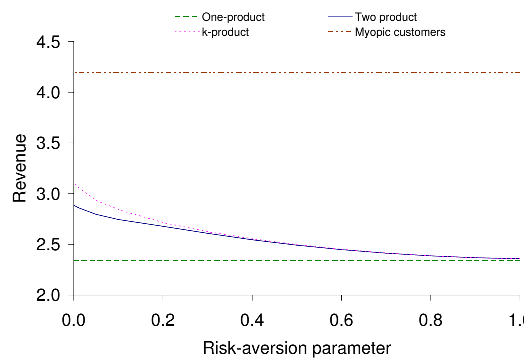
(a) Ratio of two-product to optimal revenue



(b) % scenarios where at most two products suffice



(c) Optimal number of products to offer



(d) Revenue at Capacity/Market-size = 0.75

Figure 2.2: This figure shows how the two-product solution compares to the k-product solution as a function of risk-aversion for different capacity to market-size ratios. Figures (a), (b) and (c) show that the two-product solution approaches the optimal solution as risk-aversion parameter approaches 1. These results are averaged over 50 demand scenarios. Figure (d) examines one such demand scenario in detail and shows that the k-product revenue decreases monotonically and approaches the two-product revenue as risk-aversion approaches one. The maximum revenue is obtained with myopic customers, followed by the k-product, two-product and one-product revenue.

observe that this fraction is non-monotonic, but the overall trend suggests that it increases as γ_1 increases. It equals one in the limit of risk-neutral customers, but for other risk-aversion values, it can be much less one. From Figure 2.2 (a), we know that the two product revenue is close to the k-product revenue, and together this implies that even though the two-product solution might be suboptimal in a large fraction of cases, the two-product solution is close to the k-product solution and hence the suboptimality gap is small. Figure 2.2 (c) shows the optimal number of products to offer as a function of γ_1. Again, while non-monotonic, the overall trend suggests that this number decreases as γ_1 increases. For the risk-neutral case, this number lies between one and two, consistent with our result that at most two products need to be offered in this case. Figure 2.2 (d) shows the one-product, two-product and k-product revenue as a function of γ_1 for the case where capacity to market-size ratio is fixed at 0.75. It also plots the optimal achievable revenue if the customers were all myopic. We observe that the highest revenue is achieved when customers are myopic, followed by the k-product solution, the two-product solution, and the one-product solution, respectively. Both the revenue with myopic customers and the one-product revenue do not depend on customer risk-aversion. The k-product revenue dominates the two-product revenue, and approaches it as γ_1 approaches one. Also, the k-product revenue decreases as γ_1 increases, and as expected, exceeds the revenue with risk-neutral customers.

2.6 Conclusion

In this chapter, we studied the dynamic pricing problem faced by a monopolist firm selling a homogeneous good to a market of strategic customers. Using a discrete valuation distribution, we showed how the dynamic pricing problem can be reformulated as a static mechanism design problem that leads to the same solution, and is more amenable to analysis. While the general problem is hard, we showed this problem can be solved efficiently solved in several special cases: two customer types, two products only, exogenously fixed prices, and risk-neutral customers. This generalizes and unifies several previously published results. We also showed that for the low-risk aversion case, it is asymptotically optimal to offer only two products, and hence the optimal solution to this problem can be obtained efficiently as well. The mechanism design framework that we proposed is general and can be extended to examine several other cases of interest, including myopic customers, customers with time-sensitive utility, and customers in markets with outside opportunities.

Appendix A

Proof of lemma 1 From (2.1), all type i customers select the same product variant, $\chi(i,p,r)$. Since there are N types, there can be at most N distinct products that generate non-zero demand. Hence, it suffices to offer at most N distinct products. $\square$

Proof of lemma 2 We first show that higher prices are associated with higher fill-rates. IC conditions for types i and j (that prefer product i and j, respectively) imply that, $\left(\frac{v_i-p_i}{v_i-p_j}\right)^{\gamma_i} \geq \frac{r_j}{r_i}$ and $\left(\frac{v_j-p_j}{v_j-p_i}\right)^{\gamma_j} \geq \frac{r_i}{r_j}$, respectively. There are three cases to consider. 1) Suppose $p_i = p_j$: then, from the IC condition we obtain $r_i = r_j$. 2) $p_i > p_j$: then, the IC condition for type i implies that $\frac{r_j}{r_i} \leq \left(\frac{v_i-p_i}{v_i-p_j}\right)^{\gamma_i} < 1$. So, $r_j < r_i$. 3) $p_i < p_j$: this case is symmetric to 2). Similarly, one can verify that higher fill-rates are associated with higher prices.

Next we show that "higher" types prefer higher priced products. Suppose that there exist $i < j$, such that $p_i < p_j$. IC conditions for customer types i and j imply that $\left(\frac{v_i-p_i}{v_i-p_j}\right)^{\gamma_i} \geq \frac{r_j}{r_i} \geq \left(\frac{v_j-p_i}{v_j-p_j}\right)^{\gamma_j}$. Also, $p_i < p_j$ and $v_i > v_j$ imply that $1 < \frac{v_i-p_i}{v_i-p_j} < \frac{v_j-p_i}{v_j-p_j}$, and since $\gamma_i \leq \gamma_j$, $\left(\frac{v_i-p_i}{v_i-p_j}\right)^{\gamma_i} < \left(\frac{v_j-p_i}{v_j-p_j}\right)^{\gamma_j}$. The latter implies that the IC conditions cannot hold, and this leads to a contradiction. $\square$

Proof of lemma 3 The proof comprises of two parts. First we show that customer type $i-1$ would rather make the choice made by type i than by $i+1$. Then we show that customer type $i+1$ would rather make the choice made by type i than by $i-1$. Together they imply that IC conditions are transitive.

Step 1: We will show that if type i chooses product i over product $i+1$, then type $i-1$ will also choose product i over product $i+1$. Following proposition 2, $p_{i-1} \geq p_i \geq p_{i+1}$. Consequently, $1 \geq \frac{v_{i-1}-p_i}{v_{i-1}-p_{i+1}} \geq \frac{v_i-p_i}{v_i-p_{i+1}}$, and so $\gamma_{i-1} \leq \gamma_i$ implies that $\left(\frac{v_{i-1}-p_i}{v_{i-1}-p_{i+1}}\right)^{\gamma_{i-1}} \geq \left(\frac{v_i-p_i}{v_i-p_{i+1}}\right)^{\gamma_i}$. The IC condition that guarantees that type i customers choose product i over product $i+1$ implies that $\left(\frac{v_i-p_i}{v_i-p_{i+1}}\right)^{\gamma_i} \geq \frac{r_{i+1}}{r_i}$, thereby leading to the desired inequality.

Step 2: Similarly, one can show that if type i chooses product i over product $i-1$, then type $i+1$ will also choose product i over product $i-1$ (details are omitted). $\square$

Proof of lemma 4 Suppose the seller decides to offer only $k \leq N$ products. Then, we will show that if customer types $i-1$ and $i+1$ choose product l, then customer type i chooses product l, this would imply that offered products partition the customer types into contiguous sets. IC conditions for type $i-1$ and type $i+1$ customers imply that, for any $m \neq l$, $(\frac{v_{i-1}-p_l}{v_{i-1}-p_m})^{\gamma_{i-1}} \geq \frac{r_m}{r_l}$ and $\left(\frac{v_{i+1}-p_l}{v_{i+1}-p_m}\right)^{\gamma_{i+1}} \geq \frac{r_m}{r_l}$. Consider all products m s.t. $v_i > p_m > p_l$. Then, $1 < \frac{v_{i-1}-p_l}{v_{i-1}-p_m} < \frac{v_i-p_l}{v_i-p_m}$. Next $\gamma_{i-1} \leq \gamma_i$ implies that $\left(\frac{v_i-p_l}{v_i-p_m}\right)^{\gamma_i} > \frac{r_m}{r_l}$, and so type i customers also prefer product l over all products m with $v_i > p_m > p_l$. Now consider all products m s.t. $p_m < p_l$. Then, $1 > \frac{v_i-p_l}{v_i-p_m} > \frac{v_{i+1}-p_l}{v_{i+1}-p_m}$. Next $\gamma_i \leq \gamma_{i+1}$ implies that $1 > \left(\frac{v_i-p_l}{v_i-p_m}\right)^{\gamma_i} > \frac{r_m}{r_l}$, so that type i customers also prefer product l over all products m with $p_m < p_l$.

Parts a) and b) of the lemma proceed as follows. a) Since $p_1 \geq p_2 \geq ... \geq p_k$ following lemma 2, setting r_1 to the highest possible value is optimal. This is $\min\left(1, \frac{C}{\Sigma_{l=i_0+1}^{i_1} \pi_l}\right)$. b) Suppose $p_j < v_{i_j}$ where $j = \max\{1 \geq l \leq k | r_l > 0\}$. Then by increasing p_j to v_{i_j}, we increase revenues, while none of the customer types that were buying a product switch classes or discontinue to buy the product. Hence $p_j < v_{i_j}$ cannot be optimal. $\square$

Proof of lemma 1 From lemma 4, we know that $r^* = r_1 = \min\left(1, \frac{C}{\Sigma_{k=1}^{i} \pi_i}\right)$, where customer types $1, .., i$ buy product 1. Also the optimal price to offer the product at when up to type i are being served the first product is v_i, and hence we can search for the optimal value of i by evaluating the revenue at each of the N possible price points $v_1, v_2, ..., v_N$. $\square$

Proof of proposition 1 Suppose the optimal solution to formulation (2.6), (2.8)-(2.12) involves offering $k \leq N$ distinct products, where the products partition the customer types as in lemma 4. Given this partitioning, it suffices to impose downstream IC constraints for types $i_1, i_2, ..., i_{k-1}$ and upstream IC constraints for types $i_1+1, i_2+1, ..., i_{k-1}+1$. Just as in the case of individual customer classes, transitivity across groups also holds.

We next determine the necessary conditions that the optimal prices and the fill-rates must satisfy. Since k distinct products are offered, $0 < v_{i_k} < p_j < v_{i_j},\ j = 1, ..., k-1,\ r_j > 0,\ j = 1, ..., k$, and the Lagrange multipliers with the associated bounding constraints $v_i \geq p_i,\ i = 1, ..., i_k - 1$, $p_i \geq 0,\ i = 1, ..., i_k,\ 1 \geq r_i,\ i = i_1 + 1, ..., i_k,\ r_i > 0,\ i = 1, ..., i_k$ are zero. Moreover, since the solution is optimal, $p_k = v_{i_k}$ and $r_1 = 1$, so that we have $2k$ - 2 optimization variables. We can write the Lagrangian as follows (fulfilment of the constraint qualification condition is shown in Appendix B):

$$\begin{aligned} L = {} & \Sigma_{j=1}^{k}(\Sigma_{l=i_{j-1}+1}^{i_j}\pi_l)p_j r_j + \Sigma_{j=1}^{k-1}\mu_j\left((v_{i_j} - p_j)^{\gamma_{i_j}} r_j - (v_{i_j} - p_{j+1})^{\gamma_{i_j}} r_{j+1}\right) \\ & + \Sigma_{j=1}^{k-1}\zeta_j\left((v_{i_j+1} - p_{j+1})^{\gamma_{i_j+1}} r_{j+1} - (v_{i_j+1} - p_j)^{\gamma_{i_j+1}} r_j\right) + \lambda(C - \Sigma_{j=1}^{k}\Sigma_{l=i_{j-1}}^{i_j}\pi_l r_j). \end{aligned} \tag{2.37}$$

Note that $\mu_j\zeta_j = 0$, since exactly one of the respective constraints is tight; otherwise we can increase revenues by changing the price or the fill-rate. Differentiating with respect to p_1, we obtain

$$\frac{\partial L}{\partial p_1} = \Sigma_{l=1}^{i_1}\pi_l - \mu_1\gamma_{i_1}(v_{i_1} - p_1)^{\gamma_{i_1}-1} + \zeta_1\gamma_{i_1+1}(v_{i_1+1} - p_1)^{\gamma_{i_1+1}-1} = 0,$$

implying that $\mu_1 > 0, \zeta_1 = 0$. Differentiating with respect to p_u, we get that

$$\begin{aligned} \frac{\partial L}{\partial p_u} = {} & (\Sigma_{l=i_{u-1}+1}^{i_u}\pi_l)r_u - \mu_u\gamma_{i_u}(v_{i_u} - p_u)^{\gamma_{i_u}-1}r_u + \mu_{u-1}\gamma_{i_{u-1}}(v_{i_{u-1}} - p_u)^{\gamma_{i_{u-1}}-1}r_u \\ & -\zeta_{u-1}\gamma_{i_{u-1}+1}(v_{i_{u-1}+1} - p_u)^{\gamma_{i_{u-1}+1}-1}r_u + \zeta_u\gamma_{i_u+1}(v_{i_u+1} - p_u)^{\gamma_{i_u+1}-1}r_u = 0. \end{aligned}$$

Now using the induction hypothesis that $\mu_{u-1} > 0, \eta_{u-1} = 0$, we find that $\mu_u > 0, \eta_u = 0$. This implies that all the downstream constraints are tight, while all the upstream constraints are slack. Hence, given any partitioning, we can drop the upstream constraints. Moreover, we can set the downstream constraints to be tight. Since, the choice of partitioning does not matter, this holds for all partitions, and in particular the optimal partition, and hence formulation (2.6), (2.8)-(2.11) leads to the same optimal solution as formulation (2.6), (2.8)-(2.12).

Proof of corollary 2 a) This follows directly from lemma 4. b) The expression for r follows from the tightness of constraint (2.11) in formulation (2.6), (2.8)-(2.11), the second part of lemma 4, and the non-negativity of fill-rates. $\square$

Proof of corollary 3 Note that the risk-aversion parameter enters formulation (2.6), (2.8)-(2.11) only via constraints (2.11). Next, denote the optimal solution to formulation (2.6), (2.8)-(2.11) when customers are risk-neutral by $(\overline{p}, \overline{r})$. Then $\left(\frac{v_i - \overline{p}_i}{v_i - \overline{p}_{i+1}}\right)^{\gamma_i} \geq \left(\frac{v_i - \overline{p}_i}{v_i - \overline{p}_{i+1}}\right) \geq \frac{\overline{r}_{i+1}}{\overline{r}_i}$, and hence $(\overline{p}, \overline{r})$ is feasible for formulation (2.6), (2.8)-(2.11) with γ as risk-aversion parameter vector. Hence the revenue with $(\overline{p}, \overline{r})$ serves as a lower bound for the optimal revenue to the risk-averse problem. (Note that $(\overline{p}, \overline{r})$ might not be feasible for formulation (2.6), (2.8)-(2.12) with γ as risk-aversion parameter vector.) Similarly, one can also show that the optimal revenue with risk-aversion parameter γ serves as a lower bound for revenue with risk-aversion parameter γ', if $\gamma' \leq \gamma$. $\square$

Proof of proposition 2 Define $x_i := p_i r_i,\ i = 1, ..., N$, and $y_i := r_i - r_{i+1},\ i = 1, ..., N-1,\ y_N = r_N$. Then the IR condition (equation (2.8)), the IC condition (equation (2.11)), and the capacity constraint (equation (2.9)) can be written as $v_i \Sigma_{l=i}^N y_l \geq x_i$, $x_i - x_{i+1} \leq v_i y_i$ and $\Sigma_{i=1}^N (\Sigma_{l=i}^N y_l)\pi_i \leq C$, respectively, while the objective (equation (2.6)) becomes $\Sigma_{i=1}^N \pi_i x_i$, which are all linear in the variables x_i, y_i, thereby leading to an LP. $\square$

Proof of proposition 3 Suppose customers are risk-neutral and the firm decides to offer $k > 1$ distinct products such that they partition customer types as in lemma 4. Using proposition 1, we can write the Lagrangian as

$$\begin{aligned} L = &\Sigma_{j=1}^k (\Sigma_{l=i_{j-1}+1}^{i_j} \pi_l) p_j r_j + \Sigma_{j=1}^{k-1} \mu_j \left((v_{i_j} - p_j) r_j - (v_{i_j} - p_{j+1}) r_{j+1}\right) \\ &+ \lambda (C - \Sigma_{j=1}^k \Sigma_{l=i_{j-1}+1}^{i_j} \pi_l r_j). \end{aligned} \tag{2.38}$$

Using lemma 4 and differentiating with respect to p_1 yields $\Sigma_{l=0}^{i_1} \pi_l - \mu_1 = 0$. Differentiating with respect to p_j gives $\Sigma_{l=i_{j-1}+1}^{i_j} \pi_l - \mu_j + \mu_{j-1} = 0,\ j = 2, ..., k-1$. Together these imply that $\mu_j = \Sigma_{l=1}^{i_j} \pi_l$. Differentiating with respect to r_j and using the above we get $\mu_j v_{i_j} - \mu_{j-1} v_{i_{j-1}} -$

$\lambda\Sigma_{l=i_{j-1}+1}^{i_j}\pi_l = 0, j = 2, ..., k-1$. Using $\mu_j = \Sigma_{l=1}^{i_j}\pi_l$ gives $(\Sigma_{l=i_{j-1}+1}^{i_j}\pi_l)v_{i_j} - (\Sigma_{l=i_{j-2}+1}^{i_{j-1}}\pi_l)(v_{i_{j-1}} - v_{i_j}) - \lambda\Sigma_{l=i_{j-1}+1}^{i_j}\pi_l = 0$. There are two cases to consider.

a) $\Sigma_{l=1}^{i_1}\pi_l v_{i_1} = \Sigma_{l=1}^{i_2}\pi_l v_{i_2} = ... = \Sigma_{l=1}^{i_k}\pi_l v_{i_k}$, when capacity is unconstrained and $k > 1$,

b) $\lambda = \frac{\Sigma_{l=1}^{i_2}\pi_l v_{i_2} - \Sigma_{l=1}^{i_1}\pi_l v_{i_1}}{\Sigma_{l=i_1+1}^{i_2}\pi_l} = \frac{\Sigma_{l=1}^{i_3}\pi_l v_{i_3} - \Sigma_{l=1}^{i_2}\pi_l v_{i_2}}{\Sigma_{l=i_2+1}^{i_3}\pi_l} = ... = \frac{\Sigma_{l=1}^{i_k}\pi_l v_{i_k} - \Sigma_{l=1}^{i_{k-1}}\pi_l v_{i_{k-1}}}{\Sigma_{l=i_{k-1}+1}^{i_k}\pi_l} \geq 0$, if capacity is scarce and $k \geq 2$.

The remainder of this proof verifies (details are omitted) that the revenue in a) is given by $\Sigma_{l=1}^{i_1}\pi_l v_{i_1}$, and is achieved by offering a single product at price $p_1 = v_{i_1}$, $r_1 = 1$, and that the revenue in case b) is given by $\Sigma_{l=1}^{i_1}\pi_l v_{i_1} + \lambda(C - \Sigma_{l=1}^{i_1}\pi_l)$, and is achieved by offering two distinct products at the following prices and fill-rates. Define $u := \max\{j \mid \Sigma_{l=1}^{i_j}\pi_l < C\}$, then $p_1 = v_{i_u} - (v_{i_u} - v_{i_{u+1}})\frac{C-\Sigma_{l=1}^{i_u}\pi_l}{\Sigma_{l=i_u+1}^{i_{u+1}}\pi_l}$, $r_1 = 1$, $p_2 = v_{i_{u+1}}$ and $r_2 = \frac{C-\Sigma_{l=1}^{i_u}\pi_l}{\Sigma_{l=i_u+1}^{i_{u+1}}\pi_l}$. $\square$

Proof of corollary 4 Suppose $r_1 < 1$. There are two cases to consider.

a) $k > 1$: Consider setting $r_1' = \min\left(1, \frac{\Sigma_{l=1}^{i_1}\pi_l}{C}\right)$, $r_j' = \min\left(r_j, \frac{C-\Sigma_{u=1}^{j-1}(\Sigma_{l=i_{u-1}+1}^{i_u}\pi_l)r_i'}{\Sigma_{l=i_{j-1}+1}^{i_j}\pi_l}\right)$, $j > 1$, and increasing p_1' s.t. the downstream IC constraint $(v_{i_1} - p_1')r_1' \geq (v_{i_1} - p_2')r_2'$ for type i_1 customers is tight. Then, no type chooses to buy a different product, but the revenues strictly increase. This leads to a contradiction. If $r_1' < 1$, then it implies that only one product is being offered and hence this reduces to case b).

b) $k = 1$: Suppose $r_1 < 1$. This implies that $\Sigma_{l=1}^{i_1}\pi_l > C$ and $\pi_1 v_1 < C v_{i_1} < (\Sigma_{l=1}^{i_1}\pi_l)v_{i_1}$. Consider the following two-product offering: $p_1' = v_1 - (v_1 - p_2)r_2$, $r_1' = 1$, $p_2' = v_{i_1}$, $r_2' = \frac{C-\pi_1}{\Sigma_{l=i_1+1}^{i_1}\pi_l}$. Then, the new revenue equals $\pi_1 v_1 + \frac{(\Sigma_{l=1}^{i_1}\pi_l)v_{i_1} - \pi_1 v_1}{(\Sigma_{l=2}^{i_1}\pi_l)}(C - \pi_1) > C v_{i_1}$, again implying that the original one product revenue was suboptimal, thereby leading to a contradiction. $\square$

Proof of proposition 4: We will first prove the following:

i) There exists $M \in \mathbb{N}$ sufficiently large s. t. $(\overline{p}, \overline{r}) \in \mathcal{S}^n, \forall n \geq M$

ii) There exist $p^1 \in \overline{\mathcal{S}}$, $M \in \mathbb{N}$ s. t. $|p_i^n - p_i^1| < c(1 - \gamma_1^n), \forall n \geq M$, where c is a constant

i) Suppose the optimal risk-neutral solution involves offering $k \leq N$ distinct products, where the products partition the customer types as in lemma 4. Then, following proposition 1, $(\overline{p}, \overline{r})$

satisfies the following constraints.

$$(v_{i_j} - \overline{p}_j)\overline{r}_j = (v_{i_j} - \overline{p}_{j+1})\overline{r}_{j+1}, \quad j = 1, ..., k-2,$$
$$(v_{i_j+1} - \overline{p}_{j+1})\overline{r}_{j+1} > (v_{i_j+1} - \overline{p}_j)\overline{r}_j, \quad j = 2, ..., k-1.$$

The feasible sets $\mathcal{S}^n$ and $\overline{\mathcal{S}}$ of problems $\mathcal{P}^n$ and $\overline{\mathcal{P}}$, respectively, only differ in their IC constraints. Hence, in order to establish claim i), it suffices to show that $(\overline{p}, \overline{r})$ satisfies the IC constraints of $\mathcal{P}^n$, for n sufficiently large. Note that

$$1 > (\frac{v_{i_j} - \overline{p}_j}{v_{i_j} - \overline{p}_{j+1}})^{\gamma^n_{i_j}} > \frac{v_{i_j} - \overline{p}_j}{v_{i_j} - \overline{p}_{j+1}} = \frac{\overline{r}_{j+1}}{\overline{r}_j},$$

implying that the downstream IC condition for problem $\mathcal{P}^n$ is satisfied by $(\overline{p}, \overline{r})$. If $(v_{i_j+1} - \overline{p}_j) \leq 0$, the upstream IC condition is satisfied as well, otherwise, consider the following. Define $\epsilon_j, \quad j = 2, ..., k-1$ such that

$$\epsilon_j \overline{r}_{j+1}(v_{i_j+1} - \overline{p}_j) = (v_{i_j+1} - \overline{p}_{j+1})\overline{r}_{j+1} - (v_{i_j+1} - \overline{p}_j)\overline{r}_j > 0.$$

We want to show that for sufficiently large n,

$$(v_{i_j+1} - \overline{p}_{j+1})^{\gamma^n_{i_j+1}}\overline{r}_{j+1} \geq (v_{i_j+1} - \overline{p}_j)^{\gamma^n_{i_j+1}}\overline{r}_j,$$
$$\Leftrightarrow \quad \left(\frac{v_{i_j+1} - \overline{p}_{j+1}}{v_{i_j+1} - \overline{p}_j}\right)^{\gamma^n_{i_j+1}} \geq \frac{\overline{r}_{j+1}}{\overline{r}_j} = \left(\frac{v_{i_j+1} - \overline{p}_{j+1}}{v_{i_j+1} - \overline{p}_j}\right) - \epsilon_j.$$

Substituting $\gamma^n_{i_j+1} = 1 - x_{i_j+1}$ and using the Taylor expansion, this condition can be written as

$$\left(\frac{v_{i_j+1} - \overline{p}_{j+1}}{v_{i_j+1} - \overline{p}_j}\right) - c_1 x_{i_j+1} + O((x_{i_j+1})^2) \geq \left(\frac{v_{i_j+1} - \overline{p}_{j+1}}{v_{i_j+1} - \overline{p}_j}\right) - \epsilon_j,$$

where c_1 is a constant. The latter is satisfied if $c_1 x_{i_j+1} \leq \epsilon_j + O((x_{i_j+1})^2)$, from which it follows that for any $\epsilon_j > 0$, there exists a M sufficiently large such that for all $n \geq M$, $(\overline{p}, \overline{r})$ satisfies both upstream and downstream IC constraints, and this completes the proof of claim i).

ii) Suppose the optimal solution (p^n, r^n) involves offering $k \leq N$ distinct products, where the products partition the customer types as in lemma 4. Since (p^n, r^n) is the optimal solution for $\mathcal{P}^n$, it satisfies the following constraints

$$(v_{i_j} - p_j^n)^{\gamma_{i_j}^n} r_j^n = (v_{i_j} - p_{j+1}^n)^{\gamma_{i_j}^n} r_{j+1}^n, \quad j = 1, ..., k-1, \tag{2.39}$$

$$(v_{i_j+1} - p_{j+1}^n)^{\gamma_{i_j+1}^n} r_{j+1}^n > (v_{i_j+1} - p_j^n)^{\gamma_{i_j+1}^n} r_j^n, \quad j = 2, ..., k-1. \tag{2.40}$$

We will construct a new solution $(p', r') \in \overline{\mathcal{S}}$, where $r' = r^n$, and

$$(v_{i_j} - p_j')r_j' = (v_{i_j} - p_{j+1}')r_{j+1}' \ \ [\Longrightarrow \ (v_{i_j+1} - p_{j+1}')r_{j+1}' > (v_{i_j+1} - p_j')r_j'],$$

such that $|p_i^n - p_i| < c(1 - \gamma_i^n)$, for n sufficiently large and some constant c. Consider the $k - 1^{th}$ downstream IC constraint. From (2.39), it follows that

$$(v_{i_{k-1}} - p_{k-1}^n)r_{k-1}^n < (v_{i_{k-1}} - p_k^n)r_k^n.$$

Set $p_k' = p_k^n$, $p_{k-1}' = p_{k-1}^n - \epsilon_{k-1}$, $\epsilon_{k-1} > 0$, s.t.

$$(v_{i_{k-1}} - p_{k-1}')r_{k-1}' = (v_{i_{k-1}} - p_k')r_k'.$$

This requires that

$$\begin{aligned}
\epsilon_{k-1} &= (v_{i_{k-1}} - p_k^n)\frac{r_k'}{r_{k-1}'} - (v_{i_{k-1}} - p_{k-1}^n), \\
&= (v_{i_{k-1}} - p_k^n)\left(\frac{v_{i_{k-1}} - p_{k-1}^n}{v_{i_{k-1}} - p_k^n}\right)^{\gamma_{i_{k-1}}^n} - (v_{i_{k-1}} - p_{k-1}^n), \\
&= c_{k-1}(1 - \gamma_{i_{k-1}}^n) + O((1 - \gamma_{i_{k-1}}^n)^2),
\end{aligned}$$

following a Taylor expansion, where c_{k-1} is a constant. Note that $\epsilon_{k-1} > 0$ since $p_{k-1} > p_k$. Next consider the $k-2^{th}$ downstream IC constraint. Set $p'_{k-2} = p^n_{k-2} - \epsilon_{k-2}, \epsilon_{k-2} > 0$, s.t.

$$(v_{i_{k-1}} - p'_{k-2})r'_{k-2} = (v_{i_{k-1}} - p'_{k-1})r'_{k-1}.$$

This requires that

$$\begin{aligned}
\epsilon_{k-2} &= (v_{i_{k-2}} - p'_{k-1})\frac{r'_{k-1}}{r'_{k-2}} - (v_{i_{k-2}} - p^n_{k-2}), \\
&= (v_{i_{k-2}} - p'_{k-1})\left(\frac{v_{i_{k-2}} - p^n_{k-2}}{v_{i_{k-2}} - p^n_{k-1}}\right)^{\gamma^n_{i_{k-2}}} - (v_{i_{k-2}} - p^n_{k-2}), \\
&= (v_{i_{k-2}} - p^n_{k-1} + \epsilon_{k-1})\left(\frac{v_{i_{k-2}} - p^n_{k-2}}{v_{i_{k-2}} - p^n_{k-1}}\right)^{\gamma^n_{i_{k-2}}} - (v_{i_{k-2}} - p^n_{k-2}), \\
&= c^1_{k-2}(1-\gamma^n_{i_{k-2}}) + c^2_{k-2}(1-\gamma_{i_{k-1}}) + O((1-\gamma^n_{i_{k-1}})^2) + O((1-\gamma^n_{i_{k-2}})^2), \\
&\le c_{k-2}(1-\gamma^n_{i_{k-2}}) + O((1-\gamma^n_{i_{k-1}})^2),
\end{aligned}$$

following a Taylor expansion, where c_{k-2}, c^1_{k-2} and c^2_{k-2} are constants. Note that $\epsilon_{k-2} > 0$. Proceeding in a similar fashion, one can construct $p'_1, ..., p'_{k-3}$ as well, wherein $p'_i - p^n_i \le c_i(1-\gamma^n_{i_1}) + O((1-\gamma_{i_1})^2)$, c_i constant. Finally $p'_1 > p'_2 > ... > p'_k$ is ensured if $\epsilon_i < p^n_i - p^n_{i-1}$, $i = 1, .., k-1$, which is guaranteed for n sufficiently large. Hence $|p^n_i - p'_i| < c(1-\gamma^n_{i_1})$, $i = 1, .., k$, and this completes the proof of claim ii).

We will now prove the statement of the proposition. Denote the feasible set in equations (2.7), (2.8)-(2.11) for the problem with risk-aversion parameter γ_n as $\mathcal{T}^n$, and for the risk-neutral case as $\overline{\mathcal{T}}$. Following proposition 1, $p^n \in \mathcal{T}^n$ and $(\overline{p}, \overline{r}) \in \overline{\mathcal{T}}$. Consider $n > m$ so that $\gamma^m < \gamma^n$. We will show that $\mathcal{T}^n \subset \mathcal{T}^m$. Consider any $p \in \mathcal{T}^n$. Then it satisfies constraint (2.11). However, $\gamma^m < \gamma^n$ implies that $1 \ge \left(\frac{v_i - p_i}{v_i - p_{i+1}}\right)^{\gamma^m_i} \ge \left(\frac{v_i - p_i}{v_i - p_{i+1}}\right)^{\gamma^n_i} \ge \frac{r_{i+1}}{r_i}$ implying that $p \in \mathcal{T}^m$. This implies $\{p^n\}_{n \ge m} \subset \mathcal{T}^m$. $\mathcal{T}^m$ is a compact set, implying that there exists a subsequence $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}, m \le n_1 < n_2 < ...$ s.t. $p^{n_k} \to \widetilde{p}$, $\widetilde{p} \in \mathcal{T}^m$.

Suppose $\widetilde{p} \in \overline{\mathcal{S}}$ and $\widetilde{p} \neq \overline{p}$. Then $\forall \epsilon > 0$, $\exists M$ s.t. $\forall k \geq M$, $|p_i^{n_k} - \widetilde{p}_i| < \epsilon$. This implies that the optimal revenue for problem $\mathcal{P}^{n_k}$, $R^{n_k}(p^{n_k}) \leq \overline{R}(\widetilde{p}) + c\epsilon$, where $R^n(\cdot)$ denotes the revenue with risk-aversion parameter γ^n and $\overline{R}(\cdot)$ denotes the revenue for the risk-neutral case (under feasible price vectors). Now since the optimal solution to the risk-neutral problem P is unique, $\overline{R}(\overline{p}) > \overline{R}(\widetilde{p})$ and for ϵ small enough, $\overline{R}(\overline{p}) > R^{n_k}(p^{n_k})$. However, this would violate the optimality of p^{n_k}, since for n_k large enough, $\overline{p}$ is feasible for $\mathcal{S}^{n_k}$. Then, from assumption 1, it follows that $\widetilde{p} \in \overline{\mathcal{S}} \implies \widetilde{p} = \overline{p}$.

Now suppose $\widetilde{p} \notin \overline{\mathcal{S}}$. Then let $\delta = \min_{p' \in \overline{\mathcal{S}}} ||\widetilde{p} - p'||_2 > 0$. Note the for k large enough, $||p^{n_k} - \widetilde{p}||_2 < \epsilon_1$, and $\exists p' \in \overline{\mathcal{S}}$ s.t. $||p^{n_k} - p'||_2 < \epsilon_2$. Now using the triangle inequality, $||p^{n_k} - \widetilde{p}||_2 \geq ||\widetilde{p} - p'||_2 - ||p^{n_k} - p'||_2 \geq \delta - \epsilon_2$. Hence choosing ϵ_1, ϵ_2 to be such that $\epsilon_1 + \epsilon_2 < \delta$, we will achieve a contradiction. Hence $\widetilde{p} \in \overline{\mathcal{S}}$ and consequently $\widetilde{p} = \overline{p}$. In a similar fashion, it is also possible to show that $\{p^n\}$ has a unique limit point. Moreover, this directly implies that $R^{n_k}(p^{n_k}) \to \overline{R}(\overline{p})$ (Actually we know that $R^{n_k}(p^{n_k}) \geq \overline{R}(\overline{p})$ since $\overline{p} \in \mathcal{S}^{n_k}$, for k large, or alternatively, by using corollary 3 directly). $\square$

Proof of proposition 5 It is easy to verify that the following assignment,

$$
\begin{aligned}
\rho_i &= 0, i = 1, 2, ..., N, \\
\delta_{i_{l-1}+1} &= \delta_{i_{l-1}+2} = ... = \delta_{i_l}, l = 1, 2, ..., k, \\
\delta_j &= \delta_{j+1} = ... = \delta_N = 0, \\
\delta_{i_l} &= \frac{\delta_{i_l+1}\overline{r}_{l+1} + w_{i_l,i_l+1}x_{i_l} - w_{i_l,i_l}x_{i_l} - \epsilon_{i_l}}{\overline{r}_l}, l = 1, 2, ..., k,
\end{aligned}
$$

is feasible for the LP (2.30)-(2.36). Constraints (2.31)-(2.32) and (2.34)-(2.36) are tight, while $\delta_{i_k} = -\frac{\epsilon_{i_k}}{\overline{r}_k}$ implies that constraint (2.33) is also satisfied.

We will next show that (2.30)-(2.36) has a finite optimal solution by establishing that its dual is itself feasible. The dual to (2.30)-(2.36) can be written as follows.

$$
\min \quad \Sigma_{i=1}^{i_k} \mu_i[(w_{i,i+1} - w_{i,i})x_i - \epsilon_i] + \Sigma_{i=j}^{N-1} \mu_i(-\epsilon_i) \tag{2.41}
$$

$$s.t. \ \ \mu_i \geq 0, \ i = 1, 2, ..., N-1, \ \eta_i \geq 0, \ i < i_k, \ \theta_i \geq 0, \ i \leq i_k, \tag{2.42}$$

$$\lambda \geq 0, \ \alpha_1, ..., \alpha_{i_1} \geq 0, \ \beta_{i_k} \geq 0, \ \phi_j, ..., \phi_N \geq 0, \ \nu_j, ...\nu_N \geq 0, \tag{2.43}$$

$$\pi_1 \lambda - u_{1,1}\mu_1 - \eta_1 1_{1 \notin I} + \alpha_1 = \pi_1 p_1, \tag{2.44}$$

$$\pi_i \lambda - u_{i,i}\mu_i + u_{i-1,i}\mu_{i-1} - \eta_i 1_{i \notin I} + \eta_{i-1} 1_{i-1 \notin I} + \alpha_i 1_{i \leq i_1} = \pi_i p_i, \quad i = 2, 3, ..., i_k - 1, \tag{2.45}$$

$$\pi_{i_k} \lambda - u_{i_k,i_k}\mu_{i_k} + u_{i_k-1,i_k}\mu_{i_k-1} + \eta_{i_k-1} 1_{i_k-1 \notin I} + \alpha_{i_k} 1_{i_k \leq i_1} = \pi_{i_k} p_{i_k}, \tag{2.46}$$

$$\pi_j \lambda - u_{j,j}\mu_j + u_{j-1,j}\mu_{j-1} - \phi_j = 0, \tag{2.47}$$

$$\pi_i \lambda - u_{i,i}\mu_j + u_{i-1,i}\mu_{i-1} - \phi_i = 0, \quad i = j+1, ..., N-1, \tag{2.48}$$

$$\pi_N \lambda + u_{N-1,N}\mu_{N-1} - \phi_N = 0, \tag{2.49}$$

$$\mu_1 r_1 - \theta_1 1_{1 \notin I} = \pi_1 r_1, \tag{2.50}$$

$$-\mu_{i-1} r_i + \mu_i r_i + \theta_{i-1} 1_{i-1 \notin I} - \theta_i 1_{i \notin I} = \pi_i r_i, \quad i = 2, ..., i_k - 1, \tag{2.51}$$

$$-\mu_{i_k-1} r_{i_k} + \mu_{i_k} r_{i_k} + \theta_{i_k-1} 1_{i_k-1 \notin I} - \theta_{i_k} 1_{i_k \notin I} + \beta = \pi_{i_k} r_{i_k}, \tag{2.52}$$

$$\theta_{j-1} 1_{j-1 \notin I, j-1 \leq i_k} + \nu_j = 0, \tag{2.53}$$

$$\nu_i = 0, \quad i = j+1, ..., N. \tag{2.54}$$

Here λ is the dual variable associated with the capacity constraint (2.31), μ_i is the dual variable associated with constraint (2.36), $i = 1, ..., N-1$, η_i is the dual variable associated with constraint $\rho_i \geq \rho_{i+1}$, $i = 1, ..., N-1$, θ_i is the dual variable associated with constraint $\delta_i \geq \delta_{i+1}$, $i = 1, ..., i_k$. α_i is the dual variable associated with the constraint $\rho_i \leq 0$, $i = 1, ..., i_1$, β is the dual variable associated with the constraint $\delta_{i_k} \leq 0$, ϕ_i is the dual variable associated with the constraint $\rho_i \geq 0$, $i = j, j+1, ..., N$, and ν_i is the dual variable associated with the constraint $\delta_i \leq 0$, $i = j, j+1, ..., N$.

Following proposition 3, without loss of generality, we restrict attention to the case where the optimal number of products to offer to risk-neutral customers $\overline{k} \leq 2$. By brute-force once can verify that the following assignment of variables is feasible for (2.41)-(2.54), and therefore that

(2.30)-(2.36) has a finite feasible solution (details are omitted):

$$\mu_i = \Sigma_{l=1}^{i} \pi_l, \quad i = 1, ..., N, \tag{2.55}$$

$$\theta_i = 0, \quad i = 1, ..., N-1, \tag{2.56}$$

$$\nu_i = 0, \quad i = j, j+1, ..., N, \ \beta = 0, \tag{2.57}$$

$$\eta_1 = 0, \quad \alpha_1 = \pi_1(v_1 - \lambda), \tag{2.58}$$

$$\alpha_i - \eta_i = -\lambda\pi_i + \mu_i v_i - \mu_{i-1} v_{i-1} - \eta_{i-1}, \quad \alpha_i \geq 0, \quad \eta_i \geq 0, \quad \alpha_i \eta_i = 0, \quad 1 < i < i_1, \tag{2.59}$$

$$\eta_{i_1} = 0, \quad \alpha_{i_1} = -\lambda\pi_{i_1} + \mu_{i_1} v_{i_1} - \mu_{i_1-1} v_{i_1-1} - \eta_{i_1-1}, \tag{2.60}$$

$$\eta_i = \lambda(\Sigma_{l=i_1+1}^{i} \pi_l) + \mu_{i_i} v_{i_1} - \mu_i v_i, \quad i_1 < i < i_2, \tag{2.61}$$

$$\phi_i = \lambda\pi_i + \mu_{i-1}(v_{i-1} - v_i), \quad i = j, j+1, ..., N, \tag{2.62}$$

$$\lambda = \begin{cases} 0, & \text{if } \overline{k} = 1, \\ \frac{(\Sigma_{l=1}^{i_2} \pi_l) v_{i_2} - (\Sigma_{l=1}^{i_1} \pi_l) v_{i_1}}{\Sigma_{l=i_1+1}^{i_2} \pi_l}, & \text{if } \overline{k} = 2. \end{cases} \tag{2.63}$$

We next compute the optimal solution to (2.30)-(2.36). We consider the one-product and the two-product case separately.

i) $\overline{k} = 1$: In this case, $w_{i,i+1} - w_{i,i} = 0, \ i = 1, ..., N-1$. Also, $\epsilon_i = 0, \ i = 1, ..., N-1$ would ensure that the IC conditions are not violated. Together these imply that zero is feasible revenue for the dual problem (2.41)-(2.54). However, this is also attained by setting $\delta_i = \rho_i = 0, \ i = 1, ..., N$ in the primal problem. Hence by strong duality, this must be the optimal solution.

ii) $\overline{k} = 2$: In this case, $w_{i,i+1} - w_{i,i} = 0, \ i = 1, ..., i_1 - 1, i_1 + 1, ..., N-1$. Also, we can set $\epsilon_i = 0, \ i = 1, ..., i_1 - 1, i_1 + 1, ..., N-1$. This implies that a feasible dual revenue is given by $(\Sigma_{l=1}^{i_1} \pi_l)[(w_{i_1,i_1+1} - w_{i_1,i_1})x_{i_1} - \epsilon_{i_1}]$. However, this is also attained by setting $\delta_1 = \delta_2 = ... = \delta_{i_1} = (w_{i_1,i_1+1} - w_{i_1,i_1})x_{i_1} - \epsilon_{i_1}, \ \delta_i = 0, \ i > i_1, \ \rho_i = 0, \ i = 1, ..., N$, in the primal solution. Hence, by strong duality, this must be the optimal solution. $\square$

Appendix B

Existence of optimal solution and constraint qualification: For completeness, we justify the use of the Lagrangian approach. We observe that the objective (2.6) is continuous, and the feasible sets (2.7), (2.8)-(2.12) and (2.7), (2.8)-(2.11) are compact. Hence, following Weierstrass theorem, a maxima exists. Suppose the optimal solution to formulation (2.6), (2.8)-(2.12) involves offering $k \leq N$ distinct products, where the products partition the customer types as in lemma 4. Define $a_l = \gamma_{i_l}(v_{i_l} - p_l)^{\gamma_{i_l}-1} r_l$, $b_l = \gamma_{i_l}(v_{i_l} - p_{l+1})^{\gamma_{i_l}-1} r_{l+1}$, $c_l = (v_{i_l} - p_l)^{\gamma_{i_l}}$, $d_l = (v_{i_l} - p_{l+1})^{\gamma_{i_l}}$, $l = 1, ..., k-1$. Also define $e_l = \Sigma_{l=i_{l-1}+1}^{i_l} \pi_l$, $l = 1, ..., k$. Then the matrix obtained by differentiating the $k-1$ downstream IC conditions and the capacity constraint is given as follows.

$$\begin{pmatrix} -a_1 & b_1 & 0 & 0 & \dots & 0 & -d_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -a_2 & b_2 & 0 & \dots & 0 & c_2 & -d_2 & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & & & \vdots & & & \ddots & & & \vdots \\ & & & & & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & -a_{k-1} & 0 & 0 & 0 & \dots & c_{k-1} & -d_{k-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & -e_2 & -e_3 & & \dots & & -e_k \end{pmatrix}$$

The first $k-1$ rows in this matrix correspond to the $k-1$ downstream constraints, and the k^{th} corresponds to the capacity constraint. The first $k-1$ columns correspond to the variables $p_1, ..., p_{k-1}$, while the next $k-1$ columns correspond to variables $r_2, ..., r_k$. Note that this matrix has rank k (since there are k linearly independent rows (the matrix is in row-echelon form, and can easily be converted into reduced row-echelon form with k non-zero rows)), and hence the constraint qualification condition is met for the Lagrangian in equation (2.38). In case we also add the upstream constraints to the Lagrangian, as in equation (2.37), we observe that either the upstream or the downstream constraint can be tight, but not both, and since the derivative of any upstream constraint would lead to the same non-zero entries as the derivative of the corresponding downstream constraint, the constraint qualification condition would be met.

Proof of algorithm 1: For risk-neutral customers, the required conditions are obtained from the proof of proposition 3. For risk-averse customers, the Lagrangian can be written as follows:

$$
\begin{aligned}
L = &(\Sigma_{l=1}^{i_1}\pi_l)p_1 + (\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}r_2 + \mu\left((v_{i_1}-p_1)^{\gamma_{i_1}} - (v_{i_1}-v_{i_2})^{\gamma_{i_1}}r_2\right)\\
&+ \lambda\left(C - \Sigma_{l=1}^{i_1}\pi_l - (\Sigma_{l=i_1+1}^{i_2}\pi_l)r_2\right).
\end{aligned}
$$

Differentiating with respect to p_1 and r_2 respectively, yields the following:

$$
\begin{aligned}
&(\Sigma_{l=1}^{i_1}\pi_l) - \mu\gamma_{i_1}(v_{i_1}-p_1)^{\gamma_{i_1}-1} = 0,\\
&(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2} - \lambda(\Sigma_{l=i_1+1}^{i_2}\pi_l) - \mu(v_{i_1}-v_{i_2})^{\gamma_{i_1}} = 0.
\end{aligned}
$$

There are two cases to consider: $\lambda = 0$ and $\lambda > 0$. $\lambda = 0$ implies that

$$
p_1 = v_{i_1} - \left(\frac{(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}}{(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2})^{\gamma_{i_1}}}\right)^{\frac{1}{1-\gamma_{i_1}}} \quad r_2 = \left(\frac{(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}}{(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2})}\right)^{\frac{\gamma_{i_1}}{1-\gamma_{i_1}}}.
$$

The conditions $p_1 > v_{i_2}$ and $C \geq (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)r_2$ require that

$$
\begin{aligned}
&(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2}) > \gamma_{i_1}(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2},\\
&C \geq (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)\left(\frac{(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}}{(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2})}\right)^{\frac{\gamma_{i_1}}{1-\gamma_{i_1}}}.
\end{aligned}
$$

under which the optimal revenue is given by

$$
R = (\Sigma_{l=1}^{i_1}\pi_l)v_{i_1} - (\Sigma_{l=1}^{i_1}\pi_l)\left(\frac{(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}}{(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2})^{\gamma_{i_1}}}\right)^{\frac{1}{1-\gamma_{i_1}}}
$$

$$+ v_{i_2}(\Sigma_{l=i_1+1}^{i_2}\pi_l)\left(\frac{(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}}{\pi_1(v_{i_1}-v_{i_2})}\right)^{\frac{\gamma_{i_1}}{1-\gamma_{i_1}}}.$$

and which exceeds the one product revenue at price v_{i_1} and v_{i_2} under the sufficient condition $(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2}) > (\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}$. $\lambda > 0$ implies that

$$p_1 = v_{i_1} - (v_{i_1}-v_{i_2})\left(\frac{C-(\Sigma_{l=1}^{i_1}\pi_l)}{(\Sigma_{l=i_1+1}^{i_2}\pi_l)}\right)^{\frac{1}{\gamma_{i_1}}}, \quad r_2 = \frac{C-(\Sigma_{l=1}^{i_1}\pi_l)}{(\Sigma_{l=i_1+1}^{i_2}\pi_l)}.$$

The conditions $p_1 > v_{i_2}$ and $C \leq (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)r_2$ require that

$$C < (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l), \quad C < (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)\left(\frac{(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}}{(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2})}\right)^{\frac{\gamma_{i_1}}{1-\gamma_{i_1}}}.$$

under which the optimal revenue is given by

$$(\Sigma_{l=1}^{i_1}\pi_l)v_{i_1} - (\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2})\left(\frac{C-(\Sigma_{l=1}^{i_1}\pi_l)}{(\Sigma_{l=i_1+1}^{i_2}\pi_l)}\right)^{\frac{1}{\gamma_{i_1}}} + (\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\left(\frac{C-(\Sigma_{l=1}^{i_1}\pi_l)}{(\Sigma_{l=i_1+1}^{i_2}\pi_l)}\right),$$

and which exceeds the one product revenue at price v_{i_1} and v_{i_2}.

That the constraint qualification condition is met follows from 2.6. To see that the proposed solution is indeed a maxima, write the Lagrangian as

$$\begin{aligned} L =& (\Sigma_{l=1}^{i_1}\pi_l)p_1 + (\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\left(\frac{v_{i_1}-p_1}{v_{i_1}-v_{i_2}}\right)^{\gamma_{i_1}} \\ &+ \lambda\left(C - \Sigma_{l=1}^{i_1}\pi_l - (\Sigma_{l=i_1+1}^{i_2}\pi_l)\left(\frac{v_{i_1}-p_1}{v_{i_1}-v_{i_2}}\right)^{\gamma_{i_1}}\right). \end{aligned}$$

Differentiating with respect to p_1, we obtain that

$$(\Sigma_{l=1}^{i_1}\pi_l) - \frac{(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}(v_{i_1}-p_1)^{\gamma_{i_1}-1}}{(v_1-v_{i_2})^{\gamma_{i_1}-1}} + \frac{\lambda(\Sigma_{l=1}^{i_1}\pi_l)\gamma_{i_1}(v_{i_1}-p_1)^{\gamma_{i_1}-1}}{(v_1-v_{i_2})^{\gamma_{i_1}-1}} = 0.$$

If $\lambda = 0$, then it is easy to verify that p_1 is the same as obtained earlier and the second derivative with respect to p_1 is negative. If $\lambda > 0$, the tightness of the IC condition implies that we obtain the same solution as before. Solving for λ and substituting to calculate the second derivative with respect to p_1, we find it to be negative, implying that the method does yield revenue-maximizing solution.

Chapter 3

Product design with threshold preferences

We study the product design problem of a revenue-maximizing monopolist firm that serves a market where customers are heterogeneous with respect to their valuations and desire for a quality attribute, and are characterized by a perhaps novel model of customer choice behavior. Specifically, instead of optimizing the net utility that results from an appropriate combination of prices and quality levels, customers are "satisficers" in that they seek to buy the cheapest product with quality above a certain customer-specific threshold. This model dates back to Simon's work in the 1950's ([59, 60]) and can be thought of as a model of bounded rationality for customer choice. We characterize the structural properties of the optimal product menu for this model, consider examples where such preferences might arise, and identify the optimal product menu in each case.

3.1 Introduction

Consider a monopolist firm that sells a good or service to a market of heterogeneous customers. The good or service is characterized by a quality attribute, and customers are sensitive to both product price and quality. The firm's goal is to maximize its profits by offering the optimal product menu, characterized by the number of products offered, and the price and quality of each product. This classic problem has been extensively studied in the literature. In most cases, this body of work has assumed that customer preferences vary continuously over the quality attribute. In this chapter, we consider the possibility that customer preferences over the quality attribute are discrete. In particular, we assume that each customer, in addition to its valuation, is endowed with a quality threshold, and seeks to buy the cheapest product such that the quality of this product equals or exceeds the customer-specific quality threshold. If there is no such product or if the desired product's price exceeds valuation, the customer does not buy from the firm.

To be more concrete, consider the following setting. Suppose customer valuations v and quality threshold θ are distributed according to the distribution $F(v, \theta)$ in a market with size Λ. Suppose the firm has a capacity C, and offers a menu of M products to this market. The product menu is denoted by $(p_j, q_j),\ j = 1, ..., M$, where p_j and q_j denote, respectively, the price and quality of the j^{th} product. For convenience, we define $p_0 := 0$. Then, under the proposed model of customer choice behavior, a customer with valuation v and quality threshold θ chooses to buy product $\chi_{v,\theta}(p, q)$ defined as follows:

$$\chi_{v,\theta}(p, q) = \begin{cases} \operatorname{argmin} p_j, & \exists\ q_j \geq \theta,\ p_j \leq v, \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

where $\chi_{v,\theta}(p, q) = 0$ implies that the customer does not buy from the firm. We call such preferences with respect to the quality attribute as *threshold preferences*. Under threshold preferences, the product design of the firm can be written as follows:

$$\max_{p,\ q,\ M} \int_{v,\ \theta} p_{\chi_{v,\theta}(p,q)}\ \Lambda\ dF(v,\theta) \tag{3.2}$$

$$s.t. \int_{v,\ \theta} 1_{\{\chi_{v,\theta}(p,q)\neq 0\}}\ \Lambda\ dF(v,\theta) \leq C, \tag{3.3}$$

$$0 \leq p \leq \infty,\ q \text{ feasible}, \tag{3.4}$$

$$1 \leq M < \infty,\ M \text{ integer}. \tag{3.5}$$

The objective in equation (3.2) is the sum of revenues over customer valuations, where the revenue from valuation v, threshold θ customers is given by the product of price $p_{\chi_{v,\theta_v}(p,q)}$ charged to valuation v, threshold θ customers, and their size $\Lambda\ dF(v,\theta)$. Equation (3.3) enforces that firm sales do not exceed capacity, and equation (3.4) enforces that set of offered prices is finite and non-negative, and the set of offered qualities feasible. Equation (3.5) enforces that at least one product is offered, and that the number of offered products is finite and integral. The firm's product design problem is to identify M, the number of products to offer, and the corresponding prices and qualities, that maximize its revenues.

The objective of this chapter is to study the firm's product design problem when customers have threshold preferences. In particular, we seek to determine the structure of the optimal product menu, and understand how it differs from the optimal product menu that we obtain under the traditional model of customer behavior. We seek to compute the optimal product prices and determine the optimal product qualities to offer, as well as understand how the prices should be set when the firm is constrained to offer a small number of products.

The proposed model of customer choice behavior differs from the classic rational model of customer choice behavior in the following respect. Under the classic model of customer choice behavior, customer utility is continuous and increasing in the quality attribute. For example, it is typical to assume that the utility $u(\cdot)$ to a customer with valuation v, upon purchasing a product with price p and quality q, is of the form $u(v,p,q) = (v-p)g(q)$, where $v-p$ denotes the consumer surplus if the customer buys this product, and $g(q)$ is a non-negative, continuous, increasing function of quality. This is a natural extension of the classic single quality case where customer utility (surplus) is given by $v-p$. Hence every customer prefers to buy a product notwithstanding its quality, given that the customer's valuation is at least as large as the product price. Under the threshold model of customer preferences, we assume that the function $g(q)$ is of the form $1_{\{q\geq\theta\}}$, where θ is the customer-specific quality threshold. In such a setting, all products with quality at θ or higher are acceptable to the customer and all products below quality θ are unacceptable. Among the acceptable products, if any, the customer buys the cheapest product, provided that its price does not exceed customer valuation.

Though previously not examined in revenue management literature to the best of our knowledge, this model of customer preferences is well-known in marketing and psychology literature. In psychology parlance, researchers refer to individuals that exhibit the above discussed threshold behavior as "satisficers" (as different from "maximizers" whose utility is a continuous function of product quality and who seek to maximize their utility over both product price and quality, Iyengar [30], Schwartz [54]). The threshold model of customer choice behavior can be motivated as an example of the "simple pay-off" function as discussed in Simon [59]. Alternatively, this functional form can be motivated as the limiting case of the S-shaped utility functions, discussed for example in Kahneman and Tversky [32] and Maggi [41]. For example, a utility function that would approximate $w(q)$ is the exponential S-shaped utility function

$$\tilde{w}(q) = \begin{cases} \frac{1}{\beta} + \frac{\beta-1}{\beta}(1-e^{-\alpha(q-\overline{q})}), & \text{if } q \geq \overline{q}, \\ \frac{1}{\beta}e^{-\alpha(\overline{q}-q)}, & \text{if } q < \overline{q}, \end{cases} \tag{3.6}$$

where $\alpha > 0$, $\beta \geq 1$, with the approximation becoming exact when $\beta = 1$, and α grows large.

Having threshold preferences implies that a customer with quality threshold θ chooses the cheapest product with quality $q \geq \theta$. Unlike the maximizing model of customer preferences, under the satisficing model, a customer does not consider buying any product with $q < \theta$, even though her/his willingness to pay might exceed the price. Similarly, given multiple products with $q \geq \theta$, the customer does not optimize over both price and quality, but simply chooses the cheapest product that satisfies her/his quality constraint.

The model discussed in this chapter can also be thought of as a special case of horizontal differentiation. To be more precise, suppose customers differ in their preferences over a single-dimensional quality attribute θ, with one quality extreme denoted by $\overline{\theta}$ and the other extreme by $\underline{\theta}$, with $\underline{\theta} < \overline{\theta}$. Then under the traditional model of horizontal differentiation, the cost $c(\cdot)$ to a customer with preference θ, upon purchasing a product with quality q and price p, is given by

$$c(p, q) = \begin{cases} p + t_1(\theta - q), & q < \theta, \\ p + t_2(q - \theta), & q \geq \theta. \end{cases} \tag{3.7}$$

This reduces to the threshold model of customer preferences under the assumption that $t_1 = \infty$ and $t_2 = 0$.

The remainder of the chapter is organized as follows: this section concludes with a brief literature review. §3.2 introduces the model, and §3.3 discusses examples where modeling customer behavior via threshold preferences seems to be appealing. §3.4 characterizes the structure of the optimal solution to the product design problem, and §3.5 extends this to the examples discussed in §3.3. §3.6 discusses three extensions to the original model. A qualitative discussion of the results ensues in §3.7. Finally, we conclude with some brief remarks in §3.8.

Literature Survey: Our work builds upon several different areas of revenue management. The primary motivation for our work arises from the overlapping area between marketing, psychology and prospect theory focusing on customer behavior models. In his classic papers [59, 60], Simon questioned the pervasive assumption of agent rationality made in economic models. Citing constraints on information availability and computational capacities of individuals, in [59] Simon proposed "simple payoff functions" such as the one considered in our chapter as an approximation to model complex agent utility. In [60], Simon introduced the idea of "satisficing" to model the behavior of an organism facing multiple goals. In more recent research in psychology, researchers distinguish between "maximizers" and "satisficers", as discussed in Iyengar [30] and Schwartz et. al [54]. Wieczorkowska and Burnstein [70] refer to individuals exhibiting satisficing behavior as adopting an "interval" strategy as opposed to a "point" strategy (maximizing). Schwartz et. al [54] mentions that indeed individuals might not be maximizers or satisficers along all dimensions. In our case, customers satisfice with respect to quality while they maximize with respect to price.

In their famous paper [32], Kahneman and Tversky propose that individual utility is concave for gains, while being convex for losses. Some such utility functions are discussed in Maggi [41]. Our utility function for quality attribute can be thought of as the limiting case for the S-shaped exponential utility function discussed here. The deadline delay cost structure discussed in Dewan and Mendelson [18] prescribes zero cost for delay below a certain delay threshold and linear delay costs thereafter. Delay here corresponds to our notion of quality, with lower delay implying higher quality. Our delay cost function, like Dewan and Mendelson [18], posits a zero cost for delay below a customer delay threshold and infinite (or large enough to deter customer from buying this product) costs thereafter.

We study the classic product design problem for a monopolist firm under the threshold preferences based model of customer behavior. This is the second stream of research that we build upon. The problem of second-degree price discrimination by a monopolist facing customers that differ in their preference for a quality attribute is presented in two classic papers Mussa [46] and Moorthy [44]. In Mussa [46], customer utility is linear in quality, and quality is continuous. In Moorthy [44], customer utility is allowed to be non-linear, but quality is discrete. In both cases, customers are maximizers. Both [44, 46] discuss strategic degradation of quality by the monopolist to maximize revenues. This idea of intentionally degrading product quality when offering a product to less quality sensitive customers so as to achieve differentiation is well-known and also discussed in Afeche [1] and Varian [56] among other places.

Extensive literature exists on economic analysis of queues. We consider revenue maximization in a single server queue as a setting for studying revenue management problem revenue management problem for a firm that faces a market of customers with threshold preferences. Papers that adopt a rational model of customer behavior and discuss revenue maximization in queues include Katta and Sethuraman [33], Afeche [1] and Maglaras and Zeevi [42, 43].

Revenue management problems involving strategic customer behavior constitute an active area of current research. This growing body of work motivates another setting where we study the product design problem for a monopolist firm facing strategic customers with threshold preferences. The idea of strategic rationing to induce early customer purchases has been discussed in Cachon and Swinney [13], Liu and van Ryzin [39], Su [63] and Bansal and Maglaras [7]. An extensive survey of this work is available in Shen and Su [58]. All of these assume a rational model of customer behavior. We discuss rationing under the proposed threshold preferences based model of customer behavior.

Versioning of information goods under the classic model of customer choice behavior is studied in Bhargava and Chaudhary [10] and Ghose and Sundararajan [22], and motivates another setting where we model customer choice behavior using threshold preferences. Varian [56] presents several examples of versioning of information goods.

Several researchers have addressed the problem of identifying the optimal inventory policy in presence of multiple demand classes, that differ in their tolerance for the minimum fill-rate or the maximum leadtime they are willing to accept. Such a specification of acceptable quality levels closely mirrors our model of threshold based preferences, and is considered, for example, in Klejin and Dekker [35].

Kim and Chajjed [34] study the product design problem of a monopolist firm offering a product with multiple quality attributes to a market of customers under the classic model of customer choice. The market consists of two customer segments, so at most two products need to be offered.

3.2 Model

A monopolist firm sells a good or service in a market of heterogeneous customers. The good or service is characterized by a one-dimensional quality attribute, and to maximize revenues, the firm seeks to discriminate customers by creating multiple qualities and offering them at different prices. We assume that differentiation does not entail any cost. (We consider the case with increasing marginal cost with respect to quality in section 3.5.6.) The firm offers M products, with p_j and q_j denoting respectively, the price and quality of product j. The capacity available to the firm is denoted by C, and we assume linear costs to capacity. Hence the objectives of profit and revenue maximization are the same. We assume that the firm does not offer any products to the market that generate zero demand.

Customers are heterogeneous, and are characterized by their valuation, their service requirement, and their quality threshold. Service requirements are assumed to be homogeneous across customers (heterogeneous service requirements are considered in section 3.6.1). Each customer requires c units of the product. Without loss of generality, we assume $c = 1$. Customer quality thresholds are assumed to lie in the discrete set $\mathbb{L} = \{\theta_1, \theta_2, ..., \theta_N\}$, with $\overline{\theta} > \theta_1 > \theta_2 > ... > \theta_N > \underline{\theta} > 0$, with a higher value implying the desire for a better quality. We refer to customers having θ_i as their quality threshold as being class i customers. Valuations of class i customers are assumed to have a strictly positive density $f_i(.)$ over the interval $[0, \overline{v}_i]$. The corresponding cumulative distribution function (cdf) is denoted by $F_i(.)$ and the corresponding complementary cdf (ccdf) by $\overline{F}_i(.)$. Both are assumed to be continuous. We assume that $\infty \geq \overline{v}_1 \geq \overline{v}_2 \geq ... \geq \overline{v}_N > 0$, i.e., customers having higher quality thresholds have higher maximum valuations. We denote the aggregate population of class i customers by Λ_i. We assume that $\lim_{p \to \infty} p\overline{F_i}(p) = 0,\ i = 1, ..., N$, i.e., the revenue from any customer class goes to 0 as price goes to infinity (this holds trivially for class i if $\overline{v}_i < \infty$).

Customers have threshold preferences, i.e., a customer buys the cheapest product whose quality exceeds the quality threshold of this customer. So, customer class i chooses product i given by

$$\chi_i(p,q) = \begin{cases} \operatorname{argmin} p_j, & \exists\ q_j \geq \theta_i, \\ 0, & \text{otherwise}, \end{cases} \tag{3.8}$$

where p_j and q_j denote respectively the price and the quality of the j^{th} product offered. Note that all customers belonging to the same class buy the same product, if any, and subject to their valuation exceeding the price. If $\chi_i(p,q) = l, \quad l \geq 1$, the demand from class i customers for this product is given by $\Lambda_i \overline{F}_i(p_l)$, and the revenue by $\Lambda_i p_l \overline{F}_i(p_l)$.

Problem Formulation: The firm's revenue maximization problem can be formulated as follows.

$$\max_{p,q,M} \ \Sigma_{j=1}^{M} \ \Sigma_{i=1}^{N} \ p_j \ \Lambda_i \ \overline{F}_i \ (p_j) \ 1_{\{\chi_i(p,q)=j\}} \tag{3.9}$$

$$s.t. \ \Sigma_{j=1}^{M} \ \Sigma_{i=1}^{N} \ \Lambda_i \ \overline{F}_i \ (p_j) \ 1_{\{\chi_i(p,q)=j\}} \leq C, \tag{3.10}$$

$$0 \leq p < \infty, \quad 0 \leq q < \infty, \tag{3.11}$$

$$1 \leq M < \infty, \ M \ integer. \tag{3.12}$$

The objective in equation (3.9) is the sum of revenues across the M products, where revenue for product j equals the price of the product multiplied by the number of customers that buy it. Equation (3.10) is the capacity constraint, restricting the volume sold across customer classes to capacity C. Equation (3.11) enforces non-negativity of prices and quality, while equation (3.12) restricts the total number of products offered, M, to be a finite positive integer. The optimization decisions for the firm here involve deciding upon the number of products to offer M, and the price p_j, and quality level q_j for each product. For convenience, we will assume that $p_0 = 0$, and $q_0 = 0$. In addition, we will make the following assumptions to analyze the product design problem of the firm.

1. For each class, the hazard rates of the valuation distributions are decreasing in desired quality levels, i.e $\frac{f_i(v)}{\overline{F}_i(v)} < \frac{f_{i+1}(v)}{\overline{F}_{i+1}(v)}$, $\forall v \in [0, \overline{v}_{i+1}]$, $1 \leq i < N$.

2. (a) Hazard rates are non-decreasing, i.e., $p_1 > p_2 \Rightarrow \frac{f_i(p_1)}{\overline{F}_i(p_1)} \geq \frac{f_i(p_2)}{\overline{F}_i(p_2)}$, or

   (b) Hazard rates are decreasing and bounded above, i.e., $p_1 > p_2 \Rightarrow \frac{f_i(p_1)}{\overline{F}_i(p_1)} < \frac{f_i(p_2)}{\overline{F}_i(p_2)}$ and $\frac{f_i(p)}{\overline{F}_i(p)} \geq K$, $K > 0$ constant, or

   (c) Hazard rates are decreasing with slope less than 1, i.e., $p_1 > p_2 \Rightarrow \frac{f_i(p_1)}{\overline{F}_i(p_1)} < \frac{f_i(p_2)}{\overline{F}_i(p_2)}$ and $\frac{\partial \frac{f_i(p)}{\overline{F}_i(p)}}{\partial p} < 1$, $i = 1, ..., N$.

3. $\lambda_i \overline{F}_i^{-1}\left(\frac{\lambda_i}{\Lambda_i}\right)$ is a concave function, $i = 1, ..., N$, where $\lambda_i = \Lambda_i \overline{F}_i(p_{\chi_i(p,q)})$.

Discretization of type space: In our formulation, we have assumed that the set of threshold quality levels is discrete. In reality, customer thresholds might be continuous, and for simplicity and tractability, it might be desirable to discretize these thresholds. Towards this end, suppose that customer thresholds have a distribution $l \sim G(\cdot)$ over the set $[\theta_1, \theta_{N+1})$, where $\theta_{N+1} := \underline{\theta}$, and $\theta_1 > \theta_2 > ... > \theta_N$ is a discrete grid on the support of $G(\cdot)$. Then assuming that customers with quality thresholds $[\theta_{i-1}, \theta_i)$ have quality threshold θ_{i-1}, $1 < i \leq N+1$, the problem with continuous thresholds can be mapped to our problem with discrete thresholds.

Economic assumptions: Our assumption that hazard rates are monotone decreasing with respect to quality is equivalent to assuming that $\eta_i(v) < \eta_{i+1}(v), \forall v \in [0, \overline{v}_{i+1}]$, $1 \leq i < N$, where $\eta_i = \frac{v f_i(v)}{\overline{F}_i(v)}$ is the demand elasticity of customer class i. This assumption that customers that desire higher quality levels are more inelastic than customers that desire lower quality levels implies that the former are less likely to go away from purchasing the product as its price is raised than the latter. The assumptions that hazard rates are monotonic and that the per class revenue in terms of arrival rates is concave are not restrictive. Distributions that satisfy these assumptions include the uniform distribution, exponential distribution, pareto distribution, half-logistic distribution and rayleigh distribution.

3.3 Applications: examples

We present instances of product design problems where threshold preferences arise naturally. These examples are chosen to illustrate the variety of situations in which modeling customer behavior using threshold preferences is appealing. In the following, we will assume that the notation and assumptions of section 3.2 continue to hold.

3.3.1 Queueing service

Consider a service provider (SP) that operates an M/M/1 or M/M/C system offering a product to a market of price and delay sensitive customers. Customers are heterogeneous in their valuations and have threshold preferences with respect to the average delay they experience. As in Section 3.2, there are N customer classes. Class i customers have valuations for the product that are i.i.d. draws from the distribution $F_i(.)$, and are willing to tolerate any delay up to their class specific threshold θ_i. Without loss of generality, we assume that $\theta_1 < \theta_2 < ... < \theta_N$. More precisely, let d_j denote the expected waiting time associated with product variant j, then customers of class i with delay threshold θ_i are willing to purchase any product variant j for which $d_j \leq \theta_i$. Service times of customers are assumed to be homogeneous with mean service time one. Requests for product j are stored in a product-specific queue, and requests within a queue are served in First-In-First-Out manner. Since interarrival and service times are stochastic, there's a queueing element to the problem and hence internal system dynamics affect the deliverable delay to the customers. The SP controls the pricing of each product variant as well as the capacity allocation policy that decides which request to serve next in an M/M/1 queueing system, or which request to be routed to the first available unit of capacity in an M/M/C queue. Through its capacity allocation policy the SP controls indirectly the vector of expected delays for the various product variants that, in turn, affect the equilibrium vector of arrival rates into the system. The goal of the SP is to identify the pricing and capacity allocation policy that maximizes its total revenues. This problem is a variation to the model recently advanced by Afeche [1], where we have replaced rational customer choice behavior by threshold customer choice behavior. The SP's problem can be written as follows:

$$\max_{p,d,M} \ \Sigma_{i=1}^{N} \ \Sigma_{j=1}^{M} \ p_j \ \Lambda_i \ \overline{F}_i \ (p_j) \ 1_{\{\chi_i(p,d)=j\}} \tag{3.13}$$

$$s.t. \ \Sigma_{i=1}^{N} \ \Sigma_{j=1}^{M} \ \Lambda_i \ \overline{F}_i \ (p_j) \ 1_{\{\chi_i(p,d)=j\}} \leq C, \tag{3.14}$$

$$0 \leq p < \infty, \quad 0 \leq d < \infty, \tag{3.15}$$

$$1 \leq M < \infty, \ M \ integer, \tag{3.16}$$

together with some additional conditions that govern the stochastic system dynamics and relate the expected delay vector d to the capacity allocation rule and the vector of arrival rates where M is the number of products offered by the SP, and p_i denotes the price associated with the i^{th} product variant. The presence of these system dynamics distinguishes problem (3.13)-(3.16) from the general problem (3.9)-(3.12).

3.3.2 ISP bandwidth allocation

Consider an Internet Service Provider (ISP) that offers downstream bandwidth to end-users. Customers are heterogeneous in their valuations and have threshold preferences with respect to capacity, i.e., the minimum bandwidth they can download at. As in Section 3.2, there are N customer classes, with class i customer valuations distributed as $F_i(\cdot)$, and class i having a capacity threshold θ_i, the minimum capacity that they desire. We assume that $\theta_1 > \theta_2 > ... > \theta_N$. Then, denoting as c_j the capacity associated with product j offered by the firm, class i customers seek to buy the cheapest product j such that $c_j \geq \theta_i$.

Such preferences with respect to capacity arise, for example, in the context of applications such as Voice over IP, and Video on Demand, where a certain threshold amount of bandwidth is needed for this service to be useful, and capacity in excess to this threshold does not matter. In this case, capacity corresponds to our notion of quality, with higher download capacity corresponding to a better quality.

The firm's optimization problem can be stated as follows.

$$\max_{p,c,M} \; \Sigma_{i=1}^{N} \; \Sigma_{l=1}^{M} \; p_i \; \Lambda_i \; \overline{F}_i \; (p_i) \; 1_{\{\chi_i(p,c)=l\}} \tag{3.17}$$

$$s.t. \; \Sigma_{i=1}^{N} \; \Sigma_{l=1}^{M} \; \Lambda_i \; \overline{F}_i \; (p_i) \; c_i \; 1_{\{\chi_i(p,c)=l\}} \leq C, \tag{3.18}$$

$$0 \leq p < \infty, \quad 0 \leq c < \infty, \tag{3.19}$$

$$1 \leq M < \infty, \; M \; integer. \tag{3.20}$$

Equations (3.17), (3.19)-(3.20) are the same as equations (3.9), (3.11)-(3.12) in the general problem, where c_i now denotes the quality of product i. Since offered qualities affect the available capacity in this example, the capacity constraint, equation (3.18) is different from equation (3.10) in the general problem.

3.3.3 Dynamic pricing with strategic customers

Consider a monopolist firm that seeks to sell a homogeneous product to a market of heterogeneous, strategic customers. Customers vary in their valuations and degree of risk-aversion, and the firm seeks to differentiate customers by creating rationing risk, i.e., by offering the product at different prices and with different fill-rates at different times in the market (e.g., as discussed in Liu and van Ryzin [39], Bansal and Maglaras [7], Su [63], and Cachon and Swinney [13]). Customers have threshold preferences with respect to fill-rates, with each customer possessing a threshold that corresponds to the minimum acceptable fill-rate that the customer is willing to buy at. Customers are also strategic, observe the entire product menu offered by the firm, and make the optimal timing decision to enter the market. The firm's product design problem is to identify the optimal the number of products to offer to this market, along with their prices and fill-rates. Fill-rates r satisfy $0 \leq r \leq 1$, and a fill-rate of r implies that only a proportion r of customer requests are fulfilled by the firm. Fill-rate here corresponds to our notion of quality, with a higher fill-rate implying a better quality. There are N types and type i customers have a fill-rate threshold θ_i, implying that type i customers prefer the cheapest product j with fill-rate $r_j > \theta_i$ (notice we assume that the inequality is strict). We assume $1 > \theta_1 > \theta_2 > ... > \theta_N > 0$, and all assumptions in section 3.2 hold.

To motivate such customer behavior, suppose customers have a limit on the maximum payoff variability they are willing to tolerate. The expected payoff to a customer with valuation v upon deciding to purchase a product with price p and fill-rate r, is given by $(v-p)r$, and the variance of this payoff is given by $(v-p)^2r(1-r)$. Let A denote the customer threshold for the variability the customer is willing to tolerate. Hence this customer would seek to purchase the cheapest product such that

$$\frac{\text{stdev}}{\text{mean}} = \frac{\sqrt{(v-p)^2r(1-r)}}{(v-p)r} \leq A, \tag{3.21}$$

where A is a fixed fraction. This reduces to $r \geq \frac{1}{1+A^2}$, implying that customer has threshold preferences with respect to the rationing risk where the rationing threshold is given by $\frac{1}{1+A^2}$. Also, a low desire for variability leads to a higher rationing threshold, which is intuitive.

The optimization problem that the firm faces can be expressed as follows:

$$\max_{p,r,M} \ \Sigma_{i=1}^{N} \ \Sigma_{l=1}^{M} \ p_i \ \Lambda_i \ \overline{F}_i \ (p_i) \ r_i \ 1_{\{\chi_i(p,r)=l\}} \tag{3.22}$$

$$s.t. \ \Sigma_{i=1}^{N} \ \Sigma_{l=1}^{M} \ \Lambda_i \ \overline{F}_i \ (p_i) \ r_i \ 1_{\{\chi_i(p,r)=l\}} \leq C, \tag{3.23}$$

$$0 \leq p < \infty, \quad 0 \leq r \leq 1, \tag{3.24}$$

$$1 \leq M < \infty, \; M \; integer. \tag{3.25}$$

The objective (3.22) is the sum of revenue over the N classes, where class i revenue is the product of price p_i, the number of class i customers that are willing to buy at this price, $\Lambda_i \overline{F}_i \; (p_i)$, and the fill-rate associated with this product, r_i. Equation (3.23) enforces the constraint that available capacity does not exceed sales, where the volume sold to class i customers is the product of class i demand and the fill-rate corresponding to the product they purchase. The presence of the quality attribute r in the objective (3.22) and the capacity constraint (3.23) distinguishes this problem from the general problem (3.9)-(3.12).

3.3.4 Versioning of information goods

Consider a monopolistic software firm that serves a market of heterogeneous customers. To differentiate customers, the firm creates several versions of the software, and sells better versions at higher prices. Higher priced versions may have more features, a better user interface, and faster speed. Customers do not necessarily desire the fastest version, or the version with the most features, rather they seek to buy the cheapest product that satisfies their product and computational requirements. In such a setting it might be realistic to model customer choice behavior using threshold preferences. We will assume that the software product being sold by the firm is characterized by a one-dimensional quality attribute. As in Section 3.2, we will assume that there are N classes of customers, class i customers having valuations distributed as $F_i(\cdot)$, and having quality threshold θ_i. As is typical for software and other information goods, given a high quality version of the product, it is easy to produce inferior versions. Hence, we will assume that there is the marginal cost with respect to quality is zero, and that the capacity C of the firm is infinite. All other modeling assumptions in section 3.2 continue to hold. Then, the firm's revenue maximization problem can be formulated as equations (3.9), (3.11)-(3.12).

3.3.5 Time-sensitive retail customers

Consider a retail firm selling fashion goods over a season. Customers are heterogeneous in their valuations and are time-sensitive. In particular, they have threshold preferences with respect to time into the season by which they must procure the fashion good. The firm seeks to differentiate customers by dynamically adjusting the price of the good over the season, with a higher price being offered early in the season and the price being lowered as the season progresses. As in Section 3.2, we assume that there are N classes of customers, with class i customers having valuations distributed as $F_i(\cdot)$, and having waiting threshold θ_i. Hence class i customers seek to buy the cheapest product offered at any time $t \leq \theta_i$ into the season. We will assume that the firm has a capacity C, and there are no holding costs. Then, the product design problem of the firm can be formulated as equations (3.9)-(3.12).

3.3.6 Seller of mp3 players

Consider a firm that sells mp3 music players in a market where customers differ in their valuations and preferences for the player's storage capacity (as an example, consider Apple that sells Nano in 4 and 8 GB versions). In particular, customers have threshold preferences with respect to the mp3 player's storage capacity, and seek to purchase the cheapest mp3 player with capacity above their specific threshold. The storage capacity of a mp3 player is a measure of the number of songs it can store, and customers that desire to carry along a larger number of songs have higher thresholds. The seller seeks to differentiate customers by selling mp3 players with different storage capacities, with a higher price being charged for players with larger storage capacities. The storage capacity of the player is a measure of its quality in this case. High capacity players have a higher associated with them and the presence of increasing marginal costs with respect to quality distinguish this problem from the general problem in Section 3.2. As in Section 3.2, we will assume that there are N classes of customers, with class i customers having valuations distributed as $F_i(\cdot)$, and having capacity threshold θ_i. We will assume that the seller has a capacity C. We will denote the marginal cost of a product with quality θ_i as s_i. Then, the seller's product design problem can be formulated as follows:

$$\max_{p,q,M} \; \Sigma_{j=1}^{M} \, \Sigma_{i=1}^{N} \, (p_j - s_j) \; \Lambda_i \; \overline{F}_i \; (p_j) \; 1_{\{\chi_i(p,q)=j\}} \tag{3.26}$$

$$s.t. \; \Sigma_{j=1}^{M} \, \Sigma_{i=1}^{N} \, \Lambda_i \; \overline{F}_i \; (p_j) \; 1_{\{\chi_i(p,q)=j\}} \leq C, \tag{3.27}$$

$$0 \leq p < \infty, \quad 0 \leq q < \infty, \tag{3.28}$$

$$1 \leq M < \infty, \;\; M \; integer. \tag{3.29}$$

Formulation (3.26)-(3.29) is the same as (3.9)-(3.12), except for the objective, which is modified to reflect that the profit upon selling a unit of product j changes from p_j to $p_j - s_j$.

3.3.7 Postal service provider

Consider a postal service provider, such as Fedex, that services a market where customers have heterogeneous valuations for its service, and differ in the maximum shipping time they are willing to tolerate. In particular, customers have threshold preferences with respect to the maximum shipping time that is acceptable to them, and seek to buy the cheapest product with delivery time smaller than their threshold. We consider a single-leg problem, wherein the service provider has made several fixed cost investments that guarantee different maximum delays, e.g., shipping via air, land or rail. The service provider seeks to maximize its revenue by offering services that guarantee lower maximum shipping time at higher prices. This problem differs from the general problem in that different types of capacities are available to the service provider, and demand for certain quality levels can be fulfilled using only a subset of the available quality types. We will assume that $m \leq N$ capacity types are available, and classes $1, ..., i_l$ can be serviced only using capacity types $1, ..., l$, where $1 \leq i_1 \leq ... \leq i_m = N$. The service provider's product design problem can be formulated as follows:

$$\max_{p,q,M} \ \Sigma_{j=1}^{M} \Sigma_{i=1}^{N} p_j \ \Lambda_i \ \overline{F}_i \ (p_j) \ 1_{\{\chi_i(p,q)=j\}} \tag{3.30}$$

$$s.t. \ \Sigma_{j=1}^{M} \Sigma_{i=1}^{i_l} \Lambda_i \ \overline{F}_i \ (p_j) \ 1_{\{\chi_i(p,q)=j\}} \leq C_l, \ l = 1, ..., m, \tag{3.31}$$

$$0 \leq p < \infty, \quad 0 \leq q < \infty, \tag{3.32}$$

$$1 \leq M < \infty, \ M \ integer. \tag{3.33}$$

The presence of multiple capacity types, and the constraint that qualities $\theta_1, ..., \theta_{i_l}$ can be provided only through capacity types $1, ..., l$, $l = 1, ..., m$, distinguishes this problem from the general problem (3.9)-(3.12).

3.3.8 Other examples

In addition to the examples discussed above, threshold preferences arise in several other contexts. We briefly mention some of these below.

Electronic goods: Threshold preferences arise naturally in the context of electronic goods like hard-disks or digital cameras where quality attributes might include pixel resolution, storage capacity, screen size etc. In such cases, it is conceivable that customers exhibit threshold preferences, e.g., they might want to buy the cheapest hard-disk with storage capacity $\geq$ 100 GB, or the cheapest digital camera with resolution $\geq$ 5 megapixels.

Equipment maintenance services: Consider a equipment service provider that provides maintenance service to heterogeneous customers when their equipment suffers a breakdown. In such a setting, customers might have threshold preferences with respect to the downtime they are willing to tolerate, or the probability of service provider stocking out of the required component.

Seller of hardcover and paperback books: Consider a book seller that seeks to publish a title. In this setting, customers can have threshold preferences with respect to book-format, wherein some customers would buy only the hardcover edition, while others would buy the cheaper of the paperback and hardcover edition.

Airline tickets: Customers willing to tolerate zero, or at most one, or at most two stops provide an example of how threshold preferences can arise in the context of an airline offering non-stop and indirect flights between two airports.

Additional services: Several online service providers such as www.yousendit.com or Turbotax provide the basic service for free, but charge for additional services such as delivery confirmation, live personal support, and so on. In such settings, some customers might have threshold preferences with respect to the degree of support they desire from the application.

Car-rental services: Customers seeking to rent a car might exhibit threshold preferences with respect to car-size, e.g., a customer might have a midsize car as her threshold, and might be willing to rent a SUV but not a compact car.

Financial news providers: Several online financial news providers such as Bloomberg offer delayed content for free, while charging for live content. Depending upon whether they use this information in a time-critical or offline fashion, customers might exhibit threshold preferences with respect delay in the updates.

Products with multiple quality attributes: Threshold preferences can arise naturally when product quality is multi-dimensional. In such a setting, customers are likely to exhibit threshold preferences, i.e., ensure that product quality meets their specific criterion along each of the quality attributes, rather than maximizing over a utility function that results from combining the various quality attributes. Examples would include digital cameras, printers, etc.

3.4 Analysis of the general model

In this section, we analyze the model presented in Section 3.2. We characterize the structure of the optimal solution, and show how to compute the optimal solution. We also consider the problem of identifying the optimal prices and fill-rates when the number of products that can be offered to the market is small.

3.4.1 Structural results

We present structural results that characterize the optimal solution. Our first result shows that without loss of generality, the firm only needs to offer products with quality levels in the set $\{\theta_1, \theta_2, ..., \theta_N\}$.

Lemma 5 *The following hold:*

a) It suffices to offer quality levels that lie in the set $\{\theta_1, \theta_2, ..., \theta_N\}$.

b) For any two distinct products (p_i, q_i) *and* (p_j, q_j), $q_i > q_j \Leftrightarrow p_i > p_j$.

Proof We prove each part of the lemma in sequence.

a) Since any customer class i is indifferent between the quality levels that lie in the interval $(\theta_{l-1}, \theta_l]$, $l = 1, ..., N$, and $\theta_0 := \overline{\theta}$, at most one price can be charged for any quality level in $(\theta_{l-1}, \theta_l]$, $l = 1, ..., N_i$. Hence, offering at most quality level in $(\theta_{l-1}, \theta_l]$, $l = 1, ..., N$ suffices, which without loss of generality, we can fix to θ_l.

b) Following a), the quality levels q_i and q_j lie in the set $\{\theta_1, \theta_2, ..., \theta_N\}$. Suppose $q_i > q_j$ while $p_i \leq p_j$. Then, product i strictly dominates product j in the sense that every customer of class l with $\theta_l \leq q_j$ would strictly prefer product i over product j. Since, no customer type l with $\theta_l > q_j$ buys product j, it would then mean that nobody buys product j. Hence the firm can drop product j from its product line without affecting its revenues. This would contradict our assumption that the firm only offers products that generate non-zero demand, and so $p_i > p_j$. Suppose now that $p_i > p_j$

but $q_i \leq q_j$. In this case, any customer belonging to class l with $\theta_l \leq q_i$ strictly prefers product j to product i. Since no customer type l with $\theta_l > q_i$ buys product i, the firm can drop product i from its product line without affecting its revenues. Hence $q_i > q_j$. Finally, $q_i = q_j \implies p_i = p_j$ and vice-versa or else the product with the lower quality or higher price respectively can be dropped as it will generate zero demand. □

Lemma 5 shows that there exists a strict correspondence between prices and quality levels, and higher quality levels correspond to higher prices and vice versa. As a consequence, customers cannot be indifferent between any two products offered by the firm. Lemma 5 also shows that the firm needs to offer at most N different products, each product being offered at a different quality in the set $\{\theta_1, \theta_2, ..., \theta_N\}$. We also have the following corollary.

Corollary 5 *Suppose the firm offers* $1 \leq k \leq N$ *distinct products at qualities* $\theta_{i_1}, ..., \theta_{i_k}$, $1 \leq i_1 < i_2 < ... < i_k \leq N$, *at prices* $p_{i_1}, p_{i_2}, ..., p_{i_k}$, *respectively. Then* $p_{i_1} > p_{i_2} > ... > p_{i_k}$ *and*
a) $p_{i_1} < \overline{v}_{i_1}$,
b) $p_{i_k} > 0$.

Proof The monotonicity of prices, $p_{i_1} > p_{i_2} > ... > p_{i_k}$, follows from lemma 5. We will prove the two parts in sequence.
a) If $p_{i_1} \geq \overline{v}_{i_1}$, then $\overline{F}_{i_1}(p_{i_1}) = 0$, implying that nobody this product and it can be dropped, violating our assumption that only products that offer a non-zero demand are offered. Hence, $p_{i_1} < \overline{v}_{i_1}$.
b) Suppose $p_{i_k} = 0$. Consider setting p_{i_k} to $0.5 \min\{p_{i_{k-1}}, \overline{v}_{i_k}\} > 0$ wherein the aggregate demand decreases while revenues increase. Note that $p_{i_{k-1}} > 0$, since products k and $k-1$ are distinct. Hence $p_{i_k} > 0$ in the optimal solution. □

Lemma 6 *Any* $k \leq N$ *products partition the* N *customer classes into contiguous sets, i.e., if class* $i-1$ *and* $i+1$ *customers buy product* j*, then so do class* i *customers.*

Proof Since type $i-1$ buys product j, $q_j \geq \theta_{i-1} > \theta_i$, i.e., the quality of product j is higher than the quality threshold for type i. Since type $i+1$ buys product j, $p_j = \min_{q_l \geq \theta_{i+1}} p_l \leq \min_{q_l \geq \theta_i} p_l$, i.e., product j is the cheapest product offered by the firm with quality greater than or equal to θ_i. Hence it is optimal for type i to buy product j. $\square$ Our next result shows that it is always optimal to offer the highest quality product.

Lemma 7 *The highest quality θ_1 is always offered.*

Proof Let θ_k, $k > 1$ be the highest quality offered to the market. Denote the price of this product by p. Also suppose that customers from subtypes l, $k \leq l \leq i$ are currently buying this product. Consider increasing the quality of the offered highest quality product from θ_k to θ_1 and increasing its price from p to $p+\epsilon$, $\epsilon > 0$ and such that $\Sigma_{l=1}^{i} \overline{F}_l(p+\epsilon)\Lambda_l = \Sigma_{l=k}^{i} \overline{F}_l(p)\Lambda_l$. Since the demand does not change while revenues increase (we increased the price), for the proof to be complete we need to show that such an ϵ exists. Define $A := \Sigma_{l=1}^{i} \overline{F}_l(p+\epsilon)\Lambda_l$, $B := \Sigma_{l=k}^{i} \overline{F}_l(p)\Lambda_l$. There are 4 cases to consider.

Case a) $\overline{v}_i = \infty$: This implies that $\overline{v}_1 = \overline{v}_2 = ... = \overline{v}_{i-1} = \infty$. Such an ϵ exists, since at $\epsilon = 0$, A > B, and since the A is decreasing and continuous in ϵ, with value 0 at $\epsilon = \infty$.

Case b) $\infty > \overline{v}_1 = \overline{v}_2 = ... = \overline{v}_i$: Such an ϵ exists, since at $\epsilon = 0$, A > B, and since the A is decreasing and continuous in ϵ, with value 0 at $\epsilon = \overline{v}_1 - p > 0$ (following lemma 5 and above, $p < \overline{v}_1$).

Case c) $\infty = \overline{v}_1 > \overline{v}_i$: Such an ϵ exists, since at $\epsilon = 0$, A > B, and since A is decreasing in ϵ and continuous, with value 0 at $\epsilon = \infty$. Note that in this case, the new price $p+\epsilon$ might be greater than $\overline{v}_l$ for some $1 < l \leq i$, in which one of the customer subtypes that were earlier buying the product would not buy it at the increased price.

Case d) $\infty > \overline{v}_1 > \overline{v}_i$: Such an ϵ exists, since at $\epsilon = 0$, A > B, and since A is decreasing in ϵ and continuous, with value 0 at $\epsilon = \overline{v}_1 - p > 0$ (remember from lemma 5 and above, $p < \overline{v}_1$). Again note that in this case, the new price $p+\epsilon$ might be greater than $\overline{v}_l$ for some $1 < l \leq i$, in which one of the customer subtypes that were earlier buying the product would not buy it at the increased price.

In each of the four cases, we were able to construct a solution where revenues increase while the same capacity is being utilized. Hence the highest quality product is always offered. □ Lemma 7 leads to the following formulation for the firm's one-product problem.

Corollary 6 *If a single product is offered by the firm, then it is offered at the highest quality θ_1. Consequently, the firm's one-product problem can be formulated as follows.*

$$\max_{p_1} \; \{\Sigma_{i=1}^{N} \; p_1 \; \Lambda_i \; \overline{F}_i(p_1) \;\; : \;\; \Sigma_{i=1}^{N} \; \Lambda_i \; \overline{F}_i(p_1) \; \leq \; C\}. \tag{3.34}$$

Corollary 7 *If $\overline{v}_i = \overline{v}$, $i = 1, \ldots, N$, then all classes buy a product from the firm.*

Proof Since at least one product is offered, following corollary 6, the highest quality product is offered. Let p_1 denote its price. Then $p_1 < \overline{v}$, and at least some customers from each class buy from the firm. □ Lemma 5-7 lead to the following formulation of the firm's revenue maximization problem.

Proposition 6 *The general problem (3.9)-(3.12) can be formulated as follows.*

$$\max_{p} \; \Sigma_{i=1}^{N} \; p_i \; \Lambda_i \; \overline{F}_i \; (p_i) \tag{3.35}$$

$$s.t. \; \Sigma_{i=1}^{N} \; \Lambda_i \; \overline{F}_i \; (p_i) \leq C, \tag{3.36}$$

$$p_N \leq p_{N-1} \leq ... \leq p_1, \quad i = 1, 2, ..., N, \tag{3.37}$$

$$p_i \leq \overline{v}_i \quad i = 1, 2, ..., N. \tag{3.38}$$

where p_i denotes the price of the product being offered at quality θ_i.

Proof Following lemma 7, quality θ_1 is always offered. Hence, all customer classes $1, ..., N$ would buy a product from the firm, subject to their valuations exceeding the price p_1. If $k < N$ products are offered in the optimal solution at qualities $\theta_{m_1}, \theta_{m_2}, ..., \theta_{m_k}$, with $m_1 < m_2 < ... < m_k \leq N$, $m_1 = 1, m_{k+1} := N+1$, and prices $p_{m_1} > p_{m_2} > ... > p_{m_k}$, then setting prices to be $p_j = p_{m_i}, m_i \leq j < m_{i+1}$, $i = 1, ..., k$, in the above formulation would lead to the same solution. Finally, any solution of equations (3.35)-(3.37) is consistent with customer behavior in that type i customers would buy the product priced at p_i. Hence the formulation is correct. $\square$

Proposition 6 considerably simplifies that firm's product design problem. The firm no longer needs to optimize over M and q, the number of qualities to offer and the vector of qualities respectively, making the formulation (3.35)-(3.37) more amenable to direct analysis. (In the above analysis, we haven't made any use of assumptions 1-3, and hence the above formulation is quite general.)

Lemma 8 *Suppose qualities θ_m and θ_n are offered in the optimal solution, with $m+1 < n$. Then qualities θ_l, $m+1 \leq l \leq n-1$ are also offered.*

Proof Suppose $k < N$ products are being offered by the firm (lemma holds trivially if $k = N$). Then there exist indices $1 \leq i_1 < i_2 < ... < i_k \leq N$ such that product l, $1 \leq l \leq k$ is being offered at quality θ_{i_l}. (From lemma 7, $i_1 = 1$.) Suppose there exist indices m, n such that $m+1 < n$, $i_l = m, i_{l+1} = n$ for some $1 \leq l \leq k-1$. These correspond to products with qualities θ_m and θ_n respectively. In case such indices do not exist (since $k < N$ and $i_1 = 1$, this case occurs only when the k products are offered at qualities $\theta_1, ...\theta_k$), the proposition in the lemma is already satisfied. Even then for the first case of the following two, we set $m = k, n = N+1, p_{N+1} = 0, \theta_{N+1} = 0$. For the second case, we consider only the possibility where such indices do exist. Let us denote the prices of these two products by p_m and p_n respectively, with $p_m > p_n$ (since $\theta_m > \theta_n$ and following lemma 5). There are two cases to consider here.

Case a: $p_m < \overline{v}_{m+1}$: Consider the following sequence of actions: adding a product at quality level θ_{m+1} and price $p_m - \delta$, $\delta > 0$, $p_m - \delta > p_n$, and increasing the price of the product being offered at quality θ_m from p_m to $p_m + \epsilon$, $\epsilon > 0$, $p_m + \epsilon < \overline{v}_m$, $p_m + \epsilon < p_{i_{l-1}}$, where $p_{i_{l-1}}$ is the price of the θ_{l-1} best quality product offered by the firm, if any, and ∞ otherwise. The change in demand, $\Delta D = \Lambda_m \overline{F}_m(p_m+\epsilon)+\Sigma_{u=m+1}^{n-1}\Lambda_u\overline{F}_u(p_m-\delta)-\Lambda_m\overline{F}_m(p_m)-\Sigma_{u=m+1}^{n-1}\Lambda_u\overline{F}_u(p_m)$. Using the first order Taylor expansion, we can write $\Delta D = -\epsilon\Lambda_m f_m(p_m) + \delta\Sigma_{u=m+1}^{n-1}\Lambda_u f_u(p_m) + o(\epsilon) + o(\delta)$. Similarly, the change in revenue, $\Delta R = \Lambda_m(p_m+\epsilon)\overline{F}_m(p_m+\epsilon) + (p_m-\delta)\Sigma_{u=m+1}^{n-1}\Lambda_u\overline{F}_u(p_m - \delta) - \Lambda_m p_m\overline{F}_m(p_m) - p_m\Sigma_{u=m+1}^{n-1}\Lambda_u\overline{F}_u(p_m)$. Again, using the first order Taylor expansion, $\Delta R = \epsilon\Lambda_m(\overline{F}_m(p_m) - p_m f_m(p_m)) + \delta\Sigma_{u=m+1}^{n-1}\Lambda_u(p_m f_u(p_m) - \overline{F}_u(p_m)) + o(\epsilon) + o(\delta)$.

We want to show that there exist $\delta > 0, \epsilon > 0$, both small (so that first order term in the Taylor expansion dominates, as well as $p_m + \epsilon < \overline{v}_m$, $p_m + \epsilon < p_{i_l - 1}$ and $p_m - \delta > p_n$), such that $\Delta D < 0$, $\Delta R > 0$. To this end, let's choose δ such that $\delta\Sigma_{u=m+1}^{n-1}\Lambda_u f_u(p_m) = \gamma\epsilon\Lambda_m f_m(p_m)$ holds, where $0 < \gamma < 1$. This implies that $\Delta D = -\epsilon\Lambda_m f_m(p_m)(1-\gamma) + o(\epsilon) = -\delta\frac{1-\gamma}{\gamma}\Sigma_{u=m+1}^{n-1}\Lambda_u f_u(p_m) + o(\delta)$, which is < 0 when ϵ (or equivalently δ) is small enough.

Substituting this value of δ, we get

$$
\begin{aligned}
\Delta R =& \epsilon[\Lambda_m(\overline{F}_m(p_m) - p_m f_m(p_m)) \\
&+ \frac{\gamma\Lambda_m f_m(p_m)}{\Sigma_{u=m+1}^{n-1}\Lambda_u f_u(p_m)}\Sigma_{u=n-1}^{m+1}\Lambda_u(p_m f_u(p_m) - \overline{F}_u(p_m))] + o(\epsilon), \\
&= \epsilon\Lambda_m f_m(p_m)p_m[(\frac{\overline{F}_m(p_m)}{p_m f_m(p_m)} - 1) \\
&+ \frac{\gamma}{\Sigma_{u=m+1}^{n-1}\Lambda_u f_u(p_m)}\Sigma_{u=m+1}^{n-1}\Lambda_u f_u(p_m)(1 - \frac{\overline{F}_u(p_m))}{p_m f_u(p_m)}] + o(\epsilon), \\
&= \frac{\epsilon\Lambda_m f_m(p_m)p_m}{\Sigma_{u=m+1}^{n-1}\Lambda_u f_u(p_m)}[\Sigma_{u=m+1}^{n-1}\Lambda_u f_u(p_m)(\frac{1}{\eta_m(p_m)} - 1 + \gamma - \frac{\gamma}{\eta_u(p_m)})] + o(\epsilon).
\end{aligned}
$$

For $\Delta R > 0$, we need ϵ small enough so that the first order term dominates and we need to verify that $\frac{1}{\eta_m(p_m)} - \frac{\gamma}{\eta_u(p_m)} > 1 - \gamma$ for $m+1 \le u \le n-1$. Now $\eta_m(p_m) < \eta_{m+1}(p_m) \le \eta_u(p_m) \Leftrightarrow \frac{1}{\eta_m(p_m)} > \frac{1}{\eta_{m+1}(p_m)} \ge \frac{1}{\eta_u(p_m)}$. Hence, it suffices to show that $\frac{1}{\eta_m(p_m)} - 1 > \gamma(\frac{1}{\eta_{m+1}(p_m)} - 1)$, which holds from above and the fact that we can choose any γ that satisfies $0 < \gamma < 1$.

Case b: $p_m > \overline{v}_{m+1}$: Note here that in the existing product offering, subtypes $m+1 \leq u < n$ do not buy any product. Consider the following sequence of actions: adding a product at quality level θ_{m+1} and price $\overline{v}_{m+1} - \epsilon$, $\epsilon > 0$, and increasing the price of the product offered at θ_n from p_n to $p_n + \delta$, $\delta > 0$, $p_n + \delta < \overline{v}_n$, $\overline{v}_{m+1} - \epsilon > p_n + \delta$. Let θ_r be the next best quality after θ_n that is offered by the firm (We set it to $r = N+1$, $\theta_{N+1} = 0$, $p_{N+1} = 0$, as mentioned earlier, if there's none). The change in demand $\Delta D = \Sigma_{u=m+1}^{n-1} \overline{F}_u(\overline{v}_{m+1} - \epsilon)\Lambda_u + \Sigma_{u=n}^{r-1} \overline{F}_u(p_n + \delta)\Lambda_u t_j - \Sigma_{u=n}^{r-1} \overline{F}_u(p_n)\Lambda_u$. Using the first order Taylor expansion as above, we get $\Delta D = \epsilon \Sigma_{u=m+1}^{n-1} f_u(\overline{v}_{m+1})\Lambda_u - \delta \Sigma_{u=n}^{r-1} f_u(p_n)\Lambda_u + o(\epsilon) + o(\delta)$. Similarly, the change in Revenue is given by $\Delta R = \Sigma_{u=m+1}^{n-1} \overline{F}_u(\overline{v}_{m+1} - \epsilon)\Lambda_u(\overline{v}_{m+1} - \epsilon) + \Sigma_{u=n}^{r-1} \overline{F}_u(p_n + \delta)\Lambda_u(p_n + \delta) - \Sigma_{u=n}^{r-1} \overline{F}_u(p_n)\Lambda_u p_n$. The first order approximation is given by $\Delta R \approx \overline{v}_{m+1}\epsilon \Sigma_{u=m+1}^{n-1} f_u(\overline{v}_{m+1})\Lambda_u - \delta \Sigma_{u=n}^{r-1} \Lambda_u(p_n f_u(p_n) - \overline{F}_u(p_n)) + o(\epsilon) + o(\delta)$.

Again, choose $\epsilon \Sigma_{u=m+1}^{n-1} f_u(\overline{v}_{m+1})\Lambda_u = \gamma \delta \Sigma_{u=n}^{r-1} f_u(p_n)\Lambda_u$, with $0 < \gamma < 1$. Substituting this value of ϵ, we get that

$$\Delta R = \delta \Sigma_{u=n}^{r-1} f_u(p_n)\Lambda_u p_n \left(\frac{\overline{v}_{m+1}\gamma}{p_n} - 1 + \frac{\overline{F}_u(p_n)}{p_n f_u(p_n)} \right) + o(\delta).$$

So, if we choose delta small enough for the first order term to dominate the sign of change ΔR, a sufficient condition for $\Delta R > 0$ is that $\frac{\overline{v}_{m+1}\gamma}{p_n} > 1$, which is true if we choose $\gamma > \frac{p_n}{\overline{v}_{m+1}}$. Note that this is possible, since the only condition on our choice of γ was $0 < \gamma < 1$, and $0 < \frac{p_n}{\overline{v}_{m+1}} < 1$ (since $p_n < \overline{v}_n \leq \overline{v}_{m+1}$).

Hence in both cases, through this introduction of a new product and by appropriately modifying prices, we can obtain a first order increase in revenues while simultaneously obtaining a first order decrease in required capacity. Hence, the construction works in both capacitated and uncapacitated cases. □

Intuitively, increasing the price of a higher quality product increases revenues more than the decrease in revenues that results by introducing a lower quality product at a lower price.

Our next result shows that furthermore, it is optimal to offer exactly N products.

Corollary 8 *If $\overline{v}_1 = \overline{v}_2 = ... = \overline{v}_N$, then it is optimal to offer exactly N products.*

Proof In the proof of lemma 8, under the assumption that $\overline{v}_i$ = constant for all $i = 1, ..., N$, the second case in the proof never arises. Note that the proof of the first part works for all k product offerings $1 \leq k < N$, irrespective of whether there are holes in the product offering or not. Hence we know that $\forall k < N$, offering $k+1$ products over k products increases revenues. Also from lemma 5, we know that it suffices to offer at most N products. Hence, it is optimal to offer exactly N products. □

3.4.2 Computation

In this subsection, we show how the optimal solution to the revenue maximization problem can be computed. Under our assumption that $F_i(.)$ is continuous, we observe that this problem involves maximizing a continuous function over a compact set, and hence by Weierstrass theorem, an optimal solution exists.

Instead of proceeding with a direct analysis of (3.35)-(3.37), we will first restate the problem in terms of the demand rate vector as the optimization variable; this is typical in the revenue management literature. Specifically, for each class i, define $\lambda_i = \Lambda_i \overline{F}_i(p_i)$, so that $p_i = \overline{F}_i^{-1}(\frac{\lambda_i}{\Lambda_i})$. We will also drop the monotonicity constraint (3.37), and show that as a consequence of lemma 5 and our assumptions in section 3.2, it will be automatically satisfied by the optimal solution. Then the product design problem (3.35)-(3.38) reduces to

$$\max_{\lambda} \ \Sigma_{i=1}^{N} \lambda_i \ \overline{F}_i^{-1}\left(\frac{\lambda_i}{\Lambda_i}\right) \tag{3.39}$$

$$s.t. \ \Sigma_{i=1}^{N} \ \lambda_i \ \leq C, \tag{3.40}$$

$$0 \leq \lambda_i \leq \Lambda_i, \ i = 1, ..., N, \tag{3.41}$$

which is a concave maximization problem over a polyhedron. Hence the first-order conditions are both necessary and sufficient. This leads to the following characterization of the optimal prices.

Proposition 7 *The optimal solution to the product design problem (3.35)-(3.38) is given by*

$$p_i = \frac{\overline{F}_i(p_i)}{f_i(p_i)} + \mu - \frac{\eta_i}{f_i(p_i)}, \tag{3.42}$$

$$\mu \ \left(C - \Sigma_{i=1}^{N} \ \Lambda_i \ \overline{F}_i(p_i)\right) = 0, \tag{3.43}$$

$$\mu \geq 0, \quad C - \Sigma_{i=1}^{N} \ \Lambda_i \ \overline{F}_i(p_i) \geq 0, \tag{3.44}$$

$$\eta_i \ (\overline{v}_i - p_i) = 0, \quad \eta_i \geq 0, \quad \overline{v}_i - p_i \geq 0. \tag{3.45}$$

Here μ is the Lagrange multiplier associated with the capacity constraint (3.36), and η_i is the Lagrange multiplier associated with the constraint $p_i \leq \overline{v}_i$. Following the assumptions in section 3.2, the optimal prices satisfy the monotonicity constraint (3.37).

3.4.3 $k < N$ products

We next consider the case where the firm seeks to offer a small number of products to the market, less than the number of customer classes. Such a strategy might be attractive when some customer classes are similar or when administrative costs (not considered in our model) are high. It may also be driven by branding considerations (for instance, in the rationing example, the firm may not want to offer more than 2 products, so that customers that are rationed out do not discover that the product is available in a later period). We will assume that the firm seeks to offer $k < N$ products at qualities $\theta_{m_1}, \theta_{m_2}, ..., \theta_{m_k}$, with $1 \leq m_1 < m_2 < ... < m_k \leq N$, $m_{k+1} := N + 1$. Then, in a manner similar to lemma 5-7, it can be shown that it is optimal to set $m_1 = 1$, and $p_1 > p_2 > ... > p_k$. The firm's product design problem can be formulated as follows:

$$\max_{p} \ \Sigma_{l=1}^{k} \ p_l \ \Sigma_{j=m_l}^{m_{l+1}-1} \ \Lambda_j \ \overline{F}_j \ (p_l) \tag{3.46}$$

$$s.t. \ \Sigma_{l=1}^{k} \ \Sigma_{j=m_l}^{m_{l+1}-1} \ \Lambda_i \ \overline{F}_i \ (p_l) \ c \leq C, \tag{3.47}$$

$$0 \leq p_k \leq p_{k-1} \leq ... \leq p_1, \tag{3.48}$$

$$p_j \leq \overline{v}_{m_{j+1}-1},\ j = 1, ..., k. \tag{3.49}$$

We will make the following assumptions, analogous to those made in Section 3.2, to solve for the optimal prices.

1. $\frac{\Sigma_{j=m_l}^{m_{l+1}-1}\ \overline{F}_j(v)\ \Lambda_j}{\Sigma_{j=m_l}^{m_{l+1}-1}\ f_j(v)\ \Lambda_j} < \frac{\Sigma_{j=m_{l+1}}^{m_{l+2}-1}\ \overline{F}_j(v)\ \Lambda_j}{\Sigma_{j=m_{l+1}}^{m_{l+2}-1}\ f_j(v)\ \Lambda_j}, \forall v \in [0, \overline{v}_{m_{l+1}}],\ 1 \leq l < k.$

2. (a) $p > p' \Rightarrow \frac{\Sigma_{j=m_l}^{m_{l+1}-1}\ f_j(p)\ \Lambda_j}{\Sigma_{j=m_l}^{m_{l+1}-1}\ \overline{F}_j(p)\ \Lambda_j} \geq \frac{\Sigma_{j=m_l}^{m_{l+1}-1}\ f_j(p')\ \Lambda_j}{\Sigma_{j=m_l}^{m_{l+1}-1}\ \overline{F}_j(p')\ \Lambda_j}$, or

   (b) $p > p' \Rightarrow \frac{\Sigma_{j=m_l}^{m_{l+1}-1}\ f_j(p)\ \Lambda_j}{\Sigma_{j=m_l}^{m_{l+1}-1}\ \overline{F}_j(p)\ \Lambda_j} < \frac{\Sigma_{j=m_l}^{m_{l+1}-1}\ f_j(p')\ \Lambda_j}{\Sigma_{j=m_l}^{m_{l+1}-1}\ \overline{F}_j(p')\ \Lambda_j}$ and $\frac{\Sigma_{j=m_l}^{m_{l+1}-1}\ f_j(p)\ \Lambda_j}{\Sigma_{j=m_l}^{m_{l+1}-1}\ \overline{F}_j(p)\ \Lambda_j} \geq K,\ K > 0$ constant, or

   (c) $p > p' \Rightarrow \frac{\Sigma_{j=m_l}^{m_{l+1}-1}\ f_j(p)\ \Lambda_j}{\Sigma_{j=m_l}^{m_{l+1}-1}\ \overline{F}_j(p)\ \Lambda_j} < \frac{\Sigma_{j=m_l}^{m_{l+1}-1}\ f_j(p')\ \Lambda_j}{\Sigma_{j=m_l}^{m_{l+1}-1}\ \overline{F}_j(p')\ \Lambda_j}$ and $\frac{\partial \frac{\Sigma_{j=m_l}^{m_{l+1}-1}\ f_j(p)\ \Lambda_j}{\Sigma_{j=m_l}^{m_{l+1}-1}\ \overline{F}_j(p)\ \Lambda_j}}{\partial p} < 1,\ j = 1, ..., k.$

3. $\lambda_i \overline{F}_i^{-1}\left(\frac{\lambda_i}{\Lambda_i}\right)$, $i = 1, ..., N$, is concave.

An example of a distribution that satisfies the above constraints is the exponential distribution with parameters $\alpha_1 < \alpha_2 < ... < \alpha_N$. Next, formulating the problem in terms of arrival rates, under the above assumptions, we obtain a concave maximization problem on a convex set, leading to the following characterization of optimal prices for the k-product problem.

Proposition 8 *The optimal prices are characterized by*

$$p_l = \frac{\Sigma_{j=m_l}^{m_{l+1}-1}\ \overline{F}_j(p_l)\ \Lambda_j}{\Sigma_{j=m_l}^{m_{l+1}-1}\ f_j(p_l)\ \Lambda_j} + \mu - \frac{\eta_l}{\Sigma_{j=m_l}^{m_{l+1}-1}\ f_j(p_l)\ \Lambda_j}, \tag{3.50}$$

$$\mu\ (C - \Sigma_{l=1}^{k}\ \Sigma_{j=m_l}^{m_{l+1}-1}\ \overline{F}_j(p_l)\ \Lambda_j) = 0, \tag{3.51}$$

$$\mu \geq 0, \quad C - \Sigma_{l=1}^{k}\ \Sigma_{j=m_l}^{m_{l+1}-1}\ \overline{F}_j(p_l)\ \Lambda_j\ \geq 0, \tag{3.52}$$

$$\eta_l\ (v_{i_l-1} - p_l) = 0, \quad \eta_l \geq 0, \quad v_{i_l-1} - p_l \geq 0. \tag{3.53}$$

Here μ is the Lagrange multiplier associated with the capacity constraint (3.47), and η_l is the Lagrange multiplier associated with the constraint (3.49). The monotonicity of prices in equation (3.48) is ensured by our assumptions about hazard rates.

We observe that while we can characterize the optimal prices given the qualities that the firm decides to offer, we still need to solve a combinatorial problem if we seek to identify the optimal k qualities to offer. Since $m_1 = 1$ following lemma 7, identifying the optimal k product solution requires solving $\binom{N-1}{k-1}$ problems. This can be computationally expensive for k large, however, solving the $k = 2$ problem requires solving $N - 1$ problems to identify m_2, and is hence easily done.

3.5 Applications: analysis

In light of the structural and computational results obtained above, we now revisit the examples that we presented in Section 3.3.

3.5.1 Queueing Example

A direct analysis of the problem (3.13)-(3.16), along with the constraints for system dynamics, involves a detailed treatment of the stochastic effects of the service system, and may be complex. Following the approach described in Maglaras and Zeevi [42, 43], one can first study a tractable deterministic relaxation of the problem that disregards the stochastic effects of the queueing system, and subsequently analyze the effect of stochastic fluctuation on the deterministic solution. Their key result is that in the context of systems with large processing capacity that serve large markets, the deterministic solution characterizes the "near-optimal" product design and the associated sequencing rule for the queueing facility. In common to Afeche [1], the analysis in Maglaras and Zeevi [42, 43] is limited to two customer types. While the framework on these references is more general and could extend to multiple customer classes, the details of the analysis become significantly more complex.[1] Katta and Sethuraman [33] study the N customer types problem under the simplifying assumption that all class i customers have the same valuation given by some constant R_i.

We adopt the approach of [42, 43], but for purposes of this chapter we will only analyze the analysis the deterministic relaxation of the SP's product design problem. The deterministic relaxation of the SP's problem is given exactly by equations (3.9)-(3.12), and hence following the results in Section 3.4, can be reformulated as (3.35)-(3.38). This formulation essentially removes the delay decision from the SP's problem, and as a result the above problem, which is in terms of the price vector alone, can be reformulated in terms of the demand rate vector and be readily solved, as in section 3.4.2. Hence, defining $\theta_0 = 0^-$, d_i can be chosen to be any element in the set $(\theta_{i-1}, \theta_i]$, while the optimal prices are given by equations (3.42)- (3.45). Furthermore, all the results in section 3.4 apply. In particular, the following hold.

- *Differentiation:* If capacity is ample, it is optimal to offer N products. Alternatively, if the support of customer valuations is unbounded, or common across classes, again it is optimal to differentiate and offer exactly N products. If capacity is scarce and customers with high delay thresholds have low maximum valuations, it may to only offer $k < N$ products targeting the classes with the k lowest k delay thresholds.

- Two products: If the SP decides to only offer two products, i.e., a low delay and a high delay product, then the two delay values $0 < d_1 < d_2 < \infty$ divide the market into three contiguous sets with delay thresholds $\theta_i \leq d_1$, $\theta_i \in (d_1, d_2]$ and $\theta_i > d_2$. The solution to the optimal two product design problem follows the simple procedure outlined in §3.4.3.

- *Capacity allocation rule:* The solution of the deterministic relaxation above specifies indirectly a candidate capacity allocation (or sequencing) policy that will strive to achieve the target

[1] First, the objective function, which is expressed in terms of the price and delay variables, need not be unimodal anymore. Second, the IC conditions need not describe a convex set even if the cost function is convex.

expected delay vector d. Roughly speaking this policy aims to maintain the various queue lengths at fixed relative ratios to each other that are proportional to the target d_i's for each respective product variant. This raises the question of whether a) the delay vector d is achievable and, if so, whether it is achieved via a rule of the form given above? and c) whether the capacity allocation is non-idling (and as a result tries to minimize some aggregate notion of delay)?

A set of partial answers to the above questions can be obtained if one focuses on an asymptotic analysis of the underlying system in setting where its processing capacity and potential market sizes grow proportionally large. a) if the solution of the deterministic relaxation utilizes all of the system's capacity, then the delay vector d is asymptotically achievable under the policy sketched above and the policy is non-idling; and b) if instead the solution of the deterministic relaxation does not utilize all of the system's capacity, then the delays that arise naturally due to the system's congestion effects (under any non-idling policy) are asymptotically negligible and the SP has to introduce tactical or preventable delay into the various product variants so as to inflate their associated delays and achieve the target vector d; this is the effect that was first recognized in a queueing context by Afeche [1].

- *Capacity investment:* If the capacity cost is linear or convex increasing, then the SP will choose a capacity that will be fully utilized in the solution of the associated deterministic relaxation. As a result, the optimal capacity allocation will be non-idling. The use of preventable delay (mentioned above) may still be justified if the system operates in a non-stationary environment whereat the market size fluctuates over time and the system fluctuates accordingly from being capacitated to being uncapacitated.

- *Real-time delay:* A similar analysis applies to the case where customers are sensitive to real-time delay as opposed to steady-state expected delay information.

3.5.2 Bandwidth Example

For solving problem (3.17)-(3.20), in addition to the assumptions in Section 3.2, we will assume that valuations v_i have support in $[0, \infty]$. In this setting, it is easier to verify that lemma 5-6 and their associated corollaries hold. For lemma 7 and 8, we need to slightly modify the proofs as follows.

Lemma 9 *It is always optimal to offer the highest capacity product.*

Proof Suppose that the highest quality product is being offered at θ_k and at price p_k, where $k > 1$ (else the lemma holds). Note that $p_k < \infty$. Also suppose that the next highest quality product was being offered at θ_m, $m > k$. (Set $m = N + 1$ if no other product is offered.) Consider introducing an additional product at capacity θ_1 and price p_1, $p_1 = p_k + \epsilon$, $\epsilon > 0$, $p_1 > p_k \frac{\theta_1}{\theta_k}$. Also, we increase the price of the k^{th} product to $p_k + \delta$, $\delta > 0$, δ small and such that $\Delta D = \theta_1 \Sigma_{l=1}^{k-1} \Lambda_l \overline{F}_l(p_k + \epsilon) + \theta_k \Sigma_{l=k}^{m-1} \Lambda_l [\overline{F}_l(p_k + \delta) - \overline{F}_l(p_k)] = 0$. Note that first term is positive and decreases as ϵ increases, while the second term is negative, and decreases as δ increases. So, there exist $\epsilon > 0$, $\delta > 0$, $p_k + \delta < \overline{v}_k$ and $p_k + \epsilon < \overline{v}_1$ such that $\Delta D = 0$. Now notice that the demand is unchanged, while the cost per unit capacity for products sold to types $\leq k$ has increased from $\frac{p_k}{\theta_k}$ to at least $\min(\frac{p_1}{\theta_1}, \frac{p_k+\delta}{\theta_k})$. Hence the total revenues increases via the introduction of this product at θ_1. $\square$

Note the result here is slightly different from that we obtained earlier in lemma 7. In particular, we can no longer say that the optimal single product offering involves selling the highest capacity product. However, adding the highest capacity product to the existing product offering certainly increases revenues. Hence in the optimal product menu unconstrained by the number of products that are offered, the highest capacity product will always be offered.

Lemma 10 *Suppose the firm offers products at capacities θ_m and θ_n, where $m + 1 < n$. Then it is optimal for the firm to offer products at capacity θ_l, $m + 1 \leq l \leq n$.*

Proof As in lemma 8, consider two indices m, n where $m+1 < n$, such that products are offered at θ_m and θ_n, but none in between. Note that $p_m < \infty$. Consider adding a product at θ_{m+1} and price $p_m - \delta$ while increasing the price of the product with capacity θ_m to $p_m + \epsilon$. Then, $\Delta D = \Sigma_{l=m+1}^{n-1} \Lambda_l \overline{F}_l(p_m - \delta)\theta_{m+1} + \Lambda_m \overline{F}_m(p_m + \epsilon)\theta_m - \Sigma_{l=m}^{n-1} \Lambda_l \overline{F}_l(p_m)\theta_m = -\epsilon\theta_m \Lambda_m f_m(p_m) + \delta\theta_{m+1}\Sigma_{l=m+1}^{n-1}\Lambda_l f_l(p_m) + (\theta_{m+1} - \theta_m)\Sigma_{l=m+1}^{n-1}\Lambda_l \overline{F}_l(p_m) + o(\epsilon) + o(\delta)$. Note $\theta_{m+1} < \theta_m$ and so we can choose δ small enough so that $\Delta D < 0$. Similarly, $\Delta R = \Sigma_{l=m+1}^{n-1}\Lambda_l \overline{F}_l(p_m - \delta)(p_m - \delta) + \Lambda_m \overline{F}_m(p_m + \epsilon)(p_m + \epsilon) - \Sigma_{l=m}^{n-1}\Lambda_l \overline{F}_l(p_m)p_m = \epsilon\Lambda_m \overline{F}_m(p_m) - \epsilon\Lambda_m p_m f_m(p_m) + \Sigma_{l=m+1}^{n-1}\Lambda_l \delta[-\overline{F}_l(p_m) + p_m f_l(p_m)] + o(\epsilon) + o(\delta)$. Hence $\Delta R > 0$ if
$\epsilon\Lambda_m \overline{F}_m(p_m)(1 - \eta_m(p_m)) > \delta\Sigma_{l=m+1}^{n-1}\Lambda_l \overline{F}_l(p_m)(1 - \eta_l(p_m))$ and ϵ, δ are small enough so that the first order terms in the Taylor expansion dominate.

Now from our assumption on elasticities $\eta_m < \eta_l, m \le l \le n$. Define $A = \Lambda_m \overline{F}_m(p_m)(1 - \eta_m(p_m))$, and $B = \Sigma_{l=m+1}^{n-1}\Lambda_l \overline{F}_l(p_m)(1 - \eta_l(p_m))$. Then there are three possibilities,

i) $A > 0, B > 0$: Choose δ small (compared to ϵ), $\implies \Delta R > 0$.

ii) $A > 0, B < 0$: $\implies \Delta R > 0$.

iii) $A < 0, B < 0$: Choose ϵ small (compared to δ), $\implies \Delta R > 0$.

In each of the cases, $\Delta R > 0$ while $\Delta D < 0$. Hence introducing a product at θ_{m+1} increases revenues. $\square$

Together lemma 9-10 imply that if θ_k is the lowest capacity that is offered by the firm, then it is optimal to offer products with capacities $\theta_1, ..., \theta_{k-1}$. The following corollary shows that in fact it is optimal to offer all N products at capacities $\theta_1, ..., \theta_N$.

Corollary 9 *It is optimal to offer N products.*

Proof In the proof above, substituting $n = N+1$, and introducing a dummy product with $\theta_{N+1} = 0$, $p_{N+1} = 0$ does not affect line of argument. Hence we conclude that if only first k products are being offered, introducing a product at θ_{k+1} also increases revenues. Applying this argument iteratively, we conclude that it is optimal to offer exactly N products. $\square$ Following corollary 9, we know that the optimal number of products as well as the optimal capacity level associated with each of them. Hence the service provider's revenue maximization problem can be reformulated as follows.

$$\max_{p} \ \Sigma_{i=1}^{N} \ p_i \ \Lambda_i \ \overline{F}_i \ (p_i) \tag{3.54}$$

$$s.t. \ \Sigma_{i=1}^{N} \ \Lambda_i \ \overline{F}_i \ (p_i) \ \theta_i \leq C, \tag{3.55}$$

$$0 \leq p_N < p_{N-1} < ... < p_1 < \infty. \tag{3.56}$$

We can solve the firm's revenue-maximization problem (3.54)-(3.56) by reformulating it in terms of arrival rates, wherein we obtain a concave maximization problem over a polyhedron. The first-order conditions lead to the following characterization of the optimal prices.

Lemma 11 *The optimal prices are given by* $p_i = \frac{\overline{F}_i(p_i)}{f_i(p_i)} + \mu\theta_i - \frac{\eta_i}{f_i(p_i)}, \ i = 1, ..., N,$ *where* μ *is the Lagrange multiplier associated with the capacity constraint.*

The assumption on elasticities and monotonicity of hazard rate implies that prices are non-decreasing in capacity of the product (even when $\mu = 0$, i.e., surplus capacity). Finally, the $k < N$ products problem can also be solved in a similar fashion, though solving it now requires $\binom{N}{k}$ effort.

3.5.3 Rationing Example

For formulation (3.22)-(3.25), one can again verify that lemma 5-6 and their associated corollaries hold. Similarly, it can be shown that $r_1^* = 1$, a result in the spirit of lemma 7. As a consequence, problem (3.22)- (3.25) can be formulated as follows.

$$\max_{p,r} \ \Sigma_{i=1}^{N} \ p_i \ \Lambda_i \ \overline{F}_i \ (p_i) \ r_i \tag{3.57}$$

$$s.t. \ \Sigma_{i=1}^{N} \ \Lambda_i \ \overline{F}_i \ (p_i) \ r_i \leq C, \tag{3.58}$$

$$0 \leq p_N \leq p_{N-1} \leq ... \leq p_1 < \infty, \tag{3.59}$$

$$0 \leq r_N \leq r_{N-1} \leq ... \leq r_1 = 1. \tag{3.60}$$

However, problem (3.57)- (3.60) is hard to solve because quality enters the objective (3.57) and the constraint (3.58) in a multiplicative fashion. Some special cases can be easily solved though. The N product problem is easily solved by observing that $r_1^* = 1$, $r_i^* = \theta_{i-1}, i = 2, ..., N$. Then fixing these optimal values, the above problem (3.57)-(3.60) is exactly the same as the problem (3.35)-(3.38) discussed above (with Λ_i by $\Lambda_i r_i^*$), and hence all the structural and computational results discussed above apply. Similarly, the two product problem is easily solved. It is also easy to solve the problem where the firm fixes the fill-rates to be offered exogenously. For example, suppose the firm decides to fix fill-rates to be $r_1 > r_2 > ... > r_k$, $k \leq N$. Note that it is optimal to set $r_1 = 1$, and $r_i \in \{\theta_1, ..., \theta_N\}, 2 \leq i \leq k$. So, suppose $r_j = \theta_{i_j}, 1 \leq j \leq k$. Following corollary 5 and lemma 6, this partitions the customer types into contiguous sets, so that it becomes equivalent to solving the k product problem.

3.5.4 Information good example

As observed in Section 3.3.4, the firm's revenue maximization problem can be formulated as equations (3.9), (3.11)-(3.12), and therefore all results in sections 3.4.1-3.4.2 apply. In particular, it is possible to solve for the optimal product menu.

3.5.5 Time-sensitive customers

As observed in Section 3.3.5, the firm's revenue maximization problem can be formulated as equations (3.9)-(3.12), and therefore all results in sections 3.4.1-3.4.2 apply. In particular, it is possible to solve for the optimal product menu.

3.5.6 Seller of mp3 players

To solve problem (3.26)-(3.29), in addition to the assumptions in Section 3.2, we will assume that $s_1 > s_2 > ... > s_N > 0$, implying that the marginal cost increases with respect to quality. We still assume the fixed costs to be given and constant. Define indices $i(1), ..., i(N)$ such that $\overline{v}_{i(1)} - s_{i(1)} \geq \overline{v}_{i(2)} - s_{i(2)} \geq ... \geq \overline{v}_{i(N)} - s_{i(N)}$. Notice that lemma 5 and 6, and corollary 5 continue to hold. Also, we have the following lemma.

Lemma 12 *The following hold:*

i) Quality $i(1)$ is always offered.

ii) If quality $i(l)$ is offered, then qualities $i(1), i(2), ..., i(l-1)$ are also offered. In particular, there are no holes when classes are relabeled according to the $i(\cdot)$ indices.

iii) If class l buys any product, then it is optimal to offer quality θ_l.

Proof i) Observe that $i(1) = \arg\max_l v_l - s_l$ and hence leads to the highest profit per unit capacity. Hence it must be optimal to service this class.

ii) Again, $v_{i(1)} - s_{i(1)} \geq v_{i(2)} - s_{i(2)} \geq ... \geq v_{i(l)} - s_{i(l)}$. Hence if any of classes $i(1), ..., i(l-1)$ are not served, then profits can be increased by serving them since they lead to higher profit per unit capacity.

iii) Consider the highest class l that buys a product that is not offered at its quality threshold θ_l. Then it buys a product at quality θ_m, where $m < l$. This implies that $s_m > s_l$. Suppose the firm offers a new product at quality θ_l and price $p_l = p_m - \epsilon$, $\epsilon < s_m - s_l$, and raises the price of product offered at θ_m to $p_m + \delta$. Note that any classes $u > l$ that might be buying product m earlier now switch to product m. Denote the set of these classes by $\mathcal{U}$. Also, $s_u < s_l < s_m$. The change in demand $\Delta D = \Sigma_{u \in \mathcal{U}} \Lambda_u \overline{F}_u(p_m - \epsilon) + \Lambda_m \overline{F}(p_m + \delta) - \Sigma_{u \in \mathcal{U}} \Lambda_u \overline{F}_u(p_m) - \Lambda_m \overline{F}(p_m)$. Using the first order Taylor expansion, $\Delta D \sim \epsilon \Sigma_{u \in \mathcal{U}} \Lambda_u f_u(p_m) - \delta \Lambda_m f_m(p_m)$. Similarly, the change in profit $\Delta P = \Sigma_{u \in \mathcal{U}} \Lambda_u \overline{F}_u(p_m - \epsilon)(p_m - \epsilon - s_l) + \Lambda_m \overline{F}(p_m + \delta)(p_m + \delta - s_m) - \Sigma_{u \in \mathcal{U}} \Lambda_u \overline{F}_u(p_m)(p_m - s_m) - \Lambda_m \overline{F}(p_m)(p_m - s_m)$, and $\Delta P \sim \Sigma_{u \in \mathcal{U}} \Lambda_u \overline{F}_u(p_m)(s_m - s_l - \epsilon) - \Lambda_m \overline{F}_m(p_m)\delta + \Sigma_{u \in \mathcal{U}} \Lambda_u f_u(p_m)(p_m -$

$s_l)\epsilon - \Lambda_m f_m(p_m)(p_m - s_m)\delta$. Set $\epsilon = \frac{\gamma\delta\Lambda_m f_m(p_m)}{\Sigma_{uin\mathcal{U}}\Lambda_u f_u(p_m)}, 0 < \gamma < 1$, so that $\Delta D < 0$ and $\Delta P = \Sigma_{uin\mathcal{U}}\Lambda_u \overline{F}_u(p_m)(s_m - s_l - \epsilon) + \epsilon\Sigma_{u\in\mathcal{U}}\Lambda_u f_u(p_m)(p_m - s_l)\delta\overline{F}_m(p_m)\Lambda_m - (p_m - s_m)\frac{\epsilon}{\gamma}\Sigma_{u\in\mathcal{U}}\Lambda_u f_u(p_m)$. Now choosing γ such that $\epsilon(p_m - s_l) > \frac{1}{\gamma}(p_m - s_m)$ i.e., $\gamma > \frac{p_m - s_m}{p_m - s_l}(< 1)$ which is feasible leads to $\Delta P > 0$ as well. $\square$ As a result of increasing marginal cost with quality, it might no longer be optimal to offer the highest quality product. Lemma 12 leads to the following formulation of the general problem.

$$\max \ \Sigma_{i=1}^{N} \ (p_i - s_i) \ \Lambda_i \ \overline{F}_i(p_i) \tag{3.61}$$

$$s.t. \ \Sigma_{i=1}^{N} \ \Lambda_i \ \overline{F}_i(p_i) \le C, \tag{3.62}$$

$$p_1 \ge p_2 \ge ... \ge p_N, \tag{3.63}$$

$$s_i \le p_i \le \overline{v}_i, \ \ i = 1, ..., N. \tag{3.64}$$

As before, we can drop the monotonicity constraint (3.63) and formulate the problem (3.61)-(3.64) in terms of arrival rates, wherein we obtains concave maximization problem over a convex set. This leads to the following characterization of optimal prices.

Proposition 9 *The optimal solution to (3.61)-(3.64) is given by:*

$$p_i = \frac{\overline{F}_i(p_i)}{f_i(p_i)} + s_i + \mu - \eta_i, \quad (C - \Sigma_{i=1}^{N}\Lambda_i\overline{F}_i(p_i))\mu = 0, \tag{3.65}$$

$$\Sigma_{i=1}^{N}\Lambda_i\overline{F}_i(p_i) \le C, \quad \mu \ge 0, \quad \eta_i(v_i - p_i) = 0, \quad \eta_i \ge 0, \quad v_i - p_i \ge 0. \tag{3.66}$$

We first note that given our assumptions about elasticities and hazard rates, the optimal prices are indeed monotonic, since $s_i \downarrow i$. We also note that if s_i is high enough (so that $p_i \ge v_i$), then class i customers would not buy any product. This is in contrast to our earlier result without marginal costs where some high class customers always purchased a product if any lower class customers did. This also makes the $k < N$ product problem harder, since now the number of possible combinations to be searched over becomes $\binom{N}{k}$ instead of $\binom{N-1}{k-1}$.

3.5.7 Postal service provider

In this setting, it is easy to verify that lemma 5-8 and their associated corollaries continue to hold. Then the firm's revenue maximization problem (3.30)-(3.33) can be written as follows:

$$\max_{p} \ \Sigma_{i=1}^{N} \ p_i \ \Lambda_i \ \overline{F}_i \ (p_i) \tag{3.67}$$

$$s.t. \ \Sigma_{i=1}^{i_l} \ \Lambda_i \ \overline{F}_i \ (p_i) \leq C_l, \quad l = 1, ..., m \tag{3.68}$$

$$p_N \leq p_{N-1} \leq ... \leq p_1, \quad i = 1, 2, ..., N, \tag{3.69}$$

$$p_i \leq \overline{v}_i, \quad i = 1, 2, ..., N. \tag{3.70}$$

As before, dropping the monotonicity constraint, and reformulating the problem in arrival-rate domain, we obtain a concave-maximization problem over a convex set. This leads to the following characterization of the optimal solution.

Lemma 13 *The optimal solution to (3.61)-(3.64) is given by:*

$$p_i = \frac{\overline{F}_i(p_i)}{f_i(p_i)} + \Sigma_{j=u(i)}^{l} \ \mu_j - \frac{\eta_i}{f_i(p_i)}, \tag{3.71}$$

$$(C_l - \Sigma_{i=1}^{i_l} \ \Lambda_i \ \overline{F}_i(p_i)) \ \mu_l = 0, \ l = 1, ..., m \tag{3.72}$$

$$\Sigma_{i=1}^{i_l} \Lambda_i \ \overline{F}_i(p_i) \leq C_l, \ \mu_l \geq 0, \ l = 1, ..., m, \tag{3.73}$$

$$\eta_i \ (v_i - p_i) = 0, \ \eta_i \geq 0, \ v_i - p_i \geq 0. \tag{3.74}$$

where $u(i) = \min_v i_v \geq i$.

Unlike the general model, the one-product solution does not necessarily involve offering the highest quality (e.g., when capacity C_1 is very small). As a consequence, solving the k product problem, requires $\binom{N}{k}$ work.

3.6 Extensions

We now discuss some extensions to the basic model presented in section 3.2, allowing for heterogeneous service times, multiple quality attributes, and the setting of a duopoly.

3.6.1 Heterogeneous Service Times

We now relax the assumption on homogeneous customer service requirements. We will assume that there are N types, type i having service requirement t_i. Without loss of generality, we will assume that $t_1 < t_2 < ... < t_N$. Further, we will assume that type i comprises of multiple classes differing in their quality threshold levels. Each type here corresponds to a collection of multiple classes as in section 3.2.

Within each of the N types, we refer to customers having quality threshold θ_j as type i, class j customers. Different types need not have customer classes with the same quality thresholds. We denote the number of classes of type i as N_i, and the corresponding quality thresholds as $\theta_{i,1}, ..., \theta_{i,N_i}$, with $\theta_{i,1} > \theta_{i,2} > ... > \theta_{i,N_i}$. We assume that these thresholds lie in the set $\mathbb{L}$ of qualities, though except their ordering, how these quality thresholds map onto this set is not important in the subsequent analysis. All assumptions made in section 3.2 with respect to the valuation distributions are assumed to hold for classes within a customer type. No ordering is assumed across types.

Since there are N different service requirements, the firm potentially offers products with service times $t_1, t_2, ..., t_N$. We denote the product with service time t_i as type i product. Our standing assumption in this subsection is that the firm can observe the actual service time of the customer and imposes a prohibitive penalty to customers that invaricate about their service requirement. Hence it is optimal for customers to report their true service times.

So, customer type i, subtype j chooses the variant of product i given by

$$\chi_{i,j}(p,q) = \begin{cases} \operatorname{argmin} p_{i,l}, & \exists\ q_{i,l} \geq \theta_{i,j}, \\ 0, & \text{otherwise}, \end{cases}$$

where $p_{i,l}$ and $q_{i,l}$ denote respectively the price and the quality of the l^{th} product offered at service time c_i.

Suppose the firm offers M_i products at service time c_i. Then, given the above customer choice behavior, the firm's revenue optimization problem can be formulated as follows.

$$\max_{p,q,M} \ \Sigma_{i=1}^{N} \Sigma_{j=1}^{N_i} \ \Sigma_{l=1}^{M_i} \ p_{i,l} \ \Lambda_{ij} \ \overline{F}_{i,j} \ (p_{i,l}) \ 1_{\{\chi_{i,j}(p,q)=l\}} \tag{3.75}$$

$$s.t. \ \Sigma_{i=1}^{N} \Sigma_{j=1}^{N_i} \ \Sigma_{l=1}^{M_i} \ \Lambda_{i,j} \ \overline{F}_{i,j} \ (p_{i,l}) \ t_i \ 1_{\{\chi_{i,j}(p,q)=l\}} \leq C, \tag{3.76}$$

$$0 \leq p < \infty, \quad 0 \leq q < \infty, \tag{3.77}$$

$$1 \leq M_i < \infty, \ M_i \ integer, i = 1, ..., N. \tag{3.78}$$

The objective in equation (3.75) is the sum of revenues across the N types, wherein for class i, the revenue obtained is the sum of revenues across the N_i classes. Equation (3.76) represents the capacity constraint, while equation (3.77) enforces the non-negativity and finiteness of prices and quality levels. Equation (3.78) ensures that a finite, positive and integral number of products are offered. The optimization decisions for the firm here involve deciding upon the number of products, M_i, to offer at each service time c_i, and the price $p_{i,l}$, and quality level $q_{i,l}$ for each product.

Since type i customers always choose type i product, this problem decomposes into N simpler problems linked via the common capacity constraint. Hence, for each type, all the results discussed in lemmas 5-8, proposition 6 and their corollaries hold. Moreover, proceeding in a manner similar to in propositions 7 and 8, we have the following characterization of the optimal prices for (3.75)-(3.78), and the corresponding k-product problem.

Proposition 10 *The optimal prices can be characterized using the following set of equations,*

$$p_{k,i} = \frac{\overline{F}_{k,i}(p_{k,i})}{f_{k,i}(p_{k,i})} + \mu t_k - \frac{\eta_{k,i}}{f_{k,i}(p_{k,i})},$$
$$\mu \ \left(C - \Sigma_{k=1}^{N} \Sigma_{i=1}^{N_k} \ \Lambda_{k,i} \ \overline{F}_{k,i}(p_{k,i}) \ t_k\right) = 0,$$
$$\mu \geq 0, \quad C - \Sigma_{k=1}^{N} \Sigma_{i=1}^{N_k} \ \Lambda_{k,i} \ \overline{F}_{k,i}(p_{k,i}) \ t_k \geq 0,$$
$$\eta_{k,i} \ (v_{k,i} - p_{k,i}) = 0, \quad \eta_{k,i} \geq 0, \quad v_{k,i} - p_{k,i} \geq 0,$$

where μ is the Lagrange multiplier associated with the capacity constraint (3.76), and $\eta_{k,i}$ is the Lagrange multiplier associated with the constraint $p_{k,i} \leq v_{k,i}$.

Proposition 11 *: Suppose $k_i < N_i$ products are offered to type i at $\theta_{i,m_1}, \theta_{i,m_2}, ..., \theta_{i,m_k}$, with $m_1 < m_2 < ... < m_k \leq N$, $m_1 = 1, m_{k+1} := N+1$. Then optimal prices are characterized by*

$$p_{i,l} = \frac{\Sigma_{j=m_l}^{m_{l+1}-1} \ \overline{F}_{i,j}(p_{i,l}) \ \Lambda_{i,l}}{\Sigma_{j=m_l}^{m_{l+1}-1} \ f_{i,j}(p_{i,l}) \ \Lambda_{i,l}} + \mu t_i - \frac{\eta_{i,l}}{\Sigma_{j=m_l}^{m_{l+1}-1} \ f_{i,j}(p_{i,l}) \ \Lambda_{i,l}},$$
$$\mu \ \left(C - \Sigma_{l=1}^{k} \ \Sigma_{j=m_l}^{m_{l+1}-1} \ \overline{F}_{i,j}(p_{i,l}) \ \Lambda_{i,l} \ t_i\right) = 0,$$
$$\mu \geq 0, \quad C - \Sigma_{l=1}^{k} \ \Sigma_{j=m_l}^{m_{l+1}-1} \ \overline{F}_{i,j}(p_{i,l}) \ \Lambda_{i,l} \ t_i \geq 0,$$
$$\eta_{i,l} \ (v_{i,m_{l+1}-1} - p_l) = 0, \quad \eta_{i,l} \geq 0, \quad v_{i,m_{l+1}-1} - p_l \geq 0,$$

where μ is the Lagrange multiplier associated with the capacity constraint (3.76), and $\eta_{i,l}$ is the Lagrange multiplier associated with the constraint $p_l \leq v_{i,m_{l+1}-1}$.

3.6.2 Multiple quality attributes

Our results extend naturally to the case where customers have threshold preferences with respect to more than one quality attribute. For simplicity, we discuss the two attribute case here. The analysis extends to more than two quality attributes with some additional notation. Let us denote the two quality attributes using θ_i and α_j, $i = 1, 2, ..., N_1$, $j = 1, 2, ..., N_2$. Without loss of generality, let us assume that $\theta_1 > \theta_2 > ... > \theta_{N_1}$, $\alpha_1 > \alpha_2 > ... > \alpha_{N_2}$, with higher values again denoting a desire for higher qualities. Let us denote a customer class having θ_i and α_j as its quality thresholds as class (i, j). Suppose the firm offers M products, product l having price p_l, and quality attributes, q_l^1 and q_l^2, respectively. Then, the customer choice behavior of class (i, j) customers is given by the following:

$$\chi_{i,j}(p, q^1, q^2) = \begin{cases} \min_l p_l, & q_l^1 \geq \theta_i,\ q_l^2 \geq \alpha_j, \\ 0, & \text{otherwise.} \end{cases} \tag{3.79}$$

Analogous to assumptions 1-3 in section 3.2, assume that $v_{1,j} > v_{2,j} > ... > v_{N,j}, \forall j$ and $v_{i,1} > v_{i,2} > ... > v_{i,M}, \forall i$. Also assume that $\frac{f_{i,1}(v)}{\overline{F}_{i,1}(v)} < \frac{f_{i,2}(v)}{\overline{F}_{i,2}(v)} < ... < \frac{f_{i,M}(v)}{\overline{F}_{i,M}(v)}, \forall i$, $\frac{f_{1,j}(v)}{\overline{F}_{1,j}(v)} < \frac{f_{2,j}(v)}{\overline{F}_{2,j}(v)} < ... < \frac{f_{N,j}(v)}{\overline{F}_{N,j}(v)}, \forall j$, hazard rates $\frac{f_{i,j}(p)}{\overline{F}_{i,j}(p)}$ are monotonic, and $\lambda_{i,j}\overline{F}_{i,j}(\frac{\lambda_{i,j}}{\Lambda_{i,j}})$ is concave. Then results in lemma 5-8, proposition 7 and their associated corollaries can be extended in a straightforward manner. As in proposition 8, the k product problem can also be addressed, though the problem complexity increases significantly now (there are $\binom{N_1 N_2}{k}$ ways to choose k product quality combinations).

3.6.3 Duopoly

We next consider the case where two firms compete in a market where customers have threshold preferences. We will make the same assumptions regarding the market as in Section 3.2. As in Moorthy [45], Shaked and Sutton [55], and Wauthy [68], we restrict attention to the case where each firm can offer only a single product. We examine the cases of simultaneous and sequential entry in order. In a manner similar to Moorthy [45] and Shaked and Sutton [55], we study a two-stage non-cooperative game. In the first stage, firms choose the quality level at which they seek to offer a product. In the second stage, given their and the competitor's quality, firms choose the prices at which to sell their product at. As in the above papers, we investigate the set of perfect Nash equilibria.

We begin by analyzing the second stage of the game, the price equilibrium. We first note that the two firms won't offer the same quality, else it will lead to a Bertrand game wherein profits would be zero. Hence we assume that firm 1 offers quality θ_i and firm 2 offers quality θ_j, $i < j$. Then following lemma 5, $p_i > p_j$ for two products to be offered. Then, the optimization problem for the firm offering quality θ_i can be written as follows:

$$\max_{p_i}\{p_i \ \Sigma_{l=i}^{j-1} \ \overline{F}_l(p_i) \ \Lambda_l \mid \Sigma_{l=i}^{j-1} \ \overline{F}_l(p_i) \ \Lambda_l \leq C, \ p_i \geq 0\}. \tag{3.80}$$

Define p_i^* to be the optimal price in equation (3.80). The optimization problem for firm offering quality θ_j can be written as follows:

$$\max_{p_j}\{p_j \ \Sigma_{l=j}^{N} \ \overline{F}_l(p_j) \ \Lambda_l \mid \Sigma_{l=j}^{N} \ \overline{F}_l(p_j) \ \Lambda_l \leq C, \ p_i^* > p_j \geq 0\}. \tag{3.81}$$

Then we have the following result.

Lemma 14 *Equations (3.80)-(3.81) define a Nash equilibrium in prices (given fixed qualities).*

Proof Observe that firm 1 has no incentive to change its price, since given the quality θ_i of its product, this is the optimal price for it to charge subject to its capacity. Firm 2 needs to offer a lower price than firm 1 to be able to generate non-zero revenues, since $\theta_i > \theta_j$. Hence, given its quality θ_j and capacity C, p_2^* is the optimal price for firm 2 to charge. Finally note that the resulting customer choice behavior is consistent with the formulation in equations (3.80) and (3.81). □

Next, we compute the product equilibrium. We have the following proposition for the simultaneous entry case.

Proposition 12 *The only possible product equilibrium occurs at i = 1, j = 2 and under the con-*

dition that

$$\begin{aligned} &\max_{p_1}\{p_1\ \overline{F}_1(p_1)\ \Lambda_1 \mid \overline{F}_1(p_1)\ \Lambda_1 \le C\} \ge \\ &\max_{p_3}\{p_3\ \Sigma_{l=3}^N\ \overline{F}_l(p_3)\ \Lambda_l \mid \Sigma_{l=3}^N\ \overline{F}_l(p_3)\ \Lambda_l \le C, p_2^* > p_3 \ge 0\}, \end{aligned} \tag{3.82}$$

where $p_2^* = \arg\max_{p_2}\{p_2\ \overline{F}_2(p_2)\ \Lambda_2 \mid \overline{F}_2(p_2)\ \Lambda_2 \le C\}$.

Proof Suppose the product equilibrium occurs at $1 < i < j \le N$. Then given choice of quality θ_j by a firm, its competitor until then offering quality θ_i, will find it advantageous to offer quality θ_1, for it increases revenues when the price equilibrium with product qualities fixed is considered. Hence, in the Nash equilibrium, $i = 1$. Next consider the case where $j > 1$. in this case, given that its competitor chooses to offer quality θ_1, a firm would find its revenues increased if it offers quality θ_2 instead of θ_j, given the price equilibrium that would occur with these qualities. Hence $j = 2$ in the Nash equilibrium. Next we consider if $i = 1, j = 2$ constitutes a Nash equilibrium. Clearly, firm offering θ_2 does not have an incentive to deviate. As for the firm offering θ_1, the best alternative is to offer quality θ_3 instead. This happens only if $\max_{p_1}\{p_1\ \overline{F}_1(p_1)\ \Lambda_1 \mid \overline{F}_1(p_1)\ \Lambda_1 \le C\} < \max_{p_3}\{p_3\ \Sigma_{l=3}^N\ \overline{F}_l(p_3)\ \Lambda_l \mid \Sigma_{l=3}^N\ \overline{F}_l(p_3)\ \Lambda_l \le C, p_2^* > p_3 \ge 0\}$, where
$p_2^* = \arg\max_{p_2}\{p_2\ \overline{F}_2(p_2)\ \Lambda_2 \mid \overline{F}_2(p_2)\ \Lambda_2 \le C\}$. $\square$

Let us define the following to proceed with the sequential entry case.

$$R^1(l,p) = \{p\ \overline{F}_l(p)\ \Lambda_l \mid \overline{F}_l(p)\ \Lambda_l \le C\}, \tag{3.83}$$

$$R^1(l) = \max R^1(l,p), \quad p_1^l = \arg\max R^1(l,p), \tag{3.84}$$

$$\overline{R}^1(l,p) = \{\Sigma_{u=l+1}^N\ p\ \overline{F}_u(p)\ \Lambda_u \mid \Sigma_{u=l+1}^N\ \overline{F}_u(p)\ \Lambda_u \le C, \quad p < p_1^l\}, \tag{3.85}$$

$$\overline{R}^1(l) = \max \overline{R}^1(l,p), \quad \overline{p}_1^l = \arg\max \overline{R}^1(l,p), \tag{3.86}$$

$$R^2(l,p) = \{\Sigma_{u=l}^N\ p\ \overline{F}_u(p)\ \Lambda_u \mid \Sigma_{u=l}^N\ \overline{F}_u(p)\ \Lambda_u \le C, \quad p < \overline{p}_2^l\}, \tag{3.87}$$

$$R^2(l) = \max R^2(l,p), \quad p_2^l = \arg\max R^2(l,p), \tag{3.88}$$

$$\overline{R}^2(l,p) = \{\Sigma_{u=1}^{l-1}\ p\ \overline{F}_u(p)\ \Lambda_u \mid \Sigma_{u=1}^{l-1}\ \overline{F}_u(p)\ \Lambda_u \leq C\}, \tag{3.89}$$

$$\overline{R}^2(l) = \max \overline{R}^2(l,p), \quad \overline{p}_2^l = \arg\max \overline{R}^2(l,p). \tag{3.90}$$

$R^1(l,p)$ denotes the revenue achieved by firm 1, if it offers quality θ_l at price p and firm 2 decides to offer quality θ_{l+1}. $R^1(l)$ is the optimal revenue achieved in this case, and p_1^l denotes the revenue-maximizing price. $\overline{R}^1(l,p)$ denotes the revenue achieved by firm 2, if firm 1 offers quality θ_l at price p_1^l, and firm 2 offers quality θ_{l+1} at price p. $\overline{R}^1(l)$ denotes the optimal revenue achieved in this case, and $\overline{p}_1^l$ denotes the corresponding revenue-maximizing price. $R^2(l,p)$ denotes the revenue achieved by firm 1, if it offers quality l at price $p < \overline{p}_2^l$ and firm 2 decides to offer quality θ_1 at price $\overline{p}_2^l$. $R^2(l)$ is the optimal revenue achieved in this case, and p_2^l denotes the revenue-maximizing price. $\overline{R}^2(l,p)$ denotes the revenue achieved by firm 2, if firm 1 offers quality θ_l, and firm 2 offers quality θ_1 at price p. $\overline{R}^2(l)$ denotes the optimal revenue achieved in this case, and $\overline{p}_2^l$ denotes the corresponding revenue-maximizing price. We have the following proposition characterizing the optimal qualities to offer.

Proposition 13 *The first entrant chooses to offer quality*

$$i = \arg \max_{l=1,2,\ldots,N} R(l), \tag{3.91}$$

$$R(l) = \begin{cases} R^1(l), & \text{if } \overline{R}^1 \geq \overline{R}^2, \\ R^2(l) & \text{otherwise.} \end{cases} \tag{3.92}$$

The quality chosen by the second entrant then is θ_1 if $\overline{R}_i^1 < \overline{R}_i^2$, and θ_{i+1} otherwise.

Proof Since firm 1 chooses quality first, and with the knowledge that firm 2 will subsequently choose the optimal quality to offer following firm 1's choice, there are two situations to consider. Given firm 1's choice of quality θ_l, firm 2 would either offer a better quality, in which case it is optimal for firm 2 to offer θ_1, or it will offer a worse quality, in which case it is optimal for firm 2 to offer θ_{l+1}. The revenues resulting for firm 2 in these two situations are denoted by $\overline{R}^1(l)$ and $\overline{R}^2(l)$ for firm 2, respectively. Firm 2 chooses quality θ_1 if $\overline{R}^1(l) \geq \overline{R}^2(l)$, in which case, the revenue achieved by firm 1 is given by $R^1(l)$ in equilibrium. If $\overline{R}^1(l) < \overline{R}^2(l)$, then firm 2 chooses quality θ_{l+1}, and consequently, firm 1 obtains $R^2(l)$ in revenue in equilibrium. This leads to equation (3.92). Given the optimal revenue achievable by firm 1 if it offers quality θ_1 to the market, firm 1 then optimizes over qualities $\theta_l,\ l = 1, ..., N$ to identify the optimal quality to offer, as summarized by equation (3.91). □

Comparing with two-product monopoly solution, we observe that while in the simultaneous case the two best quality products are offered if an equilibrium exists, in the sequential entry case, neither of the two best qualities may be offered. This is in contrast with the optimal two product solution of a firm, where the first product is always offered at the best quality, while the quality of the second product depends upon the problem parameters.

3.7 Interpretation of results

The analysis in Sections 3.4-3.6 leads to several interesting results. Some of the results, e.g., those about monotonicity of prices, contiguous partitioning of customer classes and the optimality of offering the highest quality product, are similar to ones that would be obtained under the classic model of customer choice behavior. However, several of the results are different as well. First, in the threshold model, all customers belonging to the same class prefer the same product, irrespective of their valuations. Hence a maximum of one product is offered to each class in this model, notwithstanding how valuations in this class are distributed. This is different from the traditional model where the optimal number of products to offer depends on the distribution of valuations, for example, as in Bhargava [10] and Dana [2]. Second, in our model, differentiation is driven by differences in elasticity of the customer types. In the traditional model, differentiation is driven by both valuation distribution and sensitivity to the quality attribute, where different sensitivities correspond to different customer classes. For example, Akshay and Sethuraman [33] discuss this problem in a queueing context where customer valuations are discrete and where customers that have higher delay sensitivities have larger valuation to sensitivity ratio. Mussa [46] considers a problem where valuation is fixed but both sensitivity to quality and the quality itself are continuous. Finally, the elasticity condition results in a monotonic ordering of prices with respect to quality, and along with the threshold preferences, completely separates customer classes so that no Incentive Compatibility (IC) conditions need to be imposed. This simplifies the problem significantly. In the traditional model of customer preferences, typically IC conditions need to be imposed to ensure that the problem formulation is consistent with customer preferences. This can make the analysis with multiple customer classes hard.

3.8 Conclusion

In this chapter, we have analyzed the product design problem for a monopolist firm under an alternative model of customer choice behavior based on threshold preferences. Several results such that it suffices to offer at most N products, prices are monotonically increasing with respect to quality, offered products partition customer types into contiguous sets, and the highest quality product is always offered, typically hold under the classic model of customer choice behavior. Using these results, we formulated the general problem, characterized the optimal set of qualities to offer, and solved for the optimal prices in closed-form. The k product problem, where $k < N$, was also solved. The general problem was then extended to include heterogeneous service requirements, multiple quality attributes, and the case of a duopoly, illustrating the tractability of the model. With heterogeneous service times, the opportunity cost of capacity enters the product price and is proportional to the capacity used. The case of multiple quality attributes was considered next, wherein the results for the general case carry over, under assumptions analogous to the one-dimensional quality case. Modeling customer choice behavior using threshold preferences resulted in a more tractable formulation, as well as avoided the necessity to map customer utility over multiple quality attributes into a one-dimensional entity. Finally, we formulated the product design decision problem in the setting of a duopoly. We considered the cases of simultaneous and sequential entry, and showed how the resulting solution differs from the monopoly two-product solution.

We considered several examples where threshold preferences arise, including a delay service rendered via a stylized queue where system dynamics affect deliverable product quality, a data service where quality is completely determined by the firm and where customer have heterogeneous service requirements, a dynamic pricing problem with strategic customers where service requirement is homogeneous and quality is completely controlled by the firm, a seller of software goods that seeks to differentiates customers via versioning, a retailer selling fashion goods to a market of time-sensitive customers, a seller of electronic goods such as mp3 players where customers have threshold preferences with respect of player storage capacity, and a provider of shipping services where customers have threshold preferences with respect to the maximum shipping time that is acceptable. In each of these examples, we derived the optimal product menu comprising of the optimal number of products, and the price and quality of each product, that should be offered by the firm. A key observation is that if customers that prefer better quality are more inelastic with respect to prices, then it is optimal to differentiate along the quality attribute. Moreover, in the optimal product offering, customers having higher quality preferences are always served before those with lower preferences, and at higher prices.

Chapter 4

Optimal putting strategies in golf

We develop a model of golfer putting skill and combine it with putt trajectory and holeout models to identify a golfer's optimal putting strategy. By optimal strategy, we mean the golfer's choice of the target velocity and direction to putt the ball in order to minimize the expected number of putts needed to achieve a holeout. A putting skill model reflects golfer execution errors, i.e., that golfers cannot hit the ball at exactly their intended velocity and direction. A green reading skill model reflects a golfer's inability to perfectly estimate the slope or contour of the putting surface. The model is calibrated to professional and amateur putting data. The problem of identifying the optimal putting strategy is shown to be equivalent to solving a two-dimensional stochastic shortest path problem. Quasi-Monte Carlo methods, dimensionality reduction of the optimization problem, and symmetry of the optimal policy are used to speed up computations. The model is used to identify optimal putting strategies and show how these strategies vary by the distance from the hole, the golfer's putting skill and green reading ability. The model can be used to quantify the total number of putts per round lost by employing a suboptimal putting strategy. The relative difficulty of downhill, sidehill, and uphill putts is also quantified.

4.1 Introduction

The game of golf is played by over fifty million golfers across the world, and billions of dollars are spent each year on greens fees, equipment and instruction [71, 49]. Putts, i.e., strokes on a putting surface called a green, are an important part of the game, representing 35-45% of the strokes in an 18-hole round. While thousands of instructional books, magazine articles and videos are devoted to putting, most focus on putting technique with relatively little attention given to putting strategy.

Previous putting research has addressed the physics of the path of a putt on the green (Perry [51], Vanderbei [67]) and the physics of a holeout, i.e., whether a ball rolling on the green will be captured by the golf hole (Holmes [28], Hubbard and Smith [29], Penner [50]). Simple models of putting skill were developed by Gelman and Nolan [21], Hoadley [26], and Tierney and Coop [66], but these models only consider level putting surfaces with simplified holeout models.

In this paper, we develop models of golfer putting and green reading skill and combine them with physical putt trajectory and holeout models for sloped green surfaces in order to determine optimal putting strategies. A putting strategy refers to the decision of the golfer to putt the ball in a given direction at a given speed. In choosing a strategy, a golfer should consider the likelihood of a holeout, how far a putt might finish from the hole in case it misses, whether the next putt will be an uphill or a downhill putt, among other factors. The decision is further complicated since there are typically many velocity-direction combinations which result in a holeout. One strategy is to attempt to hit the putt so that the ball dies into the hole, i.e., its velocity is nearly zero as it passes the front edge of the hole. By following this strategy, velocity execution errors will lead to approximately half of the putts stopping short of the hole, thus requiring at least one more putt until a holeout. Attempting to hit the ball harder will increase the probability of a one-putt, but missed putts will end farther from the hole. Another consideration in choosing a strategy is the slope of the green, which causes a putt to follow a curved trajectory, referred to as the *break* of the putt. Hitting the ball with a larger initial velocity causes the putt to break less initially, and so it is less affected by green reading error, i.e., errors in the estimate of the green slope. However, a larger initial velocity will lead to misses that tend to be further from the hole, resulting in a greater chance of a three-putt (i.e., taking three putts to holeout).

We model the two most important aspects of a golfer's putting skill. The first component is a physical skill model which incorporates execution errors in velocity and direction. The second component is a green reading model, which reflects errors in the golfer's estimate of the slope of the green. Although green reading skill plays a crucial role in the determination of a golfer's optimal putting strategy, this is, to the best of our knowledge, the first attempt to model this important aspect of the problem. Equations from Newtonian physics are used to determine the trajectory of a putt on the green. Physics principles are also used to determine whether the trajectory of a putt will lead to a holeout or a miss. Together with a specified putting strategy, the golfer skill, trajectory and holeout models determine the putting performance of the golfer. The model is calibrated to amateur and professional putting data collected under actual playing conditions in regular and tournament play.

Within this framework, we can answer many questions about how optimal putting strategies vary by the golfer skill level and the ball position relative to the hole . In particular, we address the relative difficulty of downhill, uphill and sidehill putts, and identify strategies to minimize the expected number of putts and to maximize the probability of a one-putt for both professional and amateur golfers. The impact of doubling the size of the hole upon the performance of professional and amateur players was studied by us in [6].

The trajectory model used in this paper is from Vanderbei [67] and Renshaw [53], who model the movement of the ball as sliding on a surface with friction. Perry [51] develops a trajectory model that considers both sliding and rolling effects. His model is more realistic, but the extra level of complexity is not necessary for our purposes.

Holmes [28] and Hubbard and Smith [29] derive equations of motion for the ball interacting with a hole on a level green. They determine the maximum velocity that will lead to a holeout as a function of the distance of the ball from the center of the hole. Penner [50] extends the analysis to a green with an uphill or sidehill slope. Our holeout model is based directly on Holmes [28] and Penner [50] with an extension to consider the case of a green with both uphill and sidehill slopes.

Tierney and Coop [66] and Hoadley [26] consider the problem of finding the optimal putting strategy on level greens. They do not use a trajectory nor a holeout model, but instead assume that the ending position of a putt has a bivariate normal distribution around a target point on the green.

The remainder of the paper is organized as follows. In Section 4.2, we present our model for the trajectory, the putting green, and the golfer, and discuss two golfer objectives: maximizing the probability of a one-putt and minimizing the expected number of putts. We discuss numerical algorithms to compute these objectives in Section 4.3. Numerical results, including calibration and characterization of the optimal putting strategies for the professional and the amateur golfer, are presented in Section 4.4. Concluding remarks are given in Section 4.5.

4.2 Model

In this section, we describe our model for the ball trajectory, and the model for determining whether the trajectory leads to a holeout. Then we describe our model for the green and the golfer, and discuss the two golfer objectives that we consider in this paper.

4.2.1 Trajectory model

We first specify the model for the trajectory of ball, i.e., the path followed by the ball on a green given an initial velocity and direction. Our trajectory model is based on Newtonian laws of motion for a body that moves on a curved surface under the forces of gravity and friction. In reality, after being struck, the ball is airborne for a short period, hits the green and skids for some distance, and then begins to roll (see, e.g., Cochran and Stobbs [16]). This level of detail is, however, not necessary for our purposes, and so, as in Vanderbei [67], we assume that the ball slides along the green. We work in Cartesian coordinates, with gravity acting along the negative z-axis. The acceleration, a, acting on the ball is given by the equation

$$ma = N + F - mge_z, \tag{4.1}$$

where m is the mass of the ball, N is the normal force exerted by the surface on the ball, F is the frictional force, g is acceleration due to gravity and $e_z = (0, 0, 1)$ is a unit vector along the z-axis. The frictional force is

$$F = -\eta||N||(v/||v||), \tag{4.2}$$

where $|| \cdot ||$ indicates the L_2 norm, v is the velocity of the ball, and η is the coefficient of friction. Equating the forces acting on the ball in the direction normal to the surface gives

$$||N|| - mg(e_z \cdot N/||N||) = m(a \cdot N/||N||), \tag{4.3}$$

where $\cdot$ indicates dot product. This can be solved for N_z, the normal force in the z-direction, by observing that at the point (x, y, z), the tangent to the surface along the x-axis is given by $(1, 0, dz/dx)$, the tangent to the surface along the y-axis is given by $(0, 1, dz/dy)$, the outward normal to the surface is given by $(-dz/dx, -dz/dx, 1)$, and using

$$N_x = -(dz/dx)N_z, \quad N_y = -(dz/dy)N_z. \tag{4.4}$$

Following a change of coordinates, the system of equations (4.1)–(4.4) is solved numerically (as shown in the Appendix) to obtain the ball trajectory.

4.2.2 Holeout model

Given a trajectory that the ball follows on the green, we need to determine if it results in a holeout. Clearly, if the ball falls short of the hole, or if the trajectory does not intersect with the hole, a holeout cannot occur. Even if the trajectory of the ball passes through the hole, it might not result in a holeout, e.g., if the speed of the ball is too high, or if the point of contact is too far from the center. On the other hand, a holeout can occur if the ball falls into the hole, or if the ball hits the back of the hole and drops in, or if the ball rolls along the rim and eventually falls into the hole. Holmes [28] derives critical ball velocities, which account for all of these holeout possibilities, as a function of the distance of the ball from the center of the hole for a planar green with no slope. Penner [50] extends the holeout criterion for a planar green with an uphill or sidehill slope. We extend this criterion to a planar green with both uphill and sidehill slopes. The details and the derivation are presented in the Appendix.

4.2.3 Green model

The two main characteristics of our model for the putting green are its slope and its speed. In reality, greens are curved surfaces where the slope varies from one point to the next. However, many greens are nearly flat surfaces and the hole is almost always positioned on a flat portion of the green. For these reasons and for simplicity, we assume that the entire green has a fixed slope, i.e., we model the green to be a planar surface in three-dimensional Euclidean space given by $z = ax + by + c$, where a and b are constants. With this green specification, it suffices to denote any point (x, y, z) on the green as (x, y). We will assume that the center of the hole lies at position (0, 0) on the green and will denote the radius of the hole by r_h, where $r_h = 2.125$ inches. We will refer to a planar green with $a = b = 0$ as being *level*. The coefficients a and b are referred to as the *grade*. Typically, we report the green slopes $\tan^{-1}(a)$ and $\tan^{-1}(b)$ in degrees.

The speed of a putting green is defined to be the distance a golf ball travels on the green when rolled off a Stimpmeter [3] onto a level portion of the green. The Stimpmeter is a device designed to release a golf ball from a length of 30 inches along an inclined plane making an angle of 20° with respect to the green. As shown by Holmes [27], the initial velocity of ball rolling off a Stimpmeter is 1.83 m/s. If the ball rolls d feet off the Stimpmeter, the *green speed* is said to be d feet. The speed of a green is determined by the height, type and grain of the grass on the green, the wetness and hardness of the green, and other physical features which cause friction between the ball and the green. We assume that the entire green has a constant coefficient of friction denoted η. For a level green, the equations of motion in Section 4.2.1 can be solved to give

$$d = \frac{v^2}{2\eta g}, \tag{4.5}$$

where d is the distance traveled from the initial position, v is the initial velocity of the ball, and g is acceleration due to gravity. Table 4.1 shows the coefficient of friction (η) values corresponding to some commonly observed green speeds.

Table 4.1: Green speed and corresponding friction (η) values.

Green speed (feet)	η
7	0.0801
9	0.0623
11	0.0510
13	0.0431

This table shows the coefficient of friction η corresponding to various green speeds computed with equation (4.5).

Putting greens are not perfectly smooth surfaces (e.g., as in a pool table). In particular, there may be inconsistencies in the surface of a green that we'll call *bumps*. Bumps are caused by grass growing at different speeds, footprints left by golfers walking on the green, unrepaired or improperly repaired pitch marks caused by balls landing on the green, etc. The effect of bumps is to cause putts started with the same initial velocity and in the same direction, to follow different trajectories and end at different stop points. Care and maintenance of greens has improved tremendously in recent years, so that greens at PGA tournament courses are typically very smooth. Greens at public and private courses tend to be much smoother than in years past as well. Although we do not model bumps on the green explicitly, our golfer skill model that incorporates green reading error (discussed in Section 4.2.4) can reproduce the effect of bumps.

4.2.4 Golfer skill model

We model three different aspects of golfer putting ability: errors in putting the ball with a desired velocity, errors in putting the ball in a desired direction, and errors in estimating the slope of a green. We refer to these errors as velocity error, direction error, and green reading error, respectively.

The trajectory model requires the ball's initial velocity. On a level green the distance a ball travels is proportional to the square of the initial velocity (see equation (4.5)), so our primitive variable will be $\widetilde{v}^2$, the ball's random initial velocity squared. We assume that

$$\widetilde{v}^2 \sim \mathcal{N}(\mu_v^2, g(\mu_v)^2), \tag{4.6}$$

i.e., $\widetilde{v}^2$ is normally distributed with a mean μ_v^2, where μ_v is the target velocity chosen by the golfer, and $\widetilde{v} - \mu_v$ is the *velocity error*. The variance of $\widetilde{v}^2$ is denoted $g(\mu_v)^2$, and we need to specify a functional form for g. Differences between a ball's initial velocity and the golfer's target velocity contribute to distance errors, i.e., the realized length of the putt is different from the target length. Putting data shows that distance errors are roughly proportional to the length of the putt, which implies $g(\mu_v)$ should be roughly proportional to μ_v^2. However, 20-foot putts on a fast level green will typically have greater distance errors than 20-foot putts on a slow level green. This implies that lower velocities will have slightly higher relative errors than larger velocities. Similarly, shorter putts tend to have slightly larger relative distance errors than longer putts on the same green. These considerations suggest that $g(\mu_v)$ is a convex increasing function of μ_v^2. We assume that

$g(\mu_v)$ is a piecewise-linear convex function given by:

$$g(\mu_v) = \begin{cases} \beta_2 v_\beta^2 - \beta_0(v_\beta^2 - \mu_v^2), & \mu_v^2 \leq v_\beta^2 \\ \beta_2 v_\beta^2 + \beta_1(\mu_v^2 - v_\beta^2), & \mu_v^2 > v_\beta^2 \end{cases} \tag{4.7}$$

where v_β is termed the *breakpoint velocity*, and β_0, β_1 and β_2 determine how distance error changes with velocity. We impose $0 \leq \beta_0 \leq \beta_2$, $\beta_0 \leq \beta_1$ to ensure non-negativity and convexity of $g(\mu_v)$. As a special case, taking $\beta_0 = \beta_1 = \beta_2$ leads to $g(\mu_v) = \beta_2 \mu_v^2$, which implies relative distance error, i.e., distance error normalized by length of the putt, is constant on level greens. To emphasize the dependence on parameters, we will sometimes denote $\widetilde{v}^2$ by $\widetilde{v}(\mu_v)^2$ or $\widetilde{v}(\mu_v, \beta_0, \beta_1, \beta_2, v_\beta)^2$.

Direction errors occur because golfers are unable to putt the ball in exactly the desired target direction. Given a target angle of μ_α (measured relative to the ball-hole line), we assume that golfer putts the ball at a random angle $\widetilde{\alpha}$ which follows a normal distribution:

$$\widetilde{\alpha} \sim \mathcal{N}(\mu_\alpha, \sigma_\alpha^2). \tag{4.8}$$

We sometimes denote $\widetilde{\alpha}$ by $\widetilde{\alpha}(\mu_\alpha)$ or $\widetilde{\alpha}(\mu_\alpha, \sigma_\alpha)$. We assume that direction and velocity are independent.

Green reading error occurs because golfers cannot estimate green slopes perfectly, i.e., the golfer's estimate of the green slope is different from the actual green slope. Velocity and green reading error contribute to distance error. Similarly, velocity, direction and green reading error all contribute to the overall variability in the ball's stop point. Suppose the golfer estimates the green slopes to be $\theta = (\theta_x, \theta_y)$, where θ_x and θ_y are the slope estimates along the x-axis and y-axis, respectively. The actual slope is randomly chosen by nature and its distribution is given by

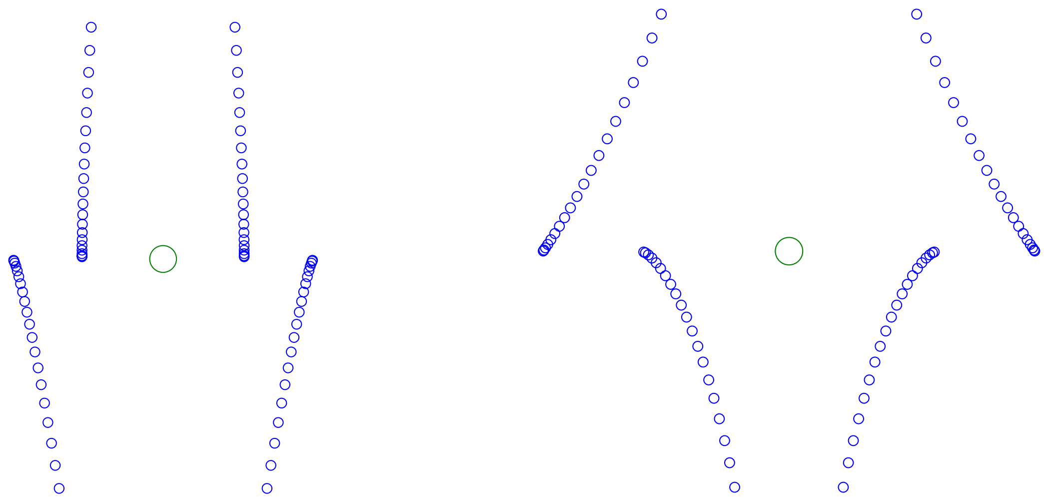
$$(\widetilde{\theta}_x, \widetilde{\theta}_y) = (\theta_x, \theta_y) + (\sigma_g Z \cos(2\pi U), \sigma_g Z \sin(2\pi U)), \tag{4.9}$$

where $Z \sim \mathcal{N}(0,1)$, $U \sim U[0,1]$ and Z and U are independent. The green reading skill parameter is σ_g and increasing the value of σ_g implies greater errors in the golfer's estimates of the green slopes. To motivate equation (4.9), observe that θ_x and θ_y can be represented as a point in two-dimensional space. Adding $(\sigma_g Z \cos(2\pi U), \sigma_g Z \sin(2\pi U))$ leads to green slopes that are uniformly distributed on a circle centered at (θ_x, θ_y) with radius $\sigma_g|Z|$.

Since we restrict our analysis to planar greens, without loss of generality, we change coordinates so that the golfer's green slope estimate is zero along the x-axis, i.e., we set $\theta_x = 0$. In other words, the negative y-axis is the downhill direction to the hole, also called the *fall line*. We use the notation $K = (\beta_0, \beta_1, \beta_2, \upsilon_\beta, \sigma_\alpha, \sigma_g)$ to denote a golfer's putting skill parameters.

4.2.5 Illustrations of velocity, direction, and green reading error

Figure 4.1 compares the effect of direction and green reading errors on uphill and downhill putts. For a 20-foot putt, with a direction error of 1°, the ball would miss the center of the hole target by 4.7 inches on a level surface. For a downhill putt the error would be 2.7 inches, while for an uphill putt it would be 5.0 inches. Hence, for the same direction error, downhill putts lead to smaller deviations than uphill putts. Essentially, the ball "breaks" downhill, i.e., the putt trajectory curves towards the hole for downhill putts so gravity reduces the effect direction errors. Gravity magnifies direction errors on uphill putts. In contrast, green reading errors are magnified on downhill putts compared to uphill putts. Suppose a golfer starts a putt directly at the hole, but because of a green reading error, the putt breaks left-right. Again because of gravity, the ball will veer farther away from the hole on downhill putts than uphill putts. For example, using equation (4.9) with $\sigma_g = 0.15$, a one standard deviation green reading error on a 20-foot putt leads to an 8.3 inch miss for a downhill putt, but only a 4.9 inch miss for an uphill putt. Because of the opposite effects of these factors, it is important to include both direction error and green reading error in the golfer model.



(a) Direction error (b) Green reading error

Figure 4.1: This figure shows the last 5 feet of 20-foot uphill (90°) and downhill (−90°) putts on a green with slope of 1.5° and green speed of 11 feet. The green slope of 1.5° is along the y-axis. The x- and y-axis scales are different to better illustrate the results. The trajectories in (a) correspond to direction errors of ±1° while aiming straight at the hole. The trajectories in (b) correspond to a putt starting straight towards the hole on a green with slopes ±0.15° and 1.5° along the x- and y-axes, respectively. Direction error leads to a greater deviation for uphill putts, while green reading error leads to a greater deviation for downhill putts.

Next we examine how convexity in golfer velocity model and green reading ability affect relative distance error. Relative distance error is defined to be the standard deviation of the distance of the stop point of a putt from the center of the hole divided by the putt-length (assuming that the hole is covered, so that there are no holeouts). Table 4.2 shows that the linear velocity error model leads to constant relative distance error. The convex velocity model leads to relative distance errors that decrease with putt length, and are larger for downhill putts than for uphill putts. Green reading error leads relative distance errors that are independent of the putt length. Green reading errors lead to greater relative distance errors for downhill putts than uphill putts. Combining the convex velocity model with green reading error leads to relative distance error that decrease with putt length, and are greater for downhill putts than for uphill putts.

Table 4.2: Effect of convexity in player velocity model and green reading ability on relative distance error

Velocity error model	Sloped green	Error	Length of putt 3	6	10	15	20	30	50
Linear	No	velocity	6.7%	6.7%	6.7%	6.7%	6.7%	6.7%	6.7%
Linear	No	green	4.4%	4.4%	4.4%	4.4%	4.4%	4.4%	4.4%
Linear	No	both	7.9%	7.9%	7.9%	7.9%	7.9%	7.9%	7.9%
Convex	No	velocity	11.7%	8.7%	7.5%	6.9%	6.7%	6.7%	6.7%
Convex	No	green	4.4%	4.4%	4.4%	4.4%	4.4%	4.4%	4.4%
Convex	No	both	12.4%	9.6%	8.6%	8.0%	7.9%	7.9%	7.9%
Convex	Up	velocity	9.7%	7.8%	7.0%	6.8%	6.8%	6.8%	6.8%
Convex	Up	green	3.5%	3.5%	3.5%	3.5%	3.5%	3.5%	3.5%
Convex	Up	both	10.1%	8.3%	7.5%	7.4%	7.4%	7.4%	7.4%
Convex	Down	velocity	18.0%	11.8%	9.3%	8.0%	7.4%	6.8%	6.6%
Convex	Down	green	8.3%	8.3%	8.3%	8.3%	8.3%	8.3%	8.3%
Convex	Down	both	20.0%	14.5%	12.5%	11.6%	11.2%	10.8%	10.6%

This table shows how convexity in the golfer velocity model and green reading ability impact relative distance error as a function of the length of the putt. Relative distance error is defined as the standard deviation of the distance from the center of the hole to the stop-point of the putt (assuming that the hole is covered) divided by the initial putt-length. The uphill (Up) and downhill (Down) putts are assumed to be putt directly along the slope of the green. For these putts, the slope is set to be 1.5°. The model parameters used are $\beta_0 = 5.5\%$, $\beta_1 = 6.5\%$, $\beta_2 = 6.5\%$, $v_\beta = 15$ feet, $\sigma_\alpha = 1.0$, $\sigma_g = 0.15$.

4.2.6 Golfer objectives

We consider two golfer objectives: maximizing the probability of a one-putt and minimizing the expected number of putts to holeout. Suppose the golfer starts at $I = (x, y)$ and putts until the ball falls in the hole. The golfer's slope estimates are $(0, \theta_y)$ and the random realized green slope is $(\widetilde{\theta}_x, \widetilde{\theta}_y)$ defined in equation (4.9). Suppose the golfer chooses a target velocity μ_v and a target angle μ_α for the putt. The realized velocity $\widetilde{v}(\mu_v)$ and the realized angle $\widetilde{\alpha}(\mu_\alpha)$ are given by equations (4.6) and (4.8), respectively, and the random trajectory of the putt (given by equations (4.28)–(4.31) in the Appendix) starting at position I is:

$$\widetilde{\mathcal{T}}(I, \mu) = \widetilde{\mathcal{T}}\left(I, \widetilde{\theta}, \eta, \widetilde{v}(\mu_v), \widetilde{\alpha}(\mu_\alpha)\right),$$

where η is the friction coefficient, and $\mu = (\mu_v, \mu_\alpha)$. The stopping point of trajectory $\mathcal{T}$ will be denoted by $\mathcal{S}(\mathcal{T}) = (\mathcal{S}_x(\mathcal{T}), \mathcal{S}_y(\mathcal{T}))$, where we assume that the hole is covered, so that trajectories for putts that would otherwise lead to holeout do not necessarily end at the hole.

The holeout function $h(\mathcal{T})$ maps a trajectory to its outcome:

$$h(\mathcal{T}) = \begin{cases} 1 & \text{if } \mathcal{T} \text{ leads to a holeout} \\ 0 & \text{otherwise} \end{cases}$$

The probability of a one-putt depends on the target velocity and angle μ, the initial position I, the slope estimate θ, the friction coefficient η, and the golfer skill parameters K:

$$P_1(\mu, I, \theta, \eta, K) = E[h(\widetilde{\mathcal{T}}(I, \widetilde{\theta}, \eta, \widetilde{v}(\mu), \widetilde{\alpha}(\mu)))]. \tag{4.10}$$

We often abbreviate the one-putt probability as $P_1(\mu, I)$.

We optimize a given objective over the set of velocity-angle combinations defined by:

$$\mathcal{U} = \{(\mu_v, \mu_\alpha) | \underline{\mu}_v \leq \mu_v \leq \overline{\mu}_v, \underline{\mu}_\alpha \leq \mu_\alpha \leq \overline{\mu}_\alpha\}, \tag{4.11}$$

where $\underline{\mu}_v$ and $\overline{\mu}_v$ are the smallest and largest candidate velocities, respectively, and $\underline{\mu}_\alpha$ and $\overline{\mu}_\alpha$ are the smallest and largest candidate angles, respectively, that are considered for optimization. The one-putt probability maximizing velocity and angle are given by:

$$\mu^{(1)}(I, \theta, \eta, K) = \arg\max_{\mu \in \mathcal{U}} P_1(\mu, I, \theta, \eta, K). \tag{4.12}$$

The expected number of putts depends on the ball's initial position (I) and the golfer strategy ($\mu(I)$), in addition to the golfer slope estimates (θ), the friction coefficient (η) and golfer skill parameters (K). The result of the first putt is either a holeout or a second putt which begins from the stopping point of the first putt. This leads to the recursion:

$$N(I,\mu) = E[1 + N(\mathcal{S}(\widetilde{\mathcal{T}}(I,\mu))(1 - h(\widetilde{\mathcal{T}}(I,\mu)))] \tag{4.13}$$

The Bellman equation for the optimal expected number of putts is:

$$N^*(I) = \min_{\mu \in \mathcal{U}} E[1 + N^*(\mathcal{S}(\widetilde{\mathcal{T}}(I,\mu))(1 - h(\widetilde{\mathcal{T}}(I,\mu)))]. \tag{4.14}$$

Denote the optimal choice of target velocity and angle in equation (4.14) by $\mu^*(I,\theta,\eta,K)$. Sufficient conditions for the existence of a solution to equation (4.14) and its optimality are discussed in the Appendix.

4.3 Computational methods

In this section, we show how the optimization problems in equations (4.12) and (4.14) are solved to identify the optimal strategies for a given golfer. Both the state and control spaces in equations (4.12) and (4.14) are continuous, so we discretize these to proceed with the computation.

4.3.1 State space discretization

The state space $I \subset \mathbb{R}^2$ is continuous, so to solve equations (4.12) and (4.14), we discretize I. It is convenient to denote the position of the ball on the green $I = (x,y)$ in polar coordinates as (d,γ), where $d = \sqrt{x^2+y^2}$ and $\gamma = \tan^{-1}(y/x), \gamma \in [0,2\pi)$. We discretize the (d,γ)-space into a finite number of points $I_{ij} = (d_i,\gamma_j)$, $i = 1,\ldots,n_d$, $j = 1,\ldots,n_\gamma$. Here $d \in (0,\overline{d}]$, where $\overline{d} < \infty$ is the length of the longest putt we consider. We assume that the probability of a one-putt from any point on the green, for any golfer, is strictly positive.

4.3.2 Control space discretization

The set of feasible controls $\mathcal{U}$ in equations (4.12) and (4.14) is continuous. Since a closed-form solution to the objectives in these equations is not available, we also discretize $\mathcal{U}$. We optimize over the discrete set $\widehat{\mathcal{U}}$ using a *grid search* procedure detailed in the Appendix.

4.3.3 Probability estimation

To estimate the one-putt probability, $P_1(\mu, I)$, we generate n samples, $(\widetilde{\theta}^{(k)}, \widetilde{v}^{(k)}(\mu), \widetilde{\alpha}^{(k)}(\mu))$, $k = 1, \ldots, n$. Then,

$$\widehat{P}_1(\mu, I) = \frac{1}{n} \sum_{k=1}^{n} h(\widetilde{\mathcal{T}}^{(k)}) \tag{4.15}$$

gives an estimate of $P_1(\mu, I)$, where $\widetilde{\mathcal{T}}^{(k)} = \widetilde{\mathcal{T}}^{(k)}(I, \widetilde{\theta}^{(k)}, \eta, \widetilde{v}^{(k)}(\mu_v), \widetilde{\alpha}^{(k)}(\mu_\alpha))$.

To identify $\widehat{\mu}^{(1)}(I)$, the estimate of the velocity-angle combination that maximizes the probability of a one-putt, we perform a grid search as described in Section 4.3.2, thereby obtaining

$$\widehat{\mu}^{(1)}(I) = \arg\min_{\mu \in \widehat{\mathcal{U}}} \widehat{P}_1(\mu, I) \tag{4.16}$$

4.3.4 Expected putts estimation

We next describe how we solve equation (4.14) to find the strategy that minimizes the expected number of putts. We observe that this is an instance of a two-dimensional stochastic shortest path problem, also known as a transient program or a first-passage problem, in which both the state and control space are continuous. These are discussed, for example, in Bertsekas [8] and Whittle [69]. We use policy iteration to solve for the optimal expected number of putts. We first discuss the policy-iteration algorithm for the continuous state and control space case, and then we show how to implement it after discretizing the state and control space. Convergence issues are discussed in the Appendix.

The one-putt probability maximizing strategy, $\mu^{(1)}(I)$, is the solution of a simple numerical optimization procedure, i.e., one that does not require a recursive dynamic programming algorithm. Furthermore, for short putts, maximizing the one-putt probability is nearly equivalent to minimizing the expected number of putts, since the expected number of putts is approximately $2 - P_1$ when the probability of three or more putts is nearly zero. For these reasons, we use $\mu^{(1)}(\cdot)$ as the initial policy in the policy iteration algorithm for expected putt minimization.

The expected number of putts starting from initial position I, and using policy $\mu^{(p)}(\cdot)$ for the initial putt and the continuation strategy, is denoted $N^{(p)}(I)$. The number of putts until a holeout occurs is the smallest m for which the putt m results in a holeout, so $N^{(p)}(I)$ can be written as

$$N^{(p)}(I) = E[\min\{m = 1, 2, \dots \mid h(\widetilde{\mathcal{T}}(I_m, \mu^{(p)}(I_m))) = 1\}] \tag{4.17}$$

where $I_1 = I$, and $I_m = \mathcal{S}(\widetilde{\mathcal{T}}(I_{m-1}, \mu^{(p)}(I_{m-1})))$, i.e., the initial position of putt m is the stop point of putt $m-1$ (if putt $m-1$ does not end in a holeout). Given a policy $\mu^{(p)}(\cdot)$, equation (4.17) defines the policy evaluation step. Under our assumption that the probability of a one-putt is strictly positive, $N^{(p)}(\cdot)$ is finite with probability one.

The policy improvement step is:

$$\mu^{(p+1)}(I) = \arg\min_{\mu \in \mathcal{U}} E[1 + N^{(p)}(\mathcal{S}(\widetilde{\mathcal{T}}(I, \mu)))(1 - h(\widetilde{\mathcal{T}}(I, \mu)))]. \tag{4.18}$$

Equation (4.18) states that $\mu^{(p+1)}(I)$, the optimal policy given an initial position I, is given by the target velocity-angle combination $\mu \in \mathcal{U}$ that minimizes the expected number of putts to holeout starting from position I, when μ is used for the first putt, and policy $\mu^{(p)}(\cdot)$ is used for subsequent putts, if any. Starting with $p = 1$, we can iterate between equations (4.17) and (4.18) until the policy converges, i.e., until $|\mu^{(p)}(I) - \mu^{(p+1)}(I)| < \epsilon$, for all I, and for some fixed $\epsilon > 0$.

Since the state space I is continuous, we show how to proceed with the computations in equations (4.17) and (4.18) after the state and control spaces are discretized. For each I_{ij}, $i = 1, \ldots, n_d$, $j = 1, \ldots, n_\gamma$, we solve equation (4.16) to find $\widehat{\mu}^{(1)}(I_{ij})$, the strategy that maximizes the probability of one-putt from I_{ij}. Next we solve equation (4.17). To estimate $N^{(1)}(I_{ij})$, the objective in equation (4.17) for $p = 1$, simulate n trials, each trial consisting of a sequence of putts until holeout occurs. Suppose trial k requires $\widetilde{m}(k)$ putts, i.e., $h(\widetilde{\mathcal{T}}_{u,k}) = 0$, $u = 1, \ldots, \widetilde{m}(k) - 1$, $h(\widetilde{\mathcal{T}}_{\widetilde{m}(k),k}) = 1$ and $\widetilde{\mathcal{T}}_{u,k}$ denotes the trajectory of putt u for trial k. Then

$$\widehat{N}^{(1)}(I_{ij}) = \frac{1}{n} \sum_{k=1}^{n} \widetilde{m}(k) \tag{4.19}$$

gives an estimate of $N^{(1)}(I_{ij})$, $i = 1, \ldots, n_d, j = 1, \ldots, n_\gamma$. For each simulation trial, the initial position of putt u is the stop point of putt $u-1$, given that it didn't result in a holeout. The target strategy is the one-putt probability maximizing strategy from the stopping point of putt $u - 1$. Since the stopping point will not, in general, coincide with a grid point, we interpolate to obtain the continuation strategy $\widehat{\mu}^{(1)}$ as discussed in the Appendix.

To identify $\widehat{\mu}^{(p+1)}(I_{ij})$, the policy at iteration $p + 1$, we perform a grid search:

$$\widehat{\mu}^{(p+1)}(I_{ij}) = \arg\min_{\mu \in \widehat{\mathcal{U}}} E\left[1 + \widehat{N}^{(p)}(\mathcal{S}(\widetilde{\mathcal{T}}(I_{ij}, \mu)))(1 - h(\widetilde{\mathcal{T}}(I_{ij}, \mu)))\right]. \tag{4.20}$$

For $p = 1$, we can thereby obtain $\widehat{\mu}^{(2)}(I_{ij})$, for $i = 1, \ldots, n_d$, $j = 1, \ldots, n_\gamma$. We repeat this procedure to determine $\widehat{N}^{(2)}(I_{ij})$, following which we determine $\widehat{\mu}^{(3)}$ and so on until the policy converges, i.e., until $|\widehat{\mu}^{(p)}(I_{ij}) - \widehat{\mu}^{(p+1)}(I_{ij})| < \epsilon$ for all I_{ij} for some fixed $\epsilon > 0$.

4.3.5 Computational speedups

We now discuss some techniques and observations that enable us to considerably speed up the computation of optimal putting strategies.

Quasi-Monte Carlo: For variance reduction, we use the Sobol sequence [52] to generate samples in equations (4.6), (4.8)–(4.9), (4.15)–(4.16) and (4.19)–(4.20). Low-discrepancy methods or Quasi-Monte Carlo methods, of which the Sobol sequence is an example, seek to achieve variance reduction by generating samples that are evenly distributed. A detailed discussion can be found in Glasserman [23].

We generate a four-dimensional Sobol sequence to estimate the one-putt probability in equation (4.15). The first two dimensions are used to generate velocity and angle samples using equations (4.6) and (4.8), respectively, while the third and the fourth dimensions are used to generate green slopes using equation (4.9). To estimate the expected number of putts in equation (4.19), we use a 10-dimensional Sobol sequence, where the third and the fourth dimension are used to generate green slopes using equation (4.9), and dimensions 1-2 and 5-10 are used to generate velocity and angle realizations for putts 1 through 5, if needed (in our experiments, we have observed 5-putts to lead to a holeout in almost all cases, and so we have capped the number of putts to be 5 in our implementation).

Reducing the dimensionality of optimization: The optimization over μ_v and μ_α in equations (4.16) and (4.20) can be CPU intensive. All optimal solutions obtained with this two-dimensional optimization procedure were found to possess the property that the ball trajectory at the optimal target velocity-angle combination passes through the center of the hole, a property which makes intuitive sense. In the Appendix we prove that this property holds in the special cases of a level green, and for straight uphill and downhill putts. If this property holds in general, then one-dimensional optimization can be used to identify the optimal solution. In particular, suppose that the optimal strategy for the golfer is to target a distance d feet beyond the hole ($d \geq 0$) and that the trajectory corresponding to the optimal velocity-angle combination passes through the center of the hole. Instead of a two-dimensional search over (μ_v, μ_α), we perform a one-dimensional search over $d \geq 0$, using a root-finding procedure to solve for the velocity-angle combination $(\mu_v(d), \mu_\alpha(d))$ that leads to a stop point d feet beyond the hole and passes through its center. The CFSQP code [36] was used for the root-finding routine that maps d_t to $(\mu_v(d_t), \mu_\alpha(d_t))$. We compared the results from two-dimensional optimization with the results from one-dimensional optimization for several putts, and found that the results matched. Following this observation, all our results have been computed with one-dimensional optimization.

Symmetry of optimal policy: Since we only consider planar greens, putts started on either side of the fall line to the hole will follow symmetric trajectories. In particular, consider symmetric starting positions $I = (d\cos\gamma, d\sin\gamma)$ and $\overline{I} = (d\cos(180-\gamma), d\sin(180-\gamma))$, for $\gamma \in [-90, 90]$. Then for $\mu_v(I) = \mu_v(\overline{I})$ and $\mu_\alpha(I) = -\mu_\alpha(\overline{I})$ the putt trajectories will also be symmetric (defined in the obvious manner). Because of the symmetry of the trajectory model, the holeout model, the green model, and the golfer model with respect to initial positions I and $\overline{I}$, it follows that $\mu_v^*(I) = \mu_v^*(\overline{I})$ and $\mu_\alpha^*(I) = -\mu_\alpha^*(\overline{I})$. This symmetry also holds for the one-putt probability maximizing strategy $\mu^{(1)}$. Hence we only need to find the optimal solution to equations (4.16) and (4.20) for $\gamma \in [-90, 90]$ for any d.

One-putt probability maximizing strategy for continuation: For short putts, where one-putt probability is close to one, we anticipate the expected-putts minimizing strategy to be close to the one-putt probability maximizing strategy. Similarly, for second putts that start in the vicinity of the hole, we expect the continuation strategy to be close to the one-putt probability maximizing strategy. This motivates the following approximation towards computing the expected-putts minimizing strategy.

$$\widehat{\mu}^*(I) = \arg\min_{\mu\in\mathcal{U}} E[1 + (1 - h(\widetilde{\mathcal{T}}))(1 \cdot P_1(\mu^{(1)}, \mathcal{S}) + 2 \cdot (1 - P_1(\mu^{(1)}, \mathcal{S}))]. \tag{4.21}$$

where $\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}(I, \mu)$ is the trajectory generated from starting point I and upon using the strategy μ, and $\mathcal{S} = \mathcal{S}(\widetilde{\mathcal{T}}(I, \mu)))$ is the stop-point of this trajectory. Equation (4.21) states that $\widehat{\mu}^*$, an approximation to the expected-putts minimizing strategy μ^*, can be obtained by solving for the strategy μ that minimizes the expected number of putts that result from using strategy μ for the first putt, and where any putts subsequent to the initial putt are approximated as a one-putt with the optimal one-putt probability from the stop point of the first putt, and as a two-putt with one minus this optimal one-putt probability. Following this, $\widehat{N}^*(I)$, an approximation to the optimal expected number of putts $N^*(I)$, can be written as

$$\widehat{N}^*(I) = \min_{\mu \in \mathcal{U}} E[1 + (1 - h(\widetilde{\mathcal{T}}))(1 \cdot P_1(\mu^{(1)}, \mathcal{S}) + 2 \cdot (1 - P_1(\mu^{(1)}, \mathcal{S}))] \tag{4.22}$$

Equation (4.21) reduces the computational complexity in Equation (4.18) in two ways. First, instead of solving for the optimal continuation strategy μ^* via policy iteration, it uses the one-putt probability maximizing strategy $\mu^{(1)}$ as the continuation strategy. Second, instead of simulating any second or subsequent putts using the one-putt probability maximizing continuation strategy, it assumes that the ball holes out in at most two putts and directly uses the one-putt probability resulting from the strategy $\mu^{(1)}$ towards the estimation of remaining number of putts to holeout. For these reasons, if $\widehat{\mu}^*$ is a good approximation to μ^*, it can result in significant computational savings. We compared the expected putts $\widehat{N}^*(I)$ obtained from the above approximate strategy with the approximation to the optimal expected putts $N^*(I)$ obtained using policy iteration, and found the differences to be small (within 1e-4 putts) across different starting positions I and for both the professional and the amateur players. We also analyzed the impact of interpolating one-putt probabilities versus interpolating v, α combination ($\widehat{\mu}^*$ was used as the continuation strategy in each case), and found that the differences in expected putts were small (within 1e-3 putts) across different initial putt-lengths and angles and for both the professional and the amateur players. Following this, in the below we always solve for the strategy $\widehat{\mu}^*$ and interpolate probabilities for finding the expected-putts minimizing strategy.

4.4 Numerical results

In this section we present numerical results for one-putt probability and expected number of putt optimization for professional and amateur golfers. We first discuss calibrating the model parameters to the data. Next we describe the steps we undertook to ensure that the numerical error in our estimates remains controlled. Then we illustrate the optimal strategies corresponding to the objective of expected-putts minimization. We also compare the results to the one-putt probability-maximization strategy and the *1.5-foot strategy*. Under the latter strategy, the golfer attempts to putt the ball along the trajectory that passes through the center of the hole and comes to a stop 1.5 feet beyond the hole (assuming the hole is covered). We begin by describing the data set and some of its features.

Data: We use amateur and professional golfer data collected under actual playing conditions from regular play and tournaments. PGA TOUR data was collected with their ShotLink™ system. The database contains the start and stop points of approximately 15,000 putts hit by over 100 different golfers. The database is contained in the Golfmetrics program which is further described in [11]. From this data the fraction of one-putts, three-putts, and average number of putts are computed as a function of the initial distance from the hole. The database does not contain information about the exact slope of the green corresponding to each individual putt. Soley [61] gives results for a putting machine which we use to quantify the impact of bumps and other green imperfections.

Parameter choices: Public and private courses typically have green speeds in the 7-10 foot Stimpmeter range. Green speeds at PGA tournaments are typically in the 9-13 foot range. For our numerical experiments, we use a green speed of 11 feet for professional golfers, and a green speed of 9 feet for amateur golfers. We measured green slope values at different points on several greens and found the average to be about $1.5°$. The range of slopes was about $0.5°$ to $2.5°$, with values as high as $7°$ observed at green "humps" and near the edges of greens. For our experiments, we use a constant green slope of $1.5°$.

Calibration: Calibrating the golfer model to the available data is challenging because evaluating results for a given set of parameters takes a significant amount of CPU time. To address this, we adopt the following procedure.

1. Define the parameter space: In general, the player parameters are $\sigma_\alpha, \sigma_g, \beta_0, \beta_1, \beta_2, d_\beta$, which denote, angle error, green-reading error, slope-down velocity error, limiting velocity error, velocity error, and breakpoint distance (or equivalently the breakpoint velocity), respectively.

2. Sample from the parameter space: We sample from the parameter space using in two ways. First, given the probability and expected-putt data we want to calibrate to, we identify bounds on the support of the parameters and sample randomly from the resulting set. For example, we know that $0 \leq \sigma_\alpha \leq 3$ for the calibration to have a chance to match the given data. Second, based on results from previous iterations (as described later), we identify parameter combinations that have lead to a good fit and sample randomly in the neighbourhood of these parameter sets. The constraints imposed on the parameters are $\sigma_\alpha \geq 0$, $\sigma_g \geq 0$, $\beta_2 \geq \beta_0 \geq 0$, $\beta_1 \geq \beta_0$, $50 \geq d_\beta \geq 0$.

3. Generate results for probability optimization and expected putts optimization for each of the sampled parameter-sets. This gives us, for each of the parameter sets, the error between the model and the data probability and expected putts for each distance.

4. For each distance, we use a quadratic polynomial (without cross-terms) in the underlying parameters to capture the probability error as a function of the parameters. Similarly, for each distance, we use a quadratic polynomial (without cross-terms) to capture the expected-putt error as a function of the parameters.

5. For a given set of parameters, using the calibrated regressions, we compute the root mean-squared probability error across distances as well as the root mean-squared expected-putt error across distances. We define the sum of these two root mean-squared errors (RMSEs) to

be the objective of our optimization. Next we optimize over the underlying parameters and under the constraints identified in Step 2 to identify parameters that would minimize this objective.

6. We sample randomly in the neighbourhood of these optimal parameters obtained in Step 5, and repeat steps 3-5, until the RMSE is sufficiently small. While carrying out this the above steps, we ensure that the numerical parameters, e.g., Δ_t, ϵ, N, r-θ grid are chosen such that the error due to these numerical choices (this error quantificaion is discussed below) is an order of magnitude smaller than the error mismatch between the model and the data.

Section 4.4.2 and the Appendix discuss the results of the calibration in detail.

Error analysis: As discussed earlier, given the holeout model, the green model, and the player skill model, we use a computational approach to identify the optimal putting strategy. This computational approach involves several numerical steps which can introduce errors in the estimated quantities. In this section we discuss the different types of errors that arise in the estimates, the error analysis we performed to quantify these errors and to estimate the tradeoff between accuracy and cpu time, and the numerical parameters chosen to ensure that these errors remain controlled and thus the results we obtain are accurate (within a desired tolerance).

Given a green, an initial position, and an initial velocity, identifying the ball-trajectory requires solving a system of ODEs. As described in the Appendix, we use the Odeint wrapper to the RKQS stepper from Numerical recipes in C to simulate the trajectory. The two parameters to this function are *step_t* and *eps*, which we denote using Δ_t and ϵ, respectively. Smaller values of these parameters lead to more accurate trajectories. To estimate trajectory error, we computed trajectories under a range of settings (green slope and speed, initial putt-length and angle), and using different sets of Δ_t and ϵ values. After confirming that the error converged as a function of these parameters, for a given choice of Δ_t and ϵ, we defined trajectory error to be the difference in the trajectory when estimated using these given parameters and either the closed-form trajectory, if available (e.g., for a green with no slope, or straight uphill or downhill putts), or extremely small values of RKQS parameters ($\Delta_t = 0.0002$, $\epsilon =$ 1e-8). This difference in the trajectories was computed at the stop-point of the trajectory, and after a given time from when the ball started moving (usually the time when the ball is closest to the hole). We found that $\epsilon \leq 0.001$ and $\Delta_t \leq 0.0625$ seconds leads to errors within one-tenth of an inch even for greens with high speed ($\eta = 0.0431$) and high slopes (2°). With RKQS, the errors did not increase with putt-length, were not higher for a particular putt-angle, and were not very sensitive to the ϵ parameter. As a final check, we checked the trajectory computed using the RKQS stepper against the trajectory computed using the RKQC stepper (from Numerical Recipes in C), the trajectory computed using an Euler discretization implementation, and the trajectory computed using an Euler discretization with Richardson extrapolation implementation. The trajectories obtained from these different approaches converged, and the RKQS and RKQC implementations were found to be the most efficient. The RKQS implementation was used for generating all the results discussed below.[1]

As the ball trajectory is also used to determine whether the putt lead to a holeout, we also examined the holeout error resulting from the choice of these parameters. To quantify this error, for a given green, putt-angle and player, we computed one-putt probabilities for different putt-lengths under different choices of Δ_t, ϵ and N, the number of trials. To improve convergence, as mentioned above, we used the low-discrepancy Sobol sequence. Again, after confirming that refining the parameters led to convergence of the one-putt probability estimate, we used the estimate obtained from parameters $\Delta_t = 0.00390625$, $\epsilon = 0.0001$ and $N = 2^{24} - 1$ as a benchmark to compute the error for other parameter settings (under this choice of parameters, the standard error was within $3e-5$). We found that for Δ_t less than or equal to 0.0625, the error was primarily driven by the number of trials N, and was not sensitive to the choice of ϵ, given that $\epsilon \leq 0.0001$. In particular,

[1] It is worth noting that we use the CFSQP optimizer to map distance beyond the hole to target into (v, α) combination that would lead to a trajectory that passes through the center of the hole and would lead to a stop-point at the desired distance beyond the hole. The CFSQP optimization parameters were chosen such that the error due to this routine was less than one-tenth of an inch at the stop-point, and less than one-hundredth of an inch near the center of the hole, both within the error tolerance of the trajectory routine.

for $N > 2^{16} - 1$ and up to $2^{19} - 1$, the error in the probability estimates due to Δ_t was within $1e-4$ and less than the standard error, thereby suggesting that a choice of $\Delta_t \leq 0.0625$, and $\epsilon \leq 0.0001$ would be adequate for our experiments.

Next we sought to understand and bound the error in optimal one-putt probability estimate. In addition to errors due to Δ_t, ϵ and N in the individual estimates of probability for different values of distance beyond the hole to target, we also need to account for the error that may result when probability estimates are compared to identify the optimal one-putt probability. To control for the latter, the confidence interval of the optimal distance to target was estimated to be the range such that optimal probability within this range minus two standard errors was above the maximum probability outside this range plus two standard errors (thereby resulting in a confidence interval of 99.95% under the assumption that underestimation and overestimation are independent). We found that using $\Delta_t = 0.0625$, $\epsilon = 0.0001$, and $N = 2^{19} - 1$ for probability optimization led to errors in distance beyond hole to target of about 5-7 inches across different putt-lengths. Under this setting, the standard error and the bias in the one-putt probability estimate were each within 2e-4. While a similar relationship could be estimated for other other parameter choices, we found this level of accuracy to be sufficient.

In addition to errors in the trajectory and holeout calculation, computing expected-putts also requires interpolation to find the continuation strategy. Towards this end, we examined the effect of using different grid-sizes and interpolation approaches on the estimation of expected putts. To understand the effect of grid-granularity, we started from a very fine grid with 360 r values and 363 θ values wherein at each of these 360x363 points on the grid, we know the continuation strategy and do not need to interpolate. We set $\overline{d}$, the length of the longest putt in our experiments, to be 58 feet. For different initial putt-lengths and angles, we computed expected putts using this grid, and compared this estimate with the expected putt estimate using a coarser grid, namely, a grid with $1/k^{th}$ of the r and θ values as the finest grid, where $k = 2, 4, 8, 16, 32$. With $k \leq 8$, we found error in expected putts to be within $1e - 4$ putts. For $k = 8$, the discretized grid points corresponded to $d = \{0.25,\ 0.50,\ 0.75,\ 1.00,\ 1.25,\ 1.50,\ 1.75,\ 2.00,\ 2.25,\ 2.50,\ 2.75,\ 3.00,\ 3.25,\ 3.50,\ 3.75,\ 4.00,\ 4.25,\ 4.50,\ 4.75,\ 5.00,\ 5.50,\ 6.00,\ 6.50,\ 7.00,\ 7.50,\ 8.00,\ 8.50,\ 9.00,\ 9.50,\ 10.00,\ 11.00,\ 12.00,\ 13.00,\ 14.00,\ 15.00,\ 17.00,\ 19.00,\ 21.00,\ 23.00,\ 25.00,\ 29.00,\ 33.00,\ 37.00,\ 42.00,\ 50.00,\ 58.00\}$ feet and $\gamma = \{0°, 7.5°, 15°, 22.5°, 30°, \ldots, 360°\}$. The concentration of gridpoints near the hole allows better interpolation in this region, which is important because a majority of the putts that do not result in a holeout are likely to end near the hole. We also compared two interpolation approaches, linear interpolation and cubic spline interpolation. Both implementations were from Numerical Recipes in C. While the differences between the two approaches were within $1e-4$ putts, we found cubic splines to perform better and hence chose this approach for interpolation. Based on the above, $k = 8$, and cubic spline interpolation were set as the default settings for our experiments. Together with $\Delta_t = 0.0625$, $\epsilon = 0.0001$, and $N = 2^{19} - 1$, the error in expected putts was found to be within $5e - 4$ putts.

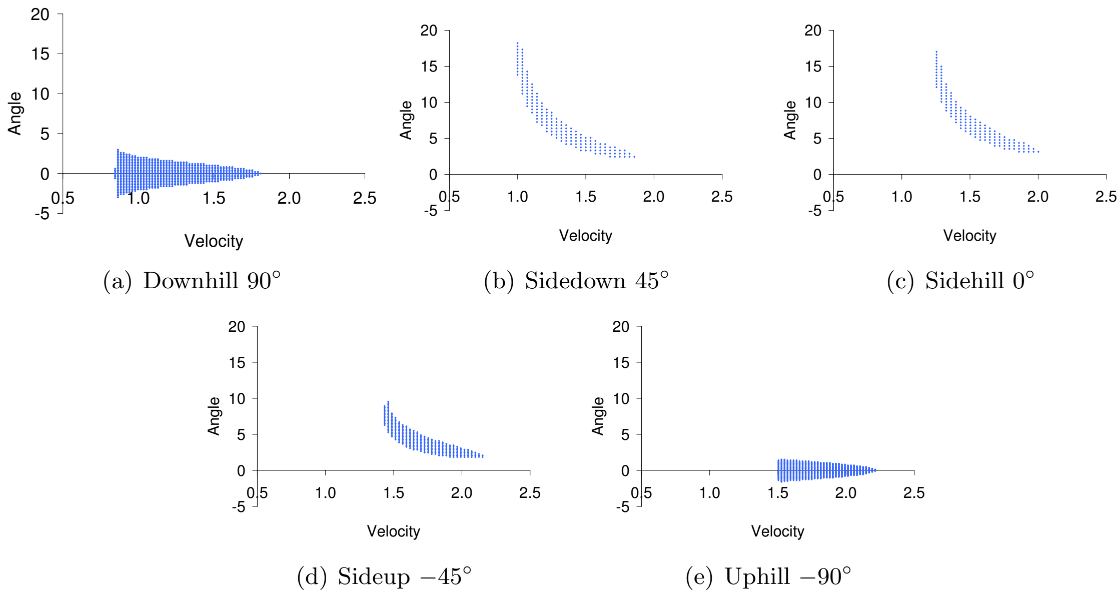
To understand and bound the error in optimal expected-putt estimates, as with probability optimization, in addition to errors due to the above parameters in the individual estimates of expected-putts for different values of distance beyond the hole to target, we also need to account for the error that may result when expected-putt estimates are compared to identify the optimal expected putt value. To control for this latter error, as with probability optimization, the confidence interval of the optimal distance to target was estimated to be the range such that optimal expected-putts within this range minus two standard errors was above the minimum expected-putts outside this range plus two standard errors (thereby resulting in a confidence interval of 99.95% under the assumption that underestimation and overestimation are independent). Using $\Delta_t = 0.0625$, $\epsilon = 1e - 4$, and $N = 2^{19} - 1$, for expected-putts optimization experiment, we found the error in the distance beyond the hole estimate to be about 6-8 inches across different putt-lengths. Under this setting, the standard error and the bias in the expected putts estimate were each within 5e-4 putts. While a similar relationship could be estimated for other other parameter choices, we found this level of accuracy to be sufficient.

Computing platform: The numerical experiments were run on Columbia Business School's Research Grid, which provides the Sun Grid Engine interface for computing. Nodes on the cluster are equipped with high-performance CPUs, e.g., Intel Xeon X5365 processor, and we used 4GB of memory for each job we ran. Using a C/C++ implementation, and setting the numerical parameters as mentioned above, it took approximately 1.5 to 3.0 hours to to solve equation (4.16) to find the one-putt maximizing strategy, and approximately 1.5 to 3.0 hours to solve equation (4.20) to find the expected-putts minimizing strategy. We ran the optimization for each given putt-length d (across angles γ) as a separate job on the grid. As we needed to perform the optimizations for 25 angles for a given putt-length, it took about 36-72 hours to identify the one-putt probability maximizing or expected-putts minimizing strategy for a given d. We ran about 15-30 of such jobs simultaneously on the grid (the number of jobs was dynamically assigned by the grid scheduler depending on the grid workload) to solve for the optimal strategy corresponding to each of the 45 d values.

4.4.1 Holeout region

Given an initial ball position I, green slope vector θ, and friction coefficient η, we call the velocity-angle combinations that lead to a holeout the *holeout region*. Figure 4.2 shows how the holeout region varies with the ball-hole angle relative to x-axis, for a 5-foot putt. For uphill and downhill putts, the symmetry of the holeout regions implies that the optimal target angle for these putts is zero. Figure 4.3 shows how the holeout region varies as a function of distance for sidedown (45°) putts. The size of the holeout region decreases as the initial distance to the hole increases, which explains why longer putts are more difficult. In both figures the green has an uphill slope of 1.5° along the y-axis and the green speed is 11 feet on the Stimpmeter scale.

Figure 4.4 shows two trajectories that lead to a holeout for a 5-foot sidehill (0°) putt. Figure 4.4(a) corresponds to the minimum-velocity trajectory, while Figure 4.4(b) corresponds to the



(a) Downhill 90°

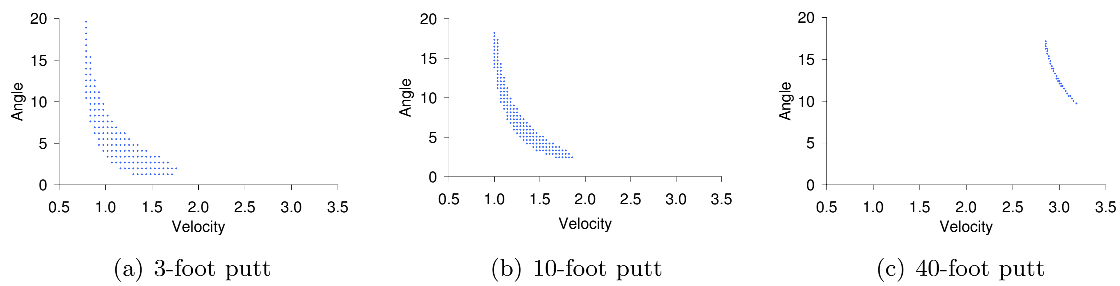
(b) Sidedown 45°

(c) Sidehill 0°

(d) Sideup −45°

(e) Uphill −90°

Figure 4.2: This figure shows how the holeout region varies with respect to the initial position for a 5-foot putt on a green that has a slope of 1.5° along the y-axis. The initial positions for downhill, sidedown, sidehill, sideup and uphill putts correspond to angles of $90°, 45°, 0°, -45°$ and $-90°$, respectively, with respect to the x-axis. The green speed is 11 feet ($\eta = 0.0510$).



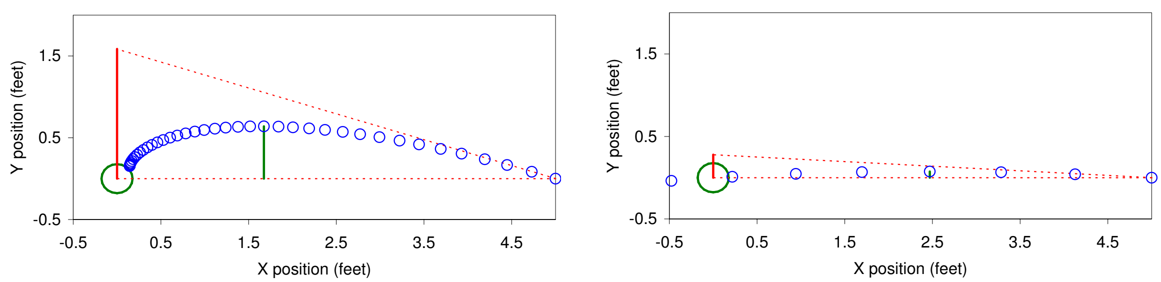
(a) 3-foot putt

(b) 10-foot putt

(c) 40-foot putt

Figure 4.3: This figure shows how the holeout region varies with distance for sidedown (45°) putts. The green has a slope of 1.5° with respect to the y-axis, and the green speed is 11 feet ($\eta = 0.0510$). Holeout regions are shown for 3-foot, 10-foot and 40-foot putts. As the length of the putt increases, fewer velocity-angle combinations lead to a holeout.

maximum-velocity trajectory. While the ball comes to rest just within the hole in the minimum-velocity case, in the maximum-velocity case the ball just avoids escaping the hole. In particular, it would come to a stop 9.7 feet beyond the hole, if the hole was covered. In general, many velocity-angle combinations lead to a holeout, but the optimal strategy must consider execution errors, green reading errors, and the stop point of the ball after a miss.



(a) $v = 1.25\text{m/s}, \alpha = 17.6°$ (b) $v = 2.01\text{m/s}, \alpha = 3.19°$

Figure 4.4: This figure shows the trajectories corresponding to the minimum and maximum velocities that lead to a holeout for a 5-foot sidehill (0°) putt. The green speed is 11 feet ($\eta = 0.0510$).

4.4.2 Player models

In this section, we separately calibrate the golfer model to professional and amateur data and identify their optimal putting strategies. Unless otherwise mentioned, for each putt-length, the results are averaged over putts with initial ball position at angles $\{0°, 7.5°, 15°, \ldots, 360°\}$ with respect to the x-axis.

Professional golfer: We calibrate the golfer model to professional golfer data using a green with a slope of 1.5°, and green speed of 11 feet. Table 4.3 shows the four parameter sets that we have calibrated for the professional player. While parameter set 1 calibrates to only one-putt probability data, parameter sets 2-4 also calibrate to expected-putts data. Parameter sets 2-4 can be thought of as representing professional players that differ in their ability to putt the ball to the desired distance and direction, as well in their green-reading skills. For brevity, we report detailed results only for the parameter set 3 of the professional player. Results when using the other parameter sets are qualitatively similar. The results of the calibration exercise for these other parameter sets, as well as a comparison between professional players with skills corresponding to Parameter Sets 3 and 4 are presented in the Appendix.

Table 4.3: Calibrated parameters for the professional golfer

	Set 1	**Set 2**	**Set 3**	**Set 4**
Calibration Target				
One-putt probability	Yes	Yes	Yes	Yes
Expected putts	No	Yes	Yes	Yes
Calibrated parameters				
β_0	4.91%	6.31%	5.90%	5.62%
β_1	13.05%	10.20%	9.20%	9.60%
β_2	5.04%	6.31%	6.14%	5.62%
σ_α	1.13	1.11	1.19	1.10
d_β	10.86	18.43	8.18	16.74
σ_g	0.147	0.124	0.085	0.148
RMSE				
One-putt probability	0.35%	0.35%	0.35%	0.37%
Expected putts	0.089	0.006	0.008	0.007

This table shows four calibrated parameter sets for the professional player. While Set 1 calibrates only to one-putt probabilities (i.e., probability root mean-squared error is minimized), Sets 2-4 calibrate to expected putts as well (i.e., sum of probability root mean-squared error and expected-putts root mean-squared error is minimized). For Parameter Set 1, the RMSE is shown under the one-putt probability maximization strategy, while for Parameter Set 2-4, the RMSE is shown under the expected-putts minimization strategy.

Table 4.4 shows the model fit to professional golfer data for parameter set 3. The overall RMSE difference between model and data one-putt probabilities is 0.35%, and between the model and data expected-putts is 0.008 putts, indicating a good fit.

Table 4.4 and Figure 4.5 also show the variation in the optimal distance to target beyond the hole d^* as a function of putt-length. Under the expected-putts minimization strategy, for putts $>$ 10 feet, d^* decreases with the length of the putt, and is less that two feet for all putts. The optimal fraction of putts to leave short increases with the length of the putt, but always remains less than 50%. These two metrics can be interpreted as measures of putting aggressiveness: targeting greater distances beyond the hole and leaving a smaller fraction of putts short of the hole indicate a more aggressive putter. For short putts, e.g., $<$ 10 feet, the optimal strategy is quite aggressive, e.g., fewer than 1% of 6-foot putts should fall short of the hole. For longer putts, the optimal strategy is more conservative, e.g., to aim less than one foot beyond the hole on all putts $>$ 30 feet, with

Table 4.4: Professional golfer: data fit and strategy comparison - Parameter Set 3

Data			Model											
			Min exp putts				Max one putt prob				1.5 feet beyond hole			
d	P_1	N^*	P_1	N^*	d^*	f_s	P_1	N^*	d^*	f_s	P_1	N^*	d^*	f_s
2	99.2%	1.01	99.8%	1.00	1.20	0.0%	99.8%	1.00	1.20	0.0%	99.7%	1.00	1.50	0.0%
3	95.5%	1.05	95.9%	1.04	1.30	0.1%	95.9%	1.04	1.33	0.1%	95.1%	1.05	1.50	0.0%
4	86.6%	1.14	86.6%	1.13	1.43	0.2%	86.6%	1.13	1.49	0.2%	85.3%	1.15	1.50	0.0%
5	75.9%	1.24	75.9%	1.24	1.51	0.4%	75.9%	1.24	1.60	0.4%	74.4%	1.26	1.50	0.1%
6	65.5%	1.35	66.1%	1.34	1.56	0.8%	66.2%	1.34	1.74	0.7%	64.7%	1.35	1.50	0.3%
7	57.1%	1.43	57.6%	1.43	1.58	1.3%	57.7%	1.43	1.82	1.2%	56.4%	1.44	1.50	0.8%
8	49.9%	1.51	50.4%	1.50	1.60	2.0%	50.6%	1.51	1.93	1.6%	49.5%	1.51	1.50	1.7%
9	44.2%	1.56	44.4%	1.56	1.62	2.8%	44.7%	1.57	2.04	2.1%	43.7%	1.57	1.50	3.0%
10	39.3%	1.61	39.3%	1.61	1.62	4.1%	39.7%	1.63	2.16	2.5%	38.8%	1.62	1.50	4.7%
15	23.0%	1.78	23.3%	1.78	1.50	14.6%	24.1%	1.81	2.54	5.5%	23.3%	1.78	1.50	14.4%
21	14.4%	1.88	14.0%	1.89	1.23	27.7%	15.3%	1.94	2.91	10.1%	14.4%	1.89	1.50	23.7%
25	10.8%	1.92	10.4%	1.93	1.05	34.1%	12.0%	2.00	3.04	13.4%	11.1%	1.94	1.50	28.1%
29	8.1%	1.97	8.0%	1.97	0.89	38.7%	9.6%	2.05	3.22	16.1%	8.8%	1.98	1.50	31.4%
33	6.7%	1.99	6.2%	2.01	0.70	42.5%	8.0%	2.09	3.32	18.8%	7.2%	2.02	1.50	33.9%
37	5.0%	2.04	5.1%	2.04	0.57	44.8%	6.7%	2.12	3.39	21.2%	6.0%	2.06	1.50	35.8%
42	4.1%	2.07	4.0%	2.07	0.38	47.1%	5.5%	2.16	3.48	23.7%	4.9%	2.09	1.50	37.7%
50	2.8%	2.15	2.9%	2.13	0.20	48.7%	4.2%	2.21	3.50	27.5%	3.7%	2.15	1.50	39.9%
		RMSE	0.35%	0.008			1.00%	0.052			0.76%	0.013		

This table shows the fit of the professional golfer data with the golfer model for different putt lengths d (in feet), when we attempt to calibrate to both one-putt probabilities and expected putts. It also compares the minimize expected number of putts, maximize one-putt probability, and aim 1.5 feet beyond the hole strategies for the professional golfer. The golfer parameters are $\beta_0 = 6.14\%$, $\beta_1 = 9.20\%$, $\beta_2 = 6.14\%$, $v_\beta = 8.1760$ feet, $\sigma_\alpha = 1.1900$, $\sigma_g = 0.0848$. We refer to this set as Parameter Set 3 for the Professional golfer. The green slope is $1.5°$, and green speed is 11 feet. The fit is good with respect to the one-putt probabilities (P_1) as well as the expected number of putts (N^*) (as shown by the root-mean-squared error (RMSE) values under the expected-putts minimization strategy). For short distances, minimizing the expected number of putts and maximizing the one-putt probabilities yield similar results. The optimal distance beyond the hole to aim at (d^*) (and consequently the fraction of short putts, f_s) depends on the length of the putt, and in general, differs from aiming 1.5 ft beyond the hole. (For each fixed distance, P_1, N^*, d^* and f_s are averaged over the angle the initial position of the putt makes with the x-axis.)

Table 4.5: Expected number of putts for professional golfer - Parameter Set 3

angle with x-axis	Length of putt (feet) 2	3	4	6	8	10	15	21	25	42	50
90	1.000	1.007	1.041	1.177	1.320	1.437	1.636	1.779	1.846	2.013	2.071
75	1.000	1.010	1.057	1.213	1.361	1.477	1.668	1.804	1.868	2.031	2.088
60	1.000	1.020	1.089	1.280	1.437	1.552	1.725	1.852	1.908	2.057	2.112
45	1.001	1.031	1.120	1.334	1.497	1.609	1.774	1.889	1.937	2.079	2.133
30	1.001	1.042	1.145	1.370	1.534	1.643	1.811	1.912	1.955	2.094	2.149
15	1.002	1.051	1.162	1.390	1.552	1.659	1.831	1.924	1.966	2.104	2.160
0	1.003	1.056	1.170	1.397	1.555	1.674	1.837	1.927	1.968	2.106	2.163
-15	1.003	1.058	1.173	1.397	1.556	1.676	1.833	1.924	1.966	2.102	2.158
-30	1.003	1.058	1.171	1.390	1.553	1.667	1.822	1.915	1.956	2.093	2.148
-45	1.003	1.055	1.163	1.379	1.542	1.653	1.809	1.903	1.945	2.081	2.135
-60	1.002	1.050	1.153	1.365	1.529	1.638	1.796	1.893	1.937	2.070	2.124
-75	1.002	1.045	1.144	1.354	1.517	1.628	1.787	1.886	1.930	2.063	2.117
-90	1.002	1.044	1.140	1.350	1.512	1.622	1.784	1.884	1.928	2.061	2.115
Avg. model	1.002	1.041	1.135	1.341	1.501	1.614	1.781	1.886	1.933	2.075	2.130
Avg. data	1.009	1.046	1.137	1.349	1.507	1.614	1.783	1.879	1.925	2.066	2.148
Difference	-0.007	-0.005	-0.002	-0.008	-0.006	0.000	-0.002	0.007	0.008	0.009	-0.018

This table shows how the expected number of putts varies for a professional golfer (parameter set 3) as a function of putt position and length of the putt. The expected number of putts to holeout increases with distance. Sidehill putts, i.e., putts from initial angles between $-30°$ and $30°$, are the most difficult.

30% to 50% of the putts finishing short of the hole. For short putts, the optimal strategy makes sure the putt is hit with enough velocity that there is a chance of a holeout. For long putts, the optimal strategy is more conservative to avoid a possible three-putt.

Comparing between strategies, we find that minimizing expected number of putts leads to similar results as maximizing the probability of one-putt for short putts, e.g., < 7 feet. For these distances, aiming 1.5 feet beyond the hole leads to higher expected number of putts than the myopic one-putt probability maximizing strategy, and smaller one-putt probability than the expected-putts minimizing strategy. This is because for short distances, aiming 1.5 feet beyond the hole leaves too many putts short for a professional golfer. We also observe that minimizing expected number of putts involves aiming shorter than for maximizing one-putt probability. For long putts, e.g., > 25 feet, the 1.5-foot strategy lies in between the one-putt probability maximizing and expected number of putts minimizing strategies, and leads to one-putt probabilities and expected number of putts that lie between the two strategies.

Table 4.5 and Figure 4.5 also show the variation in the expected-putts minimizing strategy as a function of the initial putt-angle. Graph (a) shows that sidehill putts, making an angle of $-30°$ to $30°$, are the most difficult, i.e., lead to the highest expected number of putts for a given putt-length. Graphs (c) shows that while there is some variation in the optimal distance beyond the hole to target for short putts, e.g., putts < 10 feet (where d^* is the largest for sidedown putts and the smallest for uphill putts), for longer putts, the variation in d^* across putt-angles is much smaller. Graphs (e) shows that the fraction of putts that are short increases with putt length. A comparison of these results with results that are obtained when using the one-putt probability maximizing strategy is presented in the Appendix.

Figure 4.6 shows how the optimal aim direction varies with initial putt position for 3-foot, 15-foot and 50-foot putts, and compares it with the aim direction for the 1.5-foot strategy, as well as shows the maximum and minimum angles that could lead to a holeout. The optimal aim direction becomes closer to the maximum angle that would lead to a holeout as the putt-length increases, showing that the player becomes more conservative as the length of the putt increases.

Figure 4.7 shows, for 5-foot and 25-foot sidehill putts, the target velocities and angles corresponding to the expected putts minimization (Min exp), one-putt probability maximization (Max prob), and 1.5-foot (1.5 feet) strategies. The holeout region and the 1.5 feet, 4 feet and 7 feet beyond the hole contours, as well as the contours corresponding to leaving 10% and 50% of the putts short are also shown (assuming zero green error). We observe that the optimal expected-putts minimization strategy is different from the one-putt probability maximizing and the 1.5-foot strategy, and as expected, lies in the holeout region.

Amateur golfer: We next fit the amateur data using the golfer model on a green with slope 1.5°, and a green speed of 9 feet ($\eta = 0.0623$). Table 4.6 shows the four parameter sets that we calibrated for the amateur player. As for the professional player, parameter set 1 calibrates to only one-putt probability data, while parameter sets 2-4 also calibrate to expected-putts data. Again, parameter sets 2-4 can be thought of as representing amateur players that differ in their ability to putt the ball to the desired distance and direction, as well in their green-reading skills. For brevity, we report results for only the parameter set 2 of the amateur player. Results when using the other parameter sets are qualitatively similar. The results of the calibration exercise for these other parameter sets, as well as a comparison between amateur players with skills corresponding to Parameter Sets 2 and 4 are presented in the Appendix.

Table 4.7 shows the model fit to data for parameter set 2. The overall RMSE difference between model and data one-putt probabilities is 0.35%, and between the model and data expected-putts is 0.007 putts, indicating a good fit.

Table 4.7 and Figure 4.5 also show the variation in the optimal distance to target beyond the hole d^* as a function of putt-length. Under the expected-putts minimization strategy, for putts > 10 feet, d^* decreases with the length of the putt, and is less that 1.5 feet for all putts. As in

Table 4.6: Calibrated parameters for the amateur golfer

	Set 1	**Set 2**	**Set 3**	**Set 4**
Calibration Target				
One-putt probability	Yes	Yes	Yes	Yes
Expected putts	No	Yes	Yes	Yes
Calibrated parameters				
β_0	7.66%	8.94%	8.77%	8.93%
β_1	18.88%	18.89%	18.18%	18.72%
β_2	8.47%	8.99%	8.77%	8.93%
σ_α	1.91	1.90	2.00	2.01
d_β	8.20	42.35	40.14	39.03
σ_g	0.214	0.213	0.195	0.178
RMSE				
One-putt probability	0.28%	0.35%	0.43%	0.39%
Expected putts	0.131	0.007	0.006	0.005

This table shows four calibrated parameter sets for the amateur player. While Set 1 calibrates only to one-putt probabilities (i.e., probability root mean-squared error is minimized), Sets 2-4 calibrate to expected putts as well (i.e., sum of probability root mean-squared error and expected-putts root mean-squared error is minimized). For Parameter Set 1, the RMSE is shown under the one-putt probability maximization strategy, while for Parameter Set 2-4, the RMSE is shown under the expected-putts minimization strategy.

the case of professional golfers, the fraction of putts that are short increases with the length of the putt, and is close to, though smaller than, 50% for putts longer than 30 feet.

Comparing between strategies, we again find that minimizing expected number of putts leads to similar results as maximizing the probability of one-putt for short putts, e.g., < 6 feet. As for the professional player, we also observe that minimizing expected number of putts involves aiming shorter than for maximizing one-putt probability. Unlike the professional player though, the optimal distance to target, d^*, is smaller than 1.5 feet for all putt-lengths.

Figure 4.5 shows the variation in the expected-putts minimizing strategy as a function of the initial putt-angle. While the results are qualitatively similar to that for the professional player, the optimal strategy for the amateur player is to aim a shorter distance beyond the hole and to leave a higher proportion of putts short, thereby indicating that amateur golfers are more conservative than professional golfers. This is also seen from Figure 4.6, wherein the optimal aim direction

Table 4.7: Amateur golfer: data fit and strategy comparison - Parameter Set 2

Data			Model											
			Min exp putts				Max one putt prob				1.5 feet beyond hole			
d	P_1	N^*	P_1	N^*	d^*	f_s	P_1	N^*	d^*	f_s	P_1	N^*	d^*	f_s
2	93.9%	1.06	95.0%	1.05	0.88	0.1%	95.0%	1.05	0.90	0.1%	93.6%	1.07	1.50	0.0%
3	79.2%	1.21	79.2%	1.21	0.98	0.9%	79.2%	1.21	1.04	0.7%	77.6%	1.24	1.50	0.0%
4	64.1%	1.37	63.8%	1.37	1.05	2.1%	63.9%	1.37	1.18	1.6%	62.7%	1.39	1.50	0.3%
5	52.1%	1.49	51.8%	1.49	1.09	4.0%	52.0%	1.50	1.32	2.5%	51.4%	1.52	1.50	1.2%
6	43.0%	1.58	42.8%	1.59	1.09	6.7%	43.3%	1.60	1.43	3.4%	42.8%	1.61	1.50	2.6%
7	36.0%	1.66	35.9%	1.66	1.08	9.8%	36.6%	1.68	1.56	4.3%	36.3%	1.68	1.50	4.3%
8	30.5%	1.72	30.6%	1.72	1.05	13.2%	31.5%	1.74	1.66	5.3%	31.2%	1.74	1.50	6.3%
9	26.2%	1.76	26.2%	1.77	1.02	16.5%	27.5%	1.80	1.77	6.2%	27.2%	1.79	1.50	8.5%
10	22.6%	1.80	22.8%	1.81	0.97	19.9%	24.2%	1.84	1.84	7.3%	23.9%	1.83	1.50	10.6%
15	12.2%	1.93	12.2%	1.94	0.67	34.6%	14.5%	2.01	2.20	11.9%	14.0%	1.97	1.50	19.7%
21	6.9%	2.03	7.0%	2.04	0.37	43.8%	9.1%	2.12	2.39	17.8%	8.6%	2.08	1.50	27.0%
25	5.1%	2.08	5.3%	2.09	0.24	46.5%	7.0%	2.18	2.49	20.8%	6.6%	2.13	1.50	30.3%
29	3.9%	2.13	4.2%	2.14	0.14	48.1%	5.6%	2.23	2.55	23.8%	5.2%	2.18	1.50	32.9%
33	3.0%	2.19	3.4%	2.19	0.08	49.0%	4.5%	2.27	2.50	27.4%	4.2%	2.24	1.50	35.3%
37	2.4%	2.24	2.8%	2.25	0.07	49.1%	3.6%	2.32	2.56	29.6%	3.4%	2.29	1.50	37.3%
42	1.8%	2.32	2.3%	2.31	0.07	49.2%	2.9%	2.37	2.53	32.4%	2.7%	2.34	1.50	39.2%
50	1.4%	2.39	1.7%	2.40	0.08	49.3%	2.0%	2.45	2.45	35.9%	1.9%	2.43	1.50	41.4%
		RMSE	0.35%	0.007			1.29%	0.057			1.14%	0.035		

This table shows the fit of the amateur golfer data with the golfer model for different putt lengths d (in feet), when we attempt to calibrate to both one-putt probabilities and expected putts. It also compares the minimize expected number of putts, maximize one-putt probability, and aim 1.5 feet beyond the hole strategies for the professional golfer. The golfer parameters are $\beta_0 = 8.94\%$, $\beta_1 = 18.89\%$, $\beta_2 = 8.99\%$, $v_\beta = 42.35$ feet, $\sigma_\alpha = 1.8964$, $\sigma_g = 0.2128$. We refer to this set as Parameter Set 2 for the Amateur golfer. The green slope is 1.5°, and green speed is 9 feet. The fit is good with respect to both the one-putt probabilities (P_1) and the expected number of putts (N^*) (as seen by the RMSE under the expected-putts minimization strategy). For short distances, minimizing the expected number of putts and maximizing the one-putt probabilities yield similar results. The optimal distance beyond the hole to aim at (d^*) (and consequently the fraction of short putts, f_s) depends on the length of the putt, and in general, differs from aiming 1.5 ft beyond the hole. (For each fixed distance, P_1, N^*, d^* and f_s are averaged over the angle the initial position of the putt makes with the x-axis.)

Table 4.8: Expected number of putts for amateur golfer - Parameter Set 2

angle with x-axis	Length of putt (feet) 2	3	4	6	8	10	15	21	25	42	50
90	1.022	1.134	1.275	1.503	1.655	1.758	1.911	2.019	2.078	2.273	2.345
75	1.024	1.144	1.288	1.517	1.666	1.767	1.917	2.024	2.082	2.277	2.347
60	1.031	1.167	1.320	1.547	1.691	1.788	1.932	2.035	2.092	2.285	2.355
45	1.040	1.192	1.352	1.579	1.717	1.808	1.943	2.046	2.102	2.293	2.363
30	1.048	1.213	1.378	1.603	1.736	1.823	1.953	2.053	2.109	2.300	2.369
15	1.055	1.228	1.395	1.616	1.746	1.830	1.957	2.055	2.110	2.302	2.391
0	1.060	1.238	1.403	1.622	1.748	1.830	1.955	2.052	2.107	2.304	2.413
-15	1.063	1.242	1.405	1.620	1.745	1.826	1.950	2.046	2.101	2.321	2.425
-30	1.064	1.241	1.403	1.615	1.738	1.820	1.942	2.037	2.091	2.330	2.430
-45	1.063	1.238	1.398	1.609	1.733	1.814	1.935	2.029	2.083	2.334	2.432
-60	1.061	1.234	1.393	1.604	1.726	1.807	1.929	2.022	2.075	2.334	2.431
-75	1.060	1.231	1.390	1.600	1.723	1.804	1.925	2.018	2.071	2.334	2.430
-90	1.059	1.230	1.389	1.598	1.722	1.802	1.923	2.016	2.070	2.335	2.430
Avg. model	1.050	1.211	1.370	1.589	1.720	1.807	1.937	2.035	2.091	2.310	2.397
Avg. data	1.063	1.212	1.366	1.584	1.715	1.802	1.932	2.027	2.081	2.319	2.393
Difference	-0.012	-0.001	0.004	0.005	0.005	0.005	0.005	0.008	0.010	-0.009	0.004

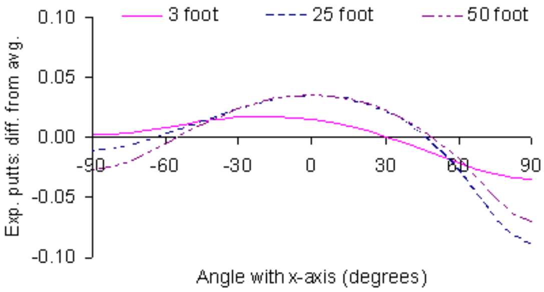
This table shows how the expected number of putts varies for a amateur golfer (parameter set 2) as a function of putt position and length of the putt. The expected number of putts to holeout increases with distance. Sidehill putts, i.e., putts from initial angles between $-30°$ and $30°$, and sideup putts, i.e., putts from initial angles between $-30°$ and $-60°$, are among the most difficult.

for the amateur player is closer to the maximum angle that leads to a holeout, for a given putt length. As with the professional player, the optimal aim direction becomes closer to the maximum angle that would lead to a holeout as the putt-length increases. A comparison of these results with results that are obtained when using the one-putt probability maximizing strategy is presented in the Appendix.

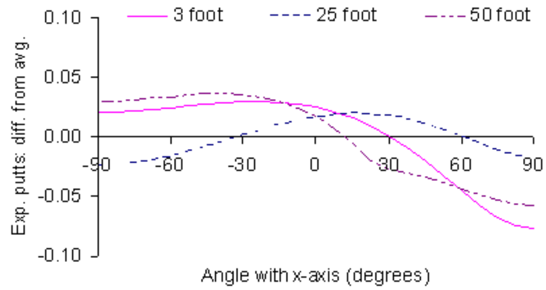
Table 4.8 shows how the expected number of putts varies as a function of the putt position for different distances. We observe that sidehill putts (between $-30°$ and $30°$), and sideup putts (between $-30°$ and $-60°$), are among the most difficult.

Putt to round summary: The number of putts per 18-hole round is given by weighting the expected number of putts for a given distance by the number of putts per round from that starting distance. We aggregate the results for different putt lengths by weighing the expected number of putts required for each putt length with the number of such putts a golfer typically hits during a 18-hole round. Table 4.9 summarizes the putts per round for the Professional players with skills corresponding to each of the four calibrated parameter sets. For each parameter set, results are shown for each of the following three strategies: minimize expected number of putts, maximize probability of one-putt, and aim 1.5 feet beyond the hole. As expected from the discussion above, expected-putts minimizing strategy for Parameter Set 1 results in more putts per round than observed in the data. However, Parameter Set 2, Parameter Set 3, and Parameter Set 4, under the expected-putts minimizing strategy, lead to putts per round that closely match the putts per round number from the data. We also observe that for a 18-hole round, the optimal strategy can save the professional golfer approximately 0.60 putts per round over the one-putt probability maximizing strategy, and approximately 0.15 putts per round over 1.5-foot strategy.

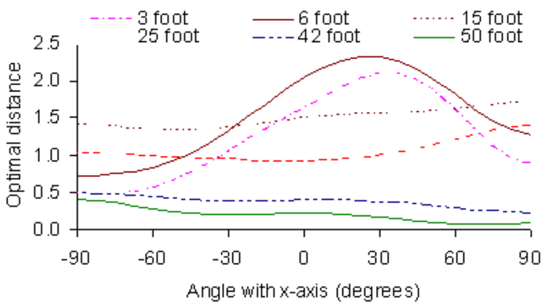
Table 4.10 summarizes the putts per round for the Amateur players with skills corresponding to each of the four calibrated parameter sets. For each parameter set, results are shown for each of the following three strategies: minimize expected number of putts, maximize probability of



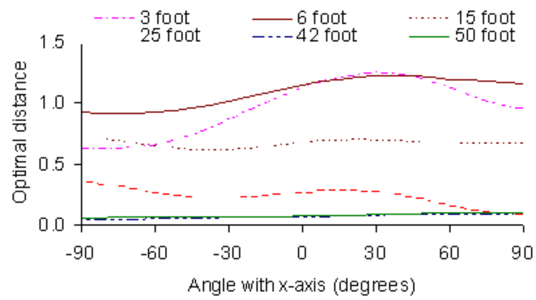
(a) Professional golfer



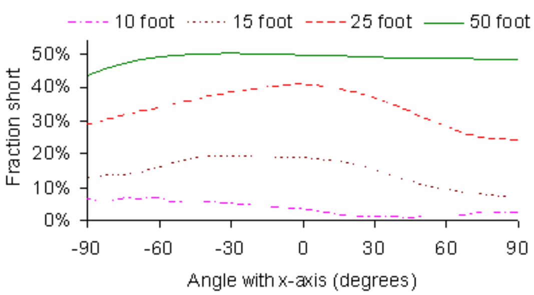
(b) Amateur golfer



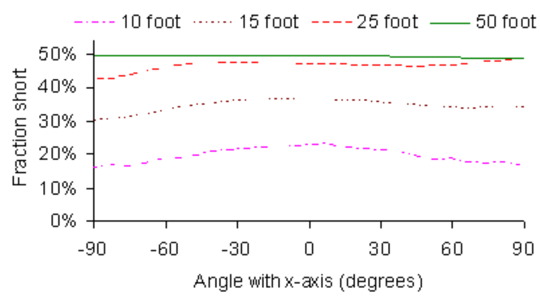
(c) Professional golfer



(d) Amateur golfer

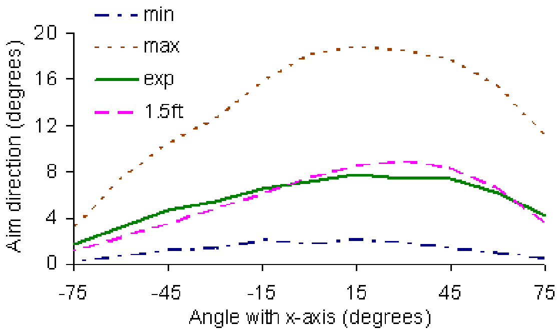


(e) Professional golfer

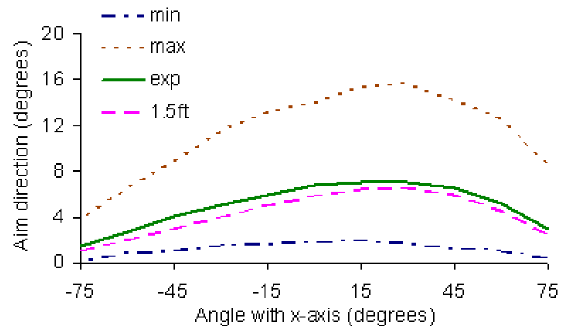


(f) Amateur golfer

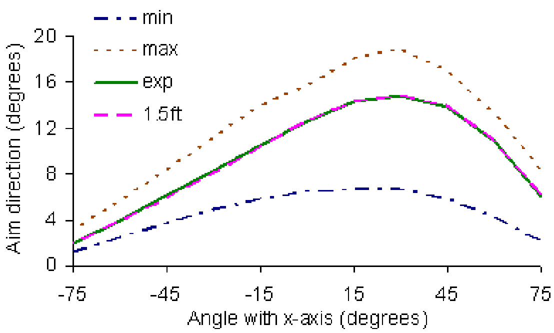
Figure 4.5: This figure shows how the optimal expected number of putts, target distance beyond the hole (in feet), and fraction of putts that are short of the hole vary as a function of initial angle of the putt and putt length, for professional and amateur golfers. Graph (a) shows that sidehill putts, making an angle of $-30°$ to $30°$ lead to the highest expected number of putts, irrespective of putt length, for the professional player. Graph (b) shows while sidehill putts continue to be among the hardest for the amateur player, for short putt-lengths, uphill putts ($-90°$ to $-60°$) are hard as well. Graphs (c) and (d) show that the professional golfer is more aggressive than the amateur golfer, i.e., aims a greater distance beyond the hole, especially for short putt lengths. For longer putts, it is optimal for golfers to aim a smaller distance beyond the hole. Graphs (e) and (f) show that the fraction of putts that are short increases with putt length for both professional and amateur golfers. Professional golfers are more aggressive than amateur golfers, and leave a smaller fraction of putts short. Parameter Set 3 for the professional player and Parameter Set 2 for the amateur player were used to generate these results.



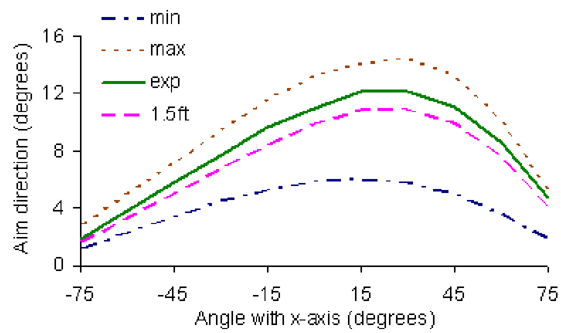
(a) Professional golfer: 3-foot putt



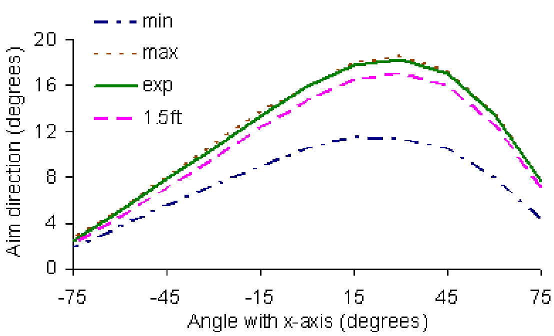
(b) Amateur golfer: 3-foot putt



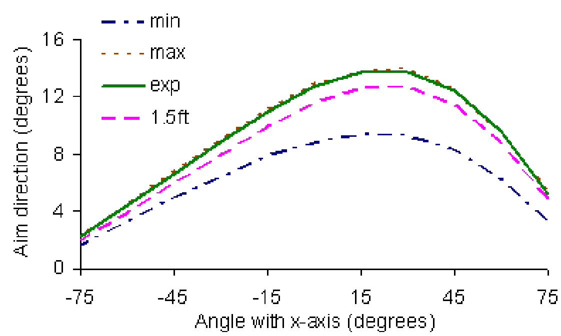
(c) Professional golfer: 15-foot putt



(d) Amateur golfer: 15-foot putt

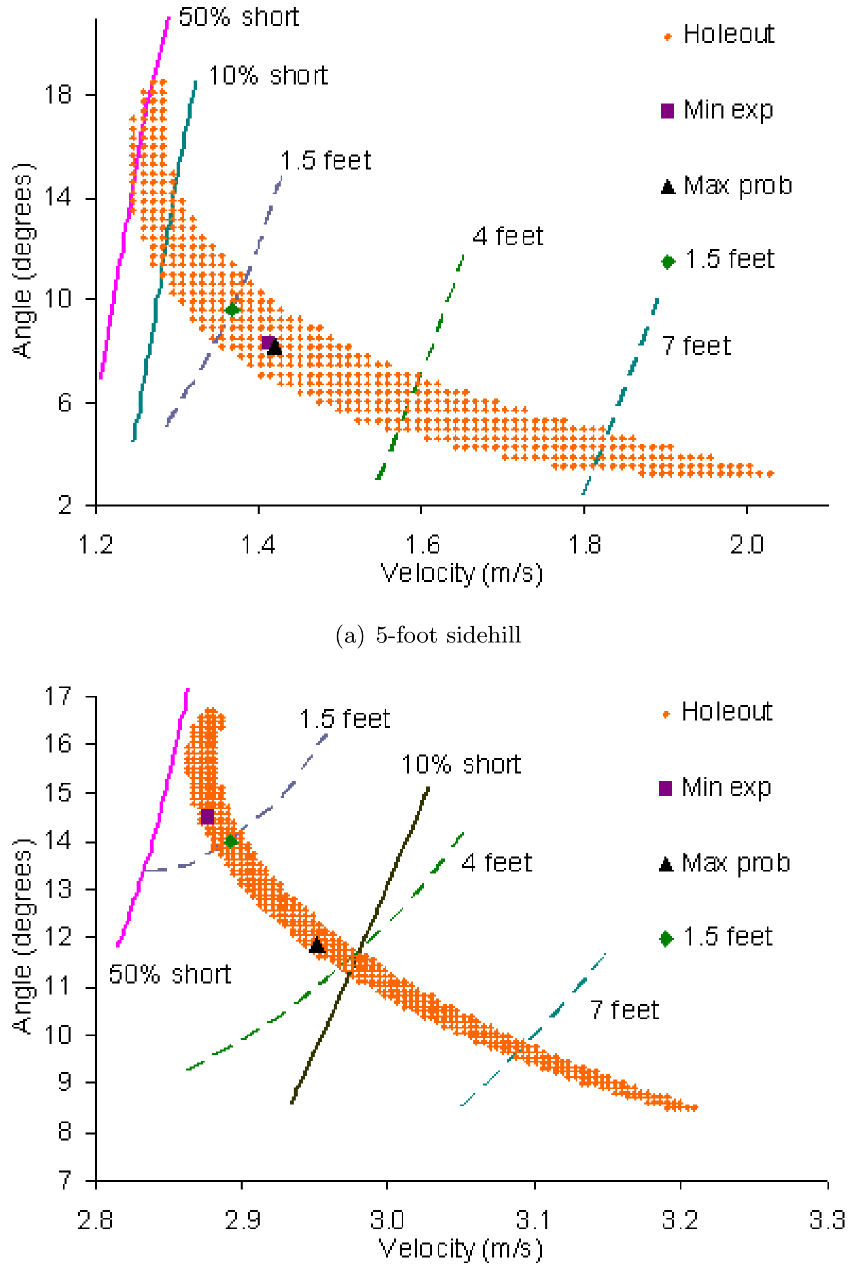


(e) Professional golfer: 50-foot putt



(f) Amateur golfer: 50-foot putt

Figure 4.6: This figure shows how the optimal aim direction changes with respect to initial position on the green for the professional and the amateur golfer for 3-foot, 15-foot and 50-foot putts. The maximum and the minimum possible angles that lead to a holeout are also shown along with the angle corresponding to strategy that aims 1.5 feet beyond the hole. These differ for professional and amateur golfers because of different green speeds (11 feet and 9 feet for professional and amateur golfer, respectively). As putt length increases, both professional and amateur golfers become more conservative, allowing for more break (curvature) in the putts. Parameter Set 3 for the professional player and Parameter Set 2 for the amateur player were used to generate these results.



(a) 5-foot sidehill

(b) 25-foot sidehill

Figure 4.7: This figure shows the holeout region the target velocities and angles corresponding to the expected putts minimization (Min exp), one-putt probability maximization (Max prob), and aiming 1.5 feet beyond the hole (1.5 feet) strategies for a 5-foot and 25-foot sidehill putt on a green with slope 1.5°, and green speed 11 feet. The holeout region and the 1.5 feet, 4 feet and 7 feet beyond the hole contours are shown assuming zero green error. Parameter Set 3 for the professional player was used to generate these results.

one-putts, and aim 1.5 feet beyond the hole. As expected, expected-putts minimizing strategy for Parameter Set 1 results in more putts per round than observed in the data. However, Parameter Set 2, Parameter Set 3, and Parameter Set 4, under the expected-putts minimizing strategy, lead to putts per round that closely match the putts per round number from the data. We observe that for a 18-hole round, the optimal strategy can save the amateur golfer approximately 0.8 putts per round over the one-putt probability maximizing strategy, and approximately 0.6 putts per round over the 1.5-foot strategy. Thus the savings for the amateur player upon following the optimal strategy are much higher than that for the professional player.

4.5 Conclusion

We developed a model of golfer putting ability, along with a model for putt trajectory and holeout. We modeled the two main aspects of a golfer's putting skill: the physical skill, which reflects the golfer's ability to putt with the desired target velocity and angle, and the green reading skill, which reflects the golfer's ability to estimate the slope of the green. Direction error was found to have a greater effect on uphill putts, while green reading error had a greater affect on downhill putts. The model was calibrated to real-world professional and amateur golfer data. The problem of finding the optimal golfer putting strategy was formulated as a two-dimensional stochastic shortest path problem, and solved using approximate dynamic programming. Two other golfer strategies were also considered: the myopic strategy that seeks to maximize the probability of one-putt, and the static strategy that would lead to a trajectory that passes through the center of the hole and would stop at a distance 1.5 feet beyond the hole, if the hole were covered.

For long putts, e.g., > 10 feet, we found that the optimal distance beyond the hole to aim at decreased as the putt-length increased. Golfers became more conservative as putt-lengths increased, in that the optimal aim angle became closer to the maximum angle that would lead to a holeout. As a result, trajectories corresponding to the optimal strategy for long putts also curved more than

Table 4.9: Putts per round for the professional golfer

Professional golfer														
			Parameter Set 1			Parameter Set 2			Parameter Set 3			Parameter Set 4		
	Avg. no.		Min	Max	1.5	Min	Max	1.5	Min	Max	1.5	Min	Max	1.5
d	per round	Data	Exp	Prob	feet	Exp	Prob	feet	Exp	Prob	feet	Exp	Prob	feet
2	1.00	1.01	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
3	1.51	1.05	1.04	1.04	1.05	1.04	1.04	1.04	1.04	1.04	1.05	1.04	1.04	1.04
4	0.78	1.14	1.14	1.14	1.15	1.13	1.13	1.15	1.13	1.13	1.15	1.13	1.13	1.15
5	0.78	1.24	1.25	1.25	1.26	1.25	1.25	1.26	1.24	1.24	1.26	1.25	1.25	1.26
6	0.78	1.35	1.35	1.35	1.36	1.35	1.35	1.36	1.34	1.34	1.35	1.35	1.35	1.36
7	0.69	1.43	1.43	1.44	1.44	1.44	1.44	1.45	1.43	1.43	1.44	1.44	1.44	1.45
8	0.69	1.51	1.50	1.51	1.51	1.51	1.52	1.52	1.50	1.51	1.51	1.51	1.52	1.52
9	0.69	1.56	1.57	1.57	1.57	1.57	1.58	1.58	1.56	1.57	1.57	1.57	1.58	1.58
10	1.59	1.61	1.62	1.63	1.62	1.62	1.63	1.62	1.61	1.63	1.62	1.62	1.63	1.62
15	2.11	1.78	1.80	1.83	1.80	1.78	1.82	1.78	1.78	1.81	1.78	1.78	1.81	1.78
21	1.53	1.88	1.91	1.97	1.92	1.88	1.94	1.89	1.89	1.94	1.89	1.89	1.94	1.89
25	1.48	1.92	1.97	2.04	1.98	1.93	2.00	1.94	1.93	2.00	1.94	1.93	2.00	1.94
29	1.75	1.97	2.01	2.10	2.03	1.97	2.05	1.98	1.97	2.05	1.98	1.97	2.05	1.98
42	1.31	2.07	2.15	2.23	2.17	2.08	2.16	2.10	2.07	2.16	2.09	2.08	2.17	2.10
50	1.05	2.15	2.22	2.29	2.24	2.14	2.23	2.16	2.13	2.21	2.15	2.14	2.22	2.16
	Total	29.06	29.45	30.08	29.63	29.06	29.71	29.22	29.04	29.65	29.20	29.08	29.72	29.24
		Diff	0.39	1.03	0.57	0.01	0.65	0.16	-0.02	0.60	0.14	0.03	0.67	0.18

This table summarizes the results of the three strategies: minimize the expected number of putts (Min exp), maximize one-putt probability (Max Prob), and aiming 1.5 feet beyond the hole upon every putt (1.5 feet), for the four calibrated professional player parameter sets. The putts per round from the data and for each strategy are shown in the row labeled ‘Total’. The difference between putts per round from each strategy and the data are shown in the row labeled ‘Diff’. We observe that the strategy minimizing the expected number of putts can save the professional golfer approximately 0.15 shots over the aim 1.5 feet beyond the hole strategy.

Table 4.10: Putts per round for the amateur golfer

Amateur golfer														
			Parameter Set 1			Parameter Set 2			Parameter Set 3			Parameter Set 4		
	Avg. no.		Min	Max	1.5	Min	Max	1.5	Min	Max	1.5	Min	Max	1.5
d	per round	Data	Exp	Prob	feet	Exp	Prob	feet	Exp	Prob	feet	Exp	Prob	feet
2	1.15	1.06	1.05	1.05	1.07	1.05	1.05	1.07	1.05	1.05	1.08	1.05	1.05	1.08
3	0.46	1.21	1.22	1.21	1.24	1.21	1.21	1.24	1.22	1.22	1.25	1.22	1.22	1.25
4	0.86	1.37	1.37	1.37	1.39	1.37	1.37	1.39	1.37	1.38	1.41	1.37	1.37	1.41
5	0.86	1.49	1.49	1.49	1.51	1.49	1.50	1.52	1.50	1.50	1.53	1.49	1.50	1.52
6	0.86	1.58	1.59	1.60	1.61	1.59	1.60	1.61	1.59	1.60	1.62	1.59	1.59	1.62
7	0.89	1.66	1.67	1.68	1.69	1.66	1.68	1.68	1.66	1.67	1.69	1.66	1.67	1.69
8	0.89	1.72	1.74	1.75	1.76	1.72	1.74	1.74	1.72	1.74	1.75	1.72	1.73	1.74
9	0.89	1.76	1.79	1.81	1.81	1.77	1.80	1.79	1.77	1.79	1.79	1.76	1.79	1.79
10	1.76	1.80	1.84	1.87	1.86	1.81	1.84	1.83	1.81	1.84	1.83	1.80	1.83	1.83
15	2.62	1.93	2.01	2.07	2.04	1.94	2.01	1.97	1.93	2.00	1.97	1.93	1.99	1.97
21	1.75	2.03	2.15	2.22	2.18	2.04	2.12	2.08	2.03	2.12	2.08	2.03	2.11	2.08
25	1.25	2.08	2.23	2.29	2.26	2.09	2.18	2.13	2.09	2.17	2.13	2.08	2.17	2.13
29	1.47	2.13	2.30	2.34	2.32	2.14	2.23	2.18	2.14	2.22	2.19	2.14	2.22	2.19
42	1.28	2.32	2.46	2.49	2.48	2.31	2.37	2.34	2.31	2.37	2.35	2.31	2.38	2.35
50	0.93	2.39	2.53	2.56	2.55	2.40	2.45	2.43	2.40	2.45	2.43	2.40	2.44	2.43
	Total	32.90	34.19	34.76	34.60	32.96	33.77	33.48	32.94	33.74	33.58	32.91	33.68	33.56
		Diff	1.30	1.86	1.71	0.06	0.88	0.58	0.05	0.85	0.68	0.02	0.78	0.66

This table summarizes the results of the three strategies: minimize the expected number of putts (Min exp), maximize one-putt probability (Max Prob), and aiming 1.5 feet beyond the hole upon every putt (1.5 feet), for the four calibrated bogey player parameter sets. The putts per round from the data and for each strategy are shown in the row labeled ‘Total’. The difference between putts per round from each strategy and the data are shown in the row labeled ‘Diff’. We observe that the strategy minimizing the expected number of putts can save the bogey golfer approximately 0.6 shots over the aim 1.5 feet beyond the hole strategy.

trajectories for short putts. The optimal strategy for professional golfers involved putting with a greater velocity, and with less break, as compared to amateur golfers. This is expected, since professional golfers are better putters, and not only achieve a greater one-putt probability, but are also better at second putts. For short putts, e.g., < 6 feet, the target distance beyond the hole also depended upon whether the putt was an uphill putt, a downhill putt, or a sidehill putt, with the distance being the largest for sidedown putts and least for uphill putts. Sidehill and sideup putts were found to be among the hardest for both professional and amateur golfers.

While the expected putt minimizing and the one-putt probability maximizing strategies led to similar results for short putts, for long putts, the one-putt probability maximizing strategy led to a larger expected number of putts. The 1.5-foot strategy was also found to be suboptimal, especially for long putts where it is optimal to aim a shorter distance beyond the hole. For professional golfers, the optimal strategy resulted in a saving of approximately 0.15 putts per round over the 1.5-foot strategy, and about 0.60 putts per round over the one-putt probability maximizing strategy. The corresponding numbers for the amateur golfer were 0.60 putts per round, and 0.80 putts per round, respectively.

Acknowledgments

We are very thankful to Tony Renshaw for helping us with the details of dynamics of the ball's motion [53].

Appendix

The Appendix comprises of the following discussion and results. First, we present the equations of motion used to simulate the ball-trajectory as well as discuss their implementation. Second, given a trajectory, we provide details on how the inputs to the Penner holeout criterion are computed and how the criterion can be extended for a surface with both uphill (or downhill) and sidehill elevation. Third, we discuss how the expected-putts minimization problem formulated in Section 4.2 is an instance of the two-dimensional stochastic shortest-path problem. Fourth, we provide some details on the interpolation approach used to obtain the continuation strategy for second or subsequent putts. Fifth, we provide implementation details for the grid-search approach used for optimization and discuss the optimality of aiming straight at the hole in the case of a level green (with no slope).

Next, we present several results. First, we show the sink-zones for uphill, sidehill and downhill putts, and motivate the reason for simulating trajectories to identify holeouts rather than assuming a bivariate-normal distribution for the sink-zone directly to determine holeouts. Second, we present the calibration results for the remaining parameter sets for the professional and the amateur golfers. These results are analogous to the results provided in Section 4.4, and include a comparison between the expected-putts minimization strategy, the one-putt probability maximization strategy, and the 1.5-foot strategy. In addition, we compare uphill, downhill and sidehill putts, identify the optimal aim-direction for the expected-putts minimization strategy and compare it with the minimum and maximum angles that lead to a holeout, as well as the aim-direction for the one-putt probability maximization strategy. We find that sidehill putts are amongst the hardest, and that downhill putts are easier than uphill putts for one-putt probability maximization. The one-putt probability maximization strategy is similar to the expected-putts minimization strategy for short putt-lengths, but is more aggressive (putts are hit flatter and with more velocity) for longer putt-lengths. To understand the difference between players with different green-reading abilities, we compare the strategies obtained from two of the calibrated parameter sets for the professional player. We find that the player with larger direction error but smaller green-reading error aims a smaller distance beyond the hole for one-putt probability maximization and for expected-putt minimization for short putt-lengths. For long putt-lengths, it is hard to differentiate between the two players in terms of distance beyond the hole to aim at. A similar comparison is performed for two of the parameter sets corresponding to the amateur golfer. Finally, as for the professional player in Section 4.4, we present the holeout map together with the contour lines corresponding to leaving 10% and 50% of the putts short, contour lines corresponding to aiming 1.5 feet, 4 feet and 7 feet beyond the hole, as well as the optimal strategy for one-putt probability maximization, expected-putts minimization and the 1.5-foot strategy, for 5-foot and 25-foot sidehill putts for the amateur player.

Trajectory simulation

We simulate the ball trajectory under the assumptions that the green is flat (constant slope) and that the ball always rests on the surface of the green (no jumps, hops). We also neglect the rotational inertia. We assume that the x-y plane is the flat plane (along which the ball moves), and the z-axis is normal to it. The plane is inclined at an angle α, such that the ball would roll along the x-axis in a straight line.[2] Then the gravitational force can be written as $(g \sin\alpha, 0, g\cos\alpha)$. We also assume that $\tan\alpha < \eta$, so that the ball, if it comes to rest, remains at rest. We denote the position of the center of the ball by $x(t), y(t), z(t)$, and the force exerted by the plane on the ball as (N_x, N_y, N_z). We denote the mass of the ball by m.

Then, the equations of motion can be written as follows.

$$ma_x = N_x + mg\sin\alpha, \tag{4.23}$$

$$ma_y = N_y, \tag{4.24}$$

$$ma_z = N_z - mg\cos\alpha \tag{4.25}$$

where (a_x, a_y, a_y) denotes the acceleration of the ball, and m denotes the mass of the ball. As we assume that the ball does not leave the surface of the plane, $a_z = 0$, and so $N_z = mg\cos\alpha$. Also,

[2] Note that this co-ordinate system is different from the co-ordinate system discussed in Section 4.2.1 as we now assume that the plane on which the ball moves is itself the x-y plane. Also, as the z-axis is normal to this plane, gravity does not act along the negative z-axis. The transformation needed to switch between these co-ordinate systems is discussed later.

under the sliding frictional force model, we have

$$N_x = -\mu N_z \frac{v_x}{\sqrt{v_x^2 + v_y^2}}, \tag{4.26}$$

$$N_y = -\mu N_z \frac{v_y}{\sqrt{v_x^2 + v_y^2}}, \tag{4.27}$$

where (v_x, v_y, v_z) denote the velocity of ball, and under the assumption above $v_z = 0$.

Then, the equations of motion can be re-written as follows.

$$\frac{dx}{dt} = v_x, \tag{4.28}$$

$$\frac{dy}{dt} = v_y, \tag{4.29}$$

$$\frac{dv_x}{dt} = g \sin \alpha - \mu g \cos \alpha \frac{v_x}{\sqrt{v_x^2 + v_y^2}}, \tag{4.30}$$

$$\frac{dv_y}{dt} = -\mu g \cos \alpha \frac{v_y}{\sqrt{v_x^2 + v_y^2}}, \tag{4.31}$$

We use the 'odeint' wrapper and the 'rkqs' routine from Numerical Recipes in C to solve this system of ODEs. The boundary conditions are $x(0) = x_0,\ y(0) = y_0,\ z(0) = 0,\ v_x(0) = u_x,\ v_y(0) = u_y,\ v_z(0) = 0$, and $a_x(0) = a_y(0) = a_z(0) = 0$, and $v_x(T) = 0,\ v_y(T) = 0,\ v_z(T) = 0$, where $(u_x, u_y, 0)$ is the initial velocity of the ball (velocity along the z direction is zero given our assumption that the ball does not leave the surface of the green), and T is the time until which the ball comes to rest and is unknown. When simulating the trajectory, we set the odeint option to record intermediate points to be true. The *step_t* and the *eps* parameters are set based on results from error analysis, described in Section 4.4.

Change of coordinates

The formulation above makes specific assumptions about the coordinate system, e.g., the x-y plane represents the surface of the green, and that the slope is only along the x-axis. In this section, we start with the usual assumptions of x-y plane being the flat green with no slope, and gravitational force action along the negative z-axis, and present equations for change of coordinates that can be used to arrive to the co-ordinate system described in the previous subsection, wherein the above system of ODEs can be solved to find the trajectory of the ball.

We assume that x-y plane represents the flat surface with no slope, and gravitational force action along the negative z-axis. The actual green is inclined at a slope of $\frac{dz}{dx}$ along the x-axis, and slope of $\frac{dz}{dy}$ along the y-axis. We will use the notation (x, y, z) to denote a point in this co-ordinate system, and the notation (x', y', z') to denote the corresponding point in the new co-ordinate system, where the ball moves on the x-y plane and the z-axis is normal to it.

There are two possibilities.

Case a: $\frac{dz}{dy} = 0$.

The following equations denote the change of coordinates.

$$\alpha = \tan^{-1}(\frac{dz}{dx}), \tag{4.32}$$

$$x' = \frac{x}{\cos\alpha}, \tag{4.33}$$

$$y' = y, \tag{4.34}$$

$$v'_x = \frac{v_x}{\cos\alpha}, \tag{4.35}$$

$$v'_y = v_y \tag{4.36}$$

To get to the old set of coordinates, given a position in the new co-ordinate system, the following equations can be used.

$$x = x' \cos\alpha, \tag{4.37}$$

$$y = y', \tag{4.38}$$

$$v_x = v'_x \cos\alpha, \tag{4.39}$$

$$v_y = v'_y \tag{4.40}$$

Case b: $\frac{dz}{dy} \neq 0$.

The following equations denote the change of coordinates.

$$\alpha = \tan^{-1}((\frac{dz}{dx})^2 + (\frac{dz}{dy})^2), \tag{4.41}$$

$$\theta = \tan^{-1}(\frac{\frac{dz}{dy}}{\frac{dz}{dx}}) \tag{4.42}$$

$$x' = \frac{x\cos\theta + y\sin\theta}{\cos\alpha}, \tag{4.43}$$

$$y' = y\cos\theta - x\sin\theta, \tag{4.44}$$

$$v'_x = \frac{v_x\cos\theta + v_y\sin\theta}{\cos\alpha}, \tag{4.45}$$

$$v'_y = v_y\cos\theta - v_x\sin\theta \tag{4.46}$$

To get to the old set of coordinates, given a position in the new co-ordinate system, the following equations can be used.

$$x = x'\cos\alpha\cos\theta - y'\sin\theta, \tag{4.47}$$

$$y = x'\cos\alpha\sin\theta + y'\cos\theta, \tag{4.48}$$

$$v_x = v'_x\cos\alpha\cos\theta - v'_y\sin\theta, \tag{4.49}$$

$$v_y \quad = \quad v'_x \cos\alpha \sin\theta + v'_y \cos\theta \tag{4.50}$$

We apply the above change of coordinates to translate from the original co-ordinate system (x, y, z) to the co-ordinate system (x', y', z'), where we simulate the ball trajectory using the odeint wrapper to the RKQS stepper, and then translate the trajectory back into the original co-ordinate system. The trajectory in the original co-ordinate system is used for determining the holeout according the Penner criterion, as discussed below.

Computing inputs to Penner's holeout criterion

Given a trajectory (as computed above), we next describe the steps that we perform to determine whether or not a holeout occurs. For the discussion in this section, x-y plane represents the flat green with no slope, and gravitational force action along the negative z-axis. The actual green may be inclined with respect to both the x and y axis. We would use the index i to reference the i^{th} point of the trajectory, with index 0 denoting the initial position. The x and y positions of the i^{th} point would be denoted as $x(i)$ and $y(i)$, respectively, and its velocity in the x and y direction would be denoted by $v_x(i)$ and $v_y(i)$, respectively. Similar indexing would be used for other quantities point such as z position, velocity v and acceleration a. So, for example $v(0)$ will denote the initial velocity of the ball. The total number of points would be denoted by n, so that the last index would be $n - 1$.

In the below, we describe the steps performed to compute the inputs to Penner holeout criterion which is used to determine whether a holeout has occurred. The inputs used by the Penner's criterion are $v_{contact}$ (the velocity of the ball when it first touches the rim of the hole), δ (the distance between the line denoting the instantaneous trajectory of the ball when it first touches the hole, and the line parallel to the instantaneous trajectory and that passes through the center of the hole), $\alpha_{contact}$ (the angle with the y-axis that the instantaneous trajectory of the ball makes when it first touches the hole), and green-slopes along the x-axis and the y-axis. These are illustrated in Figure 4.8.

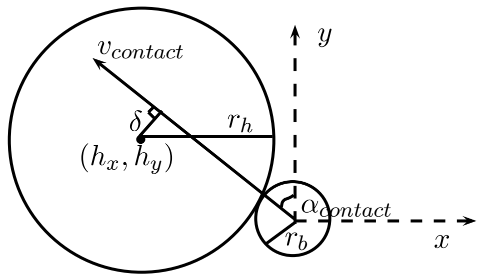


Figure 4.8: Illustration of various inputs to the Penner holeout criterion

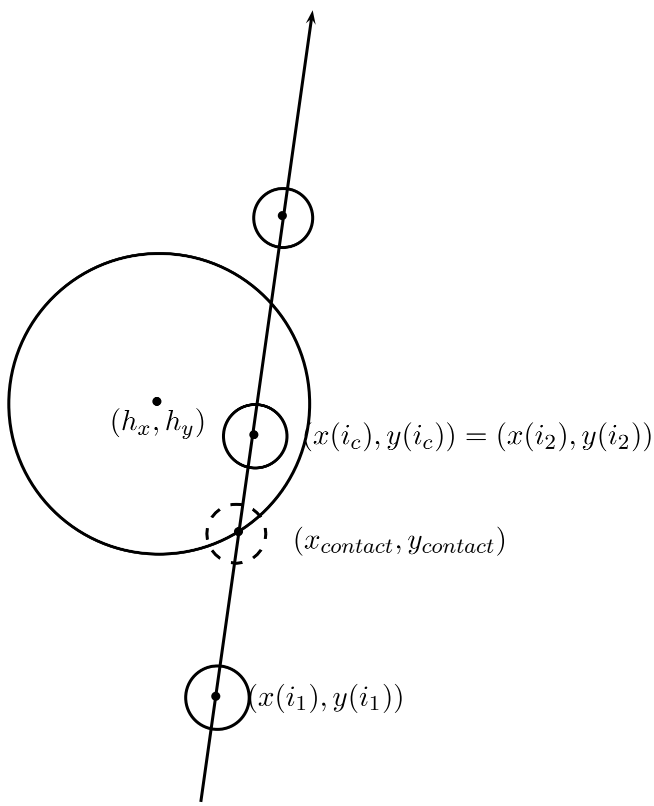


Figure 4.9: Schematic representing a trajectory where a point on the trajectory actually lies inside the hole. The inputs used by the holeout routine to identify the two points between which trajectory first crosses the hole are also labelled.

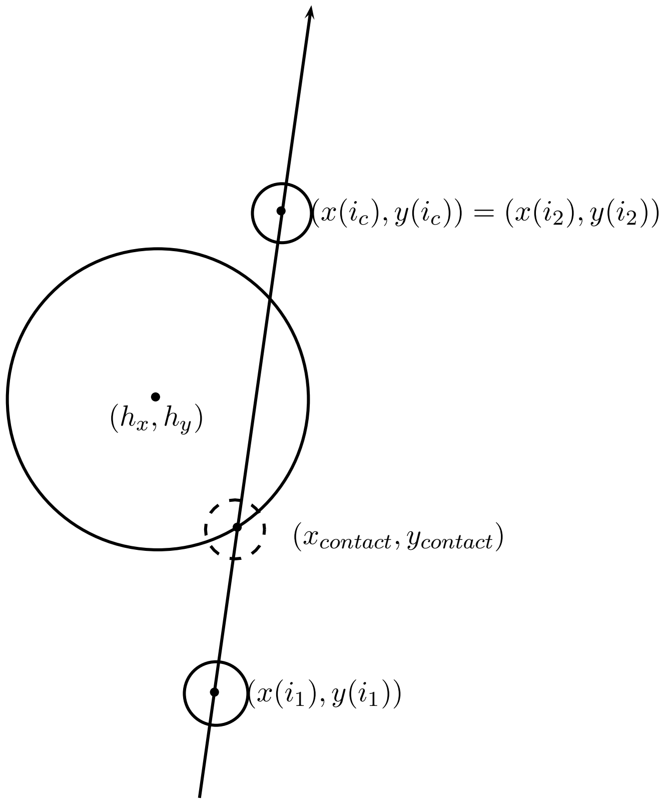


Figure 4.10: Schematic representing a trajectory where no point on the trajectory lies inside the hole. The inputs used by the holeout routine to identify the two points between which trajectory first crosses the hole are also labelled.

Given a trajectory, we perform the following steps to determine whether a holeout has occurred. In the following, we would use (h_x, h_y) to denote the (x, y) position of the hole on the green, r_h to denote the radius of the hole, and r_b to denote the radius of the ball. We would denote the indices for the two points between which the trajectory first crosses the hole (if at all) as i_{fc} and $i_{fc}+1$, respectively.

1. If $r_b > r_h$, then no holeout is possible.

2. If $n \leq 1$, then no holeout is possible (as we assume that the ball does not start in the hole).

3. If $\min_{i\in(0,1,\ldots,n-1)}(|x(i) - h_x|, |y(i) - h_y|) > r_h$, then a holeout cannot occur.

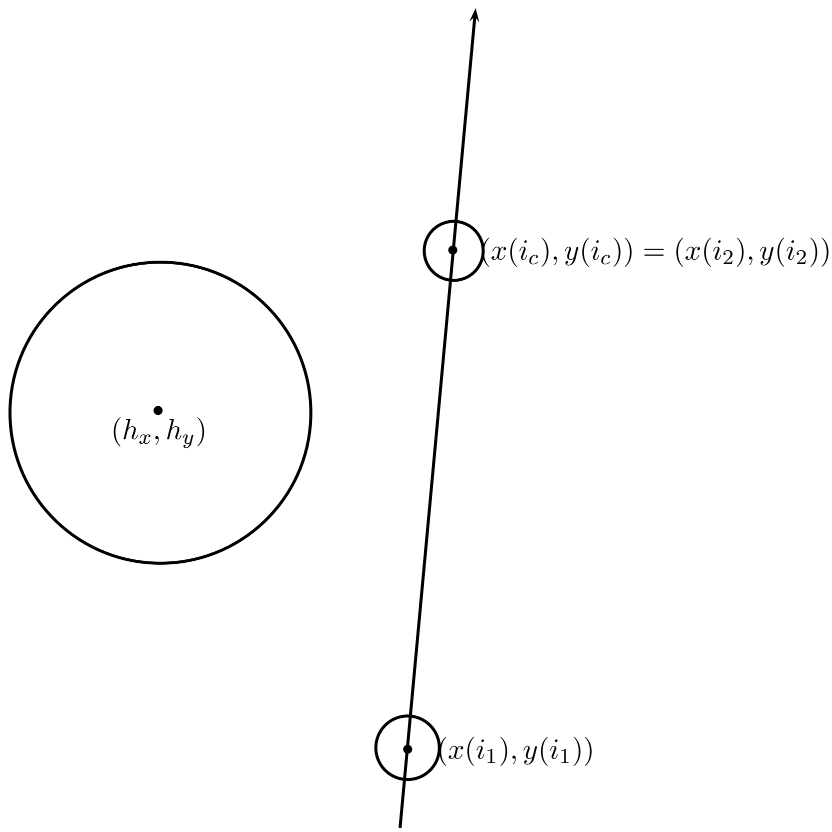


Figure 4.11: Schematic representing a trajectory where there is no intersection between the trajectory and the hole. The inputs used by the holeout routine to identify if there is a contact between the trajectory and the hole are also labelled.

4. Compute $d_i = \sqrt{(x(i) - h_x)^2 + (y(i) - h_y)^2}$, for $i = 0, 1, 2, ..., n - 1$. There are two possibilities.

   Case a: If $\min_i d_i < r_h$, then $i_{fc} + 1 = \arg\min_i d_i \leq r_h$, and denotes the first trajectory point which lies inside the hole. In this case, we know that the trajectory intersects that hole and that the actual point of contact can be determined using points i_{fc} and $i_{fc} + 1$, as explained later. One such trajectory is illustrated in Figure 4.9.

   Case b: If $\min_i d_i \geq r_h$, we adopt the following approach to determine if the trajectory crosses the hole as well as to identify the two trajectory points between which the trajectory crosses the hole for the first time. Two trajectories for which such a scenario would arise are shown in Figures 4.10 and 4.11.

(a) Let $i_c = \arg\min d_i$ denote the index of the closest point to the hole. There are three possibilities.

Case i:If $i_c == 0$, then potential contact between the trajectory and the hole occurs between points 0 and 1.

Case ii: If $i_c == n-1$, then potential contact between the trajectory and the hole occurs between points $n-2$ and $n-1$.

Case iii: The potential first point of contact lies either between $i_c - 1$ and i_c, or i_c and $i_c + 1$. We try each of these pair of points in order to see if the trajectory intersects the hole. As we are interested in the first crossing of the trajectory with the hole, we choose pair $(i_c - 1, i_c)$ if it leads to an intersection, else we try pair $(i_c, i_c + 1)$.

(Strictly speaking, it is possible that the trajectory intersects the hole such the neither of the two points that straddle the point of contact are closest to the hole. However, the probability of such a scenario is small, goes down as Δ_t parameter decreases and would show up as error due to a large value of Δ_t. (Note that the fact that we visit this condition implies that none of the points on the trajectory lies inside the hole, the probability of which also decreases as Δ_t becomes smaller.))

(b) Denote the index of the two points between which the trajectory potentially first crosses the hole as i_1, i_2, where $i_1 + 1 = i_2$. We next determine the smallest distance between the line determined by this pair of points and the center of the hole. There are three possibilities.

Case i: If $y(i_1) == y(i_2)$ (i.e., the ball is travelling horizontally, then the point that lies on the line determined by the two points indexed by i_1 and i_2, and is nearest to the center of the hole (i.e., lies on the perpendicular to line between points indexed by i_1 and i_2, and passes through the center of the hole) is given by $(x_\perp, y_\perp) = (h_x, y(i_1))$.

Case ii: If $x(i_1) == x(i_2)$ (i.e., the ball is travelling vertically, then the point that lies on the line determined by the two points and is nearest to the center of the hole is given by $(x_\perp, y_\perp) = (x(i_1), h_y)$.

Case iii: In this case, we can explicitly compute the point of intersection between the line determined by the pair of given points, and the line perpendicular to it and passing through the center of the hole. The equation of the first line is given by $y - y(i_1) = m(x - x(i_1))$, where $m = \frac{y(i_2)-y(i_1)}{x(i_2)-x(i_1)}$. The equation to the second line is given by $y - h.y = m'(x - h_x)$, where $mm' = -1$ (as the two lines are perpendicular). Solving we get $x_\perp = \frac{y(i_1)-h_y-mx(i_1)+m'h_x}{m'-m}$, and $y_\perp = h_y + m'(x_\perp - h_x)$.

(c) We rule out contact between the trajectory and the hole if the point of intersection between the two lines (the first line determined by the pair of points i_1, i_2 and the second line perpendicular to the first line and passing through the center of the hole) does not occur between the two points denoted by i_1, i_2, i.e., if any of the following conditions are met: $x_\perp > x(i_1)$ and $x_\perp > x(i_2)$, or $x_\perp < x(i_1)$ and $x_\perp < x(i_2)$ or $y_\perp > y(i_1)$ and $y_\perp > y(i_2)$, or $y_\perp < y(i_1)$ and $y_\perp < y(i_2)$.

(d) We also rule out contact between the trajectory and the hole if the distance $\delta = \sqrt{(x_\perp - h_x)^2 + (y_\perp - h_y)^2} \geq r_h$.

5. Based on the above, if no contact occurred between the trajectory and the hole, then there cannot be a holeout. Else, using the notation above, the two points between which the trajectory first intersects the hole are i_{fc} and $i_{fc} + 1$. Also, if we have not computed δ so far, we need to compute δ. For this, we follow the following steps. There are three possibilities.

   Case i: If $y(i_{fc}) == y(i_{fc} + 1)$ (i.e., the ball is travelling horizontally), then $\delta = |h_y - y(i_{fc})|$.

   Case ii: If $x(i_{fc}) == x(i_{fc} + 1)$ (i.e., the ball is travelling vertically), then $\delta = |h_x - x(i_{fc})|$.

   Case iii: If neither of the above are true, then $\delta = \frac{|-mh_x+h_y-c|}{\sqrt{1+m^2}}$, where $m = \frac{y(i_{fc}+1)-y(i_{fc})}{x(i_{fc}+1)-x(i_{fc})}$ and $c = y(i_{fc} + 1) - mx(i_{fc})$.

6. Check if $\delta \geq r_h$. If not, then there cannot be any holeout. Else, we need to compute the

velocity and direction of the ball when it first touches the rim of the hole. For this, we need to find where the line determined by points $(i_{fc}, i_{fc}+1)$ intersects the rim of the hole described by the equation $(x-h_x)^2+(y-h_y)^2 = r_h^2$. We would denote the two potential points of intersection as (x_{c1}, y_{c1}) and (x_{c2}, y_{c2}). The point where the trajectory first touches the rim of the hole would be denoted as $(x_{contact}, y_{contact})$. The following cases are possible.

Case i:If $y(i_{fc}) == y(i_{fc}+1)$ (i.e., the ball is travelling horizontally), then $y_{contact} = y_{c1} = y_{c2} = y(i_{fc})$ and $x_{c1} = h_x + \sqrt{r_h^2 - (h_y - y_{contact})^2}$ while $x_{c2} = h_x - \sqrt{r_h^2 - (h_y - y_{contact})^2}$. Of the two potential values of x at intersection, the correct one is the one closer to the first point, i.e., $x_{contact} = x_{c1}$ if $|x(i_{fc}) - x_{c1}| \leq |x(i_{fc}) - x_{c2}|$, else $x_{contact} = x_{c2}$.

Case ii:If $x(i_{fc}) == x(i_{fc}+1)$ (i.e., the ball is travelling vertically), then $x_{contact} = x_{c1} = x_{c2} = x(i_{fc})$ and $y_{c1} = h_y + \sqrt{r_h^2 - (h_x - x_{contact})^2}$ while $y_{c2} = h_y - \sqrt{r_h^2 - (h_x - x_{contact})^2}$. Of the two potential values of y at intersection, the correct one is the one closer to the first point, i.e., $y_{contact} = y_{c1}$ if $|y(i_{fc}) - y_{c1}| \leq |y(i_{fc}) - y_{c2}|$, else $y_{contact} = y_{c2}$.

Case iii: If either of the above conditions isn't true, then we proceed as follows. The equation of the line is given by $y = mx + c$, where $m = \frac{y(i_{fc}+1)-y(i_{fc})}{x(i_{fc}+1)-x(i_{fc})}$ and $c = y(i_{fc}) - mx(i_{fc})$, while the equation of the rim is given by $(x-h_x)^2 + (y-h_y)^2 = r_h^2$. Substituting the first into the second, we obtain

$x_{c1} = -\frac{-h_x - mh_y + mc}{1+m^2} + \sqrt{(\frac{-h_x - mh_y + mc}{1+m^2})^2 + \frac{r_h^2 - h_x^2 - c^2 - h_y^2 + 2h_y c}{1+m^2}}$, $y_{c1} = mx_{c1} + c$,

$x_{c2} = -\frac{-h_x - mh_y + mc}{1+m^2} - \sqrt{(\frac{-h_x - mh_y + mc}{1+m^2})^2 + \frac{r_h^2 - h_x^2 - c^2 - h_y^2 + 2h_y c}{1+m^2}}$, $y_{c2} = mx_{c2} + c$. Of these two potential points where the trajectory intersects the hole, we choose the point that is closer to the point $(x(i_{fc}), y(i_{fc}))$.

7. Given the points $(x(i_{fc}), y(i_{fc}))$ and $(x_{contact}, y_{contact})$, we use the equation $v^2 = u^2 + 2as$ to solve for $v_{x,contact}$ and $v_{y,contact}$. In particular, we solve $v_{x,contact}^2 = v_x(i_{fc})^2 + 2a_x(i_{fc})(x_{contact} - x(i_{fc}))$, and $v_{y,contact}^2 = v_y(i_{fc})^2 + 2a_y(i_{fc})(y_{contact} - y(i_{fc}))$. The angle with respect to the

x-axis at which the intersection occurs is given by $\alpha_{contact} = \tan^{-1}\frac{v_{y,contact}}{v_{x,contact}}$. The velocity of the ball at intersection is given by
$v_{contact} = \sqrt{v_{x,contact}^2 + v_{y,contact}^2 + v_{z,contact}^2}$, where
$v_{z,contact} = v_{x,contact}\frac{dz}{dx} + v_{y,contact}\frac{dz}{dy}$.

These inputs are used in the Penner holeout criterion described below to determine whether or not the given trajectory results in a holeout.

Holeout criterion

For level putts and for a ball passing through the center of the hole, Holmes [28] derives critical velocity, the maximum velocity that leads to a hole out, $v_c = ((r_h^2 - \delta^2)^{1/2} + ((r_h - r_b)^2 - \delta^2)^{1/2})\sqrt{(g/2r_b)}$, where r_b is the radius of the ball, and r_h is the radius of the hole and δ is the perpendicular distance of line representing the direction of motion of the ball at the time it first touches the hole from the center of the hole. Penner [50] approximates it as $v_c(\delta) = 1.63 - 1.63(\delta/r_h)^2$. Penner [50] then extends to take into account uphill (and downhill) putts using $v_f(\delta) = (1 - \cos(\beta)\sin(\phi))^{-(1/2)}v_c(\delta)$ and to sidehill putts using the formula $v_f(\delta) = (1+\sin(\beta)\sin(\theta))^{-(1/2)}v_c(\delta)$, where $\theta = \tan^{-1}(dz/dx)$, $\phi = \tan^{-1}(dz/dy)$ and β being the angle the trajectory of the ball makes with the y-axis when the ball first touches the hole.

In the spirit of Penner, we extend his derivation to address the case when the ball is putt on a surface that has both an uphill (or downhill) and sidehill elevation, and derive $v_h(\delta) = (1 - \sin(\lambda))^{-(1/2)}\ v_c(\delta)$, where $v_h(\delta)$ is the velocity of a successful putt, and $\lambda = \tan^{-1}(-a\sin(\beta) + b\cos(\beta))$, is the angle that the instantaneous direction of motion of the ball (along the surface of the green) makes with the x-y plane, $a = \frac{dz}{dx}$ and $b = \frac{dz}{dy}$. The derivation is as follows. As in Penner [50], assuming that the ball passes through the center of the hole, a holeout occurs if $(1/2)gt^2 - v_zt > r_b - \Delta z$, where $\Delta z = (2r_h - r_b)\sin(\lambda)$, $v_z = v\sin(\lambda)$ and $t = (2r_h - r_b/v)$. Substituting values, this condition evaluates to $v_f < \sqrt{g/2r_b}(2r_h - r_b)(1 - \sin(\lambda))^{-1/2}$, wherein from Penner, we substitute $\sqrt{g/2r_b}(2r_h - r_b) = v_c(\delta)$, noting that the δ adjustment takes care of putts where the trajectory of the ball does not pass through the center of the hole. As in Penner [50], since θ, ϕ and λ are small (we assume that $\max(\tan(\theta), \tan(\phi), \tan(\lambda)) < \eta$), we can approximate $\tan(\lambda) = -\sin(\theta)\sin(\beta) + \sin(\phi)\cos(\beta)$, wherein if only sidehill or uphill slope is present, we get the results in Penner [50]. Note that in our experiments, (without loss of generality) we always set $\frac{dz}{dx} = 0$, and have slope only along the y-axis.

Bellman equation and convergence of policy iteration

We observe that the expected-putts minimization problem is an instance of the two dimensional stochastic shortest path problem, also sometimes referred to as a transient program or first-passage problem. They are discussed, for example, in Section 2.1 of Bertsekas [8] and Chapter 25 of Whittle [69]. The event of a holeout leads to the destination state, which is absorbing and entails zero further cost. The cost of a transition from any state is 1. The next state is determined by the stopping point of current trajectory. Unlike the stochastic shortest path problem discussed in [8] though, both the state and control space in our setting are both continuous. Hence, we will follow the approach in [69], wherein for the proof of convergence of the policy iteration algorithm, we will make the following two assumptions.

A1) We assume that the green is contained within a region of radius $\overline{d}$ from the center of the hole, and given golfer skill parameters K and any initial position I on the green, $P_1(I, \mu, K) > \epsilon_1$, $\epsilon_1 > 0$ being a positive constant.

A2) We assume that given a state I, the set of feasible controls $\mathcal{U}(I) = \{\mu \mid P_1(I, \mu, K) > \epsilon_2\}$, $\epsilon_2 > 0$ being a positive constant (with $\epsilon_2 < \epsilon_1$).

Together, these assumptions ensure that $N(I)$ is bounded by $1/(1 - \epsilon_2)$ for any feasible policy, and hence the three conditions in Chapter 25, Section 6 of [69] are satisfied, namely that, the process is a time-homogeneous Markov process, the cost per stage is uniformly bounded, and the termination time has an expectation bounded uniformly in the policy and initial state. Following the results from Theorem 6.1 and the discussion in Section 9 in [69], we conclude that policy iteration converges to the optimal policy for our problem.

Choosing ϵ_1 *and* ϵ_2: Assumption A1) is relatively benign, because given any $\overline{d}$ and golfer skill K, one can choose an ϵ_1 such that the assumption is satisfied. Assumption A2) however, restricts the set of feasible controls to target velocities and angles that lead to a one-putt probability greater than ϵ_2. Again, following assumption A1), ϵ_2 can be set to be sufficiently small so that all realistic controls qualify. In particular, the expected putts minimizing controls become feasible. However, setting ϵ_2 too aggressively might lead to a suboptimal solution. Following the results in Section 4.4, we observe that for settings of our interest, the "optimal" policy leads to a positive one-putt probability, implying that ϵ_2 can be chosen conveniently.

State and control space discretization: As discussed in Section 4.3, to proceed with the computation, we discretize the state and control space. Since policy iteration converges to the optimal solution when state and control space are continuous, we expect policy iteration to converge to the optimal solution following the discretization of control and state space as well. Numerical results suggest that policy iteration converges after two iterations ($p = 2$), with error in expected number of putts being around 0.005.

Interpolation to obtain the continuation strategy

Suppose we need the continuation strategy when we follow policy $\mu^{(p)}(\cdot)$ at point $I = (d, \gamma)$, $I \neq I_{ij}$, $i = 1, \ldots, n_d$, $j = 1, \ldots, n_\gamma$. Let $j = \max\{v \mid \gamma_v < \gamma\}$ and $w = j + 1 \mod n_\gamma$. There are three cases to consider. If $d < d_1$, then $\mu^{(p)}(I)$ is obtained by interpolating between four points, the first two of which are zero (corresponding to continuation strategy from the center of the hole), and the last two $\mu^{(p)}(I_{1j})$, and $\mu^{(p)}(I_{1w})$. If $d > d_{n_d}$, then $\mu^{(p)}(I)$ is obtained by interpolating between the two points $\mu^{(p)}(I_{n_d j})$, and $\mu^{(p)}(I_{n_d w})$. Since $\overline{d} = d_{n_d}$ is the length of the longest putt we consider, and we need to interpolate the strategy only for second or subsequent putts, this case is unlikely to arise. Finally, if $d_i < d \leq d_{i+1}, i = 1, \ldots, n_d - 1$, then $\mu^{(p)}(I)$ is obtained by interpolating between the four points $\mu^{(p)}(I_{ij})$, $\mu^{(p)}(I_{i+1j})$, $\mu^{(p)}(I_{iw})$ and $\mu^{(p)}(I_{i+1w})$. As mentioned in Section 4.4, we use the bicubic spline interpolation implementation in [52] for interpolation.

Grid search procedure for sampling the control space

When optimizing over both (v, α), we sample from $\mathcal{U}$ using a two-dimensional Sobol sequence [52] as follows. Specifically, $\widehat{\mathcal{U}} = \{(\mu_v, \mu_\alpha) \mid \mu_v = \underline{\mu}_v + \mathrm{s}(i,1)\Delta_v, \mu_\alpha = \underline{\mu}_\alpha + \mathrm{s}(i,2)\Delta_\alpha, i = 1, \ldots, m\}$, where $\Delta_v = (\overline{\mu}_v - \underline{\mu}_v)/m$, $\Delta_\alpha = (\overline{\mu}_\alpha - \underline{\mu}_\alpha)/m$, m is the number of Sobol points we want to use in the optimization, and $\mathrm{s}(i,j)$ is a routine that returns the j^{th} dimension of the i^{th} point in the Sobol sequence.

Since we approximate the continuous set $\mathcal{U}$ by the discrete set $\widehat{\mathcal{U}}$, the accuracy of our optimization depends on m, the number of Sobol points chosen for the discretization. A large value of m however results in higher computational requirements as well. To increase the accuracy of our solution while avoiding the computational overhead due to a large m, we recursively refine the grid (while fixing the value of m) as follows. Let $\widehat{\mu}^* = \arg\min_{\mu \in \widehat{\mathcal{U}}} f(\mu)$. Let $\widehat{\sigma}^*$ denote the standard error of $f(\widehat{\mu}^*)$. Next define $\widehat{\mathcal{U}}^* = \{(\mu_v, \mu_\alpha) \in \widehat{\mathcal{U}} \mid f(\mu) \leq f(\widehat{\mu}^*) + k\widehat{\sigma}^*\}$, where k is a scaling parameter. Now set $\underline{\mu}_v = \min_{\mu \in \widehat{\mathcal{U}}^*} \mu_v$, $\overline{\mu}_v = \max_{\mu \in \widehat{\mathcal{U}}^*} \mu_v$, $\underline{\mu}_\alpha = \min_{\mu \in \widehat{\mathcal{U}}^*} \mu_\alpha$, $\overline{\mu}_\alpha = \max_{\mu \in \widehat{\mathcal{U}}^*} \mu_\alpha$, and define the new set $\mathcal{U} = \{(\mu_v, \mu_\alpha) | \underline{\mu}_v \leq \mu_v \leq \overline{\mu}_v, \underline{\mu}_\alpha \leq \mu_\alpha \leq \overline{\mu}_\alpha\}$, over which we can again search for the optimal solution.

When optimizing over d alone (following the computational speed-up discussed in Section 4.3.5), we follow the same procedure adjusted for the fact that the control space becomes one-dimensional.

$\mu_\alpha^{(1)} = 0$ for level greens

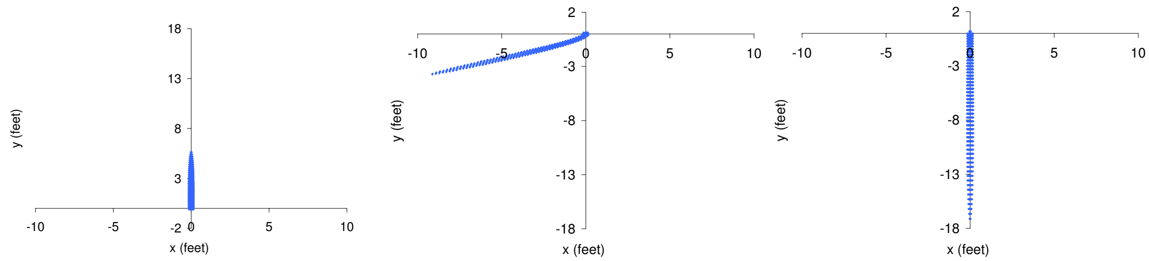
We now show that in the special case of level greens (i.e., $\theta_x = 0,\ \theta_y = 0$), the choice of aim direction that maximizes the one-putt probability is zero (i.e., $\mu_\alpha^{(1)} = 0$), irrespective of the golfer skill or putt-length. Consider any fixed velocity v and suppose the ball is hit at an angle α to the line joining the initial position of the ball to the hole. Let d denotes the length of the putt. Two cases are possible: i) $v^2/(2\eta g) < d - r_h$, ii) $v^2/(2\eta g) \geq d - r_h$. In case i), the ball falls short of the hole and the choice of α does not matter. In case ii), the trajectory intersects with the hole only if $\sin(\alpha) \leq r_h/d$, in which case the velocity at the first point of contact is given by $v^2 - 2\eta g \left(d\cos(\alpha) - \sqrt{r_h^2 - d^2\sin^2(\alpha)}\right)$. A holeout occurs if

$$v^2 - 2\eta g \left(d\cos(\alpha) - \sqrt{r_h^2 - d^2\sin^2(\alpha)}\right) \leq 1.63 - 1.63\frac{\sin^2(\alpha)}{r_h^2}$$

Note that if angle α leads to a holeout, then so does angle $-\alpha$, thereby implying that the range of angles that lead to holeout for a fixed velocity v is symmetric about zero. Given that the realized angle $\widetilde{\alpha} \sim \mathcal{N}(\mu_\alpha, \sigma_\alpha^2)$, to maximize the probability of one-putt given a fixed velocity v, it is therefore optimal to set $\mu_\alpha^{(1)} = 0$. Since this holds for any fixed v, it holds for all v, and in particular, for any distribution $\widetilde{v}$ that is uncorrelated with $\widetilde{\alpha}$. The same argument of symmetry applies to straight uphill and straight downhill putts as well. Hence if the target distance d_t lies beyond the hole, i.e., $d_t \geq 0$, the trajectory $\mathcal{T}(I, \mu_v^{(1)}, \mu_\alpha^{(1)})$ passes through the center of the hole.

Sink-zones

Figure 4.12 shows the sinkzones for uphill, sidehill and downhill putts. Notice that while the sinkzones for uphill and downhill putts look similar to those in Tierney [66] and Hoadley [26], the sinkzone for sidehill putts, as suggested in Chapter 30 of Cochran [16], curves to the side. In particular, it does not align with the line joining the initial position of the ball with the hole. Hence



(a) Uphill (b) Sidehill (c) Downhill

Figure 4.12: This figure shows how the sinkzone varies with respect to the slope of the putt. Uphill, sidehill and downhill putts are aimed at from an angle of $90°$, $0°$ and $-90°$ with respect to the x-axis. Downhill putts have the longest sink-zones. Unlike uphill and downhill putts, the sinkzone for sidehill putts does not lie along the line joining the initial putting position to the center of the hole.

any model assuming that golfer putts stop in region centered $r \geq 0$ feet beyond the hole and that this region can be approximated using a bivariate normal distribution will be incorrect. Even for the uphill/downhill case, the sink-zone is not symmetric (along the line parallel to the x-axis and passing through the median of stop points on the y-axis), which a normal distribution for the stop points would imply. It therefore motivates our golfer model, wherein we compute ball trajectories, than model their stopping points. In passing, we also note that sinkzone does not depend on the length of the putt.

Professional Player Calibration

We calibrate the golfer model to professional golfer data on a green with a slope of $1.5°$ and green speed of 11 feet. We calibrate 4 parameter sets. To obtain Parameter Set 1, we calibrate to one-putt probabilities only. To obtain Parameter Set 2, Parameter Set 3 and Parameter Set 4, we calibrate to both one-putt probabilities and expected-putts. These different parameter sets reflect professional players with different distance, direction and green-reading skills.

In Section 4.4, we presented calibration results for Parameter Set 3. We present calibration results for the remaining parameter sets below. We find that calibration to one-putt probabilities alone does not necessarily lead to a good fit to expected-putts data, and hence joint calibration to both one-putt probability data and expected-putts data is necessary. For all parameter sets, the 1.5 ft strategy is suboptimal for expected-putts minimization, though for long putt-lengths (e.g. $>$ 10 feet), it is better than the one-putt probability-maximization strategy. For all parameter sets and across putt-lengths, sidehill putts are found to be among the hardest putts. We discuss these results in detail below.

Table 4.11 shows the fit for Parameter Set 1 where we attempt to calibrate to one-putt probabilities alone. The professional player parameters thus obtained are $\beta_0 = 4.91\%$, $\beta_1 = 13.05\%$, $\beta_2 = 5.04\%$, $v_\beta = 10.86$ feet, $\sigma_\alpha = 1.134$, $\sigma_g = 0.147$. The overall RMSE between model and data is 0.35% for one-putt probabilities and 0.089 for expected putts. As we calibrate to one-putt probabilities, the probability error is small; however, the expected putts error is large. This suggests calibrating to both one-putt probabilities and expected putts to minimize the calibration error.

For comparison, in addition to the one-putt probability maximizing strategy, results obtained using these parameters and the expected-putts minimizing strategy and the 1.5-foot strategy are also shown. The expected-putts minimizing strategy involves aiming a shorter distance beyond the hole than the one-putt probability maximizing strategy, and as expected, leads to fewer expected putts. We also observe that the optimal distance to target to minimize expected putts or to maximize the one-putt probability, in general, differ from the 1.5-foot strategy. For long putts, e.g., $>$ 10 feet, the 1.5-foot strategy results in fewer expected putts to holeout than the one-putt probability maximizing strategy.

Table 4.12 shows how the expected number of putts varies as a function of the putt position (initial putt-length and putt-angle) for a player with skill determined by parameter set 1. As expected, the expected number of putts needed to holeout increases with putt-length. Sidehill putts ($-30°$ to $30°$) are the most difficult, i.e., they lead to the largest expected number of putts for a given putt-length. Finally, downhill putts ($75°$ to $90°$) are easier than uphill putts ($-75°$ to

-90°).

Tables 4.13, 4.4, and 4.15, show the fit for Parameter Set 2, Parameter Set 3, and Parameter Set 4, respectively, wherein we attempt to calibrate to both one-putt probabilities and expected-putts for the professional player. Unlike the case with Parameter Set 1, a good fit to both one-putt probabilities and expected-putts is achieved for these parameter sets. The player parameters are $\beta_0 = 6.31\%$, $\beta_1 = 10.20\%$, $\beta_2 = 6.31\%$, $v_\beta = 18.4258$ feet, $\sigma_\alpha = 1.1146$, $\sigma_g = 0.124$ for Parameter Set 2, $\beta_0 = 6.14\%$, $\beta_1 = 9.20\%$, $\beta_2 = 6.14\%$, $v_\beta = 8.1760$ feet, $\sigma_\alpha = 1.1900$, $\sigma_g = 0.0848$ for Parameter Set 3, and $\beta_0 = 5.62\%$, $\beta_1 = 9.60\%$, $\beta_2 = 5.62\%$, $v_\beta = 16.7406$ feet, $\sigma_\alpha = 1.0998$, $\sigma_g = 0.1480$ for Parameter Set 4. We note that among these parameter sets, Parameter Set 3 has lowest green error and the highest angle error, while Parameter Set 4 has the highest green error and the lowest angle error. Again, for comparison, in addition to the one-putt probability maximizing strategy, results obtained using these parameters and the expected-putts minimizing strategy and the 1.5-foot strategy are also shown. As in the case for Parameter Set 1, we observe that the expected-putt minimizing strategy involves aiming a shorter distance beyond the hole compared to the one-putt probability maximizing strategy, and typically is different that the 1.5-foot strategy. Also, for long putt-lengths, e.g., > 10 feet, the 1.5-foot strategy leads to fewer expected putts than the one-putt probability maximizing strategy.

Tables 4.14, 4.5, and 4.16, show how the expected number of putts varies as a function of the putt position for Parameter Set 2, Parameter Set 3, and Parameter Set 4, respectively. As with Parameter Set 1, we observe that sidehill putts are the most difficult for these players and downhill putts are easier than uphill putts. We also observe that the difference between the downhill and uphill expected putts is smaller for Parameter Set 4, which has the highest green error and the lowest angle error among these parameter sets.

Table 4.11: Professional golfer: data fit and strategy comparison - Parameter Set 1

Data			Model											
			Min exp putts				Max one putt prob				1.5 feet beyond hole			
d	P_1	N^*	P_1	N^*	d^*	f_s	P_1	N^*	d^*	f_s	P_1	N^*	d^*	f_s
2	99.2%	1.01	99.8%	1.00	1.22	0.0%	99.8%	1.00	1.23	0.0%	99.8%	1.00	1.50	0.0%
3	95.5%	1.05	95.9%	1.04	1.47	0.0%	95.9%	1.04	1.52	0.0%	95.2%	1.05	1.50	0.0%
4	86.6%	1.14	86.3%	1.14	1.52	0.1%	86.4%	1.14	1.64	0.1%	85.1%	1.15	1.50	0.0%
5	75.9%	1.24	75.4%	1.25	1.57	0.3%	75.5%	1.25	1.72	0.2%	74.0%	1.26	1.50	0.1%
6	65.5%	1.35	65.5%	1.35	1.56	0.5%	65.6%	1.35	1.78	0.5%	64.1%	1.36	1.50	0.3%
7	57.1%	1.43	57.1%	1.43	1.59	0.8%	57.2%	1.44	1.87	0.7%	55.9%	1.44	1.50	0.7%
8	49.9%	1.51	50.1%	1.50	1.59	1.3%	50.4%	1.51	1.96	1.0%	49.1%	1.51	1.50	1.5%
9	44.2%	1.56	44.1%	1.57	1.59	2.4%	44.5%	1.57	2.05	1.7%	43.3%	1.57	1.50	2.6%
10	39.3%	1.61	39.0%	1.62	1.60	3.8%	39.4%	1.63	2.12	2.5%	38.4%	1.62	1.50	4.2%
15	23.0%	1.78	22.1%	1.80	1.49	15.6%	23.0%	1.83	2.57	6.4%	22.0%	1.80	1.50	15.4%
21	14.4%	1.88	12.2%	1.91	1.10	31.4%	13.8%	1.97	2.98	12.0%	12.8%	1.92	1.50	25.7%
25	10.8%	1.92	8.7%	1.97	0.87	37.9%	10.4%	2.04	3.16	16.1%	9.5%	1.98	1.50	30.3%
29	8.1%	1.97	6.3%	2.01	0.65	42.8%	8.1%	2.10	3.37	19.4%	7.3%	2.03	1.50	33.9%
33	6.7%	1.99	4.8%	2.06	0.48	45.6%	6.5%	2.15	3.49	22.3%	5.8%	2.08	1.50	36.4%
37	5.0%	2.04	3.8%	2.10	0.32	47.6%	5.4%	2.19	3.62	24.5%	4.7%	2.12	1.50	38.3%
42	4.1%	2.07	3.0%	2.15	0.20	48.8%	4.3%	2.23	3.69	27.4%	3.8%	2.17	1.50	40.0%
50	2.8%	2.15	2.2%	2.22	0.12	49.3%	3.2%	2.29	3.72	31.0%	2.8%	2.24	1.50	42.0%
		RMSE	1.11%	0.037			0.35%	0.089			1.05%	0.049		

This table shows the fit of the professional golfer data with the golfer model for different putt lengths d (in feet), when we attempt to calibrate to one-putt probabilities alone. It also compares the minimize expected number of putts, maximize one-putt probability, and aim 1.5 feet beyond the hole strategies for the professional golfer. The golfer parameters are $\beta_0 = 4.91\%$, $\beta_1 = 13.05\%$, $\beta_2 = 5.04\%$, $v_\beta = 10.86$ feet, $\sigma_\alpha = 1.134$, $\sigma_g = 0.147$. We refer to this set as Parameter Set 1 for the Professional golfer. The green slope is $1.5°$, and green speed is 11 feet. The fit is good with respect to the one-putt probabilities (P_1); however the expected number of putts (N^*) error is large (as shown by the root-mean-squared error (RMSE) values under the one-putt probability maximization strategy). For short distances, minimizing the expected number of putts and maximizing the one-putt probabilities yield similar results. The optimal distance beyond the hole to aim at (d^*) (and consequently the fraction of short putts, f_s) depends on the length of the putt, and in general, differs from aiming 1.5 ft beyond the hole. (For each fixed distance, P_1, N^*, d^* and f_s are averaged over the angle the initial position of the putt makes with the x-axis.)

Table 4.12: Expected number of putts for professional golfer Parameter Set 1

angle with x-axis	Length of putt (feet) 2	3	4	6	8	10	15	21	25	42	50
90	1.001	1.023	1.091	1.268	1.418	1.530	1.713	1.836	1.891	2.081	2.157
75	1.001	1.024	1.098	1.284	1.437	1.550	1.729	1.847	1.900	2.096	2.171
60	1.001	1.029	1.112	1.316	1.475	1.588	1.760	1.868	1.930	2.123	2.196
45	1.001	1.034	1.128	1.346	1.510	1.622	1.784	1.897	1.960	2.145	2.217
30	1.001	1.041	1.142	1.367	1.531	1.642	1.797	1.925	1.981	2.162	2.233
15	1.002	1.046	1.151	1.378	1.541	1.649	1.816	1.941	1.994	2.173	2.243
0	1.002	1.049	1.157	1.381	1.540	1.645	1.835	1.949	2.000	2.176	2.246
-15	1.002	1.051	1.159	1.379	1.534	1.637	1.838	1.949	1.999	2.173	2.242
-30	1.002	1.051	1.157	1.375	1.526	1.639	1.834	1.943	1.993	2.166	2.233
-45	1.002	1.049	1.153	1.366	1.517	1.638	1.823	1.932	1.983	2.155	2.222
-60	1.002	1.046	1.147	1.357	1.507	1.631	1.813	1.923	1.975	2.147	2.213
-75	1.002	1.044	1.141	1.349	1.500	1.623	1.805	1.917	1.969	2.140	2.207
-90	1.002	1.043	1.138	1.346	1.498	1.620	1.803	1.915	1.967	2.138	2.204
Avg. model	1.002	1.041	1.137	1.348	1.504	1.618	1.798	1.912	1.966	2.145	2.215
Avg. data	1.009	1.046	1.137	1.349	1.507	1.614	1.783	1.879	1.925	2.066	2.148
Difference	-0.007	-0.005	0.001	-0.001	-0.003	0.004	0.015	0.033	0.041	0.079	0.068

This table shows how the expected number of putts varies for a professional golfer (parameter set 1) as a function of putt position and length of the putt. The expected number of putts to holeout increases with distance. Sidehill putts, i.e., putts from initial angles between $-30°$ and $30°$, are the most difficult.

Table 4.13: Professional golfer: data fit and strategy comparison - Parameter Set 2

Data			Model											
			Min exp putts				Max one putt prob				1.5 feet beyond hole			
d	P_1	N^*	P_1	N^*	d^*	f_s	P_1	N^*	d^*	f_s	P_1	N^*	d^*	f_s
2	99.2%	1.01	99.9%	1.00	1.25	0.0%	99.9%	1.00	1.26	0.0%	99.8%	1.00	1.50	0.0%
3	95.5%	1.05	96.2%	1.04	1.54	0.0%	96.2%	1.04	1.59	0.0%	95.6%	1.04	1.50	0.0%
4	86.6%	1.14	86.8%	1.13	1.62	0.1%	86.8%	1.13	1.70	0.1%	85.5%	1.15	1.50	0.0%
5	75.9%	1.24	75.7%	1.25	1.68	0.3%	75.7%	1.25	1.86	0.3%	74.2%	1.26	1.50	0.1%
6	65.5%	1.35	65.5%	1.35	1.72	0.6%	65.7%	1.35	1.98	0.5%	64.0%	1.36	1.50	0.4%
7	57.1%	1.43	56.9%	1.44	1.73	0.9%	57.1%	1.44	2.10	0.8%	55.5%	1.45	1.50	1.1%
8	49.9%	1.51	49.8%	1.51	1.74	1.4%	50.1%	1.52	2.19	1.1%	48.6%	1.52	1.50	2.1%
9	44.2%	1.56	43.9%	1.57	1.73	2.2%	44.3%	1.58	2.32	1.4%	42.9%	1.58	1.50	3.4%
10	39.3%	1.61	39.0%	1.62	1.71	3.3%	39.6%	1.63	2.36	1.8%	38.2%	1.62	1.50	4.9%
15	23.0%	1.78	23.7%	1.78	1.56	12.4%	24.7%	1.82	2.74	4.2%	23.4%	1.78	1.50	13.4%
21	14.4%	1.88	14.3%	1.88	1.29	26.1%	15.7%	1.94	3.00	8.8%	14.6%	1.89	1.50	22.7%
25	10.8%	1.92	10.6%	1.93	1.08	33.2%	12.2%	2.00	3.15	12.0%	11.2%	1.94	1.50	27.3%
29	8.1%	1.97	8.0%	1.97	0.87	38.5%	9.8%	2.05	3.27	15.1%	8.9%	1.98	1.50	30.7%
33	6.7%	1.99	6.3%	2.00	0.70	42.2%	8.0%	2.09	3.35	18.0%	7.2%	2.02	1.50	33.4%
37	5.0%	2.04	5.0%	2.04	0.56	44.7%	6.7%	2.12	3.45	20.4%	6.0%	2.05	1.50	35.4%
42	4.1%	2.07	3.9%	2.08	0.38	47.0%	5.5%	2.16	3.43	23.9%	4.8%	2.10	1.50	37.5%
50	2.8%	2.15	2.8%	2.14	0.17	48.9%	4.1%	2.23	3.69	26.8%	3.6%	2.16	1.50	39.9%
		RMSE	0.35%	0.006			1.06%	0.053			1.02%	0.014		

This table shows the fit of the professional golfer data with the golfer model for different putt lengths d (in feet), when we attempt to calibrate to both one-putt probabilities and expected putts. It also compares the minimize expected number of putts, maximize one-putt probability, and aim 1.5 feet beyond the hole strategies for the professional golfer. The golfer parameters are $\beta_0 = 6.31\%$, $\beta_1 = 10.20\%$, $\beta_2 = 6.31\%$, $v_\beta = 18.4258$ feet, $\sigma_\alpha = 1.1146$, $\sigma_g = 0.124$. We refer to this set as Parameter Set 2 for the Professional golfer. The green slope is $1.5°$, and green speed is 11 feet. The fit is good with respect to the one-putt probabilities (P_1) as well as the expected number of putts (N^*) (as shown by the root-mean-squared error (RMSE) values under the expected-putts minimization strategy). For short distances, minimizing the expected number of putts and maximizing the one-putt probabilities yield similar results. The optimal distance beyond the hole to aim at (d^*) (and consequently the fraction of short putts, f_s) depends on the length of the putt, and in general, differs from aiming 1.5 ft beyond the hole. (For each fixed distance, P_1, N^*, d^* and f_s are averaged over the angle the initial position of the putt makes with the x-axis.)

Table 4.14: Expected number of putts for professional golfer - Parameter Set 2

angle with x-axis	Length of putt (feet)										
	2	3	4	6	8	10	15	21	25	42	50
90	1.000	1.014	1.067	1.226	1.375	1.492	1.686	1.818	1.877	2.024	2.082
75	1.000	1.016	1.076	1.252	1.407	1.525	1.712	1.838	1.892	2.033	2.094
60	1.000	1.022	1.098	1.302	1.468	1.586	1.760	1.869	1.917	2.054	2.117
45	1.001	1.030	1.123	1.349	1.519	1.634	1.796	1.893	1.934	2.076	2.139
30	1.001	1.039	1.144	1.381	1.550	1.665	1.815	1.904	1.944	2.093	2.157
15	1.001	1.046	1.158	1.397	1.565	1.675	1.823	1.908	1.947	2.105	2.169
0	1.002	1.050	1.164	1.401	1.566	1.673	1.820	1.905	1.953	2.109	2.172
-15	1.002	1.052	1.165	1.397	1.557	1.663	1.811	1.904	1.953	2.105	2.168
-30	1.002	1.051	1.162	1.387	1.544	1.648	1.796	1.899	1.946	2.097	2.158
-45	1.002	1.048	1.154	1.374	1.527	1.631	1.781	1.890	1.938	2.087	2.146
-60	1.002	1.043	1.143	1.357	1.511	1.615	1.771	1.882	1.930	2.076	2.135
-75	1.001	1.039	1.133	1.343	1.498	1.605	1.765	1.876	1.924	2.069	2.128
-90	1.001	1.037	1.129	1.338	1.494	1.601	1.763	1.874	1.922	2.067	2.125
Avg. model	1.001	1.038	1.133	1.349	1.509	1.619	1.779	1.883	1.930	2.078	2.139
Avg. data	1.009	1.046	1.137	1.349	1.507	1.614	1.783	1.879	1.925	2.066	2.148
Difference	-0.007	-0.008	-0.003	0.000	0.002	0.005	-0.004	0.004	0.005	0.012	-0.009

This table shows how the expected number of putts varies for a professional golfer (parameter set 2) as a function of putt position and length of the putt. The expected number of putts to holeout increases with distance. Sidehill putts, i.e., putts from initial angles between $-30°$ and $30°$, are the most difficult.

Table 4.15: Professional golfer: data fit and strategy comparison - Parameter Set 4

Data			Model											
			Min exp putts				Max one putt prob				1.5 feet beyond hole			
d	P_1	N^*	P_1	N^*	d^*	f_s	P_1	N^*	d^*	f_s	P_1	N^*	d^*	f_s
2	99.2%	1.01	99.9%	1.00	1.24	0.0%	99.9%	1.00	1.29	0.0%	99.8%	1.00	1.50	0.0%
3	95.5%	1.05	96.2%	1.04	1.58	0.0%	96.2%	1.04	1.62	0.0%	95.5%	1.04	1.50	0.0%
4	86.6%	1.14	86.7%	1.13	1.63	0.1%	86.7%	1.13	1.73	0.1%	85.5%	1.15	1.50	0.0%
5	75.9%	1.24	75.6%	1.25	1.67	0.3%	75.6%	1.25	1.86	0.3%	74.1%	1.26	1.50	0.1%
6	65.5%	1.35	65.4%	1.35	1.68	0.5%	65.6%	1.35	1.94	0.4%	64.0%	1.36	1.50	0.4%
7	57.1%	1.43	56.8%	1.44	1.69	0.8%	57.1%	1.44	2.05	0.7%	55.5%	1.45	1.50	1.0%
8	49.9%	1.51	49.8%	1.51	1.69	1.3%	50.1%	1.52	2.16	0.9%	48.6%	1.52	1.50	1.9%
9	44.2%	1.56	43.9%	1.57	1.68	2.1%	44.4%	1.58	2.24	1.2%	43.0%	1.58	1.50	3.0%
10	39.3%	1.61	39.1%	1.62	1.68	3.0%	39.6%	1.63	2.33	1.5%	38.3%	1.62	1.50	4.3%
15	23.0%	1.78	23.7%	1.78	1.56	11.9%	24.6%	1.81	2.69	4.0%	23.4%	1.78	1.50	12.7%
21	14.4%	1.88	14.2%	1.89	1.28	25.7%	15.6%	1.94	2.96	8.6%	14.5%	1.89	1.50	22.1%
25	10.8%	1.92	10.4%	1.93	1.06	33.2%	12.1%	2.00	3.17	11.6%	11.1%	1.94	1.50	26.8%
29	8.1%	1.97	7.9%	1.97	0.87	38.5%	9.7%	2.05	3.29	14.5%	8.8%	1.98	1.50	30.3%
33	6.7%	1.99	6.2%	2.01	0.69	42.2%	7.9%	2.09	3.43	17.2%	7.1%	2.02	1.50	33.1%
37	5.0%	2.04	4.9%	2.04	0.50	45.1%	6.6%	2.13	3.51	19.8%	5.9%	2.06	1.50	35.2%
42	4.1%	2.07	3.8%	2.08	0.37	47.0%	5.4%	2.17	3.60	22.7%	4.7%	2.10	1.50	37.3%
50	2.8%	2.15	2.8%	2.14	0.17	48.8%	4.0%	2.22	3.68	26.6%	3.5%	2.16	1.50	39.7%
		RMSE	0.37%	0.007			1.01%	0.055			1.00%	0.015		

This table shows the fit of the professional golfer data with the golfer model for different putt lengths d (in feet), when we attempt to calibrate to both one-putt probabilities and expected putts. It also compares the minimize expected number of putts, maximize one-putt probability, and aim 1.5 feet beyond the hole strategies for the professional golfer. The golfer parameters are $\beta_0 = 5.62\%$, $\beta_1 = 9.60\%$, $\beta_2 = 5.62\%$, $v_\beta = 16.7406$ feet, $\sigma_\alpha = 1.0998$, $\sigma_g = 0.1480$. We refer to this set as Parameter Set 4 for the Professional golfer. The green slope is $1.5°$, and green speed is 11 feet. The fit is good with respect to the one-putt probabilities (P_1)as well as the expected number of putts (N^*) (as shown by the root-mean-squared error (RMSE) values under the expected-putts minimization strategy). For short distances, minimizing the expected number of putts and maximizing the one-putt probabilities yield similar results. The optimal distance beyond the hole to aim at (d^*) (and consequently the fraction of short putts, f_s) depends on the length of the putt, and in general, differs from aiming 1.5 ft beyond the hole. (For each fixed distance, P_1, N^*, d^* and f_s are averaged over the angle the initial position of the putt makes with the x-axis.)

Table 4.16: Expected number of putts for professional golfer - Parameter Set 4

angle with x-axis	Length of putt (feet)										
	2	3	4	6	8	10	15	21	25	42	50
90	1.000	1.021	1.087	1.263	1.413	1.528	1.713	1.839	1.895	2.043	2.106
75	1.000	1.022	1.094	1.280	1.435	1.550	1.733	1.852	1.905	2.053	2.116
60	1.001	1.026	1.108	1.316	1.479	1.595	1.769	1.877	1.924	2.070	2.134
45	1.001	1.032	1.126	1.350	1.519	1.633	1.796	1.895	1.938	2.087	2.150
30	1.001	1.039	1.141	1.374	1.544	1.657	1.812	1.903	1.944	2.098	2.162
15	1.001	1.043	1.152	1.386	1.554	1.664	1.816	1.904	1.951	2.105	2.167
0	1.002	1.047	1.157	1.390	1.553	1.660	1.811	1.906	1.954	2.105	2.167
-15	1.002	1.049	1.158	1.384	1.544	1.650	1.800	1.904	1.951	2.099	2.160
-30	1.002	1.049	1.156	1.378	1.533	1.637	1.785	1.896	1.944	2.090	2.149
-45	1.002	1.046	1.150	1.366	1.520	1.624	1.779	1.888	1.934	2.078	2.137
-60	1.002	1.043	1.141	1.354	1.506	1.611	1.771	1.880	1.927	2.069	2.127
-75	1.001	1.040	1.134	1.342	1.496	1.602	1.766	1.875	1.923	2.063	2.120
-90	1.001	1.039	1.131	1.338	1.492	1.599	1.764	1.874	1.921	2.061	2.118
Avg. model	1.001	1.038	1.134	1.350	1.509	1.618	1.780	1.885	1.933	2.080	2.141
Avg. data	1.009	1.046	1.137	1.349	1.507	1.614	1.783	1.879	1.925	2.066	2.148
Difference	-0.007	-0.007	-0.002	0.000	0.002	0.004	-0.003	0.006	0.008	0.013	-0.007

This table shows how the expected number of putts varies for a professional golfer (parameter set 4) as a function of putt position and length of the putt. The expected number of putts to holeout increases with distance. Sidehill putts, i.e., putts from initial angles between $-30°$ and $30°$, are the most difficult.

Professional player - uphill and downhill putts

Table 4.17 shows how the expected number of putts, one-putt probabilities, fraction of putts that are short, and the optimal aim distance beyond the hole varies with putt length for the professional player. Parameter Set 3 was used for generating these results. We observe that downhill putts lead to higher one-putt probability as well as fewer expected putts. The difference between downhill and uphill expected putts is the largest for intermediate putt-lengths, e.g., 5-15 foot putts. For longer putt-lengths, while downhill putts are still easier, the difference is expected putts is smaller. For putts < 35 feet, downhill putts involve aiming further beyond the hole and leaving fewer putts short. However, for putts > 40 feet, downhill putts involve aiming a shorter distance beyond the hole and leaving a larger fraction of the putts short.

Professional player - optimal aim direction

Table 4.18 shows how the optimal angle to aim at varies as a function of the initial putting position for a professional golfer. Parameter Set 3 was used to generate these results. The optimal aim direction (exp angle) is compared with the maximum (max angle) and the minimum (min angle) possible angles that could lead to a holeout, as well as the angle corresponding to the strategy that aims 1.5 feet beyond the hole (1.5 ft). Under the optimal (expected-putt minimizing) strategy, the player is aggressive and aims straighter for short putts. However, as putt length increases, the player becomes more conservative and allows for more curvature in the trajectory. While the optimal strategy is comparable to the 1.5-foot strategy for short putts, e.g., 3 feet, for longer putts, e.g., > 15 feet, the optimal strategy becomes more conservative than the 1.5-foot strategy. The difference in the two strategies is more noticeable for sidehill putts.

Table 4.17: Comparing uphill and downhill putts for the professional golfer

	Uphill putts				Downhill putts			
d	P_1	N^*	d^*	f_s	P_1	N^*	d^*	f_s
2	99.8%	1.00	0.5	0.0%	100.0%	1.00	1.5	0.0%
3	95.5%	1.05	0.5	0.3%	99.0%	1.01	1.1	0.0%
4	85.6%	1.14	0.6	0.6%	94.3%	1.06	1.1	0.1%
5	74.6%	1.25	0.7	1.7%	86.9%	1.13	1.3	0.2%
6	64.6%	1.35	0.7	2.6%	78.7%	1.21	1.3	0.6%
7	55.6%	1.44	0.9	3.3%	71.0%	1.29	1.4	1.0%
8	48.4%	1.52	1.0	4.5%	64.1%	1.36	1.4	1.6%
9	42.3%	1.58	1.1	5.4%	58.0%	1.42	1.5	2.4%
10	37.3%	1.63	1.2	7.2%	52.7%	1.48	1.5	3.0%
15	22.1%	1.79	1.4	13.4%	34.8%	1.67	1.7	7.9%
21	13.1%	1.89	1.2	25.8%	22.6%	1.80	1.5	17.9%
25	9.8%	1.93	1.1	31.1%	17.1%	1.87	1.4	25.5%
29	7.4%	1.97	0.8	37.4%	13.3%	1.92	1.2	31.1%
33	5.9%	2.00	0.7	40.5%	10.4%	1.96	0.9	37.7%
37	4.8%	2.03	0.6	43.4%	8.2%	1.99	0.5	43.5%
42	3.8%	2.06	0.4	45.2%	6.4%	2.03	0.2	47.5%
50	2.8%	2.12	0.4	46.8%	5.0%	2.09	0.1	48.8%

This table compares uphill and downhill putting ability for the professional golfer. Downhill putts lead to a higher one-putt probability (P_1), and result in fewer expected number of putts (N^*). The difference between downhill and uphill expected putts is the largest for intermediate putt-lengths, e.g., 5-15 foot putts. For longer putt-lengths, while downhill putts are still easier, the difference is expected putts is smaller. Also, for putt-lengths shorter than 35 feet, uphill putts involve aiming a shorter distance beyond the hole (d^*) than downhill putts, so that if the first putt does not lead to a holeout, the second putt is easier. The fraction of putts that are short, f_s, shows that for putt-lengths less than 35 feet, the optimal strategy for downhill putts involves leaving less putts short. Uphill and downhill putts are aimed at from an angle of $-75°$ and $75°$, respectively, with respect to the x-axis. Parameter set 3 for the professional player was used to generate these results.

Table 4.18: Professional golfer: strategy comparison

angle	3 foot				15 foot				25 foot				50 foot			
with x-axis	max angle	min angle	exp angle	1.5ft angle	max angle	min angle	exp angle	1.5ft angle	max angle	min angle	exp angle	1.5ft angle	max angle	min angle	exp angle	1.5ft angle
75	11.3	0.5	4.2	3.6	8.5	2.3	6.0	6.1	7.8	3.2	6.7	6.6	7.7	4.4	7.6	7.1
60	15.4	0.9	6.2	6.6	13.4	4.3	10.8	11.0	14.0	5.8	12.1	11.9	13.7	8.0	13.5	12.6
45	17.7	1.4	7.3	8.3	16.9	5.8	13.8	13.9	17.5	7.9	15.5	15.0	17.3	10.5	17.0	15.9
30	18.5	1.9	7.4	8.9	18.9	6.6	14.8	14.9	18.7	8.7	16.7	16.0	18.5	11.4	18.3	17.1
15	18.8	2.1	7.7	8.5	18.0	6.6	14.3	14.3	18.1	8.9	16.2	15.4	17.9	11.4	17.7	16.5
0	18.1	1.8	7.1	7.4	15.7	6.4	12.5	12.6	16.3	8.3	14.5	13.7	15.9	10.5	15.8	14.7
-15	15.9	2.0	6.6	6.1	14.0	5.8	10.4	10.5	13.7	7.1	12.1	11.4	13.7	8.9	13.3	12.3
-30	12.6	1.3	5.5	4.8	11.2	4.8	8.2	8.1	11.0	5.8	9.4	8.9	10.8	7.3	10.5	9.6
-45	10.4	1.2	4.6	3.5	8.4	3.7	6.1	5.9	8.1	4.5	6.9	6.5	8.1	5.5	7.8	7.0
-60	7.5	0.6	3.2	2.3	5.6	2.5	3.9	3.8	5.4	3.0	4.5	4.2	5.1	3.7	5.1	4.6
-75	3.1	0.2	1.7	1.1	3.1	1.2	1.9	1.9	2.6	1.5	2.2	2.1	2.6	1.8	2.4	2.2

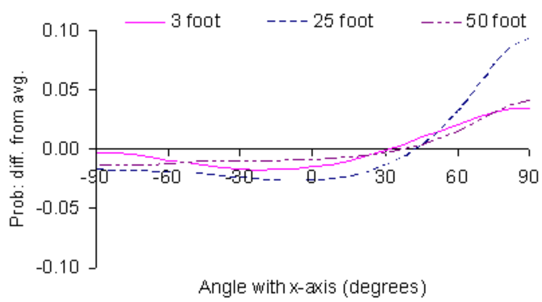
This table shows how the optimal angle to aim at varies as a function of the initial putting position (angle with respect to the x-axis) and distance for the professional golfer. The optimal aim direction (exp) is compared with the maximum (max angle) and the minimum (min angle) possible angles that would lead to a holeout, as well as the angle corresponding to the strategy that aims 1.5 feet beyond the hole (1.5 ft). For short putts, the golfer is aggressive (and aims straighter), while as putt length increases, the golfer becomes more conservative. While the optimal strategy and the aim 1.5-foot strategy are similar for short putts, as putt length increases, the optimal strategy becomes more conservative than the aim 1.5 feet beyond the hole strategy, especially for sidehill putts. The green slope is 1.5° along the y-axis, and green speed is 11 feet ($\eta = 0.0510$). Parameter Set 3 for the professional player is used to generate these results.

Professional player - comparison with probability optimization

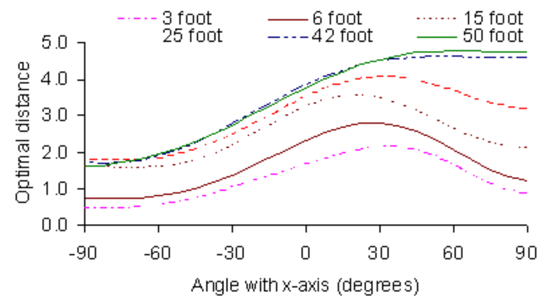
Figure 4.13 illustrates several aspects of the one-putt probability maximizing strategy for the professional player (parameter set 3 was used to generate these results). Figure (a) compares one-putt probability deviations from the average probability for putt-lengths 3 feet, 25 feet and 50 feet, as a function of the putt-angle (the one-putt probability maximizing strategy was used). We observe that downhill putts lead to higher optimal one-putt probability than uphill putts, and that sidehill putts are among the hardest for the professional player. Figure (b) shows that the distance to target beyond the hole to maximize the one-putt probability increases with putt-length for most putt-angles, and thus differs significantly from the expected-putts minimization strategy discussed above wherein the optimal distance beyond the hole to target decreases for long putts (see Figure 4.5). Similarly, Figure (c) shows that a lot less putts are left short with the one-putt probability maximizing strategy as compared to the expected-putts minimizing strategy. Finally, Figure (d) shows that one-putt probability maximization strategy is much flatter (more aggressive) than the 1.5-foot strategy or the expected putts minimization strategy (see Figure 4.6).

Professional player - comparison of calibrated parameter sets

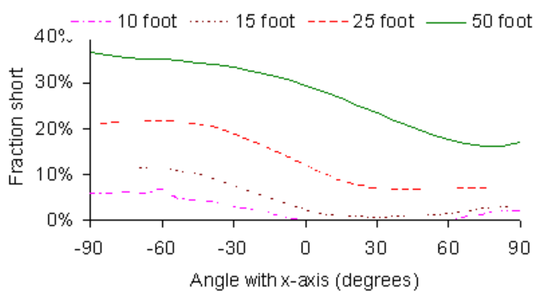
Figure 4.14 compares two of the calibrated parameter sets for the Professional player: Parameter Set 3 and Parameter Set 4. While both sets calibrate to both one-putt probabilities and expected putts, Parameter Set 3 is characterized by higher angle error and lower green error compared to Parameter Set 4. Figures (a) and (b), respectively, show how the average distance beyond the hole to target varies as a function of putt-length when one-putt probability maximization and expected putts minimization strategies are employed. The player with the higher green error, i.e., Parameter Set 4, aims further beyond the hole for maximizing one-putt probability. For expected putts, while this player aims further for short putt-lengths, for longer putts, the optimal distance to target is hard to differentiate between the two players. Figure (c) shows how the optimal one-putt probabilities



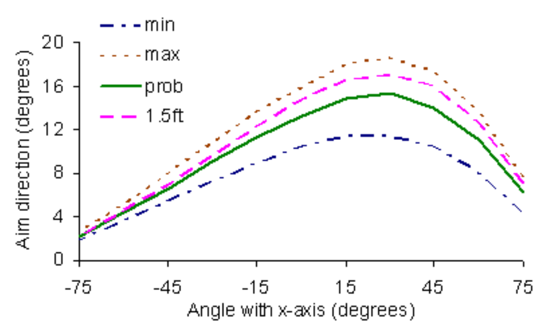
(a) One-putt probability



(b) Optimal distance



(c) Fraction short



(d) Angle to aim - 50 foot putt

Figure 4.13: This figure illustrates aspects of the one-putt probability maximizing strategy for the professional player. Figure (a) compares one-putt probability deviations from the average probability as a function of the putt-angle (the one-putt probability maximizing strategy was used) Figures (b) and (c), respectively, show that the optimal distance to target beyond the hole (in feet), and the fraction of putts left short, to maximize the one-putt probability for different putt-lengths. Graph (d) compares the one-putt probability maximization strategy with the 1.5 feet beyond the hole strategy. Parameter Set 3 for the professional player was used to generate these results.

vary as a function of the putt-angle for these two players for a 25 foot putt. While the player with higher angle error has lower one-putt probabilities for uphill and sidehill putts, this player with less green error has higher one-putt probabilities for straight and close-to-straight downhill putts. This is consistent with the observations in Section 4.2.5. Figure (d) shows the optimal distance beyond the hole to target (for maximizing one-putt probability) for these two players for a 25 foot putt. While the aim distances are hard to distinguish for uphill and sidehill putts, for downhill putts, the player with higher green error, i.e., Parameter Set 4, aims further beyond the hole. Figure (e) shows how the minimum expected putts vary as a function of the putt-angle for these two players for a 3 foot putt. The results are similar to those observed in Figure (c), in that, Parameter Set 3 performs better for downhill and close-to-downhill putts. Figure (f) shows the optimal distance beyond the hole to target (for minimizing expected putts) for these two players for a 3 foot putt. As with Figure (d), we observe that the player with higher green error, i.e., Parameter Set 4, aims further beyond the hole. As suggested by Figures (a) and (b), while this difference in strategy for maximizing one-putt probability for downhill putts can be observed for short as well as long putt-lengths (e.g., 3 foot, 25 foot), the difference in strategy for minimizing expected number of putts for downhill putts diminishes as putt length increases.

Amateur player calibration

We calibrate the golfer model to amateur golfer data on a green with a slope of 1.5° and green speed of 9 feet. As for the professional golfer, we calibrate 4 parameter sets. To obtain Parameter Set 1, we calibrate to one-putt probabilities only. To obtain Parameter Set 2, Parameter Set 3 and Parameter Set 4, we calibrate to both one-putt probabilities and expected-putts. As with the professional players, Parameter sets 2-4 reflect amateur players with varying distance, direction and green-reading skills.

In Section 4.4, we presented calibration results for Parameter Set 2. We present calibration results for the remaining parameter sets below. As for the professional player, we find that calibration to one-putt probabilities alone does not necessarily lead to a good fit to expected-putts data, and hence joint calibration to both one-putt probability data and expected-putts data is necessary. As expected, the 1.5 ft strategy is suboptimal for expected-putts minimization, though for long putt-lengths (e.g. > 10 feet), it is better than the one-putt probability-maximization strategy. For all parameter sets and across putt-lengths, sidehill putts and side-up are found to be among the hardest putts. We discuss these results in detail below.

Table 4.19 shows the fit for Parameter Set 1 where we attempt to calibrate to one-putt probabilities alone. The amateur player parameters thus obtained are $\beta_0 = 7.76\%$, $\beta_1 = 18.88\%$, $\beta_2 = 8.47\%$, $v_\beta = 8.1985$ feet, $\sigma_\alpha = 1.9089$, $\sigma_g = 0.2136$. The overall RMSE between model and data is 0.28% for one-putt probabilities and 0.131 for expected putts. As we calibrate to one-putt probabilities, the probability error is small; however, the expected putts error is large, suggesting that the player parameters need to be calibrated to both one-putt probabilities and expected-putts.

For comparison, results obtained using these parameters and the expected-putts minimizing strategy as well as the 1.5-foot strategy are also shown. As for the professional player, the expected-putts minimizing strategy for the amateur player involves aiming a shorter distance beyond the hole, and as expected, leads to fewer expected putts. Again, the optimal distance to target to minimize expected putts or to maximize the one-putt probability, in general, differs from aiming 1.5 feet beyond the hole.

Table 4.20 shows how the expected number of putts varies as a function of the putt position (initial putt-length and putt-angle) for an amateur player with skill determined by parameter set 1. As expected, we observe that the expected number of putts needed to holeout increases with putt-length. We also observe that sidehill putts ($-30°$ to $30°$) and sideup putts ($-60°$ to $-30°$) are the most difficult, i.e., they lead to the largest expected number of putts for a given putt-length. Finally, we observe that downhill putts ($75°$ to $90°$) are easier than uphill putts ($-75°$ to $-90°$).

Tables 4.7, 4.21, and 4.23, show the fit for Parameter Set 2, Parameter Set 3, and Parameter Set 4, respectively, wherein we attempt to calibrate to both one-putt probabilities and expected-putts for the amateur player. Unlike the case with Parameter Set 1, a good fit to both one-putt probabilities and expected-putts is achieved for these parameter sets. The player parameters are $\beta_0 = 8.94\%$, $\beta_1 = 18.89\%$, $\beta_2 = 8.99\%$, $v_\beta = 42.35$ feet, $\sigma_\alpha = 1.8964$, $\sigma_g = 0.2128$ for Parameter Set 2, $\beta_0 = 8.77\%$, $\beta_1 = 18.18\%$, $\beta_2 = 8.77\%$, $v_\beta = 40.1426$ feet, $\sigma_\alpha = 2.0008$, $\sigma_g = 0.1949$ for Parameter Set 3, and $\beta_0 = 8.93\%$, $\beta_1 = 18.72\%$, $\beta_2 = 8.93\%$, $v_\beta = 39.0349$ feet, $\sigma_\alpha = 2.0085$, $\sigma_g = 0.1781$ for Parameter Set 4. Among these parameter sets, Parameter Set 4 has lowest green error and the highest angle error, while Parameter Set 2 has the highest green error and the lowest angle error. Again, for comparison, in addition to the one-putt probability maximizing strategy, results obtained using these parameters and the expected-putts minimizing strategy and the 1.5-foot strategy are also shown. As in the case for Parameter Set 1, we observe that the expected-putt minimizing strategy involves aiming a shorter distance beyond the hole compared to the one-putt probability maximizing strategy, and typically is different that the strategy that aims 1.5 feet beyond the hole.

Tables 4.8, 4.22, and 4.24, show how the expected number of putts varies as a function of the putt position for Parameter Set 2, Parameter Set 3, and Parameter Set 4 for the amateur player. As with Parameter Set 1, we observe that sidehill and sideup putts are the most difficult for these players and that downhill putts are easier than uphill putts. As with the professional player, we also observe that the difference between the downhill and the uphill expected putts is smaller for the parameter set which has the highest green error and the lowest angle error, i.e., Parameter Set 2.

Table 4.19: Amateur golfer: data fit and strategy comparison - Parameter Set 1

Data			Model											
			Min exp putts				Max one putt prob				1.5 feet beyond hole			
d	P_1	N^*	P_1	N^*	d^*	f_s	P_1	N^*	d^*	f_s	P_1	N^*	d^*	f_s
2	93.9%	1.06	94.8%	1.05	0.94	0.1%	94.8%	1.05	0.96	0.1%	93.4%	1.07	1.50	0.0%
3	79.2%	1.21	79.0%	1.22	0.99	0.8%	79.0%	1.21	1.04	0.7%	77.3%	1.24	1.50	0.0%
4	64.1%	1.37	63.9%	1.37	1.03	2.0%	64.0%	1.37	1.17	1.4%	62.7%	1.39	1.50	0.3%
5	52.1%	1.49	52.2%	1.49	1.06	3.8%	52.4%	1.49	1.27	2.4%	51.5%	1.51	1.50	1.0%
6	43.0%	1.58	42.9%	1.59	1.05	7.2%	43.2%	1.60	1.38	4.0%	42.8%	1.61	1.50	2.7%
7	36.0%	1.66	35.4%	1.67	1.04	11.3%	36.1%	1.68	1.48	6.0%	35.9%	1.69	1.50	5.2%
8	30.5%	1.72	29.5%	1.74	1.00	16.1%	30.4%	1.75	1.58	7.9%	30.3%	1.76	1.50	8.3%
9	26.2%	1.76	24.7%	1.79	0.94	20.8%	26.0%	1.81	1.69	9.7%	25.8%	1.81	1.50	11.4%
10	22.6%	1.80	20.9%	1.84	0.87	25.3%	22.4%	1.87	1.78	11.6%	22.2%	1.86	1.50	14.5%
15	12.2%	1.93	9.7%	2.01	0.48	41.5%	11.8%	2.07	2.16	20.2%	11.4%	2.04	1.50	26.7%
21	6.9%	2.03	5.0%	2.15	0.17	48.2%	6.7%	2.22	2.35	27.4%	6.3%	2.18	1.50	34.4%
25	5.1%	2.08	3.7%	2.23	0.09	49.1%	4.9%	2.29	2.35	31.1%	4.7%	2.26	1.50	37.2%
29	3.9%	2.13	2.9%	2.30	0.07	49.4%	3.7%	2.34	2.30	34.3%	3.6%	2.32	1.50	39.3%
33	3.0%	2.19	2.3%	2.35	0.07	49.4%	2.9%	2.40	2.29	36.4%	2.8%	2.38	1.50	40.7%
37	2.4%	2.24	1.9%	2.41	0.07	49.5%	2.4%	2.44	2.24	38.2%	2.3%	2.43	1.50	41.9%
42	1.8%	2.32	1.6%	2.46	0.08	49.5%	1.9%	2.49	2.27	39.7%	1.8%	2.48	1.50	43.0%
50	1.4%	2.39	1.2%	2.53	0.08	49.6%	1.4%	2.56	2.31	41.4%	1.3%	2.55	1.50	44.2%
		RMSE	1.12%	0.100			0.28%	0.131			0.067%	0.118		

This table shows the fit of the amateur golfer data with the golfer model for different putt lengths d (in feet), when we attempt to calibrate to one-putt probabilities alone. It also compares the minimize expected number of putts, maximize one-putt probability, and aim 1.5 feet beyond the hole strategies for the amateur golfer. The golfer parameters are $\beta_0 = 7.76\%$, $\beta_1 = 18.88\%$, $\beta_2 = 8.47\%$, $v_\beta = 8.1985$ feet, $\sigma_\alpha = 1.9089$, $\sigma_g = 0.2136$. We refer to this set as Parameter Set 1 for the Amateur golfer. The green slope is $1.5°$, and green speed is 9 feet. The fit is good with respect to the one-putt probabilities (P_1); however the expected number of putts (N^*) error is large (as seen from the root-mean-squared error (RMSE) values under the one-putt probability maximization strategy). For short distances, minimizing the expected number of putts and maximizing the one-putt probabilities yield similar results. The optimal distance beyond the hole to aim at (d^*) (and consequently the fraction of short putts, f_s) depends on the length of the putt, and in general, differs from aiming 1.5 ft beyond the hole. (For each fixed distance, P_1, N^*, d^* and f_s are averaged over the angle the initial position of the putt makes with the x-axis.)

Table 4.20: Expected number of putts for amateur golfer - Parameter Set 1

angle with x-axis	Length of putt (feet)										
	2	3	4	6	8	10	15	21	25	42	50
90	1.024	1.141	1.285	1.515	1.666	1.768	1.929	2.075	2.159	2.413	2.492
75	1.027	1.152	1.299	1.528	1.675	1.776	1.939	2.087	2.171	2.421	2.499
60	1.035	1.176	1.330	1.556	1.699	1.794	1.963	2.113	2.194	2.437	2.512
45	1.045	1.201	1.361	1.586	1.722	1.811	1.991	2.138	2.218	2.454	2.528
30	1.053	1.221	1.385	1.605	1.737	1.831	2.016	2.160	2.237	2.468	2.541
15	1.059	1.234	1.398	1.615	1.743	1.856	2.032	2.173	2.250	2.478	2.550
0	1.063	1.240	1.402	1.617	1.757	1.870	2.038	2.180	2.256	2.481	2.553
-15	1.065	1.243	1.402	1.613	1.768	1.875	2.041	2.180	2.256	2.480	2.551
-30	1.066	1.240	1.399	1.608	1.771	1.874	2.038	2.179	2.253	2.477	2.548
-45	1.065	1.237	1.394	1.613	1.769	1.870	2.034	2.173	2.248	2.472	2.542
-60	1.063	1.234	1.390	1.614	1.766	1.866	2.029	2.169	2.245	2.467	2.537
-75	1.061	1.232	1.388	1.613	1.763	1.863	2.027	2.167	2.242	2.464	2.534
-90	1.061	1.231	1.387	1.613	1.763	1.862	2.026	2.166	2.242	2.463	2.533
Avg. model	1.053	1.215	1.372	1.593	1.739	1.841	2.009	2.152	2.230	2.460	2.533
Avg. data	1.063	1.212	1.366	1.584	1.715	1.802	1.932	2.027	2.081	2.319	2.393
Difference	-0.009	0.003	0.006	0.010	0.024	0.038	0.077	0.125	0.149	0.142	0.140

This table shows how the expected number of putts varies for a amateur golfer (parameter set 1) as a function of putt position and length of the putt. The expected number of putts to holeout increases with distance. Sidehill putts, i.e., putts from initial angles between $-30°$ and $30°$, are the most difficult.

Table 4.21: Amateur golfer: data fit and strategy comparison - Parameter Set 3

Data			Model											
			Min exp putts				Max one putt prob				1.5 feet beyond hole			
d	P_1	N^*	P_1	N^*	d^*	f_s	P_1	N^*	d^*	f_s	P_1	N^*	d^*	f_s
2	93.9%	1.06	94.6%	1.05	0.71	0.2%	94.6%	1.05	0.71	0.2%	92.6%	1.08	1.50	0.0%
3	79.2%	1.21	78.5%	1.22	0.84	1.0%	78.5%	1.22	0.87	0.9%	76.2%	1.25	1.50	0.0%
4	64.1%	1.37	63.2%	1.37	0.94	2.3%	63.2%	1.38	1.04	1.6%	61.7%	1.41	1.50	0.2%
5	52.1%	1.49	51.4%	1.50	1.00	4.1%	51.5%	1.50	1.17	2.7%	50.6%	1.53	1.50	0.8%
6	43.0%	1.58	42.5%	1.59	1.02	6.6%	42.8%	1.60	1.30	3.6%	42.4%	1.62	1.50	2.0%
7	36.0%	1.66	35.8%	1.66	1.03	9.5%	36.3%	1.67	1.41	4.7%	36.0%	1.69	1.50	3.5%
8	30.5%	1.72	30.4%	1.72	1.01	12.8%	31.2%	1.74	1.52	5.6%	31.1%	1.75	1.50	5.4%
9	26.2%	1.76	26.2%	1.77	0.99	16.0%	27.3%	1.79	1.62	6.5%	27.1%	1.79	1.50	7.5%
10	22.6%	1.80	22.7%	1.81	0.94	19.6%	24.0%	1.84	1.74	7.2%	23.8%	1.83	1.50	9.6%
15	12.2%	1.93	12.2%	1.93	0.66	34.4%	14.4%	2.00	2.08	12.1%	14.0%	1.97	1.50	18.7%
21	6.9%	2.03	7.0%	2.03	0.37	43.7%	9.1%	2.12	2.38	16.9%	8.7%	2.08	1.50	26.3%
25	5.1%	2.08	5.2%	2.09	0.21	46.8%	7.0%	2.17	2.44	20.4%	6.7%	2.13	1.50	29.7%
29	3.9%	2.13	4.1%	2.14	0.10	48.6%	5.6%	2.22	2.49	23.7%	5.3%	2.19	1.50	32.6%
33	3.0%	2.19	3.4%	2.19	0.08	49.1%	4.5%	2.27	2.43	27.4%	4.2%	2.24	1.50	35.1%
37	2.4%	2.24	2.8%	2.25	0.07	49.2%	3.6%	2.32	2.46	29.9%	3.4%	2.29	1.50	37.1%
42	1.8%	2.32	2.3%	2.31	0.07	49.3%	2.9%	2.37	2.49	32.2%	2.7%	2.35	1.50	39.0%
50	1.4%	2.39	1.7%	2.40	0.08	49.4%	2.0%	2.45	2.42	35.9%	1.9%	2.43	1.50	41.3%
		RMSE	0.43%	0.006			1.26%	0.056			1.45%	0.040		

This table shows the fit of the amateur golfer data with the golfer model for different putt lengths d (in feet), when we attempt to calibrate to both one-putt probabilities and expected putts. It also compares the minimize expected number of putts, maximize one-putt probability, and aim 1.5 feet beyond the hole strategies for the amateur golfer. The golfer parameters are $\beta_0 = 8.77\%$, $\beta_1 = 18.18\%$, $\beta_2 = 8.77\%$, $v_\beta = 40.1426$ feet, $\sigma_\alpha = 2.0008$, $\sigma_g = 0.1949$. We refer to this set as Parameter Set 3 for the Amateur golfer. The green slope is $1.5°$, and green speed is 9 feet. The fit is good with respect to both the one-putt probabilities (P_1) and the expected number of putts (N^*) (as seen by the RMSE values under the expected-putts minimization strategy). For short distances, minimizing the expected number of putts and maximizing the one-putt probabilities yield similar results. The optimal distance beyond the hole to aim at (d^*) (and consequently the fraction of short putts, f_s) depends on the length of the putt, and in general, differs from aiming 1.5 ft beyond the hole. (For each fixed distance, P_1, N^*, d^* and f_s are averaged over the angle the initial position of the putt makes with the x-axis.)

Table 4.22: Expected number of putts for amateur golfer - Parameter Set 3

angle with x-axis	Length of putt (feet) 2	3	4	6	8	10	15	21	25	42	50
90	1.022	1.135	1.276	1.502	1.652	1.755	1.907	2.012	2.069	2.262	2.334
75	1.025	1.144	1.289	1.515	1.662	1.762	1.912	2.016	2.073	2.266	2.337
60	1.031	1.166	1.317	1.542	1.685	1.782	1.925	2.028	2.083	2.276	2.346
45	1.040	1.191	1.349	1.574	1.711	1.802	1.938	2.039	2.094	2.284	2.354
30	1.049	1.213	1.376	1.597	1.730	1.818	1.947	2.046	2.101	2.291	2.365
15	1.057	1.230	1.395	1.613	1.741	1.825	1.952	2.049	2.104	2.295	2.396
0	1.063	1.242	1.405	1.620	1.746	1.828	1.951	2.048	2.102	2.311	2.417
-15	1.067	1.248	1.410	1.622	1.745	1.825	1.947	2.043	2.097	2.328	2.429
-30	1.069	1.250	1.411	1.619	1.741	1.821	1.942	2.036	2.090	2.337	2.434
-45	1.070	1.249	1.409	1.616	1.737	1.816	1.936	2.029	2.083	2.340	2.436
-60	1.070	1.247	1.406	1.613	1.733	1.811	1.931	2.022	2.076	2.342	2.435
-75	1.069	1.247	1.405	1.610	1.730	1.809	1.927	2.018	2.072	2.341	2.434
-90	1.068	1.246	1.405	1.609	1.730	1.807	1.925	2.017	2.070	2.342	2.434
Avg. model	1.054	1.217	1.374	1.590	1.720	1.806	1.935	2.032	2.086	2.309	2.397
Avg. data	1.063	1.212	1.366	1.584	1.715	1.802	1.932	2.027	2.081	2.319	2.393
Difference	-0.008	0.005	0.008	0.007	0.005	0.004	0.003	0.004	0.005	-0.010	0.003

This table shows how the expected number of putts varies for a amateur golfer (parameter set 3) as a function of putt position and length of the putt. The expected number of putts to holeout increases with distance. Sidehill putts, i.e., putts from initial angles between $-30°$ and $30°$, and sideup putts, i.e., putts from initial angles between $-30°$ and $-60°$, are among the most difficult.

Table 4.23: Amateur golfer: data fit and strategy comparison - Parameter Set 4

Data			Model											
			Min exp putts				Max one putt prob				1.5 feet beyond hole			
d	P_1	N^*	P_1	N^*	d^*	f_s	P_1	N^*	d^*	f_s	P_1	N^*	d^*	f_s
2	93.9%	1.06	94.7%	1.05	0.71	0.2%	94.7%	1.05	0.70	0.2%	92.6%	1.08	1.50	0.0%
3	79.2%	1.21	78.7%	1.22	0.83	1.0%	78.7%	1.22	0.86	1.0%	76.3%	1.25	1.50	0.0%
4	64.1%	1.37	63.4%	1.37	0.93	2.4%	63.4%	1.37	1.02	1.8%	61.8%	1.41	1.50	0.2%
5	52.1%	1.49	51.6%	1.49	0.99	4.2%	51.7%	1.50	1.15	2.9%	50.8%	1.52	1.50	0.8%
6	43.0%	1.58	42.7%	1.59	1.02	6.7%	43.0%	1.59	1.29	3.7%	42.5%	1.62	1.50	2.0%
7	36.0%	1.66	35.9%	1.66	1.02	9.7%	36.4%	1.67	1.39	4.9%	36.2%	1.69	1.50	3.6%
8	30.5%	1.72	30.6%	1.72	1.01	12.9%	31.4%	1.73	1.52	5.7%	31.2%	1.74	1.50	5.4%
9	26.2%	1.76	26.4%	1.76	0.99	15.9%	27.4%	1.79	1.61	6.6%	27.2%	1.79	1.50	7.6%
10	22.6%	1.80	22.9%	1.80	0.94	19.6%	24.2%	1.83	1.70	7.6%	24.0%	1.83	1.50	9.7%
15	12.2%	1.93	12.3%	1.93	0.66	34.4%	14.5%	1.99	2.04	12.5%	14.1%	1.97	1.50	18.7%
21	6.9%	2.03	7.0%	2.03	0.36	43.8%	9.1%	2.11	2.30	17.6%	8.7%	2.08	1.50	26.3%
25	5.1%	2.08	5.3%	2.08	0.23	46.7%	7.1%	2.17	2.42	20.5%	6.7%	2.13	1.50	29.7%
29	3.9%	2.13	4.2%	2.14	0.10	48.8%	5.6%	2.22	2.37	24.9%	5.3%	2.19	1.50	32.8%
33	3.0%	2.19	3.4%	2.19	0.07	49.2%	4.5%	2.27	2.44	27.6%	4.2%	2.24	1.50	35.3%
37	2.4%	2.24	2.8%	2.25	0.07	49.3%	3.7%	2.32	2.45	30.1%	3.4%	2.29	1.50	37.3%
42	1.8%	2.32	2.3%	2.31	0.07	49.3%	2.9%	2.38	2.47	32.6%	2.7%	2.35	1.50	39.2%
50	1.4%	2.39	1.7%	2.40	0.08	49.4%	2.0%	2.44	2.25	37.0%	1.9%	2.43	1.50	41.4%
		RMSE	0.39%	0.005			1.30%	0.053			1.45%	0.039		

This table shows the fit of the amateur golfer data with the golfer model for different putt lengths d (in feet), when we attempt to calibrate to both one-putt probabilities and expected putts. It also compares the minimize expected number of putts, maximize one-putt probability, and aim 1.5 feet beyond the hole strategies for the amateur golfer. The golfer parameters are $\beta_0 = 8.93\%$, $\beta_1 = 18.72\%$, $\beta_2 = 8.93\%$, $v_\beta = 39.0349$ feet, $\sigma_\alpha = 2.0085$, $\sigma_g = 0.1781$. We refer to this set as Parameter Set 4 for the Amateur golfer. The green slope is 1.5°, and green speed is 9 feet. The fit is good with respect to both the one-putt probabilities (P_1) and the expected number of putts (N^*) (as seen by the RMSE values under the expected-putts minimization strategy). For short distances, minimizing the expected number of putts and maximizing the one-putt probabilities yield similar results. The optimal distance beyond the hole to aim at (d^*) (and consequently the fraction of short putts, f_s) depends on the length of the putt, and in general, differs from aiming 1.5 ft beyond the hole. (For each fixed distance, P_1, N^*, d^* and f_s are averaged over the angle the initial position of the putt makes with the x-axis.)

Table 4.24: Expected number of putts for amateur golfer - Parameter Set 4

angle with x-axis	Length of putt (feet)										
	2	3	4	6	8	10	15	21	25	42	50
90	1.020	1.128	1.267	1.493	1.643	1.746	1.899	2.004	2.060	2.253	2.325
75	1.022	1.137	1.280	1.506	1.654	1.755	1.905	2.009	2.065	2.257	2.328
60	1.029	1.161	1.311	1.536	1.679	1.776	1.919	2.022	2.076	2.267	2.337
45	1.039	1.188	1.345	1.569	1.706	1.799	1.934	2.034	2.088	2.278	2.348
30	1.048	1.212	1.374	1.596	1.728	1.814	1.944	2.043	2.098	2.286	2.367
15	1.057	1.230	1.393	1.612	1.740	1.823	1.951	2.047	2.102	2.292	2.400
0	1.063	1.242	1.405	1.620	1.746	1.827	1.950	2.046	2.101	2.317	2.422
-15	1.068	1.248	1.411	1.621	1.745	1.825	1.947	2.042	2.096	2.334	2.435
-30	1.070	1.251	1.412	1.620	1.741	1.821	1.942	2.036	2.089	2.343	2.441
-45	1.070	1.250	1.409	1.616	1.736	1.816	1.936	2.029	2.083	2.348	2.442
-60	1.070	1.248	1.407	1.613	1.733	1.811	1.930	2.022	2.076	2.349	2.442
-75	1.069	1.247	1.405	1.611	1.730	1.808	1.927	2.018	2.072	2.348	2.441
-90	1.068	1.246	1.404	1.609	1.729	1.807	1.925	2.017	2.070	2.349	2.440
Avg. model	1.054	1.216	1.372	1.588	1.717	1.803	1.932	2.029	2.084	2.310	2.398
Avg. data	1.063	1.212	1.366	1.584	1.715	1.802	1.932	2.027	2.081	2.319	2.393
Difference	-0.009	0.003	0.006	0.004	0.002	0.001	0.000	0.002	0.003	-0.009	0.005

This table shows how the expected number of putts varies for a amateur golfer (parameter set 4) as a function of putt position and length of the putt. The expected number of putts to holeout increases with distance. Sidehill putts, i.e., putts from initial angles between $-30°$ and $30°$, and sideup putts, i.e., putts from initial angles between $-30°$ and $-60°$, are among the most difficult.

Table 4.25: Comparing uphill and downhill putts for the amateur golfer

	Uphill putts				Downhill putts			
d	P_1	N^*	d^*	f_s	P_1	N^*	d^*	f_s
2	94.0%	1.06	0.5	0.4%	97.6%	1.02	0.9	0.1%
3	76.9%	1.23	0.6	1.8%	85.9%	1.14	1.0	0.6%
4	61.1%	1.39	0.7	3.3%	71.9%	1.29	1.1	2.2%
5	49.2%	1.51	0.9	4.8%	60.0%	1.41	1.1	3.8%
6	40.5%	1.60	0.9	7.1%	50.4%	1.52	1.2	6.0%
7	33.9%	1.67	1.0	8.6%	43.0%	1.60	1.2	8.2%
8	28.8%	1.72	1.0	11.4%	36.7%	1.67	1.1	12.5%
9	24.7%	1.77	1.0	14.0%	31.8%	1.72	1.1	14.8%
10	21.5%	1.80	1.0	16.4%	27.9%	1.77	1.1	17.5%
15	11.6%	1.92	0.7	30.8%	15.0%	1.92	0.7	34.2%
21	6.6%	2.02	0.4	41.3%	8.7%	2.02	0.3	44.1%
25	5.0%	2.07	0.3	43.9%	6.6%	2.08	0.1	48.1%
29	3.9%	2.12	0.2	45.9%	5.5%	2.13	0.1	48.3%
33	2.9%	2.19	0.0	49.1%	4.7%	2.18	0.1	48.4%
37	2.2%	2.26	0.0	49.2%	4.0%	2.23	0.1	48.5%
42	1.7%	2.33	0.1	49.3%	3.3%	2.28	0.1	48.6%
50	1.1%	2.43	0.1	49.4%	2.6%	2.35	0.1	48.8%

This table compares uphill and downhill putting ability for the amateur golfer. Downhill putts lead to a higher one-putt probability (P_1), and result in fewer expected number of putts (N^*). Uphill and downhill putts are aimed at from an angle of $-75°$ and $75°$, respectively, with respect to the x-axis. Parameter set 2 for the amateur player was used to generate these results.

Amateur player - uphill and downhill putts

Table 4.25 shows how the expected number of putts, one-putt probabilities, fraction of putts that are short, and the optimal aim distance beyond the hole compares for uphill and downhill putts for the amateur player. Parameter Set 2 was used for generating these results. We observe that downhill putts lead to higher one-putt probability and usually fewer expected putts. The difference between downhill and uphill expected putts is the largest for short putts, e.g., < 7 feet, and for long putts, e.g., > 35 feet. For short uphill putts, e.g., < 7 feet, the optimal strategy involves aiming a smaller distance beyond the hole than for downhill putts, and leaves a greater proportion of the putts short. For long putts, e.g., < 35 feet, the differences in the optimal aim distance and fraction of putts left short are relatively small.

Amateur player - optimal aim direction

Table 4.26 shows how the optimal angle to aim at varies as a function of the initial putting position for a amateur golfer. Parameter Set 2 was used to generate these results. As for the professional golfer, the optimal aim direction (exp angle) is compared with the maximum (max angle) and the minimum (min angle) possible angles that would lead to a holeout, as well as the angle corresponding to the strategy that aims 1.5 feet beyond the hole (1.5 ft). Under the optimal (expected-putt minimizing) strategy, the player is aggressive and aims straighter for short putts. However, as putt length increases, the player becomes more conservative and allows for more curvature in the trajectory. As compared with the 1.5-foot strategy, the optimal strategy is more conservative and allows for more curvature.

Amateur player - comparison with probability optimization

Figure 4.15 illustrates several aspects of the one-putt probability maximizing strategy for the amateur player (parameter set 2 was used to generate these results). Figure (a) compares one-putt probability deviations from the average probability for putt-lengths 3 feet, 25 feet and 50 feet, as a function of the putt-angle (the one-putt probability maximizing strategy was used). We observe that downhill putts lead to higher optimal one-putt probability than uphill putts, and that sidehill and sideup putts are among the hardest for the amateur player. Figure (b) shows that the distance to target beyond the hole to maximize the one-putt probability increases with putt-length for most putt-angles, and thus differs significantly from the expected-putts minimization strategy discussed above wherein the optimal distance beyond the hole to target decreases for long putts (see Figure 4.5). Similarly, Figure (c) shows that a lot less putts are left short with the one-putt probability maximizing strategy as compared to the expected-putts minimizing strategy. Finally, Figure (d) shows that one-putt probability maximization strategy is much flatter (more aggressive) than the 1.5 feet beyond the hole strategy or the expected putts minimization strategy (see Figure 4.6 for

Table 4.26: Amateur golfer: strategy comparison

angle	3 foot				15 foot				25 foot				50 foot			
with x-axis	max angle	min angle	exp angle	1.5ft angle	max angle	min angle	exp angle	1.5ft angle	max angle	min angle	exp angle	1.5ft angle	max angle	min angle	exp angle	1.5ft angle
75	8.7	0.3	2.9	2.5	5.4	1.9	4.7	4.2	5.6	2.6	5.2	4.5	5.5	3.4	5.2	4.9
60	12.5	1.0	5.1	4.5	10.2	3.6	8.6	7.6	10.1	4.8	9.3	8.3	9.5	6.2	9.6	8.9
45	14.2	1.2	6.5	5.9	13.3	5.0	11.1	9.9	12.8	6.4	12.1	10.7	12.5	8.3	12.4	11.5
30	15.6	1.6	7.0	6.5	14.5	5.7	12.1	10.9	14.1	7.4	13.4	11.8	13.9	9.3	13.7	12.7
15	15.3	1.8	7.0	6.4	14.0	6.0	12.2	10.8	13.7	7.5	13.4	11.7	13.7	9.4	13.7	12.6
0	13.9	1.7	6.7	5.8	13.2	5.8	11.0	9.9	13.1	7.2	12.2	10.7	12.9	8.8	12.7	11.6
-15	13.1	1.6	5.9	5.0	11.6	5.3	9.7	8.4	11.2	6.4	10.6	9.2	11.1	7.9	10.9	9.8
-30	11.6	1.4	5.0	4.0	9.4	4.5	7.7	6.8	9.1	5.3	8.5	7.4	9.1	6.4	8.9	8.0
-45	8.8	1.1	4.0	3.0	7.1	3.4	5.8	5.0	7.0	4.1	6.5	5.5	6.8	5.0	6.7	6.0
-60	6.5	0.8	2.6	2.0	5.0	2.3	3.8	3.3	4.8	2.7	4.2	3.6	4.5	3.3	4.4	3.9
-75	3.8	0.1	1.4	1.0	2.7	1.1	1.9	1.7	2.3	1.4	2.1	1.8	2.3	1.6	2.2	2.0

This table shows how the optimal angle to aim at varies as a function of the initial putting position (angle with respect to the x-axis) and distance for the amateur golfer. The optimal aim direction (exp) is compared with the maximum (max angle) and the minimum (min angle) possible angles that would lead to a holeout, as well as the angle corresponding to the strategy that aims 1.5 feet beyond the hole (1.5 ft). For short putts, the golfer is aggressive (and aims straighter), while as putt length increases, the golfer becomes more conservative. While the optimal strategy and the aim 1.5-foot strategy are similar for short putts, as putt length increases, the optimal strategy becomes more conservative than the aim 1.5 feet beyond the hole strategy, especially for sidehill putts. The green slope is 1.5° along the y-axis, and green speed is 9 feet ($\eta = 0.0623$). Parameter Set 2 for the amateur player is used to generate these results.

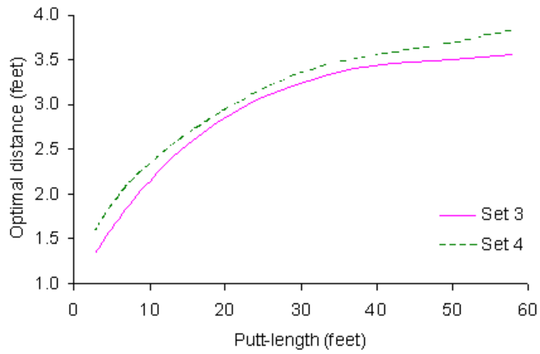
comparison).

Amateur player - comparison of calibrated parameter sets

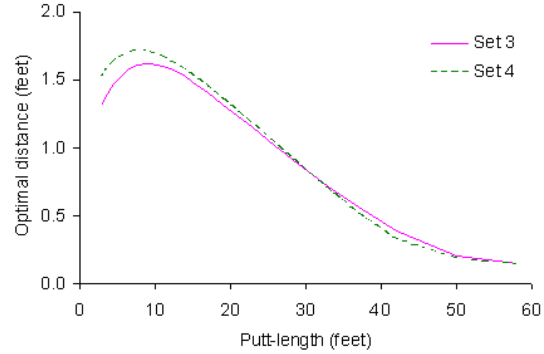
Figure 4.16 compares two of the calibrated parameter sets for the Amateur player: Parameter Set 2 and Parameter Set 4. While both sets calibrate to both one-putt probabilities and expected putts, Parameter Set 4 is characterized by higher angle error and lower green error compared to Parameter Set 2. Figures (a) and (b), respectively, show how the average distance beyond the hole to target varies as a function of putt-length when one-putt probability maximization and expected putts minimization strategies are employed. The amateur player with the higher green error, i.e., Parameter Set 2, aims further beyond the hole for maximizing one-putt probability. For expected putts, while this player aims further for short putt-lengths, for longer putts, the optimal distance to target is essentially the same for the two players. Figure (c) shows how the optimal one-putt probabilities vary as a function of the putt-angle for these two players for a 25 foot putt. While the player with higher angle error has lower one-putt probabilities for uphill and sidehill putts, this player with less green error has higher one-putt probabilities for straight and close-to-straight downhill putts. This is consistent with the observations in Section 4.2.5. Figure (d) shows the optimal distance beyond the hole to target (for maximizing one-putt probability) for these two players for a 25 foot putt. While the aim distances are hard to distinguish for uphill and sidehill putts, for downhill putts, the player with higher green error, i.e., Parameter Set 2, aims further beyond the hole. Figure (e) shows how the minimum expected putts vary as a function of the putt-angle for these two players for a 3 foot putt. The results are similar to those observed in Figure (c), in that, Parameter Set 4 with lower green error, performs better for downhill and close-to-downhill putts. Figure (f) shows the optimal distance beyond the hole to target (for minimizing expected putts) for these two players for a 3 foot putt. As with Figure (d), we observe that the player with higher green error, i.e., Parameter Set 2, aims further beyond the hole. As suggested by Figures (a) and (b), while this difference in strategy for maximizing one-putt probability for downhill putts can be observed for short as well as long putt-lengths (e.g., 3 foot, 25 foot), the difference in strategy for minimizing expected number of putts for downhill putts diminishes as putt length increases. These results are consistent with the results for the professional player examined above in Figure 4.14.

Amateur player - sidehill putts

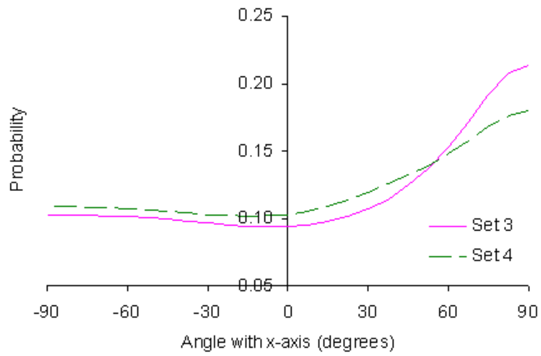
Figure 4.17 shows, for 5-foot and 25-foot sidehill putts, the target velocities and angles corresponding to the expected putts minimization (Min exp), one-putt probability maximization (Max prob), and aiming 1.5 feet beyond the hole (1.5 feet) strategies for Parameter Set 2 of the amateur player. The holeout region and the 1.5 feet, 4 feet and 7 feet beyond the hole contours, as well as the contours corresponding to leaving 10% and 50% of the putts short are also shown (assuming zero green error). We observe that the optimal expected-putts minimization strategy is more conservative than the one-putt probability-maximization and the 1.5-foot strategy, and as expected, lies in the holeout region.



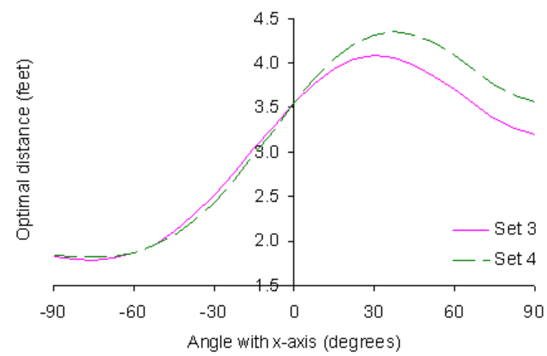
(a) Probability optimization - optimal distance versus putt-length



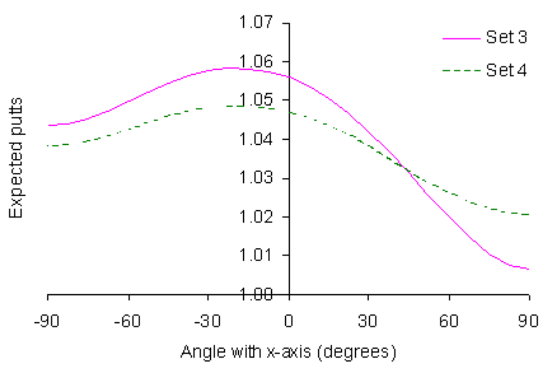
(b) Expected putts optimization - optimal distance versus putt-length



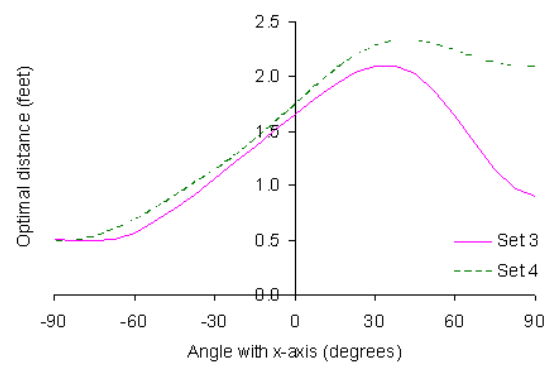
(c) Probability optimization, 25 foot putt - Optimal probability



(d) Probability optimization, 25 foot putt - Optimal distance

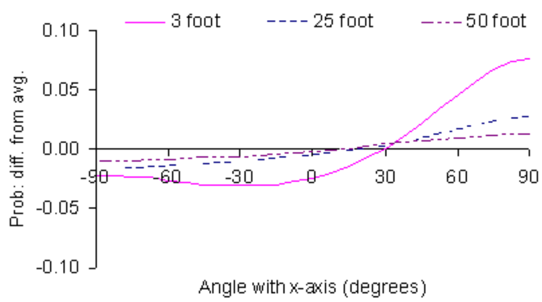


(e) Expected putts optimization, 3 foot putt - Optimal expected putts

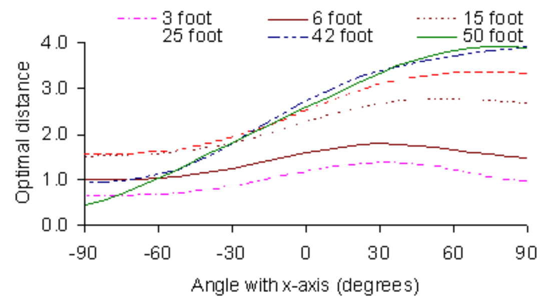


(f) Expected putts optimization, 3 foot putt - Optimal distance

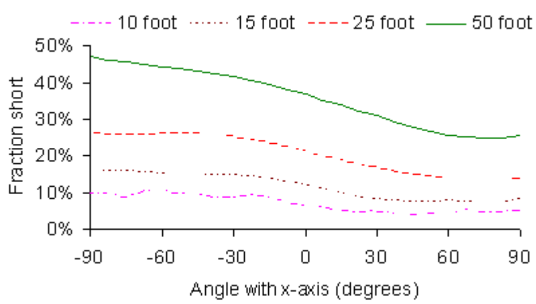
Figure 4.14: This figure compares two calibrated parameter sets for the Professional player: Parameter Set 3 and Parameter Set 4. Parameter Set 3 is characterized by higher angle error and lower green error compared to Parameter Set 4. Figures (a) and (b), respectively, show how the average distance beyond the hole to target varies as a function of putt-length when one-putt probability maximization and expected putts minimization strategies are employed. Figures (c) and (d), respectively, show how the optimal one-putt probabilities and distance beyond the hole to target vary as a function of the putt-angle for these two players for a 25 foot putt. Figures (e) and (f), respectively, show how the minimum expected putts and optimal distance beyond the hole to target vary as a function of the putt-angle for these two players for a 3 foot putt.



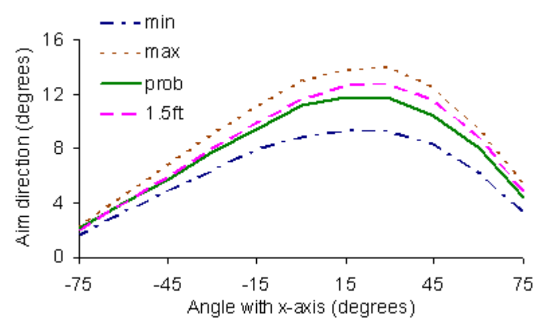
(a) One-putt probability



(b) Optimal distance

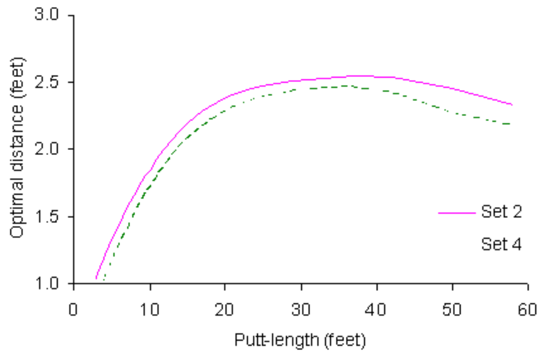


(c) Fraction short

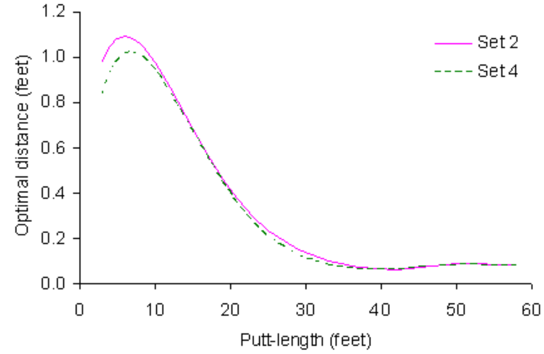


(d) Angle to aim - 50 foot putt

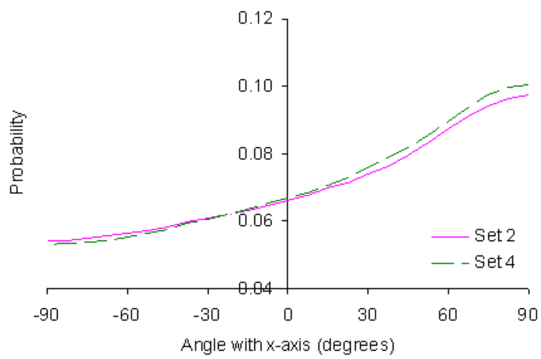
Figure 4.15: This figure illustrates aspects of the one-putt probability maximizing strategy for the amateur player. Figure (a) compares one-putt probability deviations from the average probability as a function of the putt-angle (the one-putt probability maximizing strategy was used) Figures (b) and (c), respectively, show that the optimal distance to target beyond the hole (in feet), and the fraction of putts left short, to maximize the one-putt probability for different putt-lengths. Graph (d) compares the one-putt probability maximization strategy with the 1.5 feet beyond the hole strategy. Parameter Set 2 for the amateur player was used to generate these results.



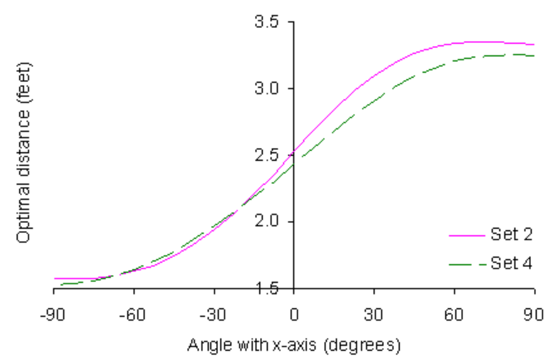
(a) Probability optimization - optimal distance versus putt-length



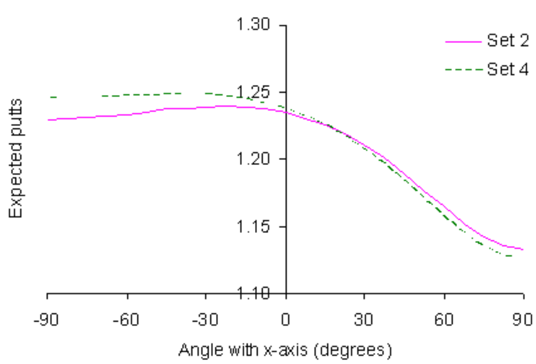
(b) Expected putts optimization - optimal distance versus putt-length



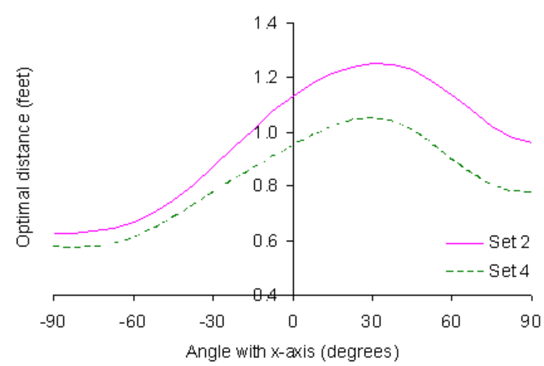
(c) Probability optimization, 25 foot putt - Optimal probability



(d) Probability optimization, 25 foot putt - Optimal distance

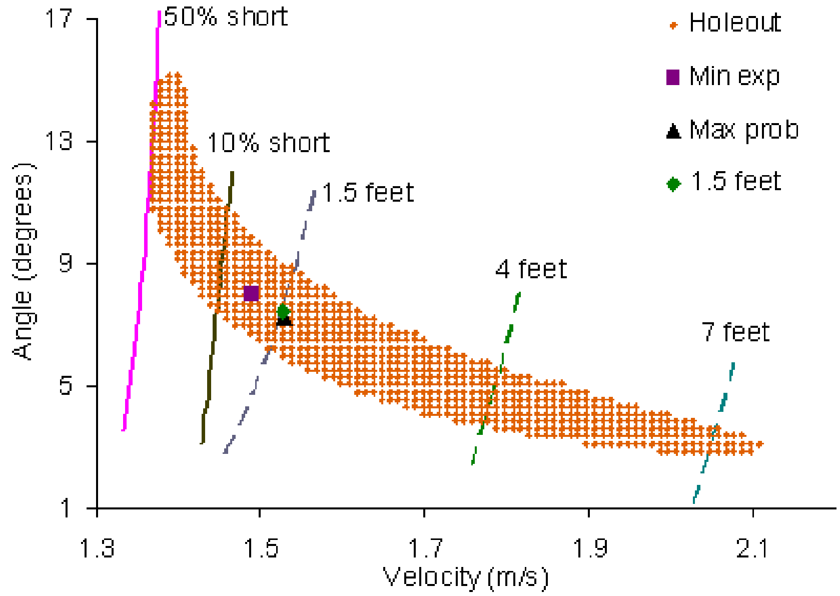


(e) Expected putts optimization, 3 foot putt - Optimal expected putts

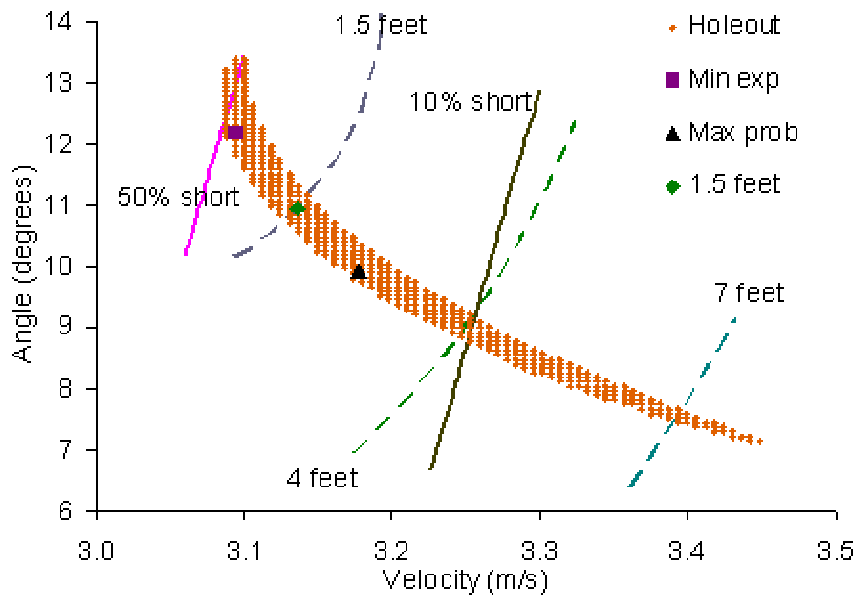


(f) Expected putts optimization, 3 foot putt - Optimal distance

Figure 4.16: This figure compares two calibrated parameter sets for the Amateur player: Parameter Set 2 and Parameter Set 4. Parameter Set 4 is characterized by higher angle error and lower green error compared to Parameter Set 2. Figures (a) and (b), respectively, show how the average distance beyond the hole to target varies as a function of putt-length when one-putt probability maximization and expected putts minimization strategies are employed. Figures (c) and (d), respectively, show how the optimal one-putt probabilities and distance beyond the hole to target vary as a function of the putt-angle for these two players for a 25 foot putt. Figures (e) and (f), respectively, show how the minimum expected putts and optimal distance beyond the hole to target vary as a function of the putt-angle for these two players for a 3 foot putt.



(a) 5-foot sidehill



(b) 25-foot sidehill

Figure 4.17: This figure shows the holeout region the target velocities and angles corresponding to the expected putts minimization (Min exp), one-putt probability maximization (Max prob), and aiming 1.5 feet beyond the hole (1.5 feet) strategies for a 5-foot and 25-foot sidehill putt on a green with slope 1.5°, and green speed 9 feet. The holeout region and the 1.5 feet, 4 feet and 7 feet beyond the hole contours are shown assuming zero green error. Parameter Set 2 for the amateur player was used to generate these results.

Chapter 5

Conclusion

We studied three problems in this thesis. We first considered the dynamic pricing problem of a monopolist selling a homogeneous good to a market of risk-averse, strategic customers with heterogeneous valuations. Using a discrete model of customer valuations, we showed how the dynamic pricing problem can be reformulated as a mechanism design problem, when fill-rate are interpreted as product quality. The mechanism design formulation was analyzed to characterize the structure of the optimal solution. The general problem remained hard to solve, and two special cases, the case of risk-neutral customers, and the case where at most two products are offered to the market were solved. Using asymptotic analysis for the low risk-aversion case, it was shown that it is asymptotically optimal to offer at most two products, the prices and fill-rates for which are easily determined. Numerical results showed that the two-product revenue approached the optimal revenue as customer risk-aversion approached the risk-neutral case. The mechanism design formulation for the dynamic pricing problem with strategic customers is general, and can also be extended to include, for example, myopic customers, heterogeneous outside opportunities, and time-sensitivity.

We next studied the product design problem of a monopolist in a market where customers have threshold preferences. We characterized the structure of the optimal solution, and discussed several settings where such preferences seem to be appealing: a delay service rendered via a stylized M/M/1 queue, an ISP offering download bandwidth, a retailer pricing a homogeneous good dynamically while varying its availability via rationing, a seller of software goods that uses versioning, a seller in a market of time-sensitive customers, a seller of a mp3 players facing increasing marginal cost to quality, and a postal service provider with several dedicated resources to serve customer requests. The proposed model of customer choice behavior is easily extended to several other settings, including products with multiple quality attributes, and the case of a duopoly. In addition to being intuitive and realistic for several settings, the threshold model of customer preferences often leads to formulations that are more tractable as compared to the classical model of customer choice behavior, and is therefore appealing.

Finally, we considered the problem of identifying the optimal putting strategy for a golfer. We developed a model of golfer putting skill that combines golfer's physical putting ability with golfer's ability to estimate putting green slopes. The golfer's problem of minimizing the expected number of putts to holeout was shown to be equivalent to a two-dimensional stochastic shortest path problem, and was solved using approximate dynamic programming. We calibrated the model using data for professional and amateur golfers from tournaments and regular play, and used the calibrated model to characterize the optimal putting strategy for professional and amateur golfers. For long putts, e.g., > 10 feet, we found that the optimal distance beyond the hole to aim at decreased as the putt-length increased. Golfers became more conservative as putt-lengths increased, in that the optimal aim angle became closer to the maximum angle that would lead to a holeout. The optimal strategy for professional golfers involved putting with a greater velocity, and with less break, as compared to amateur golfers. Sidehill and sideup putts were found to be among the hardest for both professional and amateur golfers. While we considered only professional and amateur players, the proposed model can also be used to identify the putting strategies for golfers with other skill levels, and under various settings.

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