# Forbidden Substructures in Graphs and Trigraphs, and Related Coloring Problems

**Irena Penev**

Submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2012

# Abstract

## Forbidden Substructures in Graphs and Trigraphs, and Related Coloring Problems

**Irena Penev**

Given a graph $G$, $\chi(G)$ denotes the chromatic number of $G$, and $\omega(G)$ denotes the clique number of $G$ (i.e. the maximum number of pairwise adjacent vertices in $G$). A graph $G$ is *perfect* provided that for every induced subgraph $H$ of $G$, $\chi(H) = \omega(H)$. This thesis addresses several problems from the theory of perfect graphs and generalizations of perfect graphs.

The *bull* is a five-vertex graph consisting of a triangle and two vertex-disjoint pendant edges; a graph is said to be *bull-free* provided that no induced subgraph of it is a bull. The first result of this thesis is a structure theorem for bull-free perfect graphs. This is joint work with Chudnovsky, and it first appeared in [12].

The second result of this thesis is a decomposition theorem for bull-free perfect graphs, which we then use to give a polynomial time combinatorial coloring algorithm for bull-free perfect graphs. We remark that de Figueiredo and Maffray [33] previously solved this same problem, however, the algorithm presented in this thesis is faster than the algorithm from [33]. We note that a decomposition theorem that is very similar (but slightly weaker) than the one from this thesis was originally proven in [52], however, the proof in this thesis is significantly different from the one in [52]. The algorithm from this thesis is very similar to the one from [52].

A class $\mathcal{G}$ of graphs is said to be $\chi$-*bounded* provided that there exists a function $f$ such that for all $G \in \mathcal{G}$, and all induced subgraphs $H$ of $G$, we have that $\chi(H) \leq f(\omega(H))$. $\chi$-bounded classes were introduced by Gyárfás [41] as a generalization of the class of perfect graphs (clearly, the class of perfect graphs is $\chi$-bounded by the identity function). Given a graph $H$, we denote by $\text{Forb}^*(H)$ the class of all graphs that do not contain any subdivision of $H$ as an induced subgraph. In [57], Scott proved that $\text{Forb}^*(T)$ is $\chi$-bounded for every tree $T$, and he conjectured that $\text{Forb}^*(H)$ is $\chi$-bounded for every graph $H$. Recently, a group of authors constructed a counterexample to Scott's conjecture [51]. This raises the following question: for which graphs $H$ is Scott's conjecture true? In this thesis, we present the proof of Scott's conjecture for the cases when $H$ is the *paw* (i.e. a four-vertex graph consisting of a triangle and a pendant edge), the bull, and a *necklace* (i.e. a graph obtained from a path by choosing a matching such that no edge of the matching is incident with an endpoint of the path, and for each edge of the matching, adding a vertex adjacent to the ends of this edge). This is joint work with Chudnovsky, Scott, and Trotignon, and it originally appeared in [13].

Finally, we consider several operations (namely, "substitution," "gluing along a clique," and "gluing along a bounded number of vertices"), and we show that the closure of a $\chi$-bounded class under any one of them, as well as under certain combinations of these three operations (in particular, the combination of substitution and gluing along a clique, as well as the combination of gluing along a clique and gluing along a bounded number of vertices) is again $\chi$-bounded. This is joint work with Chudnovsky, Scott, and Trotignon, and it originally appeared in [14].

# Contents


**1 Introduction** **1**

1.1 Bull-Free Perfect Graphs . . . . . . . . . . . . . . . . . . . . . . . . . . . 2

1.2 $\chi$-Bounded Classes . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9

1.3 Outline . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 12

**2 Definitions: Graphs and Trigraphs** **15**

2.1 Definitions: Graphs . . . . . . . . . . . . . . . . . . . . . . . . . . . . 15

2.2 Definitions: Trigraphs . . . . . . . . . . . . . . . . . . . . . . . . . . . 24

**3 The Structure of Bull-Free Perfect Graphs** **34**

3.1 Structure Theorem for Bull-Free Berge Trigraphs . . . . . . . . . . . . . . . 35

3.2 Good Homogeneous Pairs . . . . . . . . . . . . . . . . . . . . . . . . . . 37

3.3 Elementary Expansions . . . . . . . . . . . . . . . . . . . . . . . . . . . 39

3.4 Class $\mathcal{T}_1$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 44

3.5 Class $\mathcal{T}_2$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 67

3.6 The Main Theorem . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 75

**4 A Decomposition Theorem for Bull-Free Perfect Graphs** **77**

4.1 Transitively Orientable Trigraphs . . . . . . . . . . . . . . . . . . . . . . 80

4.2 Some Lemmas from [7] and [8] . . . . . . . . . . . . . . . . . . . . . . . 87

4.3 A "Preliminary" Decomposition Theorem for Bull-Free Berge Trigraphs . . 89

4.4 Homogeneous Pairs and the Proof of the Main Decomposition Theorem for Bull-Free Berge Trigraphs . . . 103

**5 Coloring Bull-Free Perfect Graphs 119**

5.1 A Decomposition Theorem for Bull-Free Perfect Graphs and an Outline of the Chapter . . . 122

5.2 Reducible Homogeneous Pairs . . . 123

5.3 Reducing Homogeneous Sets and Reducible Homogeneous Pairs . . . 134

5.4 The Algorithm . . . 142

5.5 Complexity Analysis . . . 144

**6 Excluding Induced Subdivisions of the Bull and Related Graphs 148**

6.1 Subdivisions of the Paw . . . 150

6.2 Decomposing Bull*-Free Graphs . . . 152

6.3 A $\chi$-Bounding Function for Forb*(bull) . . . 161

6.4 Necklaces . . . 164

**7 Substitution and $\chi$-Boundedness 177**

7.1 Substitution . . . 180

7.1.1 Substitution Depth and $\chi$-Boundedness . . . 181

7.1.2 Polynomial $\chi$-Bounding Functions . . . 183

7.1.3 Faster Growing $\chi$-Bounding Functions . . . 192

7.2 Small Cutsets, Substitution, and Cliques . . . 194

7.2.1 Gluing Operations . . . 194

7.2.2 Substitution and Gluing along a Clique . . . 201

7.3 Open Questions . . . 206

**Bibliography 208**

# Acknowledgments

First and foremost, I would like to thank my advisor, Maria Chudnovsky, for everything that she has taught me, for all the ideas that she has shared with me, and for her support and nearly inexhaustible patience. I would also like to thank Alex Scott and Nicolas Trotignon with whom I have had the privilege and pleasure of working, and from whom I have learned a lot; some of our joint work appears in this thesis. Furthermore, I would like to thank András Sebö for his help with the complexity analysis from chapter 5 of this thesis. I am grateful to my fellow graduate students from the Math Department for all their help, especially during my first year of graduate school; a special thanks goes to Ina Petkova for convincing me that graph theory would be a fun topic to study. I am also grateful to George Weaver who first introduced me to research when I was an undergraduate at Bryn Mawr College. Finally, I would like to thank the Math Department staff, especially Terrance Cope and Mary Young.

# Chapter 1

# Introduction

Unless specified otherwise, all graphs in this thesis are finite and simple. The vertex-set and the edge-set of a graph $G$ are denoted by $V_G$ and $E_G$, respectively. A *proper coloring* of a graph $G$ is an assignment of colors to the vertices of $G$ (one color for each vertex) in such a way that whenever two vertices are adjacent, they receive a different color. (When clear from the context, we often say simply "coloring" instead of "proper coloring."). The *chromatic number* of a graph $G$, denoted by $\chi(G)$, is the smallest number of colors needed to properly color $G$. A *stable set* in a graph $G$ is any set of pairwise non-adjacent vertices of $G$. Since in a coloring of $G$, adjacent vertices cannot receive the same color, one can think of a coloring of $G$ simply as a partition of $V_G$ into stable sets (sometimes called *color classes*); clearly, the chromatic number of $\chi(G)$ is the smallest number of stable sets needed to partition $V_G$.

A *clique* in a graph $G$ is any set of pairwise adjacent vertices of $G$, and a *complete graph* on $n$ vertices, denoted by $K_n$, is a graph whose vertex-set is a clique consisting of $n$ vertices. The *clique number* of $G$, denoted by $\omega(G)$, is the maximum number of vertices in a clique in $G$. Since adjacent vertices of $G$ must receive distinct colors in any coloring of $G$, it is clear that $\omega(G) \leq \chi(G)$. This raises a natural question: is it possible to obtain an upper bound for $\chi(G)$ in terms of $\omega(G)$? In general, the answer is "no." A *triangle* in $G$ is a three-vertex clique, and a graph is *triangle-free* provided it contains no triangles; thus, triangle-free graphs are precisely those graphs $G$ that satisfy $\omega(G) \leq 2$. Mycielski [49] proved the following famous (and perhaps surprising) theorem.

**1.0.1** (Mycielski [49]). *There exist triangle-free graphs of arbitrarily large chromatic number.*

We note that Mycielski's proof of 1.0.1 consists in giving a recursive (and fairly simple) construction of a sequence of triangle-free graphs with strictly increasing chromatic numbers. We also note that 1.0.1 is an immediate consequence of a stronger result due to Erdős and Hajnal [31], which states that there exist graphs of arbitrarily large girth and chromatic number. However, for some special classes of graphs, it is indeed possible to obtain an upper bound for the chromatic number in terms of the clique number, and these graphs are the subject of this thesis. In particular, we are interested in "perfect graphs" (see section 1.1), and in "$\chi$-bounded classes" (see section 1.2).

## 1.1 Bull-Free Perfect Graphs

Given that the clique number is a trivial lower bound for the chromatic number, one might ask whether there is anything special about graphs whose chromatic number is equal to their clique number. The answer here is "not really." Indeed, consider any graph $G$, and form a graph $G'$ by taking the disjoint union of $G$ and the complete graph on $\chi(G)$ vertices; by construction, $\chi(G') = \omega(G')$, and yet, apart from the fact that $G'$ contains a "large" clique, we have little insight into the structure of $G'$ (and we have no insight whatsoever into the structure of $G$). However, if we restrict our attention to a smaller class of graphs, more interesting questions arise. Given graphs $H$ and $G$, we say that $H$ is an *induced subgraph* of $G$ provided that $V_H \subseteq V_G$, and that for all distinct $u, v \in V_H$, $u$ and $v$ are adjacent in $H$ if and only if $u$ and $v$ are adjacent in $G$. A graph $G$ is said to be *perfect* provided that for every induced subgraph $H$ of $G$, $\chi(H) = \omega(H)$; a graph is said to be *imperfect* if it is not perfect. We remark that graphs from many important classes are perfect; among others, all of the following are perfect: bipartite graphs, line graphs of bipartite graphs, transitively orientable graphs (also known as comparability graphs), chordal graphs, split graphs, interval graphs (see, for instance, [37] or chapters 65 and 66 of [55]).

Perfect graphs were introduced by Berge [2] in the 1960s; at this time, he also made a couple of famous conjectures concerning perfect graphs. The complement of a graph $G$, denoted by $\overline{G}$, is the graph whose vertex-set is $V_G$, and that satisfies that property that for all distinct $u, v \in V_G$, $u$ and $v$ are adjacent in $\overline{G}$ if and only if $u$ and $v$ are non-adjacent in $G$. Berge's first conjecture, known as the Weak Perfect Graph Conjecture, states that a graph is perfect if and only if its complement is perfect. Berge's second conjecture, known as the Strong Perfect Graph Conjecture, states that a graph $G$ is perfect provided that neither $G$ nor $\overline{G}$ contains an odd cycle of length at least five as an induced subgraph; graphs $G$ that satisfy the property that neither $G$ nor $\overline{G}$ contains an induced odd cycle of length at least five are now known as *Berge*, and so the Strong Perfect Graph Conjecture states that a graph is perfect if and only if it is Berge. Both of these conjectures have been proven by now. The Weak Perfect Graph Conjecture was proven by Lovász [48] in the 1970's (see also a simpler – and more recent – proof due to Gasparyan [36]), and the Strong Perfect Graph Conjecture was proven only a few years ago by Chudnovsky, Robertson, Seymour, and Thomas [16]. We state both of these theorems below for future reference.

**1.1.1** (Lovász [48]). *A graph is perfect if and only if its complement is perfect.*

**1.1.2** (Chudnovsky, Robertson, Seymour, and Thomas [16]). *A graph $G$ is perfect if and only if it is Berge.*

1.1.1 is now known as the Weak Perfect Graph Theorem, and 1.1.2 is known as the Strong Perfect Graph Theorem. The proof of Berge's Strong Perfect Graph Conjecture settled the most important question from the theory of perfect graphs, and yet, a number of natural questions (some raised by this proof itself) remain, and some of these questions are the subject of this thesis.

The proof of the Weak Perfect Graph Theorem 1.1.1 (especially the proof due to Gasparyan [36]) is short and relatively simple, and one direction of the proof of the Strong Perfect Graph Theorem 1.1.2 ("if a graph is perfect, then it is Berge") is an easy exercise. However, the proof of the other direction of 1.1.2 ("if a graph is Berge, then it is perfect") is well over a hundred pages long and higly complicated; the outline of this "hard" direction is as follows. Let us say that a graph $G$ is *basic* if at least one of $G$ and $\overline{G}$ is either a bipartite graph, or the line graph of a bipartite graph, or a double split graph (the first two classes are well-known classes of graphs and can be found in any text on graph theory, and we refer the reader to [16] for the definition of a double split graph). It was shown in [16] that if $G$ is a Berge graph, then either $G$ is basic, or one of $G$ and $\overline{G}$ admits a proper 2-join, or $G$ admits a proper homogeneous pair, or $G$ admits a balanced skew partition. (We refer the reader to [16] for the definitions of these decompositions.) In addition, Chudnovksy [6] proved that homogeneous pairs are in fact unnecessary, which proves the following decomposition theorem.

**1.1.3** (Chudnovsky [6])**.** *Let $G$ be a Berge graph. Then either $G$ is basic, or one of $G$ and $\overline{G}$ admits a 2-join, or $G$ admits a balanced skew partition.*

Using elementary methods and classical results, one can easily show that all graphs in the basic classes are perfect, and one can also show that the minimum counterexample to the Strong Perfect Graph Conjecture cannot admit any of the decompositions from 1.1.3. (A "minimum counterexample" to the Strong Perfect Graph Conjecture is an imperfect Berge graph $G$ that satisfies the property that every Berge graph that has fewer than $|V_G|$ vertices is perfect.) Thus, 1.1.3 suffices to prove 1.1.2. In another sense, however, this decomposition theorem is unsatisfying. One might like to prove a theorem (a "structure theorem") of the following form: every Berge graph either belongs to one of several well-understood basic classes of Berge graphs, or it can be "built" from smaller Berge graphs by a sequence of operations that preserve the property of being Berge. The decomposition theorem 1.1.3 falls short here: while the 2-join decomposition can be turned into an operation that preserves the property of being Berge, the balanced skew partition is a true decomposition in the sense that it decomposes Berge graphs into smaller Berge graphs, but it cannot be turned into an operation that builds larger graphs from smaller ones, while preserving the property of being Berge. Thus, finding a structure theorem for Berge graphs remains an open question. In the general case, not much progress has been made in this direction, however, results have been obtained in some special cases. In this thesis in particular, we have studied "bull-free" Berge (equivalently: perfect) graphs (more on this below).

We remark here that in order to obtain 1.1.3, Chudnovsky [6] introduced objects called "trigraphs," which are a generalization of graphs. Trigraphs are introduced formally in chapter 2, and they are used in chapters 3 and 4, but let us give an informal introduction here. A *homogeneous set* in a graph $G$ is a non-empty set $S \subseteq V_G$ such that for every vertex $u \in V_G \smallsetminus S$, either $u$ is adjacent to every vertex in $S$, or $u$ is non-adjacent to every vertex in $S$. Thus, globally, a homogeneous set can be thought of as behaving like a vertex (indeed, in various applications, it is often convenient to "contract" a homogeneous set to a vertex in order to obtain a smaller graph). A *homogeneous pair* in a graph $G$ is a pair $(A, B)$ of non-empty, disjoint subsets of $V_G$ such that for every vertex $u \in V_G \smallsetminus (A \cup B)$, both for the following hold:

- either $u$ is adjacent to every vertex in $A$, or $u$ is non-adjacent to every vertex in $A$;
- either $u$ is adjacent to every vertex in $B$, or $u$ is non-adjacent to every vertex in $B$.

Thus, just as a homogeneous set can be thought of as behaving like a vertex, a homogeneous pair $(A, B)$ can be thought of as behaving like a pair of vertices. However, if there are both edges and non-edges between $A$ and $B$ in $G$, it is unclear whether this pair of vertices should be an edge or a non-edge. This motivates the definition of a trigraph. Informally, a trigraph is a graph in which some pairs of vertices (which we call "semi-adjacent pairs") are neither edges nor non-edges, but are instead left "undecided." Thus, just as a homogeneous set can conveniently be "contracted" to a vertex in certain situations, a homogeneous pair can conveniently be "contracted" to a semi-adjacent pair. We observe that every graph can be thought of as a trigraph in a natural way: a graph is simply a trigraph with no semi-adjacent pairs. While we do not define such a thing as a "perfect trigraph" (this is because there is no convenient way to define a "trigraph coloring"), there is a natural way to define a "Berge trigraph." Berge trigraphs were first introduced in [6] (indeed, 1.1.3 is a consequence of an analogous theorem for trigraphs), and we give a formal definition in chapter 2.

The algorithmic aspects of perfect graphs have also received a good deal of attention. One important result has been the recognition algorithm for Berge graphs due to Chudnovsky, Cornuéjols, Liu, Seymour, and Vušković [11]; by the Strong Perfect Graph Theorem 1.1.2, this is in fact a recognition algorithm for perfect graphs. Another important result has been the polynomial time coloring algorithm for perfect graphs due to Grötschel, Lovász, and Schrijver [39]. However, this coloring algorithm is based on the ellipsoid method, and it remains an open problem to find a polynomial time combinatorial coloring algorithm for perfect graphs. So far, research in this direction has focused on special cases of perfect graphs, and one class that has received attention is the class of "bull-free" perfect graphs, to which we now turn.

The *bull* is the graph with vertex-set $\{x_1, x_2, x_3, y_1, y_2\}$ and edge-set $\{x_1x_2, x_2x_3, x_3x_1, x_1y_1, x_2y_2\}$. A graph is said to be *bull-free* provided that it does not contain a bull as an induced subgraph. Bull-free graphs were originally studied in the context of perfect graphs. Many years before the Strong Perfect Graph Conjecture was proven [16], Chvátal and Sbihi [24] showed that bull-free Berge graphs are perfect. Similarly, Reed and Sbihi [54] gave a polynomial time recognition algorithm for bull-free perfect graphs long before the recognition algorithm for perfect graphs [11] was obtained. Furthermore, the class of bull-free perfect graphs is one of the subclasses of the class of perfect graphs for which a combinatorial polynomial time coloring algorithm has been constructed [33] (more on this below). Recently, in a series of papers [7, 8, 9, 10], Chudnovsky gave a structure theorem for bull-free graphs; in fact, the structure theorem from [7, 8, 9, 10] is a structure theorem for bull-free trigraphs, but since every graph can be thought of as a trigraph in a natural way, this is also a structure theorem for bull-free graphs. Together with Chudnovsky, we used the structure theorem from [7, 8, 9, 10] to obtain a structure theorem for bull-free Berge trigraphs [12]; since every graph can be thought of as a trigraph, and since Berge graphs are perfect (by the Strong Perfect Graph Theorem 1.1.2), this is also a structure theorem for bull-free perfect graphs. Chapter 3 of this thesis is closely based on [12].

We now return to the question of coloring bull-free perfect graphs. As mentioned above, de Figueiredo and Maffray [33] gave a combinatorial polynomial time coloring algorithm for bull-free perfect graphs. In fact, in [33], de Figueiredo and Maffray obtained stronger results and solved four optimization problems (described below) for the class of bull-free perfect graphs.

In this thesis, a *weighted graph* is a graph $G$ such that every vertex $v$ of $G$ is assigned a positive integer *weight*, denoted by $w_G(v)$. Given a set $S \subseteq V_G$, the *weight* of $S$, denoted by $w_G(S)$, is the sum of the weights of the vertices in $S$; the weight of the empty set is assumed to be zero. A *maximum weighted clique* (respectively: *maximum weighted stable set*) in $G$ is a clique (respectively: stable set) that has the maximum weight among all the cliques (respectively: stable sets) in $G$. We now describe the four optimization problems mentioned above. First, the *maximum weighted clique problem* (respectively: *maximum weighted stable set problem*) is the problem of finding a maximum weighted clique (respectively: maximum weighted stable set) in a weighted graph. Next, the *minimum weighted coloring problem* is the problem of finding stable sets $S_1, ..., S_t$ in a weighted graph $G$, and positive integers $\lambda_1, ..., \lambda_t$, such that $\Sigma_{S_i \ni v} \lambda_i \geq w_G(v)$ for all $v \in V_G$, and with the property that $\Sigma_{i=1}^t \lambda_i$ is minimum. Finally, the *minimum weighted clique covering problem* is the problem of finding cliques $C_1, ..., C_t$ in a weighted graph $G$, and positive integers $\lambda_1, ..., \lambda_t$, such that $\Sigma_{C_i \ni v} \lambda_i \geq w_G(v)$ for all $v \in V_G$, and with the property that $\Sigma_{i=1}^t \lambda_i$ is minimum. De Figueiredo and Maffray found combinatorial algorithms that solve these four optimization problems for the class of bull-free perfect graphs in polynomial time. In [52], we gave different (and faster) polynomial time combinatorial algorithms that solve these same four problems.

The algorithms from [52] are based on a decomposition theorem for bull-free Berge trigraphs that was derived from the structure theorem for bull-free Berge trigraphs from [12] (this is the structure theorem from chapter 3 of this thesis). However, the structure theorem from [12] is unnecessarily strong for the purposes of deriving the decomposition theorem from [52]. Accordingly, in chapter 4, we present a new proof of the decomposition theorem that appears in [52] (in fact, the decomposition theorem from chapter 4 is slightly stronger than the one from [52]). The proof of the decomposition theorem from chapter 4 is obtained by imitating the proof of a decomposition theorem for bull-free trigraphs from [8], only not for general bull-free trigraphs, but for bull-free Berge trigraphs; under the assumption that our trigraphs are Berge, we are able to obtain results that are a bit stronger than the ones from [8]. It is worth remarking that, while the structure theorem for bull-free Berge trigraphs from chapter 3 does not imply a structure theorem for bull-free Berge (equivalently: perfect) graphs that can be stated solely in terms of graphs (i.e. with no mention of trigraphs), the decomposition theorem for bull-free Berge trigraphs from chapter 4 immediately implies a decomposition theorem for bull-free Berge (equivalently: perfect) graphs that can be stated using only graph (rather than trigraph) terminology. This last theorem is then used in chapter 5 to give combinatorial polynomial time algorithms that solve the four optimization problems mentioned above. Chapter 5 is closely based on [52].

## 1.2 $\chi$-Bounded Classes

As remarked above (see 1.0.1), the chromatic number of a graph cannot, in general, be bounded above in terms of its clique number. However, for some classes of graphs, this is indeed possible; let us say that a class $\mathcal{G}$ of graphs is *$\chi$-bounded* provided that there exists a function $f$ such that for all graphs $G \in \mathcal{G}$, and for all induced subgraphs $H$ of $G$, we have that $\chi(H) \leq f(\omega(H))$; under these circumstances, we say that $G$ is *$\chi$-bounded by the function $f$*, and that $f$ is a *$\chi$-bounding function* for the class $\mathcal{G}$.

$\chi$-bounded classes were introduced by Gyárfás in [41] as a generalization of the class of perfect graphs. Indeed, the class of perfect graphs is (by definition) $\chi$-bounded by the identity function. We also observe that the Strong Perfect Graph Theorem 1.1.2 states that the class of Berge graphs is $\chi$-bounded by the identity function; it is interesting to note that the Strong Perfect Graph Theorem 1.1.2 is the only known proof of the fact that the class of Berge graphs is $\chi$-bounded (by any function). It is also worth stressing that while graphs can be perfect or imperfect, it makes no sense to talk about $\chi$-bounded graphs, only $\chi$-bounded classes, and the classes that are interesting in this context are those classes that contain infinitely many (isomorphism types of) graphs. Indeed, if $\mathcal{G}$ is a class of graphs containing only finitely many pairwise non-isomorphic graphs, then $\mathcal{G}$ is (trivially) $\chi$-bounded by the constant function $f(n) = \max(\{\chi(G) \mid G \in \mathcal{G}\} \cup \{0\})$.

The classes of graphs that have received the most attention in this context have been the classes defined by forbidding induced subgraphs (or families of induced subgraphs). Given a graph $H$, we denote by $\text{Forb}(H)$ the class of all graphs that do not contain (an isomorphic copy of) $H$ as an induced subgraph. For which graphs $H$ is the class $\text{Forb}(H)$ $\chi$-bounded? The *girth* of a graph $G$ that contains at least one cycle is the length of the shortest cycle in $G$; note that if the girth of $G$ is greater than three, then $\omega(G) = 2$. Clearly, if $H$ is a graph that contains a cycle of length $k$, then $\text{Forb}(H)$ contains all graphs whose girth is greater than $k$. Erdős and Hajnal [31] showed that there exist graphs of arbitrarily large girth and chromatic number; consequently, if $H$ is a graph that contains at least one cycle, then Forb($H$) is not $\chi$-bounded. The graphs $H$ that remain to be considered are the forests. Gyárfás [40] and Sumner [60] independently conjectured that for any tree $T$, the class Forb($T$) is $\chi$-bounded. (We remark that this conjecture for trees is in fact equivalent to the analogous - and seemingly stronger - conjecture for forests. This easily follows from the observation that every forest is an induced subgraph of some tree.) Gyárfás' and Sumner's conjecture has been proven for trees of radius 2 and a few trees of larger radius (see [41], [42], [44], [45], [57]); however, for the general case, the conjecture remains open.

Progress has, however, been made in a slightly different direction. Given a graph $H$, we denote by Forb$^*(H)$ the class of all graphs that do not contain any subdivision of $H$ as an induced subgraph. Scott [57] proved what can conveniently be thought of as the "topological" version of Gyárfás' and Sumner's conjecture; we state this result below.

**1.2.1** (Scott [57])**.** *For every tree $T$, the class* Forb$^*(T)$ *is $\chi$-bounded.*

We note that 1.2.1 immediately implies that for every forest $F$, Forb$^*(F)$ is $\chi$-bounded (this is because every forest is an induced subgraph of some tree). In view of 1.2.1, Scott conjectured in [57] that for every graph $H$, Forb$^*(H)$ is $\chi$-bounded. (We remark that Scott's conjecture generalizes a still-open conjecture of Gyárfás [41], that the class Forb$^*(C_n)$ is $\chi$-bounded for every $n$, where $C_n$ is the chordless cycle of length $n$; see also [56].) Recently, a group of authors constructed a counterexample to Scott's conjecture [51]. This raises the following question: for which graphs $H$ is Scott's conjecture true? Scott's conjecture has been proven for a number of special cases (see below), and in this thesis, we present the proof of this conjecture for several new graphs $H$.

The *paw* is the graph with vertex-set $\{x_1, x_2, x_3, y\}$ and edge-set $\{x_1x_2, x_2x_3, x_3x_1, x_1y\}$. In [13], we proved (together with Chudnovsky, Scott, and Trotignon) that the classes

Forb$^*$(paw) and Forb$^*$(bull) are $\chi$-bounded. In [13], we also considered a family of graphs called "necklaces," and showed that Scott's conjecute is true for every necklace. Chapter 6 of this thesis is based on [13]. (The definition of a necklace is relatively complicated, and so we omit it here and refer the reader to chapter 6. We remark though that the bull is a special kind of necklace.)

We remark that the fact that Forb$^*$(paw) is $\chi$-bounded, together with previously known results, implies that if $H$ is a graph on at most four vertices, then Forb$^*(H)$ is $\chi$-bounded. Indeed, if $H$ is a forest, then (as explained above) the result follows from 1.2.1. If $H$ is the complete graph on three vertices, then Forb$^*(H)$ is the class of all forests. If $H$ is the graph with vertex-set $\{x, y, z, w\}$ and edge-set $\{xy, yz, zx\}$, then any graph $G$ in Forb$^*(H)$ can be partitioned into a forest and a graph whose clique number is smaller than $\omega(G)$ (indeed, take any vertex $v$ of $G$, and note that the subgraph of $G$ induced $v$ and its non-neighbors is a forest, while the subgraph of $G$ induced by the neighbors of $v$ has clique number smaller than $\omega(G)$), and consequently, Forb$^*(H)$ is $\chi$-bounded by the function $f(n) = 2n$. If $H$ is the *diamond* (i.e. the graph obtained by deleting an edge from the complete graph on four vertices), then the result follows from a theorem of Trotignon and Vušković (see [61]). If $H$ is the complete graph on four vertices, Scott's conjecture follows from the work of several authors, see [47]. Finally, if $H$ is the *square* (i.e. the chordless cycle of length four), then Forb$^*(H)$ is the famous class of *chordal* graphs (see [53]).

As mentioned in the previous section, the Strong Perfect Graph Theorem 1.1.2 was proven using the following approach: it was shown that every Berge graph either belongs to a "basic" class, or it admits a "decomposition" that "cuts it up" into smaller Berge graphs. Various decompositions (some of them mentioned in the previous section) were studied in this context, among them: clique cutsets [35], substitutions [48], amalgams [4], 2-joins [27], homogeneous pairs [24], star cutsets and skew partitions [23]. We remark that some (though not all) of these decompositions are in fact operations that allow one to build bigger graphs from smaller ones, while preserving the property of being Berge; for instance, substitution is such an operation, and the clique cutset decomposition can easily be turned into such an operation, namely, the "gluing along a clique" operation (see chapter 2 for the definition of substitution, and chapter 7 for the definition of gluing along a clique).

These same decompositions and operations can also be studied in the context of $\chi$-bounded classes. In [14], together with Chudnovsky, Scott, and Trotignon, we considered the two operations (substitution and gluing along a clique) mentioned above, as well the operation of "gluing along a bounded number of vertices" (see chapter 7 for the definition of this operation) and we showed that the closure of a $\chi$-bounded class under any one of these three operations is again $\chi$-bounded. We also showed that the closure of a $\chi$-bounded class under the operations of substitution and gluing along a clique together is $\chi$-bounded, as is the closure of a $\chi$-bounded class under the operations of gluing along a clique and gluing along a bounded number of vertices together. Finally, we showed that if a class $\mathcal{G}$ is $\chi$-bounded by a polynomial (respectively: exponential) function, then the closure of $\mathcal{G}$ under substitution is $\chi$-bounded by (some other) polynomial (respectively: exponential) function. We remark that the fact that the closure of a $\chi$-bounded class under the operation of gluing along a bounded number of vertices is again $\chi$-bounded follows from an earlier (and more general) result due to a group of authors [1]. However, the proof presented in this thesis is significantly different, and furthermore, the $\chi$-bounding function that we obtained is better than the one that can be derived from [1]. Chapter 7 is closely based on [14].

## 1.3 Outline

In the previous two sections, we gave a (somewhat spread-out) outline of this thesis. In this section, we provide a brief summary, for the reader's convenience.

- In chapter 2, we give some basic definitions that will appear throughout this thesis. This chapter includes a formal definition of a trigraph as well as a few elementary results about trigraphs.

- In chapter 3, we prove a structure theorem for bull-free Berge trigraphs. This is joint work with Chudnovsky, and it is based on [12].

- In chapter 4, we prove a decomposition theorem for bull-free Berge trigraphs. We note that we proved a very similar (though slightly weaker) decomposition theorem in [52], however, the proof given in this thesis differs substantially from the one given in [52].

- In chapter 5, we state a decomposition theorem for bull-free perfect graphs (this theorem is an immediate consequence of the decomposition theorem for bull-free Berge trigraphs from chapter 4), and we use it in order to give combinatorial polynomial time algorithms that solve the following four optimization problems for the class of bull-free perfect graphs: the maximum weighted clique problem; the maximum weighted stable set problem; the minimum weighted coloring problem; and the minimum weighted clique covering problem. This chapter is based on [52].

- In chapter 6, we prove that the class of graphs obtained by excluding the induced subdivisions of any one graph from a certain family of graphs (namely, the paw, the bull, and any necklace) is $\chi$-bounded. This is joint work with Chudnovsky, Scott, and Trotignon, and it is based on [13].

- In chapter 7, we prove that the closure of a $\chi$-bounded class under any one of the following operations: substitution, gluing along a clique, and gluing along a bounded number of vertices, is again $\chi$-bounded. We also prove that the closure of a $\chi$-bounded class under the operations of substitution and gluing along a clique together is $\chi$-bounded, as is the closure of a $\chi$-bounded class under the operations of gluing along a clique and gluing along a bounded number of vertices together.

Finally, we show that if a class $\mathcal{G}$ is $\chi$-bounded by a polynomial (respectively: exponential) function, then the closure of $\mathcal{G}$ under substitution is $\chi$-bounded by (some other) polynomial (respectively: exponential) function. This is joint work with Chudnovsky, Scott, and Trotignon, and it is based on [14].

# Chapter 2

# Definitions: Graphs and Trigraphs

In this chapter, we give some basic definitions (and prove a few elementary results) concerning graphs and trigraphs. In section 2.1, we introduce some terminology and notation concerning graphs; this terminology and notation will be used throughout this thesis. Some of these definitions were already given in the Introduction, but we repeat them here for the reader's convenience. In section 2.2, we give the formal definition of a trigraph, and we define some basic concepts concerning trigraphs. In this section, we also prove a few elementary results about trigraphs. Trigraphs will appear in chapters 3 and 4.

Before moving on to graphs and trigraphs, we establish some non-graph theoretic notation that we will need. In this thesis, $\mathbb{N}$ is the set of all positive integers, $\mathbb{N}_0$ is the set of all non-negative integers, $\mathbb{Z}$ is the set of all integers, $\mathbb{Z}_k$ is the cyclic group of order $k$, and $\mathbb{R}$ is the set of all real numbers. Given a real number $r$, we denote by $\lfloor r \rfloor$ the largest integer that is no greater than $r$.

## 2.1 Definitions: Graphs

Unless specified otherwise, all graphs in this thesis are finite and simple. The vertex-set and the edge-set of a graph $G$ are denoted by $V_G$ and $E_G$, respectively. Graph isomorphism is defined in the usual way. A graph is said to be *trivial* if it contains exactly one vertex; a *non-trivial* graph is a graph that has at least two vertices. The *empty* graph, is the graph whose vertex-set is empty; a graph is *non-empty* if its vertex-set is non-empty. (Note that, according to our definition, the empty graph is neither trivial nor non-trivial.)

Given graphs $H$ and $G$, we say that $H$ is a *subgraph* of $G$ provided that $V_H \subseteq V_G$, and that for all distinct $u, v \in V_H$, if $u$ and $v$ are adjacent in $H$, then $u$ and $v$ are adjacent in $G$.

Given a graph $G$ and a vertex $v \in V_G$, we denote by $\Gamma_G(v)$ the set of all neighbors of $v$ in $G$; in particular, $v \notin \Gamma_G(v)$.

Given a graph $G$ and a set $S \subseteq V_G$, we define the *subgraph of* $G$ *induced by* $S$, denoted by $G[S]$, to be the graph whose vertex-set is $S$, and that satisfies the property that for all distinct $u, v \in S$, $u$ and $v$ are adjacent in $G[S]$ if and only if $u$ and $v$ are adjacent in $G$; if $S = \{v_1, ..., v_n\}$, we sometimes write $G[v_1, ..., v_n]$ instead of $G[S]$. Given a graph $G$ and a set $S \subseteq V_G$, we denote by $G \smallsetminus S$ the graph $G[V_G \smallsetminus S]$; given a vertex $v \in V_G$, we often write $G \smallsetminus v$ instead of $G \smallsetminus \{v\}$.

Given graphs $G$ and $H$, we say that $H$ is an *induced subgraph* of $G$, or that $G$ *contains* $H$ *as an induced subgraph*, provided that there exists some set $S \subseteq V_G$ such that $H = G[S]$ (however, we typically relax this condition and say that $H$ is an induced subgraph of $G$ provided that there exists some set $S \subseteq V_G$ such that $H$ is isomorphic to $G[S]$). Given graphs $G$ and $H$, we say that $G$ is $H$-*free* provided that $G$ does not contain $H$ as an induced subgraph; we denote by $\mathrm{Forb}(H)$ is the class of all $H$-free graphs. In this thesis, we deal almost exclusively with induced subgraphs, and subgraphs that are not necessarily induced appear only sporadically.

A class $\mathcal{G}$ of graphs is said to be *hereditary* provided that $\mathcal{G}$ is closed under isomprhism and induced subgraphs (this means that for all $G \in \mathcal{G}$, if $G'$ is isomorphic to $G$, then $G' \in \mathcal{G}$, and if $H$ is an induced subgraph of $G$, then $H \in \mathcal{G}$). Clearly, for every graph $H$, the class Forb($H$) is hereditary.

The *complement* of a graph $G$, denoted by $\overline{G}$, is the graph whose vertex-set is $V_G$ and that satisfies the property that for all distinct $u, v \in V_G$, $u$ and $v$ are adjacent in $\overline{G}$ if and only if $u$ and $v$ are non-adjacent in $G$. A class $\mathcal{G}$ of graphs is said to be *self-complementary* provided that for all graphs $G \in \mathcal{G}$, we have that $\overline{G} \in \mathcal{G}$.

A *clique* in a graph $G$ is a set of pairwise adjacent vertices of $G$; a *stable set* in $G$ is a set of pairwise non-adjacent vertices of $G$. The *size* of a clique or a stable set is the number of vertices that it contains. A clique of size three is called a *triangle*, and a stable set of size three is called a *triad*; a graph is said to be *triangle-free* (respectively: *triad-free*) provided that it contains no triangle (respectively: triad). The *clique number* of a graph $G$, denoted by $\omega(G)$, is the maximum size of a clique in $G$.

A *proper coloring* of a graph $G$ is a function that assigns a color to each vertex of $G$ in such a way that whenever two vertices are adjacent, they receive distinct colors; we often say simply "coloring" instead of "proper coloring." The *chromatic number* of a graph $G$, denoted by $\chi(G)$, is the smallest number of colors necessary to (properly) color $G$. An *optimal coloring* of a graph $G$ is a proper coloring of $G$ that uses only $\chi(G)$ colors. Since in a coloring of a graph $G$, adjacent vertices always receive distinct colors, we can think of a coloring simply as a partition of the vertex-set of $G$ into stable sets; thus, the chromatic number of $G$ is the smallest number of stable sets needed to partition $V_G$. Clearly, $\omega(G) \leq \chi(G)$ for all graphs $G$.

A graph $G$ is said to be *perfect* provided that for all induced subgraphs $H$ of $G$, we have that $\chi(H) = \omega(H)$. A graph is *imperfect* if it is not perfect.

A class $\mathcal{G}$ of graphs is said to be *$\chi$-bounded* provided that there exists a function $f : \mathbb{N}_0 \to \mathbb{R}$ such that for all graphs $G \in \mathcal{G}$, and all induced subgraphs $H$ of $G$, $\chi(H) \leq f(\omega(H))$. Under these circumstances, we say that the class $\mathcal{G}$ is *$\chi$-bounded* by the function $f$, and that $f$ is a *$\chi$-bounding function* for $\mathcal{G}$. (Clearly, the class of perfect graphs is $\chi$-bounded by the identity function.) Note that if $f$ is a $\chi$-bounding function for $\mathcal{G}$, then so is the function $g : \mathbb{N}_0 \to \mathbb{R}$ given by $n \mapsto \lfloor \max\{f(0), ..., f(n)\} \rfloor$. Thus, we may assume that every $\chi$-bounding function is non-decreasing, and (when convenient) that it is integer-valued. We also remark that if $\mathcal{G}$ is a hereditary class, then $\mathcal{G}$ is $\chi$-bounded if and only if there exists a function $f : \mathbb{N}_0 \to \mathbb{R}$ such that for all $G \in \mathcal{G}$, $\chi(G) \leq f(\omega(G))$.

Let $G$ be a graph. Given a set $S \subseteq V_G$ and a vertex $v \in V_G \smallsetminus S$, we say that $v$ is *complete* (respectively: *anti-complete*) to the set $S$ or to the induced subgraph $G[S]$ of $G$ provided that $v$ is adjacent (respectively: non-adjacent) to every vertex in $S$; we say that $v$ is *mixed* on $S$ or on $G[S]$ provided that $v$ is neither complete nor anti-complete to $S$. If $v$ is complete (respectively: anti-complete) to $S$ in $G$, we also say that $v$ is a *center* (respectively: *anti-center*) for $S$ or $G[S]$ in $G$. Given disjoint sets $A$ and $B$ in $G$, we say that $A$ or $G[A]$ is *complete* (respectively: *anti-complete*) to $B$ or $G[B]$ provided that every vertex in $A$ is complete (respectively: anti-complete) to $B$.

Given a graph $G$ and a non-empty set $S \subseteq V_G$, we say that $S$ is a *homogeneous set* in $G$ provided that no vertex in $V_G \smallsetminus S$ is mixed on $S$; a homogeneous set $S$ in $G$ is said to be *proper* provided that $2 \leq |S| \leq |V_G| - 1$, and $G$ is said to admit a *homogeneous set decomposition* provided that $G$ contains a proper homogeneous set. Given a homogeneous set $S$ in $G$, the *partition* of $G$ associated with $S$ is the triple $(S, X, Y)$, where $X$ is the set of all vertices in $V_G \smallsetminus S$ that are complete to $S$, and $Y$ is the set of all vertices in $V_G \smallsetminus S$ that are anti-complete to $S$. We remark that if $S$ is a homogeneous set in a graph $G$, and if $(S, X, Y)$ is the associated partition of $G$, then $S$ is also a homogeneous set in $\overline{G}$, and

$(S, Y, X)$ is the associated partition of $\overline{G}$.

If $S$ is a homogeneous set in a graph $G$, then a *reduction* of the ordered pair $(G, S)$ is an ordered pair $(H, s)$, where:

- $H$ is a graph;
- $s \notin V_G$;
- $V_H = (V_G \smallsetminus S) \cup \{s\}$;
- $H \smallsetminus s = G \smallsetminus S$;
- for all $v \in V_G \smallsetminus S$, if $v$ is complete to $S$ in $G$, then $v$ is adjacent to $s$ in $H$, and if $v$ is anti-complete to $S$ in $G$, then $v$ is non-adjacent to $s$ in $H$.

Intuitively, we can think of the graph $H$ as being obtained from $G$ by "contracting" the homogeneous set $S$ to a vertex. Clearly, $H$ is (isomorphic to) an induced subgraph of $G$.

Given non-empty graphs $G_1$ and $G_2$ with disjoint vertex-sets, and a vertex $v \in V_{G_1}$, we say that a graph $G$ is obtained by *substituting* $G_2$ *for* $v$ *in* $G_1$ provided that the following four conditions hold:

- $V_G = (V_{G_1} \smallsetminus \{v\}) \cup V_{G_2}$;
- $G[V_{G_1} \smallsetminus \{v\}] = G_1 \smallsetminus v$;
- $G[V_{G_2}] = G_2$;
- for all $u \in V_{G_1} \smallsetminus \{v\}$, if $u$ is adjacent to $v$ in $G_1$, then $u$ is complete to $V_{G_2}$ in $G$, and if $u$ is non-adjacent to $v$ in $G_1$, then $u$ is anti-complete to $V_{G_2}$ in $G$.

We say that $G$ is obtained by *substitution from smaller graphs* provided that there exist graphs $G_1$ and $G_2$ with disjoint vertex-sets, and a vertex $v \in V_{G_1}$ such that $|V_{G_1}| \geq 2$, $|V_{G_2}| \geq 2$, and $G$ is obtained by substituting $G_2$ for $v$ in $G_1$. We remark that a graph $G$ is obtained by substitution from smaller graphs if and only if $G$ admits a homogeneous set decomposition.

As usual, a class $\mathcal{G}$ of graphs is said to be *closed under substitution* provided that for all non-empty graphs $G_1, G_2 \in \mathcal{G}$ with disjoint vertex-sets, and all vertices $v \in V_{G_1}$, if $G$ is obtained by substituting $G_2$ for $v$ in $G_1$, then $G \in \mathcal{G}$.

Given a graph $G$ and non-empty, disjoint sets $A, B \subseteq V_G$, we say that $(A, B)$ is a *homogeneous pair* in $G$ provided that no vertex in $V_G \smallsetminus (A \cup B)$ is mixed on $A$, and no vertex in $V_G \smallsetminus (A \cup B)$ is mixed on $B$. A homogeneous pair $(A, B)$ in $G$ is *tame* provided that $A$ is neither complete nor anti-complete to $B$, and $3 \leq |A \cup B| \leq |V_G| - 3$. We say that a graph $G$ admits a *homogeneous pair decomposition* provided that $G$ contains a tame homogeneous pair. If $(A, B)$ is a homogeneous pair in $G$, then the *partition* of $G$ associated with $(A, B)$ is the six-tuple $(A, B, C, D, E, F)$, where:

- $C$ is the set of all vertices $v \in V_G \smallsetminus (A \cup B)$ such that $v$ is complete to $A$ and anti-complete to $B$;
- $D$ is the set of all vertices $v \in V_G \smallsetminus (A \cup B)$ such that $v$ is complete to $B$ and anti-complete to $A$;
- $E$ is the set of all vertices $v \in V_G \smallsetminus (A \cup B)$ such that $v$ is complete to $A \cup B$;
- $F$ is the set of all vertices $v \in V_G \smallsetminus (A \cup B)$ such that $v$ is anti-complete to $A \cup B$.

We note that if $(A, B)$ is a homogeneous set in $G$, and $(A, B, C, D, E, F)$ is the associated partition, then: $A$, $B$, $C$, $D$, $E$, and $F$ are pairwise disjoint; $V_G = A \cup B \cup C \cup D \cup E \cup F$; $A$ and $B$ are both non-empty, but none of $C$, $D$, $E$, and $F$ need be non-empty. However, if $(A, B)$ is tame, then $|A \cup B| \geq 3$ and $|C \cup D \cup E \cup F| \geq 3$. We remark that if $(A, B)$ is a homogeneous pair in a graph $G$, and if $(A, B, C, D, E, F)$ is the associated partition of $G$, then $(A, B)$ is also a homogeneous pair in $\overline{G}$, and $(A, B, D, C, F, E)$ is the associated partition of $\overline{G}$. In particular then, $G$ admits a homogeneous pair decomposition if and only if $\overline{G}$ does.

A *path* is a graph whose vertex-set can be ordered as $\{p_0, p_1, ..., p_k\}$ (for some $k \geq 0$) so that for all distinct $i, j \in \{0, ..., k\}$, $p_i$ is adjacent to $p_j$ if and only if $|i - j| = 1$; such a path is often denoted by $p_0 - ... - p_k$, and we say that $p_0$ and $p_k$ are the *endpoints* of this path, and that $p_0 - ... - p_k$ is a path from $p_0$ to $p_k$ (or between $p_0$ and $p_k$). The *length* of a path is the number of edges that it contains; a *$k$-edge path* is a path of length $k$. A *path* in a graph $G$ is a (not necessarily induced) subgraph $P$ of $G$ such that $P$ is a path. An *induced path* in $G$ is an induced subgraph $P$ of $G$ such that $P$ is a path. A path (induced or not) in $G$ is often denoted by $p_0 - p_1 - ... - p_k$, consistently with the notation introduced above.

A graph $G$ is *connected* if for all distinct vertices $u, v \in V_G$, there is a path in $G$ between $u$ and $v$. A *component* of a non-empty graph $G$ is a maximal connected subgraph of $G$. A graph $G$ is said to be *anti-connected* if $\overline{G}$ is connected. An *anti-component* of a non-empty graph $G$ is a maximal anti-connected subgraph of $G$. A component or an anti-component of $G$ is said to be *trivial* if it contains only one vertex, and it is said to be *non-trivial* if it contains at least two vertices.

A *cycle* is a graph whose vertex-set can be ordered as $\{c_1, .., c_k\}$ (where $k \geq 3$) so that for all distinct $i, j \in \{1, ..., k\}$, $c_i$ is adjacent to $c_j$ if and only if $|i - j| = 1$ or $|i - j| = k - 1$; such a cycle is often denoted by $c_1 - c_2 - ... - c_k - c_1$. The *length* of a cycle is the number of vertices that it contains. A *square* is a cycle of length four. A *cycle* in a graph $G$ is a (not necessarily induced) subgraph $C$ of $G$ such that $C$ is a cycle. An *induced cycle* (or a *chordless cycle*) in a graph $G$ is an induced subgraph $C$ of $G$ such that $C$ is a cycle.

A *hole* in a graph $G$ is an induced cycle of length at least four in $G$, and an *anti-hole* in $G$ is an induced subgraph of $G$ whose complement is a hole in $\overline{G}$. The *length* of a hole or an anti-hole in $G$ is the number of vertices in this hole or anti-hole. (We remark that any hole of length five is also an anti-hole of length five.) An *odd* hole or anti-hole is a hole or anti-hole of odd length. We often denote a hole $H$ of length $k$ in $G$ by $h_1 - h_2 - ... - h_k - h_1$, where $V_H = \{h_1, h_2, ..., h_k\}$, and for all distinct $i, j \in \{1, 2, ..., k\}$, $h_i$ and $h_j$ are adjacent if and only if $|i - j| = 1$ or $|i - j| = k - 1$. Similarly, we often denote an anti-hole $H$ of length $k$ in $G$ by $h_1 - h_2 - ... - h_k - h_1$, where $V_H = \{h_1, h_2, ..., h_k\}$, and for all distinct $i, j \in \{1, 2, ..., k\}$, $h_i$ and $h_j$ are non-adjacent if and only if $|i - j| = 1$ or $|i - j| = k - 1$. A graph is *odd hole-free* if it contains no odd holes, and it is *odd anti-hole-free* provided that it contains no odd anti-holes. A graph is *Berge* provided that it contains no odd holes and no odd anti-holes. Clearly, the class of all Berge graphs is self-complementary.

A graph $G$ is *bipartite* provided that its vertex-set can be partitioned into (possibly empty) stable sets $A$ and $B$; under these circumstances, $(A, B)$ is said to be a *bipartition* of the bipartite graph $G$. $G$ is said to be *complement bipartite* provided that its vertex-set can be partitioned into (possibly empty) cliques $A$ and $B$; under these circumstances, $(A, B)$ is said to be a *bipartition* of the complement-bipartite graph $G$.

The *bull* is the graph with vertex-set $\{x_1, x_2, x_3, y_1, y_2\}$ and edge-set $\{x_1x_2, x_2x_3, x_3x_1, x_1y_1, x_2y_2\}$. A graph is said to be *bull-free* provided that it does not contain the bull as an induced subgraph. Consistently with the terminology introduced at the beginning of this section, a graph is said to be *bull-free* provided that it does not contain the bull as an induced subgraph. We remark that the complement of a bull is again a bull; consequently, the class Forb(bull), the class of all bull-free graphs, is self-complementary. Furthermore, the bull does not contain a proper homogeneous set, and so the class of bull-free graphs is closed under substitution. We state these two facts below for future reference.

**2.1.1.** *The class of bull-free graphs is self-complementary and closed under substitution.*

Note that the Weak Perfect Graph Theorem 1.1.1 states that the class of perfect graphs is self-complementary. In the paper in which he proved the Weak Perfect Graph Theorem (namely [48]), Lovász also proved that the class of perfect graphs is closed under substitution. We state these results below for future reference.

**2.1.2** (Lovász [48])**.** *The class of perfect graphs is self-complementary and closed under substitution.*

We remark that 2.1.2 is also an easy consequence of the Strong Perfect Graph Theorem 1.1.2. Indeed, the class of Berge graphs is self-complementary (by definition), and since cycles of length at least five and their complements do not admit a homogeneous set decomposition, the class of Berge graphs is closed under substitution. Now 2.1.2 follows from the Strong Perfect Graph Theorem 1.1.2.

In this thesis, a *directed graph* is an ordered pair $\vec{G} = (V_G, A_G)$, where $V_G$ is a non-empty set (called the *vertex-set* of $\vec{G}$), and $A_G$ (called the *arc set* of $G$) is an irreflexive, asymmetric binary relation on $V_G$; members of $V_G$ are called the *vertices* of the directed graph $\vec{G}$, and members of $A_G$ are called the *arcs* of $\vec{G}$. A directed graph $\vec{G} = (V_G, A_G)$ is a said to be *transitive* provided that for all $u, v, w \in V_G$, if $(u, v), (v, w) \in A_G$ then $(u, w) \in A_G$. A directed graph $\vec{G}$ is said to be an *orientation* of a graph $G$ provided that:

- the vertex-sets of the directed graph $\vec{G}$ and the graph $G$ are identical;
- for all adjacent vertices $u$ and $v$ of $G$, exactly one of $(u, v)$ and $(v, u)$ is an arc of $\vec{G}$;
- for all non-adjacent vertices $u$ and $v$ of $G$, $(u, v)$ is not an arc of $\vec{G}$.

A graph $G$ is said to be *transitively orientable* provided that some orientation of $G$ is a transitive directed graph. (Transitively orientable graphs are also called *comparability graphs*.) It is a well-known (and easy to prove) fact that transitively orientable graphs are perfect (see, for instance, [37]).

## 2.2 Definitions: Trigraphs

In this section, we introduce "trigraphs." As mentioned in the Introduction, a trigraph is a generalization of a graph, and indeed, every graph can be thought of as a particular kind of trigraph. Many (though not all) graph-theoretic concepts can readily be generalized to trigraphs. Of particular relevance for this thesis is the fact that the concepts of "bull-free graphs" and "Berge graphs" are readily generalized to trigraphs (see below), however, there is no convenient way to define a "trigraph coloring," and so we do not define such a thing as a "perfect trigraph."

A trigraph $G$ is an ordered pair $(V_G, \theta_G)$, where $V_G$ is a finite set, called the *vertex-set* of $G$, and $\theta_G : V_G \times V_G \to \{-1, 0, 1\}$ is a map, called the *adjacency function* of $G$, satisfying the following:

- for all $v \in V_G$, $\theta_G(v, v) = 0$;
- for all $u, v \in V_G$, $\theta_G(u, v) = \theta(v, u)$;
- for all $u \in V_G$, there exists at most one $v \in V_G \smallsetminus \{u\}$ such that $\theta_G(u, v) = 0$.

Members of $V_G$ are called the *vertices* of $G$. Let $u, v \in V_G$ be distinct. We say that $uv$ is a *strongly adjacent pair*, or that $u$ and $v$ are *strongly adjacent*, or that $u$ is *strongly adjacent* to $v$, or that $u$ is a *strong neighbor* of $v$, provided that $\theta_G(u, v) = 1$. We say that $uv$ is a *strongly anti-adjacent pair*, or that $u$ and $v$ are *strongly anti-adjacent*, or that $u$ is *strongly anti-adjacent* to $v$, or that $u$ is a *strong anti-neighbor* of $v$, provided that $\theta_G(u, v) = -1$. We say that $uv$ is a *semi-adjacent pair*, or that $u$ and $v$ are *semi-adjacent*, or that $u$ is *semi-adjacent* to $v$, provided that $\theta_G(u, v) = 0$. (Note that we do not say that a vertex $w \in V_G$ is semi-adjacent to itself even though $\theta_G(w, w) = 0$.) If $uv$ is a strongly adjacent pair or a semi-adjacent pair, then we say that $uv$ is an *adjacent pair*, or that $u$ and $v$ are *adjacent*, or that $u$ is *adjacent* to $v$, or that $u$ is a *neighbor* of $v$. If $uv$ is a strongly anti-adjacent pair or a semi-adjacent pair, then we say that $uv$ is an *anti-adjacent pair*, or that $u$ and $v$ are *anti-adjacent*, or that $u$ is *anti-adjacent* to $v$, or that $u$ is an *anti-neighbor* of $v$. Thus, if $uv$ is a semi-adjacent pair, then $uv$ is simultaneously an adjacent pair and an anti-adjacent pair. The *endpoints* of the pair $uv$ (regardless of adjacency) are $u$ and $v$. Given distinct vertices $u$ and $v$ of $G$, we do not distinguish between pairs $uv$ and $vu$. However, we will sometimes need to maintain the asymmetry between the endpoints of a semi-adjacent pair $uv$, and in those cases, we will use the ordered pair notation and write $(u, v)$ or $(v, u)$, as appropriate, rather than $uv$.

Intuitively, one can think of the strongly adjacent pairs in a trigraph $G$ as "edges," the strongly anti-adjacent pairs as "non-edges," and the semi-adjacent pairs as "undecided." Note that each (finite and simple) graph can be thought of as a trigraph in a natural way: graphs are simply trigraphs with no semi-adjacent pairs.

Given a trigraph $G$ and graph $\hat{G}$, we say that $\hat{G}$ is a *realization* of $G$ provided that all of the following hold:

- the vertex-sets of $G$ and $\hat{G}$ are identical;
- for all $u, v \in V_G$, if $uv$ is an edge in $\hat{G}$, then $uv$ is an adjacent pair in $G$;
- for all $u, v \in V_G$, if $uv$ is a non-edge in $\hat{G}$, then $uv$ is an anti-adjacent pair in $G$.

Thus, a realization of $G$ is obtained by turning all the strongly adjacent pairs of $G$ into edges, all the strongly anti-adjacent pairs of $G$ into non-edge, and all the semi-adjacent pairs of $G$ (arbitrarily and independently of each other) into edges or non-edges. Note that this means that if a trigraph $G$ has $n$ semi-adjacent pairs, then $G$ has $2^n$ distinct (though not necessarily non-isomorphic) realizations.

The *complement* of a trigraph $G$ is the trigraph $\overline{G}$ with vertex-set $V_{\overline{G}} = V_G$ and adjacency function $\theta_{\overline{G}} = -\theta_G$. Given a set $S \subseteq V_G$, $G[S]$ is the trigraph with vertex-set $S$ and adjacency function $\theta_G \upharpoonright S \times S$ (the restriction of $\theta_G$ to $S \times S$); we call $G[S]$ the *subtrigraph of $G$ induced by $S$*; if $S = \{v_1, ..., v_n\}$, we sometimes write $G[v_1, ..., v_n]$ instead of $G[S]$. Isomorphism between trigraphs is defined in the natural way; if trigraphs $G_1$ and $G_2$ are isomorphic, then we write $G_1 \cong G_2$. Given trigraphs $G$ and $H$, we say that $H$ is an *induced subtrigraph* of $G$ (or that $G$ *contains $H$ as an induced subtrigraph*) if there exists some $X \subseteq V_G$ such that $H = G[X]$. (However, when convenient, we relax this condition and say that $H$ is an induced subtrigraph of $G$, or that $G$ contains $H$ as an induced subtrigraph, if there exists some $X \subseteq V_G$ such that $H \cong G[X]$.) Given a trigraph $G$ and a set $X \subseteq V_G$, we denote by $G \smallsetminus X$ the trigraph $G[V_G \smallsetminus X]$; given a vertex $v \in V_G$, we often write $G \smallsetminus v$ instead of $G \smallsetminus \{v\}$.

We remark that we do not define such a thing as "subtrigraph" (only an "induced subtrigraph"). Accordingly, when we say that a trigraph $G$ *contains* a trigraph $H$, we always mean that $G$ contains $H$ as an induced subtrigraph.

A trigraph $P$ is said to be a *path* provided that its vertex-set can be ordered as $\{p_0, ..., p_k\}$ (where $k \geq 0$), so that for all distinct $i, j \in \{0, ..., k\}$, if $|i - j| = 1$ then $p_i p_j$ is an adjacent pair, and if $|i - j| > 1$ then $p_i p_j$ is an anti-adjacent pair. Under these circumstances, we say that $p_0$ and $p_k$ are the *endpoints* of the path $P$, and that $P$ is a path *between* $p_0$ and $p_k$. We sometimes denote such a path $P$ by $p_0 - ... - p_k$. The *length* of a path $p_0 - ... - p_k$ is $k$, and a *$k$-edge path* is a path of length $k$. (In particular, a *three-edge path* is a path that has exactly four vertices.) A *path* in a trigraph $G$ is an induced subtrigraph $P$ of $G$ such that $P$ is a path. A trigraph $G$ is said to be *connected* provided that for all distinct $u, v \in V_G$, there is a path in $G$ between $u$ and $v$. A *component* of a non-empty trigraph $G$ is a maximal non-empty connected induced subtrigraph of $G$. A component $C$ of a trigraph $G$ is said to be *non-trivial* provided that $|V_C| \geq 2$.

A trigraph $P$ is said to be an *anti-path* provided that its vertex-set can be ordered as $\{p_0, ..., p_k\}$ (where $k \geq 0$), so that for all distinct $i, j \in \{0, ..., k\}$, if $|i - j| = 1$ then $p_i p_j$

is an anti-adjacent pair, and if $|i-j|>1$ then $p_ip_j$ is an adjacent pair. Under these circumstances, we say that $p_0$ and $p_k$ are the *endpoints* of the anti-path $P$, and that $P$ is an anti-path *between* $p_0$ and $p_k$. We sometimes denote such an anti-path $P$ by $p_0 - ... - p_k$. The *length* of an anti-path $p_0 - ... - p_k$ is $k$. An *anti-path* in a trigraph $G$ is an induced subtrigraph $P$ of $G$ such that $P$ is an anti-path. A trigraph $G$ is said to be *anti-connected* provided that for all distinct $u, v \in V_G$, there is an anti-path in $G$ between $u$ and $v$. An *anti-component* of a non-empty trigraph $G$ is a maximal non-empty anti-connected induced subtrigraph of $G$. An anti-component $C$ of a trigraph $G$ is said to be *non-trivial* provided that $|V_C| \geq 2$.

We remark that the complement of a path is an anti-path. Further, a trigraph $G$ is anti-connected if and only if $\overline{G}$ is connected, and $C$ is an anti-component of $G$ is and only if $\overline{C}$ is a component of $G$.

An induced subtrigraph $H$ of a trigraph $G$ is a *hole* in $G$ provided that the vertex-set of $H$ can be ordered as $\{h_1, ..., h_k\}$ (with $k \geq 4$), so that for all distinct $i, j \in \{1, ..., k\}$, if $|i-j| = 1$ or $|i-j| = k-1$ then $h_ih_j$ is an adjacent pair, and if $1 < |i-j| < k-1$ then $h_ih_j$ is an anti-adjacent pair; such a hole is often denoted by $h_1 - h_2 - ... - h_k - h_1$. An induced subtrigraph $H$ of a trigraph $G$ is an *anti-hole* in $G$ provided that $\overline{H}$ is a hole in $\overline{G}$; we sometimes denote an anti-hole $H$ in $G$ by $h_1 - h_2 - ... - h_k - h_1$ when $\overline{H}$ is a hole of the form $h_1 - h_2 - ... - h_k - h_1$ in $\overline{G}$. The *length* of a hole or anti-hole is the number of vertices that it contains; a hole or anti-hole is said to be *odd* if it has an odd number of vertices. A trigraph $G$ is said to be *odd hole-free* provided that it contains no odd holes; $G$ is said to be *odd anti-hole-free* provided that it contains no odd anti-holes. A trigraph is said to be *Berge* if it contains neither an odd hole nor an odd anti-hole (in other words, a trigraph is Berge provided that it is both odd hole-free and odd anti-hole-free).

A trigraph is called a *bull* provided that its vertex-set is $\{x_1, x_2, x_3, y_1, y_2\}$, with adjacency as follows: $x_1x_2, x_2x_3, x_3x_1, x_1y_1, x_2y_2$ are adjacent pairs, and $x_1y_2, x_2y_1, x_3y_1, x_3y_2, y_1y_2$ are anti-adjacent pairs. (In other words, a trigraph is a bull provided that some realization of it is a bull.) A trigraph is said to be *bull-free* provided that no induced subtrigraph of it is a bull.

We observe that every bull-free (respectively: Berge) graph can be seen as a bull-free (respectively: Berge) trigraph that has no semi-adjacent pairs. We remark that a trigraph $G$ is bull-free (respectively: odd hole-free, odd anti-hole-free, Berge) if and only if every realization of $G$ is bull-free (respectively: odd hole-free, odd anti-hole free, Berge).

Let us say that a class $\mathcal{G}$ of trigraphs is *self-complementary* provided that for all $G \in \mathcal{G}$, we have that $\overline{G} \in \mathcal{G}$. We now prove an easy lemma that will be used repeatedly in chapters 3 and 4.

**2.2.1.** *The class of bull-free trigraphs is self-complementary. The class of Berge trigraphs is self-complementary.*

*Proof.* Note that the complement of a bull is again a bull; thus, the class of bull-free trigraphs is self-complementary. The class of Berge trigraphs is self-complementary by the definition of a Berge trigraph. $\square$

A *clique* (respectively: *stable set*) in a trigraph is a set of pairwise adjacent (respectively: anti-adjacent) vertices; a *strong clique* (respectively: *strongly stable set*) is a set of pairwise strongly adjacent (respectively: strongly anti-adjacent) vertices. A clique (respectively: strong clique) that contains exactly three vertices is called a *triangle* (respectively: *strong triangle*), and a stable set (respectively: strongly stable set) that contains exactly three vertices is called a *triad* (respectively: *strong triad*). A trigraph is said to be *triangle-free* (respectively: *triad-free*) provided that it contains no triangle (respectively: triad).

A trigraph $G$ is said to be *bipartite* if its vertex-set can be partitioned into two (possibly empty) strongly stable sets, $A$ and $B$; under these circumstances, we say that $(A, B)$ is a *bipartition* of the bipartite trigraph $G$. A trigraph $G$ is said to be *complement-bipartite* provided that its vertex-set can be partitioned into two (possibly empty) strong cliques, $A$ and $B$; under these circumstances, we say that $(A, B)$ is a *bipartition* of the complement-bipartite trigraph $G$. We remark that a trigraph $G$ is bipartite (respectively: complement-bipartite) with bipartition $(A, B)$ if and only if every realization of $G$ is bipartite (respectively: complement-bipartite) with bipartition $(A, B)$.

Given a trigraph $G$, a vertex $a \in V_G$, and a set $B \subseteq V_G \smallsetminus \{a\}$, we say that $a$ is *strongly complete* (respectively: *strongly anti-complete*, *complete*, *anti-complete*) to $B$ provided that $a$ is strongly adjacent (respectively: strongly anti-adjacent, adjacent, anti-adjacent) to every vertex in $B$; we say that $a$ is *mixed* on $B$ provided that $a$ is neither strongly complete nor strongly anti-complete to $B$. If $a$ is complete (respectively: anti-complete) to $B$ in $G$, we also say that $a$ is a *center* (respectively: *anti-center*) for $B$ in $G$. Given disjoint sets $A, B \subseteq V_G$, we say that $A$ is *strongly complete* (respectively: *strongly anti-complete*, *complete*, *anti-complete*) to $B$ provided that for every $a \in A$, $a$ is strongly complete (respectively: strongly anti-complete, complete, anti-complete) to $B$.

Given a trigraph $G$, a non-empty set $S \subseteq V_G$ is said to be a *homogeneous set* in $G$ provided that for every $v \in V_G \smallsetminus S$, $v$ is either strongly complete to $S$ or strongly anti-complete to $S$. A homogeneous set $S$ in $G$ is said to be *proper* provided that $2 \leq |S| \leq |V_G| - 1$. We say that a trigraph $G$ admits a *homogeneous set decomposition* provided that $G$ contains a proper homogeneous set. We observe that if $S$ is a homogeneous set in $G$, and $uv$ is a semi-adjacent pair in $G$, then either $u, v \in S$ or $u, v \in V_G \smallsetminus S$. Given a homogeneous set $S$ in a trigraph $G$, the *partition* of $G$ associated with the homogeneous set $S$ is the ordered triple $(S, X, Y)$, where $X$ is the set of all vertices in $V_G \smallsetminus S$ that are strongly complete to $S$, and $Y$ is the set of all vertices in $V_G \smallsetminus S$ that are strongly anti-complete to $S$. Clearly, if $(S, X, Y)$ is the partition of a trigraph $G$ associated with a homogeneous set $S$, then $S$, $X$, and $Y$ are pairwise disjoint, and $V_G = S \cup X \cup Y$. Note that if $S$ is a homogeneous set in a trigraph $G$, and $(S, X, Y)$ is the associated partition of $G$, then $S$ is also a homogeneous set in $\overline{G}$, and $(S, Y, X)$ is the associated partition of $\overline{G}$.

If $S$ is a homogeneous set in a trigraph $G$, then the *reduction* of the ordered pair $(G, S)$ is an ordered pair $(H, s)$, where:

- $H$ is a trigraph;
- $s \notin V_G$;
- $V_H = (V_G \smallsetminus S) \cup \{s\}$;
- $H \smallsetminus s = G \smallsetminus S$;
- for all $v \in V_G \smallsetminus S$, if $v$ is strongly complete to $S$ in $G$, then $vs$ is a strongly adjacent pair in $H$, and if $v$ is strongly anti-complete to $S$ in $G$, then $vs$ is a strongly anti-adjacent pair in $H$.

Intuitively, we can think of the trigraph $H$ as being obtained from $G$ by "contracting" the homogeneous set $S$ to a vertex. Clearly, $H$ is (isomorphic to) an induced subtrigraph of $G$.

Let $G_1$ and $G_2$ be non-empty trigraphs with disjoint vertex sets, let $v \in V_{G_1}$, and assume that $v$ is not an endpoint of any semi-adjacent pair in $G_1$. We then say that a trigraph $G$ is obtained by *substituting* $G_2$ *for* $v$ *in* $G_1$ provided that all of the following hold:

- $V_G = (V_{G_1} \smallsetminus \{v\}) \cup V_{G_2}$;
- $G[V_{G_1} \smallsetminus \{v\}] = G_1 \smallsetminus v$;
- $G[V_{G_2}] = G_2$;

- for all $v_1 \in V_{G_1} \smallsetminus \{v\}$ and $v_2 \in V_{G_2}$, $v_1v_2$ is a strongly adjacent pair in $G$ if $v_1v$ is a strongly adjacent pair in $G_1$, and $v_1v_2$ is a strongly anti-adjacent pair in $G$ if $v_1v$ is a strongly anti-adjacent pair in $G_1$.

We say that a trigraph $G$ is obtained by *substitution from smaller trigraphs* provided that there exist non-empty trigraphs $G_1$ and $G_2$ with disjoint vertex-sets satisfying $|V_{G_1}| < |V_G|$ and $|V_{G_2}| < |V_G|$ (or equivalently: $|V_{G_1}| \geq 2$ and $|V_{G_2}| \geq 2$) and some $v \in V_{G_1}$ that is not an endpoint of any semi-adjacent pair in $G_1$, such that $G$ is obtained by substituting $G_2$ for $v$ in $G_1$. We observe that a trigraph $G$ admits a homogeneous set decomposition if and only if $G$ is obtained from smaller trigraphs by substitution. We will use the following result several times in this thesis.

**2.2.2.** *Let $G_1$ and $G_2$ be non-empty trigraphs with disjoint vertex sets, let $v \in V_{G_1}$, and assume that $v$ is not an endpoint of any semi-adjacent pair in $G_1$. Assume that a trigraph $G$ is obtained by substituting $G_2$ for $v$ in $G_1$. Then all of the following hold:*

- *$G$ is bull-free if and only if both $G_1$ and $G_2$ are bull-free;*
- *$G$ is odd hole-free if and only if both $G_1$ and $G_2$ are odd hole-free;*
- *$G$ is odd anti-hole-free if and only if both $G_1$ and $G_2$ are odd anti-hole-free;*
- *$G$ is Berge if and only if $G_1$ and $G_2$ are both Berge.*

*Proof.* This is an easy consequence of the fact that bulls, holes of length at least five, and anti-holes of length at least five do not admit a homogeneous set decomposition. □

Next, let $G$ be a trigraph, and let $A$ and $B$ be non-empty, disjoint subsets of $V_G$. We say that $(A, B)$ is a *homogeneous pair* in $G$ provided that no vertex in $V_G \smallsetminus (A \cup B)$ is mixed on $A$, and no vertex in $V_G \smallsetminus (A \cup B)$ is mixed on $B$. We observe that if $(A, B)$ is a homogeneous pair in $G$, and $uv$ is a semi-adjacent pair in $G$, then either $u, v \in A \cup B$ or $u, v \in V_G \smallsetminus (A \cup B)$. If $(A, B)$ is a homogeneous pair in $G$, and $C$ is the set of all vertices in $V_G \smallsetminus (A \cup B)$ that are strongly complete to $A$ and strongly anti-complete to $B$, $D$ is the set of all vertices in $V_G \smallsetminus (A \cup B)$ that are strongly complete to $B$ and strongly anti-complete to $A$, $E$ is the set of all vertices in $V_G \smallsetminus (A \cup B)$ that are strongly complete to $A \cup B$, and $F$ is the set of all vertices in $V_G \smallsetminus (A \cup B)$ that are strongly anti-complete to $A \cup B$, then we say that $(A, B, C, D, E, F)$ is the *partition* of $G$ associated with the homogeneous pair $(A, B)$. Note that if $(A, B)$ is a homogeneous pair in $G$, and if $(A, B, C, D, E, F)$ is the associated partition of $G$, then $(A, B)$ is a homogeneous pair in $\overline{G}$, and $(A, B, D, C, F, E)$ is the associated partition of $\overline{G}$.

A homogeneous pair $(A, B)$ in a trigraph $G$ is said to be *tame* provided that the following two conditions are satisfied:

- $A$ is neither strongly complete nor strongly anti-complete to $B$;
- $3 \leq |A \cup B| \leq |V_G| - 3$

A trigraph $G$ is said to admit a *homogeneous pair decomposition* provided that it contains a tame homogeneous pair. Clearly a trigraph $G$ admits a homogeneous pair decomposition if and only if $\overline{G}$ does.

If $(A, B)$ is a homogeneous pair in a trigraph $G$, and if $(A, B, C, D, E, F)$ is the associated partition of $G$, then the *semi-adjacent reduction* of the triple $(G, A, B)$ is a triple $(H, a, b)$ such that:

- $H$ is a trigraph;
- $a, b \notin V_G$;
- $V_H = \{a, b\} \cup C \cup D \cup E \cup F$;
- $ab$ is a semi-adjacent pair in $H$;
- $H[C \cup D \cup E \cup F] = G[C \cup D \cup E \cup F]$;
- $a$ is strongly complete to $C \cup E$ and strongly anti-complete to $D \cup F$ in $G$;

- $b$ is strongly complete to $D \cup E$ and strongly anti-complete to $C \cup F$ in $G$.

Intuitively, we can think of the trigraph $H$ as being obtained from $G$ by "contracting" the homogeneous pair $(A, B)$ to a semi-adajcent pair $ab$. Note that if $A$ is neither strongly complete nor strongly anti-complete to $B$ in $H$, then every realization of $H$ is (isomorphic to) an induced subgraph of some realization of $G$.

# Chapter 3

# The Structure of Bull-Free Perfect Graphs

In this chapter, we use the structure theorem for bull-free trigraphs due to Chudnovsky [7, 8, 9, 10] to derive a structure theorem for bull-free Berge trigraphs. Since every graph can be thought of as a trigraph in a natural way (indeed, a graph is simply a trigraph with no semi-adjacent pairs), this is implicitly a structure theorem for bull-free Berge graphs, and therefore (by the Strong Perfect Graph Theorem 1.1.2) it is a structure theorem for bull-free perfect graphs.

This chapter is organized as follows. In section 3.1, we define "elementary trigraphs," and we use a result from [7] to reduce our problem to finding the structure of all elementary bull-free Berge trigraphs. We then cite the structure theorem for elementary bull-free trigraphs from [10]; this theorem states that every bull-free trigraph $G$ is either obtained from smaller bull-free trigraphs by substitution, or $G$ or its complement is an "elementary expansion" (this is defined later, in section 3.3) of a trigraph in one of two basic classes (classes $\mathcal{T}_1$ and $\mathcal{T}_2$). We complete section 3.1 by stating our main theorem (3.1.4), the structure theorem for all bull-free Berge trigraphs; however, we do not prove this theorem in section 3.1 (we only do this in section 3.6), and we also postpone defining certain terms used in the theorem. In section 3.2, we introduce "good homogeneous pairs," and we prove a useful lemma about them; good homogeneous pairs appear in sections 3.3 and 3.5. In section 3.3, we study "elementary expansions." Informally, an elementary expansion of a trigraph $H$ is the trigraph obtained by expanding some semi-adjacent pairs of $H$ to homogeneous pairs of a certain kind. We show that if $G$ is an elementary expansion of a trigraph $H$, then $G$ is Berge if and only if $H$ is. In section 3.4, we give the definition of the class $\mathcal{T}_1$ from [10] and derive the class $\mathcal{T}_1^*$ of all Berge trigraphs in $\mathcal{T}_1$. In section 3.5, we define the class $\mathcal{T}_2$ from [10], and prove that every trigraph in $\mathcal{T}_2$ is Berge. Finally, in section 3.6, we prove our main theorem.

## 3.1 Structure Theorem for Bull-Free Berge Trigraphs

Following [7], we call a bull-free trigraph $G$ *elementary* provided that it contains no three-edge path $P$ such that some vertex of $G$ is a center for $P$, and some vertex of $G$ is an anti-center for $P$. A bull-free trigraph that is not elementary is said to be *non-elementary.* We now state a decomposition theorem from [7] (this is 3.3 from [7]; we remark that not all terms from the statement of this theorem have been defined in this thesis).

**3.1.1** (Chudnovsky [7])**.** *Let $G$ be a non-elementary bull-free trigraph. Then at least one of the following holds:*

- *$G$ or $\overline{G}$ belongs to $\mathcal{T}_0$;*
- *$G$ or $\overline{G}$ contains a homogeneous pair of type zero;*
- *$G$ admits a homogeneous set decomposition.*

We omit the definitions of the class $\mathcal{T}_0$ and of a homogeneous pair of type zero, and instead refer the reader to [7]. What we need here is the fact (easy to check) that every trigraph in $\mathcal{T}_0$ contains a hole of length five, as does every trigraph that contains a homogeneous pair of type zero. Now 3.1.1 implies that every non-elementary bull-free trigraph that does not contain a hole of length five (and in particular, every non-elementary bull-free Berge trigraph) admits a homogeneous set decomposition; we state this result below for future reference.

**3.1.2.** *Every non-elementary bull-free trigraph that does not contain a hole of length five admits a homogeneous set decomposition. In particular, every non-elementary bull-free Berge trigraph admits a homogeneous set decomposition.*

While the proof of 3.1.1 is relatively involved, if we restrict our attention to non-elementary bull-free trigraphs $G$ that do not contain a hole of length five, only a couple of pages are needed to prove that $G$ admits a homogeneous set decomposition (we refer the reader to the proof of 5.2 from [7]). We also remark that for the case of graphs (rather than trigraphs), a result analogous to 3.1.2 was originally proven in [34].

Recall that a trigraph $G$ admits a homogeneous set decomposition if and only if it can be obtained from smaller trigraphs by substitution. Since the class of bull-free Berge trigraphs is closed under substitution (by 2.2.2), we need only consider bull-free Berge trigraphs that do not admit a homogeneous set decomposition, and by 3.1.2, all such trigraphs are elementary. Thus, the rest of the chapter deals with bull-free Berge trigraphs that are elementary.

We now state the structure theorem for elementary bull-free trigraphs. (We note that some terms used in the statement of this theorem have not yet been defined.) The following is an immediate consequence of 6.1 and 5.5 from [10].

**3.1.3** (Chudnovsky [10])**.** *Let $G$ be an elementary bull-free trigraph that is not obtained from smaller bull-free trigraphs by substitution. Then at least one of the following holds:*

- *$G$ or $\overline{G}$ is an elementary expansion of a member of $\mathcal{T}_1$;*
- *$G$ is an elementary expansion of a member of $\mathcal{T}_2$.*

*Conversely, if $H$ is a trigraph such that either one of $H$ and $\overline{H}$ is an elementary expansion* *of a member of* $\mathcal{T}_1$*, or* $H$ *is an elementary expansion of a trigraph in* $\mathcal{T}_2$*, then* $H$ *is an elementary bull-free trigraph.*

We note that some trigraphs $H$ that satisfy the hypotheses of 3.1.3 admit a homogeneous set decomposition (that is, they can be obtained by substitution from smaller bull-free trigraphs).

The definitions of classes $\mathcal{T}_1$ and $\mathcal{T}_2$, as well as of elementary expansions, are long and complicated, and we do not give them in this section. Instead, we give the definition of an elementary expansion of a trigraph in section 3.3; we prove there that if $G$ is an elementary expansion of a trigraph $H$, then $G$ is Berge if and only if $H$ is. In section 3.4, we give the definition of the class $\mathcal{T}_1$, and we derive the class $\mathcal{T}_1^*$ of all Berge trigraphs in $\mathcal{T}_1$. In section 3.5, we give the definition of the class $\mathcal{T}_2$, and we prove that every trigraph in $\mathcal{T}_2$ is Berge. In section 3.6 (the final section), we put all of this together to derive the structure theorem for Berge bull-free trigraphs, which we state below.

**3.1.4.** *Let* $G$ *be a trigraph. Then* $G$ *is bull-free and Berge if and only if at least one of the following holds:*

- $G$ *is obtained from smaller bull-free Berge trigraphs by substitution;*
- $G$ *or* $\overline{G}$ *is an elementary expansion of a trigraph in* $\mathcal{T}_1^*$*;*
- $G$ *is an elementary expansion of a trigraph in* $\mathcal{T}_2$*.*

## 3.2 Good Homogeneous Pairs

We say that a homogeneous pair $(A, B)$ in a trigraph $G$ is *good* provided that the following three conditions hold:

- neither $G[A]$ nor $G[B]$ contains a three-edge path;
- there does not exist a path $v_1 - v_2 - v_3 - v_4$ in $G$ such that $v_1, v_4 \in A$ and $v_2, v_3 \in B$;

- there does not exist a path $v_1 - v_2 - v_3 - v_4$ in $G$ such that $v_1, v_4 \in B$ and $v_2, v_3 \in A$.

We observe that if $(A, B)$ is a good homogeneous pair in a trigraph $G$, then $(A, B)$ is a good homogeneous pair in $\overline{G}$ as well; this follows from the fact that the complement of a three-edge path is again a three-edge path. Good homogeneous pairs will appear in sections 3.3 and 3.5 below. There, we will need the following lemma.

**3.2.1.** *Let $G$ be a trigraph, let $(A, B)$ be a good homogeneous pair in $G$, and let $W$ be the vertex-set of an odd hole or an odd anti-hole in $G$. Then $|W \cap A| \leq 1$ and $|W \cap B| \leq 1$.*

*Proof.* First, by passing to $\overline{G}$ if necessary, we may assume that $W$ is the vertex-set of an odd hole in $G$. Next, let $\hat{G}$ be a realization of $G$ in which $W$ is the vertex-set of an odd hole. Finally, let $(A, B, C, D, E, F)$ be the partition of $G$ associated with $(A, B)$.

We begin by proving that $W \not\subseteq A \cup B$. Suppose otherwise. Since the number of edges in $\hat{G}[W]$ with one endpoint in $A$ and the other one in $B$ is even, exactly one of $\hat{G}[W \cap A]$ and $\hat{G}[W \cap B]$ contains an odd number of edges; by symmetry, we may assume that $\hat{G}[B]$ contains an odd number of edges. Since $\hat{G}[W \cap B]$ contains no induced three-edge path, and since $\hat{G}[W]$ is a chordless cycle of length at least five, we know that $\hat{G}[W \cap B]$ contains an edge $b_1b_2$ that meets no other edges in $\hat{G}[W \cap B]$. Since $\hat{G}[W]$ is a chordless cycle of length at least five, there exist some $a_1, a_2 \in W$ such that $a_1 - b_1 - b_2 - a_2$ is an induced three-edge path in $\hat{G}[W]$ (and therefore in $G[W]$ as well). Since the edge $b_1b_2$ meets no other edges in $\hat{G}[W \cap B]$, we know that $a_1, a_2 \in A$. But then the path $a_1 - b_1 - b_2 - a_2$ contradicts the fact that $(A, B)$ is good.

We next show that $|W \cap A| \leq 2$ and $|W \cap B| \leq 2$. Suppose otherwise. By symmetry, we may assume that $|W \cap A| \geq 3$. Then $W \cap (C \cup E) = \emptyset$, for otherwise, some vertex in $\hat{G}[W]$ would be of degree at least three. Since $W \not\subseteq A \cup B$, $W$ intersects $D \cup F$; and since $\hat{G}[W]$ is connected, $W \cap D \neq \emptyset$. Now, fix some $a \in W \cap A$ and $d \in W \cap D$. Note that there are two paths in $\hat{G}[W]$ between $a$ and $d$ that meet only at their endpoints; both of these paths pass through $B$, and so $|W \cap B| \geq 2$. Fix distinct $b_1, b_2 \in W \cap B$. Since $B$ is complete to $D$ in $\hat{G}$, and since $\hat{G}[W]$ is a chordless cycle of length at least five, it follows that $W \cap B = \{b_1, b_2\}$ and $W \cap D = \{d\}$. It then easily follows that $W \setminus \{d\} \subseteq A \cup B$. Then $\hat{G}[W \cap A]$ is an odd path, and so since $|W \cap A| \geq 3$, we get that $\hat{G}[A]$ (and therefore $G[A]$) contains an induced three-edge path, contrary to the fact that $(A, B)$ is good. Thus, $|W \cap A| \leq 2$ and $|W \cap B| \leq 2$.

Finally, suppose that $|W \cap A| = 2$; set $W \cap A = \{a_1, a_2\}$. Since $\hat{G}[W]$ is a chordless cycle of length at least five, there exist some $b_1, b_2 \in W \setminus \{a_1, a_2\}$ such that $a_1b_1$, $a_2b_2$ are edges, and $a_1b_2$, $a_2b_1$ are non-edges in $\hat{G}$. Since $b_1$ and $b_2$ are both mixed on $A$, it follows that $b_1, b_2 \in B$; since $|W \cap B| \leq 2$, this means that $W \cap B = \{b_1, b_2\}$. Since $(A, B)$ is good, and since $\hat{G}[W]$ contains no cycles of length four, we know that both $a_1a_2$ and $b_1b_2$ are non-edges. Note that $W \cap E = \emptyset$, for otherwise, some vertex in $W$ would be of degree at least four in $\hat{G}[W]$. Thus, all neighbors of $a_1$ in $\hat{G}[W]$ lie in $C \cup \{b_1\}$; since $a_1$ has at least two neighbors in $W$, this means that $W \cap C \neq \emptyset$. Similarly, $W \cap D \neq \emptyset$. But if $c \in W \cap C$ and $d \in W \cap D$, then $c - a_1 - b_1 - d - b_2 - a_2 - c$ is a (not necessarily induced) cycle of length six in $\hat{G}[W]$, which is impossible. Thus, $|W \cap A| \leq 1$. In an analogous way, we get that $|W \cap B| \leq 1$. This completes the argument. $\square$

## 3.3 Elementary Expansions

Our goal in this section is to prove that if a trigraph $G$ is an elementary expansion of a trigraph $H$, then $G$ is Berge if and only if $H$ is Berge. Informally, a trigraph $G$ is said to be an "elementary expansion" of a trigraph $H$ provided that $G$ can be obtained by "expanding" some semi-adjacent pairs of a certain kind to homogeneous pairs of a corresponding kind. We start by defining the two kinds of semi-adjacent pair and the two kinds of homogeneous pair that we will need. After that, we define elementary expansions.

**Semi-adjacent pairs of type one and two.** Let $G$ be a trigraph, let $(a, b)$ be a semi-adjacent pair in $G$, and let $(\{a\}, \{b\}, C, D, E, F)$ be the partition of $G$ associated with the homogeneous pair $(\{a\}, \{b\})$.

We say that $(a, b)$ is a semi-adjacent pair of *type one* provided all of the following hold:

- $C$, $D$, and $F$ are non-empty;
- $E$ is empty;
- neither $C$ nor $D$ is strongly anti-complete to $F$.

We say that $(a, b)$ is a semi-adjacent pair of *type two* provided all of the following hold:

- $C$, $D$, and $F$ are non-empty;
- $E$ is empty;
- $C$ is not strongly anti-complete to $F$;
- $D$ is strongly anti-complete to $F$.

Finally, a semi-adjacent pair $(a, b)$ in a trigraph $G$ is said to be of *complement type one* or of *complement type two* in $G$ provided that $(a, b)$ is a semi-adjacent pair of type one or two, respectively, in $\overline{G}$.

**Closures of rooted forests.** We say that a trigraph $T$ is a *forest* provided that there are neither triangles nor holes in $T$. (Thus, for any two vertices of $T$, there is at most one path between them.) A connected forest is called a *tree*. A rooted forest is a $(k+1)$-tuple $\mathfrak{T} = (T, r_1, ..., r_k)$, where $T$ is a forest with components $T_1, ..., T_k$ such that $r_i \in V_{T_i}$ for all $i \in \{1, ..., k\}$. Given distinct $u, v \in V_T$, we say that $u$ is a *descendant* of $v$, or that $v$ is an *ancestor* of $u$, provided that $u, v \in V_{T_i}$ for some $i \in \{1, ..., k\}$, and that if $P$ is the (unique) path from $u$ to $r_i$ then $v \in V_P$. We say that $u$ and $v$ are *comparable* in $\mathfrak{T}$ provided that $u$ is either an ancestor or a descendant of $v$. We say that $u$ is a *child* of $v$, or that $v$ is the *parent* of $u$, provided that $u$ and $v$ are adjacent, and that $u$ is a descendant of $v$. A vertex $v \in V_T$ is a *leaf* in $\mathfrak{T}$ provided that $v$ has no descendants. We say that the rooted forest $\mathfrak{T}$ is *good* provided that for all semi-adjacent $u, v \in V_T$, one of $u$ and $v$ is a leaf in $\mathfrak{T}$. Finally, we say that the trigraph $T'$ is the *closure* of the rooted forest $\mathfrak{T} = (T, r_1, ..., r_k)$ provided that:

- $V_{T'} = V_T$;
- for all distinct $u, v \in V_{T'}$, $uv$ is an adjacent pair in $T'$ if and only if $u$ and $v$ are comparable in $\mathfrak{T}$;
- for all distinct $u, v \in V_{T'}$, $uv$ is a semi-adjacent pair in $T'$ if and only if $uv$ is a semi-adjacent pair in $T$.

**Homogeneous pairs of type one and two.** A tame homogeneous pair $(A, B)$ in a trigraph $G$ is said to be of *type one* in $G$ provided that the associated partition $(A, B, C, D, E, F)$ of $G$ satisfies all of the following:

(1) $A$ and $B$ are strongly stable sets;

(2) $C$, $D$, and $F$ are all non-empty;

(3) $E$ is empty;

(4) neither $C$ nor $D$ is strongly anti-complete to $F$.

A tame homogeneous pair $(A, B)$ in a trigraph $G$ is said to be of *type two* in $G$ provided there exists a good rooted forest $\mathfrak{T} = (T, r_1, ..., r_k)$ such that the partition $(A, B, C, D, E, F)$ of $G$ associated with $(A, B)$ satisfies all of the following:

(1) $A$ is a strongly stable set;

(2) $G[B]$ is the closure of $\mathfrak{T}$;

(3) if $a \in A$ is adjacent to $b \in B$, then $a$ is strongly adjacent to every descendant of $b$ in $\mathfrak{T}$;

(4) if all of the following hold:

- $u, v \in B$ and $u, v \in V_{T_i}$ for some $i \in \{1, ..., k\}$,
- $u$ is a child of $v$ in $\mathfrak{T}$,
- $P$ is the (unique) path in $T_i$ between $r_i$ and $v$,
- $X$ is the component of $T_i \setminus (V_P \setminus \{v\})$ that contains $u$ and $v$,
- $Y$ is the set of vertices of $X$ that are semi-adjacent to $v$,
- $a \in A$ is adjacent to $u$ and anti-adjacent to $v$;

then $a$ is strongly complete to $Y$ and to $B \setminus (V_X \cup V_P)$, and strongly anti-complete to $V_P \setminus \{v\}$;

(7) $C$, $D$, and $F$ are all non-empty;

(8) $E$ is empty;

(9) $C$ is not strongly anti-complete to $F$;

(10) $D$ is strongly anti-complete to $F$.

We will need the following result.

**3.3.1.** *Let $G$ be a trigraph, and let $(A, B)$ be a homogeneous pair of type one or two in one of $G$ and $\overline{G}$. Then $(A, B)$ is a good homogeneous pair in $G$.*

*Proof.* Recall that $(A, B)$ is a good homogeneous pair in $G$ if and only if $(A, B)$ is a good homogeneous pair in $\overline{G}$. So we may assume that $(A, B)$ is a homogeneous pair of type one or two in $G$. Now, we need to prove the following:

- neither $G[A]$ nor $G[B]$ contains a three-edge path;
- there does not exist a path $v_1 - v_2 - v_3 - v_4$ in $G$ such that $v_1, v_4 \in A$ and $v_2, v_3 \in B$;
- there does not exist a path $v_1 - v_2 - v_3 - v_4$ in $G$ such that $v_1, v_4 \in B$ and $v_2, v_3 \in A$.

If $(A, B)$ is a homogeneous pair of type one, then $A$ and $B$ are both stable, and the result is immediate. So assume that $(A, B)$ is a homogeneous pair of type two. Then $A$ is stable, and so $G[A]$ contains no three-edge path. Furthermore, there is no path $v_1 - v_2 - v_3 - v_4$ in $G$ with $v_1, v_4 \in B$ and $v_2, v_3 \in A$. Let $\mathfrak{T} = (T, r_1, ..., r_k)$ be a good rooted forest such that $G[B]$ is the closure of $\mathfrak{T}$, as in the definition of a homogeneous pair of type two.

Suppose that $v_1 - v_2 - v_3 - v_4$ is a three-edge path in $G[B]$; then $v_1, v_2, v_3, v_4 \in V_{T_i}$ for some component $T_i$ of $T$. Since $v_1 - v_2 - v_3$ is a path, $v_2$ is comparable to both $v_1$ and $v_3$ in $\mathfrak{T}$, and either $v_1$ and $v_3$ are not comparable in $\mathfrak{T}$ or there exist distinct $i, j \in \{1, 2\}$ such that $v_i$ is a leaf in $\mathfrak{T}$ and $v_i$ is a child of and is semi-adjacent to $v_j$; it then easily follows that $v_2$ is an ancestor of both $v_1$ and $v_3$. Similarly, since $v_2 - v_3 - v_4$ is a path, $v_3$ is an ancestor of both $v_2$ and $v_4$. But then $v_2$ is an ancestor of $v_3$, and $v_3$ is an ancestor of $v_2$, which is impossible. Thus, $G[B]$ contains no three-edge path.

Suppose now that $v_1 - v_2 - v_3 - v_4$ is a three-edge path in $G$ with $v_1, v_4 \in A$ and $v_2, v_3 \in B$. Then $v_2$ and $v_3$ are comparable in $\mathfrak{T}$; by symmetry, we may assume that $v_3$ is a descendant of $v_2$. But then the fact that $v_1$ is adjacent to $v_2$ implies that $v_1$ is strongly adjacent to $v_3$, which contradicts the fact that $v_1 - v_2 - v_3 - v_4$ is a path. $\square$

We now give the definition of an elementary expansion of a trigraph, and prove the main result of this section.

**Elementary expansions.** Let $H$ and $G$ be trigraphs. We say that $G$ is an *elementary expansion* of $H$ provided that $V_G = \bigcup_{v \in V_H} X_v$, where the $X_v$'s are non-empty and pairwise disjoint, and all of the following hold:

(1) if $u, v \in V_H$ are strongly adjacent, then $X_u$ is strongly complete to $X_v$;

(2) if $u, v \in V_H$ are strongly anti-adjacent, then $X_u$ is strongly anti-complete to $X_v$;

(3) if $v \in V_H$ is not an endpoint of any semi-adjacent pair of type one or two, or of complement type one or two, then $|X_v| = 1$;

(4) if $u, v \in V_H$ are semi-adjacent, and neither $(u, v)$ nor $(v, u)$ is a semi-adjacent pair of type one or two, or of complement type one or two, then the unique vertex of $X_u$ is semi-adjacent to the unique vertex of $X_v$;

(5) if $(u, v)$ is a semi-adjacent pair of type one or two in $H$, then either $|X_u| = |X_v| = 1$ and the unique vertex of $X_u$ is semi-adjacent to the unique vertex of $X_v$, or $(X_u, X_v)$ is a homogeneous pair of type one or two, respectively, in $G$;

(6) if $(u, v)$ is a semi-adjacent pair of complement type one or two in $H$, then either $|X_u| = |X_v| = 1$ and the unique vertex of $X_u$ is semi-adjacent to the unique vertex of $X_v$, or $(X_u, X_v)$ is a homogeneous pair of type one or two, respectively, in $\overline{G}$;

Note that every trigraph is an elementary expansion of itself.

**3.3.2.** *Let $G$ and $H$ be trigraphs, and assume that $G$ is an elementary expansion of $H$. Then $G$ is Berge if and only if $H$ is Berge.*

*Proof.* The 'only if' part follows from the fact that every realization of $H$ is an induced subgraph of some realization of $G$. To prove the 'if' part, we assume that $H$ is Berge. Suppose that $G$ is not Berge, and let $W$ be the vertex-set of an odd hole or an odd anti-hole in $G$. By 3.3.1 and 3.2.1, we have that $|W \cap X_v| \leq 1$ for all $v \in V_H$. But then $\{v \in V_H \mid W \cap X_v \neq \emptyset\}$ is the vertex-set of an odd hole or an odd anti-hole in $H$, which contradicts the assumption that $H$ is Berge. $\square$

## 3.4 Class $\mathcal{T}_1$

In this section, we state the definition of the class $\mathcal{T}_1$ from [10], and we derive the class $\mathcal{T}_1^*$ of all Berge trigraphs in $\mathcal{T}_1$. The section is organized as follows. We first define 'clique connectors' and 'tulips.' Clique connectors can conveniently be thought of as the basic 'building blocks' of trigraphs in $\mathcal{T}_1$ and $\mathcal{T}_1^*$. A clique connector consists of a bipartite trigraph and a strong clique that 'attaches' to the bipartite trigraph in a certain specified way; a tulip is a special kind of clique connector. We next introduce trigraphs called 'tulip beds,' which consist of a bipartite trigraph and an unlimited number of strong cliques that 'attach' to the bipartite trigraph as partially overlapping tulips. We prove that each tulip bed is Berge (see 3.4.6). We then define 'melts' (which are tulip beds and therefore Berge), and trigraphs that 'admit an $H$-structure' for some 'usable' graph $H$. The class $\mathcal{T}_1$ is defined to be the collection of all melts and all trigraphs that admit an $H$-structure for some usable graph $H$. Finally, we define the subclass $\mathcal{T}_1^*$ of $\mathcal{T}_1$, and to complete the section, we prove that every trigraph in $\mathcal{T}_1^*$ is a tulip bed (and therefore Berge), and that every Berge trigraph in $\mathcal{T}_1$ is in $\mathcal{T}_1^*$. (However, we note that not every tulip bed is bull-free, and consequently, the class $\mathcal{T}_1^*$ is only a proper subclass of the class of all tulip beds.)

**Clique connectors.** Let $G$ be a trigraph such that $V_G = K \cup A \cup B \cup C \cup D$, where $K$, $A$, $B$, $C$, and $D$ are pairwise disjoint. Assume that $K = \{k_1, ..., k_t\}$ is a strong clique, and that $A$, $B$, $C$, and $D$ are strongly stable sets. Let $A = \bigcup_{i=1}^{t} A_i$, $B = \bigcup_{i=1}^{t} B_i$, $C = \bigcup_{i=1}^{t} C_i$, and $D = \bigcup_{i=1}^{t} D_t$, and assume that $A_1, ..., A_t$, $B_1, ..., B_t$, $C_1, ..., C_t$, $D_1, ..., D_t$ are pairwise disjoint. Assume that for all $i \in \{1, ..., t\}$, the following hold:

(1) $A_i$ is strongly complete to $\{k_1, ..., k_{i-1}\}$;

(2) $A_i$ is complete to $\{k_i\}$;

(3) $A_i$ is strongly anti-complete to $\{k_{i+1}, ..., k_t\}$;

(4) $B_i$ is strongly complete to $\{k_{t-i+2}, ..., k_t\}$;

(5) $B_i$ is complete to $\{k_{t-i+1}\}$;

(6) $B_i$ is strongly anti-complete to $\{k_1, ..., k_{t-i}\}$.

For each $i \in \{1, ..., t\}$, let $A_i'$ be the set of all vertices in $A_i$ that are semi-adjacent to $k_i$, and let $B_i'$ be the set of all vertices in $B_i$ that are semi-adjacent to $k_{t-i+1}$ (thus, $|A_i'| \leq 1$

and $|B_i'| \leq 1$). Next, assume that:

(7) if there exist some $i, j \in \{1, ..., t\}$ such that $i + j \neq t$ and $A_i$ is not strongly complete to $B_j$, then $|K| = |A| = |B| = 1$, and the unique vertex of $A$ is semi-adjacent to the unique vertex of $B$;

(8) for all $i \in \{1, ..., t\}$, $A_i'$ is strongly complete to $B_{t-i}$, $B_{t-i}'$ is strongly complete to $A_i$, and the adjacency between $A_i \setminus A_i'$ and $B_{t-i} \setminus B_{t-i}'$ is arbitrary;

(9) $A \cup K$ is strongly anti-complete to $D$, and $B \cup K$ is strongly anti-complete to $C$;

(10) for all $i \in \{1, ..., t\}$, $C_i$ is strongly complete to $\bigcup_{j=1}^{i-1} A_j$ and strongly anti-complete to $\bigcup_{j=i+1}^{t} A_j$;

(11) for all $i \in \{1, ..., t\}$, $C_i$ is strongly complete to $A_i'$, every vertex of $C_i$ has a neighbor in $A_i$, and otherwise the adjacency between $C_i$ and $A_i \setminus A_i'$ is arbitrary;

(12) for all $i \in \{1, ..., t\}$, $D_i$ is strongly complete to $\bigcup_{j=1}^{i-1} B_j$ and strongly anti-complete to $\bigcup_{j=i+1}^{t} B_j$;

(13) for all $i \in \{1, ..., t\}$, $D_i$ is strongly complete to $B_i'$, every vertex of $D_i$ has a neighbor in $B_i$, and otherwise the adjacency between $D_i$ and $B_i \setminus B_i'$ is arbitrary;

(14) for all $i, j \in \{1, ..., t\}$, if $i + j > t$ then $C_i$ is strongly complete to $D_j$, and otherwise the adjacency between $C_i$ and $D_j$ is arbitrary;

(15) $A_t$ and $B_t$ are both non-empty.

We then say that $G$ is a $(K, A, B, C, D)$-*clique connector*. If for all $i, j \in \{1, ..., t\}$ such that $i + j \neq t$ we have that $A_i$ is strongly complete to $B_j$, then we say that $G$ is a *non-degenerate* $(K, A, B, C, D)$-clique connector; otherwise, we say that $G$ is *degenerate*. If $C$ and $D$ are both empty, and for all $i, j \in \{1, ..., t\}$ such that $i + j \neq t$ we have that $A_i$ is strongly complete to $B_j$, then we say that $G$ is a $(K, A, B)$-*tulip*.

We say that a trigraph $G$ is a *clique connector* provided that there exist some $K, A, B, C, D$ such that $G$ is a $(K, A, B, C, D)$-clique connector; we say that $G$ is a *degenerate* (respectively: *non-degenerate*) clique connector provided that there exist some $K, A, B, C, D$ such that $G$ is a degenerate (respectively: non-degenerate) $(K, A, B, C, D)$-clique connector. We say that $G$ is a *tulip* if there exist some $K, A, B$ such that $G$ is a $(K, A, B)$-tulip.

We observe that $G$ is a $(K, A, B, C, D)$-clique connector if and only if $G$ is a $(K, B, A, D, C)$-clique connector; similarly, $G$ is a $(K, A, B)$-tulip if and only if $G$ is a $(K, B, A)$-tulip; we will exploit this symmetry throughout the section. We also note that if $G$ is a $(K, A, B, C, D)$-clique connector, then $G[A \cup B \cup C \cup D]$ is a bipartite trigraph with bipartition $(A \cup D, B \cup C)$. Finally, we note that $G$ is a $(K, A, B)$-tulip if and only if $G$ is a non-degenerate $(K, A, B, \emptyset, \emptyset)$-clique connector.

All non-degenerate clique connectors (and therefore, all tulips) are Berge, as we will see in a slightly more general setting later in the section (see 3.4.6 and the comment after it). For now, we prove three results about clique connectors and tulips. The first (3.4.1) gives a necessary and sufficient condition for a degenerate clique connector to be Berge; the second (3.4.2) states that each Berge degenerate clique connector becomes non-degenerate after relabeling; and the third (3.4.3) is a technical lemma about tulips that will be used throughout this section.

**3.4.1.** *Let $G$ be a degenerate $(K, A, B, C, D)$-clique connector. Then $G$ is Berge if and only if at least one of $C$ and $D$ is empty.*

*Proof.* Since $G$ is degenerate, we can set $K = \{k_1\}$, $A = A_1 = \{a\}$, and $B = B_1 = \{b\}$, with $a$ and $b$ semi-adjacent. Furthermore, by axiom (14) from the definition of a clique connector, we know that $C$ is strongly complete to $D$. Now, for the 'only if' part, we observe that if both $C$ and $D$ are non-empty with some $c \in C$ and $d \in D$, then $k_1 - a - c - d - b - k_1$ is an odd hole in $G$, and so $G$ is not Berge. For the 'if' part, suppose that at least one of $C$ and $D$ is empty. If both $C$ and $D$ are empty, then $|V_G| = 3$ and $G$

is Berge. So suppose that exactly one of $C$ and $D$ is empty; by symmetry, we may assume that $C \neq \emptyset$ and $D = \emptyset$. Now, we claim that $C$ is a homogeneous set in $G$. First, we know by axiom (11) from the definition of a $(K, A, B, C, D)$-clique connector that every vertex in $C$ has a neighbor in $A$; since $A = \{a\}$ and $a$ is semi-adjacent to $b \notin C$, it follows that $C$ is strongly complete to $A$. Second, by axiom (9), we know that $C$ is strongly anti-complete to $K \cup B$. Thus, $C$ is a homogeneous set in $G$, as claimed. Since $|K| = |A| = |B| = 1$, and since $D = \emptyset$, it follows that $G$ is obtained by substituting the trigraph $G[C]$ for a vertex in a 4-vertex trigraph. $G[C]$ is Berge because $C$ is a strongly stable set in $G$, and clearly, every 4-vertex trigraph is Berge. By 2.2.2 then, $G$ is Berge. □

**3.4.2.** *If $G$ is a degenerate $(K, A, B, C, \emptyset)$-clique connector, then $G$ is a non-degenerate $(B, A, K, C, \emptyset)$-clique connector, and if $G$ is a degenerate $(K, A, B, \emptyset, D)$-clique connector, then $G$ is a non-degenerate $(A, K, B, \emptyset, D)$-clique connector.*

*Proof.* This is immediate from the definition. □

**3.4.3.** *Let $G$ be a $(K, A, B)$-tulip, and let $p_1 - p_2 - p_3 - p_4$ be a path in $G$ such that $p_2, p_3 \in K$. Then either $p_1 \in A$ and $p_4 \in B$, or $p_1 \in B$ and $p_4 \in A$.*

*Proof.* Since $K$ is a strong clique, we know that $p_1, p_4 \notin K$; thus, $p_1, p_4 \in A \cup B$. Now, suppose that neither of the stated outcomes holds. By symmetry then, we may assume that $p_1, p_4 \in A$. Set $K = \{k_1, ..., k_t\}$ as in the definition of a tulip, and set $p_2 = k_i$ and $p_3 = k_j$; by symmetry, we may assume that $i < j$. Since $p_4$ is adjacent to $p_3 = k_j$, we know that $p_4$ is strongly complete to $\{k_1, ..., k_{j-1}\}$, and so in particular, $p_4$ is strongly adjacent to $p_2 = k_i$, which is a contradiction. □

**Tulip beds.** We say that a trigraph $G$ is a *tulip bed* provided that either $G$ is bipartite, or $V_G$ can be partitioned into (non-empty) sets $F_1, F_2, Y_1, ..., Y_s$ (for some integer $s \geq 1$) such that all of the following hold:

(1) $F_1$ and $F_2$ are strongly stable sets;

(2) $Y_1, ..., Y_s$ are strong cliques, pairwise strongly anti-complete to each other;

(3) for all $v \in F_1 \cup F_2$, $v$ has neighbors in at most two of $Y_1, ..., Y_s$;

(4) for all adjacent $v_1 \in F_1$ and $v_2 \in F_2$, $v_1$ and $v_2$ have common neighbors in at most one of $Y_1, ..., Y_s$;

(5) for all $l \in \{1, ..., s\}$, if $X_l$ is the set of all vertices in $F_1 \cup F_2$ with a neighbor in $Y_l$, then $G[Y_l \cup X_l]$ is a $(Y_l, X_l \cap F_1, X_l \cap F_2)$-tulip.

As we stated at the beginning of this section, not all tulip beds are bull-free (for example, a bull that contains no semi-adjacent pairs is easily seen to be a tulip bed). However, all tulip beds are Berge, and we now turn to proving this fact. We begin with some technical lemmas.

**3.4.4.** *Let $G$ be a tulip bed, and let $F_1, F_2, Y_1, ..., Y_s, X_1, ..., X_s$ be as in the definition of a tulip bed. Let $v_1 \in F_1$, $v_2 \in F_2$, and $l \in \{1, ..., s\}$, and assume that $v_1$ and $v_2$ have a common neighbor in $Y_l$. Then both of the following hold:*

- *$v_1v_2$ is a strongly adjacent pair;*
- *$v_1$ and $v_2$ have no common anti-neighbor in $Y_l$.*

*Proof.* Clearly, $v_1, v_2 \in X_l$. Set $K = Y_l$, $A = X_l \cap F_1$, and $B = X_l \cap F_2$. Now $G[Y_l \cup X_l]$ is a $(K, A, B)$-tulip, $v_1 \in A$, $v_2 \in B$, and $v_1$ and $v_2$ have a common neighbor in $K$. Set $K = \{k_1, ..., k_t\}$, $A = \bigcup_{i=1}^t A_i$, and $B = \bigcup_{i=1}^t B_i$ as in the definition of a $(K, A, B)$-tulip. Fix $i \in \{1, ..., t\}$ such that $k_i$ is a common neighbor of $v_1$ and $v_2$. Fix $p, q \in \{1, ..., t\}$ such that $v_1 \in A_p$ and $v_2 \in B_q$. Since $v_1 \in A_p$ is adjacent to $k_i$, we know by axioms (1), (2), and (3) from the definition of a $(K, A, B)$-tulip that $i \leq p$; and since $v_2 \in B_q$ is adjacent to $k_i$, we know by axioms (4), (5), and (6) that $t - q + 1 \leq i$. Thus, $t - q + 1 \leq p$, and so $p + q \geq t + 1$. In particular, $p + q \neq t$, and so $A_p$ is strongly complete to $B_q$ (this follows from the fact that tulips are non-degenerate clique connectors). Since $v_1 \in A_p$ and $v_2 \in B_q$, it follows that $v_1v_2$ is a strongly adjacent pair.

It remains to show that $v_1$ and $v_2$ do not have a common anti-neighbor in $Y_l$. Suppose otherwise; fix $j \in \{1, ..., t\}$ such that $k_j$ is anti-adjacent to both $v_1$ and $v_2$. Since $k_j$ is anti-adjacent to $v_1 \in A_p$, we know by axioms (1), (2), and (3) from the definition of a $(K, A, B)$-tulip that $p \leq j$; and since $k_j$ is anti-adjacent to $v_2 \in B_q$, we know by axioms (4), (5), and (6) that $j \leq t-q+1$. But now $p \leq t-q+1$, and so $p+q \leq t+1$. We showed before that $p+q \geq t+1$, and so it follows that $p+q = t+1$, and consequently, that $j = p = t-q+1$. Since $k_j = k_p$ is anti-adjacent to $v_1 \in A_p$, axiom (2) from the definition of a $(K, A, B)$-tulip implies that $k_j$ is semi-adjacent to $v_1$; similarly, since $k_j = k_{t-q+1}$ is anti-adjacent to $v_2 \in B_q$, axiom (5) implies that $k_j$ is semi-adjacent to $v_2$. But now $k_j$ is semi-adjacent to both $v_1$ and $v_2$, which is impossible by the definition of a trigraph. □

We remark that a result very similar to 3.4.4 was proven in [8] (see the proof of 3.1, statements (1) and (3), from [8]).

**3.4.5.** *No tulip bed contains a three-edge path with a center.*

*Proof.* Let $G$ be a tulip bed, and suppose that $p_1 - p_2 - p_3 - p_4$ is a path with a center $p_c$ in $G$. Since $G$ contains a triangle, $G$ is not bipartite. Then let $F_1, F_2, Y_1, ..., Y_s, X_1, ..., X_s$ be as in the definition of a tulip bed.

Our first goal is to show that $p_c \notin F_1 \cup F_2$. Suppose otherwise. Since $\{p_c, p_2, p_3\}$ is a triangle, we know that $p_2$ and $p_3$ cannot both lie in $F_1 \cup F_2$; by symmetry, we may assume that $p_2 \notin F_1 \cup F_2$; thus, $p_2 \in Y_l$ for some $l \in \{1, ..., s\}$. We claim that $p_3 \in Y_l$. Suppose otherwise. Since $p_2p_3$ is an adjacent pair and the strong cliques $Y_1, ..., Y_s$ are strongly anti-complete to each other, this means that $p_3 \in F_1 \cup F_2$. Now, $\{p_c, p_3, p_4\}$ is a triangle, and so $p_4 \notin F_1 \cup F_2$. Since $p_c, p_3 \in F_1 \cup F_2$ are adjacent with a common neighbor $p_2 \in Y_l$, axiom (4) from the definition of a tulip bed implies that all common neighbors of $p_c$ and $p_3$ lie in $Y_l$, and so $p_4 \in Y_l$. But then $p_2, p_4 \in Y_l$, which is impossible since $p_2p_4$ is an anti-adjacent pair and $Y_l$ is a strong clique. Thus, $p_3 \in Y_l$. Now, $p_2, p_3 \in Y_l$, $Y_l$ is a strong clique, and $p_1p_3$ and $p_2p_4$ are anti-adjacent pairs; thus, $p_1, p_4 \notin Y_l$, and therefore, $p_1, p_4 \in F_1 \cup F_2$. Clearly, $p_c, p_1, p_4 \in X_l$; by symmetry, we may assume that $p_c \in X_l \cap F_1$. Since $p_c$ is complete to $\{p_1, p_4\}$ and $F_1$ is strongly stable, it follows that $p_1, p_4 \in X_l \cap F_2$. But then the path $p_1 - p_2 - p_3 - p_4$ contradicts 3.4.3. This proves that $p_c \notin F_1 \cup F_2$.

Let $l \in \{1, ..., s\}$ be such that $p_c \in Y_l$. Since $p_c \in Y_l$ is complete to $\{p_1, p_2, p_3, p_4\}$, we know that $p_1, p_2, p_3, p_4 \in Y_l \cup X_l$. Since $p_1p_4$ is an anti-adjacent pair, $p_1$ and $p_4$ cannot both lie in $Y_l$; by symmetry, we may assume that $p_1 \in X_l \cap F_1$. Since $p_1p_2$ is an adjacent pair, there are two cases to consider: when $p_2 \in Y_l$, and when $p_2 \in X_l \cap F_2$. Suppose first that $p_2 \in Y_l$. Since $p_2p_4$ is an anti-adjacent pair, this means that $p_4 \notin Y_l$; since $p_1p_4$ is an anti-adjacent pair with a common neighbor in $Y_l$, and since $p_1 \in X_l \cap F_1$, 3.4.4 implies that $p_4 \in X_l \cap F_1$. Since $p_3p_4$ is an adjacent pair, we know that $p_3 \notin X_l \cap F_1$; and since $p_1 \in X_l \cap F_1$ and $p_3$ are anti-adjacent with a common neighbor in $Y_l$, we know by 3.4.4 that $p_3 \notin X_l \cap F_2$. Thus, $p_3 \in Y_l$. But then the path $p_1 - p_2 - p_3 - p_4$ contradicts 3.4.3. Thus, $p_2 \in X_l \cap F_2$. The fact that $p_1p_4$ is an anti-adjacent pair with a common neighbor $p_c \in Y_l$, together with the fact that $p_1 \in X_l \cap F_1$, implies (by 3.4.4) that $p_4 \notin X_l \cap F_2$. Similarly, since $p_2p_4$ is an anti-adjacent pair with a common neighbor $p_c \in Y_l$, and since $p_2 \in X_l \cap F_2$, we have that $p_4 \notin X_l \cap F_1$. Finally, since $p_1 \in X_l \cap F_1$ and $p_2 \in X_l \cap F_2$ have a common neighbor $p_c \in Y_l$, 3.4.4 implies that $p_1$ and $p_2$ have no common anti-neighbor in $Y_l$, and so $p_4 \notin Y_l$. But then $p_4 \notin Y_l \cup X_l$, which is a contradiction. □

**3.4.6.** *Each tulip bed is Berge.*

*Proof.* Let $G$ be a tulip bed. Since every anti-hole of length at least seven contains a three-edge path with a center, 3.4.5 implies that $G$ contains no anti-hole of length at least seven. Since each anti-hole of length five is also a hole of length five, this reduces our problem to proving that $G$ contains no odd holes. If $G$ is bipartite, then the result is immediate; so assume that $G$ is not bipartite. Now let $F_1, F_2, Y_1, ..., Y_s, X_1, ..., X_s$ be as in the definition of a tulip bed.

Suppose that $w_0 - w_1 - ... - w_{2k} - w_0$ (with indices in $\mathbb{Z}_{2k+1}$ for some integer $k \geq 2$) is an odd hole in $G$, and set $W = \{w_0, w_1, ..., w_{2k}\}$. We will obtain a contradiction by showing that $G[W]$ is bipartite. Note that it suffices to show that for all $l \in \{1, ..., s\}$, $G[W \cap (Y_l \cup F_1 \cup F_2)]$ is bipartite with some bipartition $(F_1^l, F_2^l)$ such that $W \cap F_1 \subseteq F_1^l$ and $W \cap F_2 \subseteq F_2^l$, for then the fact that $Y_1, ..., Y_s$ are pairwise strongly anti-complete to each other will imply that $G[W]$ is bipartite with bipartition $(\bigcup_{l=1}^s F_1^l, \bigcup_{l=1}^s F_2^l)$.

We begin by showing that for all $l \in \{1, ..., s\}$ and $i \in \mathbb{Z}_{2k+1}$ such that $w_i \in Y_l$, $w_i$ is strongly anti-complete to at least one of $W \cap F_1$ and $W \cap F_2$. Suppose otherwise. Fix some $l \in \{1, ..., s\}$ and $i \in \mathbb{Z}_{2k+1}$ such that $w_i \in Y_l$, and $w_i$ has neighbors in both $W \cap F_1$ and $W \cap F_2$. First, note that $w_i$ is anti-complete to at least one of $W \cap F_1$ and $W \cap F_2$; indeed if there existed some $i_1, i_2 \in \mathbb{Z}_{2k+1}$ such that $w_{i_1} \in W \cap F_1$, $w_{i_2} \in W \cap F_2$, and $w_i$ is strongly adjacent to both $w_{i_1}$ and $w_{i_2}$, then (by 3.4.4) $w_{i_1}$ and $w_{i_2}$ would be strongly adjacent, and $\{w_i, w_{i_1}, w_{i_2}\}$ would be a strong triangle in $G[W]$, which is impossible. Now suppose that there exist some $i_1, i_2 \in \mathbb{Z}_{2k+1}$ such that $w_{i_1} \in W \cap F_1$, $w_{i_2} \in W \cap F_2$, $w_i$ is adjacent to both $w_{i_1}$ and $w_{i_2}$ and semi-adjacent to one of them. By symmetry, we may assume that $w_i$ is strongly adjacent to $w_{i_1}$ and semi-adjacent to $w_{i_2}$; thus, $w_i$ is anti-complete to $W \cap F_2$. Since $w_{i_1} \in F_1$ and $w_{i_2} \in F_2$ have a common neighbor $w_i \in Y_l$, 3.4.4 implies that $w_{i_1}w_{i_2}$ is a strongly adjacent pair. Since $w_iw_{i_1}$ and $w_{i_1}w_{i_2}$ are strongly adjacent pairs, by symmetry, we may assume that $w_{i_1} = w_{i+1}$ and $w_{i_2} = w_{i+2}$. Since $w_iw_{i+2}$ is a semi-adjacent pair, $w_{i-1}w_i$ is a strongly adjacent pair; as $w_i$ is anti-complete to $W \cap F_2$, it follows that $w_{i-1} \notin F_2$. Next, the fact that $w_{i+1} \in F_1$ and $w_{i+2} \in F_2$ have a common neighbor in $Y_l$ implies (by 3.4.4) that $w_{i+1}$ and $w_{i+2}$ do not have a common anti-neighbor in $Y_l$; thus, the fact that $w_{i-1}$ is anti-adjacent to both $w_{i+1}$ and $w_{i+2}$ implies that $w_{i-1} \notin Y_l$. It follows that $w_{i-1} \in F_1$. Then since $w_{i-1} \in F_1$ and $w_{i+2} \in F_2$ have a common neighbor $w_i \in Y_l$, we know (by 3.4.4) that $w_{i-1}w_{i+2}$ is a strongly adjacent pair, which is impossible. Thus, $w_i$ is strongly anti-complete to at least one of $W \cap F_1$ and $W \cap F_2$.

Next, fix $l \in \{1, ..., s\}$. We need to show that $G[W \cap (Y_l \cup F_1 \cup F_2)]$ is bipartite with some bipartition $(F_1^l, F_2^l)$ such that $W \cap F_1 \subseteq F_1^l$ and $W \cap F_2 \subseteq F_2^l$. Since $Y_l$ is a strong clique, we know that $|W \cap Y_l| \leq 2$. If $W \cap Y_l = \emptyset$, then $G[W \cap (Y_l \cup F_1 \cup F_2)]$ is bipartite with bipartition $(W \cap F_1, W \cap F_2)$, and we are done. So assume that $1 \leq |W \cap Y_l| \leq 2$.

Suppose first that $|W \cap Y_l| = 1$, say $W \cap Y_l = \{w_i\}$. By the above, $w_i$ is strongly anti-complete to at least one of $W \cap F_1$ and $W \cap F_2$. By symmetry, we may assume that $w_i$ is strongly anti-complete to $W \cap F_1$. But then $G[W \cap (Y_l \cup F_1 \cup F_2)]$ is bipartite with bipartition $((W \cap F_1) \cup \{w_i\}, W \cap F_2)$.

Suppose now that $|W \cap Y_l| = 2$; since $Y_l$ is a strong clique, this means that $W \cap Y_l = \{w_i, w_{i+1}\}$ for some $i \in \mathbb{Z}_{2k+1}$. Clearly then, $w_{i-1}, w_{i+2} \in X_l$. Now, $w_{i-1}-w_i-w_{i+1}-w_{i+2}$ is a three-edge path with $w_i, w_{i+1} \in Y_l$ and $w_{i-1}, w_{i+2} \in F_1 \cup F_2$; it then follows from 3.4.3 that either $w_{i-1} \in F_1$ and $w_{i+2} \in F_2$, or $w_{i-1} \in F_2$ and $w_{i+2} \in F_1$; by symmetry, we may assume that the former holds. Then since each of $w_i$ and $w_{i+1}$ is strongly anti-complete to at least one of $W \cap F_1$ and $W \cap F_2$, it follows that $w_i$ is strongly anti-complete to $W \cap F_2$, and $w_{i+1}$ is strongly anti-complete to $W \cap F_1$. Thus, $F[W \cap (Y_l \cup F_1 \cup F_2)]$ is bipartite with bipartition $((W \cap F_1) \cup \{w_{i+1}\}, (W \cap F_2) \cup \{w_i\})$. This completes the argument. $\square$

We observe that every tulip and every non-degenerate clique-connector is a tulip bed and therefore Berge.

**Melts.** Let $G$ be a trigraph. Assume that $V_G = K \cup M \cup A \cup B$, where $K$ and $M$ are strong cliques, $A$ and $B$ are strongly stable sets, and $K$, $M$, $A$, and $B$ are pairwise disjoint. Assume that $|A| \geq 2$ and $|B| \geq 2$, and that $K = \{k_1, ..., k_m\}$ and $M = \{m_1, ..., m_n\}$. Let $A = \bigcup_{i=0}^{m} \bigcup_{j=0}^{n} A_{i,j}$, where the $A_{i,j}$'s are pairwise disjoint; and let $B = \bigcup_{i=0}^{m} \bigcup_{j=0}^{n} B_{i,j}$, where the $B_{i,j}$'s are pairwise disjoint. Assume that $A_{0,0} = B_{0,0} = \emptyset$. Assume that for all $i \in \{1, ..., m\}$, $A_{i,0} = \bigcup_{j=0}^{n} A_{i,0}^j$, where the $A_{i,0}^j$'s are pairwise disjoint, and assume that for all $j \in \{1, ..., n\}$, $A_{0,j} = \bigcup_{i=0}^{m} A_{0,j}^{i}$, where the $A_{0,j}^{i}$'s are pairwise disjoint. Similarly, assume that for all $i \in \{1, ..., m\}$, $B_{i,0} = \bigcup_{j=0}^{n} B_{i,0}^{j}$, where the $B_{i,0}^{j}$'s are pairwise disjoint, and assume that for all $j \in \{1, ..., n\}$, $B_{0,j} = \bigcup_{i=0}^{m} B_{0,j}^{i}$, where the $B_{0,j}^{i}$'s are pairwise disjoint. Assume also that:

(1) $K$ is strongly anti-complete to $M$;

(2) for all $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$, $A_{i,j}$ is:

- strongly complete to $\{k_1, ..., k_{i-1}\} \cup \{m_{n-j+2}, ..., m_n\}$,
- complete to $\{k_i, m_{n-j+1}\}$,
- strongly anti-complete to $\{k_{i+1}, ..., k_m\} \cup \{m_1, ..., m_{n-j}\}$;

(3) for all $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$, $B_{i,j}$ is:

- strongly complete to $\{k_{m-i+2}, ..., k_m\} \cup \{m_1, ..., m_{j-1}\}$,
- complete to $\{k_{m-i+1}, m_j\}$,
- strongly anti-complete to $\{k_1, ..., k_{m-i}\} \cup \{m_{j+1}, ..., m_n\}$;

(4) for all $i \in \{1, ..., m\}$, $A_{i,0}$ is:

- strongly complete to $\{k_1, ..., k_{i-1}\}$,
- complete to $\{k_i\}$,
- strongly anti-complete to $\{k_{i+1}, ..., k_m\} \cup M$;

(5) for all $j \in \{1, ..., n\}$, $A_{0,j}$ is:

- strongly complete to $\{m_{n-j+2}, ..., m_n\}$,
- complete to $\{m_{n-j+1}\}$,
- strongly anti-complete to $K \cup \{m_1, ..., m_{n-j}\}$;

(6) for all $i \in \{1, ..., m\}$, $B_{i,0}$ is:

- strongly complete to $\{k_{m-i+2}, ..., k_m\}$,
- complete to $\{k_{m-i+1}\}$,
- strongly anti-complete to $\{k_1, ..., k_{m-i}\} \cup M$;

(7) for all $j \in \{1, ..., n\}$, $B_{0,j}$ is:

- strongly complete to $\{m_1, ..., m_{j-1}\}$,
- complete to $\{m_j\}$,
- strongly anti-complete to $K \cup \{m_{j+1}, ..., m_n\}$;

(8) the sets $\bigcup_{j=0}^{n} A_{m,j}$, $\bigcup_{i=0}^{m} A_{i,n}$, $\bigcup_{j=0}^{n} B_{m,j}$, and $\bigcup_{i=0}^{m} B_{i,n}$ are all non-empty;

(9) for all $i, i' \in \{0, ..., m\}$ and $j, j' \in \{0, ..., n\}$ such that $i < i'$ and $j < j'$, at least one of the sets $A_{i,j}$ and $A_{i',j'}$ is empty, and at least one of the sets $B_{i,j}$ and $B_{i',j'}$ is empty;

(10) for all $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$, $A_{i,j}$ is strongly complete to $B$, and $B_{i,j}$ is strongly complete to $A$;

(11) for all $i, i' \in \{1, ..., m\}$, $A_{i,0}$ is strongly complete to $B_{i',0}$;

(12) for all $j, j' \in \{1, ..., n\}$, $A_{0,j}$ is strongly complete to $B_{0,j'}$;

(13) for all $i \in \{1, ..., m\}$, $A^0_{i,0}$ is strongly anti-complete to $\bigcup_{j=1}^{n} B_{0,j}$, and every vertex of $A^0_{i,0}$ has a neighbor in $\bigcup_{i'=1}^{m} \bigcup_{j=1}^{n} B_{i',j}$;

(14) for all $j \in \{1, ..., n\}$, $A^0_{0,j}$ is strongly anti-complete to $\bigcup_{i=1}^{m} B_{i,0}$, and every vertex of $A^0_{0,j}$ has a neighbor in $\bigcup_{i=1}^{m} \bigcup_{j'=1}^{n} B_{i,j'}$;

(15) for all $i \in \{1, ..., m\}$, $B^0_{i,0}$ is strongly anti-complete to $\bigcup_{j=1}^{n} A_{0,j}$, and every vertex of $B^0_{i,0}$ has a neighbor in $\bigcup_{i'=1}^{m} \bigcup_{j=1}^{n} A_{i',j}$;

(16) for all $j \in \{1, ..., n\}$, $B^0_{0,j}$ is strongly anti-complete to $\bigcup_{i=1}^{n} A_{i,0}$, and every vertex of $B^0_{0,j}$ has a neighbor in $\bigcup_{i=1}^{m} \bigcup_{j'=1}^{n} A_{i,j'}$;

(17) for all $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$:

- every vertex of $A^i_{0,j}$ has a neighbor in $B_{i,0}$,
- $A^i_{0,j}$ is strongly complete to $\bigcup_{i'=1}^{i-1} B_{i',0}$,
- $A^i_{0,j}$ is strongly anti-complete to $\bigcup_{i'=i+1}^{m} B_{i',0}$,

- every vertex of $A^j_{i,0}$ has a neighbor in $B_{0,j}$,
- $A^j_{i,0}$ is strongly complete to $\bigcup_{j'=1}^{j-1} B_{0,j'}$,
- $A^j_{i,0}$ is strongly anti-complete to $\bigcup_{j'=j+1}^{n} B_{0,j'}$,

- every vertex of $B^i_{0,j}$ has a neighbor in $A_{i,0}$,
- $B^i_{0,j}$ is strongly complete to $\bigcup_{i'=1}^{i-1} A_{i',0}$,
- $B^i_{0,j}$ is strongly anti-complete to $\bigcup_{i'=i+1}^{m} A_{i',0}$,

- every vertex of $B^j_{i,0}$ has a neighbor in $A_{0,j}$,
- $B^j_{i,0}$ is strongly complete to $\bigcup_{j'=1}^{j-1} A_{0,j'}$,
- $B^j_{i,0}$ is strongly anti-complete to $\bigcup_{j'=j+1}^{n} A_{0,j'}$.

For all $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$:

(18) let $A'_{i,0}$ be the set of all vertices in $A_{i,0}$ that are semi-adjacent to $k_i$;

(19) let $A'_{0,j}$ be the set of all vertices of $A_{0,j}$ that are semi-adjacent to $m_{n-j+1}$;

(20) let $B'_{i,0}$ be the set of all vertices of $B_{i,0}$ that are semi-adjacent to $k_{m-i+1}$;

(21) let $B'_{0,j}$ be the set of all vertices of $B_{0,j}$ that are semi-adjacent to $m_j$.

Assume that:

(22) for all $i \in \{1, ..., m\}$, $A'_{i,0}$ is strongly complete to $\bigcup_{j=1}^{n} B^i_{0,j}$;

(23) for all $j \in \{1, ..., n\}$, $A'_{0,j}$ is strongly complete to $\bigcup_{i=1}^{m} B^j_{i,0}$;

(24) for all $i \in \{1, ..., m\}$, $B'_{i,0}$ is strongly complete to $\bigcup_{j=1}^{n} A^i_{0,j}$;

(25) for all $j \in \{1, ..., n\}$, $B'_{0,j}$ is strongly complete to $\bigcup_{i=1}^{m} A^j_{i,0}$.

Finally, assume that:

(26) there exist some $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$ such that at least one of $A_{i,j}$ and $B_{i,j}$ is non-empty;

(27) for all $i, i' \in \{1, ..., m\}$ and $j, j' \in \{1, ..., n\}$, if $i + i' \geq m + 1$ and $j + j' \geq n + 1$, then at least one of $A_{i,j}$ and $B_{i',j'}$ is empty.

Under these circumstances, we say that $G$ is a *melt*. We say that $G$ is an $A$-*melt* if $B_{i,j} = \emptyset$ for all $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$. We say that $G$ is a $B$-*melt* if $A_{i,j} = \emptyset$ for all $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$. We say that $G$ is a *double melt* if there exist $i, i' \in \{1, ..., m\}$ and $j, j' \in \{1, ..., n\}$ such that $A_{i,j} \neq \emptyset$ and $B_{i',j'} \neq \emptyset$.

**3.4.7.** *Every melt is a tulip bed, and consequently, every melt is Berge.*

*Proof.* Let $G$ be a melt; we use the notation from the definition of a melt. Set $F_1 = A$, $F_2 = B$, $Y_1 = K$, and $Y_2 = M$. Further, note that $\bigcup_{i=1}^{m} \bigcup_{j=0}^{n} (A_{i,j} \cup B_{i,j})$ is the set of all vertices in $F_1 \cup F_2 = A \cup B$ with a neighbor in $Y_1 = K$; set $X_1 = \bigcup_{i=1}^{m} \bigcup_{j=0}^{n} (A_{i,j} \cup B_{i,j})$. Similarly, note that $\bigcup_{i=0}^{m} \bigcup_{j=1}^{n} (A_{i,j} \cup B_{i,j})$ is the set of all vertices in $F_1 \cup F_2 = A \cup B$ with a neighbor in $Y_2 = M$, and set $X_2 = \bigcup_{i=0}^{m} \bigcup_{j=1}^{n} (A_{i,j} \cup B_{i,j})$. With this setup, it is easy to check that $G$ is a tulip bed, and we leave the details to the reader. Since each tulip bed is Berge (by 3.4.6), this implies that $G$ is Berge. □

In fact, it is possible to get a slightly stronger result: if $G$ is a melt, and $K$, $M$, $A$, and $B$ are as in the definition, then $G \smallsetminus K$ and $G \smallsetminus M$ are both non-degenerate clique connectors, as the reader can check. However, 3.4.7 is sufficiently strong for the purposes of this thesis.

**The class $\mathcal{T}_1$.** The *degree* of a vertex $v$ of a loopless graph $H$ ($H$ may possibly have parallel edges), denoted by $deg_H(v)$, is the number of edges of $H$ that are incident with $v$;

$v$ is an *isolated vertex* in $H$ provided that $deg_H(v) = 0$. We say that a (possibly empty) graph $H$ is *usable* provided that $H$ is loopless (possibly with parallel edges) and triangle-free, and that no vertex in $H$ is of degree greater than two.

Let $H$ be a usable graph, and let $G$ be a trigraph. Assume that there exists some $L \subseteq V_G$ and a map

$$h : V_H \cup E_H \cup (E_H \times V_H) \to 2^{V_G \smallsetminus L}$$

such that all of the following hold:

(1) for all distinct $x, y \in V_H \cup E_H \cup (E_H \times V_H)$, $h(x)$ and $h(y)$ are disjoint;

(2) $V_G \smallsetminus L = \bigcup h[V_H \cup E_H \cup (E_H \times V_H)]$;

(3) for every isolated vertex $v \in V_H$, $h(v) \neq \emptyset$;

(4) for every $e \in E_H$, $h(e) \neq \emptyset$;

(5) for every $e \in E_H$ and $v \in V_H$, $h(e, v) \neq \emptyset$ if and only if $e$ is incident with $v$;

(6) for all distinct $u, v \in V_H$, $h(u)$ is strongly anti-complete to $h(v)$;

(7) for all $v \in V_H$, $h(v)$ is a (possibly empty) strong clique;

(8) every vertex in $L$ has a neighbor in at most one set $h(v)$ with $v \in V_H$;

(9) $G[L \cup \bigcup_{e \in E_H} h(e)]$ is triangle-free;

(10) for every $e \in E_H$ and $a \in L$, $a$ is either strongly complete or strongly anti-complete to $h(e)$;

(11) for all distinct $e, f \in E_H$, $h(e)$ is either strongly complete or strongly anti-complete to $h(f)$, and if $e$ and $f$ share an endpoint, then $h(e)$ is strongly complete to $h(f)$;

(12) for every $e \in E_H$ and $v \in V_H$, $h(e)$ is strongly anti-complete to $h(v)$;

(13) for every $v \in V_H$, let $S_v$ be the set of all vertices in $L$ with a neighbor in $h(v)$, and let $T_v$ be the set of all vertices in $(L \cup \bigcup_{e \in E_H} h(e)) \setminus S_v$ with a neighbor in $S_v$; then either:

- $h(v) = \emptyset$ (in which case $S_v = T_v = \emptyset$), and we set $A_v = B_v = C_v = D_v = \emptyset$, or
- there exist pairwise disjoint $A_v, B_v, C_v, D_v$ such that $S_v = A_v \cup B_v$, $T_v = C_v \cup D_v$, and $G[h(v) \cup S_v \cup T_v]$ is a $(h(v), A_v, B_v, C_v, D_v)$-clique connector;

(14) for every $v \in V_H$, if $deg_H(v) \geq 1$ and $h(v) \neq \emptyset$, then $G[h(v) \cup S_v \cup T_v]$ is a non-degenerate $(h(v), A_v, B_v, C_v, D_v)$-clique connector;

(15) for all distinct $e, f \in E_H$ and $v \in V_H$, $h(e, v)$ is strongly complete to $h(f, v)$;

(16) for all (not necessarily distinct) $e, f \in E_H$ and distinct $u, v \in V_H$, $h(e, u)$ is strongly anti-complete to $h(f, v)$.

(17) for all $e \in E_H$ and $v \in V_H$, $h(e, v)$ is strongly complete to $h(v)$;

(18) for all $e \in E_H$ and distinct $u, v \in V_H$, $h(e, v)$ is strongly anti-complete to $h(u)$;

(19) for all distinct $e, f \in E_H$ and $v \in V_H$, $h(e, v)$ is strongly anti-complete to $h(f)$;

(20) for all $e \in E_H$ and $v \in V_H$, $h(e, v)$ can be partitioned into a (possibly empty) strong clique $h^c(e, v)$ and a (possibly empty) strongly stable set $h^s(e, v)$;

(21) for all $e \in E_H$ with (distinct) endpoints $u, v \in V_H$, $G[h(e) \cup h(e, v) \cup h(e, u)]$ is an $h(e)$-melt such that $h(e) = A$, $h^c(e, v) = K$, $h^c(e, u) = M$, and $h^s(e, v) \cup h^s(e, u) = B$, with $h^s(e, v) = \bigcup_{i=1}^{m} B_{i,0}$ and $h^s(e, u) = \bigcup_{j=1}^{n} B_{0,j}$, where $K$, $M$, $A$, $B$, $m$, and $n$ are as in the definition of an $A$-melt;

(22) for all $e \in E_H$ with (distinct) endpoints $u, v \in V_H$, either all of the following hold, or they all hold with the roles of $(A_u, A_v)$ and $(B_u, B_v)$ switched:

- $h(e)$ is strongly complete to $B_u \cup B_v$,

– $h(e,v)$ is strongly complete to $A_v$ and strongly anti-complete to $L \setminus A_v$,

– every vertex of $(L \cup \bigcup_{f \in E_H \setminus \{e\}} h(f)) \setminus (A_u \cup A_v)$ with a neighbor in $A_u \cup A_v$ is strongly complete to $h(e)$.

We then say that $G$ admits an *$H$-structure*.

We define $\mathcal{T}_1$ to be the class of all trigraphs $G$ such that either $G$ is a double melt or $G$ admits an $H$-structure for some usable graph $H$. We observe that all triangle-free trigraphs are in $\mathcal{T}_1$ (a triangle-free trigraph admits an $H$-structure for the empty graph $H$), as are all clique-connectors (a clique connector admits an $H$-structure for the single-vertex graph $H$), and all melts (double melts are in $\mathcal{T}_1$ by definition, and an $A$-melt admits an $H$-structure for the complete graph $H$ that consists of a single edge).

It is easy to see that not all trigraphs in $\mathcal{T}_1$ are Berge. First of all, if $G$ admits an $H$-structure for some usable graph $H$, then $G[L \cup \bigcup_{e \in E_H} h(e)]$ may contain odd holes. Second, if $v$ is an isolated vertex in $H$, then $G[h(v) \cup S_v \cup T_v]$ may be a degenerate clique connector, and as we saw in 3.4.1, not all degenerate clique connectors are Berge. It turns out that these two are the only 'anomalies' whose presence can prevent a trigraph in $\mathcal{T}_1$ from being Berge. The definition of the class $\mathcal{T}_1^*$, to which we turn next, eliminates these anomalies. In addition, we note that if a trigraph $G$ admits an $H$-structure for some usable graph $H$, and $e, f \in E_H$ are distinct edges, then $h(e)$ and $h(f)$ are strongly complete to each other if $e$ and $f$ share an endpoint, but the converse need not hold: $h(e)$ and $h(f)$ may be strongly complete to each other even if $e$ and $f$ do not share an endpoint. In the definition of $\mathcal{T}_1^*$, we use 'usable 4-tuples,' defined below, instead of usable graphs, in order to 'encode' the adjacency in $L \cup \bigcup_{e \in E_H} h(e)$ more precisely.

**The class $\mathcal{T}_1^*$.** Let $H$ be a bipartite graph (possibly empty and possibly with parallel edges), none of whose vertices are of degree greater than two. Let $L$ be a (possibly empty) set such that $V_H \cap L = \emptyset$. Let $H'$ be a bipartite trigraph such that $V_{H'} = E_H \cup L$. Assume that for all distinct $e, f \in E_H$ that share at least one endpoint, $ef$ is a strongly adjacent pair in $H'$; assume also that every semi-adjacent pair in $H'$ has both of its endpoints in $L$. Let $(E'_1, E'_2)$ be a bipartition of the bipartite trigraph $H'$. Under these circumstances, we say that $(H, H', E'_1, E'_2)$ is a *usable* 4*-tuple*.

Let $(H, H', E'_1, E'_2)$ be a usable 4-tuple, let $L = V_{H'} \smallsetminus E_H$, let $G$ be a trigraph, and let $E_1, E_2 \subseteq V_G$. We then say that $(G, E_1, E_2)$ admits an $(H, H', E'_1, E'_2)$-structure provided that $L \subseteq V_G$, and that there exists a map

$$h : V_H \cup E_H \cup (E_H \times V_H) \to 2^{V_G \smallsetminus L}$$

such that all of the following hold:

(1) for all distinct $x, y \in V_H \cup E_H \cup (E_H \times V_H)$, $h(x)$ and $h(y)$ are disjoint;

(2) $V_G = L \cup \bigcup_{v \in V_H} h(v) \cup \bigcup_{e \in E_H} h(e) \cup \bigcup_{(e,v) \in E_H \times V_H} h(e, v)$;

(3) for all $v \in V_H$, $h(v)$ is a (possibly empty) clique;

(4) for all isolated vertices $v \in V_H$, $h(v) \neq \emptyset$;

(5) for all $e \in E_H$, $h(e)$ is a (non-empty) strongly stable set;

(6) for all $e \in E_H$ and $v \in V_H$, $h(e, v) \neq \emptyset$ if and only if $e$ is incident with $v$;

(7) for all $e \in E_H$ and $v \in V_H$, $h(e, v)$ can be partitioned into a (possibly empty) strong clique $h^c(e, v)$ and a (possibly empty) strongly stable set $h^s(e, v)$;

(8) for all $e \in E_H$ and $v \in V_H$, if $e$ is incident with $v$ then $h^c(e, v)$ and $h^s(e, v)$ are both non-empty, and if $e$ is not incident with $v$ then $h^c(e, v) = h^s(e, v) = \emptyset$;

(9) $E_1 \cup E_2 = L \cup \bigcup_{e \in E_H} h(e) \cup \bigcup_{e \in E_H} \bigcup_{v \in V_H} h^s(e, v)$;

(10) $E_1 \cap E_2 = \emptyset$;

(11) for all $x \in L$ and $i \in \{1, 2\}$, if $x \in E_i'$ then $x \in E_i$;

(12) for all $e \in E_H$ and $i \in \{1, 2\}$, if $e \in E_i'$ then $h(e) \subseteq E_i$;

(13) for all $e \in E_H$, $v \in V_H$, and all distinct $i, j \in \{1, 2\}$, if $e \in E_i'$ then $h^s(e, v) \subseteq E_j$;

(14) $H'[L] = G[L]$;

(15) for all $x \in L$ and $e \in E_H$, if $xe$ is a strongly adjacent pair in $H'$ then $x$ is strongly complete to $h(e)$, and if $xe$ is a strongly anti-adjacent pair in $H'$ then $x$ is strongly anti-complete to $h(e)$;

(16) for all distinct $e, f \in E_H$, if $ef$ is a strongly adjacent pair in $H'$ then $h(e)$ is strongly complete to $h(f)$, and if $ef$ is a strongly anti-adjacent pair in $H'$ then $h(e)$ is strongly anti-complete to $h(f)$;

(17) for all $v \in V_H$, if $S_v$ is the set of all vertices in $L$ that have a neighbor in $h(v)$, and $T_v$ is the set of all vertices in $(L \cup \bigcup_{e \in E_H} h(e)) \setminus S_v$ that have a neighbor in $S_v$, then either $h(v) = \emptyset$ (in which case $S_v = T_v = \emptyset$) or $G[h(v) \cup S_v \cup T_v]$ is a non-degenerate $(h(v), S_v \cap E_1, S_v \cap E_2, T_v \cap E_2, T_v \cap E_1)$-clique connector;

(18) for all distinct $e, f \in E_H$ and $v \in V_H$, $h(e, v)$ is strongly complete to $h(f, v)$;

(19) for all (not necessarily distinct) $e, f \in E_H$ and distinct $u, v \in V_H$, $h(e, u)$ is strongly anti-complete to $h(f, v)$;

(20) for all $e \in E_H$ and $v \in V_H$, $h(e, v)$ is strongly complete to $h(v)$;

(21) for all $e \in E_H$ and distinct $u, v \in V_H$, $h(e, v)$ is strongly anti-complete to $h(u)$;

(22) for all distinct $e, f \in E_H$ and $v \in V_H$, $h(e, v)$ is strongly anti-complete to $h(f)$;

(23) for all $e \in E_H$ with (distinct) endpoints $u, v \in V_H$, $G[h(v) \cup h(e, v) \cup h(e, u)]$ is an $h(e)$-melt such that $h(e) = A$, $h^c(e, v) = K$, $h^c(e, u) = M$, and $h^s(e, v) \cup h^s(e, u) = B$, with $h^s(e, v) = \bigcup_{i=1}^{m} B_{i,0}$ and $h^s(e, u) = \bigcup_{j=1}^{n} B_{0,j}$, where $K$, $M$, $A$, $B$, $m$, and $n$ are as in the definition of an $A$-melt;

(24) for all $e \in E_H$ with (distinct) endpoints $u, v \in V_H$, and all distinct $i, j \in \{1, 2\}$ such that $e \in E'_i$, all of the following hold:

- $h(e)$ is strongly complete to $(S_u \cup S_v) \cap E_j$,
- $h(e, v)$ is strongly complete to $S_v \cap E_i$ and strongly anti-complete to $L \smallsetminus (S_v \cap E_i)$,
- every vertex of $(L \cup \bigcup_{f \in E_H \smallsetminus \{e\}} h(f)) \smallsetminus ((S_u \cup S_v) \cap E_i)$ with a neighbor in $(S_u \cup S_v) \cap E_i$ is strongly complete to $h(e)$.

We leave it to the reader to check that if $(H, H', E'_1, E'_2)$ is a usable 4-tuple and $(G, E_1, E_2)$ admits an $(H, H', E'_1, E'_2)$-structure, then $H$ is a usable graph, $G$ admits an $H$-structure, and $E_1$ and $E_2$ are both (possibly empty) strongly stable sets.

We say that a trigraph $G$ belongs to the *class* $\mathcal{T}_1^*$ provided that either $G$ is a double melt, or there exist $E_1, E_2 \subseteq V_G$ and a usable 4-tuple $(H, H', E'_1, E'_2)$ such that $(G, E_1, E_2)$ admits an $(H, H', E'_1, E'_2)$-structure.

We observe that all bipartite trigraphs are in $\mathcal{T}_1^*$, as are all non-degenerate clique connectors (and therefore all tulips), and all melts. Further, we remind the reader that the class $\mathcal{T}_1$ consists of trigraphs $G$ such that either $G$ is a double melt or there exists a usable graph $H$ such that $G$ admits an $H$-structure. Thus, the class $\mathcal{T}_1^*$ is a subclass of the class $\mathcal{T}_1$.

Our goal for the remainder of this section is to establish that each trigraph in $\mathcal{T}_1^*$ is a tulip bed and therefore Berge, and that each Berge trigraph in $\mathcal{T}_1$ is in $\mathcal{T}_1^*$.

**3.4.8.** *Each trigraph in $\mathcal{T}_1^*$ is a tulip bed. Consequently, each trigraph in $\mathcal{T}_1^*$ is Berge.*

*Proof.* By 3.4.6, it suffices to prove the first statement. Let $G \in \mathcal{T}_1^*$. If $G$ is a double melt, then we are done by 3.4.7. So assume that there exists some usable 4-tuple $(H, H', E'_1, E'_2)$ and some $E_1$ and $E_2$ such that $(G, E_1, E_2)$ admits an $(H, H', E'_1, E'_2)$-structure. If $H$ is the empty graph, then $G$ is bipartite and therefore a tulip bed; so assume that $H$ is not empty. We use the notation from the definition of a triple $(G, E_1, E_2)$ that admits an $(H, H', E'_1, E'_2)$-structure. Set $F_1 = E_1$ and $F_2 = E_2$. We may assume that the vertex-set of $H$ is $\{v_1, ..., v_s\}$ for some integer $s \geq 1$; then for each $l \in \{1, ..., s\}$, set $Y_l = h(v_l) \cup \bigcup_{e \in E_H} h^c(e, v_l)$. We observe that $Y_1, ..., Y_s$ partition $V_G \setminus (F_1 \cup F_2)$ into non-empty strong cliques, pairwise strongly anti-complete to each other. Next, for each $l \in \{1, ..., s\}$ and $e \in E_H$, let $h^l(e)$ be the set of all vertices in $h(e)$ with a neighbor in $h^c(e, v_l)$; then $S_{v_l} \cup \bigcup_{e \in E_H} h^l(e) \cup \bigcup_{e \in E_H} h^s(e, v_l)$ is the set of all vertices in $F_1 \cup F_2 = E_1 \cup E_2$ with a neighbor in $Y_l$, and so we set $X_l = S_{v_l} \cup \bigcup_{e \in E_H} h^l(e) \cup \bigcup_{e \in E_H} h^s(e, v_l)$. With this setup, it is easy to see that $G$ is a tulip bed. $\square$

As with melts (see the comment after 3.4.7), it is possible to get a slightly stronger result than the one that we stated in 3.4.8. The reader can check that if $(G, E_1, E_2)$ admits an $(H, H', E'_1, E'_2)$-structure for some usable 4-tuple $(H, H', E'_1, E'_2)$, and if $Y_1, ..., Y_s$ and $X_1, ..., X_s$ are constructed as in the proof above, then for each $l \in \{1, ..., s\}$, $G[Y_l \cup X_l \cup Z_l]$ is a $(Y_l, X_l \cap E_1, X_l \cap E_2, Z_l \cap E_2, Z_l \cap E_1)$-clique connector, where $Z_l$ is the set of all vertices in $(E_1 \cup E_2) \setminus X_l$ with a neighbor in $X_l$. But we do not need this stronger result, and so we omit the proof.

It remains to show that every Berge trigraph in $\mathcal{T}_1$ is in $\mathcal{T}_1^*$. We begin with a technical lemma.

**3.4.9.** *Let $H$ be a usable graph, and let $G$ be a Berge trigraph that admits an $H$-structure. Then the set $L$ and the function $h$ from the definition of a trigraph that admits an $H$-structure can be chosen so that for all isolated vertices $v \in V_H$, $G[h(v) \cup S_v \cup T_v]$ is a non-degenerate $(h(v), A_v, B_v, C_v, D_v)$-clique connector (where $S_v, T_v, A_v, B_v, C_v, D_v$ are as in the definition).*

*Proof.* Let $L \subseteq V_G$ and $h : V_H \cup E_H \cup (E_H \times V_H) \to 2^{V_G \setminus L}$ satisfy the properties laid out in the definition of a trigraph that admits an $H$-structure. Since $G$ is Berge, by 3.4.1

we have that for all isolated vertices $v \in V_H$ such that $G[h(v) \cup S_v \cup T_v]$ is a degenerate $(h(v), A_v, B_v, C_v, D_v)$-clique connector, at least one of $C_v$ and $D_v$ is empty. After possibly relabeling, we may assume that for all isolated vertices $v \in V_H$ such that $G[h(v) \cup S_v \cup T_v]$ is a degenerate $(h(v), A_v, B_v, C_v, D_v)$-clique connector, we have that $D_v = \emptyset$. Now, let $V_H^d$ be set of all isolated vertices in $V_H$ such that $G[h(v) \cup S_v \cup T_v]$ is a degenerate $(h(v), A_v, B_v, C_v, \emptyset)$-clique connector, and for all $v \in V_H^d$, set $B_v = \{b_v\}$. Set $\hat{L} = (L \smallsetminus \bigcup_{v \in V_H^d} \{b_v\}) \cup \bigcup_{v \in V_H^d} h(v)$. Next, we define $\hat{h} : V_H \cup E_H \cup (E_H \times V_H) \to 2^{V_G \smallsetminus \hat{L}}$ to be the map that satisfies all of the following:

- for all $v \in V_H^d$, $\hat{h}(v) = \{b_v\}$;
- for all $v \in V_H \smallsetminus V_H^d$, $\hat{h}(v) = h(v)$;
- for all $e \in E_H$, $\hat{h}(e) = h(e)$;
- for all $e \in E_H$ and $v \in V_H$, $\hat{h}(e, v) = h(e, v)$.

Using 3.4.2, we easily get that $\hat{L}$ and $\hat{h}$ satisfy the requirements from the statement of the theorem. □

**3.4.10.** *Let $H$ be a usable graph, and let $G$ be a Berge trigraph that admits an $H$-structure. Then $H$ is a bipartite graph. Furthermore, there exists a bipartite trigraph $H'$ such that for every bipartition $(E'_1, E'_2)$ of $H'$, $(H, H', E'_1, E'_2)$ is a usable 4-tuple, and there exist some $E_1, E_2 \subseteq V_G$ such that $(G, E_1, E_2)$ admits an $(H, H', E'_1, E'_2)$-structure.*

*Proof.* Let $L$ and $h$ be chosen as in 3.4.9. We construct $H'$ as follows. The vertex-set of $H'$ is $E_H \cup L$. Set $H'[L] = G[L]$. For all $x \in L$ and $e \in E_H$, we let $xe$ be a strongly adjacent pair in $H'$ if $x$ is strongly complete to $h(e)$ in $G$, and we let $xe$ be a strongly anti-adjacent pair in $H'$ if $x$ is strongly anti-complete to $h(e)$ in $G$; since for all $x \in L$ and $e \in E_H$, $x$ is either strongly complete or strongly anti-complete to $h(e)$ in $G$, this completely defines the adjacency between $L$ and $E_H$ in $H'$. Finally, for all distinct $e, f \in E_H$, we let $ef$ be a strongly adjacent pair in $H'$ if $h(e)$ is strongly complete to $h(f)$, and $ef$ is a strongly anti-adjacent pair in $H'$ if $h(e)$ is strongly anti-complete to $h(f)$ in $G$; since for all distinct $e, f \in E_H$, we have that $h(e)$ is either strongly complete or strongly anti-complete to $h(f)$, this completely defines adjacency in $H'[E_H]$. We observe that if distinct $e, f \in E_H$ share an endpoint, then $e$ and $f$ are adjacent in $H'$.

Note that $H'$ contains no odd holes and no triangles, for otherwise, we would immediately get an odd hole or a triangle, respectively, in $G[L \cup \bigcup_{e \in E_H} E_H]$, which is impossible. Since every realization of $H'$ contains the line graph of $H$ as a (not necessarily induced) subgraph, this implies that $H$ is bipartite. Let $(E'_1, E'_2)$ be any bipartition of the bipartite trigraph $H'$. Clearly, $(H, H', E'_1, E'_2)$ is a usable 4-tuple.

Next, we set:

$$E_1 = (L \cap E'_1) \cup \textstyle\bigcup_{e \in E_H \cap E'_1} h(e) \cup \bigcup_{(e,v) \in (E_H \cap E'_2) \times V_H} h^s(e, v);$$

$$E_2 = (L \cap E'_2) \cup \textstyle\bigcup_{e \in E_H \cap E'_2} h(e) \cup \bigcup_{(e,v) \in (E_H \cap E'_1) \times V_H} h^s(e, v).$$

By construction, $(E_1, E_2)$ is a partition of the set

$$L \cup \textstyle\bigcup_{e \in E_H} h(e) \cup \bigcup_{(e,v) \in E_H \times V_H} h^s(e, v).$$

To show that $(G, E_1, E_2)$ admits an $(H, H', E'_1, E'_2)$-structure, it suffices to show that for all $v \in V_H$, either $A_v \cup D_v \subseteq E_1$ and $B_v \cup C_v \subseteq E_2$, or $A_v \cup D_v \subseteq E_2$ and $B_v \cup C_v \subseteq E_1$, for then the result will easily follow from the appropriate definitions (together with the choice of $L$ and $h$). So fix $v \in V_H$; if $h(v) = \emptyset$, then we are done, and so assume that $h(v) \neq \emptyset$. Then by the definition of a clique connector, there exist some $a \in A_v$ and $b \in B_v$ such that $a$ is strongly complete to $B_v$ and $b$ is strongly complete to $A_v$. In particular, $ab$ is an adjacent pair, and so $a \in E'_i$ and $b \in E'_j$ for some distinct $i, j \in \{1, 2\}$. Since $a$ is strongly complete to $B_v$, this implies that $B_v \subseteq E'_j$; and similarly, $A_v \subseteq E'_i$. By the construction of $E_1$ and $E_2$ then, we get that $A_v \subseteq E_i$ and $B_v \subseteq E_j$. It remains to show that $C_v \subseteq E_j$ and that $D_v \subseteq E_i$; by symmetry, it suffices to prove the former. By definition, there exist some

$L_{C_v} \subseteq L$ and $E_{C_v} \subseteq E_H$ such that $C_v = L_{C_v} \cup \bigcup_{e \in E_{C_v}} h(e)$. Now, in $H'$, each member of $L_{C_v} \cup E_{C_v}$ has a neighbor in $A_v \subseteq E'_i$, and so $L_{C_v} \cup E_{C_v} \subseteq E'_j$. It then easily follows that $C_v \subseteq E_j$. □

We can now finally prove the main result of this section.

**3.4.11.** *$\mathcal{T}_1^*$ is the class of all Berge trigraphs in $\mathcal{T}_1$.*

*Proof.* It is easy to check that $\mathcal{T}_1^* \subseteq \mathcal{T}_1$. By 3.4.8, every trigraph in $\mathcal{T}_1^*$ is Berge. Now, suppose that $G$ is a Berge trigraph in $\mathcal{T}_1$. If $G$ is a double melt, then $G \in \mathcal{T}_1^*$ by definition. So suppose that $G$ admits an $H$-structure for some usable graph $H$. Then by 3.4.10, there exist some $E_1, E_2 \subseteq V_G$ and some usable 4-tuple $(H, H', E'_1, E'_2)$ such that $(G, E_1, E_2)$ admits an $(H, H', E'_1, E'_2)$-structure, and consequently, $G \in \mathcal{T}_1^*$ by the definition of the class $\mathcal{T}_1^*$. □

## 3.5 Class $\mathcal{T}_2$

In this section, we give the definition of the class $\mathcal{T}_2$ from [10], and we prove that each trigraph in $\mathcal{T}_2$ is Berge. Prior to giving the definition of the class $\mathcal{T}_2$, we note that, by 5.5 from [10], the class $\mathcal{T}_2$ is self-complementary; we state this result below for future reference.

**3.5.1** (Chudnovsky [10])**.** *The class $\mathcal{T}_2$ is self-complementary, that is, for all $G \in \mathcal{T}_2$, we have that $\overline{G} \in \mathcal{T}_2$.*

Thus, in order to show that each trigraph in $\mathcal{T}_2$ is Berge, it suffices to show that each trigraph in $\mathcal{T}_2$ is odd hole-free.

Informally, trigraphs in the class $\mathcal{T}_2$ are obtained from some basic 'building blocks' (namely, "1-thin trigraphs," "2-thin trigraphs," and bipartite and complement-bipartite trigraphs of a certain kind; we define each of these below) by "composing along doubly dominating semi-adjacent pairs" (this operation is also defined below). We will show that if a trigraph

$G$ is obtained from two trigraphs that do not contain any odd holes by composing along doubly dominating semi-adjacent pairs, then $G$ does not contain any odd holes. We will then show that none of our basic 'building blocks' contain an odd hole. This will prove that no trigraph in $\mathcal{T}_2$ contains an odd hole, and therefore (by 3.5.1) that each trigraph in $\mathcal{T}_2$ is Berge.

We begin with some definitions. We say that a homogeneous pair $(A, B)$ in a trigraph $G$ is *doubly dominating* provided that there exist non-empty sets $C, D \subseteq V_G$ such that $(A, B, C, D, \emptyset, \emptyset)$ is the partition of $G$ associated with $(A, B)$. We say that a semi-adjacent pair $ab$ in $G$ is *doubly dominating* provided that $(\{a\}, \{b\})$ is a doubly dominating homogeneous pair in $G$. Next, let $G_1$ and $G_2$ be trigraphs with disjoint vertex-sets, and for each $i \in \{1, 2\}$, let $a_ib_i$ be a doubly dominating semi-adjacent pair in $G_i$. For each $i \in \{1, 2\}$, let $(\{a_i\}, \{b_i\}, A_i, B_i, \emptyset, \emptyset)$ be the partition of $G$ associated with $(\{a_i\}, \{b_i\})$. We then say that a trigraph $G$ is obtained from $G_1$ and $G_2$ by *composing along* $(a_1, b_1, a_2, b_2)$ provided all of the following hold:

- $V_G = A_1 \cup B_1 \cup A_2 \cup B_2$;
- for each $i \in \{1, 2\}$, $G[A_i \cup B_i] = G_i[A_i \cup B_i]$;
- $A_1$ is strongly complete to $A_2$ and strongly anti-complete to $B_2$;
- $B_1$ is strongly complete to $B_2$ and strongly anti-complete to $A_2$.

**3.5.2.** *Let $G_1$ and $G_2$ be odd hole-free trigraphs with disjoint vertex-sets, and for each $i \in \{1, 2\}$, let $a_ib_i$ be a doubly dominating semi-adjacent pair in $G_i$. Let $G$ be the trigraph obtained by composing $G_1$ and $G_2$ along $(a_1, b_1, a_2, b_2)$. Then $G$ is odd hole-free.*

*Proof.* Suppose otherwise. Let $W$ be the vertex-set of an odd hole in $G$, and let $\hat{G}$ be a realization of $G$ such that $W$ is the vertex-set of an odd hole in $\hat{G}$. First, note that $G \smallsetminus A_1$ is obtained by substituting $G_1[B_1]$ for the vertex $b_2$ in $G_2 \smallsetminus a_2$. Since neither $G_1$ nor $G_2$ contains an odd hole, by 2.2.2, this means that $G \smallsetminus A_1$ contains no odd hole. Thus, $W$

intersects $A_1$. In an analogous manner, we get that $W$ intersects $B_1$, $A_2$, and $B_2$ as well. Next, since $A_1$ is complete to $A_2$ in $\hat{G}$, and since $\hat{G}[W]$ is a chordless cycle of length at least five and therefore contains no vertices of degree greater than two and no (not necessarily induced) cycles of length 4, we know that $|W \cap (A_1 \cup A_2)| \leq 3$; similarly, $|W \cap (B_1 \cup B_2)| \leq 3$. Since $|W|$ is odd, and since $W$ intersects each of $A_1$, $B_1$, $A_2$, and $B_2$, this means that we may assume by symmetry that $|W \cap A_1| = 2$ and $|W \cap B_1| = |W \cap A_2| = |W \cap B_2| = 1$. Set $W \cap A_1 = \{\hat{a}_1, \hat{a}_1'\}$, $W \cap B_1 = \{\hat{b}_1\}$, $W \cap A_2 = \{\hat{a}_2\}$, and $W \cap B_2 = \{\hat{b}_2\}$. Since $\hat{a}_2$ is complete to $\{\hat{a}_1, \hat{a}_1'\}$ in $\hat{G}$, we know that $\hat{a}_2$ is non-adjacent to $\hat{b}_2$ in $\hat{G}$. But then the only neighbor of $\hat{b}_2$ in $\hat{G}[W]$ is $\hat{a}_2$, which is impossible since $\hat{G}[W]$ is a cycle. $\square$

We note that in [27], Cornuéjols and Cunningham proved a result similar to 3.5.2. They showed that a graph operation whose special case is very similar to our operation of composing along doubly dominating semi-adjacent pairs preserves perfection.

**Triangle-patterns and triad-patterns.** Given a graph $H$, we say that a trigraph $G$ is an *$H$-pattern* provided that $V_G$ can be partitioned into sets $\{a_v \mid v \in V_H\}$ and $\{b_v \mid v \in V_H\}$ such that all of the following hold:

- $a_v b_v$ is a semi-adjacent pair for all $v \in V_H$;
- if $u, v \in V_H$ are adjacent, then $a_u a_v$ and $b_u b_v$ are strongly adjacent pairs, and $a_u b_v$ and $a_v b_u$ are strongly anti-adjacent pairs;
- if $u, v \in V_H$ are non-adjacent, then $a_u a_v$ and $b_u b_v$ are strongly anti-adjacent pairs, and $a_u b_v$ and $a_v b_u$ are strongly adjacent pairs.

We observe that $a_v b_v$ is a doubly dominating semi-adjacent pair in $G$ for all $v \in V_H$. Let $K_n$ denote the complete graph on $n$ vertices. We say that a trigraph $G$ is a *triangle-pattern* provided that $G$ is a $K_3$-pattern, we say that $G$ is a *triad-pattern* provided that $G$ is a $\overline{K}_3$-pattern.

We note that triangle-patterns are complement-bipartite and triad-patterns are bipartite; thus, triangle-patterns and triad-patterns are Berge.

**1-thin trigraphs.** Let $G$ be a trigraph, and let $a, b \in V_G$ be distinct. Let $A = \{a_1, ..., a_n\}$ and $B = \{b_1, ..., b_m\}$ be disjoint, non-empty subsets of $V_G \setminus \{a, b\}$ such that $V_G \setminus \{a, b\} = A \cup B$. Assume that all of the following hold:

(1) $ab$ is a semi-adjacent pair;

(2) $a$ is strongly complete to $A$ and strongly anti-complete to $B$;

(3) $b$ is strongly complete to $B$ and strongly anti-complete to $A$;

(4) for all $i, j \in \{1, ..., n\}$ such that $i < j$, if $a_i a_j$ is an adjacent pair, then $a_i$ is strongly complete to $\{a_{i+1}, ..., a_{j-1}\}$, and $a_j$ is strongly complete to $\{a_1, ..., a_{i-1}\}$;

(5) for all $i, j \in \{1, ..., m\}$ such that $i < j$, if $b_i b_j$ is an adjacent pair, then $b_i$ is strongly complete to $\{b_{i+1}, ..., b_{j-1}\}$, and $b_j$ is strongly complete to $\{b_1, ..., b_{i-1}\}$;

(6) for all $p \in \{1, ..., n\}$ and $q \in \{1, ..., m\}$, if $a_p b_q$ is an adjacent pair, then $a_p$ is strongly complete to $\{b_{q+1}, ..., b_m\}$, and $b_q$ is strongly complete to $\{a_{p+1}, ..., a_n\}$.

We then say that $G$ is 1-*thin with base* $(a, b)$, or simply that $G$ is 1-*thin*.

Note that if a trigraph $G$ is 1-thin with base $(a, b)$, then $ab$ is a doubly dominating semi-adjacent pair in $G$. Note also that a trigraph $G$ is 1-thin with base $(a, b)$ if and only if $G$ is 1-thin with base $(b, a)$. Further, the complement of a 1-thin trigraph with base $(a, b)$ is again 1-thin with base $(b, a)$ (or equivalently, with base $(a, b)$). Indeed if $G$ is 1-thin, then setting $\overline{a} = b$, $\overline{b} = a$, $\overline{a}_i = a_{n-i+1}$ for all $i \in \{1, ..., n\}$, and $\overline{b}_i = b_{m-i+1}$ for all $i \in \{1, ..., m\}$, we immediately get that $\overline{G}$ is 1-thin with base $(\overline{a}, \overline{b})$, that is, with base $(b, a)$.

**3.5.3.** *Let $G$ be a 1-thin trigraph. Then $G$ is odd hole-free.*

*Proof.* Let $a$, $b$, $A = \{a_1, ..., a_n\}$, and $B = \{b_1, ..., b_m\}$ be as in the definition of a 1-thin trigraph. We begin by showing that $(A, B)$ is a good homogeneous pair in $G$. This means that we need to prove the following:

- neither $G[A]$ nor $G[B]$ contains a three-edge path;
- there does not exist a path $v_1 - v_2 - v_3 - v_4$ in $G$ such that $v_1, v_4 \in A$ and $v_2, v_3 \in B$;
- there does not exist a path $v_1 - v_2 - v_3 - v_4$ in $G$ such that $v_1, v_4 \in B$ and $v_2, v_3 \in A$.

We begin by proving the second claim. Suppose that $i, l \in \{1, ..., m\}$ and $j, k \in \{1, ..., n\}$ are such that $a_i - b_j - b_k - a_l$ is a three-edge path in $G$. Since $a_ib_j$ is an adjacent pair, while $a_ib_k$ is an anti-adjacent pair, we know that $k < j$. But then since $a_lb_k$ is an adjacent pair, $a_lb_j$ must be a strongly adjacent pair, which is a contradiction. An analogous argument establishes that the third claim holds as well.

Before tackling the first claim, we establish an auxiliary result. Let $i, j, k \in \{1, ..., n\}$ be such that $a_i - a_j - a_k$ is a path in $G[A]$; we claim that $j < \min\{i, k\}$. Suppose otherwise. By symmetry, we may assume that $i < j$. Then if $k < i < j$, the fact that $a_ja_k$ is an adjacent pair implies that $a_ia_k$ is a strongly adjacent pair; if $i < k < j$, then the fact that $a_ia_j$ is an adjacent pair implies that $a_ia_k$ is a strongly adjacent pair; and if $i < j < k$, then the fact that $a_ja_k$ is an adjacent pair implies that $a_ia_k$ is a strongly adjacent pair. But since $a_i - a_j - a_k$ is a path, $a_ia_k$ is an anti-adjacent pair, which is a contradiction.

Now, suppose that $i, j, k, l \in \{1, ..., n\}$ are such that $a_i - a_j - a_k - a_l$ is a path in $G$. Then since $a_i - a_j - a_k$ is a path in $G[A]$, we have by the above that $j < \min\{i, k\}$; similarly, since $a_j - a_k - a_l$ is a path in $G[A]$, we have that $k < \min\{j, l\}$. But it then follows that $j < k$ and that $k < j$, which is impossible. Thus, $G[A]$ contains no three-edge paths. We get in an analogous fashion that $G[B]$ contains no three-edge paths, and so $(A, B)$ is a good homogeneous pair.

Now, suppose that $G$ contains an odd hole, and let $W$ be the vertex-set of an odd hole in $G$. By 3.2.1, we get that $|W \cap A| \leq 1$ and $|W \cap B| \leq 1$. But since $V_G = \{a, b\} \cup A \cup B$, this means that $|W| \leq 4$, which is impossible since an odd hole must have at least five vertices. □

**2-thin trigraphs.** Let $G$ be a trigraph. Let $V_G = A \cup B \cup K \cup M \cup \{x_{AK}, x_{AM}, x_{BK}, x_{BM}\}$, where $x_{AK}, x_{AM}, x_{BK}, x_{BM}$ are pairwise distinct vertices, and $A$, $B$, $K$, $M$, and $\{x_{AK}, x_{AM}, x_{BK}, x_{BM}\}$ are pairwise disjoint sets. Let $t, s \geq 0$, and let $K = \{k_1, ..., k_t\}$ and $M = \{m_1, ..., m_s\}$ (so if $t = 0$ then $K = \emptyset$, and if $s = 0$ then $M = \emptyset$). Let $A$ be the disjoint union of sets $A_{i,j}$ and let $B$ be the disjoint union of the sets $B_{i,j}$, where $i \in \{0, ..., t\}$ and $j \in \{0, ..., s\}$. Assume that:

(1) $A$ and $B$ are (possibly empty) strongly stable sets;

(2) $K$ and $M$ are (possibly empty) strong cliques;

(3) $A$ is strongly complete to $B$;

(4) $K$ is strongly anti-complete to $M$;

(5) $A$ is strongly complete to $\{x_{AK}, x_{AM}\}$ and strongly anti-complete to $\{x_{BK}, x_{BM}\}$;

(6) $B$ is strongly complete to $\{x_{BK}, x_{BM}\}$ and strongly anti-complete to $\{x_{AK}, x_{AM}\}$;

(7) $K$ is strongly complete to $\{x_{AK}, x_{BK}\}$ and strongly anti-complete to $\{x_{AM}, x_{BM}\}$;

(8) $M$ is strongly complete to $\{x_{AM}, x_{BM}\}$ and strongly anti-complete to $\{x_{AK}, x_{BK}\}$;

(9) $x_{AK}x_{BM}$ and $x_{AM}x_{BK}$ are semi-adjacent pairs;

(10) $x_{AK}x_{BK}$ and $x_{AM}x_{BM}$ are strongly adjacent pairs;

(11) $x_{AK}x_{AM}$ and $x_{BK}x_{BM}$ are strongly anti-adjacent pairs;

(12) for all $i, i' \in \{0, ..., t\}$ and $j, j' \in \{0, ..., s\}$, if $i < i'$ and $j < j'$, then at least one of the sets $A_{i,j}$ and $A_{i',j'}$ is empty, and at least one of the sets $B_{i,j}$ and $B_{i',j'}$ is empty;

(13) for all $i \in \{0, ..., t\}$ and $j \in \{0, ..., s\}$, all of the following hold:

- $A_{i,j}$ is strongly complete to $\{k_1, ..., k_{i-1}\} \cup \{m_{s-j+2}, ..., m_s\}$,
- $A_{i,j}$ is complete to $\{k_i, m_{s-j+1}\}$,
- $A_{i,j}$ is strongly anti-complete to $\{k_{i+1}, ..., k_t\} \cup \{m_1, ..., m_{s-j}\}$,

- $B_{i,j}$ is strongly complete to $\{k_{t-i+2}, ..., k_t\} \cup \{m_1, ..., m_{j-1}\}$,
- $B_{i,j}$ is complete to $\{k_{t-i+1}, m_j\}$,
- $B_{i,j}$ is strongly anti-complete to $\{k_1, ..., k_{t-i}\} \cup \{m_{j+1}, ..., m_s\}$.

Then we say that $G$ is 2-*thin with base* $(x_{AK}, x_{BM}, x_{BK}, x_{AM})$, or simply that $G$ is 2-*thin*. We call $(A, B, K, M)$ the *partition of $G$ with respect to the base* $(x_{AK}, x_{BM}, x_{BK}, x_{AM})$.

Suppose that $G$ is a 2-thin trigraph with base $(x_{AK}, x_{BM}, x_{BK}, x_{AM})$. It is then easy to see that $G$ is 2-thin with base $(x_{BK}, x_{AM}, x_{AK}, x_{BM})$; it is also easy to see that $\overline{G}$ is 2-thin with base $(x_{AK}, x_{BM}, x_{AM}, x_{BK})$. Next, note $x_{AK}x_{BM}$ and $x_{BK}x_{AM}$ are both doubly dominating semi-adjacent pairs, and $G$ contains no other doubly dominating semi-adjacent pairs. We also observe that $G$ is 1-thin with base $(x_{AK}, x_{BM})$, and also with base $(x_{BK}, x_{AM})$. By 3.5.3 then, 2-thin trigraphs are odd hole-free.

**The class $\mathcal{T}_2$.** Let $k \geq 1$ be an integer, and let $G'_1, ..., G'_k$ be trigraphs, such that for all $i \in \{1, ..., k\}$, $G'_i$ is either a triangle-pattern or a triad-pattern or a 2-thin trigraph. For each $i \in \{2, ..., k\}$, let $c_i d_i$ be a doubly dominating semi-adjacent pair in $G'_i$. For each $j \in \{1, ..., k-1\}$, let $x_j y_j$ be a doubly dominating semi-adjacent pair in $G'_q$ for some $q \in \{1, ..., j\}$. Assume that $\{c_2, d_2\}, ..., \{c_k, d_k\}, \{x_1, y_1\}, ..., \{x_{k-1}, y_{k-1}\}$ are pairwise distinct (and therefore pairwise disjoint). Let $G_1 = G'_1$, and for each $i \in \{1, ..., k-1\}$, let $G_{i+1}$ be the trigraph obtained by composing $G_i$ and $G'_{i+1}$ along $(x_i, y_i, c_{i+1}, d_{i+1})$. Let $G = G_k$. We call such a trigraph $G$ a *skeleton*. We observe that a semi-adjacent pair $uv$ in $G$ is doubly dominating in $G$ if and only if $uv$ is a doubly dominating semi-adjacent pair in $G'_i$ for some $i \in \{1, ..., k\}$ and $\{u, v\}$ is not among $\{c_2, d_2\}, ..., \{c_k, d_k\}, \{x_1, y_1\}, ..., \{x_{k-1}, y_{k-1}\}$.

The class $\mathcal{T}_2$ consists of all skeletons, and of all trigraphs $G$ that can be obtained as follows. Let $G'_0$ be a skeleton, and let $n \geq 1$ be an integer. Let $a_1b_1, ..., a_nb_n$ be doubly dominating semi-adjacent pairs in $G'_0$ such that $\{a_1, b_1\}, ..., \{a_n, b_n\}$ are pairwise distinct (and therefore pairwise disjoint). For each $i \in \{1, ..., n\}$, let $G'_i$ be a trigraph such that:

(1) $V_{G'_i} = A_i \cup B_i \cup \{a'_i, b'_i\}$;

(2) the sets $A_i, B_i, \{a'_i, b'_i\}$ are all non-empty and pairwise disjoint;

(3) $a'_i$ is strongly complete to $A_i$ and strongly anticomplete to $B_i$;

(4) $b'_i$ is strongly complete to $B_i$ and strongly anticomplete to $A_i$;

(5) $a'_i$ is semi-adjacent to $b'_i$, and either

- both $A_i, B_i$ are strong cliques, and there do not exist $a \in A_i$ and $b \in B_i$, such that $a$ is strongly anti-complete to $B_i \setminus \{b\}$, $b$ is strongly anti-complete to $A_i \setminus \{a\}$, and $a$ is semi-adjacent to $b$, or
- both $A_i, B_i$ are strongly stable sets, and there do not exist $a \in A_i$ and $b \in B_i$, such that $a$ is strongly complete to $B_i \setminus \{b\}$, $b$ is strongly complete to $A_i \setminus \{a\}$, and $a$ is semi-adjacent to $b$, or
- $G'_i$ is a 1-thin trigraph with base $(a'_i, b'_i)$, and $G'_i$ is not a 2-thin trigraph.

We observe that for all $i \in \{1, ..., n\}$, $a'_ib'_i$ is a doubly dominating semi-adjacent pair in $G'_i$, and if $uv$ is a doubly dominating semi-adjacent pair in $G'_i$, then $\{u, v\} = \{a'_i, b'_i\}$. Now, let $G_0 = G'_0$, and for $i \in \{1, ..., n\}$, let $G_i$ be obtained by composing $G_{i-1}$ and $G'_i$ along $(a_i, b_i, a'_i, b'_i)$. Let $G = G_n$.

**3.5.4.** *Every trigraph in $\mathcal{T}_2$ is Berge.*

*Proof.* By 3.5.1, it suffices to show that every trigraph in $\mathcal{T}_2$ is odd hole-free. Recall that 2-thin trigraphs are 1-thin, that triangle-patterns are complement-bipartite, and that triad-patterns are bipartite. Thus, each trigraph in $\mathcal{T}_2$ is obtained by successively composing 1-thin trigraphs, bipartite trigraphs, and complement bipartite-trigraphs along doubly dominating semi-adjacent pairs. Since bipartite and complement-bipartite trigraphs are odd hole-free, the result follows form 3.5.2 and 3.5.3. □

We end this section by stating a few results from [9] that help us understand the structure of trigraphs in the class $\mathcal{T}_2$. By 6.7 from [9], all 1-thin trigraphs (and therefore, all 2-thin trigraphs) are in $\mathcal{T}_2$. By 6.6 from [9], all bipartite trigraphs with a doubly dominating semi-adjacent pair are in $\mathcal{T}_2$; and since $\mathcal{T}_2$ is closed under complementation, it follows that all complement-bipartite trigraphs with a doubly dominating semi-adjacent pair are in $\mathcal{T}_2$. Finally, by 6.8 from [9], the class $\mathcal{T}_2$ is closed under composing along doubly dominating semi-adjacent pairs.

## 3.6 The Main Theorem

In this section, we restate and prove 3.1.4, the structure theorem for bull-free Berge trigraphs.

**3.1.4.** *Let $G$ be a trigraph. Then $G$ is a bull-free Berge trigraph if and only if at least one of the following holds:*

- *$G$ is obtained from smaller bull-free Berge trigraphs by substitution;*
- *$G$ or $\overline{G}$ is an elementary expansion of a trigraph in $\mathcal{T}_1^*$;*
- *$G$ is an elementary expansion of a trigraph in $\mathcal{T}_2$.*

*Proof.* We first prove the 'if' part. If $G$ is obtained by substitution from smaller bull-free Berge trigraphs, then $G$ is bull-free and Berge by 2.2.2. Next, suppose that $G$ or $\overline{G}$ is an elementary expansion of a trigraph in $\mathcal{T}_1^*$; since $G$ is bull-free and Berge if and only if $\overline{G}$

is, we may assume that $G$ is an elementary expansion of a trigraph in $\mathcal{T}_1^*$. Since $\mathcal{T}_1^*$ is a subclass of $\mathcal{T}_1$, $G$ is an elementary expansion of a trigraph in $\mathcal{T}_1$, and so $G$ is bull-free by 3.1.3; $G$ is Berge by 3.3.2 and 3.4.11. Finally, suppose that $G$ is an elementary expansion of a trigraph in $\mathcal{T}_2$. Then $G$ is bull-free by 3.1.3 and Berge by 3.3.2 and 3.5.4. This proves the 'if' part.

To prove the 'only if' part, suppose that $G$ is a bull-free Berge trigraph. If $G$ contains a proper homogeneous set, then $G$ is obtained by substitution from smaller bull-free Berge trigraphs, and we are done. So assume that $G$ contains no proper homogeneous sets. Then by 3.1.2 and 3.1.3, one of the following holds:

- $G$ or $\overline{G}$ is an elementary expansion of a trigraph in $\mathcal{T}_1$;
- $G$ is an elementary expansion of a trigraph in $\mathcal{T}_2$.

If the latter outcome holds, then we are done. So assume that $G$ or $\overline{G}$ is an elementary expansion of a trigraph $H \in \mathcal{T}_1$. Since $G$ (and therefore $\overline{G}$ as well) is Berge, by 3.3.2, $H$ is Berge. By 3.4.11 then, $H \in \mathcal{T}_1^*$. This completes the argument. $\square$

# Chapter 4

# A Decomposition Theorem for Bull-Free Perfect Graphs

The main goal of this chapter is to derive a decomposition theorem for bull-free Berge trigraphs. The results of this chapter will be used in the coloring algorithm for bull-free perfect graphs in chapter 5.

Let us begin with a couple of definitions. First, a tame homogeneous pair $(A, B)$ in a trigraph $G$ is said to be *reducible* provided that the associated partition $(A, B, C, D, E, F)$ of $G$ satisfies the following:

- either
  - $|B| \geq 3$, or
  - $|B| = 2$ and there exist distinct vertices $a, a' \in A$ such that $a$ and $a'$ are both mixed on $B$;
- $C$ and $D$ are both non-empty.

We observe that if $(A, B)$ is a reducible homogeneous pair in a trigraph $G$, then $|A \cup B| \geq 4$ and $|C \cup D \cup E \cup F| \geq 3$ (the latter is a consequence of the fact that $(A, B)$ is tame). We remark that if $(A, B)$ is a reducible homogeneous pair in a trigraph $G$, and if $(A, B, C, D, E, F)$ is the associated partition of $G$, then $(A, B)$ is also a reducible homogeneous pair in $\overline{G}$, and $(A, B, D, C, F, E)$ is the associated partition of $\overline{G}$.

Next, a *directed trigraph* is an ordered pair $\vec{G} = (G, A_G)$, where $G = (V_G, \theta_G)$ is a trigraph, and $A_G \subseteq V_G \times V_G$ satisfies the following:

- for all $u \in V_G$, $(u, u) \notin A_G$;
- for all adjacent pairs $uv$ in $G$, exactly one of $(u, v)$ and $(v, u)$ is in $A_G$;
- for all strongly anti-adjacent pairs $uv$ in $G$, $(u, v) \notin A_G$.

Under these conditions, the directed trigraph $\vec{G}$ is said to be an *orientation* of the trigraph $G$; the set $A_G$ is called the *arc set* of $\vec{G}$ and an *orientation relation* for $G$. Members of $A_G$ are called the *arcs* of $\vec{G}$. An arc $(u, v)$ in $\vec{G}$ is said to be *strong* provided that $uv$ is a strongly adjacent pair in $G$. If $\vec{G}$ is an orientation of a trigraph $G$, and if $(u, v)$ is an arc in $\vec{G}$, then we say that the adjacent pair $uv$ of $G$ is *oriented from $u$ to $v$* in $\vec{G}$, or that it is *oriented as* $(u, v)$ in $\vec{G}$. If $A$ and $B$ are disjoint subsets of $V_G$, then we say that *all the adjacent pairs between $A$ and $B$ are oriented from $A$ to $B$* in $\vec{G}$ provided that for all adjacent pairs $ab$ in $G$ such that $a \in A$ and $b \in B$, the adjacent pair $ab$ is oriented from $a$ to $b$ in $\vec{G}$. Given an oriented trigraph $\vec{G} = (G, A_G)$, and a set $S \subseteq V_G$, we denote by $\vec{G}[S]$ the oriented trigraph $(G[S], A_G \cap (S \times S))$.

As in the case of (undirected) graphs, we note that every directed graph can be thought of as a directed trigraph in a natural way: a directed graph is simply a directed trigraph in which all arcs are strong.

A directed trigraph $\vec{G} = (G, A_G)$ is *transitive* provided that for all $u, v, w \in V_G$, if $(u, v)$ and $(v, w)$ are arcs in $\vec{G}$, then $(u, w)$ is a strong arc in $\vec{G}$; under these circumstances, we say that $A_G$ is a *transitive orientation relation* for $G$, and that $\vec{G}$ is a *transitive orientation* for

$G$. A trigraph $G$ is said to be *transitively orientable* provided that there exists a transitive orientation relation for it. As directed graphs are simply directed trigraphs in which all arcs are strong, it is easy to see that a directed graph $\vec{G}$ is transitive as a graph if and only if it is transitive as a trigraph; furthermore, every graph $G$ is transitively orientable as a graph if and only if it is transitively orientable as a trigraph.

The main goal of this chapter is to prove the following decomposition theorem (which we will need in chapter 5).

**4.0.1.** *Let $G$ be a bull-free Berge trigraph. Then at least one of the following holds:*

- *$G$ or $\overline{G}$ is transitively orientable;*
- *$G$ contains a proper homogeneous set;*
- *$G$ contains a reducible homogeneous pair.*

We remark that a theorem very similar to 4.0.1 was originally proven in [52]. However, in [52], reducible homogeneous pairs were not required to be tame, and so 4.0.1 is slightly stronger than the corresponding theorem from [52]. Furthermore, the proofs of the two theorems are significantly different. In particular, the proof from [52] uses the structure theorem for bull-free Berge trigraphs from [12] (this is the structure theorem from chapter 3 of this thesis), while the proof of 4.0.1 given in the present chapter does not rely on this structure theorem.

In this chapter, we will also prove the following result about reducible homogeneous pairs.

**4.0.2.** *Let $G$ be a bull-free Berge trigraph that does not contain a proper homogeneous set, and let $(A, B)$ be a reducible homogeneous pair in $G$. Then $G[A]$ and $G[B]$ are both transitively orientable.*

The bulk of this chapter will deal with the so-called "elementary" bull-free Berge trigraphs. As in chapter 3, we call a bull-free trigraph $G$ *elementary* provided that it contains no three-edge path $P$ such that some vertex of $G$ is a center for $P$, and some vertex of $G$ is an anti-center for $P$. A bull-free trigraph that is not elementary is said to be *non-elementary*. (We remark that elementary and non-elementary bull-free trigraphs were originally introduced in [7].) It was shown in chapter 3 that every non-elementary bull-free Berge trigraph admits a homogeneous set decomposition (see 3.1.2); equivalently, every bull-free Berge trigraph that does not admit a homogeneous set decomposition is elementary. We will use this result repeatedly in this chapter, and we state it below for future reference.

**4.0.3.** *Every bull-free Berge trigraph that does not admit a homogeneous set decomposition is elementary.*

This chapter is organized as follows. In section 4.1, we prove some results about transitively orientable trigraphs, and we prove 4.0.2; we remark that results very similar to the ones from section 4.1 were originally proven in [52]. In section 4.2, we state some lemmas (due to Chudnovsky) from [7] and [8] that we will need in section 4.3. In section 4.3, we prove a "preliminary" decomposition theorem for bull-free Berge trigraphs (see 4.3.2), which we will use as a "stepping stone" toward 4.0.1; we remark that in section 4.3, we make heavy use of ideas from [8], in particular, from the proof of 6.2 from [8] (this is discussed in more detail in section 4.3). Finally, in section 4.4, we study homogeneous pairs of various kinds, and we prove 4.0.1 (the main result of this chapter).

## 4.1 Transitively Orientable Trigraphs

Our main goal in this section is to prove two theorems: one about "appropriate expansions" (defined below) of transitively orientable trigraphs (4.1.1), and one about trigraphs that contain no three-edge path as an induced subtrigraph (4.1.3); these two results will be used in sections 4.3 and 4.4. At the end of the section, we use 4.1.3 and 4.0.3 to prove 4.0.2.

We begin with the definition of an "appropriate expansion." Given a trigraph $G$ and a semi-adjacent pair $ab$ in $G$, we say that $ab$ is *expandable* provided that $G$ contains at least five vertices, and there exist vertices $c, d \in V_G \setminus \{a, b\}$ such that $c$ is strongly adjacent to $a$ and strongly anti-adjacent to $b$, and $d$ is strongly adjacent to $b$ and strongly anti-adjacent to $a$. A semi-adjacent pair $ab$ in $G$ is said to be *non-expandable* provided that it is not expandable. (Note that if $G$ is a trigraph on at most four vertices, then no semi-adjacent pair in $G$ is expandable. Note also that if $ab$ is an expandable semi-adjacent pair in a trigraph $G$, then $ab$ is also expandable in $\overline{G}$.) Given trigraphs $H$ and $G$, we say that $G$ is an *appropriate expansion* of $H$ provided that there exists a family $\{X_v\}_{v \in V_H}$ of pairwise disjoint non-empty sets such that $V_G = \bigcup_{v \in V_H} X_v$, with all of the following satisfied:

- for all $v \in V_H$, if $v$ is not an endpoint of any expandable semi-adjacent pair in $H$, then $|X_v| = 1$;
- if $uv$ is a non-expandable semi-adjacent pair in $H$, then the unique vertex of $X_u$ is semi-adjacent to the unique vertex of $X_v$ in $G$;
- for all strongly adjacent pairs $uv$ in $H$, $X_u$ is strongly complete to $X_v$;
- for all strongly anti-adjacent pairs $uv$ in $H$, $X_u$ is strongly anti-adjacent to $X_v$.

We note that if a trigraph $G$ is an appropriate expansion of a trigraph $H$, then $\overline{G}$ is an appropriate expansion of the trigraph $\overline{H}$. We also note that every trigraph is an appropriate expansion of itself.

**4.1.1.** *Let $H$ be a transitively orientable trigraph, and let $G$ be an appropriate expansion of $H$. Then at least one of the following holds:*

- *$G$ is transitively orientable;*
- *$G$ contains a proper homogeneous set;*
- *$G$ contains a reducible homogeneous pair.*

*Proof.* Suppose first that $|X_v| \geq 3$ for some $v \in V_G$. Since $|X_v| > 1$, there exists some $u \in V_H$ such that $uv$ is an expandable semi-adjacent pair in $H$. If $X_u$ is strongly complete or strongly anti-complete to $X_v$, then $X_v$ is a proper homogeneous set in $G$, and we are done. So assume that $X_u$ is neither strongly complete nor strongly anti-complete to $X_v$. Let $(\{u\}, \{v\}, C, D, E, F)$ be the partition of $H$ associated with the homogeneous pair $(\{u\}, \{v\})$; since $uv$ is expandable, we know that $C$ and $D$ are non-empty, and that $|C \cup D \cup E \cup F| \geq 3$. But now $(X_u, X_v)$ is a homogeneous pair in $G$ with the associated partition

$$(X_u, X_v, \bigcup_{w \in C} X_w, \bigcup_{w \in D} X_w, \bigcup_{w \in E} X_w, \bigcup_{w \in F} X_w);$$

since $C$ and $D$ are non-empty, we know that $\bigcup_{w \in C} X_w$ and $\bigcup_{w \in D} X_w$ are both non-empty, and since $|X_v| \geq 3$ and $|C \cup D \cup E \cup F| \geq 3$, we know that $3 \leq |X_u \cup X_v| \leq |V_G| - 3$. Since $X_u$ is neither strongly complete nor strongly anti-complete to $X_v$, and since $3 \leq |X_u \cup X_v| \leq |V_G| - 3$, we know that $(X_u, X_v)$ is a tame homogeneous pair in $G$. Since $|X_v| \geq 3$, and since $\bigcup_{w \in C} X_w$ and $\bigcup_{w \in D} X_w$ are both non-empty, it follows that $(X_u, X_v)$ is a reducible homogeneous pair in $G$, and we are done.

From now on, we assume that $|X_v| \leq 2$ for all $v \in V_H$. First, suppose that for some $v \in V_H$, we have that $|X_v| = 2$, and that more than one vertex in $V_G \setminus X_v$ is mixed on $X_v$. As $|X_v| > 1$, we know that there exists some $u \in V_H$ such that $uv$ is an expandable semi-adjacent pair in $H$. By the definition of an appropriate expansion, every vertex in $V_G \setminus X_v$ that is mixed on $X_v$ is a member of $X_u$; as more than one vertex of $V_G \setminus X_v$ is mixed on $X_v$, this implies that at least two distinct vertices of $X_u$ are mixed on $X_v$. Now, let $(\{u\}, \{v\}, C, D, E, F)$ be the partition of $H$ associated with $(\{u\}, \{v\})$; as $uv$ is expandable, we know that $C$ and $D$ are non-empty, and $|C \cup D \cup E \cup F| \geq 3$. But now $(X_u, X_v)$ is a homogeneous pair in $G$ with the associated partition

$$(X_u, X_v, \bigcup_{w \in C} X_w, \bigcup_{w \in D} X_w, \bigcup_{w \in E} X_w, \bigcup_{w \in F} X_w);$$

since $C$ and $D$ are non-empty, we know that $\bigcup_{w \in C} X_w$ and $\bigcup_{w \in D} X_w$ are non-empty, and since $|X_v| = 2$ and $|C \cup D \cup E \cup F| \geq 3$, we know that $3 \leq |X_u \cup X_v| \leq |V_G| - 3$. It now follows that $(X_u, X_v)$ is a reducible homogeneous pair in $G$, and we are done.

From now on, we assume that for all $v \in V_H$ such that $|X_v| = 2$, at most one vertex in $V_G \setminus X_v$ is mixed on $X_v$. If there exists some $v \in V_H$ such that $|X_v| = 2$ and no vertex in $V_G \setminus X_v$ is mixed on $X_v$, then $X_v$ is a proper homogeneous set in $G$, and we are done. So we may assume that for all $v \in V_H$ such that $|X_v| = 2$, exactly one vertex in $V_G \setminus X_v$ is mixed on $X_v$. Our goal now is to show that $G$ is transitively orientable.

Since $H$ is transitively orientable, there exists a transitive orientation relation $A_H$ for the trigraph $H$; set $\vec{H} = (H, A_H)$, so that $\vec{H}$ is a transitive directed trigraph. Now, we define an orientation relation $A_G$ for the trigraph $G$ as follows. For all distinct $u, v \in V_H$ such that $(u, v) \notin A_H$, set $A_{u,v} = \emptyset$. For all distinct $u, v \in V_H$ such that $(u, v) \in A_H$, set $A_{u,v} = \{(\hat{u}, \hat{v}) \mid \hat{u} \in X_u, \hat{v} \in X_v, \theta_G(\hat{u}, \hat{v}) \geq 0\}$. For all $v \in V_H$, if $X_v$ is a strongly stable set, set $A_{v,v} = \emptyset$. Now, suppose that $v \in V_H$ is such that $|X_v| = 2$, say $X_v = \{\hat{v}_1, \hat{v}_2\}$, where $\hat{v}_1\hat{v}_2$ is an adjacent pair. Then there exists a unique vertex $\hat{u} \in V_G \setminus X_v$ such that $\hat{u}$ is mixed on $X_v$; fix distinct $i, j \in \{1, 2\}$ such that $\hat{u}$ is adjacent to $v_i$ and anti-adjacent to $v_j$. Next, fix $u \in V_H$ such that $\hat{u} \in X_u$; then $uv$ is an expandable semi-adjacent pair, and in particular, either $(u, v) \in A_H$ or $(v, u) \in A_H$. If $(u, v) \in A_H$, then set $A_{v,v} = \{(v_j, v_i)\}$; and if $(v, u) \in A_H$, then set $A_{v,v} = \{(v_i, v_j)\}$. Finally, set $A_G = \bigcup_{u,v \in V_G} A_{u,v}$ and $\vec{G} = (G, A_G)$. Clearly, $\vec{G}$ is a directed trigraph; we claim that $\vec{G}$ is a transitive.

Let $x, y, z \in V_G$, and assume that $(x, y)$ and $(y, z)$ are arcs in $\vec{G}$; we need to show that $(x, z)$ is a strong arc in $\vec{G}$. First, we claim that there does not exist a vertex $v \in V_H$ such that $x, z \in X_v$. Suppose otherwise. Fix $v \in V_H$ such that $x, z \in X_v$; since $|X_v| \leq 2$, this implies that $X_v = \{x, z\}$. Next, fix $u \in V_H \setminus \{v\}$ such that $y \in X_u$. Now, since $(x, y) \in A_G$, $x \in X_v$, and $y \in X_u$, we know that $(v, u) \in V_H$; on the other hand, since $(y, z) \in A_G$, $y \in X_u$, and $z \in X_v$, we know that $(u, v) \in A_H$. But then $(u, v)$ and $(v, u)$

are both in $A_H$, which is impossible. This proves our claim.

Fix distinct $u, v \in V_H$ such that $x \in X_u$ and $z \in X_v$. There are three possibilities: that $y \in X_u$; that $y \in X_v$; and that $y \in X_w$ for some $w \in V_H \setminus \{u, v\}$. We note, however, that the cases when $y \in X_u$ and when $y \in X_v$ are very similar, and so it suffices to consider only the following two cases: when $y \in X_u$, and when $y \in X_w$ for some $w \in V_H \setminus \{u, v\}$.

Suppose first that $y \in X_u$. Since $(y, z) \in A_G$, $y \in X_u$, and $z \in X_v$, it follows that $(u, v) \in A_H$. Since $x \in X_u$ and $z \in X_v$, this implies that if $xz$ is an adjacent pair in $G$ then $(x, z) \in A_G$. Thus, it suffices to show that $xz$ is a strongly adjacent pair in $G$. Since $(u, v) \in A_H$, we know that $uv$ is an adjacent pair in $G$. If $uv$ is a strongly adjacent pair in $H$, then since $x \in X_u$ and $z \in X_v$, we have that $xz$ is a strongly adjacent pair, and we are done. So assume that $uv$ is a semi-adjacent pair in $H$. Now, suppose that $xz$ is not a strongly adjacent pair in $G$; then $xz$ is an anti-adjacent pair. Since $(y, z) \in A_G$, we know that $yz$ is an adjacent pair in $G$. Now $yz$ is adjacent pair in $G$, $xz$ is an anti-adjacent pair in $G$, and $x, y \in X_u$, $z \in X_v$, and $(u, v) \in A_H$; by construction then, $(y, x) \in A_G$. But this is impossible since $(x, y) \in A_G$. Thus, $xz$ is a strongly adjacent pair, and we are done.

Suppose now that $y \in X_w$ for some $w \in V_H \setminus \{u, v\}$. Since $(x, y) \in A_G$, $x \in X_u$, and $y \in X_w$, we get that $(u, w) \in A_H$. Similarly, since $(y, z) \in A_G$, $y \in X_w$, and $z \in X_v$, we know that $(w, v) \in A_H$. Now since $(u, w), (w, v) \in A_H$, and $\vec{H}$ is transitive, we know that $uv$ is a strongly adjacent pair in $H$ and that $(u, v) \in A_H$. Since $x \in X_u$ and $z \in X_v$, it follows that $xz$ is a strongly adjacent pair, and that $(x, z) \in A_G$. This completes the argument. □

Our next goal is to prove that every trigraph that contains no three-edge path as an induced subtrigraph is transitively orientable (see 4.1.3 below). The proof of this result uses 4.5 from [10], which we state below.

**4.1.2** (Chudnovsky [10]). *Let $G$ be a trigraph that contains at least two vertices and that does not contain a three-edge path as an induced subtrigraph. Then at least one of the following holds:*

- *$G$ is not connected;*
- *$\overline{G}$ is not connected;*
- *there exist vertices $x, y \in V_G$ such that $x$ is semi-adjacent to $y$, $x$ is strongly anti-complete to $V_G \setminus \{x, y\}$, and $y$ is strongly complete to $V_G \setminus \{x, y\}$.*

**4.1.3.** *Let $G$ be a trigraph that does not contain a three-edge path as an induced subtrigraph. Then $G$ is transitively orientable.*

*Proof.* We may assume inductively that for all $X \subsetneqq V_G$, $G[X]$ is transitively orientable. Clearly, if $|V_G| \leq 2$, then $G$ is transitively orientable; so assume that $|V_G| \geq 3$. By 4.1.2, $G$ satisfies at least one of the following:

(i) $G$ is not connected;

(ii) $\overline{G}$ is not connected;

(iii) there exist vertices $x, y \in V_G$ such that $x$ is semi-adjacent to $y$, $x$ is strongly anti-complete to $V_G \setminus \{x, y\}$, and $y$ is strongly complete to $V_G \setminus \{x, y\}$.

Suppose first that (i) or (ii) holds, that is, that one of $G$ and $\overline{G}$ is not connected. Fix disjoint, non-empty sets $X, Y \subseteq V_G$ such that $V_G = X \cup Y$ and such that $X$ is either strongly complete or strongly anti-complete to $Y$. By assumption, $G[X]$ and $G[Y]$ are transitively orientable. Let $A_{G[X]}$ and $A_{G[Y]}$ be transitive orientation relations for $G[X]$ and $G[Y]$, respectively. If $X$ is anti-complete to $Y$, then it is easy to see that $A_{G[X]} \cup A_{G[Y]}$ is a transitive orientation relation for $G$, and we are done. So assume that $X$ is strongly complete to $Y$. Let $A'_G = \{(x, y) \mid x \in X, y \in Y\}$, and set $A_G = A_{G[X]} \cup A_{G[Y]} \cup A'_G$. We claim that $A_G$ is a transitive orientation relation for $G$. Fix $u, v, w \in V_G$, and assume that $(u, v), (v, w) \in A_G$; we need to show that $uv$ is a strongly adjacent pair in $G$ and that

$(u,w) \in A_G$. If $u \in X$ and $w \in Y$, then this is immediate. So assume that either $u \in Y$ or that $w \in X$. Suppose first that $u \in Y$. Since $u \in Y$ and $(u,v) \in A_G$, it follows that $v \notin X$, and consequently, $v \in Y$. Similarly, since $v \in Y$ and $(v,w) \in A_G$, it follows that $w \notin X$, and so $w \in Y$. But now $u, v, w \in Y$, and the result follows from the fact that $A_{G[Y]}$ is a transitive orientation relation for $G[Y]$. In a similar way, we get that if $w \in X$, then $u, v, w \in X$, and then the result follows from the fact that $A_{G[X]}$ is a transitive orientation relation for $G[X]$.

It remains to consider the case when (iii) holds. Set $Y = V_G \smallsetminus \{x, y\}$; then $xy$ is a semi-adjacent pair, $x$ is strongly anti-complete to $Y$, and $y$ is strongly complete to $Y$. By assumption, $G \smallsetminus y$ is transitively orientable. Let $A_{G \smallsetminus y}$ be a transitive orientation relation for $G \smallsetminus y$. Let $A_y = \{(y, y') \mid y' \in Y\}$, and set $A_G = \{(y, x)\} \cup A_y \cup A_{G \smallsetminus y}$. We claim that $A_G$ is a transitive orientation relation on $G$. Fix $u, v, w \in V_G$, and assume that $(u,v), (v,w) \in A_G$; we need to show that $uw$ is a strongly adjacent pair, and that $(u,w) \in A_G$. If $y \notin \{u, v, w\}$, then the result follows from the fact that $A_{G \smallsetminus y}$ is a transitive orientation relation for $G \smallsetminus y$. So assume that $y \in \{u, v, w\}$. By construction, we have that for all $z \in V_G \smallsetminus \{y\}$, $(z, y) \notin A_G$; thus, $y \neq v$ and $y \neq w$, and consequently, $y = u$. Note that $x$ has only one neighbor in $G$ (namely $y$), while $v$ has at least two neighbors (namely $u$ and $w$) in $G$; thus, $v \neq x$, and consequently, $v \in Y$. Since $x$ is anti-complete to $Y$, and $w$ is adjacent to $v \in Y$, it follows that $w \in Y$. But now $u = y$ and $w \in Y$, and so by construction, $uw$ is a strongly adjacent pair in $G$, and $(u,w) \in A_G$. This completes the argument. $\square$

We complete this section with the proof of 4.0.2, restated below.

**4.0.2.** *Let $G$ be a bull-free Berge trigraph that does not contain a proper homogeneous set, and let $(A, B)$ be a reducible homogeneous pair in $G$. Then $G[A]$ and $G[B]$ are both transitively orientable.*

*Proof.* First, by 4.0.3, $G$ is elementary. Let $(A, B, C, D, E, F)$ be the partition of $G$ associated with $(A, B)$. Since $(A, B)$ is reducible, we know that $C$ and $D$ are non-empty; fix $c \in C$ and $d \in D$. Since $G$ is elementary, and $c$ is a center and $d$ an anti-center for $A$, we know that $G[A]$ does not contain a three-edge path, and so by 4.1.3, $G[A]$ is transitively orientable. Similarly, since $G$ is elementary, and $d$ is a center and $c$ an anti-center for $B$, we know that $G[B]$ does not contain a three-edge path, and so by 4.1.3, $G[B]$ is transitively orientable. □

## 4.2 Some Lemmas from [7] and [8]

In this section, we state some lemmas from [7] and [8] that we will need in the proof of 4.0.1.

First, we will need 3.2 from [7] (this is the main theorem of [7]). We state this theorem below (we remark that we have not given the definition of the class $\mathcal{T}_0$ in this thesis).

**4.2.1** (Chudnovsky [7])**.** *Let $G$ be a bull-free trigraph, and let $P$ and $Q$ be three-edge paths in $G$. Assume that $G$ contains a center for $P$ and an anti-center for $Q$ in $G$. Then at least one of the following holds:*

- *$G$ admits a homogeneous set decomposition;*
- *$G$ admits a homogeneous pair decomposition;*
- *$G$ or $\overline{G}$ belongs to the class $\mathcal{T}_0$.*

We omit the definition of the class $\mathcal{T}_0$, and we refer the reader to [7]. All that we need here is the fact that every trigraph in $\mathcal{T}_0$ contains a hole of length five (this readily follows from the definition of the class $\mathcal{T}_0$), and so 4.2.1 immediately yields the following result.

**4.2.2.** *Let $G$ be a bull-free Berge trigraph, and let $P$ and $Q$ be three-edge paths in $G$. Assume that $G$ contains a center for $P$ and an anti-center for $Q$ in $G$. Then at least one of the following holds:*

- *$G$ admits a homogeneous set decomposition;*

- *$G$ admits a homogeneous pair decomposition.*

We now need some definitions from [8]. First, a bull-free trigraph that admits neither a homogeneous set decomposition nor a homogeneous pair decomposition, and that does not contain a three-edge path with a center, is called *unfriendly*. Second, a *square* in a trigraph $G$ is a hole of length four in $G$. Finally, a *prism* is a trigraph $G$ with vertex-set $V_G = \{a_1, a_2, a_3, b_1, b_2, b_3\}$, and adjacency as follows:

- $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are cliques;
- for all $i \in \{1, 2, 3\}$, $a_i b_i$ is an adjacent pair;
- for all distinct $i, j \in \{1, 2, 3\}$, $a_i b_j$ is an anti-adjacent pair.

Note that if we insist that the prism $G$ be bull-free, then $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ must in fact be strong cliques (indeed, suppose that at least one of these two cliques is not strong; then by symmetry, we may assume that $b_1 b_2$ is a semi-adjacent pair, and then $G[a_1, a_2, a_3, b_1, b_2]$ is a bull). This implies that every bull-free prism is complement-bipartite; we state this result below for future reference.

**4.2.3.** *Every bull-free prism is complement-bipartite.*

Next, the following is 4.2 from [8].

**4.2.4** (Chudnovsky [8])**.** *Let $G$ be an unfriendly bull-free trigraph, and assume that $G$ contains a prism as an induced subtrigraph. Then $G$ is a prism.*

We complete this section with some lemmas from section 5 of [8]. The following three lemmas are 5.2, 5.3, and 5.6 (in that order) from [8].

**4.2.5** (Chudnovsky [8])**.** *Let $G$ be an unfriendly bull-free trigraph that does not contain a prism as an induced subtrigraph, and let $a_1 - a_2 - a_3 - a_4 - a_1$ be a square in $G$. Let $K$ be the set of all vertices in $V_G \setminus \{a_1, a_2, a_3, a_4\}$ that are complete to $\{a_1, a_2\}$ and anti-complete to $\{a_3, a_4\}$. Then $K$ is a strong clique in $G$.*

**4.2.6** (Chudnovsky [8]). *Let $G$ be an unfriendly bull-free trigraph that does not contain a prism as an induced subtrigraph, let $a_1 - a_2 - a_3 - a_4 - a_1$ be a square in $G$, and let $c$ be a center and $a$ an anti-center for $\{a_1, a_2, a_3, a_4\}$. Then $c$ is strongly anti-adjacent to $a$.*

**4.2.7** (Chudnovsky [8]). *Let $G$ be an unfriendly bull-free trigraph that does not contain a prism as an induced subtrigraph. Then there do not exist six vertices $a, b, c, d, x, y \in V_G$ such that all of the following hold:*

- *$ab$, $cd$, and $xy$ are adjacent pairs;*
- *$\{a, b\}$ is anti-complete to $\{c, d\}$;*
- *$\{x, y\}$ is complete to $\{a, b, c, d\}$.*

## 4.3 A "Preliminary" Decomposition Theorem for Bull-Free Berge Trigraphs

The goal of this section is to prove a certain "preliminary" decomposition theorem for bull-free Bere trigraphs (see 4.3.2); this decomposition theorem will be used in the proof of 4.0.1 (the main result of this chapter) in section 4.4.

We begin with some definitions. A *frame* is a connected, bipartite triagraph that contains a three-edge path as an induced subtrigraph. A trigraph is *framed* if some induced subtrigraph of it is a frame. A *frame* for a trigraph $G$ is a subtrigraph of $G$ that is a frame. An *optimal* frame for a trigraph $G$ is a frame $F$ for $G$ such that all of the following are satisfied:

- for every frame $F'$ for $G$, $|V_{F'}| \leq |V_F|$;
- for every frame $F'$ for $G$, if $|V_{F'}| = |V_F|$, then $F$ has at least as many adjacent pairs as $F'$ does;

- for every frame $F'$ for $G$, if $|V_{F'}| = |V_F|$ and $F'$ has the same number of adjacent pairs as $F$ does, then $F$ has at least as many strongly adjacent pairs as $F'$ does.

Clearly, every framed trigraph contains an optimal frame (but this optimal frame need not be unique). Frames were originally defined in [8], however, the definition in [8] was slightly different: a frame was only required to be triangle-free, and not necessarily bipartite. However, in this chapter, we are only interested in Berge trigraphs, and a Berge trigraph is bipartite if and only if it is triangle-free; this motivated the change of the definition.

The result that follows is a technical lemma whose proof borrows heavily from the proof of 6.2 of [8] (specifically, the proofs of the first nine statements of 6.2 of [8]). However, since we are restricting our attention to Berge trigraphs here, we are able to strengthen certain claims from the proof of 6.2 of [8]. (Clearly, the conclusion of 4.3.1 is false if the assumption that the trigraph be Berge is omitted: it is easy to check that all transitively orientable trigraphs are Berge.)

**4.3.1.** *Every unfriendly framed bull-free Berge trigraph that contains no prism is transitively orientable.*

*Proof.* Let $G$ be an unfriendly framed trigraph that does not contain a prism as an induced subtrigraph. First, note that $|V_G| \geq 4$ (this is because $G$ is framed, and every frame contains a three-edge path), and so since $G$ contains no proper homogeneous set, we know that $G$ is connected. Let $F$ be an optimal frame for $G$, and let $(E_1, E_2)$ be a bipartition of the bipartite trigraph $F$. For all $v \in V_G \smallsetminus V_F$ and $i \in \{1, 2\}$, let $S_i(v)$ be the set of all neighbors of $v$ in $E_i$.

**Claim 1.** *For every vertex $v \in V_G \smallsetminus V_F$, if $v$ is not strongly anti-complete to $V_F$, then $S_1(v)$ and $S_2(v)$ are both non-empty, and $S_1(v)$ is not strongly anti-complete to $S_2(v)$.*

Fix $v \in V_G \smallsetminus V_F$ such that $v$ is not strongly anti-complete to $V_F$. Then at least one of $S_1(v)$ and $S_2(v)$ is non-empty, and we need to show that they are both non-empty.

Suppose otherwise; by symmetry, we may assume that $S_1(v) \neq \emptyset$ and $S_2(v) = \emptyset$. Then $G[V_F \cup \{v\}]$ is a connected bipartite subtrigraph of $G$ with bipartition $(E_1, E_2 \cup \{v\})$, and clearly, $G[V_F \cup \{v\}]$ contradicts the optimality of $F$. This proves that $S_1(v)$ and $S_2(v)$ are both non-empty. It remains to show that $S_1(v)$ is not strongly anti-complete to $S_2(v)$. Suppose otherwise. Let $P$ be a path in $F$ such that one endpoint of $P$ is in $S_1(v)$ and the other is in $S_2(v)$, and assume that $P$ is of minimum length among all such paths. Since $S_1(v)$ is strongly anti-complete to $S_2(v)$, $P$ is of length greater than one; and since $F$ is bipartite, $P$ is of odd length. But now $G[V_P \cup \{v\}]$ is an odd hole in $G$, contrary to the fact that $G$ is Berge.

**Claim 2.** *For every vertex $u \in V_G \smallsetminus V_F$, $S_1(u)$ and $S_2(u)$ are both non-empty, and $S_1(u)$ is not strongly anti-complete to $S_2(u)$.*

In view of Claim 1, it suffices to show that every vertex in $V_G \smallsetminus V_F$ has a neighbor in $V_F$. Suppose otherwise. Since $G$ is connected, there exist adjacent vertices $u, v \in V_G \smallsetminus V_F$ such that $u$ is strongly anti-complete to $V_F$, while $v$ has a neighbor in $V_F$. By Claim 1, $S_1(v)$ and $S_2(v)$ are both non-empty, and they are not strongly anti-complete to each other. Let $C$ be the vertex-set of a non-trivial component of $G[S_1(v) \cup S_2(v)]$. Since $G$ is unfriendly and $v$ is complete to $C$, we know that $G[C]$ does not contain a three-edge path. Since $F$ contains a three-edge path, it follows that $C \subsetneqq V_F$. Since $F$ is connected, this implies that there exist adjacent vertices $c \in C$ and $w \in V_F \smallsetminus (S_1(v) \cup S_2(v))$. Since $G[C]$ is connected and $|C| \geq 2$, there exists some $c' \in C \smallsetminus \{c\}$ such that $cc'$ is an adjacent pair. Since $F$ contains no triangles (because it is bipartite), we know that $c'w$ is a strongly anti-adjacent pair. Now $G[u, v, c, c', w]$ is a bull, which is a contradiction.

**Claim 3.** *For all $v \in V_G \smallsetminus V_F$, if $C$ is the vertex-set of a component of $G[S_1(v) \cup S_2(v)]$, then $C \cap S_1(v)$ is complete to $C \cap S_2(v)$.*

Let $v \in V_G \smallsetminus V_F$, let $C$ be the vertex-set of a component of $G[S_1(v) \cup S_2(v)]$, and assume that $C \cap S_1(v)$ is not complete to $C \cap S_2(v)$. Since $S_1(v)$ and $S_2(v)$ are both stable, we know that any path in $G[C]$ with one endpoint in $C \cap S_1(v)$ and the other one in

$C \cap S_2(v)$ is of odd length. Now, fix strongly anti-adjacent vertices $c_1 \in C \cap S_1(v)$ and $c_2 \in C \cap S_2(v)$. Since $G[C]$ is connected, there is a path in $G[C]$ between them; this path is of odd length greater than one. Now $G[C]$ contains a three-edge path. But $v$ is a center for this three-edge path, contrary to the fact that $G$ is unfriendly.

**Claim 4.** *For all $v \in V_G \setminus V_F$, $G[S_1(v) \cup S_2(v)]$ is connected.*

Fix $v \in V_G \setminus V_F$. Let $C$ be the vertex-set of a non-trivial component of $G[S_1(v) \cup S_2(v)]$ (the existence of such a component follows from Claim 2). If $C = S_1(v) \cup S_2(v)$, then we are done; so assume that $C \subsetneqq S_1(v) \cup S_2(v)$, and set $C' = (S_1(v) \cup S_2(v)) \setminus C$. Set $C_1 = C \cap S_1(v)$ and $C_2 = C \cap S_2(v)$; clearly, $C_1$ and $C_2$ are both strongly stable, and so since $G[C]$ is connected and has at least two vertices, it follows that $C_1$ and $C_2$ are both non-empty. Furthermore, by Claim 3, $C_1$ is complete to $C_2$.

First, we claim that either $C_1$ is strongly anti-complete to $E_2 \setminus C_2$, or $C_2$ is strongly anti-complete to $E_1 \setminus C_1$. Suppose otherwise. Fix adjacent $c_1 \in C_1$ and $e_2 \in E_2 \setminus C_2$, and fix adjacent $c_2 \in C_2$ and $e_1 \in E_1 \setminus C_1$. Since $C_1$ is complete to $C_2$, $c_1c_2$ is an adjacent pair. Since $C' \neq \emptyset$, there exists some $u \in C'$; by symmetry, we may assume that $u \in E_1$. But now $G[v, c_1, c_2, e_1, u]$ is a bull, which is a contradiction. This proves our claim. By symmetry, we may assume that $C_2$ is strongly anti-complete to $E_1 \setminus C_1$.

Let $F_2$ be the set of all vertices in $E_2 \setminus S_2(v)$ that have a neighbor in $C_1$; since $F$ is connected, $F_2$ is non-empty. Then $F_2$ must be strongly complete to $C'$. (Indeed, suppose that there were some anti-adjacent $f \in F_2$ and $c' \in C'$. Fix some $c_1 \in C_1$ such that $c_1 f$ is an adjacent pair. Since $C_1$ is complete to $C_2$, and $C_2$ is non-empty, there exists some $c_2 \in C_2$ such that $c_1c_2$ is an adjacent pair. Since $E_2$ is stable, $c_2 f$ is an anti-adjacent pair. But now $G[v, c_1, c_2, f, c']$ is a bull, which is a contradiction.) In particular then, $C' \subseteq E_1$.

If $|C_2| = 1$, say $C_2 = \{c_2\}$, then it is easy to see that $G[(V_F \setminus \{c_2\}) \cup \{v\}]$ contradicts the optimality of $F$. Thus, $|C_2| \geq 2$. If $|C_1| \geq 2$, then since $C_1$ is complete to $C_2$, we know that $G[C]$ contains a square. Now $v$ is a center for this square and any vertex $c' \in C'$ is an anti-center for it. Then by 4.2.6, $vc'$ must be a strongly anti-adjacent pair, which is a contradiction because $v$ is complete to $C'$. It follows that $|C_1| = 1$, say $C_1 = \{c_1\}$. Since every vertex in $F_2$ has a neighbor in $C_1$, this implies that $c_1$ is complete to $F_2$. Fix some $f \in F_2$ and $c' \in C'$. Now $v - c_1 - f - c' - v$ is a square. By 4.2.5, the set of all vertices in $G$ that are complete to $\{v, c_1\}$ and anti-complete to $\{f, c'\}$ is a strong clique. But every vertex in $C_2$ is complete to $\{v, c_1\}$ and anti-complete to $\{f, c'\}$, and so $C_2$ is a strong clique. Since $C_2$ is also a strongly stable set (because $C_2 \subseteq E_2$ and $E_2$ is strongly stable), we get that $|C_2| = 1$. But this contradicts the fact that $|C_2| \geq 2$.

**Claim 5.** *For all $v \in V_G \smallsetminus V_F$, $S_1(v)$ is strongly complete to $S_2(v)$.*

Let $v \in V_G \smallsetminus V_F$. By Claim 2, $S_1(v)$ and $S_2(v)$ are both non-empty. By Claim 4, $G[S_1(v) \cup S_2(v)]$ is connected, and so by Claim 3, $S_1(v)$ is complete to $S_2(v)$. Thus, we just need to show that there are no semi-adjacent pairs in $G$ with one endpoint in $S_1(v)$ and the other in $S_2(v)$.

Suppose first that $|S_1(v)| \geq 2$ and $|S_2(v)| \geq 2$, and suppose that some $s_1 \in S_1(v)$ and $s_2 \in S_2(v)$ are semi-adjacent. Fix $s_1' \in S_1(v) \smallsetminus \{s_1\}$ and $s_2' \in S_2(v) \smallsetminus \{s_2\}$. Then $s_1 - s_2' - s_1' - s_2$ is a three-edge path, and $v$ is a center for it, contrary to the fact that $G$ is unfriendly. This proves that if $|S_1(v)| \geq 2$ and $|S_2(v)| \geq 2$, then $S_1(v)$ is strongly complete to $S_2(v)$. From now on, we assume that one of $S_1(v)$ and $S_2(v)$ contains only one vertex; by symmetry, we may assume that $|S_1(v)| = 1$, say $S_1(v) = \{s_1\}$.

Next, suppose that $|S_2(v)| = 1$, say $S_2(v) = \{s_2\}$. If $s_1 s_2$ is a strongly adjacent pair, then we are done; so assume that $s_1 s_2$ is a semi-adjacent pair. (Note that this means that $v$ is strongly adjacent to both $s_1$ and $s_2$.) Suppose first that $s_1$ has a neighbor $s_2' \in E_2 \smallsetminus \{s_2\}$, and $s_2$ has a neighbor $s_1' \in E_1 \smallsetminus \{s_1\}$. But now if $s_1' s_2'$ is an adjacent pair, then $v - s_1 - s_2' - s_1' - s_2 - v$ is a hole of length five (contrary to the fact that $G$ is Berge), and if $s_1's_2'$ is an anti-adjacent pair, then $G[v, s_1, s_2, s_1', s_2']$ is a bull (contrary to the fact that $G$ is bull-free). We may now assume by symmetry that $s_1$ is strongly anti-compete to $E_2 \smallsetminus \{s_2\}$. But now $G[(V_F \smallsetminus \{s_1\}) \cup \{v\}]$ contradicts the optimality of $F$ (this is because $G[(V_F \smallsetminus \{s_1\}) \cup \{v\}]$ has the same number of vertices and the same number of adjacent pairs as $F$ does, but $G[(V_F \smallsetminus \{s_1\}) \cup \{v\}]$ has one more strongly adjacent pair than $F$ does).

From now on, we assume that $|S_2(v)| \geq 2$. If $s_1$ is strongly complete to $S_2(v)$, then we are done, so assume that $s_1$ is semi-adjacent to some $s_2 \in S_2(v)$.

First, we claim that $s_1$ is strongly anti-complete to $E_2 \smallsetminus S_2(v)$. Suppose otherwise; fix some $e_2 \in E_2 \smallsetminus S_2(v)$ such that $s_1e_2$ is an adjacent pair. Fix some $s_2' \in S_2(v) \smallsetminus \{s_2\}$. But now $G[v, s_1, s_2, s_2', e_2]$ is a bull, which is a contradiction. Thus, $s_1$ is strongly anti-complete to $E_2 \smallsetminus S_2(v)$.

Next, $v$ must be semi-adjacent to some $s_2' \in S_2(v) \smallsetminus \{s_2\}$, for otherwise, $G[(V_F \smallsetminus \{s_1\}) \cup \{v\}]$ would contradict the optimality of $F$. Now, suppose that $|S_2(v)| \geq 3$; fix some $s_2'' \in S_2(v) \smallsetminus \{s_2, s_2'\}$. Then $G[v, s_1, s_2, s_2', s_2'']$ is a bull, which is a contradiction. It follows that $S_2(v) = \{s_2, s_2'\}$. Next, suppose that $S_2(v)$ is strongly anti-complete to $E_1 \smallsetminus \{s_1\}$. Then since $F$ is connected, it follows that $V_F = \{s_1, s_2, s_2''\}$, which contradicts the fact that $F$ contains a three-edge path. Now, fix some $e_1 \in E_1 \smallsetminus \{s_1\}$ such that $e_1$ has a neighbor in $\{s_2, s_2'\}$. If $e_1$ is adjacent to exactly one of $s_2$ and $s_2'$, then $G[v, s_1, s_2, s_2', e_1]$ is a bull, contrary to the fact that $G$ is bull-free. Thus, $e_1$ is complete to $\{s_2, s_2'\}$. But now $v - s_2 - e_1 - s_2' - s_1 - v$ is a hole of length five, contrary to the fact that $G$ is Berge.

**Claim 6.** *For all adjacent $u, v \in V_G \smallsetminus V_H$, both $S_1(u) \cap S_1(v)$ and $S_2(u) \cap S_2(v)$ are non-empty.*

Suppose otherwise. Fix adjacent $u, v \in V_G \smallsetminus V_H$ such that at least one of $S_1(u) \cap S_1(v)$

and $S_2(u) \cap S_2(v)$ is empty; by symmetry, we may assume that $S_2(u) \cap S_2(v) = \emptyset$. First, we claim that $S_1(u) \cap S_1(v) = \emptyset$. Suppose otherwise; fix some $s \in S_1(u) \cap S_1(v)$. By Claim 2, $S_2(u)$ and $S_2(v)$ are both non-empty; fix some $u' \in S_2(u)$ and $v' \in S_2(v)$. By definition, $s$ is complete to $\{u, v\}$, and by Claim 5, $s$ is complete to $\{u', v'\}$. But now $u' - u - v - v'$ is a three-edge path, and $s$ is a center for it, contrary to the fact that $G$ is unfriendly. This proves that $S_1(u) \cap S_1(v) = \emptyset$.

Next, we claim that $S_1(u) \cup S_2(u) \cup S_1(v) \cup S_2(v)$ is strongly anti-complete to $V_F \smallsetminus (S_1(u) \cup S_2(u) \cup S_1(v) \cup S_2(v))$. Suppose otherwise. By symmetry, we may assume that there exist adjacent $u_1 \in S_1(u)$ and $f \in E_2 \smallsetminus (S_2(u) \cup S_2(v))$. Fix $u_2 \in S_2(u)$. But now $G[u, u_1, u_2, f, v]$ is a bull. This proves that $S_1(u) \cup S_2(u) \cup S_1(v) \cup S_2(v)$ is strongly anti-complete to $V_F \smallsetminus (S_1(u) \cup S_2(u) \cup S_1(v) \cup S_2(v))$. Since $F$ is connected, it follows that $V_F = S_1(u) \cup S_2(u) \cup S_1(v) \cup S_2(v)$, and furthermore, that $S_1(u) \cup S_2(u)$ is not strongly anti-complete to $S_1(v) \cup S_2(v)$. The latter implies that either $S_1(u)$ is not strongly anti-complete to $S_2(v)$, or $S_2(u)$ is not strongly anti-complete to $S_1(v)$. Now, if $S_1(u)$ is not strongly anti-complete to $S_2(v)$, and $S_2(u)$ is not strongly anti-complete to $S_1(v)$, then we fix some $u_1 \in S_1(u)$, $u_2 \in S_2(u)$, $v_1 \in S_1(v)$, and $v_2 \in S_2(v)$ such that $u_1v_2$ and $u_2v_1$ are adjacent pairs, and we note that $G[u, u_1, u_2, v, v_1, v_2]$ is a prism, contrary to the assumption that $G$ contains no prism. From now on, we assume (by symmetry) that $S_1(u)$ is not strongly anti-complete to $S_2(v)$, and that $S_2(u)$ is strongly anti-complete to $S_1(v)$.

We now claim that $|S_2(u)| = |S_1(v)| = 1$. By symmetry, it suffices to show that $|S_2(u)| = 1$. Fix adjacent $u' \in S_1(u)$ and $v' \in S_2(v)$. Now $u - u' - v' - v - u$ is a square, and $S_2(u)$ is complete to $\{u, u'\}$ and anti-complete to $\{v, v'\}$, and so by 4.2.5, $S_2(u)$ is a strong clique. Since $S_2(u)$ is also a (non-empty) strongly stable set, it follows that $|S_2(u)| = 1$. Similarly, $|S_1(v)| = 1$. Set $S_2(u) = \{u_2\}$ and $S_1(v) = \{v_1\}$. But now $G[(V_F \smallsetminus \{u_2, v_1\}) \cup \{u, v\}]$ contradicts the optimality of $\{u, v\}$ (using the fact that $uv$ are adjacent and $u_2v_1$ are strongly anti-adjacent, we easily infer that $G[(V_F \smallsetminus \{u_2, v_1\}) \cup \{u, v\}]$ has one more adjacent pair than $F$).

**Claim 7.** *For all anti-adjacent $u, v \in V_G \setminus V_H$, at least one of $S_1(u) \cap S_1(v)$ and $S_2(u) \cap S_2(v)$ is empty.*

Suppose otherwise. Fix anti-adjacent $u, v \in V_G \setminus V_H$ such that both $S_1(u) \cap S_1(v)$ and $S_2(u) \cap S_2(v)$ are non-empty. We begin by showing that $S_1(u) = S_1(v)$ and $S_2(u) = S_2(v)$. Suppose otherwise. By symmetry, we may assume that $S_1(u) \setminus S_1(v) \neq \emptyset$. Fix $u_1 \in S_1(u) \setminus S_1(v)$, $f_1 \in S_1(u) \cap S_1(v)$, and $f_2 \in S_2(u) \cap S_2(v)$. Now $u_1 - u - f_1 - v$ is a three-edge path, and (by Claim 5) $f_2$ is a center for it, which is impossible since $G$ is unfriendly. It follows that $S_1(u) = S_1(v)$ and $S_2(u) = S_2(v)$. Set $S_1 = S_1(u) = S_1(v)$ and $S_2 = S_2(u) = S_2(v)$. By Claim 5, $S_1$ is strongly complete to $S_2$.

First, we claim that either $S_1$ is strongly anti-complete to $E_2 \setminus S_2$, or $S_2$ is strongly anti-complete to $E_1 \setminus S_1$ (but not both). Suppose not. Then one of the following must hold:

(a) $S_1$ is strongly anti-complete to $E_2 \setminus S_2$, and $S_2$ is strongly anti-complete to $E_1 \setminus S_1$;

(b) $S_1$ is not strongly anti-complete to $E_2 \setminus S_2$, and $S_2$ is not strongly anti-complete to $E_1 \setminus S_1$.

Suppose first that (a) holds. Since $F$ is connected, this implies that $V_F = S_1 \cup S_2$. But $S_1$ and $S_2$ are both strongly stable, and they are strongly complete to each other. Thus, $F$ contains no three-edge path, which is a contradiction. Suppose now that (b) holds. Fix adjacent $s_1 \in S_1$ and $e_2 \in E_2 \setminus S_2$, and fix adjacent $s_2 \in S_2$ and $e_1 \in E_1 \setminus S_1$. Then $e_1e_2$ is a strongly adjacent pair, for otherwise, $G[u, s_1, s_2, e_1, e_2]$ would be a bull. Now $s_1 - s_2 - e_1 - e_2 - s_1$ is a square, and $\{u, v\}$ is complete to $\{s_1, s_2\}$ and anti-complete to $\{e_1, e_2\}$; by 4.2.5, $\{u, v\}$ is a strong clique, which is a contradiction because $uv$ is an anti-adjacent pair. This proves our claim. By symmetry, we may assume that $S_1$ is strongly anti-complete to $E_2 \setminus S_2$, and that $S_2$ is not strongly anti-complete to $E_1 \setminus S_1$.

Next, we claim that $|S_1| = 1$. Suppose otherwise. Fix distinct $s_1, s_1' \in S_1$, and fix adjacent $s_2 \in S_2$ and $e_1 \in E_1 \smallsetminus S_1$. Then $u - s_1 - v - s_1' - u$ is a square, $s_2$ is a center for it, and $e_1$ is an anti-center for it. By 4.2.6 then, $s_2e_1$ must be a strongly anti-adjacent pair, which is a contradiction. Thus, $|S_1| = 1$, say $S_1 = \{s_1\}$.

Now, let $C$ be the set of all vertices in $V_G \smallsetminus V_F$ that are complete to $S_1 \cup S_2$ and strongly anti-complete to $V_F \smallsetminus (S_1 \cup S_2)$; clearly, $u, v \in C$. First, we claim that $C$ is a clique. Suppose otherwise; fix strongly anti-adjacent $c_1, c_2 \in C$. But now $G[(V_F \smallsetminus \{s_1\}) \cup \{c_1, c_2\}]$ contradicts the optimality of $F$. Thus, $C$ is a clique. Since $uv$ is an anti-adjacent pair, this implies that $uv$ is a semi-adjacent pair, and that $C \smallsetminus \{u, v\}$ is strongly complete to $\{u, v\}$. Since $\{u, v\}$ is not a homogeneous set, some vertex $x \in V_G \smallsetminus C$ is mixed on $\{u, v\}$. By symmetry, we may assume that $x$ is adjacent to $u$ and anti-adjacent to $v$; since $uv$ is a semi-adjacent pair, this means that $x$ is strongly adjacent to $u$ and strongly anti-adjacent to $v$. Since $x$ is adjacent to $u$, we know that $x \notin V_F \smallsetminus (S_1 \cup S_2)$, and since $x$ is strongly anti-adjacent to $v$, we know that $x \notin S_1 \cup S_2$. It follows that $x \notin V_F$, and so $x \in V_G \smallsetminus (C \cup V_F)$. Since $u, x \in V_G \smallsetminus V_F$ are adjacent, Claim 6 implies that $S_1(u) \cap S_1(x)$ and $S_2(u) \cap S_2(x)$ are both non-empty. Since $S_1(u) = \{s_1\}$, it follows that $xs_1$ is an adjacent pair. Further, fix some $s_2 \in S_2(u)$ such that $xs_2$ is an adjacent pair.

Next, we claim that $x$ is strongly anti-complete to $V_F \smallsetminus (S_1 \cup S_2)$. Suppose not; fix some $f \in V_F \smallsetminus (S_1 \cup S_2)$ such that $xf$ is an adjacent pair. But now $G[u, v, x, s_1, f]$ is a bull, which is a contradiction (we are using the fact that $s_1$ is strongly anti-complete to $E_2 \smallsetminus S_2$). It follows that $x$ is strongly anti-complete to $V_F \smallsetminus (S_1 \cup S_2)$. But now $G[(V_F \smallsetminus \{s_1\}) \cup \{v, x\}]$ contradicts the optimality of $F$ (the fact that $G[(V_F \smallsetminus \{s_1\}) \cup \{v, x\}]$ is connected can easily be established by using the fact that $xs_2$ is an adjacent pair, and it is easy to check that $G[(V_F \smallsetminus \{s_1\}) \cup \{v, x\}]$ satisfies the other conditions from the definition of a frame).

**Claim 8.** *$G \smallsetminus V_F$ contains no semi-adjacent pairs.*

Suppose otherwise; let $u, v \in V_G \smallsetminus V_F$ be semi-adjacent. Then since $uv$ is an adjacent pair, we know by Claim 6 that $S_1(u) \cap S_1(v)$ and $S_2(u) \cap S_2(v)$ are both non-empty. But on the other hand, since $uv$ is an anti-adjacent pair, Claim 7 guarantees that at least one of $S_1(u) \cap S_1(v)$ and $S_2(u) \cap S_2(v)$ is empty, which is a contradiction.

**Claim 9.** *Every component of $G \smallsetminus V_F$ is a strong clique.*

Suppose otherwise. Let $C$ be the vertex-set of a component of $G \smallsetminus V_F$ such that $C$ is not a strong clique. By Claim 8, $G[C]$ contains no semi-adjacent pairs, and so since $C$ is not a strong clique, it follows that there exist some $x, y, z \in C$ such that $x-y-z$ is a path in $G[C]$.

First, since $xz$ is an anti-adjacent pair, we know (by Claim 7) that at least one of $S_1(x) \cap S_1(z)$ and $S_2(x) \cap S_2(z)$ is empty; by symmetry, we may assume that $S_1(x) \cap S_1(z) = \emptyset$. Since $xy$ and $yz$ are adjacent pairs, Claim 6 guarantees that both $S_1(x) \cap S_1(y)$ and $S_1(y) \cap S_1(z)$ are non-empty; fix $a \in S_1(x) \cap S_1(y)$ and $b \in S_1(y) \cap S_1(z)$; since $S_1(x) \cap S_1(z) = \emptyset$, we know that $a \notin S_1(z)$ and $b \notin S_1(x)$, and in particular, $a \neq b$. Furthermore, $a$ is complete to $\{x, y\}$ and strongly anti-adjacent to $z$, $b$ is complete to $\{y, z\}$ and strongly anti-adjacent to $x$. Now, we consider two cases: when $S_2(x) \cap S_2(y) \cap S_2(z) \neq \emptyset$, and when $S_2(x) \cap S_2(y) \cap S_2(z) = \emptyset$.

Suppose first that $S_2(x) \cap S_2(y) \cap S_2(z) \neq \emptyset$. Fix $c \in S_2(x) \cap S_2(y) \cap S_2(z)$. But now the vartices $a, x, b, z, y, c$ contradict 4.2.7.

Suppose now that $S_2(x) \cap S_2(y) \cap S_2(z) = \emptyset$. Since $xy$ and $yz$ are adjacent pairs, Claim 6 guarantees that $S_2(x) \cap S_2(y)$ and $S_2(y) \cap S_2(z)$ are both non-empty; since $S_2(x) \cap S_2(y) \cap S_2(z) = \emptyset$, we also know that $S_2(x) \cap S_2(y)$ and $S_2(y) \cap S_2(z)$ are disjoint. Now, fix $c \in S_2(x) \cap S_2(y)$ and $d \in S_2(y) \cap S_2(z)$. Since $a \in S_1(y)$ and $d \in S_2(y)$, we know by Claim 5 that $ad$ is a strongly adjacent pair. But now $x-a-d-z$ is a three-edge path, and $y$ is a center for it, contrary to the fact that $G$ is unfriendly.

**Claim 10.** *For all adjacent $u, v \in V_G \smallsetminus V_F$, and for each $i \in \{1, 2\}$, either $u$ is strongly complete to $S_i(v)$, or $v$ is strongly complete to $S_i(u)$.*

Fix adjacent $u, v \in V_G \smallsetminus V_F$. By symmetry, it suffices to show that either $u$ is strongly complete to $S_1(v)$, or $v$ is strongly complete to $S_1(u)$. Suppose otherwise. Then there exist $s_u \in S_1(u)$ and $s_v \in S_1(v)$ such that $vs_u$ and $us_v$ are anti-adjacent pairs. Since $uv$ is an adjacent pair, Claim 6 implies that $S_2(u) \cap S_2(v) \neq \emptyset$; fix some $s_2 \in S_2(u) \cap S_2(v)$. By Claim 5, we know that $s_2$ is complete to $\{s_u, s_v\}$. But now $s_u - u - v - s_v$ is a three-edge path, and $s_2$ is a center for it, contrary to the fact that $G$ is unfriendly.

**Claim 11.** *For all adjacent $u, v \in V_G \smallsetminus V_F$, at least one of the following holds:*

- *$u$ is strongly complete to $S_1(v)$, and $v$ is strongly complete to $S_2(u)$;*
- *$v$ is strongly complete to $S_1(u)$, and $u$ is strongly complete to $S_2(v)$.*

Fix adjacent $u, v \in V_G \smallsetminus V_F$. By Claim 10, either $u$ is strongly complete to $S_1(v)$, or $v$ is strongly complete to $S_1(u)$; by symmetry, we may assume that $u$ is strongly complete to $S_1(v)$. Now, if $v$ is strongly complete to $S_2(u)$, then we are done; so assume that $v$ is not strongly complete to $S_2(u)$. By Claim 10, it follows that $u$ is strongly complete to $S_2(v)$. If $v$ is strongly complete to $S_1(u)$, then we are done; so assume $v$ is not strongly complete to $S_1(u)$. We now have that $u$ is strongly complete to $S_1(v) \cup S_2(v)$, and $v$ is not strongly complete to either one of $S_1(u)$ and $S_2(u)$. Fix $s_1^u \in S_1(u)$ and $s_2^u \in S_2(u)$ such that $v$ is anti-complete to $\{s_1^u, s_2^u\}$; then $v$ must be strongly anti-adjacent to at least one of $s_1^u$ and $s_2^u$, and by symmetry, we may assume that $v$ is strongly anti-adjacent to $s_1^u$ (and so $s_1^u \in S_1(u) \smallsetminus S_1(v)$). Next, by Claim 2, $S_1(v)$ is non-empty; fix some $s_1^v \in S_1(v)$. Note that $s_1^v \in S_1(u)$ (because $u$ is strongly complete to $S_1(v)$), and $s_2^u \in S_2(u)$; thus, by Claim 5, $s_2^u s_1^v$ is a strongly adjacent pair. But now $s_1^u - s_2^u - s_1^v - v$ is a three-edge path, and $u$ is a center for it, contrary to the fact that $G$ is unfriendly.

**Claim 12.** *For every non-empty clique $C \subseteq V_G \smallsetminus V_F$, there exists a vertex $c \in C$ such that for all $c' \in C \smallsetminus \{c\}$, $c$ is strongly complete to $S_1(c')$, and $c'$ is strongly complete to*

$S_2(c)$.

Suppose that $C \subseteq V_G \setminus V_F$ is a non-empty clique. Now, fix $c \in C$ such that:

- $c$ has as many strong neighbors in $E_1$ as possible;
- subject to the above, $c$ has as many neighbors in $E_1$ as possible;
- subject to the above, $c$ has as few neighbors in $E_2$ as possible;
- subject to the above, $c$ has as few strong neighbors in $E_2$ as possible.

First, we claim that for all $c' \in C \setminus \{c\}$, $c$ is strongly complete to $S_1(c')$. Suppose otherwise. Fix some $c' \in C \setminus \{c\}$ such that $c$ is not strongly complete to $S_1(c')$. Then by Claim 10, $c'$ is strongly complete to $S_1(c)$. But now $c'$ must have more strong neighbors in $E_1$ than $c$ does, contrary to the choice $c$. This proves that for all $c' \in C \setminus \{c\}$, $c$ is strongly complete to $S_1(c')$.

Next, we claim that for all $c' \in C \setminus \{c\}$, $c'$ is strongly complete to $S_2(c)$. Suppose otherwise. Fix some $c' \in C \setminus \{c\}$ such that $c'$ is not strongly complete to $S_2(c)$. Now by Claim 11, we know that $c$ is strongly complete to $S_2(c')$, and $c'$ is strongly complete to $S_1(c)$. Thus, $c$ and $c'$ have exactly the same number of strong neighbors in $E_1$, and neither is semi-adjacent to any vertex in $E_1$. By the choice of $c$, this means that $c'$ cannot have fewer neighbors in $E_2$ than $c$ does. Since $c$ is strongly complete to $S_2(c')$, this means that $S_2(c') = S_2(c)$; set $S_2 = S_2(c) = S_2(c')$. Since $c$ is strongly complete to $S_2$, but $c'$ is not strongly complete to $S_2$, it follows that $c'$ has fewer strong neighbors in $E_2$ than $c$ does. But this is impossible by the choice of $c$.

**Claim 13.** *For all non-empty cliques $C \subseteq V_G \setminus V_F$, the vertices of $C$ can be ordered as $C = \{c_1, ..., c_k\}$ so that for all $i, j \in \{1, ..., k\}$, if $i < j$, then $c_i$ is strongly complete to $S_1(c_j)$, and $c_j$ is strongly complete to $S_2(c_i)$.*

The claim follows by an easy induction from Claim 12.

**Claim 14.** *$G$ is transitively orientable.*

If $G = F$, then $G$ is bipartite, and the result is immediate. So assume that $G \neq F$, and let $C_1, ..., C_n$ be the vertex-sets of the components of $G \setminus V_F$. By Claim 9, $C_1, ..., C_n$ are all strong cliques. We then apply Claim 13, and for each $r \in \{1, ..., n\}$, we set $C_n = \{c_1^r, ..., c_{k_r}^r\}$ so that for all $i, j \in \{1, ..., k_r\}$, if $i < j$, then $c_i^r$ is strongly complete to $S_1(c_j^r)$, and $c_j^r$ is strongly complete to $S_2(c_i^r)$.

Now, we orient $G$ as follows. For all $r \in \{1, ..., n\}$, and all $i, j \in \{1, ..., k_r\}$ such that $i < j$, we orient the strongly adjacent pair $c_j^r c_i^r$ as $(c_j^r, c_i^r)$. Next, for all adjacent $e_1 \in E_1$ and $c \in V_G \setminus V_F$, we orient $e_1 c$ as $(e_1, c)$. For all adjacent $c \in V_G \setminus V_F$ and $e_2 \in E_2$, we orient $ce_2$ as $(c, e_2)$. Finally, for all adjacent $e_1 \in E_1$ and $e_2 \in E_2$, we orient $e_1 e_2$ as $(e_1, e_2)$. This yields an oriented trigraph $\vec{G}$.

We claim that $\vec{G}$ is transitive. Suppose that $(u, v)$ and $(v, w)$ are arcs in $\vec{G}$; we need to show that $(u, w)$ is a strong arc in $\vec{G}$. Now, because of the arc $(u, v)$, we know that $u \notin E_2$ and $v \notin E_1$; and because of the arc $(v, w)$, we know that $v \notin E_2$ and $w \notin E_1$. Thus, there exists some $r \in \{1, ..., n\}$ such that $v \in C_r$, and one of the following holds:

(a) $u \in E_1$ and $w \in C_r$;

(b) $u \in E_1$ and $w \in E_2$;

(c) $u \in C_r$ and $w \in C_r$;

(d) $u \in C_r$ and $w \in E_2$.

Suppose first that (a) holds. Because of the arc $(v, w)$, we know that there exist some $i, j \in \{1, ..., k_r\}$ such that $i < j$, $v = c_j^r$, and $w = c_i^r$. Now $w$ is strongly complete to $S_1(v)$, and so $uw$ is a strongly adjacent pair. By construction, it was oriented as $(u, w)$, and we are done.

Suppose next that (b) holds. Then $u \in S_1(v)$ and $w \in S_2(v)$, and so by Claim 5, $uw$ is a strongly adjacent pair. By construction, it was oriented as $(u, w)$, and we are done.

Suppose now that (c) holds. But clearly, $\vec{G}[C]$ is transitive, so we are done.

Suppose finally that (d) holds. Then because of the arc $(u, v)$, we know that there exist some $i, j \in \{1, ..., k_r\}$ such that $i < j$, $u = c^r_j$, and $v = c^r_i$. Now $u$ is strongly complete to $S_2(v)$, and so $uw$ is a strongly adjacent pair. By construction, it was oriented as $(u, w)$, and we are done. □

We now need a definition. Let us say that a trigraph $G$ is *happy* provided that the following two conditions hold:

- $G$ is transitively orientable;
- $G$ does not contain a three-edge path $P$ such that some vertex of $G$ is a center for $P$.

We now use 4.3.1 and some lemmas from section 4.2 to prove a "preliminary" decomposition theorem for bull-free Berge trigraphs.

**4.3.2.** *Let $G$ be a bull-free Berge trigraph. Then at least one of the following holds.*

- *$G$ or $\overline{G}$ is happy;*
- *$G$ admits a homogeneous set decomposition;*
- *$G$ admits a homogeneous pair decomposition.*

*Proof.* We assume that $G$ admits neither a homogeneous set decomposition nor a homogeneous pair decomposition, for otherwise we are done. (Note that this implies that $\overline{G}$ also admits neither a homogeneous set decomposition nor a homogeneous pair decomposition.) Our goal is to show that at least one of $G$ and $\overline{G}$ is happy. By 4.2.2, at most one of the following holds:

- $G$ contains a three-edge path with a center;
- $G$ contains a three-edge path with an anti-center.

Exploiting the symmetry between $G$ and $\overline{G}$, we may assume that $G$ does not contain a three-edge path with a center. By definition then, $G$ is unfriendly.

Suppose first that $G$ contains a prism. Then by 4.2.4, $G$ is a prism. Since $G$ is bull-free, 4.2.3 guarantees that $G$ is complement-bipartite. But clearly then, $\overline{G}$ is happy, and we are done. Assume now that $G$ contains no prism. If $G$ is framed, then by 4.3.1, $G$ is transitively orientable; since (by assumption) $G$ contains no three-edge path with a center, it follows that $G$ is happy. So assume that $G$ is not framed. Then $G$ contains no three-edge path. By 4.1.3, $G$ is transitively orientable, and it follows that $G$ is happy. □

## 4.4 Homogeneous Pairs and the Proof of the Main Decomposition Theorem for Bull-Free Berge Trigraphs

In this section, we study tame homogeneous pairs in the context of bull-free Berge trigraphs, and we use the results that we obtain, as well as the decomposition theorem 4.3.2, to prove 4.0.1.

We begin by proving an easy lemma that we will use repeatedly in this section.

**4.4.1.** *Let $G$ be a trigraph, let $(A, B)$ be a tame homogeneous pair in $G$, and let $(H, a, b)$ be semi-adjacent reduction of $(G, A, B)$. Then both of the following hold:*

- *if $G$ is bull-free, then $H$ is also bull-free;*
- *if $G$ is Berge, then $H$ is also Berge.*

*Proof.* Since $(A, B)$ is tame, $A$ is neither strongly complete nor strongly anti-complete to $B$. Thus, every realization of $H$ is (isomorphic to) an induced subtrigraph of $G$, and the result follows. □

We now need a definition. Let $G$ be a trigraph, let $(A, B)$ be a tame homogeneous pair in $G$, and let $(A, B, C, D, E, F)$ be the associated partition of $G$. We say that $(A, B)$ is *degenerate* provided that $C$, $E$, and $F$ are all non-empty, and $D$ is empty. We remark that if $(A, B)$ is a degenerate homogeneous pair in $G$, and if $(A, B, C, \emptyset, E, F)$ is the associated partition of $G$, then $(B, A)$ is a degenerate homogeneous pair in $\overline{G}$, and $(B, A, C, \emptyset, F, E)$ is the associated partition of $\overline{G}$. We now prove a few lemmas about degenerate homogeneous pairs. We remark that the main idea of the proof of 4.4.2 is included in the proofs of 7.2 and 7.3 from [9].

**4.4.2.** *Let $G$ be a bull-free Berge trigraph that does not admit a homogeneous set decomposition, and let $(A, B)$ be a degenerate homogeneous pair in $G$. Then $A$ is a strongly stable set and $B$ is a strong clique.*

*Proof.* Since $(A, B)$ is a degenerate homogeneous pair in $G$, we know that $(B, A)$ is a degenerate homogeneous pair in $\overline{G}$. Thus, it suffices to show that $A$ is strongly stable in $G$, for then an analogous argument applied to the degenerate homogeneous pair $(B, A)$ in $\overline{G}$ will establish that $B$ is strongly stable in $\overline{G}$, and therefore, that $B$ is a strong clique in $G$.

Let $(A, B, C, \emptyset, E, F)$ be the partition of $G$ associated with the degenerate homogeneous pair $(A, B)$. We begin by showing that $F$ is not strongly anti-complete to $C$. Suppose otherwise. Then $F$ cannot be strongly anti-complete to $E$, for otherwise, $A \cup B \cup C \cup E$ would be a proper homogeneous set in $G$. Let $E_0$ be the set of all vertices in $E$ that have a neighbor in $F$. Then $E_0$ is strongly complete to $C$. (Indeed, suppose $E_0$ were not strongly complete to $C$. Then fix anti-adjacent $c \in C$ and $e \in E$. Fix $f \in F$ such that $ef$ is an adjacent pair. Fix adjacent $a \in A$ and $b \in B$. Then $G[a, b, c, e, f]$ is a bull.)

Next, let $E'$ be the union of the vertex-sets of all the anti-components of $G[E]$ that intersect $E_0$. First, we claim that every vertex in $E' \smallsetminus E_0$ has an anti-neighbor in $E_0$. Suppose otherwise. Then by the definition of $E_0$, there exists an anti-path $e_0 - e_1 - e_2$ such that $e_0 \in E_0$ and $e_1, e_2 \in E' \smallsetminus E_0$. Fix $a \in A$, and fix some $f \in F$ such that $e_0 f$

is an adjacent pair. Then $G[a, e_0, e_1, e_2, f]$ is a bull. This proves that every vertex in $E' \setminus E_0$ has an anti-neighbor in $E_0$. Now, we claim that $E'$ is strongly complete to $C$. Suppose otherwise. Since $E_0$ is strongly complete to $C$, and every vertex in $E' \setminus E_0$ has an anti-neighbor in $E_0$, this implies that there exist anti-adjacent vertices $e_0 \in E_0$ and $e' \in E' \setminus E_0$ such that $e'$ is not strongly complete to $C$. Now fix some $c \in C$ such that $ce'$ is an anti-adjacent pair, fix some $f \in F$ such that $e_0 f$ is an adjacent pair, and fix some $a \in A$. Then $G[a, c, e_0, e', f]$ is a bull, which is a contradiction.

Set $X = A \cup B \cup C \cup (E \setminus E')$. Now $E'$ is strongly complete to $X$, and $F$ is strongly anti-complete to $X$, and so it follows that $X$ is a proper homogeneous set in $G$, which is a contradiction. This proves that $F$ is not strongly anti-complete to $C$.

Now, suppose that $A$ is not a strongly stable set. Since $A$ is not a proper homogeneous set, this implies that there exist adjacent vertices $a, a' \in A$ such that some vertex $b \in B$ is adjacent to $a$ and anti-adjacent to $a'$. Since $F$ is not strongly anti-complete to $C$, and there exist adjacent $c \in C$ and $f \in F$. But now $G[a, a', b, c, f]$ is a bull, which is a contradiction. This completes the argument. $\square$

**4.4.3.** *Let $G$ be a bull-free Berge trigraph that does not admit a homogeneous set decomposition, and assume that $(A, B)$ is a degenerate homogeneous pair in $G$. Then there exists a transitive orientation of $G[A \cup B]$ such that all the adjacent pairs between $A$ and $B$ are oriented from $A$ to $B$.*

*Proof.* Let $(A, B, C, \emptyset, E, F)$ be the partition of $G$ associated with the degenerate homogeneous pair $(A, B)$. First, by 4.4.2, we know that $A$ is a strongly stable set, and $B$ is a strong clique. Now, we claim that for all distinct $b, b' \in B$, either $b$ is strongly adjacent to every neighbor of $b'$ in $A$, or $b'$ is strongly adjacent to every neighbor of $b$ in $A$. Suppose otherwise. Fix distinct $b, b' \in B$ and distinct $a, a' \in A$ such that $ab$ and $a'b'$ are adjacent pairs, and $ab'$ and $a'b$ are anti-adjacent pairs. Fix $c \in C$. Then $c - a - b - b' - a' - c$ is a hole of length five, contrary to the fact that $G$ is Berge. It now follows that the vertices in $B$ can be ordered as $B = \{b_1, ..., b_k\}$ so that for all $a \in A$ and $i, j \in \{1, ..., k\}$, if $i < j$ and $ab_j$ is an adjacent pair, then $ab_i$ is a strongly adjacent pair.

Now, we orient the adjacent pairs in $G[A \cup B]$ as follows:

- all the adjacent pairs between $A$ and $B$ are oriented from $A$ to $B$;
- for all $i, j \in \{1, ..., k\}$ such that $i < j$, the strongly adjacent pair $b_ib_j$ is oriented as $(b_j, b_i)$.

Clearly, this produces a desired transitive orientation of $G[A \cup B]$. $\square$

**4.4.4.** *Let $G$ be a bull-free Berge trigraph that does not admit a homogeneous set decomposition. Let $(A, B)$ be a degenerate homogeneous pair in $G$, and $(H, a, b)$ be a semi-adjacent reduction of $(G, A, B)$. Assume that $H$ is happy. Then $G$ is happy.*

*Proof.* First of all, by 4.0.3, $G$ is elementary; by 4.4.2, $A$ is a strongly stable set and $B$ is a strong clique; and by 4.4.1, $H$ is bull-free and Berge. Next, let $(A, B, C, \emptyset, E, F)$ be the partition of $G$ with respect to the degenerate homogeneous pair $(A, B)$.

We begin by showing that $G$ is transitively orientable. Since $H$ is happy, $H$ is transitively orientable; let $\vec{H}$ be a transitive orientation of $H$. We may assume that the semi-adjacent pair $ab$ is oriented as $(a, b)$ in $\vec{H}$. Now we produce an orientation $\vec{G}$ of $G$ as follows. First, we use 4.4.3 to produce a transitive orientation of $G[A \cup B]$ such that all the adjacent pairs between $A$ and $B$ are oriented from $A$ to $B$. Next, let $uv$ be an adjacent pair in $G$ such that $u$ and $v$ do not both lie in $A \cup B$. Then we orient $uv$ as follows.

- If $u, v \in C \cup E \cup F$, then $uv$ is also an adjacent pair in $H$, and we orient $uv$ in $\vec{G}$ in the same way as in $\vec{H}$.
- If $u \in A$ and $v \in C \cup E$, then we note that $uv$ is a strongly adjacent pair in $G$ and $av$ is a strongly adjacent pair in $H$, and we orient the adjacent pair $uv$ as follows:

- if the strongly adjacent pair $av$ is oriented as $(a, v)$ in $\vec{H}$, then the strongly adjacent pair $uv$ is oriented as $(u, v)$ in $\vec{G}$,
- if the strongly adjacent pair $av$ is oriented as $(v, a)$ in $\vec{H}$, then the strongly adjacent pair $uv$ is oriented as $(v, u)$ in $\vec{G}$.

- If $u \in B$ and $v \in E$, then we note that $uv$ is a strongly adjacent pair in $G$ and $bv$ is a strongly adjacent pair in $H$, and we orient the adjacent pair $uv$ as follows:
  - if the strongly adjacent pair $bv$ is oriented as $(b, v)$ in $\vec{H}$, then the strongly adjacent pair $uv$ is oriented as $(u, v)$ in $\vec{G}$,
  - if the strongly adjacent pair $bv$ is oriented as $(v, b)$ in $\vec{H}$, then the strongly adjacent pair $uv$ is oriented as $(v, u)$ in $\vec{G}$.

Clearly, the orientation $\vec{G}$ of $G$ defined in this way is transitive.

It remains to show that $G$ contains no three-edge path with a center.

First, we claim that every vertex in $C$ that has a neighbor in $E$ is strongly anti-complete to $F$. Suppose otherwise. Fix $c \in C$, $e \in E$, and $f \in F$ such that $c$ is complete to $\{e, f\}$. But then if $ef$ is an adjacent pair, then $b - a - c - f$ is a three-edge path in $H$, and $e$ is a center for it, contrary to the fact that $H$ is happy; and if $ef$ is an anti-adjacent pair, then $H[a, b, c, e, f]$ is a bull, contrary to the fact that $H$ is bull-free. This proves the claim.

Now suppose that $p_1 - p_2 - p_3 - p_4$ is a three-edge path in $G$, and that $x$ is a center for it. First, we know that $p_1, p_2, p_3, p_4$ cannot all lie in $A \cup B$, for then any vertex in $E$ would be a center for it, and any vertex in $F$ would be an anti-center for it, contrary to the fact that $G$ is elementary.

We first show that either $|A \cap \{x, p_1, p_2, p_3, p_4\}| \geq 2$ or $|B \cap \{x, p_1, p_2, p_3, p_4\}| \geq 2$. Suppose otherwise. Then $|A \cap \{x, p_1, p_2, p_3, p_4\}| \leq 1$ and $|B \cap \{x, p_1, p_2, p_3, p_4\}| \leq 1$. We now define $\hat{x}, \hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4$ as follows:

- if $x \in A$, then we set $\hat{x} = a$;
- if $x \in B$, then we set $\hat{x} = b$;
- if $x \notin A \cup B$, then we set $\hat{x} = x$;
- for all $i \in \{1, 2, 3, 4\}$, if $p_i \in A$, then we set $\hat{p}_i = a$;
- for all $i \in \{1, 2, 3, 4\}$, if $p_i \in B$, then we set $\hat{p}_i = b$;
- for all $i \in \{1, 2, 3, 4\}$, if $p_i \notin A \cup B$, then we set $\hat{p}_i = p_i$.

Since $|A \cap \{x, p_1, p_2, p_3, p_4\}| \leq 1$ and $|B \cap \{x, p_1, p_2, p_3, p_4\}| \leq 1$, we know that $\hat{x}, \hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4$ are pairwise distinct, and it easily follows that $\hat{p}_1 - \hat{p}_2 - \hat{p}_3 - \hat{p}_4$ is a three-edge path in $H$, and $\hat{x}$ is a center for it, contrary to the fact that $H$ is happy. This proves that either $|A \cap \{x, p_1, p_2, p_3, p_4\}| \geq 2$ or $|B \cap \{x, p_1, p_2, p_3, p_4\}| \geq 2$.

We next show that $x \notin A \cup B$. Suppose otherwise. Suppose first that $x \in A$. Since $A$ is strongly stable, this implies that $p_1, p_2, p_3, p_4 \in B \cup C \cup E$. Note that $p_1, p_2, p_3, p_4$ cannot all lie in $B$ because $B$ is a strong clique. Now, since $B$ is a homogeneous set in $G[B \cup C \cup E]$, and $G[p_1, p_2, p_3, p_4]$ contains no proper homogeneous set, it follows that $B$ contains at most one of $p_1, p_2, p_3, p_4$. Now $A \cap \{p_1, p_2, p_3, p_4, x\} = \{x\}$ and $|B \cap \{p_1, p_2, p_3, p_4\}| \leq 1$. But this contracts the fact that $|A \cap \{x, p_1, p_2, p_3, p_4\}| \geq 2$ or $|B \cap \{x, p_1, p_2, p_3, p_4\}| \geq 2$. Suppose now that $x \in B$. Then $p_1, p_2, p_3, p_4 \in A \cup B \cup E$. Note that $p_1, p_2, p_3, p_4$ cannot all lie in $E$, for otherwise, $p_1 - p_2 - p_3 - p_4$ would be a three-edge path in $H$, and $b$ would be a center for it, contrary to the fact that $H$ is happy. Now, since $E$ is a homogeneous set in $G[A \cup B \cup E]$ and $G[p_1, p_2, p_3, p_4]$ contains no proper homogeneous set, we know that $|E \cap \{p_1, p_2, p_3, p_4\}| \leq 1$. On the other hand, since $A \cup B$ does not contain all of $p_1, p_2, p_3, p_4$, we know that $E$ contains at least one of $p_1, p_2, p_3, p_4$. It follows that $E$ contains exactly one of $p_1, p_2, p_3, p_4$; say $p_i \in E$. But since $E$ is strongly complete to $A \cup B$, it follows that $p_i$ is strongly complete to $\{p_1, p_2, p_3, p_4\} \smallsetminus \{p_i\}$, which is impossible. This proves that $x \notin A \cup B$.

We now have that either $|A \cap \{p_1, p_2, p_3, p_4\}| \geq 2$ or $|B \cap \{p_1, p_2, p_3, p_4\}| \geq 2$. Both cannot hold because $p_1, p_2, p_3, p_4$ do not all lie in $A \cup B$. Furthermore, since $A$ is a strongly stable set and $B$ a strong clique, and since $G[p_1, p_2, p_3, p_4]$ contains neither strong triads nor strong triangles, we know that $|A \cap \{p_1, p_2, p_3, p_4\}| \leq 2$ and $|B \cap \{p_1, p_2, p_3, p_4\}| \leq 2$. This proves that exactly one of the following holds:

- $|A \cap \{p_1, p_2, p_3, p_4\}| \leq 1$ and $|B \cap \{p_1, p_2, p_3, p_4\}| = 2$;
- $|A \cap \{p_1, p_2, p_3, p_4\}| = 2$ and $|B \cap \{p_1, p_2, p_3, p_4\}| \leq 1$.

Now, we claim that $A$ and $B$ each intersect $\{p_1, p_2, p_3, p_4\}$. Suppose otherwise. Then one of the following holds:

- $A \cap \{p_1, p_2, p_3, p_4\} = \emptyset$ and $|B \cap \{p_1, p_2, p_3, p_4\}| = 2$;
- $|A \cap \{p_1, p_2, p_3, p_4\}| = 2$ and $B \cap \{p_1, p_2, p_3, p_4\} = \emptyset$.

Suppose that $A \cap \{p_1, p_2, p_3, p_4\} = \emptyset$ and $|B \cap \{p_1, p_2, p_3, p_4\}| = 2$. Now $p_1, p_2, p_3, p_4, x \in V_G \smallsetminus A$, and $B$ is a homogeneous set in $G \smallsetminus A$. This implies that $B \cap \{p_1, p_2, p_3, p_4\}$ is a proper homogeneous set in $G[p_1, p_2, p_3, p_4, x]$, which is impossible. Thus, the first outcome is impossible. A similar argument proves that the second outcome is impossible as well. This proves that exactly one of the following holds:

- $|A \cap \{p_1, p_2, p_3, p_4\}| = 1$ and $|B \cap \{p_1, p_2, p_3, p_4\}| = 2$;
- $|A \cap \{p_1, p_2, p_3, p_4\}| = 2$ and $|B \cap \{p_1, p_2, p_3, p_4\}| = 1$.

Thus, exactly three vertices among $p_1, p_2, p_3, p_4$ lie in $A \cup B$. Since the trigraph $G[p_1, p_2, p_3, p_4]$ is connected, the fourth vertex does not lie in $F$, and since $G[p_1, p_2, p_3, p_4]$ is anticonnected, the fourth vertex does not lie in $E$. Thus, the fourth vertex (call it $p_i$) lies in $C$. Furthermore, since $x \notin A \cup B$, since $x$ is a center for $p_1 - p_2 - p_3 - p_4$, and since

$\{p_1, p_2, p_3, p_4\}$ intersects both $A$ and $B$, it follows that $x \in E$. Now $p_i \in C$, $x \in E$, and $p_i x$ is an adjacent pair; thus, $p_i$ is strongly anti-complete to $F$. Fix $f \in F$. Now $x$ is a center and $f$ is an anti-center for the three-edge path $p_1 - p_2 - p_3 - p_4$ in $G$, contrary to the fact that $G$ is elementary. This proves that $G$ is happy. □

The next couple of lemmas deal with tame homogeneous pairs $(A, B)$ whose associated partitions are either of the form $(A, B, \emptyset, D, \emptyset, F)$ or of the form $(A, B, C, \emptyset, E, \emptyset)$.

**4.4.5.** *Let $G$ be a bull-free Berge trigraph that does not admit a homogeneous set decomposition. Let $(A, B)$ be a tame homogeneous pair in $G$, and assume that $(A, B, \emptyset, D, \emptyset, F)$ is the associated partition of $G$. Then both of the following hold:*

- *$(F, D)$ is a tame homogeneous pair in $G$, and $(F, D, \emptyset, B, \emptyset, A)$ is the associated partition of $G$;*
- *$B$ and $D$ are strongly stable sets.*

*Proof.* First, since $(A, B)$ is tame, we know that $|A \cup B| \geq 3$ and $|D \cup F| \geq 3$. Note that if $D = \emptyset$, then $F$ is a proper homogeneous set in $G$, and if $F = \emptyset$, then $D$ is a proper homogeneous set in $G$; since $G$ contains no proper homogeneous set, it follows that $D$ and $F$ are both non-empty. Further, $D$ is neither strongly complete nor strongly anti-complete to $F$, for otherwise, both $D$ and $F$ would be homogeneous sets in $G$, and since at least one of $D$ and $F$ has more than one vertex (because $|D \cup F| \geq 3$), at least one of $D$ and $F$ would be a proper homogeneous set in $G$, contrary to the fact that $G$ does not admit a homogeneous set decomposition. It now follows that $(F, D)$ is a tame homogeneous pair in $G$, and $(F, D, \emptyset, B, \emptyset, A)$ is the associated partition.

It remains to show that $B$ and $D$ are strongly stable sets. Because of the symmetry between the tame homogeneous pairs $(A, B)$ and $(F, D)$, it suffices to show that $B$ is strongly stable. Suppose that $B$ is not strongly stable. Then since $G$ does not contain a proper homogeneous set, there exist adjacent vertices $b, b' \in B$ and a vertex $a \in A$ such that $ab$ is an adjacent pair, and $ab'$ is an anti-adjacent pair. Next, fix adjacent $d \in D$ and $f \in F$. Then $G[a, b, b', d, f]$ is a bull, which is a contradiction. $\square$

**4.4.6.** *Let $G$ be a bull-free Berge trigraph that does not admit a homogeneous set decomposition. Let $ab$ be a semi-adjacent pair in $G$, and assume that $(\{a\}, \{b\}, \emptyset, D, \emptyset, F)$ is the partition of $G$ associated with the homogeneous pair $(\{a\}, \{b\})$. Assume that $\overline{G}$ is happy. Then $G$ is happy as well.*

*Proof.* If $|D \cup F| \leq 1$, then $G$ contains at most three vertices, and the result is immediate. So assume that $|D \cup F| \geq 2$. Now if $D = \emptyset$, then $F$ is a proper homogeneous set in $G$, and if $F = \emptyset$, then $D$ is a proper homogeneous set in $G$; since $G$ does not contain admit a homogeneous set decomposition, it follows that $D$ and $F$ are both non-empty.

We first show that $D$ a strongly stable set. Suppose otherwise. Since $D$ is not a proper homogeneous set in $G$, and since neither $a$ nor $b$ is mixed on $D$, if follows that there exist adjacent vertices $d, d' \in D$ and a vertex $f \in F$ such that $df$ is an adjacent pair, and $d'f$ is an anti-adjacent pair. But now $G[a, b, d, d', f]$ is a bull, contrary to the fact that $G$ is bull-free. Thus, $D$ is strongly stable.

We now show that $F$ is a strongly stable set. Suppose otherwise. Since $F$ is not a proper homogeneous set in $G$, and since $\{a, b\}$ is strongly anti-complete to $F$, it follows that there exist adjacent vertices $f, f' \in F$ and a vertex $d \in D$ such that $df$ is an adjacent pair and $df'$ is an anti-adjacent pair. Now $f' - f - d - b$ is a three-edge path in $G$, and $a$ is an anti-center for it. Thus, in $\overline{G}$, $f - b - f' - d$ is a three-edge path, and $a$ is a center for it, contrary to the fact that $\overline{G}$ is happy. Thus, $F$ is a strongly stable set in $G$.

Since $D$ and $F$ are both strongly stable, it follows that $G$ is bipartite with bipartition $(\{a\} \cup D, \{b\} \cup F)$, and it is now immediate that $G$ is happy. $\square$

Let $G$ be a trigraph, let $(A, B)$ be a tame homogeneous pair in $G$, and $(A, B, C, D, E, F)$

be the associated partition of $G$. We say that $(A, B)$ is *appropriate* provided that $C$ and $D$ are both non-empty. We observe that if $(A, B)$ is an appropriate homogeneous pair in a trigraph $G$, then $(A, B)$ is an appropriate homogeneous pair in $\overline{G}$. Furthermore, every reducible homogeneous pair is appropriate (however, not all appropriate homogeneous pairs are reducible). We now prove a technical lemma, and then we prove another decomposition theorem for bull-free Berge trigraphs.

**4.4.7.** *Let $(A, B)$ be a tame homogeneous pair in $G$, and let $(H, a, b)$ be a semi-adjacent reduction of $(G, A, B)$. Then both of the following hold:*

- *if $H$ contains a proper homogeneous set, then so does $G$;*
- *if $H$ contains an appropriate homogeneous pair, then so does $G$.*

*Proof.* Suppose first that $H$ contains a proper homogeneous set, call it $S$. Since $ab$ is a semi-adjacent pair in $H$, we know that either $a, b \in S$ or $a, b \notin S$. If $a, b \in S$, then clearly, $(S \smallsetminus \{a, b\}) \cup (A \cup B)$ is a proper homogeneous set in $G$, and if $a, b \notin S$, then $S$ is a proper homogeneous set in $G$.

Suppose now that $H$ contains an appropriate homogeneous pair, call it $(X, Y)$. Since $ab$ is a semi-adjacent pair in $H$, we know that either $a, b \in X \cup Y$ or $a, b \notin X, Y$. We now define sets $\hat{X}$ and $\hat{Y}$ as follows.

- if $a, b \in X$, then set $\hat{X} = (X \smallsetminus \{a, b\}) \cup (A \cup B)$ and $\hat{Y} = Y$;
- if $a, b \in Y$, then set $\hat{X} = X$ and $\hat{Y} = (Y \smallsetminus \{a, b\}) \cup (A \cup B)$;
- if $a \in X$ and $b \in Y$, then set $\hat{X} = (X \smallsetminus \{a\}) \cup A$ and $\hat{Y} = (Y \smallsetminus \{b\}) \cup B$;
- if $a \in Y$ and $b \in X$, then set $\hat{X} = (X \smallsetminus \{b\}) \cup B$ and $\hat{Y} = (Y \smallsetminus \{a\}) \cup A$;
- if $a, b \notin X \cup Y$, then set $\hat{X} = X$ and $\hat{Y} = Y$.

It now follows by routine checking that $(\hat{X}, \hat{Y})$ is an appropriate homogeneous pair in $G$. □

**4.4.8.** *For every bull-free Berge trigraph $G$, at least one of the following holds:*

*(a) $G$ or $\overline{G}$ is happy;*

*(b) $G$ contains a proper homogeneous set;*

*(c) $G$ contains an appropriate homogeneous pair.*

*Proof.* Let $G$ be a bull-free Berge trigraph, and assume inductively that the claim holds for all bull-free Berge trigraphs that have fewer than $|V_G|$ vertices. Now, by 4.3.2, we know that at least one of the following holds:

(i) $G$ or $\overline{G}$ is happy;

(ii) $G$ contains a proper homogeneous set;

(iii) $G$ contains a tame homogeneous pair.

If (i) or (ii) holds, then we are done. So assume that (iii) holds. Let $(A, B)$ be a tame homogeneous pair in $G$, and let $(A, B, C, D, E, F)$ be the associated partition of $G$. If $C$ and $D$ are both non-empty, then $(A, B)$ is an appropriate homogeneous pair, and we are done. So assume that at least one of $C$ and $D$ is empty. If $C$ and $D$ are both empty, then $A \cup B$ is a proper homogeneous set in $G$, and we are done. So assume that exactly one of $C$ and $D$ is empty. If $E \cup F = \emptyset$, then one of $C$ and $D$ is a proper homogeneous set in $G$, and we are done. So assume that $E \cup F \neq \emptyset$.

Suppose first that $E$ and $F$ are both non-empty. We know that exactly one of $C$ and $D$ is non-empty, and we may assume by symmetry that $C \neq \emptyset$ and $D = \emptyset$. Then $(A, B)$ is a degenerate homogeneous pair. Let $(H, a, b)$ be a semi-adjacent reduction of $(G, A, B)$. Then by Proposition 4.4.1, $H$ is bull-free and Berge. By the induction hypothesis, at least one of the following holds:

- $H$ or $\overline{H}$ is happy;

- $H$ contains a proper homogeneous set;
- $H$ contains an appropriate homogeneous pair.

If $H$ contains a proper homogeneous set or an appropriate homogeneous pair, then by 4.4.7, $G$ does as well. So assume that $H$ or $\overline{H}$ is happy. Since $(B, A)$ is a degenerate homogeneous pair in $\overline{G}$, and $(\overline{H}, b, a)$ is a semi-adjacent reduction of $(\overline{G}, B, A)$, there is a symmetry between $G$ and $\overline{G}$, and so we may assume that $H$ is happy. But then by Proposition 4.4.4, $G$ is happy.

Suppose now that exactly one of $E$ and $F$ is non-empty. We exploit the symmetry between $G$ and $\overline{G}$ and assume that $E = \emptyset$ and $F \neq \emptyset$. We know that exactly one of $C$ and $D$ is non-empty, and we may assume by symmetry that $C = \emptyset$ and $D \neq \emptyset$. Now $(A, B)$ is a tame homogeneous pair in $G$, and the associated partition of $G$ is $(A, B, \emptyset, D, \emptyset, F)$. By 4.4.5, we know that $(F, D)$ is a tame homogeneous pair in $G$, and that the associated partition of $G$ is $(F, D, \emptyset, B, \emptyset, A)$. We also know by 4.4.5 that both $B$ and $D$ are strongly stable. Now, let $(H_1, a, b)$ be a semi-adjacent reduction of the triple $(G, A, B)$, and let $(H_2, f, d)$ be a semi-adjacent reduction of the triple $(G, F, D)$. If one of $H_1$ and $H_2$ contains a proper homogeneous set or an appropriate homogeneous pair, then 4.4.7 implies that $G$ does as well. So assume that this is not the case. Then $H_1$ or $\overline{H}_1$ is happy, and $H_2$ or $\overline{H}_2$ is happy. By 4.4.6, if $\overline{H}_1$ is happy, then so is $H_1$, and if $\overline{H}_2$ is happy, then so is $H_2$. Thus, $H_1$ and $H_2$ are both happy. We now claim that $G$ is happy.

We first show that $G$ contains no three-edge path with a center. Suppose otherwise. Let $p_1, p_2, p_3, p_4, x \in V_G$ be such that $p_1 - p_2 - p_3 - p_4$ is a three-edge path in $G$, and $x$ is a center for it. If $x \in A$, then $p_1, p_2, p_3, p_4 \in A \cup B$, and so $H_2$ contains a three-edge path with a center, contrary to the fact that $H_2$ is happy. Thus, $x \notin A$, and similarly, $x \notin F$. Thus, $x \in B \cup D$; by symmetry, we may assume that $x \in B$. Then $p_1, p_2, p_3, p_4 \in A \cup B \cup D$. If $p_1, p_2, p_3, p_4 \in A \cup B$, then $p_1 - p_2 - p_3 - p_4$ is a three-edge path in $H_2$, and $x$ is a center for it, contrary to the fact that $H_2$ is happy; thus, $D \cap \{p_1, p_2, p_3, p_4\} \neq \emptyset$. Next, since $D$ is strongly stable and $G[p_1, p_2, p_3, p_4]$ contains no strong triads, it follows that $|D \cap \{p_1, p_2, p_3, p_4\}| \leq 2$. This proves that $1 \leq |D \cap \{p_1, p_2, p_3, p_4\}| \leq 2$. Suppose first that $|D \cap \{p_1, p_2, p_3, p_4\}| = 1$. Now, for each $i \in \{1, 2, 3, 4\}$, set $\hat{p}_i = d$ if $p_i \in D$, and set $\hat{p}_i = p_i$ if $p_i \notin D$; then $\hat{p}_1 - \hat{p}_2 - \hat{p}_3 - \hat{p}_4$ is a three-edge path in $H_2$, and $x$ is a center for it, contrary to the fact that $H_2$ is happy. Thus, $|D \cap \{p_1, p_2, p_3, p_4\}| = 2$. Since $D$ is strongly stable, we may assume by symmetry that either $D \cap \{p_1, p_2, p_3, p_4\} = \{p_1, p_3\}$ or $D \cap \{p_1, p_2, p_3, p_4\} = \{p_1, p_4\}$. In either case, neither vertex in $\{p_1, p_2, p_3, p_4\} \smallsetminus D$ is strongly anti-complete to $D \cap \{p_1, p_2, p_3, p_4\}$, and it follows that $A \cap \{p_1, p_2, p_3, p_4\} = \emptyset$. Thus, $\{p_1, p_2, p_3, p_4\} \smallsetminus D \subseteq B$. But since $B$ is strongly complete to $D$, it follows that $G[p_1, p_2, p_3, p_4]$ is not anti-connected, which is false. This proves that $G$ does not contain a three-edge path with a center.

It remains to show that $G$ is transitively orientable. Let $\vec{H}_1$ and $\vec{H}_2$ be transitive orientations of $H_1$ and $H_2$, respectively. We may assume that the semi-adjacent pair $ab$ is oriented as $(a, b)$ in $\vec{H}_1$, and that the semi-adjacent pair $df$ is oriented as $(d, f)$ in $\vec{H}_2$. Since $\vec{H}_1$ is transitive, the presence of the arc $(a, b)$ implies that all the adjacent pairs between $\{b\}$ and $D$ are oriented from $D$ to $\{b\}$ in $\vec{H}_1$; since $b$ is strongly complete to $D$ and strongly anti-complete to $F$, and since $\vec{H}_1$ is transitive, this implies that all the adjacent pairs between $D$ and $F$ in $H_1$ are oriented from $D$ to $F$. Similarly, since $\vec{H}_2$ is transitive, the presence of the arc $(d, f)$ implies that all the adjacent pairs between $B$ and $\{d\}$ are oriented from $\{d\}$ to $B$ in $\vec{H}_2$; since $d$ is strongly complete to $B$ and strongly anti-complete to $A$, and since $\vec{H}_2$ is transitive, this implies that all the adjacent pairs between $A$ and $B$ in $H_2$ are oriented from $A$ to $B$. Now we orient $G$ as follows:

- all the adjacent pairs in $G[A \cup B]$ are oriented as in $\vec{H}_2$;
- all the adjacent pairs in $G[D \cup F]$ are oriented as in $\vec{H}_1$;
- all the adjacent pairs between $B$ and $D$ in $G$ are oriented from $D$ to $B$.

It is easy to check that the resulting orientation of $G$ is transitive. Thus, $G$ is transitively orientable, and it follows that $G$ is happy. This completes the argument. $\square$

We now prove one last lemma that we need before we can prove 4.0.1. We remind the reader that "appropriate expansions" were defined in section 4.1

**4.4.9.** *Let $G$ be a bull-free Berge trigraph. Then at least one of the following holds:*

- *$G$ admits a homogeneous set decomposition;*
- *$G$ or $\overline{G}$ is an appropriate expansion of a happy bull-free Berge trigraph.*

*Proof.* We may assume by induction that the claim holds for bull-free Berge trigraphs on fewer than $|V_G|$ vertices. Now, by 4.4.8, we know that at least one the following holds:

(a) $G$ or $\overline{G}$ is happy;

(b) $G$ contains a proper homogeneous set;

(c) $G$ contains an appropriate homogeneous pair.

If (a) or (b) holds, then we are done. So assume that (c) holds. Let $(A, B)$ be an appropriate homogeneous pair in $G$, and let $(A, B, C, D, E, F)$ be the associated partition of $G$. Let $(K, a, b)$ be a semi-adjacent reduction of $(G, A, B)$. By the induction hypothesis, at least one of the following holds:

- $K$ admits a homogeneous set decomposition;
- $K$ or $\overline{K}$ is an appropriate expansion of a happy bull-free Berge trigraph.

If $K$ admits a homogeneous set decomposition, then by 4.4.7, so does $G$. So assume that $K$ or $\overline{K}$ is an appropriate expansion of a happy bull-free Berge trigraph. By symmetry, we may assume that $K$ is an appropriate expansion of a happy bull-free Berge trigraph $H$. We claim that $G$ is also an appropriate expansion of $H$.

Set $V_K = \bigcup_{v \in V_H} X_v$ as in the definition of an appropriate expansion. If there exists some $u \in V_H$ such that $a, b \in X_u$, then $u$ is an endpoint of an expandable semi-adjacent pair (because $|X_u| \geq 2$); we then set $\hat{X}_u = (X_u \smallsetminus \{a,b\}) \cup (A \cup B)$, and for all $v \in V_H \smallsetminus \{u\}$, we set $\hat{X}_v = X_v$, and clearly, this turns $G$ into an appropriate expansion of $K$. So from now on, we assume that there exist distinct $u, v \in V_H$ such that $a \in X_u$ and $b \in X_v$.

We claim that $uv$ is an expandable semi-adjacent pair in $H$. Suppose otherwise. Then since $a \in X_u$, $b \in X_v$, and $ab$ is a semi-adjacent pair in $K$, it follows that $X_u = \{a\}$, $X_v = \{b\}$, and $uv$ is a semi-adjacent pain $H$. Fix some $c \in C$ and $d \in D$, and let $c', d' \in V_H$ be such that $c \in X_{c'}$ and $d \in X_{d'}$; since $X_u = \{a\}$ and $X_v = \{b\}$, we know that $c', d' \notin \{u, v\}$. Now, $ac$ and $bd$ are adjacent pairs in $K$, and $ad$ and $bc$ are anti-adjacent pairs in $K$; since $a \in X_u$, $b \in X_v$, $c \in X_{c'}$, and $d \in X_{d'}$, this implies that $uc'$ and $vd'$ are adjacent pairs in $H$, and that $ud'$ and $vc'$ are anti-adjacent pairs in $H$. Since $uv$ is a semi-adjacent pair in $H$, it follows that $uc'$ and $vd'$ are strongly adjacent pairs in $H$, and that $ud'$ and $vc'$ are strongly anti-adjacent pairs in $H$. Since the semi-adjacent pair $uv$ is not expandable, this proves that $|V_H| \leq 4$. But then no semi-adjacent pair in $H$ is expandable, and consequently, $K$ and $H$ are isomorphic. Thus, $|V_K| \leq 4$, and so since $V_K = \{a, b\} \cup C \cup D \cup E \cup F$, we get that $|C \cup D \cup E \cup F| \leq 2$. But this contradicts the fact that $(A, B)$ is a tame homogeneous pair in $G$. Thus, $uv$ is expandable, as claimed.

Set $\hat{X}_u = (X_u \smallsetminus \{a\}) \cup A$, set $\hat{X}_v = (X_v \smallsetminus \{b\}) \cup B$, and for all $w \in V_H \smallsetminus \{u, v\}$, set $\hat{X}_w = X_w$. Now $G$ is an appropriate expansion of $H$, and we are done. $\square$

We can now finally prove 4.0.1, restated below.

**4.0.1.** *Let $G$ be a bull-free Berge trigraph. Then at least one of the following holds:*

- *$G$ or $\overline{G}$ is transitively orientable;*
- *$G$ contains a proper homogeneous set;*

- *$G$ contains a reducible homogeneous pair.*

*Proof.* By 4.4.9, we know that at least one of the following holds:

- $G$ admits a homogeneous set decomposition;
- $G$ or $\overline{G}$ is an appropriate expansion of a happy bull-free Berge trigraph.

If $G$ admits a homogeneous set decomposition, then we are done. So assume that $G$ or $\overline{G}$ is an appropriate expansion of a happy bull-free Berge trigraph. In particular then, $G$ or $\overline{G}$ is an appropriate expansion of a transitively orientable trigraph. Recall that $G$ contains a proper homogeneous set if and only if $\overline{G}$ does, and $G$ contains a reducible homogeneous pair if and only if $\overline{G}$ does. Thus, we may assume by symmetry that $G$ is an appropriate expansion of a transitively orientable trigraph. But now the result follows from 4.1.1. $\square$

# Chapter 5

# Coloring Bull-Free Perfect Graphs

In this chapter, we give combinatorial polynomial time algorithms that solve four optimization problems for the class of bull-free perfect graphs. These problems were described in the Introduction, but let us repeat them here. In this thesis, a *weighted graph* is a graph $G$ such that every vertex $v$ of $G$ is assigned a positive integer *weight*, denoted by $w_G(v)$. Given a set $S \subseteq V_G$, the *weight* of $S$, denoted by $w_G(S)$, is the sum of the weights of the vertices in $S$; the weight of the empty set is assumed to be zero. Unless stated otherwise, an induced subgraph $H$ of a weighted graph $G$ is assumed to inherit the weights from $G$, that is, it is assumed that $w_H(v) = w_G(v)$ for all $v \in V_H$. A *maximum weighted clique* (respectively: *maximum weighted stable set*) in $G$ is a clique (respectively: stable set) that has the maximum weight among all the cliques (respectively: stable sets) in $G$. We denote by $W(G)$ the maximum weight of a clique in $G$. We now describe the four optimization problems mentioned above. First, the *maximum weighted clique problem* (respectively: *maximum weighted stable set problem*) is the problem of finding a maximum weighted clique (respectively: maximum weighted stable set) in a weighted graph. Next, the *minimum weighted coloring problem* is the problem of finding stable sets $S_1, ..., S_t$ in a weighted graph $G$, and positive integers $\lambda_1, ..., \lambda_t$, such that $\Sigma_{S_i \ni v} \lambda_i \geq w_G(v)$ for all $v \in V_G$, and with the property that $\Sigma_{i=1}^{t} \lambda_i$ is minimum. Finally, the *minimum weighted clique covering problem* is the problem of finding cliques $C_1, ..., C_t$ in a weighted graph

$G$, and positive integers $\lambda_1, ..., \lambda_t$, such that $\Sigma_{C_i \ni v} \lambda_i \geq w_G(v)$ for all $v \in V_G$, and with the property that $\Sigma_{i=1}^{t} \lambda_i$ is minimum. As stated in the Introduction, de Figueiredo and Maffray gave combinatorial polynomial time algorithms that solve these four problems for the class of bull-free perfect graphs, and in this chapter, we describe faster algorithms that solve these same problems.

First, recall that a class $\mathcal{G}$ of graphs is *self-complementary* provided that for all $G \in \mathcal{G}$, we have that $\overline{G} \in \mathcal{G}$. By 2.1.1, the class of bull-free graphs is self-complementary and closed under substitution, and by 2.1.2, the class of perfect graphs is self-complementary and closed under substitution. This immediately implies the following result.

**5.0.10.** *The class of bull-free perfect graphs is self-complementary and closed under substitution.*

Let us now return to our four optimization problems. In what follows, $n$ denotes the number of vertices and $m$ the number of edges of the input graph.

Clearly, any maximum weighted clique in a graph $G$ is a maximum weighted stable set in $\overline{G}$. Thus, since the class of bull-free perfect graphs is self-complementary, it is easy to see that the maximum weighted clique problem and the maximum weighted stable set problem are equivalent for this this class of graphs in the following sense: any algorithm $A$ that finds a maximum weighted clique in any bull-free perfect graph in $O(n^k)$ time can be "turned into" an algorithm that finds a maximum weighted stable set in any bull-free perfect graph in $O(n^{\max\{k,2\}})$ time; indeed, we first take the complement of $G$, which takes $O(n^2)$ time, and then we run the algorithm $A$ on $\overline{G}$ to obtain a maximum weighted clique $C$ in $\overline{G}$, which is clearly a maximum weighted stable set in $G$. The reverse also holds: any algorithm that finds a maximum weighted stable set in any bull-free perfect graph in $O(n^k)$ time can be "turned into" an algorithm that finds a maximum weighted clique in any bull-free perfect graph in $O(n^{\max\{k,2\}})$ time. Clearly, the minimum weighted coloring and the minimum weighted clique covering problems are also equivalent for the class of bull-free perfect graphs in this same sense.

Further, it follows implicitly from the proof of Corollary 67.5c from [55] that if $\mathcal{G}$ is a class of perfect graphs, closed under taking induced subgraphs, and $A$ is any algorithm that can find a maximum weighted clique and a maximum weighted stable set in each graph in $\mathcal{G}$ in $O(n^k)$ time, then the algorithm $A$ can be used to obtain an algorithm $B$ that can find a minimum weighted coloring in each graph in $\mathcal{G}$ in $O(n^{k+2})$ time. The details can be found in [55], but let us provide a brief outline of the argument here. Suppose we are given a weighted graph $G \in \mathcal{G}$ on $n$ vertices. The first step is to find a stable set $S$ in $G$ that intersects each maximum weighted clique in $G$; this can be done after at most $O(n)$ calls to the algorithm that finds a maximum weighted clique or stable set in $G$ or its induced subgraphs (possibly with modified weights). An appropriate number of copies of $S$ become color classes of the desired minimum weighted coloring of $G$. The process is then repeated for an induced subgraph of $G$ (with reassigned weights); after at most $O(n)$ iterations, we obtain a minimum weighted coloring. All this implies that, in order to solve the four optimization problems for the class of bull-free perfect graphs in polynomial time, it suffices to construct an algorithm that solves the maximum weighted clique problem in polynomial time.

The algorithm from [33] finds a maximum weighted clique in a weighted bull-free perfect graph in time $O(n^5m^3)$, and it relies on the argument outlined above to solve the remaining three optimization problems. In this chapter, we give a polynomial time algorithm that solves the maximum weighted clique problem for weighted bull-free perfect graphs in $O(n^6)$ time. By the discussion above, this yields an algorithm for finding a maximum weighted stable set in a weighted bull-free prefect graph in $O(n^6)$ time, as well as algorithms for solving the minimum wighted coloring problem and the minimum weighted clique covering problem in such a graph in $O(n^8)$ time.

## 5.1 A Decomposition Theorem for Bull-Free Perfect Graphs and an Outline of the Chapter

We begin with a definition. A tame homogeneous pair $(A, B)$ in a graph $G$ is said to be *reducible* provided that the associated partition $(A, B, C, D, E, F)$ of $G$ satisfies the following:

- either
  - $|B| \geq 3$, or
  - $|B| = 2$ and there exist distinct vertices $a, a' \in A$ such that $a$ and $a'$ are both mixed on $B$;
- $C$ and $D$ are both non-empty.

We observe that if $(A, B)$ is a reducible homogeneous pair in a graph $G$, then $|A \cup B| \geq 4$ and $|C \cup D \cup E \cup F| \geq 3$ (the latter is a consequence of the fact that $(A, B)$ is tame). Note that if $(A, B)$ is a reducible homogeneous pair in a graph $G$, and if $(A, B, C, D, E, F)$ is the associated partition of $G$, then $(A, B)$ is also a reducible homogeneous pair in $\overline{G}$, and $(A, B, D, C, F, E)$ is the associated partition of $\overline{G}$.

Reducible homogeneous pairs were originally defined in the trigraph context (see chapter 4), but in the present chapter, we only work with graphs, and so here, we give the definition of reducible homogeneous pairs for graphs only. (However, the definition is completely analogous in the trigraph context; we refer the reader to chapter 4.)

Now, the main theoretical tools that we will need for our algorithm are the main results (namely, 4.0.1 and 4.0.2) of chapter 4. These two theorems are about bull-free Berge trigraphs, but since every graph can be thought of as a trigraph (a graph is simply a trigraph with no semi-adjacent pairs), and since every perfect graph is Berge, 4.0.1 and 4.0.2

immediately imply corresponding results about bull-free perfect graphs. First, the decomposition theorem for bull-free Berge trigraphs 4.0.1 immediately implies the following decomposition theorem for bull-free perfect graphs.

**5.1.1.** *Let $G$ be a bull-free perfect graph. Then at least one of the following holds:*

- *$G$ or $\overline{G}$ is transitively orientable;*
- *$G$ contains a proper homogeneous set;*
- *$G$ contains a reducible homogeneous pair.*

Similarly, 4.0.2 immediately implies the following result.

**5.1.2.** *Let $G$ be a bull-free perfect graph that does not contain a proper homogeneous set, and let $(A, B)$ be a reducible homogeneous pair in $G$. Then $G[A]$ and $G[B]$ are both transitively orientable.*

The remainder of the chapter is organized as follows. In section 5.2, we give an algorithm that, given a graph $G$ that does not contain a proper homogeneous set, either finds a reducible homogeneous pair in $G$, or determines that $G$ does not contain one. In section 5.3, we explain how to obtain a maximum weighted clique in a graph that contains a proper homogeneous set or a reducible homogeneous pair once we have obtained maximum weighted cliques in certain smaller graphs. In section 5.4, we describe the algorithm MWCLIQUE that, given a weighted bull-free perfect graph $G$, finds a maximum weighted clique in $G$. Finally, in section 5.5, we perform a complexity analysis, and we discuss the reasons why the algorithm MWCLIQUE is faster than the algorithm from [33].

## 5.2 Reducible Homogeneous Pairs

Our main goal in this section is to describe the algorithm REDUCIBLE, which, given a graph $G$ that does not contain a proper homogeneous set, either finds a reducible homogeneous pair in $G$, or determines that $G$ does not contain such a homogeneous pair. We begin with some definitions. Given a graph $G$, and a triple $(b, b', d)$ of pairwise distinct vertices, we say that $(b, b', d)$ is a *reducible frame* in $G$ provided that there exists a reducible homogeneous pair $(A, B)$ in $G$ with associated partition $(A, B, C, D, E, F)$ such that the following hold:

- $b, b' \in B$;
- $d \in D$;
- at least one of the following holds:
  - some vertex $b'' \in B \smallsetminus \{b, b'\}$ is mixed on $\{b, b'\}$,
  - there exist distinct vertices $a, a' \in A$ such that $a$ and $a'$ are both mixed on $\{b, b'\}$.

Under these circumstances, we also say that $(b, b', d)$ is a *frame* for the reducible homogeneous pair $(A, B)$. We observe that if $(b, b', d)$ is reducible frame, then all of the following hold:

- $d$ is complete to $\{b, b'\}$;
- at least one of the following holds:
  - there exists a vertex $b'' \in V_G \smallsetminus \{b, b', d\}$ such that $b''$ is mixed on $\{b, b'\}$ and adjacent to $d$,
  - there exist distinct vertices $a, a' \in V_G \smallsetminus \{b, b', d\}$ such that $a$ and $a'$ are both mixed on $\{b, b'\}$ and non-adjacent to $d$.

We remark that reducible frames are unrelated to the frames from chapter 4.

Reducible frames will be our main tool for detecting reducible homogeneous pairs in graphs that contain no proper homogeneous sets. But first, we need the following result.

**5.2.1.** *Let $G$ be a graph that does not contain a proper homogeneous set, and let $(A, B)$ be a reducible homogeneous pair in $G$. Then $G$ contains a frame for $(A, B)$.*

*Proof.* Let $(A, B)$ be a reducible homogeneous pair in $G$, and let $(A, B, C, D, E, F)$ be the associated partition of $G$. By the definition of a reducible homogeneous pair, we know that $D$ is non-empty.

Suppose first that $|B| = 2$, say $B = \{b, b'\}$, and that there exist distinct vertices $a, a' \in A$ such that $a$ and $a'$ are both mixed on $B$. Now, using the fact that $D$ is non-empty, we fix some $d \in D$, and observe that $(b, b', d)$ is a frame for $(A, B)$.

It remains to consider the case when $|B| \geq 3$. Suppose first that $B$ is neither a clique nor a stable set. Then there exist vertices $b, b', b''$ such that $b''$ is adjacent to $b$ and non-adjacent to $b'$. Fix some $d \in D$. Then $(b, b', d)$ is a frame for $(A, B)$.

Suppose now that $B$ is either a clique or a stable set. Since $B$ is not a homogeneous set in $G$, some vertex $a \in V_G \smallsetminus B$ is mixed on $B$; since $(A, B)$ is a homogeneous pair, we know that $a \in A$. Let $N_1(a)$ and $N_2(a)$ be the sets of neighbors and non-neghibhors, respectively, of $a$ in $B$. Since $a$ is mixed on $B$, both $N_1(a)$ and $N_2(a)$ are non-empty. Let $i, j \in \{1, 2\}$ be distinct with the property that $|N_i(a)| \geq |N_j(a)|$. Since $B$ is the disjoint union of the sets $N_i(a)$ and $N_j(a)$, and since $|B| \geq 3$, we know that $|N_i(a)| \geq 2$. Since $N_i(a)$ is not a homogeneous set in $G$, some vertex $a' \in V_G \smallsetminus N_i(a)$ is mixed on $N_i(a)$. Now, $a'$ is mixed on $N_i(a) \subseteq B$, $B$ is either a clique or a stable set, and $(A, B)$ is homogeneous pair in $G$; it follows that $a' \in A$. Fix $b_i, b_i' \in N_i(a)$ such that $a'$ is adjacent to $b_i$ and non-adjacent to $b_i'$, and fix some $b_j \in N_j(a)$. By construction, $a$ is mixed on both $\{b_i, b_j\}$ and $\{b_i', b_j\}$. Further, if $a'$ is adjacent to $b_j$, then $a'$ is mixed on $\{b_i', b_j\}$, and if $a'$ is non-adjacent to $b_j$, then $a'$ is mixed on $\{b_i, b_j\}$. Thus, we have that either $a$ and $a'$ are both mixed on $\{b_i, b_j\}$, or $a$ and $a'$ are both mixed on $\{b_i', b_j\}$. Now, fix some $d \in D$. Then at least one of $(b_i, b_j, d)$ and $(b_i', b_j, d)$ is a frame for $(A, B)$. This completes the argument. □

We remark that while reducible frames could be defined in the trigraph context in a straightforward fashion, the trigraph analog of 5.2.1 would be false. This is because a trigraph may contain a reducible homogeneous pair $(\{a\}, \{b, b', b''\})$, where $a$ is strongly adjacent to $b$, semi-adjacent to $b'$, and strongly anti-adjacent to $b''$, and where $\{b, b', b''\}$ is a strong clique or a strongly stable set; clearly, there is no frame for such a reducible homogeneous pair.

Given a graph $G$, a reducible homogeneous pair $(A, B)$ in $G$, and a frame $(b, b', d)$ for $(A, B)$, we say that $(A, B)$ is the *minimal* reducible homogeneous pair for the reducible frame $(b, b', d)$ provided that for all reducible homogeneous pairs $(A', B')$ in $G$ such that $(b, b', d)$ is a frame for $(A', B')$, we have that $A \subseteq A'$ and $B \subseteq B'$. Our next result (5.2.2) establishes that for every graph $G$, and every reducible frame $(b, b', d)$ in $G$, there exists a unique minimal reducible homogeneous pair in $G$ for $(b, b', d)$. We note that the proof of 5.2.2 can easily be turned into an algorithm that, given a graph $G$ that does not contain a proper homogeneous set, and a triple $(b, b', d)$ of pairwise distinct vertices in $G$, either returns a 6-tuple $(A, B, C, D, E, F)$ such that $(A, B)$ is the unique minimal reducible homogeneous pair in $G$ for the reducible frame $(b, b', d)$, and the associated partition of $G$ is $(A, B, C, D, E, F)$, or determines that $(b, b', d)$ is not a reducible frame in $G$. The running time of the algorithm is $O(n^2)$, where $n = |V_G|$.

**5.2.2.** *Let $G$ be a graph that does not contain a proper homogeneous set, and let $(b, b', d)$ be a triple of pairwise distinct vertices in $G$. Then if $(b, b', d)$ is a reducible frame in $G$, then there exists a unique minimal reducible homogeneous pair $(A, B)$ for $(b, b', d)$.*

*Proof.* If $d$ is not complete to $\{b, b'\}$, then $(b, b', d)$ is not a reducible frame, and there is nothing to show. So assume that $d$ is complete to $\{b, b'\}$. Next, we let $S_A$ be the set of all vertices in $V_G \smallsetminus \{b, b', d\}$ that are mixed on $\{b, b'\}$ and non-adjacent to $d$, and we let $S_B$ be the set of all vertices in $V_G \smallsetminus \{b, b', d\}$ that are mixed on $\{b, b'\}$ and adjacent to $d$. If

$|S_A| \leq 1$ and $S_B = \emptyset$, then $(b, b', d)$ is not a reducible frame, and again, there is nothing to show. So assume that either $|S_A| \geq 2$ or $S_B \neq \emptyset$. We note that if $(A', B')$ is a reducible homogeneous pair in $G$ such that $(b, b', d)$ is a frame for $(A', B')$, then the fact that every vertex in $S_A \cup S_B$ is mixed on $\{b, b'\} \subseteq B'$ implies that $S_A \cup S_B \subseteq A' \cup B'$, and then the fact that $d$ is complete to $A'$ and anti-complete $B'$ implies that $S_A \subseteq A'$ and $S_B \subseteq B'$.

Now, we construct sets $A$ and $B$, as well as the function $l : V_G \smallsetminus (A \cup B \cup \{d\}) \rightarrow \{\text{E}, \text{C}, \text{A}, \text{M}\} \times \{\text{C}, \text{A}, \text{M}\}$, as follows. (Note: "E" stands for "empty," "C" stands for "complete," "A" stands for "anti-complete," and "M" stands for "mixed.")

First, set $A_0 = S_A$ and $B_0 = \{b, b'\} \cup S_B$. (Note that $A_0$ may be empty, but $B_0$ is non-empty.) Next, define the function $l_0 : V_G \smallsetminus (A_0 \cup B_0 \cup \{d\}) \rightarrow \{\text{E}, \text{C}, \text{A}, \text{M}\} \times \{\text{C}, \text{A}, \text{M}\}$ as follows. For all $v \in V_G \smallsetminus (A_0 \cup B_0 \cup \{d\})$, set $l_0(v) = (X, Y)$, where:

- if $A_0 = \emptyset$, then we set $X = \text{E}$;
- if $A_0 \neq \emptyset$ and $v$ is complete to $A_0$, then we set $X = \text{C}$;
- if $A_0 \neq \emptyset$ and $v$ is anti-complete to $A_0$, then we set $X = \text{A}$;
- if $A_0 \neq \emptyset$ and $v$ is mixed on $A_0$, then we set $X = \text{M}$;
- if $v$ is complete to $B_0$, then we set $Y = \text{C}$;
- if $v$ is anti-complete to $B_0$, then we set $Y = \text{A}$;
- if $v$ is mixed on $B_0$, then we set $Y = \text{M}$.

Assume now that we have constructed sets $A_i$ and $B_i$, as well as a function $l_i : V_G \smallsetminus (A_i \cup B_i \cup \{d\}) \rightarrow \{\text{E}, \text{C}, \text{A}, \text{M}\} \times \{\text{C}, \text{A}, \text{M}\}$. If every vertex $u \in V_G \smallsetminus (A_i \cup B_i \cup \{d\})$ satisfies the property that $l_i(u) \in \{\text{E}, \text{C}, \text{A}\} \times \{\text{C}, \text{A}\}$, then we terminate the sequence, and we set $A = A_i$, $B = B_i$, and $l = l_i$. Suppose now that there exists some vertex $u \in V_G \smallsetminus (A_i \cup B_i \cup \{d\})$ such that at least one coordinate of $l_i(u)$ is M. In this case, we construct sets $A_{i+1}$ and

$B_{i+1}$ as follows. If $u$ is adjacent to $d$, then we set $A_{i+1} = A_i$ and $B_{i+1} = B_i \cup \{u\}$, and we define a function $l_{i+1} : V_G \smallsetminus (A_{i+1} \cup B_{i+1} \cup \{d\}) \to \{\mathrm{E}, \mathrm{C}, \mathrm{A}, \mathrm{M}\} \times \{\mathrm{C}, \mathrm{A}, \mathrm{M}\}$ in such a way that for all $v \in V_G \smallsetminus (A_{i+1} \cup B_{i+1} \cup \{d\})$ with $l_i(v) = (X, Y)$, we set $l_{i+1}(v) = (X', Y')$, where:

- $X' = X$;
- if $Y \in \{\mathrm{C}, \mathrm{M}\}$ and $v$ is adjacent to $u$, then $Y' = Y$;
- if $Y = \mathrm{A}$ and $v$ is adjacent to $u$, then $Y' = \mathrm{M}$;
- if $Y \in \{\mathrm{A}, \mathrm{M}\}$ and $v$ is non-adjacent to $u$, then $Y' = Y$;
- if $Y = \mathrm{C}$ and $v$ is non-adjacent to $u$, then $Y' = \mathrm{M}$.

On the other hand, if $u$ is non-adjacent to $d$, then we set $A_{i+1} = A_i \cup \{u\}$ and $B_{i+1} = B_i$, and we define a function $l_{i+1} : V_G \smallsetminus (A_{i+1} \cup B_{i+1} \cup \{d\}) \to \{\mathrm{E}, \mathrm{C}, \mathrm{A}, \mathrm{M}\} \times \{\mathrm{C}, \mathrm{A}, \mathrm{M}\}$ in such a way that for all $v \in V_G \smallsetminus (A_{i+1} \cup B_{i+1} \cup \{d\})$ with $l_i(v) = (X, Y)$, we set $l_{i+1}(v) = (X', Y')$, where:

- $Y' = Y$;
- if $X = \mathrm{E}$ and $v$ is adjacent to $u$, then $X' = \mathrm{C}$;
- if $X = \mathrm{E}$ and $v$ is non-adjacent to $u$, then $X' = \mathrm{A}$;
- if $X \in \{\mathrm{C}, \mathrm{M}\}$ and $v$ is adjacent to $u$, then $X' = X$;
- if $X = \mathrm{A}$ and $v$ is adjacent to $u$, then $X' = \mathrm{M}$;
- if $X \in \{\mathrm{A}, \mathrm{M}\}$ and $v$ is non-adjacent to $u$, then $X' = X$;
- if $X = \mathrm{C}$ and $v$ is non-adjacent to $u$, then $X' = \mathrm{M}$.

Now, we may assume that the construction above yields sequences of sets $A_0, ..., A_n$ and $B_0, ..., B_n$, as well as a sequence of functions $l_0 : V_G \smallsetminus (A_0 \cup B_0 \cup \{d\}) \to \{\mathrm{E}, \mathrm{C}, \mathrm{A}, \mathrm{M}\} \times \{\mathrm{C}, \mathrm{A}, \mathrm{M}\}, ..., l_{n-1} : V_G \smallsetminus (A_{n-1} \cup B_{n-1} \cup \{d\}) \to \{\mathrm{E}, \mathrm{C}, \mathrm{A}, \mathrm{M}\} \times \{\mathrm{C}, \mathrm{A}, \mathrm{M}\}, l_n : V_G \smallsetminus (A_n \cup$

$B_n \cup \{d\}) \to \{\mathrm{E}, \mathrm{C}, \mathrm{A}\} \times \{\mathrm{C}, \mathrm{A}\}$, such that $A = A_n$, $B = B_n$, and $l = l_n$. We claim that for all $i \in \{0, ..., n\}$, the following hold:

- $S_A \subseteq A_i$;
- $S_B \cup \{b, b'\} \subseteq B_i$;
- $d \notin A_i \cup B_i$;
- $d$ is anti-complete to $A_i$ and complete to $B_i$;
- for all $v \in V_G \smallsetminus (A_i \cup B_i \cup \{d\})$ with $l_i(v) = (X, Y)$, the following hold:
  - if $X = \mathrm{E}$, then $A_i$ is empty,
  - if $X = \mathrm{C}$, then $A_i$ is non-empty and $v$ is complete to $A_i$,
  - if $X = \mathrm{A}$, then $A_i$ is non-empty and $v$ is anti-complete to $A_i$,
  - if $X = \mathrm{M}$, then $A_i$ is non-empty and $v$ is mixed on $A_i$,
  - if $Y = \mathrm{C}$, then $v$ is complete to $B_i$,
  - if $Y = \mathrm{A}$, then $v$ is anti-complete to $B_i$,
  - if $Y = \mathrm{M}$, then $v$ is mixed on $B_i$;
- for all reducible homogeneous pairs $(A', B')$ such that $(b, b', d)$ is a frame for $(A', B')$, we have that $A_i \subseteq A'$ and $B_i \subseteq B'$.

We prove this by induction on $i$. For the base case, this is immediate by construction. For the induction step, we assume that the claim holds for some $i \in \{0, ..., n-1\}$, and we show that it holds for $i+1$. All requirements except for the last one are easily seen to follow from the induction hypothesis and the construction. For the last requirement, suppose that $(A', B')$ is a reducible homogeneous pair in $G$ such that $(b, b', d)$ is a frame for $(A', B')$. By the induction hypothesis, $A_i \subseteq A'$ and $B_i \subseteq B'$. Furthermore, since $(b, b', d)$ is a frame for $(A', B')$, we know that $d \notin A' \cup B'$, and that $d$ is complete to $B'$ and anti-complete to $A'$. Now, fix $u \in V_G \smallsetminus (A_i \cup B_i \cup \{d\})$ such that either $A_{i+1} = A_i$

and $B_{i+1} = B_i \cup \{u\}$, or $A_{i+1} = A_i \cup \{u\}$ and $B_{i+1} = B_i$. By construction, we know that at least one coordinate of $l_i(u)$ is M, and so by the induction hypothesis, $u$ is mixed on at least one of $A_i$ and $B_i$. Since $A_i \subseteq A'$ and $B_i \subseteq B'$, and since $(A', B')$ is a homogeneous pair, it follows that $u \in A' \cup B'$. Since $d$ is anti-complete to $A'$ and complete to $B'$, we have that if $u$ is adjacent to $d$ then $u \in B'$, and if $u$ is non-adjacent to $d$ then $u \in A'$. By construction then, we get that $A_{i+1} \subseteq A'$ and $B_{i+1} \subseteq B'$. This completes the induction. Now, by construction, we have that $A = A_n$, $B = B_n$, and $l = l_n$; furthermore, we know that $l_n(v) \in \{\mathrm{E}, \mathrm{C}, \mathrm{A}\} \times \{\mathrm{C}, \mathrm{A}\}$ for all $v \in V_G \setminus (A_n \cup B_n \cup \{d\})$. By what we just showed, this implies the following:

- $S_A \subseteq A$;
- $S_B \cup \{b, b'\} \subseteq B$;
- $d \notin A \cup B$;
- $d$ is anti-complete to $A$ and complete to $B$;
- $l : V_G \setminus (A \cup B \cup \{d\}) \to \{\mathrm{E}, \mathrm{C}, \mathrm{A}\} \times \{\mathrm{C}, \mathrm{A}\}$;
- for all $v \in V_G \setminus (A \cup B \cup \{d\})$ with $l(v) = (X, Y)$, the following hold:
  - if $X = \mathrm{E}$, then $A$ is empty,
  - if $X = \mathrm{C}$, then $A$ is non-empty and $v$ is complete to $A$,
  - if $X = \mathrm{A}$, then $A$ is non-empty and $v$ is anti-complete to $A$,
  - if $Y = \mathrm{C}$, then $v$ is complete to $B$,
  - if $Y = \mathrm{A}$, then $v$ is anti-complete to $B$,
- for all reducible homogeneous pairs $(A', B')$ such that $(b, b', d)$ is a frame for $(A', B')$, we have that $A \subseteq A'$ and $B \subseteq B'$.

Note first that the above implies that no vertex in $V_G \setminus (A \cup B)$ is mixed on either $A$ or $B$. Next, note that $A$ is non-empty and neither complete nor anti-complete to $B$, for otherwise, $B$ would be a proper homogeneous set in $G$, and by assumption, $G$ has no proper homogeneous sets; clearly, this implies that for all $v \in V_G \setminus (A \cup B \cup \{d\})$, $l(v) \in \{\mathrm{C}, \mathrm{A}\} \times \{\mathrm{C}, \mathrm{A}\}$. Now, since $S_A \subseteq A$, $S_B \cup \{b, b'\} \subseteq B$, and either $|S_A| \geq 2$ or $S_B \neq \emptyset$, we get that either

- $|B| \geq 3$, or
- $|B| = 2$ and there exist distinct vertices $a, a' \in A$ such that $a$ and $a'$ are both mixed on $B$.

Next, let $C$ be the set of all vertices $v \in V_G \setminus (A \cup B \cup \{d\})$ such that $l(v) = (\mathrm{C}, \mathrm{A})$; let $D$ be the set consisting of the vertex $d$ as well as of all the vertices $v \in V_G \setminus (A \cup B \cup \{d\})$ such that $l(v) = (\mathrm{A}, \mathrm{C})$; let $E$ be the set of all vertices $v \in V_G \setminus (A \cup B \cup \{d\})$ such that $l(v) = (\mathrm{C}, \mathrm{C})$; and let $F$ be the set of all vertices $v \in V_G \setminus (A \cup B \cup \{d\})$ such that $l(v) = (\mathrm{A}, \mathrm{A})$. Note that the fact that $d \in D$ implies that $D$ is non-empty. It now easily follows that $(A, B)$ is a homogeneous pair in $G$ with associated partition $(A, B, C, D, E, F)$. Now, we claim that if $C$ is non-empty and $|C \cup D \cup E \cup F| \geq 3$, then $(A, B)$ is the unique minimal reducible homogeneous pair for $(b, b', d)$ in $G$, and otherwise, $(b, b', d)$ is not a reducible frame.

Suppose first that $C$ is non-empty and $|C \cup D \cup E \cup F| \geq 3$. Since $A$ is neither complete nor anti-complete to $B$, it follows that $(A, B)$ is tame. Next, we showed above that either $|B| \geq 3$, or $|B| = 2$ and there exist distinct vertices $a, a' \in A$ such that $a$ and $a'$ are both mixed on $B$. By supposition, $C$ is non-empty, and since $d \in D$, $D$ is non-empty. It follows that $(A, B)$ is a reducible homogeneous pair in $G$. The fact that $(b, b', d)$ is a frame for $(A, B)$ follows from the fact that either $|S_A| \geq 2$ or $S_B \neq \emptyset$. The minimality of $(A, B)$ follows from the fact that for all reducible homogeneous pairs $(A', B')$ such that $(b, b', d)$ is a frame for $(A', B')$, we have that $A \subseteq A'$ and $B \subseteq B'$. The uniqueness of $(A, B)$ is immediate.

Suppose now that $C$ is empty or that $|C \cup D \cup E \cup F| \leq 2$. Suppose that $(b, b', d)$ is a reducible frame, and fix a reducible homogeneous pair $(A', B')$ in $G$ such that $(b, b', d)$ is a frame for $(A', B')$. Let $(A', B', C', D', E', F')$ be the partition of $G$ associated with $(A', B')$. We know that $A \subseteq A'$ and $B \subseteq B'$, and so we have that $C' \subseteq C$, $D' \subseteq D$, $E' \subseteq E$, and $F' \subseteq F$. But now if $C$ is empty, then so is $C'$, and if $|C \cup D \cup E \cup F| \leq 2$, then $|C' \cup D' \cup E' \cup F'| \leq 2$; neither outcome is possible because $(A', B')$ is a reducible homogeneous pair. It follows that $(b, b', d)$ is not a reducible frame. This completes the argument. □

We now prove an easy lemma, and then we turn to the algorithm REDUCIBLE.

**5.2.3.** *Let $G$ be a graph that does not contain a proper homogeneous set, let $(A, B)$ be a reducible homogeneous pair in $G$, and let $(b, b', d)$ be a frame for $(A, B)$. Then the following hold:*

- *if $b$ is adjacent to $b'$, then there exist vertices $a \in A$ and $b_1, b_2 \in B$ such that $ab_1$ and $b_1b_2$ are edges, and $ab_2$ is a non-edge;*
- *if $b$ is non-adjacent to $b'$, then there exist vertices $a \in A$ and $b_1, b_2 \in B$ such that $ab_1$ is an edge, and $ab_2$ and $b_1b_2$ are non-edges.*

*Proof.* If $b$ is adjacent to $b'$, then $G[B]$ contains a non-trivial component, and if $b$ is non-adjacent to $b'$, then $G[B]$ contains a non-trivial anti-component. If $bb'$ is an edge, let $W \subseteq B$ be such that $G[W]$ is a non-trivial component of $G[B]$; and if $bb'$ is a non-edge, then let $W \subseteq B$ be such that $G[W]$ is a non-trivial anti-component of $G[B]$. Since $|W| \geq 2$, and $G$ contains no proper homogeneous set, we know that some vertex $a \in A$ is mixed on $W$. If $bb'$ is an edge, so that $G[W]$ is a component of $G[B]$, then there exist adjacent vertices $b_1, b_2 \in W$ such that $a$ is adjacent to $b_1$ and non-adjacent to $b_2$. And if $bb'$ is a non-edge, so that $G[W]$ is an anti-component of $G[B]$, then there exist non-adjacent vertices $b_1, b_2 \in W$ such that $a$ is adjacent to $b_1$ and non-adjacent to $b_2$. This completes the argument. □

We now describe the algorithm REDUCIBLE that, given a graph $G$ that does not contain a proper homogeneous set, either returns a 7-tuple $(A, B, C, D, E, F, z)$ such that the following hold:

- $(A, B)$ is a reducible homogeneous pair, and $(A, B, C, D, E, F)$ is the associated partition of $G$;
- $z \in \{\mathrm{a}, \mathrm{n}\}$;
- if $z = \mathrm{a}$, then there exist vertices $a \in A$ and $b_1, b_2 \in B$ such that $ab_1$ and $b_1b_2$ are edges, and $ab_2$ is a non-edge;
- if $z = \mathrm{n}$, then there exist vertices $a \in A$ and $b_1, b_2 \in B$ such that $ab_1$ is an edge, and $ab_2$ and $b_1b_2$ are non-edges;

or determines that $G$ does not contain a reducible homogeneous pair.

We enumerate all triples $(b, b', d)$ of pairwise distinct vertices in $G$. For each such triple $(b, b', d)$, we call the algorithm from 5.2.2, and either obtain a 6-tuple $(A, B, C, D, E, F)$ such that $(A, B)$ is a reducible homogeneous pair in $G$ such that $(b, b', d)$ is a frame for $(A, B)$ and $(A, B, C, D, E, F)$ is the associated partition of $G$, or we obtain the answer that $(b, b', d)$ is not a reducible frame. If we obtain a 6-tuple $(A, B, C, D, E, F)$, then we use 5.2.3, and if $bb'$ is an edge then we set $z = \mathrm{a}$, and if $bb'$ is a non-edge then we set $z = \mathrm{n}$; we then stop, and the algorithm returns the 7-tuple $(A, B, C, D, E, F, z)$. If we obtained the answer that $(b, b', d)$ is not a reducible frame, then we move to the next triple on the list, and repeat the process. If for every triple $(b, b', d)$ on the list, the algorithm determines that $(b, b', d)$ is not a reducible frame, then by 5.2.1, it follows that $G$ does not contain a reducible homogeneous pair; in this case, we stop, and the algorithm returns the answer that $G$ contains no reducible homogeneous pair.

We observe that the running time of the algorithm REDUCIBLE is at most $O(n^5)$, where $n$ is the number of vertices of the input graph.

## 5.3 Reducing Homogeneous Sets and Reducible Homogeneous Pairs

In this section, we describe "weighted reductions" of weighted graphs with respect to proper homogeneous sets and with respect to reducible homogeneous pairs. (We remark here that these "weighted reductions" for reducible homogeneous pairs are unrelated to the semi-adjacent reductions introduced in section 2.2.) We also explain how to use these weighted reductions to "recover" a maximum weighted clique in the original graph.

We first deal with weighted reductions with respect to proper homogeneous sets. Suppose that $G$ is a weighted graph, and that $S$ is a proper homogeneous set in $G$. Let $\tilde{G}$ be the graph whose veretex set is $(V_G \smallsetminus S) \cup \{s\}$, where $s \notin V_G$, with weights assigned as follows: $w_{\tilde{G}}(v) = w_G(v)$ for all $v \in V_G \smallsetminus S$; and $w_{\tilde{G}}(s) = W(G[S])$. We refer to the weighted graph $\tilde{G}$ as the *weighted reduction of $G$ with respect to $S$*. Note that if we regard $G$ and $\tilde{G}$ as unweighted graphs, then $(\tilde{G}, s)$ is a reduction of $(G, S)$ in the sense defined in section 2.1. Note also that, as an unweighted graph, $\tilde{G}$ is isomorphic to an induced subgraph of $G$; consequently, if $G$ is bull-free and perfect, then so is $\tilde{G}$. Our next result describes how to "recover" a maximum weighted clique in $G$ from maximum weighted cliques in $\tilde{G}$ and $G[S]$.

**5.3.1.** *Let $G$ be a weighted graph, let $S$ be a proper homogeneous set in $G$. Let $\tilde{G}$ and $s$ be as in the definition of the weighted reduction of $G$ with respect to $S$. Let $\tilde{K}$ and $K_S$ be maximum weighted cliques in $\tilde{G}$ and $G[S]$, respectively. If $s \notin \tilde{K}$ then set $K = \tilde{K}$, and if $s \in \tilde{K}$ then set $K = (\tilde{K} \smallsetminus \{s\}) \cup K_S$. Then $K$ is a maximum weighted clique in $G$.*

*Proof.* First, if $s \notin \tilde{K}$ so that $K = \tilde{K}$, then it is clear that $K$ is a clique in $G$ and that $w_G(K) = w_{\tilde{G}}(\tilde{K})$. On the other hand, if $s \in \tilde{K}$, then $G[K]$ is obtained by substituting the complete graph $G[K_S]$ for $s$ in the complete graph $\tilde{G}[\tilde{K}]$, and consequently $K$ is a clique in $G$; furthermore, since $w_{\tilde{G}}(s) = W(G[S])$, we know that $w_G(K) = w_{\tilde{G}}(\tilde{K})$.

It remains to show that the clique $K$ is of maximum weight in $G$. Let $K'$ be a maximum weighted clique in $G$; we need to show that $w_G(K') \leq w_G(K)$. Let $(S, X, Y)$ be the partition of $G$ associated with the homogeneous set $S$. Suppose first that $K' \cap S = \emptyset$. Then $K'$ is a clique in $\tilde{G}$ as well, and $w_{\tilde{G}}(K') = w_G(K')$; by the maximality of $\tilde{K}$, we have that:

$$w_G(K') = w_{\tilde{G}}(K') \leq w_{\tilde{G}}(\tilde{K}) = w_G(K),$$

which is what we needed to show. Suppose now that $K' \cap S \neq \emptyset$. As $Y$ is anti-complete to $S$ in $G$, and $K'$ is a clique that intersects $S$ in $G$, we know that $K' \subseteq S \cup X$. But since $X$ is complete to $S$ (and therefore to $K_S$ as well) in $G$, and since $K_S$ is a clique in $G$, we know that $(K' \smallsetminus S) \cup K_S$ is a clique in $G$; furthermore, since $s$ is complete to $X$ in $\tilde{G}$ and $K' \subseteq S \cup X$, we know that $(K' \smallsetminus S) \cup \{s\}$ is a clique in $\tilde{G}$. Now, by the maximality of $\tilde{K}$ and $K_S$, we have the following:

$$\begin{aligned}
w_G(K') &= w_G(K' \smallsetminus S) + w_G(K' \cap S) \\
&\leq w_G(K' \smallsetminus S) + w_G(K_S) \\
&= w_{\tilde{G}}(K' \smallsetminus S) + w_{\tilde{G}}(s) \\
&= w_{\tilde{G}}((K' \smallsetminus S) \cup \{s\}) \\
&\leq w_{\tilde{G}}(\tilde{K}) \\
&= w_G(K).
\end{aligned}$$

This completes the argument. □

We now discuss weighted reductions with respect to reducible homogeneous pairs. Given a weighted graph $G$ and a reducible homogeneous pair $(A, B)$ in $G$, we define "type a weighted reduction of $G$ with respect to $(A, B)$" and "type n weighted reduction of $G$ with respect to $(A, B)$," as follows.

We first define type a weighted reductions. Let $G$ be a weighted graph, let $(A, B)$ be a reducible homogeneous pair in $G$, and let $(A, B, C, D, E, F)$ be the associated partition of $G$. Let $\tilde{G}'$ be the graph with vertex-set $\{a, b, b'\} \cup C \cup D \cup E \cup F$, where $a$, $b$, and $b'$ are pairwise distinct and do not lie in $V_G$, with adjacency as follows:

- $\tilde{G}'[C \cup D \cup E \cup F] = G[C \cup D \cup E \cup F]$;
- $a$ is complete to $C \cup E$ and anti-complete to $D \cup F$;
- $b$ and $b'$ are complete to $D \cup E$ and anti-complete to $C \cup F$;
- $a$ is adjacent to $b$ and non-adjacent to $b'$;
- $b$ is adjacent to $b'$.

Next, we assign weights to the vertices of $\tilde{G}'$ as follows. The weights of the vertices in $C \cup D \cup E \cup F$ in the graph $\tilde{G}'$ are inherited from $G$, and for the vertices $a, b, b'$, we set:

- $w_{\tilde{G}'}(a) = W(G[A])$;
- $w_{\tilde{G}'}(b) = W(G[A \cup B]) - W(G[A])$;
- $w_{\tilde{G}'}(b') = W(G[A]) + W([G[B]) - W(G[A \cup B])$.

We observe that all vertices in $C \cup D \cup E \cup F \cup \{a\}$ have positive integer weight in $\tilde{G}'$, $b$ and $b'$ have non-negative integer weights, and at most one of $b$ and $b'$ has zero weight. Furthermore, we have that:

- $w_{\tilde{G}'}(a) = W(G[A])$;
- $w_{\tilde{G}'}(a) + w_{\tilde{G}'}(b) = W(G[A \cup B])$;
- $w_{\tilde{G}'}(b) + w_{\tilde{G}'}(b') = W([G[B])$.

Finally, we define $\tilde{G}$ to be the graph obtained from $\tilde{G}'$ by deleting all vertices in $\tilde{G}'$ with weight zero. (Thus, either $\tilde{G} = \tilde{G}'$, or $\tilde{G} = \tilde{G}' \smallsetminus b$, or $\tilde{G} = \tilde{G}' \smallsetminus b'$.) We refer to the weighted graph $\tilde{G}$ as the *type* a *weighted reduction of* $G$ *with respect to* $(A, B)$.

It remains to define type n weighted reductions. Let $G$ be a weighted graph, let $(A, B)$ be a reducible homogeneous pair in $G$, and let $(A, B, C, D, E, F)$ be the associated partition of $G$. Let $\tilde{G}'$ be the graph with vertex-set $\{a, b, b'\} \cup C \cup D \cup E \cup F$, where $a$, $b$, and $b'$ are pairwise distinct and do not lie in $V_G$, with adjacency as follows:

- $\tilde{G}'[C \cup D \cup E \cup F] = G[C \cup D \cup E \cup F]$;
- $a$ is complete to $C \cup E$ and anti-complete to $D \cup F$;
- $b$ and $b'$ are complete to $D \cup E$ and anti-complete to $C \cup F$;
- $a$ is adjacent to $b$ and non-adjacent to $b'$;
- $b$ is non-adjacent to $b'$.

Next, we assign weights to the vertices of $\tilde{G}'$ as follows. The weights of the vertices in $C \cup D \cup E \cup F$ in the graph $\tilde{G}'$ are inherited from $G$, and for the vertices $b, b', d$, we set:

- $w_{\tilde{G}'}(a) = W(G[A])$;
- $w_{\tilde{G}'}(b) = W(G[A \cup B]) - W(G[A])$;
- $w_{\tilde{G}'}(b') = W(G[B])$.

We observe that all vertices in $C \cup D \cup E \cup F \cup \{a, b'\}$ have positive integer weight in $\tilde{G}'$, and that $b$ has non-negative integer weight. Furthermore, we note that:

- $w_{\tilde{G}'}(a) = W(G[A])$;
- $w_{\tilde{G}'}(a) + w_{\tilde{G}'}(b) = W(G[A \cup B])$;
- $w_{\tilde{G}'}(b') = W([G[B])$.

Now, if $w_{\tilde{G}}(b) \neq 0$ then set $\tilde{G} = \tilde{G}'$, and if $w_{\tilde{G}}(b) = 0$ then set $\tilde{G} = \tilde{G}' \smallsetminus b$; clearly, every vertex of $\tilde{G}$ has positive integer weight. We refer to the weighted graph $\tilde{G}$ as the *type* n *weighted reduction of* $G$ *with respect to* $(A, B)$.

We observe that if $G$ is a bull-free perfect graph, and $(A, B)$ is a reducible homogeneous pair in $G$, then the type a weighted reduction and the type n weighted reduction of $G$ with respect to $(A, B)$ are not necessarily bull-free and perfect. We do, however, have the following result, which will suffice for the purposes of our algorithm.

**5.3.2.** *Let $G$ be a weighted bull-free perfect graph that does not contain a proper homogeneous set. Assume that applying the algorithm REDUCIBLE to $G$ yields a 7-tuple $(A, B, C, D, E, F, z)$. Then $(A, B)$ is a reducible homogeneous pair in $G$, and the type $z$ weighted reduction of $G$ with respect to $(A, B)$ is a weighted bull-free perfect graph.*

*Proof.* If $z = \mathrm{a}$, then there exists a frame $(b, b', d)$ for $(A, B)$, with $b$ adjacent to $b'$; and if $z = \mathrm{n}$, then there exists a frame $(b, b', d)$ for $(A, B)$, with $b$ non-adjacent to $b'$. In either case, 5.2.3 implies that, as an unweighted graph, the type $z$ weighted reduction of $G$ with respect to $(A, B)$ is isomorphic to an induced subgraph of $G$, and the result follows. $\square$

We complete this section by describing how to "recover" a maximum weighted clique in a weighted graph $G$ that contains a reducible homogeneous pair $(A, B)$ from maximum weighted cliques in the weighted graphs $\tilde{G}$, $G[A]$, $G[B]$, and $G[A \cup B]$, where $\tilde{G}$ is the type a or type n weighted reduction of the graph $G$ with respect to $(A, B)$.

**5.3.3.** *Let $G$ be a weighted graph, and let $(A, B)$ be a reducible homogeneous pair in $G$. Let $\tilde{G}$, $\tilde{G}'$, $a$, $b$, and $b'$ be as in the definition of the type* a *or type* n *weighted reduction of $G$ with respect to $(A, B)$. Let $\tilde{K}$, $K_A$, $K_B$, and $K_{A \cup B}$ be maximum weighted cliques in $\tilde{G}$, $G[A]$, $G[B]$, and $G[A \cup B]$, respectively. Then exactly one of the following holds:*

- $a, b, b' \notin \tilde{K}$;
- $a \in \tilde{K}$ *and* $b, b' \notin \tilde{K}$;
- $\tilde{K}$ *intersects* $\{b, b'\}$, *and* $a \notin \tilde{K}$;
- $a, b \in \tilde{K}$ *and* $b' \notin \tilde{K}$.

*Now, define the set $K$ as follows:*

- *if* $a, b, b' \notin \tilde{K}$*, then set* $K = \tilde{K}$*;*
- *if* $a \in \tilde{K}$ *and* $b, b' \notin \tilde{K}$*, then set* $K = (\tilde{K} \setminus \{a\}) \cup K_A$*;*
- *if* $\tilde{K}$ *intersects* $\{b, b'\}$ *and* $a \notin \tilde{K}$*, then set* $K = (\tilde{K} \setminus \{b, b'\}) \cup K_B$*;*
- *if* $a, b \in \tilde{K}$ *and* $b' \notin \tilde{K}$*, then set* $K = (\tilde{K} \setminus \{a, b\}) \cup K_{A \cup B}$*.*

*Then* $K$ *is a maximum weighted clique in* $G$*.*

*Proof.* Let $(A, B, C, D, E, F)$ be the partition of $G$ associated with the homogeneous pair $(A, B)$. We note that the first claim follows from the fact that $a$ is non-adjacent to $b'$ in $\tilde{G}'$; this also implies that the set $K$ is well-defined.

Now, we claim that $K$ is a clique in $G$, and that $w_G(K) = w_{\tilde{G}}(\tilde{K})$.

Suppose first that $a, b, b' \notin \tilde{K}$. Then $K = \tilde{K}$, and by the definition of $\tilde{G}$, we have that $G[K] = \tilde{G}[\tilde{K}]$. This implies that $K$ is a clique in $G$ and that $w_G(K) = w_{\tilde{G}}(\tilde{K})$.

Suppose next that $a \in \tilde{K}$ and $b, b' \notin \tilde{K}$, so that $K = (\tilde{K} \setminus \{a\}) \cup K_A$. Then $G[K]$ is obtained by substituting the complete graph $G[K_A]$ for the vertex $a$ in the complete graph $\tilde{G}[\tilde{K}]$, and so $G[K]$ is a clique; the fact that $w_G(K) = w_{\tilde{G}}(\tilde{K})$ follows from the fact that $w_{\tilde{G}}(a) = W(G[A]) = w_G(K_A)$.

Suppose now that $\tilde{K}$ intersects $\{b, b'\}$ and $a \notin \tilde{K}$, so that $K = (\tilde{K} \setminus \{b, b'\}) \cup K_B$. Since $C \cup F$ is anti-complete to $\{b, b'\}$ in $\tilde{G}'$ and $\tilde{K}$ is a clique in $\tilde{G}$ (and therefore in $\tilde{G}'$ as well), we know that $\tilde{K} \subseteq \{b, b'\} \cup D \cup E$; but now since $D \cup E$ is complete to $B$ (and therefore to $K_B$ as well) in $G$, and $K_B$ is a clique in $G$, it follows that $K$ is a clique in $G$. It remains to show that $w_G(K) = w_{\tilde{G}}(\tilde{K})$; as $w_G(K_B) = W(G[B])$, it suffices to show that $\Sigma_{v \in \tilde{K} \cap \{b, b'\}} w_{\tilde{G}}(v) = W(G[B])$. By construction, no clique in $\tilde{G}'[b, b']$ is of weight greater than $W(G[B])$, and so $\Sigma_{v \in \tilde{K} \cap \{b, b'\}} w_{\tilde{G}}(v) \leq W(G[B])$. On the other hand, by construction,

$\tilde{G}[V_{\tilde{G}} \cap \{b, b'\}]$ contains a clique $\tilde{B}$ of weight $W(G[B])$; since $\tilde{K} \subseteq \{b, b'\} \cup D \cup E$, and $D \cup E$ is complete to $\tilde{B}$ in $\tilde{G}$, we know that $(\tilde{K} \smallsetminus \{b, b'\}) \cup \tilde{B}$ is a clique in $\tilde{G}$. Since $\tilde{K}$ is of maximum weight in $\tilde{G}$, it follows that $\Sigma_{v \in \tilde{K} \cap \{b,b'\}} w_{\tilde{G}}(v) \geq w_{\tilde{G}}(\tilde{B})$. Since $w_{\tilde{G}}(\tilde{B}) = W(G[B])$, this implies that $\Sigma_{v \in \tilde{K} \cap \{b,b'\}} w_{\tilde{G}}(v) = W(G[B])$. Thus, $w_G(K) = w_{\tilde{G}}(\tilde{K})$.

Finally, suppose that $a, b \in \tilde{K}$ and $b' \notin \tilde{K}$, so that $K = (\tilde{K} \smallsetminus \{a, b\}) \cup K_{A \cup B}$. Since $\tilde{K}$ is a clique in $\tilde{G}$ with $a, b \in \tilde{G}$, and since $a$ and $b$ are anti-complete to $C \cup F$ and $D \cup F$, respectively, in $\tilde{G}$, we get that $\tilde{K} \subseteq \{a, b\} \cup E$. But since $E$ is complete to $A \cup B$ (and therefore to $K_{A \cup B}$ as well) in $G$, and since $K_{A \cup B}$ is a clique in $G$, it easily follows that $K$ is a clique in $G$. The fact that $w_G(K) = w_{\tilde{G}}(\tilde{K})$ follows from the fact that $w_{\tilde{G}}(a) + w_{\tilde{G}}(b) = W(G[A \cup B]) = w_G(K_{A \cup B})$.

It remains to show that the clique $K$ is of maximum weight in $G$. Fix some maximum weighted clique $K'$ in $G$; we need to show that $w_G(K') \leq w_G(K)$.

Suppose first that $K' \cap (A \cap B) = \emptyset$. Then $K'$ is a clique in $\tilde{G}$, and so by the maximality of $\tilde{K}$, we have that $w_{\tilde{G}}(K') \leq w_{\tilde{G}}(\tilde{K}) = w_G(K)$, which is what we needed to show.

Suppose next that $K' \cap A \neq \emptyset$ and $K' \cap B = \emptyset$. Since $K'$ is a clique and $D \cup F$ is anti-complete to $A$ in $G$, we have that $K' \subseteq A \cup C \cup E$. Since $C \cup E$ is complete to $A$ in $G$, and since $K_A \subseteq A$, we know that $(K' \smallsetminus A) \cup K_A$ is a clique in $G$ and that $(K' \smallsetminus A) \cup \{a\}$

is a clique in $\tilde{G}$. Now by the maximality of $K_A$ and $\tilde{K}$, we get the following:

$$\begin{aligned} w_G(K') &= w_G(K' \smallsetminus A) + w_G(K' \cap A) \\ &\leq w_G(K' \smallsetminus A) + w_G(K_A) \\ &= w_{\tilde{G}}(K' \smallsetminus A) + w_{\tilde{G}}(a) \\ &= w_{\tilde{G}}((K' \smallsetminus A) \cup \{a\}) \\ &\leq w_{\tilde{G}}(\tilde{K}) \\ &= w_G(K). \end{aligned}$$

Next, suppose that $K' \cap B \neq \emptyset$ and $K' \cap A = \emptyset$. Since $K'$ is a clique and $C \cup F$ is anti-complete to $B$, we have that $K' \subseteq B \cup D \cup E$. Since $D \cup E$ is complete to $B$ in $G$, and since $K_B \subseteq B$, we know that $(K' \smallsetminus B) \cup K_B$ is a clique in $G$. By the construction of $\tilde{G}$, $\tilde{G}[V_{\tilde{G}} \cap \{b, b'\}]$ contains a clique $\tilde{B}$ of weight $W(G[B]) = w_G(K_B)$; clearly, $\tilde{B}$ is complete to $D \cup E$ in $\tilde{G}$, and so it easily follows that $(K' \smallsetminus B) \cup \tilde{B}$ is a clique in $\tilde{G}$. Now, by the maximality of $K_B$ and $\tilde{K}$, we have the following:

$$\begin{aligned} w_G(K') &= w_G(K' \smallsetminus B) + w_G(K' \cap B) \\ &\leq w_G(K' \smallsetminus B) + w_G(K_B) \\ &= w_{\tilde{G}}(K' \smallsetminus B) + w_{\tilde{G}}(\tilde{B}) \\ &= w_{\tilde{G}}((K' \smallsetminus B) \cup \tilde{B}) \\ &\leq w_{\tilde{G}}(\tilde{K}) \\ &= w_G(K). \end{aligned}$$

Suppose, finally, that $K'$ intersects both $A$ and $B$. As $C \cup F$ is anti-complete to $B$, and $D \cup F$ is anti-complete to $A$, we have that $K' \subseteq A \cup B \cup E$. As $E$ is complete to $A \cup B$ (and therefore to $K_{A \cup B}$ as well), we know that $(K' \smallsetminus (A \cup B)) \cup K_{A \cup B}$ is a clique in $G$. Now, set $H = V_{\tilde{G}} \cap \{a, b\}$; clearly, $H$ is a clique complete to $E$ in $\tilde{G}$, and so $(K' \smallsetminus (A \cup B)) \cup H$ is a clique in $\tilde{G}$. Furthermore, note that $w_{\tilde{G}}(H) = W(G[A \cup B]) = w_G(K_{A \cup B})$. Now, by the maximality of $K_{A\cup B}$ and $\tilde{K}$, we have the following:

$$\begin{aligned} w_G(K') &= w_G(K' \smallsetminus (A \cup B)) + w_G(K' \cap (A \cup B)) \\ &\leq w_G(K' \smallsetminus (A \cup B)) + w_G(K_{A\cup B}) \\ &= w_{\tilde{G}}(K' \smallsetminus (A \cup B)) + w_{\tilde{G}}(H) \\ &= w_{\tilde{G}}((K' \smallsetminus (A \cup B)) \cup H) \\ &\leq w_{\tilde{G}}(\tilde{K}) \\ &= w_G(K). \end{aligned}$$

This completes the argument. □

## 5.4 The Algorithm

In this section, we describe the algorithm MWCLIQUE whose input is a weighted bull-free perfect graph $G$, and whose output is a maximum weighted clique in $G$. We begin by discussing some previously known algorithms that we use in our algorithm MWCLIQUE.

First, given a graph $G$ on $n$ vertices, one can use the algorithm from [38] or the algorithm from [59] to check whether $G$ is transitively orientable, and if so, to find a transitive orientation for $G$; this takes at most $O(n^3)$ time. Next, given a weighted transitive directed graph $G$ on $n$ vertices, one can use the algorithm from [43] to find a maximum weighted clique in $G$ in at most $O(n^3)$ time, and one can use the algorithm from [5] to find a maximum weighted stable set in $G$ (which is a maximum weighted clique in $\overline{G}$) in at most $O(n^4)$ time. (In fact, in [5], the problem of finding a maximum weighted stable set in a weighted transitive directed graph is reduced to finding a maximum weighted stable set in a weighted bipartite graph. The latter can be done using network flows, as explained, for example, in section 2 of [32].) All of this implies that, given a weighted graph $G$ on $n$ vertices, one can determine whether at least one of $G$ and $\overline{G}$ is transitively orientable in at most $O(n^3)$ time, and if so, one can find a maximum weighted clique in $G$ in at most

$O(n^4)$ time.

Second, given a graph $G$ on $n$ vertices, one can use the algorithm from [28] or the algorithm from [29] to check whether $G$ has a proper homogeneous set, and if so, to find a proper homogeneous set in $G$. This takes at most $O(n^2)$ time.

We now turn to describing the algorithm MWCLIQUE. As stated at the beginning of this section, the input is a weighted bull-free perfect graph $G$, and the output is a maximum weighted clique in $G$. Along with the algorithm, we construct a rooted decomposition tree $T_G$ associated with $G$. The vertices of $T_G$ are the graphs constructed by the algorithm, and the root of $T_G$ is the graph $G$. In this thesis, a *leaf* of a rooted tree is a vertex of the tree that has no descendants. (In particular, every non-root vertex of degree one in a rooted tree is a leaf, and the root is a leaf if and only if the tree consists of the root only.)

Suppose that $G$ is a weighed bull-free perfect graph, and set $n = |V_G|$. By 5.1.1, at least one of the following holds:

- $G$ or $\overline{G}$ is transitively orientable;
- $G$ contains a proper homogeneous set;
- $G$ contains a reducible homogeneous pair.

The first step is to check whether at least one of $G$ and $\overline{G}$ is transitively orientable, and if so, to find a maximum weighted clique in $G$; as explained above, this takes at most $O(n^4)$ time. In this case, $G$ is a leaf of the rooted tree $T_G$.

From now on, we assume that neither $G$ nor $\overline{G}$ is transitively orientable. We then check whether $G$ contains a proper homogeneous set, and if so, we find a proper homogeneous set in $G$; as explained above, this takes at most $O(n^2)$ time. If the algorithm returns a proper homogeneous set $S$, then we call the algorithm MWCLIQUE on the graphs $G[S]$

and $\tilde{G}$, where $\tilde{G}$ is the weighted reduction of $G$ with respect to $S$, as defined in section 5.3. Once we have obtained maximum weighted cliques for $G[S]$ and $\tilde{G}$, we can find a maximum weighted clique in $G$ as outlined in 5.3.1. In this case, $G$ has two children in the tree $T_G$, namely $G[S]$ and $\tilde{G}$.

From now on, we assume that $G$ does not contain a proper homogeneous set. Then $G$ contains a reducible homogeneous pair. We now call the algorithm REDUCIBLE from section 5.2 on the graph $G$, and obtain a 7-tuple $(A, B, C, D, E, F, z)$, where $(A, B)$ is a reducible homogeneous pair in $G$, $(A, B, C, D, E, F)$ is the partition of $G$ associated with $(A, B)$, and $z \in \{\mathrm{a}, \mathrm{n}\}$; this takes at most $O(n^5)$ time. By 5.1.2, $G[A]$ and $G[B]$ are both transitively orientable, and so (as explained above) we can find a maximum weighted clique in each of them in at most $O(n^3)$ time. We then call the algorithm MWCLIQUE on the graphs $G[A \cup B]$ and $\tilde{G}$, where $\tilde{G}$ is the type $z$ weighted reduction of $G$ with respect to $(A, B)$, as defined in section 5.3; we note that the graph $\tilde{G}$ is bull-free and perfect by 5.3.2, and since $|A \cup B| \geq 4$, we know that $\tilde{G}$ has fewer vertices than $G$. Once we have obtained maximum weighted cliques for each of $G[A]$, $G[B]$, $G[A \cup B]$, and $\tilde{G}$, we can find a maximum weighted clique in $G$ as outlined in 5.3.3. In this case, $G$ has two children in the tree $T_G$, namely $G[A \cup B]$ and $\tilde{G}$.

## 5.5 Complexity Analysis

Our goal in this section is to prove the following result.

**5.5.1.** *The running time of the algorithm MWCLIQUE is at most $O(n^6)$, where $n$ is the number of vertices of the input graph.*

We observe that each step of the algorithm MWCLIQUE can be performed in at most $O(n^5)$ time, and so in order to prove 5.5.1, it suffices to show that for each weighted bull-free perfect graph $G$, the number of vertices in the decomposition tree $T_G$ is bounded by a linear function of the number of vertices of $G$. We begin with a technical lemma (5.5.2), and then we use this lemma to prove 5.5.3, which states that the number of vertices in the decomposition tree $T_G$ of a weighted bull-free perfect graph $G$ is at most $3|V_G|$. The main result of this section (5.5.1) then follows immediately.

**5.5.2.** *Let $T$ be a rooted tree with root $r$. Let $f : V_T \to \mathbb{N}$ be a function such that for all vertices $v \in V_T$ that are not leaves of $T$, if $v_1, ..., v_k \in V_T$ are the children of $v$, then $\Sigma_{i=1}^k f(v_k) < f(v)$. Then $|V_T| \leq f(r)$.*

*Proof.* We proceed by induction on $|V_T|$. If $|V_T| = 1$, then the result is immediate as $f(r)$ is a positive integer. So assume that $T$ has at least two vertices, and that the claim holds for rooted trees with fewer vertices. Let $r_1, ..., r_k$ be the children of the root $r$ in the tree $T$. For each $i \in \{1, ..., k\}$, let $T_i$ be the component of $T \smallsetminus r$ that contains $r_i$; we turn $T_i$ into a rooted tree by letting $r_i$ be the root of $T_i$. By the induction hypothesis, $|V_{T_i}| \leq f(r_i)$ for each $i \in \{1, ..., k\}$. But then note the following:

$$\begin{array}{rcl} |V_T| & = & 1 + \Sigma_{i=1}^k |V_{T_i}| \\ & \leq & 1 + \Sigma_{i=1}^k f(r_i) \\ & < & 1 + f(r). \end{array}$$

Since $|V_T|$ and $f(r)$ are both integers, it follows that $|V_T| \leq f(r)$, as we had claimed. □

**5.5.3.** *Let $G$ be a weighted bull-free perfect graph, and let $n = |V_G|$. Then the number of vertices in the decomposition tree $T_G$ is at most $3n$.*

*Proof.* If $G$ or $\overline{G}$ is transitively orientable, then the tree $T_G$ has only one vertex (namely, the root $G$), and the result is immediate. So assume that $G$ and $\overline{G}$ are not transitively orientable. It is easy to check that every graph on at most four vertices is transitively orientable; consequently, $n \geq 5$. Furthermore, $G$ has exactly two children in the tree $T_G$, and so in particular, the root $G$ is not a leaf of $T_G$. Now, let $T'_G$ be the graph obtained from $T_G$ by deleting all the leaves of $T_G$. As every vertex of $T_G$ has at most two children, it follows that $|V_{T_G}| \leq 3|V_{T'_G}|$. Thus, in order to show that $|V_{T_G}| \leq 3n$, we just have to show that $|V_{T'_G}| \leq n$. Note that no vertex $H$ of $T'_G$ is transitively orientable, for otherwise, $H$ would be a leaf of $T_G$, contrary to the fact that $T'_G$ contains no leaves of $T_G$; since every graph on at most four vertices is transitively orientable, it follows that every vertex of $T'_G$ has at least five vertices. Now, to each vertex $H$ of $T'_G$, we associate the number $f(H) = |V_H| - 4$; since every vertex in the tree $T'_G$ has at least five vertices, it follows that $f(H)$ is a positive integer for every vertex $H$ of $T'_G$.

Suppose that $H$ is not a leaf of $T'_G$, and let $H_1, ..., H_k$ be the children of $H$ in $T'_G$; we claim that $\Sigma_{i=1}^{k} f(H_i) < f(H)$. By construction, every child of $H$ has fewer vertices than $H$, and so if $k = 1$, the result is immediate. So assume that $k \geq 2$; as every vertex in $T_G$ that is not a leaf has exactly two children, it follows that $k = 2$, and we need to show that $f(H_1) + f(H_2) < f(H)$. If $H$ contains a proper homogeneous set, then we may assume that $|V_{H_1}| = p$, $|V_{H_2}| = q + 1$, and $|V_H| = p + q$. If $H$ does not contain a proper homogeneous set, then $H$ contains a reducible homogeneous pair, and we may assume that $|V_{H_1}| = p$, $|V_{H_2}| = q + 2$ or $|V_{H_2}| = q + 3$, and $|V_H| = p + q$. In any case, we may assume that $|V_{H_1}| = p$, $|V_{H_2}| = q + r$, and $|V_H| = p + q$, for some positive integers $p$ and $q$, and some $r \in \{1, 2, 3\}$. But then we have the following:

$$\begin{array}{rcl} f(H_1) + f(H_2) & = & (p - 4) + (q + r - 4) \\ & = & p + q + r - 8 \\ & < & p + q - 4 \\ & = & f(H) \end{array}$$

But now 5.5.2 implies that $T'_G$ has at most $f(G) = n - 4$ vertices, which completes the argument. $\square$

We now restate and prove the main result of the section.

**5.5.1.** *The running time of the algorithm MWCLIQUE is at most $O(n^6)$, where $n$ is the number of vertices of the input graph.*

*Proof.* Each step of the algorithm takes at most $O(n^5)$ time, and by 5.5.3, we make at most $O(n)$ calls to the algorithm. The result is then immediate. □

It is natural to ask why the algorithm MWCLIQUE is faster than the algorithm from [33] due to de Figueiredo and Maffray. One reason for this is that the weighted reductions for our homogeneous pairs (see section 5.3) use fewer new vertices than the reductions that de Figueiredo and Maffray use for their homogeneous pairs. As a result, we make only $O(n)$ recursive calls to the algorithm, whereas the algorithm from [33] makes $O(nm^2)$ calls, where $n$ is the number of vertices and $m$ is the number of edges of the input graph. The second reason that our algorithm is faster is that the decomposition theorem that the algorithm MWCLIQUE is based on is different from the one that the algorithm from [33] is based on, and so the slowest step in the algorithm MWCLIQUE takes $O(n^5)$ time, while the slowest step in the algorithm from [33] takes $O(n^4m)$ time.

# Chapter 6

# Excluding Induced Subdivisions of the Bull and Related Graphs

Recall from chapter 2 (section 2.1) that a class $\mathcal{G}$ is *hereditary* if it is closed under isomorphism and induced subgraphs, and that a hereditary class $\mathcal{G}$ is $\chi$-*bounded* if there exists a non-decreasing function $f : \mathbb{N}_0 \to \mathbb{R}$ such that for all $G \in \mathcal{G}$, $\chi(G) \leq f(\omega(G))$. We note that the definition of a $\chi$-bounded class given in the Introduction and in chapter 2 is slightly different from the one given in the present chapter; however, as discussed in section 2.1, the two definitions are equivalent in the case of hereditary classes, and in the present chapter, we are only interested in hereditary classes.

Given a graph $H$, we denote by $H^*$ any graph that is a subdivision of $H$ (in particular, the graph $H$ itself if an $H^*$). A graph $G$ is said to be $H^*$-*free* provided that $G$ does not contain any subdivision of $H$ as an induced subgraph. We denote by $\text{Forb}^*(H)$ the class of all $H^*$-free graphs; clearly $\text{Forb}^*(H)$ is hereditary for all graphs $H$. As discussed in the Introduction (section 1.2), Scott's conjecture [57] states that for every graph $H$, the class $\text{Forb}^*(H)$ is $\chi$-bounded. In general, Scott's conjecture is false: a group of authors recently constructed a counterexample [51]. However, this raises the following question: for which graphs $H$ is Scott's conjecture true? In this chapter, we prove Scott's conjecture for several particular graphs $H$.

We remind the reader that the *paw* is the graph with vertex-set $\{x_1, x_2, x_3, y\}$ and edge-set $\{x_1x_2, x_2x_3, x_3x_1, x_1y\}$. In section 6.1, we give a structural description of the class Forb$^*$(paw), which we then use to compute the best possible $\chi$-bounding function for the class (see 6.1.2). As explained in the Introduction (see section 1.2), together with previously known results, this theorem implies that the class Forb$^*(H)$ is $\chi$-bounded for all graphs $H$ on at most four vertices.

In section 6.2, we prove a decomposition theorem for bull$^*$-free graphs (see 6.2.1). In section 6.3, we use this theorem to prove that the class Forb$^*$(bull) is $\chi$-bounded by the function $f(n) = n^2$ (see 6.3.2). We note that this is the best possible polynomial $\chi$-bounding function for Forb$^*$(bull) in the following sense: there do not exist positive constants $c, r \in \mathbb{R}$, with $r < 2$, such that Forb$^*$(bull) is $\chi$-bounded by the function $f(n) = cn^r$. As Forb$^*$(bull) contains all graphs with no stable set of size three, this follows immediately from a result of Kim [46] that the Ramsey number $R(t, 3)$ has order of magnitude $\frac{t^2}{\log t}$ (in fact, it is enough that $R(t, 3) = t^{2-o(1)}$, which also follows from an earlier result of Erdős [30]).

Finally, in section 6.4, we consider graphs that we call “necklaces.” A necklace is a graph obtained from a path by choosing a matching such that no edge of the matching is incident with an endpoint of the path, and for each edge of the matching, adding a vertex adjacent to the ends of this edge (see section 6.4 for a more formal definition). We prove that for any given necklace $N$, the class Forb$^*(N)$ is $\chi$-bounded by an exponential function (see 6.4.2). We observe that the bull is a special case of a necklace, and so the results of section 6.4 imply that Forb$^*$(bull) is $\chi$-bounded; however, the $\chi$-bounding function for Forb$^*$(bull) from 6.3.2 is polynomial, whereas the one from 6.4.2 is exponential. Further, we note that for all positive integers $m$, the $m$-edge path, denoted by $P_{m+1}$, is a necklace; furthermore, since any subdivision of an $m$-edge path contains an $m$-edge path as an induced subgraph, we know that $\text{Forb}(P_{m+1}) = \text{Forb}^*(P_{m+1})$. Thus, 6.4.2 implies a result of Gyárfás (see [41]) that the class $\text{Forb}(P_{m+1})$ is $\chi$-bounded by an exponential function (we note, however, that our $\chi$-bounding function is faster growing than that of Gyárfás).

## 6.1 Subdivisions of the Paw

In this section, we give a structure theorem for paw$^*$-free graphs (6.1.1), and then use it to derive the fact that $\text{Forb}^*(\text{paw})$ is $\chi$-bounded by a linear function (6.1.2). We first need a definition: a graph is said to be *complete multipartite* if its vertex-set can be partitioned into stable sets, pairwise complete to each other.

**6.1.1.** *A graph $G$ is paw$^*$-free if and only if each of its components is either a tree, a chordless cycle, or a complete multipartite graph.*

*Proof.* The 'if' part is established by routine checking. For the 'only if' part, suppose that $G$ is a connected paw$^*$-free graph. Our goal is to show that if $G$ is both triangle-free and square-free, then $G$ is either a tree or a chordless cycle, and otherwise $G$ is a complete multipartite graph.

Suppose first that $G$ is both triangle-free and square-free. If $G$ contains no cycles, then it is a tree, and we are done. So assume that $G$ does contain a cycle, and let $v_0-v_1-...-v_{k-1}-v_0$ (with the indices in $\mathbb{Z}_k$) be a cycle in $G$ of length as small as possible; note that the minimality of $k$ implies that this cycle is induced, and the fact that $G$ is triangle-free and square-free implies that $k \geq 5$. If $V_G = \{v_0, v_1, ..., v_{k-1}\}$, then $G$ is a chordless cycle, and we are done. So assume that $\{v_0, ..., v_{k-1}\} \subsetneqq V_G$. Since $G$ is connected, there exists a vertex $v \in V_G \smallsetminus \{v_0, ..., v_{k-1}\}$ that has a neighbor in $\{v_0, ..., v_{k-1}\}$. Note that $v$ must have at least two neighbors in $\{v_0, v_1, ..., v_{k-1}\}$, for otherwise, $G[v, v_0, v_1, ..., v_{k-1}]$ would be a paw*. By symmetry, we may assume that for some $i \in \mathbb{Z}_k \setminus \{0\}$, $v$ is complete to $\{v_0, v_i\}$ and anti-complete to $\{v_1, ..., v_{i-1}\}$ in $G$. By the minimality of $k$, the cycle $v - v_0 - v_1 - ... - v_i - v$ is of length at least $k$, and so it follows that either $i = k-2$ or $i = k-1$. But then $v - v_i - v_{i+1} - ... - v_0 - v$ is a (not necessarily induced) cycle of length at most four in $G$, which contradicts the fact that $G$ is triangle-free and square-free.

It remains to consider the case when $G$ contains a triangle or a square. Let $H$ be an inclusion-wise maximal complete multipartite induced subgraph of $G$ such that $H$ contains a cycle. (The existence of such a graph $H$ follows from the fact that a triangle or a square is itself a complete multipartite graph that contains a cycle.) If $G = H$, then $G$ is complete multipartite, and we are done. So assume that this is not the case. Since $G$ is connected, there exists a vertex $v \in V_G \setminus V_H$ with a neighbor in $V_H$.

Let $H_1, H_2, ..., H_k$ be a partition of $V_H$ into stable sets, pairwise complete to each other. First, we claim that $v$ is not mixed on any set among $H_1, ..., H_k$. Suppose otherwise. By symmetry, we may assume that $v$ is adjacent to some $h_1 \in H_1$ and non-adjacent to some $h_1' \in H_1$. Then $v$ is anti-complete to $H_2 \cup ... \cup H_k$, for if $v$ had a neighbor $h \in H_2 \cup ... \cup H_k$, then $G[v, h, h_1, h_1']$ would be a paw. Now, since $H$ contains a cycle, we know that $|H_2 \cup ... \cup H_k| \geq 2$; fix distinct vertices $h, h' \in H_2 \cup ... \cup H_k$. But if $hh'$ is an edge then $G[h, h', h_1, v]$ is a paw, and if $hh'$ is a non-edge then $G[h, h', h_1, h_1', v]$ is a paw*. This proves our claim. Now $v$ is anti-complete to at least two sets among $H_1, ..., H_k$ (say $H_1$ and $H_2$), for otherwise, $G[V_H \cup \{v\}]$ would contradict the maximality of $H$. Let $h \in H_3 \cup ... \cup H_k$ be some neighbor of $v$, and fix $h_1 \in H_1$ and $h_2 \in H_2$. Then $G[h_1, h_2, h, v]$ is a paw, which is a contradiction. This completes the argument. $\square$

We note that our structure theorem for paw*-free graphs (6.1.1) is similar to the structure theorem for paw-free graphs (due to Olariu [50]), which states that a graph $G$ is paw-free if and only if every component of $G$ is either triangle-free or complete multipartite. In fact, our proof of 6.1.1 could be slightly shortened by using [50], but in order to keep the section self-contained, we include an independent proof. We now turn to proving that the class Forb*(paw) is $\chi$-bounded by a linear function.

**6.1.2.** Forb*(paw) *is $\chi$-bounded by the function $f : \mathbb{N}_0 \to \mathbb{R}$ defined by $f(2) = 3$ and for all $n \neq 2$, $f(n) = n$.*

*Proof.* Let $G \in$ Forb*(paw). We may assume that $G$ is connected (otherwise, we consider the components of $G$ separately). By 6.1.1 then, $G$ is either a tree, or a chordless cycle, or a complete multipartite graph, and in each of these cases, we have that $\chi(G) = 3$ or $\chi(G) = \omega(G)$. □

It is easy to see that the $\chi$-bounding function given in 6.1.2 is the best possible for the class Forb*(paw). Indeed, on the one hand, we have that $\omega(G) \leq \chi(G)$ for every graph $G$, and on the other hand, there exist paw*-free graphs with clique number 2 and chromatic number 3 (any chordless cycle of odd length greater than three is such a graph.)

## 6.2 Decomposing Bull*-Free Graphs

In this section, we prove a decomposition theorem for bull*-free graphs. We begin with a couple of definitions. We call a graph $G$ *basic* if it contains neither an odd hole with an anti-center nor an odd anti-hole with an anti-center. (More precisely: a graph $G$ is *basic* provided that there do not exist an induced subgraph $H$ of $G$ and a vertex $a \in V_G \smallsetminus V_H$ such that $H$ is an odd hole or an odd anti-hole of $G$, and $a$ is an anti-center for $H$.) Given a graph $G$, we say that a vertex $v \in V_G$ is a *cut-vertex* of $G$ provided that $G \smallsetminus v$ has more components than $G$. Our goal in this section is to prove the following decomposition theorem.

**6.2.1.** *Let $G \in$ Forb*(bull). Then either $G$ is basic, or it contains a proper homogeneous set or a cut-vertex.*

We will need the following result, which is an immediate consequence of 1.4 from [17].

**6.2.2** (Chudnovsky and Safra [17])**.** *Let $G \in \text{Forb}^*(\text{bull})$. If $G$ contains an odd hole with a center and an anti-center, or an odd anti-hole with a center and an anti-center, then $G$ has a proper homogeneous set.*

The proof of 6.2.1 proceeds as follows. We assume that a graph $G \in \text{Forb}^*(\text{bull})$ is not basic, and then we consider two cases: when $G$ contains an odd anti-hole of length at least seven with an anti-center; and when $G$ contains an odd hole with an anti-center. In the former case, we show that $G$ contains a proper homogeneous set (see 6.2.3 below). The latter case is more difficult, and our approach is to prove a series of lemmas that describe how vertices that lie outside of our odd hole "attach" to this odd hole and to each other, and then to use these results to prove that $G$ contains a proper homogeneous set or a cut-vertex (see 6.2.8). Since an anti-hole of length five is also a hole of length five, these two results (6.2.3 and 6.2.8) imply 6.2.1.

**6.2.3.** *Let $G \in \text{Forb}^*(\text{bull})$, let $h_0 - h_1 - ... - h_{k-1} - h_0$ (with $k \geq 7$ and the indices in $\mathbb{Z}_k$) be an odd anti-hole in $G$, and set $H = \{h_0, h_1, ..., h_{k-1}\}$. Assume that $G$ contains an anti-center for $H$. Then $G$ contains a proper homogeneous set.*

*Proof.* We may assume that $G$ is connected, for otherwise, $G$ contains a proper homogeneous set and we are done. Since $G$ is connected and contains an anti-center for $H$, there exist adjacent $a, a' \in V_G \smallsetminus H$ such that $a$ is anti-center for $H$ and $a'$ has a neighbor in $H$. Our goal is to show that $a'$ is a center for $H$, for then we are done by 6.2.2.

First, we claim that there is no index $i \in \mathbb{Z}_k$ such that $a'$ is anti-complete to $\{h_i, h_{i+1}\}$. Suppose otherwise. Since $a'$ has a neighbor in $H$, we may assume by symmetry that $a'$ is adjacent to $h_0$ and anti-complete to $\{h_1, h_2\}$. But then if $a'h_4$ is an edge, then $G[h_0, h_1, h_4, a, a']$ is a bull; and if $a'h_4$ is a non-edge, then $G[h_0, h_1, h_2, h_4, a']$ is a bull. This proves our claim.

Next, since $H$ has an odd number of vertices, there exists some $i \in \mathbb{Z}_k$ such that $a'$

is either complete or anti-complete to $\{h_i, h_{i+1}\}$; by what we just showed, the latter is impossible, and so the former must hold. Now, if $a'$ is not a center for $H$, then we may assume by symmetry that $a'$ is non-adjacent to $h_0$ and complete to $\{h_1, h_2\}$; but then $a'h_{k-1}$ is an edge (because $a'$ is not anti-complete to $\{h_{k-1}, h_0\}$), and so $G[h_0, h_2, h_{k-1}, a, a']$ is a bull. Thus, $a'$ is a center for $H$, which completes the argument. □

For the remainder of this section, we focus on graphs in Forb*(bull) that contain an odd-hole with an anti-center. We begin with some definitions. Let $G$ be a graph, let $h_0 - h_1 - ... - h_{k-1} - h_0$ (with $k \geq 5$ and the indices in $\mathbb{Z}_k$) be a hole in $G$, let $H = \{h_0, h_1, ..., h_{k-1}\}$, and let $v \in V_G \smallsetminus H$. Then for all $i \in \mathbb{Z}_k$:

- $v$ is a *leaf for $H$ at $h_i$* if $v$ is adjacent to $h_i$ and anti-complete to $H \smallsetminus \{h_i\}$;
- $v$ is a *star for $H$ at $h_i$* if $v$ is complete to $H \smallsetminus \{h_i\}$ and non-adjacent to $h_i$;
- $v$ is an *adjacent clone for $H$ at $h_i$* if $v$ is complete to $\{h_{i-1}, h_i, h_{i+1}\}$ and anti-complete to $H \smallsetminus \{h_{i-1}, h_i, h_{i+1}\}$;
- $v$ is a *non-adjacent clone for $H$ at $h_i$* if $v$ is complete to $\{h_{i-1}, h_{i+1}\}$ and anti-complete to $H \smallsetminus \{h_{i-1}, h_{i+1}\}$;
- $v$ is a *clone for $H$ at $h_i$* if $v$ is an adjacent clone or a non-adjacent clone for $H$ at $h_i$.

We say that $v$ is a *leaf* (respectively: *star*, *adjacent clone*, *non-adjacent clone*, *clone*) for $H$ if there exists some $i \in \mathbb{Z}_k$ such that $v$ is a leaf (respectively: star, adjacent clone, non-adjacent clone, clone) for $H$ at $h_i$. If $|H| = k$ is odd, then we say that a vertex $v \in V_G \smallsetminus H$ is *appropriate* for $H$ or for $G[H]$ provided that one of the following holds:

- $v$ is a center for $H$;
- $v$ is an anti-center for $H$;
- $v$ is a leaf for $H$;
- $v$ is an adjacent clone for $H$;

- $v$ is a non-adjacent clone for $H$ and $|H| = 5$;
- $v$ is a star for $H$ and $|H| = 5$.

**6.2.4.** *Let* $G \in \text{Forb}^*(\text{bull})$*, let* $h_0 - h_1 - ... - h_{k-1} - h_0$ *(with* $k \geq 5$ *and the indices in* $\mathbb{Z}_k$*) be an odd hole in* $G$*, and set* $H = \{h_0, h_1, ..., h_{k-1}\}$*. Then every vertex in* $V_G \smallsetminus H$ *is appropriate for* $H$*.*

*Proof.* Fix $v \in V_G \smallsetminus H$. We may assume that $v$ has at least two neighbors and at least one non-neighbor in $H$, for otherwise, $v$ is a center, an anti-center, or a leaf for $H$, and we are done.

Suppose first that $v$ has two adjacent neighbors in $H$. Fix a path $h_i - h_{i+1} - ... - h_j$ of maximum length in $G[H \cap \Gamma_G(v)]$; set $P = \{h_i, h_{i+1}, ..., h_j\}$. Note first that $|P| \geq 3$, for otherwise, we would have that $j = i+1$, and then $G[v, h_{i-1}, h_i, h_{i+1}, h_{i+2}]$ would be a bull. Now, we claim that $v$ is anti-complete to $H \smallsetminus P$. Suppose otherwise. Fix $h_l \in H \smallsetminus P$ such that $vh_l$ is an edge; by the maximality of $P$, we know that $l \notin \{i-1, j+1\}$. Since neither $G[v, h_{i-1}, h_i, h_{i+1}, h_l]$ nor $G[v, h_{j-1}, h_j, h_{j+1}, h_l]$ is bull, we get that $l = i-2 = j+2$, and consequently, that $|H| = |P| + 3$. Since $|H|$ is odd and $|P| \geq 3$, this means that $|P| \geq 4$, and so $G[v, h_{i-1}, h_i, h_{i+1}, h_{i+3}]$ is a bull, which is a contradiction. It follows that $v$ is anti-complete to $H \smallsetminus P$. Now, if $|P| = 3$, then $v$ is an adjacent clone for $H$ at $h_{i+1}$, and we are done. So assume that $|P| \geq 4$. Since $G[v, h_{i-1}, h_i, h_{i+1}, h_{i+3}]$ is not a bull, $h_{i+3}$ is adjacent to $h_{i-1}$, and so $|H| = 5$ and $v$ is a star for $H$ at $h_{i-1}$.

Suppose now that $H \cap \Gamma_G(v)$ is a stable set. Fix distinct $i, j \in \mathbb{Z}_k$ such that $v$ is complete to $\{h_i, h_j\}$ and the path $h_i - h_{i+1} - ... - h_j$ is as short as possible (in particular, $v$ is non-adjacent to the interior vertices of the path). Since the neighbors of $v$ in $H$ are pairwise non-adjacent, and $v$ is complete to $\{h_i, h_j\}$, we know that $v$ is anti-complete to $\{h_{i-1}, h_{j+1}\}$. Since $G[v, h_{i-1}, h_i, h_{i+1}, ..., h_j, h_{j+1}]$ is not a bull*, this implies that either $h_{i-1} = h_{j+1}$, or $h_{i-1}h_{j+1}$ is an edge, and in either case, $v$ is anti-complete to $H \smallsetminus \{h_i, h_j\}$.

We now know that the path $h_j - h_{j+1} - ... - h_i$ has at most three edges and that $v$ is adjacent to the ends of this path and non-adjacent to its interior vertices. The minimality of the path $h_i - h_{i+1} - ... - h_j$ then implies that $|H| \leq 6$. Since $|H|$ is odd and $|H| \geq 5$, it follows that $|H| = 5$. The minimality of the path $h_i - h_{i+1} - ... - h_j$ now implies that $v$ is a non-adjacent clone for $H$ at $h_{i+1}$. This completes the argument. □

Given a graph $G$ with a hole $h_0 - h_1 - ... - h_{k-1} - h_0$ (with $k \geq 5$ and the indices in $\mathbb{Z}_k$), and setting $H = \{h_0, h_1, ..., h_{k-1}\}$, we let $A_H$ denote the set of all anti-centers for $H$ in $G$, and for all $i \in \mathbb{Z}_k$:

- we let $L_H^i$ denote the set of all leaves for $H$ at $h_i$;
- we let $N_H^i$ denote the set of all non-adjacent clones for $H$ at $h_i$;
- we let $C_H^i$ denote the set of all adjacent clones for $H$ at $h_i$;
- we let $S_H^i$ denote the set of all stars for $H$ at $h_i$.

**6.2.5.** *Let $G \in \text{Forb}^*(\text{bull})$, let $h_0 - h_1 - ... - h_{k-1} - h_0$ (with $k \geq 5$ and the indices in $\mathbb{Z}_k$) be an odd hole in $G$, and set $H = \{h_0, h_1, ..., h_{k-1}\}$. Assume that $G$ contains an anti-center for $H$, and that $G$ does not contain a proper homogeneous set. Then there exists an index $i \in \mathbb{Z}_k$ such that all of the following hold:*

*(i) $L_H^i \neq \emptyset$, and for all $j \in \mathbb{Z}_k \smallsetminus \{i\}$, $L_H^j = \emptyset$;*

*(ii) $A_H$ is not anti-complete to $L_H^i$;*

*(iii) $A_H$ is anti-complete to $V_G \smallsetminus (A_H \cup L_H^i)$.*

*Proof.* First, since $G$ does not contain a proper homogeneous set and $|V_G| \geq 3$, we know that $G$ is connected. Further, since $G$ does not contain a proper homogeneous set and contains an anti-center for $H$, 6.2.2 implies that $G$ does not contain a center for $H$.

Now, we claim that every vertex in $V_G \smallsetminus (H \cup A_H)$ that has a neighbor in $A_H$ is a leaf for $H$. Suppose otherwise; fix adjacent $v \in V_G \setminus (H \cup A_H)$ and $a \in A_H$ such that $v$ is not a leaf for $H$. Since $v$ is appropriate for $H$ (by 6.2.4), and since $v$ is not a leaf, or a center, or an anti-center for $H$, we know that $v$ is either a star, or an adjacent clone, or a non-adjacent clone for $H$. Suppose first that $v$ is a star or an adjacent clone for $H$. Then there exists an index $i \in \mathbb{Z}_k$ such that $v$ is complete to $\{h_i, h_{i+1}\}$ and non-adjacent to $h_{i+2}$; but now $G[a, v, h_i, h_{i+1}, h_{i+2}]$ is a bull. Suppose now that $v$ is a non-adjacent clone for $H$. Then there exists an index $i \in \mathbb{Z}_k$ such that $v$ is complete to $\{h_{i-1}, h_{i+1}\}$ and anti-complete to $\{h_i, h_{i+2}\}$; but now $G[a, v, h_{i-1}, h_i, h_{i+1}, h_{i+2}]$ is a bull*. This proves our claim.

Since $G$ is connected and $A_H$ is non-empty, what we just showed implies that there exists an index $i \in \mathbb{Z}_k$ such that $L_H^i$ is non-empty and is not anti-complete to $A_H$. The only thing left to show is that $L_H^j = \emptyset$ for all $j \in \mathbb{Z}_k \setminus \{i\}$. Suppose otherwise. Fix some $j \in \mathbb{Z}_k \setminus \{i\}$ such that $L_H^j \neq \emptyset$. First, note that $L_H^j$ is complete to $L_H^i$, for if some $l_i \in L_H^i$ and $l_j \in L_H^j$ were non-adjacent, $G[H \cup \{l_i, l_j\}]$ would be a bull*. By symmetry and the fact that $|H|$ is odd, we may assume that the path $h_i - h_{i+1} - ... - h_j$ is shorter than the path $h_j - h_{j+1} - ... - h_i$; since $|H| \geq 5$, this means that $i - 1 \notin \{j, j+1\}$. Note furthermore that $j \neq i+1$, for otherwise, we fix some $l_i \in L_H^i$ and $l_{i+1} \in L_H^{i+1}$ and note that $G[l_i, l_{i+1}, h_{i-1}, h_i, h_{i+1}, h_{i+2}]$ is a bull*. Next, fix an anti-center $a$ for $H$ such that $a$ is adjacent to some $l_i \in L_H^i$. Fix $l_j \in L_H^j$. But then if $al_j$ is an edge, $G[a, l_i, l_j, h_i, h_j]$ is a bull; and if $al_j$ is a non-edge, then $G[a, l_i, l_j, h_{i-1}, h_i, h_{i+1}, ..., h_{j-1}, h_j]$ is a bull*. This completes the argument. $\square$

**6.2.6.** *Let $G \in$ Forb*(bull), let $h_0 - h_1 - ... - h_{k-1} - h_0$ (with $k \geq 5$ and the indices in $\mathbb{Z}_k$) be an odd hole in $G$, and set $H = \{h_0, h_1, ..., h_{k-1}\}$. Assume that $G$ contains an anti-center for $H$, and that $G$ does not contain a proper homogeneous set. Then there exists an index $i \in \mathbb{Z}_k$ such that $V_G = H \cup A_H \cup L_H^i \cup S_H^i \cup \bigcup_{j \in \mathbb{Z}_k} (N_H^j \cup C_H^j)$, where $L_H^i$ is non-empty, $L_H^i$ is anti-complete to $S_H^i$, and if $k \geq 7$, then $S_H^i$ and $\bigcup_{j \in \mathbb{Z}_k} N_H^j$ are empty.*

*Proof.* If $k \geq 7$, then the result is immediate from 6.2.2, 6.2.4, and 6.2.5. So assume that

$k = 5$. By 6.2.2, 6.2.4, and 6.2.5, we know that $V_G = H \cup A_H \cup L_H^i \cup \bigcup_{j\in\mathbb{Z}_5}(S_H^j \cup N_H^j \cup C_H^j)$, with $L_H^i \neq \emptyset$, for some $i \in \mathbb{Z}_5$. We need to show that $S_H^j = \emptyset$ for all $j \in \mathbb{Z}_5 \smallsetminus \{i\}$, and that $L_H^i$ is anti-complete to $S_H^i$.

We first show that $S_H^j = \emptyset$ for all $j \in \mathbb{Z}_5 \smallsetminus \{i\}$. By symmetry, it suffices to show that $S_H^{i+1}$ and $S_H^{i+2}$ are empty. Fix some $l_i \in L_H^i$. Suppose first that $S_H^{i+1} \neq \emptyset$, and fix $s_{i+1} \in S_H^{i+1}$. But then if $s_{i+1}l_i$ is an edge, then $G[l_i, s_{i+1}, h_{i-2}, h_i, h_{i+1}]$ is a bull; and if $s_{i+1}l_i$ is a non-edge, then $G[l_i, s_{i+1}, h_{i-1}, h_i, h_{i+2}]$ is a bull. Thus, $S_H^{i+1} = \emptyset$. Suppose now that $S_H^{i+2} \neq \emptyset$, and fix $s_{i+2} \in S_H^{i+2}$. But then if $s_{i+2}l_i$ is an edge, then $G[s_{i+2}, l_i, h_{i-2}, h_{i-1}, h_{i+2}]$ is a bull; and if $s_{i+2}l_i$ is a non-edge, then $G[s_{i+2}, l_i, h_i, h_{i+1}, h_{i+2}]$ is a bull. Thus, $S_H^{i+2} = \emptyset$.

It remains to show that $L_H^i$ is anti-complete to $S_H^i$. Suppose otherwise. By 6.2.5, $A_H$ is not anti-complete to $L_H^i$, and $A_H$ is anti-complete to $H \cup S_H^i$. We first note that every vertex in $L_H^i$ is anti-complete to at least one of $A_H$ and $S_H^i$, for otherwise, we fix some $l_i \in L_H^i$, $s_i \in S_H^i$, and $a \in A_H$ such that $l_i$ is adjacent to both $s_i$ and $a$, and we observe that $G[l_i, s_i, a, h_{i-1}, h_i, h_{i+2}]$ is a bull*. Now, fix some adjacent $l_i \in L_H^i$ and $s_i \in S_H^i$. By what we just showed, $l_i$ is anti-complete to $A_H$. Since $A_H$ is not anti-complete to $L_H^i$, there exist adjacent $a \in A_H$ and $l_i' \in L_H^i \smallsetminus \{l_i\}$. Since $l_i' \in L_H^i$ has a neighbor in $A_H$, we know that $l_i'$ is anti-complete to $S_H^i$, and in particular, that $l_i's_i$ is a non-edge. But now if $l_il_i'$ is an edge, then $G[l_i, l_i', a, s_i, h_i]$ is a bull; and if $l_il_i'$ is a non-edge, then $G[l_i, l_i', s_i, h_{i-1}, h_i, h_{i+2}]$ is a bull*. This completes the argument. □

**6.2.7.** *Let $G \in$ Forb*(bull), let $h_0 - h_1 - ... - h_{k-1} - h_0$ (with $k \geq 5$ and the indices in $\mathbb{Z}_k$) be an odd hole in $G$, and set $H = \{h_0, h_1, ..., h_{k-1}\}$. Assume that $G$ contains an anti-center for $H$, and that $G$ does not contain a proper homogeneous set. Then there exists an index $i \in \mathbb{Z}_k$ such that $V_G = H \cup A_H \cup L_H^i \cup S_H^i$, where $L_H^i$ is non-empty, $L_H^i$ is anti-complete to $S_H^i$, and if $k \geq 7$, then $S_H^i$ is empty.*

*Proof.* By 6.2.6, we just need to show that $N_H^j \cup C_H^j = \emptyset$ for all $j \in \mathbb{Z}_k$. It suffices to show that for all $j \in \mathbb{Z}_k$, $\{h_j\} \cup N_H^j \cup C_H^j$ is a homogeneous set in $G$, for then the fact that $G$ contains no proper homogeneous set will imply that $\{h_j\} \cup N_H^j \cup C_H^j$ is a singleton, and therefore, that $N_H^j \cup C_H^j = \emptyset$.

Fix $j \in \mathbb{Z}_k$, and suppose that $\{h_j\} \cup N_H^j \cup C_H^j$ is not a homogeneous set in $G$. Fix some $v \in V_G \smallsetminus (\{h_j\} \cup N_H^j \cup C_H^j)$ such that $v$ is mixed on $\{h_j\} \cup N_H^j \cup C_H^j$. Clearly, $v \notin H$. Fix some $c_j, c_j' \in \{h_j\} \cup N_H^j \cup C_H^j$ such that $v$ is adjacent to $c_j$ and non-adjacent to $c_j'$. Set $\hat{H} = (H \smallsetminus \{h_j\}) \cup \{c_j\}$ and $\hat{H}' = (H \smallsetminus \{h_j\}) \cup \{c_j'\}$. Then $G[\hat{H}]$ and $G[\hat{H}']$ are both odd holes of length $k$. Next, by 6.2.5, $A_H$ is anti-complete to $\{c_j, c_j'\}$, and so since $A_H$ is non-empty, $G$ contains an anti-center for both $\hat{H}$ and $\hat{H}'$; thus, 6.2.6 applies to both $\hat{H}$ and $\hat{H}'$. This, together with the fact that $v$ has exactly one more neighbor in $\hat{H}$ than in $\hat{H}'$, implies that either:

(a) $v$ is a leaf for $\hat{H}$ and an anti-center for $\hat{H}'$; or

(b) $k = 5$ and one of the following holds:

  (b1) $v$ is a non-adjacent clone for $\hat{H}$ and a leaf for $\hat{H}'$;

  (b2) $v$ is an adjacent clone for $\hat{H}$ and a non-adjacent clone for $\hat{H}'$;

  (b3) $v$ is a star for $\hat{H}$ and an adjacent clone for $\hat{H}'$.

Suppose that (a) holds. Since $v$ is adjacent to $c_j$, $v$ is a leaf for $\hat{H}$ at $c_j$. But now if $c_j c_j'$ is an edge, then $G[v, c_j, c_j', h_{j+1}, h_{j+2}]$ is a bull; and if $c_j c_j'$ is a non-edge, then $G[v, c_j, c_j', h_{j-1}, h_{j+1}, h_{j+2}]$ is a bull*. From now on, we assume that (b) holds, and so $k = 5$.

Suppose first that (b1) holds. Since $v$ is a non-adjacent clone for $\hat{H}$ and is adjacent to $c_j$, we know that $v$ is a non-adjacent clone for $\hat{H}$ at either $h_{j-1}$ or at $h_{j+1}$; by symmetry, we may assume that $v$ is a non-adjacent clone for $\hat{H}$ at $h_{j+1}$. But now if $c_j c_j'$ is an edge, then $G[v, c_j, c_j', h_{j-2}, h_{j-1}]$ is a bull; and if $c_j c_j'$ is a non-edge, then $G[v, c_j, c_j', h_{j-2}, h_{j-1}, h_{j+1}]$

is a bull$^*$.

Suppose next that (b2) holds. Since $v$ is a clone for both $\hat{H}$ and $\hat{H}'$, and since $v$ is adjacent to $c_j$ and non-adjacent to $c'_j$, it is easy to see that $v$ is an adjacent clone for $\hat{H}$ at $c_j$ and a non-adjacent clone for $\hat{H}'$ at $c'_j$. But now $v$ is a clone for $H$ at $h_j$, contrary to the fact that $v \in V_G \setminus (\{h_j\} \cup N^j_H \cup C^j_H)$.

Suppose finally that (b3) holds. Since $v$ is adjacent to $c_j$ and non-adjacent to $c'_j$, it is easy to see that $v$ is a star for $\hat{H}$ at either $h_{j-1}$ or $h_{j+1}$; by symmetry, we may assume that $v$ is a star for $\hat{H}$ at $h_{j+1}$. Since 6.2.6 applies to $\hat{H}$, it follows that $G$ contains a leaf $l_{j+1}$ for $\hat{H}$ at $h_{j+1}$, and that $l_{j+1}$ is non-adjacent to $v$. Since $l_{j+1}$ is appropriate for $\hat{H}'$, it is non-adjacent to $c'_j$. But now if $c_jc'_j$ is an edge, then $G[v, c_j, c'_j, l_{j+1}, h_{j+1}]$ is a bull; and if $c_jc'_j$ is a non-edge, then $G[v, c_j, c'_j, h_{j-1}, h_{j+2}]$ is a bull. This completes the argument. □

**6.2.8.** *Let $G \in$ Forb$^*$(bull), let $h_0 - h_1 - ... - h_{k-1} - h_0$ (with $k \geq 5$ and the indices in $\mathbb{Z}_k$) be an odd hole in $G$, and set $H = \{h_0, h_1, ..., h_{k-1}\}$. Assume that $G$ contains an anti-center for $H$. Then $G$ contains a proper homogeneous set or a cut-vertex.*

*Proof.* We assume that $G$ does not contain a proper homogeneous set and show that it contains a cut-vertex. By 6.2.7, there exists an index $i \in \mathbb{Z}_k$ such that $V_G = H \cup A_H \cup L^i_H \cup S^i_H$ and $L^i_H$ is non-empty and anti-complete to $S^i_H$. Now, by 6.2.5, $A_H$ is anti-complete to $S^i_H$. Thus, $A_H \cup L^i_H$ is anti-complete to $(H \setminus \{h_i\}) \cup S^i_H$. Since $V_G = H \cup A_H \cup L^i_H \cup S^i_H$, and since $h_i$ has neighbors both in $L^i_H$ and in $H \setminus \{h_i\}$, it follows that $h_i$ is a cut-vertex of $G$. □

We now restate and prove 6.2.1, the main result of this section.

**6.2.1.** *Let $G \in$ Forb$^*$(bull). Then either $G$ is basic, or it contains a proper homogeneous set or a cut-vertex.*

*Proof.* Since an anti-hole of length five is also a hole of length five, the result is immediate from 6.2.3 and 6.2.8. □

## 6.3 A $\chi$-Bounding Function for Forb*(bull)

In this section, we use 6.2.1 to prove that the class Forb*(bull) is $\chi$-bounded by the function $f(n) = n^2$. We begin with a definition. Given graphs $G_1$ and $G_2$ with $V_{G_1} \cap V_{G_2} = \{u\}$, we say that a graph $G$ is obtained by *gluing* $G_1$ *and* $G_2$ *along* $u$ provided that the following hold:

- $V_G = V_{G_1} \cup V_{G_2}$;
- for all $i \in \{1, 2\}$, $G[V_{G_i}] = G_i$;
- $V_{G_1} \setminus \{u\}$ is anti-complete to $V_{G_2} \setminus \{u\}$ in $G$.

We observe that if a graph $G$ has a cut-vertex, then $G$ is obtained by gluing smaller graphs (i.e. graphs that have strictly fewer vertices than $G$) along a vertex.

In this section, we will use the Strong Perfect Graph Theorem 1.1.2, as well as the fact that the class of perfect graphs is closed under substitution (see 2.1.2).

In this thesis, a *weighted graph* is a graph $G$ such that each vertex $v \in V_G$ is assigned a positive integer called its *weight* and denoted by $w_v$. The *weight* of a non-empty set $S \subseteq V_G$ is the sum of weights of the vertices in $S$. We denote by $W_G$ the weight of a clique of maximum weight in $G$. Given an induced subgraph $H$ of $G$, and a vertex $v \in V_G$, we say that $H$ *covers* $v$ provided that $v \in V_H$. We now prove a technical lemma, which we then use to prove the main result of this section.

**6.3.1.** *Let $G \in$ Forb*(bull) be a weighted graph. Then there exists a family $\mathcal{P}_G$ of at most $W_G$ perfect induced subgraphs of $G$ such that for every vertex $v \in V_G$, at least $w_v$ members of $\mathcal{P}_G$ cover $v$.*

*Proof.* We assume inductively that the claim holds for graphs with fewer than $|V_G|$ vertices. If $G$ is the empty graph, then there is nothing to show; so assume that $G$ is non-empty.

By 6.2.1, we know that either $G$ is basic, or $G$ contains a proper homogeneous set, or $G$ contains a cut-vertex.

Suppose first that $G$ is basic. Fix $u \in V_G$ such that $w_u$ is maximal. Let $A$ be the set of all neighbors of $u$ in $G$, and let $B$ be the set of all non-neighbors of $u$ in $G$. Since $G$ is basic, and $u$ is an anti-center for $B$, we know that $G[B]$ contains no odd holes and no odd anti-holes. Since $u$ is anti-complete to $B$, it follows that $G[B \cup \{u\}]$ contains no odd holes and no odd anti-holes, and so by the Strong Perfect Graph Theorem (1.1.2), $G[B \cup \{u\}]$ is perfect. Let $\mathcal{P}_B$ be the family consisting of $w_u$ copies of the perfect graph $G[B \cup \{u\}]$. Note that by the maximality of $w_u$, every vertex $v \in B \cup \{u\}$ is covered by at least $w_v$ graphs in $\mathcal{P}_B$. If $A = \emptyset$ (so that $V_G = B \cup \{u\}$), then we set $\mathcal{P}_G = \mathcal{P}_B$, and we are done. So assume that $A \neq \emptyset$. Now by the induction hypothesis, there exists a family $\mathcal{P}_A$ of at most $W_{G[A]}$ perfect induced subgraphs of $G[A]$ such that each vertex $v \in A$ is covered by at least $w_v$ graphs in $\mathcal{P}_A$. Since $u$ is complete to $A$, we have that $w_u + W_{G[A]} \leq W_G$. Since the family $\mathcal{P}_B$ contains exactly $w_u$ graphs, it follows that the family $\mathcal{P}_G = \mathcal{P}_A \cup \mathcal{P}_B$ contains at most $W_G$ graphs, and by construction, every vertex $v \in V_G$ is covered by at least $w_v$ graphs in $\mathcal{P}_G$.

Suppose now that $G$ contains a proper homogeneous set; let $S$ be a proper homogeneous set in $G$, let $A$ be the set of all vertices in $V_G$ that are complete to $S$, and let $B$ be the set of all vertices in $V_G$ that are anti-complete to $S$. Let $H$ be the graph whose vertex-set is $\{s\} \cup A \cup B$, with $H[A \cup B] = G[A \cup B]$, and $s$ complete to $A$ and anti-complete to $B$ in $H$. We turn $H$ into a weighted graph by letting the vertices in $A \cup B$ have the same weights in $H$ as they do in $G$, and setting $w_s = W_{G[S]}$. Clearly, $W_H = W_G$. Using the induction hypothesis, we let $\mathcal{P}_H$ be a family of at most $W_H = W_G$ perfect induced subgraphs of $H$ such that every vertex $v \in V_H$ is covered by at least $w_v$ graphs in $\mathcal{P}_H$, and we let $\mathcal{P}_{G[S]}$ be the family of at most $W_{G[S]} = w_s$ perfect inducted subgraphs of $G[S]$ such that every vertex $v \in S$ is covered by at least $w_v$ graphs in $\mathcal{P}_{G[S]}$. We may assume that the vertex $s$ is covered by exactly $w_s$ graphs in $\mathcal{P}_H$; let $P_1, ..., P_{w_s}$ be the graphs in $\mathcal{P}_H$ covering $s$, and let $\mathcal{P}'_H = \mathcal{P}_H \smallsetminus \{P_1, ..., P_{w_s}\}$. We may assume that $\mathcal{P}_{G[S]}$ contains exactly $W_{G[S]} = w_s$ graphs; say $\mathcal{P}_{G[S]} = \{Q_1, ..., Q_{w_s}\}$. Now, for each $i \in \{1, ..., w_s\}$, let $P'_i$ be the graph obtained by substituting the graph $Q_i$ for $s$ in $P_i$; by 2.1.2, the graph $P'_i$ is perfect for all $i \in \{1, ..., w_s\}$. We then set $\mathcal{P}_G = \{P'_1, ..., P'_{w_s}\} \cup \mathcal{P}'_H$. By construction, $\mathcal{P}_G$ is a family of at most $W_G$ perfect induced subgraphs of $G$ such that for every vertex $v \in V_G$, at least $w_v$ members of $\mathcal{P}_G$ cover $v$.

Suppose finally that $G$ contains a cut-vertex. Then there exist $u \in V_G$ and $C_1, C_2 \subseteq V_G \smallsetminus \{u\}$ such that $V_G = \{u\} \cup C_1 \cup C_2$, where $C_1$ and $C_2$ are non-empty, disjoint, and anti-complete to each other. For $i \in \{1, 2\}$, let $G_i = G[C_i \cup \{u\}]$. (Note that $G$ is obtained by gluing $G_1$ and $G_2$ along $u$.) Using the induction hypothesis, for each $i \in \{1, 2\}$, we get a family $\mathcal{P}_{G_i}$ of at most $W_{G_i}$ perfect induced subgraphs of $G_i$ such that each vertex $v \in V_{G_i}$ is covered by at least $w_v$ graphs in $\mathcal{P}_{G_i}$. We may assume that for all $i \in \{1, 2\}$, $\mathcal{P}_{G_i}$ contains exactly $W_{G_i}$ graphs, and that $u_i$ is covered by exactly $w_{u_i}$ graphs in $\mathcal{P}_{G_i}$. By symmetry, we may assume that $W_{G_1} \leq W_{G_2}$. For each $i \in \{1, 2\}$, let $P^i_1, ..., P^i_{w_u}$ be the graphs in $\mathcal{P}_{G_i}$ covering $u$, let $P^i_{w_u+1}, ..., P^i_{W_{G_1}}$ be $W_{G_1} - w_u$ graphs in $\mathcal{P}_{G_i}$ that do not cover $u$, and let $P^2_{W_{G_1}+1}, ..., P^2_{W_{G_2}}$ be the remaining $W_{G_2} - W_{G_1}$ graphs in $\mathcal{P}_{G_2}$. Now, for all $j \in \{1, ..., w_u\}$, let $P_j$ be the graph obtained by gluing $P^1_j$ and $P^2_j$ along $u$; for all $j \in \{w_u + 1, ...., W_{G_1}\}$, let $P_j$ be the disjoint union of $P^1_j$ and $P^2_j$; and for all $j \in \{W_{G_1} + 1, ..., W_{G_2}\}$, let $P_j = P^2_j$. It is easy to see that $P_j$ is perfect for all $j \in \{1, ..., W_{G_2}\}$. Now set $\mathcal{P}_G = \{P_1, ..., P_{W_{G_2}}\}$. Since $W_G = \max\{W_{G_1}, W_{G_2}\} = W_{G_2}$, $\mathcal{P}_G$ is a family of at most $W_G$ perfect induced subgraphs of $G$ such that for every vertex $v \in V_G$, at least $w_v$ members of $\mathcal{P}_G$ cover $v$. $\square$

**6.3.2.** *The class* Forb*(bull) *is* $\chi$*-bounded by the function* $f(n) = n^2$.

*Proof.* Let $G \in$ Forb*(bull). Using 4.3, we obtain a family $\mathcal{P}$ of at most $\omega(G)$ perfect induced subgraphs of $G$ such that each vertex in $V_G$ is covered by at least one graph in $\mathcal{P}$. Clearly, we may assume that each vertex in $V_G$ is covered by exactly one graph in $\mathcal{P}$.

Since the graphs in $\mathcal{P}$ are perfect, each graph $P \in \mathcal{P}$ can be colored with $\omega(P) \leq \omega(G)$ colors; we may assume that the sets of colors used on the graphs in $\mathcal{P}$ are pairwise disjoint. Now we take the union of the colorings of the graphs in $\mathcal{P}$ to obtain a coloring of $G$ that uses at most $\omega(G)^2$ colors. $\square$

## 6.4 Necklaces

We begin with some definitions. Let $n$ be a non-negative integer, and let $m_0, ..., m_n$ be positive integers. Let $H$ be a graph whose vertex-set is $\bigcup_{i=0}^{n}\{x_{i,0}, x_{i,1}, ..., x_{i,m_i}\} \cup \{y_1, ..., y_n\}$, with adjacency as follows:

- $x_{0,0} - ... - x_{0,m_0} - x_{1,0} - ... - x_{1,m_1} - ... - x_{n,0} - ... - x_{n,m_n}$ is a chordless path;
- $\{y_1, ..., y_n\}$ is a stable set;
- for all $i \in \{1, ..., n\}$, $y_i$ has exactly two neighbors in the set $\bigcup_{i=0}^{n}\{x_{i,0}, x_{i,1}, ..., x_{i,m_i}\}$, namely $x_{i-1,m_{i-1}}$ and $x_{i,0}$.

Under these circumstances, we say that $H$ is an $(m_0, ..., m_n)$*-necklace with base* $x_{0,0}$ *and hook* $x_{n,m_n}$, or simply that $H$ is an $(m_0, ..., m_n)$*-necklace*. If $G$ is a subdivision of $H$, then we say that $G$ is an $(m_0, ..., m_n)$*-necklace*$^*$ *with base* $x_{0,0}$ *and hook* $x_{n,m_n}$, or simply that $G$ is an $(m_0, ..., m_n)$*-necklace*$^*$. To simplify notation, given a non-negative integer $n$ and a positive integer $m$, we often write "$(m)_n$-necklace" instead of "$\underbrace{(m, ..., m)}_{n+1}$-necklace," and "$(m)_n$-necklace$^*$" instead of "$\underbrace{(m, ..., m)}_{n+1}$-necklace$^*$." (We remark that a $(1)_1$-necklace is the bull, and that for all positive integers $m$, an $(m)_0$-necklace with base $x_0$ and hook $x_m$ is a chordless $m$-edge path between $x_0$ and $x_m$.)

Our goal in this section is to prove that for all non-negative integers $n$ and positive integers $m_0, ..., m_n$, the class Forb$^*((m_0, ..., m_n)-$necklace) is $\chi$-bounded by an exponential function (see 6.4.2 below). We observe that in order to prove 6.4.2, it suffices to consider only the $(m)_n$-necklaces. Indeed, if $m = \max\{m_0, ..., m_n\}$, then an $(m)_n$-necklace is a subdivision of an $(m_0, ..., m_n)$-necklace, and consequently, Forb$^*((m_0, ..., m_n)-$necklace$) \subseteq$ Forb$^*((m)_n-$necklace$)$. Thus, it suffices to show that Forb$^*((m)_n-$necklace$)$ is $\chi$-bounded by an exponential function.

We now need some more definitions. First, in this thesis, the *local chromatic number* of a non-empty graph $G$, denoted by $\chi_l(G)$, is the number $\max_{v \in V_G} \chi(G[\Gamma_G(v)])$. Next, let $n$ be a non-negative and $m$ a positive integer. Let $G$ be a graph whose vertex-set is the disjoint union of non-empty sets $N$ and $X$, let $x_0$ and $x$ be distinct vertices in $N$, and assume that the adjacency in $G$ is as follows:

- $G[N]$ is an $(m)_n$-necklace$^*$ with base $x_0$ and hook $x$;
- $G[X]$ is connected;
- $N \smallsetminus \{x\}$ is anti-complete to $X$;
- $x$ has a neighbor in $X$.

Under these circumstances, we say that $(G, x_0, x)$ is an $(m)_n$*-alloy* or simply an *alloy*. The graph $G$ is referred to as the *base graph* of the alloy $(G, x_0, x)$, and the ordered pair $(N, X)$ is the *partition* of the alloy $(G, x_0, x)$. The *potential* of the alloy $(G, x_0, x)$ is the chromatic number of the graph $G[X]$.

We now state the main technical lemma of this section.

**6.4.1.** *Let $G$ be a connected graph, and let $x_0 \in V_G$. Let $n$ and $\beta$ be non-negative integers, and let $m$ and $\alpha$ be positive integers. Assume that $\chi_l(G) \leq \alpha$ and $\chi(G) > 2^{n+1}((m + 3)\alpha + \beta)$. Then there exists an induced subgraph $H$ of $G$ and a vertex $x \in V_G$ such that $(H, x_0, x)$ is an $(m)_n$-alloy of potential greater than $\beta$.*

Since the base graph of an $(m)_n$-alloy contains an $(m)_n$-necklace$^*$ as an induced subgraph, 6.4.1 easily implies the main result of this section (6.4.2), as we now show. (We note that our proof of 6.4.2 relies only on the special case of 6.4.1 when $\beta = 0$.)

**6.4.2.** *Let $n$ be a non-negative integer, let $m_0, ..., m_n$ be positive integers, and let $m = \max\{m_0, ..., m_n\}$. Then the class* Forb$^*((m_0, ..., m_n)-$necklace) *is $\chi$-bounded by the exponential function $f(k) = (2^{n+1}(m+3))^{k-1}$.*

*Proof.* Since an $(m)_n$-necklace is a subdivision of an $(m_0, ..., m_n)$-necklace, we know that Forb$^*((m_0, ..., m_n)-$necklace) $\subseteq$ Forb$^*((m)_n-$necklace), and so it suffices to show that Forb$^*((m)_n-$necklace) is $\chi$-bounded by the function $f$. Suppose that this is not the case; let $k \in \mathbb{N}$ be minimal with the property that there is a graph $G \in$ Forb$^*((m)_n-$necklace) such that $\omega(G) = k$ and $\chi(G) > f(k)$. Clearly, $k \geq 2$. Furthermore, we may assume that $G$ is connected, for otherwise, instead of $G$, we consider a component of $G$ with maximum chromatic number. Note that for all $v \in V_G$, we have that $\omega(G[\Gamma_G(v)]) \leq k-1$, and so by the minimality of $k$, $\chi(G[\Gamma_G(v)]) \leq f(k-1)$; thus $\chi_l(G) \leq f(k-1)$. Now, set $\alpha = f(k-1)$; then $\chi_l(G) \leq \alpha$ and $\chi(G) > 2^{n+1}(m+3)\alpha$. Fix $x_0 \in V_G$. Then 6.4.1 implies that there exists an induced subgraph $H$ of $G$ and a vertex $x \in V_G$ such that $(H, x_0, x)$ is an $(m)_n$-alloy. But then $H$ contains an $(m)_n$-necklace$^*$ as an induced subgraph, contrary to the fact that $G \in$ Forb$^*((m)_n-$necklace). □

The rest of the section is devoted to proving 6.4.1. The idea of the proof is to show that, given a connected graph $G$ whose chromatic number is sufficiently large relative to its local chromatic number, it is possible to recursively "chisel" an $(m)_n$-alloy out of the graph $G$. At each recursive step, the "length" of the alloy (i.e. the number $n$) increases, and the potential of the alloy decreases (but in a controlled fashion, so as to allow the next recursive step). We begin with a technical lemma, which we will use many times in this section.

**6.4.3.** *Let $G$ be a graph, let $x_0 \in V_G$, and let $S \subseteq V_G \smallsetminus \{x_0\}$ be such that $G[S]$ is connected and $x_0$ has a neighbor in $S$. Let $k$ be a non-negative integer, let $\alpha$ be a positive integer, and assume that $\chi_l(G) \leq \alpha$, and that $\chi(G[S]) > k\alpha$. Then there exist vertices $x_1, ..., x_k \in S$ and a set $X \subseteq S$ such that:*

*a.* $x_0 - x_1 - ... - x_k$ *is an induced path in* $G$*;*

*b.* $G[X]$ *is connected;*

*c.* $x_1, ..., x_k \notin X$*;*

*d.* $x_k$ *has a neighbor in* $X$*;*

*e. vertices* $x_0, ..., x_{k-1}$ *are anti-complete to* $X$*;*

*f.* $\chi(G[X]) \geq \chi(G[S]) - k\alpha$.

*Proof.* Let $i \in \{0, ..., k\}$ be maximal such that there exist vertices $x_1, ..., x_i \in S$ and a set $X \subseteq S$ such that:

- $x_0 - x_1 - ... - x_i$ is an induced path in $G$;
- $G[X]$ is connected;
- $x_1, ..., x_i \notin X$;
- $x_i$ has a neighbor in $X$;
- vertices $x_0, ..., x_{i-1}$ are anti-complete to $X$;
- $\chi(G[X]) \geq \chi(G[S]) - i\alpha$.

(The existence of such an index $i$ follows from the fact that $x_0$ is an induced path in $G$, $G[S]$ is connected, $x_0$ has a neighbor in $S$, and $\chi(G[S]) \geq \chi(G[S]) - 0 \cdot \alpha$.)

We need to show that $i = k$. Suppose otherwise, that is, suppose that $i < k$. Then:

$$\begin{aligned}\chi(G[X]) &\geq \chi(G[S]) - i\alpha \\ &> k\alpha - i\alpha \\ &= (k-i)\alpha \\ &\geq \alpha,\end{aligned}$$

and so $\chi(G[X]) > \alpha$. Since $\chi(G[\Gamma_G(x_i)]) \leq \alpha$ (because $\chi_l(G) \leq \alpha$), it follows that $x_i$ is not complete to $X$; let $X'$ be the vertex-set of a component of $G[X \smallsetminus \Gamma_G(x_i)]$ with maximum chromatic number. Then $\chi(G[X]) \leq \chi(G[\Gamma_G(x_i)]) + \chi(G[X'])$, and so:

$$\begin{aligned}\chi(G[X']) &\geq \chi(G[X]) - \chi(G[\Gamma_G(x_i)]) \\ &\geq (\chi(G[S]) - i\alpha) - \alpha \\ &= \chi(G[S]) - (i+1)\alpha\end{aligned}$$

Fix a vertex $x_{i+1} \in X \cap \Gamma_G(x_i)$ such that $x_{i+1}$ has a neighbor in $X'$. But now the sequence $x_1, ..., x_i, x_{i+1}$ and the set $X'$ contradict the maximality of $i$. It follows that $i = k$, which completes the argument. □

The following is an easy consequence of 6.4.3, and it will serve as the base for our recursive construction of an $(m)_n$-alloy.

**6.4.4.** *Let $G$ be a connected graph, let $x_0 \in V_G$, let $\beta$ be a non-negative integer, and let $m$ and $\alpha$ be positive integers. Assume that $\chi_l(G) \leq \alpha$, and that $\chi(G) > (m+1)\alpha + \beta$. Then there exists a vertex $x \in V_G \smallsetminus \{x_0\}$ and an induced subgraph $H$ of $G$ such that $(H, x_0, x)$ is an $(m)_0$-alloy of potential greater than $\beta$.*

*Proof.* Let $S$ be the vertex-set of a component of $G \smallsetminus x_0$ of maximum chromatic number. Clearly then, $\chi(G) \leq \chi(G[S]) + 1$, and consequently, $\chi(G[S]) > m\alpha + \beta$. Since $G$ is connected, $x_0$ has a neighbor in $S$. By 6.4.3 then, there exist vertices $x_1, ..., x_m \in S$ and a set $X \subseteq S$ such that:

- $x_0 - x_1 - ... - x_m$ is an induced path in $G$;
- $G[X]$ is connected;
- $x_1, ..., x_m \notin X$;
- $x_m$ has a neighbor in $X$;
- vertices $x_0, ..., x_{m-1}$ are anti-complete to $X$;

- $\chi(G[X]) \geq \chi(G[S]) - m\alpha$.

The fact that $\chi(G[X]) \geq \chi(G[S]) - m\alpha$ and $\chi(G[S]) > m\alpha + \beta$ implies that $\chi(G[X]) > \beta$. Now set $H = G[\{x_0, ..., x_{m_0}\} \cup X]$ and $x = x_m$. Then $(H, x_0, x)$ is an $(m)_0$-alloy of potential greater than $\beta$. □

Our goal now is to show that, given an $(m)_n$-alloy with large potential and small local chromatic number of the base graph, we can "chisel" out of this $(m)_n$-alloy an $(m)_{n+1}$-alloy of large potential. More formally, we wish to prove the following lemma.

**6.4.5.** *Let $n$ and $\beta$ be non-negative integers, and let $m$ and $\alpha$ be positive integers. Let $(G, x_0, x)$ be an $(m)_n$-alloy of potential greater than $2((m+3)\alpha + \beta)$, and let $(N, X)$ be the partition of the alloy $(G, x_0, x)$. Assume that $\chi_l(G) \leq \alpha$. Then there exist disjoint sets $N', X' \subseteq V_G$ such that $N \subseteq N'$ and $X' \subseteq X$, and a vertex $x' \in X$ such that $(G[N' \cup X'], x_0, x')$ is an $(m)_{n+1}$-alloy of potential greater than $\beta$ and with partition $(N', X')$.*

We now need some definitions. Let $n$ be a non-negative and $m$ a positive integer, and let $(G, x_0, x)$ be an $(m)_n$-alloy with partition $(N, X)$. Assume that the potential of $(G, x_0, x)$ is greater than $2\beta$ (where $\beta$ is some non-negative integer). For each $i \in \mathbb{N} \cup \{0\}$, let $S'_i$ be the set of all vertices in $\{x\} \cup X$ that are at distance $i$ from $x$ in $G[\{x\} \cup X]$; thus, $S'_0 = \{x\}$. Let $t \in \mathbb{N}$ be such that $\chi(G[S'_t])$ is as large as possible. As the sets $S_1, S_3, S_5, ...$ are pairwise anti-complete to each other, as are the sets $S_2, S_4, S_6, ...$, it is easy to see that $\chi(G[X]) \leq 2\chi(G[S'_t])$, and consequently, $\chi(G[S'_t]) > \beta$. Now, let $S_t$ be the vertex-set of a component of $G[S'_t]$ with maximum chromatic number (thus, $\chi(G[S_t]) > \beta$), and for each $i \in \{0, 1, ..., t-1\}$, let $S_i$ be an inclusion-wise minimal subset of $S'_i$ such that every vertex in $S_{i+1}$ has a neighbor in $S_i$; clearly, $S_0 = \{x\}$. Let $H = G[N \cup \bigcup_{i=1}^t S_i]$. We then say that $(H, x_0, x)$ is a *reduction* of the $(m)_n$-alloy $(G, x_0, x)$, and that $\{S_i\}_{i=0}^t$ is the *stratification* of $(H, x_0, x)$. Clearly, $(H, x_0, x)$ is itself an $(m)_n$-alloy, and $(N, \bigcup_{i=1}^t S_i)$ is the associated partition. Further, as $\chi(G[S_t]) > \beta$ and $H$ is an induced subgraph of $G$, we know that $\chi(H[S_t]) > \beta$. Next, given vertices $a \in S_p$ and $b \in S_q$ for some $p, q \in \{0, ..., t\}$, a path $P$ in $H$ between $a$ and $b$ is said to be *monotonic* provided that it has $|p - q|$ edges. This means that if $p = q$ then $a = b$, and if $p \neq q$ then all the internal vertices of the path $P$ lie in $\bigcup_{r=\min\{p,q\}+1}^{\max\{p,q\}-1} S_r$, with each set $S_r$ (with $\min\{p,q\}+1 \leq r \leq \max\{p,q\}-1$) containing exactly one vertex of the path. Clearly, every monotonic path is induced. We observe that for all $p \in \{0, ..., t\}$ and $a \in S_p$, there exists a monotonic path between $x$ and $a$.

The idea of the proof of 6.4.5 is as follows. First, we let $(H, x_0, x)$ be a reduction of the $(m)_n$-alloy $(G, x_0, x)$, and we let $\{S_i\}_{i=0}^t$ be the associated stratification. From now on, we work only with the graph $H$ (and not $G$). We find the needed vertex $x'$ in the set $S_t$, and the set $X'$ is chosen to be a suitable subset of the set $S_t$. Our proof splits into two cases. The first (and easier) case is when at least one of the sets $S_1, ..., S_{t-2}$ is not stable (in this case, we necessarily have $t \geq 3$); the second (and harder) case is when the sets $S_1, ..., S_{t-2}$ are all stable. We treat these two cases in two separate lemmas (the first case is treated in 6.4.6, and the second case in 6.4.7).

**6.4.6.** *Let $n$ and $\beta$ be non-negative integers, and let $m$ and $\alpha$ be positive integers. Let $(G, x_0, x)$ be an $(m)_n$-alloy of potential greater than $2(m\alpha + \beta)$, and let $(N, X)$ be the partition of the alloy $(G, x_0, x)$. Assume that $\chi_l(G) \leq \alpha$. Let $(H, x_0, x)$ be a reduction of the $(m)_n$-alloy $(G, x_0, x)$, and let $\{S_i\}_{i=0}^t$ be the associated stratification. Assume that $t \geq 3$ and that at least one of the sets $S_1, ..., S_{t-2}$ is not stable. Then there exist disjoint sets $N', X' \subseteq V_H$ such that $N \subseteq N'$ and $X' \subseteq S_t$, and a vertex $x' \in S_t$ such that $(H[N' \cup X'], x_0, x')$ is an $(m)_{n+1}$-alloy of potential greater than $\beta$ and with partition $(N', X')$.*

*Proof.* First, as pointed out above, we know that $\chi(H[S_t]) > m\alpha + \beta$. Now, let $r \in \{1, ..., t-2\}$ be minimal with the property that $S_r$ is not stable; fix adjacent $a, b \in S_r$. Let $p \in \{0, ..., r-1\}$ be maximal with the property that there exists some $z \in S_p$ such that for each $d \in \{a, b\}$, there exists a monotonic path $P_d$ between $z$ and $d$ (such an index $p$ and a vertex $z$ exist because $x_0 \in S_0$ and there exist monotonic paths between $x_0$ and $a$ and between $x_0$ and $b$). Since $S_0, ..., S_{r-1}$ are all stable, this means that $H[V_{P_a} \cup V_{P_b}]$ is a chordless cycle, and by construction, $(V_{P_a} \cup V_{P_b}) \cap S_p = \{z\}$ and $(V_{P_a} \cup V_{P_b}) \cap S_r = \{a, b\}$. Next, let $Q$ be a monotonic path between $x$ and $z$. By the minimality of $S_r$, there exists some $s_{r+1} \in S_{r+1}$ that is adjacent to $a$ and non-adjacent to $b$. Now, fix some $s_{t-1} \in S_{t-1}$ such that there exists a monotonic path $R$ between $s_{r+1}$ and $s_{t-1}$ (the existence of $s_{t-1}$ follows from the fact that for all $i \in \{0, ..., t-1\}$ and $v \in S_i$, $v$ has a neighbor in $S_{i+1}$). Since $s_{t-1}$ has a neighbor in $S_t$, and since $\chi(H[S_t]) > m\alpha$, we can apply 6.4.3 to the vertex $s_{t-1}$ and the set $S_t$ to obtain vertices $u_1, ..., u_m \in S_t$ and a set $X' \subseteq S_t \smallsetminus \{u_1, ..., u_m\}$ such that the following hold:

- $s_{t-1} - u_1 - ... - u_m$ is an induced path in $G$;
- $u_m$ has a neighbor in $X'$;
- vertices $s_{t-1}, u_1, ..., u_{m-1}$ are anti-complete to $X'$;
- $H[X']$ is connected;
- $\chi(H[X']) \geq \chi(H[S_t]) - m\alpha$.

Set $N' = N \cup V_Q \cup V_{P_a} \cup V_{P_b} \cup V_R \cup \{u_1, ..., u_m\}$ and $x' = u_m$. Clearly then, $(H[N' \cup X'], x_0, x')$ is an $(m)_{n+1}$-alloy with partition $(N', X')$. Since $\chi(H[X']) \geq \chi(H[S_t]) - m\alpha$ and $\chi(H[S_t]) > m\alpha + \beta$, we get that $\chi(H[X']) > \beta$. This completes the argument. □

**6.4.7.** *Let $n$ and $\beta$ be non-negative integers, and let $m$ and $\alpha$ be positive integers. Let $(G, x_0, x)$ be an $(m)_n$-alloy of potential greater than $2((m+3)\alpha + \beta)$, and let $(N, X)$ be the partition of the alloy $(G, x_0, x)$. Assume that $\chi_l(G) \leq \alpha$. Let $(H, x_0, x)$ be a reduction of the $(m)_n$-alloy $(G, x_0, x)$, and let $\{S_i\}_{i=0}^t$ be the associated stratification. Assume that the sets $S_1, ..., S_{t-2}$ are all stable. Then there exist disjoint sets $N', X' \subseteq V_H$ such that $N \subseteq N'$ and $X' \subseteq S_t$, and a vertex $x' \in S_t$ such that $(H[N' \cup X'], x_0, x')$ is an $(m)_{n+1}$-alloy of potential greater than $\beta$ and with partition $(N', X')$.*

*Proof.* First, since the potential of the alloy $(G, x_0, x)$ is greater than $2((m+3)\alpha + \beta)$, we know that $\chi(H[S_t]) > (m+3)\alpha + \beta$. Next, fix $a \in S_{t-1}$, and set $A = S_t \cap \Gamma_H(a)$. Note that $\chi(H[S_t]) > 2\alpha$, and so we can apply 6.4.3 to the vertex $a$ and the set $S_t$ in $H$ to obtain vertices $u_0', u_1' \in S_t$ and a non-empty set $C \subseteq S_t \smallsetminus \{u_0', u_1'\}$ such that $a - u_0' - u_1'$

is an induced path in $H$, $a$ and $u_0'$ are anti-complete to $C$ (note that this implies that $C \cap A = \emptyset$), $u_1'$ has a neighbor in $C$, $H[C]$ is connected, and

$$\begin{aligned}\chi(H[C]) &\geq \chi(H[S_t]) - 2\alpha \\ &> ((m+3)\alpha + \beta) - 2\alpha \\ &= (m+1)\alpha + \beta.\end{aligned}$$

Now, fix some $b \in S_{t-1}$ adjacent to $u_1'$; since $a$ is not adjacent to $u_1'$, this means that $a \neq b$. Set $B = S_t \cap \Gamma_H(b)$; clearly, $u_1' \in B$. Since $\chi(H[C]) > \alpha$ and $\chi(H[B]) \leq \alpha$, we know that $C \not\subseteq B$; let $U$ be the vertex-set of a component of $H[C \setminus B]$ with maximum chromatic number. Then

$$\begin{aligned}\chi(H[C]) &\leq \chi(H[B]) + \chi(H[U]) \\ &\leq \alpha + \chi(H[U]),\end{aligned}$$

and so $\chi(H[U]) > m\alpha + \beta$. Note that by construction, neither $A$ nor $B$ intersects $U$.

Let us define a path of *type one* in $H$ to be an induced path $u_0 - ... - u_p$ (with $p \geq 1$) in $H[S_t \setminus U]$ such that $u_0 \in A \cup B$, exactly one vertex among $u_1, ..., u_p$ is in $A \cup B$, $u_p$ has a neighbor in $U$, and $u_0, ..., u_{p-1}$ are all anti-complete to $U$. We define a path of *type two* in $H$ to be an induced path $u_0 - ... - u_p$ (with $p \geq 1$) in $H[S_t \setminus U]$ such that $u_0 = u_0'$, no vertex among $u_1, ..., u_p$ lies in $A \cup B$ (in particular, $u_1' \notin \{u_1, ..., u_p\}$), $u_p$ has a neighbor in $U$, vertices $u_0, ..., u_{p-1}$ are all anti-complete to $U$, and $u_1'$ is complete to $\{u_0, u_1\}$ and anti-complete to $\{u_2, ..., u_p\} \cup U$.

Our goal now is to show that $H$ contains a path of type one or two. Suppose that there is no path of type one in $H$. Since $H[S_t]$ is connected, and $u_0'$ is anti-complete to $U$, there exists an induced path $u_0 - ... - u_p$ (with $p \geq 1$) in $H[S_t \setminus U]$ such that $u_0 = u_0'$, $u_p$ has a neighbor in $U$, and vertices $u_0, ..., u_{p-1}$ are anti-complete to $U$. Note that $u_0 \in A$ (because $u_0 = u_0'$ and $u_0' \in A$). Clearly then, $u_1, ..., u_p \notin A \cup B$, for otherwise, at least two vertices among $u_0, u_1, ..., u_p$ would lie in $A \cup B$, and then $u_{p'} - u_{p'+1} - ... - u_p$ would be a path of type one in $H$ for $p' \in \{0, ..., p-1\}$ chosen maximal with the property that at least two vertices among $u_{p'}, u_{p'+1}, ..., u_p$ lie in $A \cup B$. Since $u_0 = u'_0$ and $u_1, ..., u_p \notin A \cup B$, we know that $u'_1 \notin \{u_0, ..., u_p\}$. Next, note that $u'_1$ is anti-complete to $U$, for otherwise, $u'_0 - u'_1$ would be a path of type one in $H$. Further, $u'_1$ is anti-complete to $\{u_2, ..., u_p\}$, for otherwise, we let $p' \in \{2, ..., p\}$ be maximal with the property that $u'_1$ is adjacent to $u_{p'}$, and we observe that $u'_0 - u'_1 - u_{p'} - u_{p'+1} - ... - u_p$ is a path of type one in $H$. Finally, $u'_1$ is adjacent to $u_1$, for otherwise, $u'_1 - u_0 - u_1 - ... - u_p$ would be a path of type one in $H$. Thus, $u_0 - ... - u_p$ is a path of type two in $H$. This proves that $H$ contains a path of type one or two.

Let $u_0 - ... - u_p$ (with $p \geq 1$) be a path of type one or two in $H$. Recall that $\chi(H[U]) > m\alpha + \beta$. We now apply 6.4.3 to the vertex $u_p$ and the set $U$ in $H$ to obtain vertices $u_{p+1}, ..., u_{p+m} \in U$ and a set $X' \subseteq U \setminus \{u_{p+1}, ..., u_{p+m}\}$ such that the following hold:

- $u_p - u_{p+1} - ... - u_{p+m}$ is an induced path in $H$;
- $u_{p+m}$ has a neighbor in $X'$;
- vertices $u_p, ..., u_{p+m-1}$ are anti-complete to $X'$;
- $H[X']$ is connected;
- $\chi(H[X']) \geq \chi(H[U]) - m\alpha$;

note that the last condition, together with the fact that $\chi(H[U]) > m\alpha + \beta$, implies that $\chi(H[X']) > \beta$. Set $x' = u_{p+m}$. Our goal is to construct a set $N'$ with $N \subseteq N'$ such that $(H[N' \cup X'], x_0, x')$ is an $(m)_{n+1}$-alloy with partition $(N', X')$. Since $\chi(H[X']) > \beta$, the potential of any such alloy is greater than $\beta$, as desired.

First, if $u_0 - ... - u_p$ is a path of type two in $H$, then we let $P$ be a monotonic path between $a$ and $x$, we set $N' = N \cup V_P \cup \{u_0, ..., u_{p+m}\} \cup \{u'_1\}$, and we are done. From now on, we assume that $u_0 - ... - u_p$ is a path of type one in $H$. Fix $l \in \{1, ..., p\}$

such that $u_l \in A \cup B$; then by the definition of a path of type one in $H$, we get that $u_0, u_l \in A \cup B$, and no other vertex on the path $u_0 - ... - u_p$ lies in $A \cup B$. If some vertex $d \in \{a, b\}$ is complete to $\{u_0, u_l\}$, then we let $P$ be a monotonic path between $x$ and $d$, we set $N' = N \cup V_P \cup \{u_0, ..., u_{p+m}\}$, and we are done. From now on, we assume that neither $a$ nor $b$ is complete to $\{u_0, u_l\}$. Then one of $a$ and $b$ is adjacent to $u_0$ and non-adjacent to $u_l$, and the other is adjacent to $u_l$ and non-adjacent to $u_0$. Now, fix maximal $q \in \{0, ..., t-2\}$ such that there exists a vertex $z \in S_q$ with the property that for each $d \in \{a, b\}$, there exists a monotonic path $P_d$ between $z$ and $d$. Since $S_0, ..., S_{t-2}$ are all stable, we get that if $a$ and $b$ are adjacent then $H[V_{P_a} \cup V_{P_b}]$ is a chordless cycle, and if $a$ and $b$ are non-adjacent then $H[V_{P_a} \cup V_{P_b}]$ is an induced path between $a$ and $b$; in either case, we have that $(V_{P_a} \cup V_{P_b}) \cap S_{t-1} = \{a, b\}$ and $(V_{P_a} \cup V_{P_b}) \cap S_q = \{z\}$. Let $Q$ be a monotonic path between $z$ and $x$. Now, if $a$ and $b$ are adjacent, then we set $N' = N \cup V_Q \cup V_{P_a} \cup V_{P_b} \cup \{u_l, u_{l+1}, ..., u_{p+m}\}$; and if $a$ and $b$ are non-adjacent, then we set $N' = V_Q \cup V_{P_a} \cup V_{P_b} \cup \{u_0, ..., u_{p+m}\}$. This completes the argument. □

We can now prove 6.4.5, restated below.

**6.4.5.** *Let $n$ and $\beta$ be non-negative integers, and let $m$ and $\alpha$ be positive integers. Let $(G, x_0, x)$ be an $(m)_n$-alloy of potential greater than $2((m+3)\alpha + \beta)$, and let $(N, X)$ be the partition of the alloy $(G, x_0, x)$. Assume that $\chi_l(G) \leq \alpha$. Then there exist disjoint sets $N', X' \subseteq V_G$ such that $N \subseteq N'$ and $X' \subseteq X$, and a vertex $x' \in X$ such that $(G[N' \cup X'], x_0, x')$ is an $(m)_{n+1}$-alloy of potential greater than $\beta$ and with partition $(N', X')$.*

*Proof.* Let $(H, x_0, x)$ be a reduction of the $(m)_n$-alloy $(G, x_0, x)$, and let $\{S_i\}_{i=0}^t$ be the associated stratification. If $t \geq 3$ and at least one of the sets $S_1, ..., S_{t-2}$ is not stable, then the result follows from 6.4.6. Otherwise, the result follows from 6.4.7. □

Finally, we use 6.4.4 and 6.4.5 to prove 6.4.1, restated below.

**6.4.1.** *Let $G$ be a connected graph, and let $x_0 \in V_G$. Let $n$ and $\beta$ be non-negative integers, and let $m$ and $\alpha$ be positive integers. Assume that $\chi_l(G) \leq \alpha$ and $\chi(G) > 2^{n+1}((m+$*

$3)\alpha+\beta)$. *Then there exists an induced subgraph $H$ of $G$ and a vertex $x \in V_G$ such that $(H, x_0, x)$ is an $(m)_n$-alloy of potential greater than $\beta$.*

*Proof.* For all $j \in \{0, ..., n\}$, set $\beta_j = \beta + (\Sigma_{i=1}^{n-j} 2^i)((m+3)\alpha+\beta)$. Our goal is to prove inductively that for all $j \in \{0, ..., n\}$, there exist disjoint sets $N_j, X_j \subseteq V_G$ and a vertex $x^j \in V_G$ such that $(G[N_j \cup X_j], x_0, x^j)$ is an $(m)_j$-alloy of potential greater than $\beta_j$. Since $\beta_n = \beta$, the result will follow.

For the base case (when $j = 0$), we observe that

$$\begin{aligned}
\chi(G) &> 2^{n+1}((m+3)\alpha+\beta) \\
&> (\Sigma_{i=0}^{n} 2^i)((m+3)\alpha+\beta) \\
&= (m+3)\alpha+\beta+(\Sigma_{i=1}^{n} 2^i)((m+3)\alpha+\beta) \\
&= (m+3)\alpha+\beta_0 \\
&> (m+1)\alpha+\beta_0,
\end{aligned}$$

and so 6.4.4 implies that there exist sets $N_0, X_0 \subseteq V_G$ and a vertex $x^0 \in V_G$ such that $(G[N_0 \cup X_0], x_0, x^0)$ is an $(m)_0$-alloy of potential greater than $\beta_0$.

For the induction step, suppose that $j \in \{0, ..., n-1\}$ and that there exist disjoint sets $N_j, X_j \subseteq V_G$ and a vertex $x^j \in V_G$ such that $(G[N_j \cup X_j], x_0, x^j)$ is an $(m)_j$-alloy of potential greater than $\beta_j$. Since

$$\begin{aligned}
\beta_j &= \beta + (\Sigma_{i=1}^{n-j} 2^i)((m+3)\alpha+\beta) \\
&\geq (\Sigma_{i=1}^{n-j} 2^i)((m+3)\alpha+\beta) \\
&= 2((m+3)\alpha+\beta+(\Sigma_{i=1}^{n-(j+1)} 2^i)((m+3)\alpha+\beta)) \\
&= 2((m+3)\alpha+\beta_{j+1}),
\end{aligned}$$

6.4.5 implies that there exist sets $N_{j+1}, X_{j+1} \subseteq V_G$ and a vertex $x^{j+1}$ such that $(G[N_{j+1} \cup X_{j+1}], x_0, x^{j+1})$ is an $(m)_{j+1}$-alloy of potential greater than $\beta_{j+1}$. This completes the induction. □

# Chapter 7

# Substitution and $\chi$-Boundedness

Recall from section 2.1 that a class $\mathcal{G}$ of graphs is said to be $\chi$-*bounded* provided that there exists a function $f : \mathbb{N}_0 \to \mathbb{R}$ such that for all graphs $G \in \mathcal{G}$, and all induced subgraphs $H$ of $G$, $\chi(H) \leq f(\omega(H))$. Under these circumstances, we say that the class $\mathcal{G}$ is $\chi$-*bounded* by the function $f$, and that $f$ is a $\chi$-*bounding function* for $\mathcal{G}$. Note that if $f$ is a $\chi$-bounding function for $\mathcal{G}$, then so is the function $g : \mathbb{N}_0 \to \mathbb{R}$ given by $n \mapsto \lfloor \max\{f(0), ..., f(n)\} \rfloor$. Thus, we may assume that every $\chi$-bounding function is non-decreasing, and (when convenient) that it is integer-valued. We also remark that if $\mathcal{G}$ is a *hereditary* class (i.e. a class closed under isomorphism and induced subgraphs), then $\mathcal{G}$ is $\chi$-bounded if and only if there exists a function $f : \mathbb{N}_0 \to \mathbb{R}$ such that for all $G \in \mathcal{G}$, $\chi(G) \leq f(\omega(G))$.

In this chapter, we consider several operations (namely, "substitution," "gluing along a clique," and "gluing along a bounded number of vertices"), and we show that the closure of a $\chi$-bounded class under any one of them (as well as under certain combinations of those operations) is again $\chi$-bounded. We begin with the precise definitions of these three operations.

The usual definitions of substitution was given in section 2.1, however, we will also need a slightly different definition, to which we now turn. Given a non-empty graph $G_0$ with vertex-set $V_G = \{v_1, ..., v_t\}$ and non-empty graphs $G_1, ..., G_t$ with pairwise disjoint vertex-sets, we say that a graph $G$ is obtained by *substituting* $G_1, ..., G_t$ for $v_1, ..., v_t$ in $G_0$ provided that the following hold:

- $V_G = \bigcup_{i=1}^{t} V_{G_i}$;
- for all $i \in \{1, ..., t\}$, $G[V_{G_i}] = G_i$;
- for all distinct $i, j \in \{1, ..., t\}$, if $v_i$ is adjacent (respectively: non-adjacent) to $v_j$ in $G_0$, then $V_{G_i}$ is complete (respectively: anti-complete) to $V_{G_j}$ in $G$.

Unless specified otherwise, "substitution" means substitution of one graph for a vertex of another graph, i.e. the kind of substitution that we defined in section 2.1. However, it is easy to see that for hereditary classes, the two kinds of substitution that we have defined are equivalent in the following sense: a hereditary class $\mathcal{G}$ is closed under one kind of substitution if and only if it is closed under the other kind of substitution. (This follows from the fact that every hereditary class $\mathcal{G}$ that contains even one non-empty graph contains all single-vertex graphs, and these may be substituted for some vertices of a graph in the kind of substitution that we just defined. This, however, is not the case for general classes of graphs.) We observe that substitution preserves hereditariness in the following sense: the closure of a hereditary class under substitution is again hereditary.

Next, we define a certain "gluing operation" as follows. Let $G_1$ and $G_2$ be non-empty graphs with inclusion-wise incomparable vertex-sets, and let $C = V_{G_1} \cap V_{G_2}$. Assume that $C$ is a proper (possibly empty) subset of both $V_{G_1}$ and $V_{G_2}$, and that $G_1[C] = G_2[C]$. Let $G$ be a graph such that $V_G = V_{G_1} \cup V_{G_2}$, with adjacency as follows:

- $G[V_{G_1}] = G_1$;
- $G[V_{G_2}] = G_2$;
- $V_{G_1} \smallsetminus C$ is anti-complete to $V_{G_2} \smallsetminus C$ in $G$.

We then say that $G$ is obtained by *gluing* $G_1$ *and* $G_2$ *along* $C$. Under these circumstances, we also say that $G$ is obtained by gluing $G_1$ and $G_2$ along $|C|$ vertices. If $C$ is, in addition, a (possibly empty) clique in both $G_1$ and $G_2$, then we say that $G$ is obtained from $G_1$ and $G_2$ by *gluing along a clique*. We observe that gluing two graphs with disjoint vertex-sets along the empty set (equivalently: along the empty clique) simply amounts to taking the disjoint union of the two graphs; thus, if a hereditary class $\mathcal{G}$ is closed under gluing along a clique, then $\mathcal{G}$ is also closed under taking disjoint unions.

Given a positive integer $k$ and a class $\mathcal{G}$ of graphs, we say that $\mathcal{G}$ is *closed under gluing along at most $k$ vertices* provided that for all non-empty graphs $G_1, G_2 \in \mathcal{G}$ with inclusion-wise incomparable vertex-sets, if $G_1[V_{G_1} \cap V_{G_2}] = G_2[V_{G_1} \cap V_{G_2}]$ and $|V_{G_1} \cap V_{G_2}| \leq k$, then the graph obtained by gluing $G_1$ and $G_2$ along $V_{G_1} \cap V_{G_2}$ is a member of $\mathcal{G}$.

We observe that (like substitution) the operation of gluing along a clique preserves hereditariness, as does the operation of gluing along a bounded number of vertices.

The chapter is organized as follows. In section 7.1, we show that the closure of a $\chi$-bounded class under substitution is again $\chi$-bounded (see 7.1.2), and we also examine the effects of substitution on $\chi$-bounding functions. In particular, we show the following: if a class $\mathcal{G}$ is $\chi$-bounded by a polynomial function $P$, then there exists a polynomial function $Q$ such that the closure of $\mathcal{G}$ under substitution is $\chi$-bounded by $Q$ (see 7.1.3). Interestingly, the degree of $Q$ cannot be bounded by any function of the degree of $P$ (see 7.1.4). Further, we prove that if a class $\mathcal{G}$ is $\chi$-bounded by an exponential function, then the closure of $\mathcal{G}$ under substitution is also $\chi$-bounded by some exponential function (see 7.1.6).

In section 7.2, we turn to the two gluing operations. It is easy to show that the closure of a $\chi$-bounded class under gluing along a clique is $\chi$-bounded (see 7.2.1). Next, we show that the closure of a $\chi$-bounded class under gluing along at most $k$ vertices (where $k$ is a fixed positive integer) is $\chi$-bounded (see 7.2.2). We note that this answers an open question from [25]. In [25], Cicalese and Milanič ask whether for some fixed $k$, the class of graphs of separability at most $k$ is $\chi$-bounded, where a graph has *separability at most* $k$ if every two non-adjacent vertices are separated by a set of at most $k$ other vertices. Since graphs of separability at most $k$ form a subclass of the closure of the class of all complete graphs under gluing along at most $k$ vertices, 7.2.2 implies that graphs of separability at most $k$ are $\chi$-bounded by the linear function $f(x) = x + 2k^2 - 1$. We also note that the fact that the closure of a $\chi$-bounded class under gluing along at most $k$ vertices is again $\chi$-bounded also follows from an earlier (and more general) result due to a group of authors [1]. However, the proof presented in this thesis is significantly different from the one given in [1], and furthermore, the $\chi$-bounding function that we obtained is better than the one that can be derived using the result from [1] (see section 7.2 for a more detailed explanation). In section 7.2, we also show that the closure of a $\chi$-bounded class under both of our gluing operations (gluing along a clique and gluing along at most $k$ vertices) together is $\chi$-bounded (see 7.2.6). At the end of the section, we prove that that the closure of a $\chi$-bounded class under substitution and gluing along a clique together is $\chi$-bounded (see 7.2.7, as well as 7.2.11 for a strengthening of 7.2.7 in some special cases).

Finally, in section 7.3, we state some open questions related to $\chi$-boundedness.

## 7.1 Substitution

Given a class $\mathcal{G}$ of graphs, we denote by $\mathcal{G}^+$ the closure of $\mathcal{G}$ under taking disjoint unions, and we denote by $\mathcal{G}^*$ the closure of $\mathcal{G}$ under taking disjoint unions and substitution. In this section, we show that if $\mathcal{G}$ is a $\chi$-bounded class, then the class $\mathcal{G}^*$ is also $\chi$-bounded (see 7.1.2). We then improve on this result in a number of special cases: when the $\chi$-bounding function for $\mathcal{G}$ is polynomial (see 7.1.3), when it is supermultiplicative (see 7.1.5), and when it is exponential (see 7.1.6).

### 7.1.1 Substitution Depth and $\chi$-Boundedness

Let $\mathcal{G}$ be a hereditary class. We note that if $\mathcal{G}$ contains even one non-empty graph, then $\mathcal{G}^+$ contains all the edgeless graphs; we also note that if $\mathcal{G}$ is $\chi$-bounded by a non-decreasing function $f$, then $\mathcal{G}^+$ is also $\chi$-bounded by $f$. We observe that every graph $G \in \mathcal{G}^* \smallsetminus \mathcal{G}^+$ can be obtained from a graph $G_0 \in \mathcal{G}^+$ with vertex-set $V_{G_0} = \{v_1, ..., v_t\}$ (where $2 \leq t \leq |V_G| - 1$) and non-empty graphs $G_1, ..., G_t \in \mathcal{G}^*$ with pairwise disjoint vertex-sets by substituting $G_1, ..., G_t$ for $v_1, ..., v_t$ in $G_0$. We now define the *substitution depth* of the graphs $G \in \mathcal{G}^*$ with respect to $\mathcal{G}$, denoted by $d_{\mathcal{G}}(G)$, as follows. If $G$ is the empty graph, then set $d_{\mathcal{G}}(G) = -1$. For all non-empty graphs $G \in \mathcal{G}^+$, set $d_{\mathcal{G}}(G) = 0$. Next, let $G \in \mathcal{G}^* \smallsetminus \mathcal{G}^+$, and assume that $d_{\mathcal{G}}(G')$ has been defined for every graph $G' \in \mathcal{G}^* \smallsetminus \mathcal{G}^+$ with at most $|V_G| - 1$ vertices. Then we define $d_{\mathcal{G}}(G)$ to be the smallest non-negative integer $r$ such that there exist non-empty graphs $G_1, ..., G_t \in \mathcal{G}^*$ (where $2 \leq t \leq |V_G| - 1$) with pairwise disjoint vertex-sets, and a graph $G_0 \in \mathcal{G}^+$ with vertex-set $V_{G_0} = \{v_1, ..., v_t\}$, where $v_1, ..., v_s$ (for some $s \in \{0, ..., t\}$) are isolated vertices in $G_0$ and each of $v_{s+1}, ..., v_t$ has a neighbor in $G_0$, such that $G$ is obtained by substituting $G_1, ..., G_t$ for $v_1, ..., v_t$ in $G_0$, and

$$r = \max(\{d_{\mathcal{G}}(G_1), ..., d_{\mathcal{G}}(G_s)\} \cup \{d_{\mathcal{G}}(G_{s+1}) + 1, ..., d_{\mathcal{G}}(G_t) + 1\}).$$

We observe that the fact that $\mathcal{G}$ is hereditary implies that $d_{\mathcal{G}}(H) \leq d_{\mathcal{G}}(G)$ for all graphs $G \in \mathcal{G}^*$, and all induced subgraphs $H$ of $G$. We now prove a technical lemma.

**7.1.1.** *Let $\mathcal{G}$ be a hereditary class, $\chi$-bounded by a non-decreasing function $f : \mathbb{N}_0 \to \mathbb{R}$. Then for all $G \in \mathcal{G}^*$, we have that $\omega(G) \geq d_{\mathcal{G}}(G) + 1$ and $\chi(G) \leq f(\omega(G))^{d_{\mathcal{G}}(G)+1}$.*

*Proof.* We proceed by induction on the number of vertices. Fix $G \in \mathcal{G}^*$, and assume that the claim holds for graphs in $\mathcal{G}^*$ that have fewer vertices than $G$. If $G \in \mathcal{G}^+$, then the result is immediate, so assume that $G \notin \mathcal{G}^+$. Fix $G_0 \in \mathcal{G}^+$ with vertex-set $V_{G_0} = \{v_1, ..., v_t\}$ (with $2 \leq t \leq |V_G| - 1$), where $v_1, ..., v_s$ (with $s \in \{0, ..., t\}$) are isolated vertices in $G_0$ and each of $v_{s+1}, ..., v_t$ has a neighbor in $G_0$, and non-empty graphs $G_1, ..., G_t \in \mathcal{G}^*$ such that $G$ is obtained by substituting $G_1, ..., G_t$ for $v_1, ..., v_t$ in $G_0$, and $d_{\mathcal{G}}(G) = \max(\{d_{\mathcal{G}}(G_1), ..., d_{\mathcal{G}}(G_s)\} \cup \{d_{\mathcal{G}}(G_{s+1}) + 1, ..., d_{\mathcal{G}}(G_t) + 1\})$.

We first show that $\omega(G) \geq d_{\mathcal{G}}(G) + 1$. We need to show that $\omega(G) \geq d_{\mathcal{G}}(G_i) + 1$ for all $i \in \{1, ..., s\}$, and that $\omega(G) \geq d_{\mathcal{G}}(G_i) + 2$ for all $i \in \{s + 1, ..., t\}$. By the induction hypothesis, we have that $\omega(G_i) \geq d_{\mathcal{G}}(G_i) + 1$ for all $i \in \{1, ..., t\}$, and so it suffices to show that $\omega(G) \geq \omega(G_i)$ for all $i \in \{1, ..., s\}$, and that $\omega(G) \geq \omega(G_i) + 1$ for all $i \in \{s+1, ..., t\}$. The former follows from the fact that $G_i$ is an induced subgraph of $G$ for all $i \in \{1, ..., s\}$. For the latter, fix $i \in \{s + 1, ..., t\}$, and let $K$ be a clique of size $\omega(G_i)$ in $G_i$. Let $v_j$ be a neighbor of $v_i$ in $G_0$. Now fix $k \in V_{G_j}$, and note that $K \cup \{k\}$ is a clique of size $\omega(G_i) + 1$ in $G$.

It remains to show that $\chi(G) \leq f(\omega(G))^{d_{\mathcal{G}}(G)+1}$. Since $\chi(H)$ is non-negative integer for every graph $H$, we know that the class $\mathcal{G}$ is $\chi$-bounded by the function given by $n \mapsto \max\{\lfloor f(n) \rfloor, 0\}$; thus, we may assume without loss of generality that $f(n)$ is a non-negative integer for all $n \in \mathbb{N}_0$. Note that $V_{G_i}$ is anti-complete to $V_G \smallsetminus V_{G_i}$ for all $i \in \{1, ..., s\}$. Thus, it suffices to show that $\chi(G_i) \leq f(\omega(G))^{d_{\mathcal{G}}(G)+1}$ for all $i \in \{1, ..., s\}$, and that $\chi(G[\bigcup_{i=s+1}^t V_{G_i}]) \leq f(\omega(G))^{d_{\mathcal{G}}(G)+1}$. The former is immediate from the induction hypothesis. For the latter, we use the induction hypothesis to assign a proper coloring $b_i : V_{G_i} \rightarrow \{1, ..., f(\omega(G))^{d_{\mathcal{G}}(G)}\}$ to $G_i$ for each $i \in \{s + 1, ..., t\}$. Next, we use the fact that $G_0 \in \mathcal{G}^+$ and that $\mathcal{G}$ (and therefore $\mathcal{G}^+$ as well) is $\chi$-bounded by $f$ in order to assign a proper coloring $b_0 : V_{G_0} \rightarrow \{1, ..., f(\omega(G))\}$ to $G_0$. Now define $b : V_{G[\bigcup_{i=s+1}^t V_{G_i}]} \rightarrow \{1, ..., f(\omega(G))\} \times \{1, ..., f(\omega(G))^{d_{\mathcal{G}}(G)}\}$ by setting $b(v) = (b_0(v_i), b_i(v))$ for all $i \in \{s + 1, ..., t\}$ and $v \in V_{G_i}$. This is clearly a proper coloring of $G[\bigcup_{i=s+1}^t V_{G_i}]$ that uses at most $f(\omega(G))^{d_{\mathcal{G}}(G)+1}$ colors. $\square$

As an immediate corollary, we have the following.

**7.1.2.** *Let $\mathcal{G}$ be a class of graphs, $\chi$-bounded by a non-decreasing function $f : \mathbb{N}_0 \to \mathbb{R}$. Then the class $\mathcal{G}^*$ is $\chi$-bounded by the function $g(k) = f(k)^k$.*

*Proof.* We may assume that $\mathcal{G}$ is hereditary, because otherwise, instead of considering $\mathcal{G}$, we consider the closure $\tilde{\mathcal{G}}$ of $\mathcal{G}$ under isomorphism and taking induced subgraphs. (We may do this because $\tilde{\mathcal{G}}$ is readily seen to be hereditary and $\chi$-bounded by $f$, and furthermore, $\mathcal{G}^* \subseteq \tilde{\mathcal{G}}^*$, and so if $\tilde{\mathcal{G}}^*$ is $\chi$-bounded by $g$, then so is $\mathcal{G}^*$.)

We may assume that $f(0) \geq 0$ and that $f(k) \geq 1$ for all $k \in \mathbb{N}$, for otherwise, $\mathcal{G}$ contains no non-empty graphs, and the result is immediate. Next, if $H$ is the empty graph, then $\chi(H) = 0 \leq 1 = f(\omega(H))^{\omega(H)}$. Finally, suppose that $G \in \mathcal{G}^*$ is a non-empty graph. Now, by 7.1.1, we have that $\chi(G) \leq f(\omega(G))^{d_{\mathcal{G}}(G)+1}$ and $d_{\mathcal{G}}(G) + 1 \leq \omega(G)$; since $f(\omega(G)) \geq 1$, it follows that $\chi(G) \leq f(\omega(G))^{\omega(G)}$. $\square$

### 7.1.2 Polynomial $\chi$-Bounding Functions

We now turn to the special case when a hereditary class $\mathcal{G}$ is $\chi$-bounded by a polynomial function.

**7.1.3.** *Let $\mathcal{G}$ be a $\chi$-bounded class. If $\mathcal{G}$ has a polynomial $\chi$-bounding function, then so does $\mathcal{G}^*$.*

*Proof.* We may assume that $\mathcal{G}$ is hereditary (otherwise, instead of $\mathcal{G}$, we consider the closure of $\mathcal{G}$ under isomorphism and taking induced subgraphs). Further, we may assume that $\mathcal{G}$ is $\chi$-bounded by the function $f(x) = x^A$ for some $A \in \mathbb{N}$. Set $g(x) = x^{3A+11}$, and set $B = 2A + 11$, so that $g(x) = x^{A+B}$. Our goal is to show that $\mathcal{G}^*$ is $\chi$-bounded by the function $g$. Fix a graph $G \in \mathcal{G}^*$, set $d_{\mathcal{G}}(G) = t$, and assume inductively that for every graph $G' \in \mathcal{G}^*$ with $d_{\mathcal{G}}(G') < t$, we have that $\chi(G') \leq g(\omega(G'))$. Set $\omega = \omega(G)$. We need to show that $\chi(G) \leq g(\omega)$. If $G$ is the empty graph, then the result is immediate; so we may assume that $G$ is a non-empty graph.

By 7.1.1, if $t \leq 2$, then $\chi(G) \leq f(\omega(G))^3 \leq g(\omega(G))$, and we are done. So from now on, we assume that $t \geq 3$. 7.1.1 then implies that $\omega \geq 4$. Next, since $d_{\mathcal{G}}(H) \leq d_{\mathcal{G}}(G)$ for every induced subgraph $H$ of $G$, we may assume that $G$ is connected (for otherwise, we deal with the components of $G$ separately). Thus, there exists a connected graph $F \in \mathcal{G}$ with vertex-set $V_F = \{v_1, ..., v_n\}$ (with $n \geq 2$), and non-empty graphs $B_1, ..., B_n \in \mathcal{G}^*$, with $d_{\mathcal{G}}(B_i) < t$ for all $i \in \{1, ..., n\}$, such that $G$ is obtained by substituting $B_1, ..., B_n$ for $v_1, ..., v_n$ in $F$. For all $i \in \{1, ..., n\}$, set $\omega_i = \omega(B_i)$. Note that by the induction hypothesis, we have that $\chi(B_i) \leq g(\omega_i)$ for all $i \in \{1, ..., n\}$. We observe that if $v_i, v_j \in V_F$ are adjacent, then $\omega_i + \omega_j \leq \omega$; since $F$ contains no isolated vertices, it follows that $\omega_i \leq \omega - 1$ for all $i \in \{1, ..., n\}$.

Fix $\alpha \in [\frac{5}{4}, \frac{3}{2}]$ such that $\alpha^m = \frac{\omega}{2}$ for some $m \in \mathbb{N}$; such an $\alpha$ exists because $\{\hat{\alpha}^k \mid k \in \mathbb{N}, \hat{\alpha} \in [\frac{5}{4}, \frac{3}{2}]\} = [\frac{5}{4}, \frac{3}{2}] \cup [\frac{25}{16}, +\infty)$ and $\frac{\omega}{2} \geq 2$. We now define:

$$
\begin{array}{lcl}
V_0 & = & \{v_i \mid \omega_i > \frac{\omega}{2}\}, \\
V_j & = & \{v_i \mid \omega_i \in (\frac{\omega}{2\alpha^j}, \frac{\omega}{2\alpha^{j-1}}]\}, \quad 1 \leq j \leq m, \\
V_{m+1} & = & \{v_i \mid \omega_i = 1\},
\end{array}
$$

so that the sets $V_0, V_1, ..., V_{m+1}$ are pairwise disjoint with $V_F = \bigcup_{j=0}^{m+1} V_j$. For each $j \in \{0, ..., m+1\}$, set $F_j = F[V_j]$, and let $G_j$ be the corresponding induced subgraph of $G$ (formally: $G_j = G[\bigcup_{v_i \in V_j} V_{B_i}]$).

Note that if $C$ is a clique in $F$, then

$$\omega \geq \Sigma_{v_i \in C}\, \omega_i. \tag{7.1}$$

In particular, $V_0$ is a stable set. Further, for all $j \in \{1, ..., m\}$, if $v_i \in V_j$ then $\omega_i \geq \frac{\omega}{2\alpha^j}$; by (7.1), this implies that $\omega \geq \omega(F_j) \cdot \frac{\omega}{2\alpha^j}$, and so $\omega(F_j) \leq 2\alpha^j$. But now for each

$j \in \{1, ..., m\}$, we have:

$$\begin{aligned} \chi(G_j) &\leq \chi(F_j) \cdot \max_{v_i \in V_j} \chi(B_i) \\ &\leq \chi(F_j) \cdot \max_{v_i \in V_j} g(\omega_i) \\ &\leq f(2\alpha^j) g(\tfrac{\omega}{2\alpha^{j-1}}). \end{aligned} \tag{7.2}$$

We also have that:

$$\chi(G_{m+1}) = \chi(F_{m+1}) \leq f(\omega). \tag{7.3}$$

We now color $G$ as follows:

- we first color each subgraph $G_j$, $j \in \{1, ..., m+1\}$, with a separate set of colors (using in each case only $\chi(G_j)$ colors);
- we then color the subgraphs $B_i$ with $v_i \in V_0$ one at a time, introducing at each step as few new colors as possible.

We need to show that this coloring of $G$ uses at most $g(\omega)$ colors.

From (7.2) and (7.3), we get that coloring the graphs $G_1, ..., G_{m+1}$ together takes at most the following number of colors:

$$\begin{aligned}
\Sigma_{j=1}^{m+1}\chi(G_j) &\leq f(\omega)+\Sigma_{j=1}^{m}f(2\alpha^j)g(\tfrac{\omega}{2\alpha^{j-1}}) \\
&= \omega^A+\Sigma_{j=1}^{m}(2\alpha^j)^A(\tfrac{\omega}{2\alpha^{j-1}})^{3A+11} \\
&= \omega^A+(\alpha\omega)^A\Sigma_{j=1}^{m}(\tfrac{\omega}{2\alpha^{j-1}})^B \\
&= \omega^{A+B}(\omega^{-B}+\tfrac{\alpha^A}{2^B}\Sigma_{j=0}^{m-1}(\alpha^{-B})^j) \\
&\leq \omega^{A+B}(\omega^{-B}+\tfrac{\alpha^A}{2^B}\tfrac{1}{1-\alpha^{-B}}) \\
&= g(\omega)(\omega^{-B}+\tfrac{\alpha^A}{2^B}\tfrac{1}{1-\alpha^{-B}}) \qquad (7.4) \\
&\leq g(\omega)(\tfrac{1}{2^B}+\tfrac{(\frac{3}{2})^A}{2^B}\tfrac{1}{1-(\frac{5}{4})^{-B}}) \\
&\leq g(\omega)(\tfrac{1}{2^B}+\tfrac{(\frac{3}{2})^A}{2^B}\tfrac{1}{1-\frac{4}{5}}) \\
&= g(\omega)\cdot\tfrac{1+5(\frac{3}{2})^A}{2^B} \\
&\leq g(\omega)\cdot\tfrac{6(\frac{3}{2})^A}{2^{2A+11}} \\
&\leq g(\omega).
\end{aligned}$$

Now consider the graphs $B_i$ with $v_i \in V_0$. These are pairwise anti-complete to each other (as $V_0$ is stable). Fix $v_i \in V_0$. It suffices to show that our coloring of $G$ used no more than $g(\omega)$ colors on $B_i$ and all the vertices with a neighbor in $B_i$. Note that if a vertex $v_j$ is adjacent to $v_i$ in $F$, then $V_{B_j}$ is complete to $V_{B_i}$ in $G$, and so $\omega_i+\omega_j \leq \omega$; thus, all neighbors of $v_i$ lie in

$$V_{m+1} \cup \{V_j \mid 1 \leq j \leq m, \frac{\omega}{2\alpha^j} < \omega-\omega_i\}.$$

Let $s_i = \min\{s \in \mathbb{N} \mid \frac{\omega}{2\alpha^s} < \omega-\omega_i\}$; $s_i$ is well-defined because $\omega_i < \omega$. Then using (7.2) and (7.3), we get that the number of colors already used in subgraphs $G_j$ that are not anti-complete to $B_i$ is at most:

$$\begin{aligned}
\chi(G_{m+1}) + \Sigma_{j=s_i}^{m}\chi(G_j) &\leq f(\omega) + \Sigma_{j=s_i}^{m} f(2\alpha^j) g(\tfrac{\omega}{2\alpha^{j-1}}) \\
&= f(\omega) + \Sigma_{j=s_i}^{m} (2\alpha^j)^A (\tfrac{\omega}{2\alpha^{j-1}})^{3A+11} \\
&= f(\omega) + (\alpha\omega)^A \Sigma_{j=s_i}^{m} (\tfrac{\omega}{2\alpha^{j-1}})^B \\
&= f(\omega) + f(\alpha\omega) \Sigma_{j=0}^{m-s_i} (\tfrac{\omega}{2\alpha^{s_i+j-1}})^B \\
&= f(\omega) + f(\alpha\omega) \Sigma_{j=0}^{m-s_i} (\tfrac{\alpha}{\alpha^j} \cdot \tfrac{\omega}{2\alpha^{s_i}})^B \\
&\leq f(\omega) + f(\alpha\omega) \Sigma_{j=0}^{m-s_i} (\tfrac{\alpha(\omega-\omega_i)}{\alpha^j})^B.
\end{aligned} \tag{7.5}$$

Set $p = 1 - \frac{\omega_i}{\omega}$; note that we then have that $p \in [\frac{1}{\omega}, \frac{1}{2})$, as $\frac{\omega}{2} < \omega_i \leq \omega - 1$. Now, we use at most $g(\omega_i) = g((1-p)\omega)$ colors on $B_i$, which together with (7.5) implies that we use at most

$$P = f(\omega) + f(\alpha\omega)\Sigma_{j=0}^{m-s_i}(\frac{\alpha p \omega}{\alpha^j})^B + g((1-p)\omega) \tag{7.6}$$

colors on $B_i$ and all the $G_j$ that are not anti-complete to $B_i$ together; our goal is to show that $P \leq g(\omega)$. Note the following:

$$\begin{aligned}
P &= f(\omega) + f(\alpha\omega)\Sigma_{j=0}^{m-s_i}(\tfrac{\alpha p\omega}{\alpha^j})^B + g((1-p)\omega) \\
&= \omega^A + \alpha^A\omega^A\Sigma_{j=0}^{m-s_i}\tfrac{\alpha^B p^B \omega^B}{\alpha^{jB}} + (1-p)^{A+B}\omega^{A+B} \\
&= \omega^{A+B}(\omega^{-B} + \alpha^{A+B}p^B\Sigma_{j=0}^{m-s_i}\tfrac{1}{(\alpha^B)^j} + (1-p)^{A+B}) \\
&\leq \omega^{A+B}(\omega^{-B} + \alpha^{A+B}p^B\Sigma_{j=0}^{\infty}\tfrac{1}{(\alpha^B)^j} + (1-p)^{A+B}) \\
&= g(\omega)(\omega^{-B} + \tfrac{\alpha^{A+B}p^B}{1-\alpha^{-B}} + (1-p)^{A+B}) \\
&\leq g(\omega)(2\tfrac{\alpha^{A+B}p^B}{1-\alpha^{-B}} + (1-p)^{A+B}).
\end{aligned}$$

(In the last step, we used the fact that $\frac{\alpha^{A+B}}{1-\alpha^{-B}} \geq 1$ and $p \geq \frac{1}{\omega}$.) Thus, in order to show that $P \leq g(\omega)$, it suffices to show that $2\frac{\alpha^{A+B}p^B}{1-\alpha^{-B}} + (1-p)^{A+B} \leq 1$. First, using the fact that $\frac{5}{4} \leq \alpha \leq \frac{3}{2}$ and $0 \leq p \leq \frac{1}{2}$ (and consequently, $\alpha p \leq \frac{3}{4}$), we get that:

$$\begin{aligned}
2\frac{\alpha^{A+B}p^B}{1-\alpha^{-B}} &= 2\alpha^A \frac{(\alpha p)^B}{1-\alpha^{-B}} \\
&\leq 2(\tfrac{3}{2})^A \frac{(\frac{3}{4})^B}{1-\frac{4}{5}} \\
&= 10(\tfrac{3}{2})^A(\tfrac{3}{4})^{2A+11} \\
&= 10(\tfrac{27}{32})^A(\tfrac{3}{4})^{11} \\
&\leq 10 \cdot (\tfrac{3}{4})^{11} \\
&\leq \tfrac{1}{2}.
\end{aligned}$$

On the other hand, we have that $(1-p)^{A+B} \leq e^{-p(A+B)}$, and so if $p \geq \frac{1}{A+B}$, then

$$2\frac{\alpha^{A+B}p^B}{1-\alpha^{-B}} + (1-p)^{A+B} \quad \leq \quad \tfrac{1}{2} + \tfrac{1}{e} < 1,$$

and we are done. So assume that $p < \frac{1}{A+B}$. Note first that:

$$\begin{aligned}
\frac{2\alpha^{A+B}}{1-\alpha^{-B}} &\leq \frac{2(\frac{3}{2})^{A+B}}{1-\frac{4}{5}} \\
&= 10(\tfrac{3}{2})^{3A+11} \\
&= 10(\tfrac{3}{2})^{11}(\tfrac{27}{8})^A \\
&\leq 4^{11} \cdot 4^A \\
&\leq 4^B.
\end{aligned}$$

Now, since $p < \frac{1}{A+B}$, we have that $4p \leq 1$ and $p(A+B) \leq 1$, and consequently, that

$(4p)^B \leq 4p$ and $(p(A+B))^2 \leq p(A+B)$. But now we have the following:

$$\begin{aligned}
2\tfrac{\alpha^{A+B}p^B}{1-\alpha^{-B}} + (1-p)^{A+B} &\leq 4^B p^B + e^{-p(A+B)} \\
&\leq (4p)^B + (1 - p(A+B) + \tfrac{(p(A+B))^2}{2}) \\
&\leq 4p + (1 - p(A+B) + \tfrac{p(A+B)}{2}) \\
&= 1 - (\tfrac{A+B-8}{2})p \\
&= 1 - \tfrac{3A+3}{2}p \\
&< 1.
\end{aligned}$$

This completes the argument. □

It is natural to ask whether 7.1.3 could be improved by bounding the degree of $g$ in terms of the degree of $f$. However, the lemma that follows (7.1.4) shows that this is not possible. We first need a definition. A *fractional coloring* of a graph $G$ is a family $(S_i, \lambda_i)_{i\in\mathcal{I}}$ such that for each $i \in \mathcal{I}$, $S_i$ is a stable set in $G$ and $\lambda_i$ is a non-negative scalar, and for each vertex $v \in V_G$, we have that $\Sigma_{S_i \ni v}\lambda_i \geq 1$. The *fractional chromatic number* of a graph $G$ is the smallest number $r$ with the property that there exists a fractional coloring $(S_i, \lambda_i)_{i\in\mathcal{I}}$ with $r = \Sigma_{i\in\mathcal{I}}\lambda_i$; we denote the fractional chromatic number of a graph $G$ by $\chi_f(G)$. The proof of 7.1.4 uses the fact that there exist triangle-free graphs of arbitrarily large fractional chromatic number; this follows immediately from the fact that the Ramsey number $R(3,t)$ satisfies $\frac{R(3,t)}{t} \to \infty$, which follows from standard probabilistic arguments (in fact, $R(3,t)$ has order of magnitude $\frac{t^2}{\log t}$, as shown in [46]).

**7.1.4.** *For every $d \in \mathbb{N}$, there is a hereditary class $\mathcal{G}$, $\chi$-bounded by a linear $\chi$-bounding function, such that every polynomial $\chi$-bounding function of $\mathcal{G}^*$ has degree greater than $d$.*

*Proof.* Fix $d \in \mathbb{N}$. Let $F$ be a graph with $\omega(F) = 2$ and $\chi_f(F) > 2^d$. Let $\mathcal{G}$ be the class that consists of all the isomorphic copies of $F$ and its induced subgraphs, as well as all the complete graphs. Then $\mathcal{G}$ is a hereditary class, $\chi$-bounded by the linear function $f(x) = x + \chi(F)$. Suppose that $\mathcal{G}^*$ is $\chi$-bounded by a polynomial function $g$ of degree at most $d$; we may assume that $g(x) = Mx^d$ for some $M \in \mathbb{N}$.

Define a sequence $F_1, F_2, ...$ as follows. Set $F_1 = F$, and for each $i \in \mathbb{N}$, let $F_{i+1}$ be the graph with vertex-set $V_F \times V_{F_i}$ in which vertices $(u_1, v_1), (u_2, v_2) \in V_{F_{i+1}}$ are adjacent if and only if either $u_1$ and $u_2$ are adjacent in $F$, or $u_1 = u_2$ and $v_1$ and $v_2$ are adjacent in $V_{F_i}$; note that this means that $F_{i+1}$ is obtained by substituting a copy $F_i^v$ of $F_i$ for every vertex $v$ of $F$, and so $F_i \in \mathcal{G}^*$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, let $\mathcal{S}_i$ be the set of all stable sets in $F_i$, and set $\mathcal{S} = \mathcal{S}_1$.

First, we note that it follows by an easy induction that $\omega(F_i) = \omega(F)^i = 2^i$ for all $i \in \mathbb{N}$. Next, we argue inductively that $\chi_f(F_i) = \chi_f(F)^i$ for all $i \in \mathbb{N}$. For $i = 1$, this is immediate. Now assume that $\chi_f(F_i) = \chi_f(F)^i$; we claim that $\chi_f(F_{i+1}) = \chi_f(F)^{i+1}$.

We begin by showing that $\chi_f(F_{i+1}) \geq \chi_f(F)^{i+1}$. Let $(S, \lambda_S)_{S \in \mathcal{S}_{i+1}}$ be a fractional coloring of $F_{i+1}$ (where each stable set $S$ is taken with weight $\lambda_S \geq 0$) with $\Sigma_{S \in \mathcal{S}_{i+1}} \lambda_S = \chi_f(F_{i+1})$. For each $X \subseteq V_{F_{i+1}}$, set $\widehat{X} = \{u \in V_F \mid (u, v) \in X \text{ for some } v \in V_{F_i}\}$. Clearly, for all $S \in \mathcal{S}_{i+1}$, we have that $\widehat{S} \in \mathcal{S}$. For all $S' \in \mathcal{S}$, let $[S']_{i+1} = \{S \in \mathcal{S}_{i+1} \mid \widehat{S} = S'\}$; note that the set $\mathcal{S}_{i+1}$ is the disjoint union of the sets $[S']_{i+1}$ with $S' \in \mathcal{S}$. For each $S' \in \mathcal{S}$, set

$$\lambda_{S'} = \frac{\Sigma_{S \in [S']_{i+1}} \lambda_S}{\chi_f(F_i)}.$$

Now, given $u \in V_F$, set $\mathcal{S}[u] = \{S \in \mathcal{S} \mid u \in S\}$ and $\mathcal{S}_{i+1}[u] = \{S \in \mathcal{S}_{i+1} \mid u \in \widehat{S}\}$, and note that for all $u \in V_F$, we have the following:

$$\begin{aligned}
\Sigma_{S' \in \mathcal{S}[u]} \lambda_{S'} &= \Sigma_{S' \in \mathcal{S}[u]} \frac{\Sigma_{S \in [S']_{i+1}} \lambda_S}{\chi_f(F_i)} \\
&= \frac{1}{\chi_f(F_i)} \Sigma_{S \in \mathcal{S}_{i+1}[u]} \lambda_S \\
&\geq \frac{\chi_f(F_i)}{\chi_f(F_i)} \\
&= 1.
\end{aligned}$$

Thus, $(S', \lambda_{S'})_{S \in \mathcal{S}}$ is a fractional coloring of $F$, and so $\Sigma_{S' \in \mathcal{S}} \lambda_{S'} \geq \chi_f(F)$. But now we have that:

$$\begin{aligned}
\chi_f(F) &\leq \Sigma_{S' \in \mathcal{S}} \lambda_{S'} \\
&= \Sigma_{S' \in \mathcal{S}} \frac{\Sigma_{S \in [S']_{i+1}} \lambda_S}{\chi_f(F_i)} \\
&= \frac{\Sigma_{S \in \mathcal{S}_{i+1}} \lambda_S}{\chi_f(F_i)} \\
&= \frac{\chi_f(F_{i+1})}{\chi_f(F_i)},
\end{aligned}$$

and so $\chi_f(F_{i+1}) \geq \chi_f(F)\chi_f(F_i) = \chi_f(F)^{i+1}$.

It remains to construct a fractional coloring of $F_{i+1}$ in which the sum of weights is equal to $\chi_f(F)^{i+1}$, the lower bound for $\chi_f(F_{i+1})$ that we just obtained. First, let $(S, \lambda_S)_{S \in \mathcal{S}}$ be a fractional coloring of $F$ with $\Sigma_{S \in \mathcal{S}} \lambda_S = \chi_f(F)$, and let $(S, \lambda_S)_{S \in \mathcal{S}_i}$ be a fractional coloring of $F_i$ with $\Sigma_{S \in \mathcal{S}_i} \lambda_S = \chi_f(F_i)$. Next, for all $S \in \mathcal{S}_{i+1}$, if there exist some $S' \in \mathcal{S}$ and $S'' \in \mathcal{S}_i$ such that $S = S' \times S''$ then we set $\lambda_S = \lambda_{S'}\lambda_{S''}$, and otherwise we set $\lambda_S = 0$. But now $(S, \lambda_S)_{S \in \mathcal{S}_{i+1}}$ is a fractional coloring of $F_{i+1}$ with $\Sigma_{S \in \mathcal{S}_{i+1}} \lambda_S = \chi_f(F)\chi_f(F_i) = \chi_f(F)^{i+1}$. This completes the induction.

Finally, from $\chi_f(F_i) \leq g(\omega(F_i))$, we get that $\chi_f(F)^i \leq M \cdot 2^{id}$ for all $i \in \mathbb{N}$. But this implies that $\chi_f(F) \leq M^{1/i} \cdot 2^d$ for all $i \in \mathbb{N}$, which is impossible since $\chi_f(F) > 2^d$ and $\lim_{i\to\infty} M^{1/i} = 1$. □

### 7.1.3 Faster Growing $\chi$-Bounding Functions

A function $f : \mathbb{N}_0 \to \mathbb{R}$ is said to be *supermultiplicative* provided that $f(m)f(n) \leq f(mn)$ for all $m, n \in \mathbb{N}$. Our next theorem (7.1.5) improves on 7.1.2 in the case when the $\chi$-bounding function of a $\chi$-bounded class $\mathcal{G}$ is supermultiplicative.

**7.1.5.** *Let $\mathcal{G}$ a class of graphs, $\chi$-bounded by a supermultiplicative non-decreasing function $f : \mathbb{N}_0 \to \mathbb{R}$. Then $\mathcal{G}^*$ is $\chi$-bounded by the function $g : \mathbb{N}_0 \to \mathbb{R}$ given by $g(0) = 0$ and $g(x) = f(x)x^{\log_2 x}$ for all $x \in \mathbb{N}$.*

*Proof.* We may assume that $\mathcal{G}$ is hereditary (otherwise, instead of $\mathcal{G}$, we consider the closure of $\mathcal{G}$ under isomorphism and induced subgraphs). Let $G \in \mathcal{G}^*$, set $d_{\mathcal{G}}(G) = t$, and assume inductively that $\chi(G') \leq g(\omega(G'))$ for all graphs $G' \in \mathcal{G}^*$ with $d_{\mathcal{G}}(G') < t$; we need to show that $\chi(G) \leq g(\omega(G))$. If $t = -1$, then $G$ is the empty graph, and the result is immediate. If $t = 0$, then $G$ is a non-empty graph in $\mathcal{G}^+$, and the result follows from the fact that $\mathcal{G}^+$ is $\chi$-bounded by $f$ and that $f(n) \leq g(n)$ for all $n \in \mathbb{N}$. So assume that $t \geq 1$. By 7.1.1, this means that $\omega(G) \geq 2$. We may assume that $G$ is connected, so that there exists a connected graph $F \in \mathcal{G}^+$ with vertex-set $V_F = \{v_1, ..., v_n\}$ (where $2 \leq n \leq |V_G| - 1$), and non-empty graphs $B_1, ..., B_n \in \mathcal{G}^*$ with pairwise disjoint vertex-sets, and each with substitution depth at most $t - 1$, such that $G$ is obtained by substituting $B_1, .., B_n$ for $v_1, ..., v_n$ in $F$. Note that by the induction hypothesis, $\chi(B_i) \leq g(\omega(B_i))$ for all $i \in \{1, ..., n\}$.

Set $\omega = \omega(G)$, and for all $i \in \{1, ..., n\}$, set $\omega_i = \omega(B_i)$. Next, for all $j \in \{1, ..., \lfloor \frac{\omega}{2} \rfloor\}$, set $W_j = \{v_i \mid \omega_i = j\}$, and set $W_\infty = \{v_i \mid \omega_i > \frac{\omega}{2}\}$. For all $j \in \{1, ..., \lfloor \frac{\omega}{2} \rfloor\}$, set $F_j = F[W_j]$ and $G_j = G[\bigcup_{v_i \in W_j} B_i]$, and set $F_\infty = F[W_\infty]$ and $G_\infty = G[\bigcup_{v_i \in W_\infty} B_i]$. Note that if $C$

is a clique in $F$, then we have that:

$$\omega \geq \Sigma_{v_i \in C}\, \omega_i. \tag{7.7}$$

Therefore, for all $j \in \{1, ..., \lfloor \frac{\omega}{2} \rfloor\}$, we have that $\omega(F_j) \leq \lfloor \frac{\omega}{j} \rfloor$. But then for all $j \in \{1, ..., \lfloor \frac{\omega}{2} \rfloor\}$,

$$\begin{aligned} \chi(G_j) &\leq \chi(F_j) \cdot \max_{v_i \in W_j} \chi(B_i) \\ &\leq f(\lfloor \tfrac{\omega}{j} \rfloor) g(j) \end{aligned} \tag{7.8}$$

Furthermore, by (7.7) again, we have that $F_\infty$ is a stable set. Since $F$ contains no isolated vertices, we get by (7.7) that for all $i \in \{1, ..., n\}$, $\omega_i \leq \omega - 1$. Thus:

$$\begin{aligned} \chi(G_\infty) &\leq \chi(F_\infty) \cdot \max_{v_i \in W_\infty} \chi(B_i) \\ &\leq g(\omega - 1) \end{aligned} \tag{7.9}$$

But now using (7.8) and (7.9), we have the following:

$$\begin{aligned} \chi(G) &\leq \chi(G_\infty) + \Sigma_{j=1}^{\lfloor \frac{\omega}{2} \rfloor} \chi(G_j) \\ &\leq g(\omega - 1) + \Sigma_{j=1}^{\lfloor \frac{\omega}{2} \rfloor} f(\lfloor \tfrac{\omega}{j} \rfloor) g(j) \\ &= g(\omega - 1) + \Sigma_{j=1}^{\lfloor \frac{\omega}{2} \rfloor} f(\lfloor \tfrac{\omega}{j} \rfloor) f(j) j^{\log_2 j} \\ &\leq g(\omega - 1) + \Sigma_{j=1}^{\lfloor \frac{\omega}{2} \rfloor} f(\lfloor \tfrac{\omega}{j} \rfloor j) j^{\log_2 j} \\ &\leq g(\omega - 1) + \Sigma_{j=1}^{\lfloor \frac{\omega}{2} \rfloor} f(\omega) j^{\log_2 j} \\ &\leq f(\omega)(\omega - 1)^{\log_2 \omega} + \tfrac{\omega}{2} f(\omega) (\tfrac{\omega}{2})^{\log_2 (\frac{\omega}{2})} \\ &= f(\omega)\omega^{\log_2 \omega}(1 - \tfrac{1}{\omega})^{\log_2 \omega} + f(\omega)(\tfrac{\omega}{2})^{\log_2 \omega} \\ &\leq f(\omega)\omega^{\log_2 \omega}(1 - \tfrac{1}{\omega}) + f(\omega)\tfrac{\omega^{\log_2 \omega}}{\omega} \\ &= f(\omega)\omega^{\log_2 \omega} \\ &= g(\omega) \end{aligned}$$

This completes the argument. □

As a corollary of 7.1.5, we have the following result.

**7.1.6.** *Let $\mathcal{G}$ be a class of graphs, $\chi$-bounded by an exponential function. Then $\mathcal{G}^*$ is also $\chi$-bounded by an exponential function.*

*Proof.* We may assume that $\mathcal{G}$ is hereditary (otherwise, instead of $\mathcal{G}$, we consider the closure of $\mathcal{G}$ under isomorphism and induced subgraphs). We may assume that $\mathcal{G}$ is $\chi$-bounded by the function $f(x) = 2^{c(x-1)}$ for some $c \in \mathbb{N}$. Then $f$ is a supermultiplicative non-decreasing function, and so by 7.1.5, $\mathcal{G}^*$ is $\chi$-bounded by the function $g : \mathbb{N}_0 \to \mathbb{R}$ given by $g(0) = 0$ and $g(x) = f(x)x^{\log_2 x}$ for all $x \in \mathbb{N}$. But now note that for all $x \in \mathbb{N}$, we have the following:

$$\begin{aligned} g(x) &= x^{\log_2 x} f(x) \\ &= 2^{(\log_2 x)^2} 2^{c(x-1)} \\ &\leq 2^x 2^{cx} \\ &= 2^{(c+1)x} \end{aligned}$$

Thus, $\mathcal{G}^*$ is $\chi$-bounded by the exponential function $h(x) = 2^{(c+1)x}$. $\square$

## 7.2 Small Cutsets, Substitution, and Cliques

In section 7.1, we saw that the closure of a $\chi$-bounded class under substitution is $\chi$-bounded. In this section, we prove analogous results for the operations of gluing along a clique (see 7.2.1) and gluing along a bounded number of vertices (see 7.2.2). We then consider "combinations" of the three operations discussed in this chapter, namely substitution, gluing along a clique, and gluing along a bounded number of vertices. In particular, we prove that the closure of a $\chi$-bounded class under gluing along a clique and gluing along a bounded number of vertices is $\chi$-bounded (see 7.2.6), as well as that the closure of a $\chi$-bounded class under gluing along a clique and substitution is $\chi$-bounded (see 7.2.7).

### 7.2.1 Gluing Operations

We begin by giving an easy proof of the fact that the closure of a $\chi$-bounded class under gluing along a clique is again $\chi$-bounded.

**7.2.1.** *Let $\mathcal{G}$ be a class of graphs, $\chi$-bounded by a non-decreasing function $f : \mathbb{N}_0 \to \mathbb{R}$. Then the closure of $\mathcal{G}$ under gluing along a clique is also $\chi$-bounded by $f$.*

*Proof.* Note that if a graph $G$ is obtained by gluing graphs $G_1$ and $G_2$ along a clique, then $\omega(G) = \max\{\omega(G_1), \omega(G_2)\}$ and $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$. The result now follows by an easy induction. $\square$

We now turn to the question of gluing along a bounded number of vertices. Given a class $\mathcal{G}$ of graphs, and a positive integer $k$, let $\mathcal{G}^k$ denote the closure of $\mathcal{G}$ under gluing along at most $k$ vertices. Our goal is to prove the following theorem.

**7.2.2.** *Let $k$ be a positive integer, and let $\mathcal{G}$ be a class of graphs, $\chi$-bounded by a non-decreasing function $f : \mathbb{N}_0 \to \mathbb{R}$. Then $\mathcal{G}^k$ is $\chi$-bounded by the function $g : \mathbb{N}_0 \to \mathbb{R}$ given by $g(n) = f(n) + 2k^2 - 1$.*

We begin with some definitions. Given a set $S$, we denote by $\mathscr{P}(S)$ the power set of $S$ (i.e. the set of all subsets of $S$). Given a graph $G$, we say that a four-tuple $(B, K, \phi_K, F)$ is a *coloring constraint* for $G$ provided that the following hold:

- $B$ is a non-empty set;
- $K \subseteq V_G$;
- $\phi_K : K \to B$ is a proper coloring of $G[K]$;
- $F : V_G \smallsetminus K \to \mathscr{P}(B)$.

$B$ should be seen as the set of colors with which we wish to color $G$, $K$ should be seen as the set of "precolored" vertices of $G$ with "precoloring" $\phi_K$, and for all $v \in V_G \smallsetminus K$, $F(v)$ should be seen a set of colors "forbidden" on $v$. Given a graph $G$ with a coloring constraint $(B, K, \phi_K, F)$, we say that a proper coloring $\phi : V_G \to B$ of $G$ is *appropriate* for $(B, K, \phi_K, F)$ provided that $\phi \restriction K = \phi_K$, and that for all $v \in V_G \smallsetminus K$, $\phi(v) \notin F(v)$. We now prove a technical lemma.

**7.2.3.** *Let $\mathcal{G}$ be a hereditary class, $\chi$-bounded by a non-decreasing function $f : \mathbb{N}_0 \to \mathbb{R}$. Then for all $G \in \mathcal{G}$ and all coloring constraints $(B, K, \phi_K, F)$ for $G$ such that $|B| \geq f(\omega(G)) + 2k^2 - 1$ and $k|K| + \Sigma_{v \in V_G \smallsetminus K}|F(v)| \leq 2k^2 - 1$, there exists a proper coloring coloring $\phi : V_G \to B$ of $G$ that is appropriate for $(B, K, \phi_K, F)$.*

*Proof.* Fix $G \in \mathcal{G}^k$, and assume inductively that the claim holds for all proper induced subgraphs of $G$. Fix a coloring constraint $(B, K, \phi_K, F)$ for $G$ such that $|B| \geq f(\omega(G)) + 2k^2 - 1$ and $k|K| + \Sigma_{v \in V_G \smallsetminus K}|F(v)| \leq 2k^2 - 1$. We need to show that there exists a proper coloring $\phi : V_G \to B$ of $G$ that is appropriate for $(B, K, \phi_K, F)$.

Suppose first that $G \in \mathcal{G}$. Since $k|K| + \Sigma_{v \in V_G \smallsetminus K}|F(v)| \leq 2k^2 - 1$, we know that $|\phi_K[K] \cup \bigcup_{v \in V_G \smallsetminus K} F(v)| \leq 2k^2 - 1$; consequently, $|B \smallsetminus (\phi_K[K] \cup \bigcup_{v \in V_G \smallsetminus K} F(v))| \leq f(\omega(G))$. Since $G \in \mathcal{G}$ and $\mathcal{G}$ is $\chi$-bounded by $f$, it follows that there exists a proper coloring $\phi' : V_G \smallsetminus K \to B \smallsetminus (\phi_K[K] \cup \bigcup_{v \in V_G \smallsetminus K} F(v))$ of $G \smallsetminus K$. Now define $\phi : V_G \to B$ by setting

$$\phi(v) \quad = \quad \begin{cases} \phi_K(v) & \text{if} \quad v \in K \\ \phi'(v) & \text{if} \quad v \in V_G \smallsetminus K \end{cases}$$

By construction, the colorings $\phi_K$ and $\phi'$ use disjoint color sets; furthermore, for all $v \in V_G \smallsetminus K$, $\phi(v) \notin F(v)$. It follows that $\phi$ is a proper coloring of $G$, appropriate for $(B, K, \phi_K, F)$.

Suppose now that $G \notin \mathcal{G}$. Then there exist graphs $G_1, G_2 \in \mathcal{G}^k$ with inclusion-wise incomparable vertex-sets such that $G$ is obtained by gluing $G_1$ and $G_2$ along at most $k$ vertices. Set $C = V_{G_1} \cap V_{G_2}$; then $|C| \leq k$, $G_1[C] = G_2[C]$, and $G$ is obtained by gluing $G_1$ and $G_2$ along $C$. Set $V_1 = V_{G_1} \smallsetminus C$ and $V_2 = V_{G_2} \smallsetminus C$. Note that $V_G = C \cup V_1 \cup V_2$; furthermore, since the vertex-sets of $G_1$ and $G_2$ are inclusion-wise incomparable, we know that $V_1$ and $V_2$ are both non-empty. By symmetry, we may assume that

$$k|K \cap V_1| + \Sigma_{v \in V_1 \smallsetminus K}|F(v)| \quad \geq \quad k|K \cap V_2| + \Sigma_{v \in V_2 \smallsetminus K}|F(v)|.$$

Our first goal is to obtain a coloring constraint for $G_1$ that "forbids" on the vertices in $C \smallsetminus K$ all the colors used by $\phi_K$ on the set $K \cap V_2$, and then to use the induction hypothesis to obtain a coloring $\phi_1$ of $G_1$ that is appropriate for this constraint. We do this as follows. First, as $k|K| + \Sigma_{v \in V_G \smallsetminus K}|F(v)| \leq 2k^2 - 1$, the inequality above implies that $k|K \cap V_2| + \Sigma_{v \in V_2 \smallsetminus K}|F(v)| \leq k^2 - 1$, and consequently, that $|K \cap V_2| \leq k - 1$. Now, set $K_1 = K \smallsetminus V_2$ and $\phi_{K_1} = \phi_K \upharpoonright K_1$. Further, define $F_1 : (V_1 \cup C) \smallsetminus K \to \mathscr{P}(B)$ by setting

$$F_1(v) \quad = \quad \begin{cases} F(v) & \text{if} \quad v \in V_1 \smallsetminus K_1 \\ F(v) \cup \phi_K[K \cap V_2] & \text{if} \quad v \in C \smallsetminus K_1 \end{cases}$$

Clearly, $(B, K_1, \phi_{K_1}, F_1)$ is a coloring constraint for $G_1$. Further, since $f$ is non-decreasing, we get that

$$|B| \quad \geq \quad f(\omega(G)) + 2k^2 - 1 \quad \geq \quad f(\omega(G_1)) + 2k^2 - 1.$$

Finally, note the following:

$$\begin{aligned} & k|K_1| + \Sigma_{v \in V_{G_1} \smallsetminus K_1}|F_1(v)| \\ \leq \quad & k|K| - k|K \cap V_2| + \Sigma_{v \in V_{G_1} \smallsetminus K_1}|F(v)| + |C \smallsetminus K||\phi_K[K \cap V_2]| \\ \leq \quad & k|K| - k|K \cap V_2| + \Sigma_{v \in V_G \smallsetminus K}|F(v)| + k|K \cap V_2| \\ = \quad & k|K| + \Sigma_{v \in V_G \smallsetminus K}|F(v)| \\ \leq \quad & 2k^2 - 1. \end{aligned}$$

Thus, by the induction hypothesis, there exists a proper coloring $\phi_1 : V_1 \cup C \to B$ of $G_1$ that is appropriate for $(B, K_1, \phi_{K_1}, F_1)$.

Our next goal is to "combine" the coloring constraint $(B, K, \phi_K, F)$ for $G$ and the coloring $\phi_1$ of $G_1$ (or more precisely, the restriction of $\phi_1$ to $C$) in order to obtain a coloring constraint for $G_2$; we then use the induction hypothesis to obtain a coloring $\phi_2$ for $G_2$ that is appropriate for this constraint, and finally, we "combine" the colorings $\phi_1$ and $\phi_2$ to obtain a proper coloring $\phi$ of $G$ that is appropriate for the coloring constraint $(B, K, \phi_K, F)$. We do this as follows. First, set $K_2 = C \cup (K \cap V_2)$, and define $F_2 = F \restriction (V_2 \smallsetminus K)$. Next, define $\phi_{K_2} : K_2 \to B$ by setting

$$\phi_{K_2}(v) \quad = \quad \begin{cases} \phi_1(v) & \text{if} \quad v \in C \\ \phi_K(v) & \text{if} \quad v \in K \cap V_2 \end{cases}$$

Since $\phi_1$ and $\phi_K$ are proper colorings of $G_1$ and $G[K]$, respectively, and since the colors used to precolor vertices in $K \cap V_2$ were forbidden on the vertices in $C \smallsetminus K$, we get that $\phi_{K_2}$ is a proper coloring of $G_2[K_2]$. Thus, $(B, K_2, \phi_{K_2}, F_2)$ is a coloring constraint for $G_2$. Since $f$ is non-decreasing, we know that

$$|B| \quad \geq \quad f(\omega(G)) + 2k^2 - 1 \quad \geq \quad f(\omega(G_2)) + 2k^2 - 1.$$

Further, since $k|K \cap V_2| + \Sigma_{v \in V_2 \smallsetminus K}|F(v)| \leq k^2 - 1$, we have the following:

$$\begin{aligned} k|K_2| + \Sigma_{v \in V_{G_2} \smallsetminus K_2}|F_2(v)| \quad &= \quad k|C| + k|K \cap V_2| + \Sigma_{v \in V_2 \smallsetminus K}|F(v)| \\ &\leq \quad k^2 + k^2 - 1 \\ &= \quad 2k^2 - 1. \end{aligned}$$

Thus, by the induction hypothesis, there exists a proper coloring $\phi_2 : V_{G_2} \to B$ of $G_2$ that is appropriate for $(B, K_2, \phi_{K_2}, F_2)$. Note that by construction, $\phi_1 \restriction C = \phi_2 \restriction C$; define $\phi : V_G \to B$ by setting

$$\phi(v) \quad = \quad \begin{cases} \phi_1(v) & \text{if} \quad v \in V_1 \cup C \\ \phi_2(v) & \text{if} \quad v \in V_2 \cup C \end{cases}$$

By construction, $\phi$ is a proper coloring of $G$, appropriate for $(B, K, \phi_K, F)$. This completes the argument. $\square$

We are now ready to prove 7.2.2, restated below.

**7.2.2.** *Let $k$ be a positive integer, and let $\mathcal{G}$ be a class of graphs, $\chi$-bounded by a non-*

*decreasing function* $f : \mathbb{N}_0 \to \mathbb{R}$*. Then* $\mathcal{G}^k$ *is* $\chi$*-bounded by the function* $g : \mathbb{N}_0 \to \mathbb{R}$ *given by* $g(n) = f(n) + 2k^2 - 1$*.*

*Proof.* We may assume that $\mathcal{G}$ is hereditary (otherwise, instead of $\mathcal{G}$, we consider the closure of $\mathcal{G}$ under isomorphism and taking induced subgraphs). Fix $G \in \mathcal{G}^k$. Let $B = \{1, ..., f(\omega(G)) + 2k^2 - 1\}$, let $K = \emptyset$, let $\phi_K$ be the empty function, and define $F : V_G \to \mathscr{P}(B)$ by setting $F(v) = \emptyset$ for all $v \in V_G$. Then $(B, K, \phi_K, F)$ is a coloring constraint for $G$ with $|B| \geq f(\omega(G)) + 2k^2 - 1$ and $k|K| + \Sigma_{v \in V_G \smallsetminus K}|F(v)| \leq 2k^2 - 1$. By 7.2.3 then, there exists a proper coloring $\phi : V_G \to B$ that is appropriate for $(B, K, \phi_K, F)$. But then $\phi$ is a proper coloring of $G$ that uses at most $g(\omega(G))$ colors. □

As remarked in the Introduction, the fact that the closure of a $\chi$-bounded class is again $\chi$-bounded follows from a more general result from [1], which we state below.

**7.2.4** (Alon, Kleitman, Saks, Seymour, and Thomassen [1])**.** *Let* $k$ *and* $m$ *be positive integers. Then every graph* $G$ *of chromatic number greater than* $\max\{100k^3, m + 10k^2\}$ *contains a* $(k + 1)$*-connected subgraph of chromatic number at least* $m$*.*

It is easy to see that 7.2.4 implies that if $\mathcal{G}$ is $\chi$-bounded by a non-decreasing function $f : \mathbb{N}_0 \to \mathbb{R}$, then $\mathcal{G}^k$ is $\chi$-bounded by the function $g : \mathbb{N}_0 \to \mathbb{R}$ defined by $g(n) = \max\{100k^3, f(n) + 10k^2 + 1\}$. (This follows from the fact that if a graph $G \in \mathcal{G}^k$ contains a $(k + 1)$-connected subgraph $H$, then there exists an induced subgraph $G'$ of $G$ such that $G' \in \mathcal{G}$ and $H$ is a subgraph of $G'$.) Note, however, that the $\chi$-bounding function from 7.2.2 is better than the $\chi$-bounding function that follows from 7.2.4.

We complete this subsection by considering "combinations" of gluing along a clique and gluing along a bounded number of vertices. Given a class $\mathcal{G}$ of graphs and a positive integer $k$, we denote by $\mathcal{G}_{cl}^k$ the closure of $\mathcal{G}$ under gluing along a clique and gluing along at most $k$ vertices. Our goal is to prove that if $\mathcal{G}$ is a $\chi$-bounded class, then for every $k \in \mathbb{N}$, the class $\mathcal{G}_{cl}^k$ is $\chi$-bounded (see 7.2.6 below). We begin with a technical lemma, which we then use to prove 7.2.6.

**7.2.5.** *Let $\mathcal{G}$ be a hereditary class, closed under gluing along a clique, and let $k$ be a positive integer. Then $\mathcal{G}^k$ is closed under gluing along a clique, and consequently, $\mathcal{G}^k = \mathcal{G}^k_{cl}$.*

*Proof.* Clearly, the second claim follows from the first, and so we just need to show that $\mathcal{G}^k$ is closed under gluing along a clique. Let $\tilde{\mathcal{G}}^k$ be the closure of $\mathcal{G}^k$ under gluing along a clique. We claim that $\tilde{\mathcal{G}}^k = \mathcal{G}^k$. Fix $G \in \tilde{\mathcal{G}}^k$, and assume inductively that for all $G' \in \tilde{\mathcal{G}}^k$ such that $|V_{G'}| < |V_G|$, we have that $G' \in \mathcal{G}^k$; we claim that $G \in \mathcal{G}^k$.

By the definition of $\tilde{\mathcal{G}}^k$, we know that either $G \in \mathcal{G}^k$, or $G$ is obtained by gluing smaller graphs in $\tilde{\mathcal{G}}^k$ along a clique. In the former case, we are done; so assume that there exist graphs $G_1, G_2 \in \tilde{\mathcal{G}}^k$ such that $|V_{G_i}| < |V_G|$ for each $i \in \{1, 2\}$, such that $C = V_{G_1} \cap V_{G_2}$ is a clique in both $G_1$ and $G_2$, and such that $G$ is obtained by gluing $G_1$ and $G_2$ along the clique $C$. By the induction hypothesis, $G_1, G_2 \in \mathcal{G}^k$. Now, if $G_1, G_2 \in \mathcal{G}$, then the fact that $\mathcal{G}$ is closed under gluing along a clique implies that $G \in \mathcal{G}$, and consequently, that $G \in \mathcal{G}^k$. So assume that at least one of $G_1$ and $G_2$ is not a member of $\mathcal{G}$; by symmetry, we may assume that $G_1 \notin \mathcal{G}$.

Since $G_1 \in \mathcal{G}^k \smallsetminus \mathcal{G}$, there exist graphs $G_1^1, G_1^2 \in \mathcal{G}^k$ such that $|V_{G_1^i}| < |V_{G_1}|$ for each $i \in \{1, 2\}$, and such that $G_1$ is obtained by gluing $G_1^1$ and $G_1^2$ along $K = V_{G_1^1} \cap V_{G_1^2}$, where $|K| \leq k$. Now, $C$ is a clique in $G_1$, and so we know that $C \subseteq V_{G_1^i}$ for some $i \in \{1, 2\}$; by symmetry, we may assume that $C \subseteq V_{G_1^1}$. If $C = V_{G_1^1}$, then set $G_1' = G_2$; and if $C \subsetneqq V_{G_1^1}$, then let $G_1'$ be the graph obtained by gluing $G_1^1$ and $G_2$ along $C$. As $G_1^1, G_2 \in \mathcal{G}^k$, we know that $G_1' \in \tilde{\mathcal{G}}^k$. Further, note that $|V_{G_1'}| < |V_G|$, and so by the induction hypothesis, $G_1' \in \mathcal{G}^k$. But now $G$ is obtained by gluing $G_1'$ and $G_1^2$ along $K$, and so since $G_1', G_1^2 \in \mathcal{G}^k$, we know that $G \in \mathcal{G}^k$. This completes the argument. □

**7.2.6.** *Let $\mathcal{G}$ be a class of graphs, $\chi$-bounded by a non-decreasing function $f : \mathbb{N}_0 \to \mathbb{R}$, and let $k$ be a positive integer. Then $\mathcal{G}^k_{cl}$ is $\chi$-bounded by the function $g : \mathbb{N}_0 \to \mathbb{R}$ given by $g(n) = f(n) + 2k^2 - 1$.*

*Proof.* We may assume that $\mathcal{G}$ is hereditary (otherwise, instead of $\mathcal{G}$, we consider the closure of $\mathcal{G}$ under isomorphism and taking induced subgraphs). Let $\tilde{\mathcal{G}}$ be the closure of $\mathcal{G}$ under gluing along a clique. Then by 7.2.5, $\tilde{\mathcal{G}}^k = \mathcal{G}_{cl}^k$ (where $\tilde{\mathcal{G}}^k$ is the closure of $\tilde{\mathcal{G}}$ under gluing along at most $k$ vertices). By 7.2.1, $\tilde{\mathcal{G}}$ is $\chi$-bounded by $f$; but then by 7.2.2, $\tilde{\mathcal{G}}^k$ is $\chi$-bounded by $g$. It follows that $\mathcal{G}_{cl}^k$ is $\chi$-bounded by $g$. □

### 7.2.2 Substitution and Gluing along a Clique

In section 7.1, we proved that the closure of a $\chi$-bounded class under substitution is $\chi$-bounded (see 7.1.2, as well as 7.1.3 and 7.1.6 for a strengthening of 7.1.2 in some special cases), and in this section, we proved an analogous result for gluing along a clique (see 7.2.1). In this subsection, we discuss "combinations" of these two operations. Given a class $\mathcal{G}$ of graphs, we denote by $\mathcal{G}^{\#}$ the closure of $\mathcal{G}$ under substitution and gluing along a clique. Our main goal is to prove the following theorem.

**7.2.7.** *Let $\mathcal{G}$ be a class of graphs, $\chi$-bounded by a non-decreasing function $f : \mathbb{N}_0 \to \mathbb{R}$. Then $\mathcal{G}^{\#}$ is $\chi$-bounded by the function $g(k) = f(k)^k$.*

We note that, as in section 7.1, we can obtain a strengthening of 7.2.7 in the case when the $\chi$-bounding function for the class $\mathcal{G}$ is polynomial or exponential (see 7.2.11). The main "ingredient" in the proof of 7.2.7 is the following lemma.

**7.2.8.** *Let $\mathcal{G}$ be a hereditary class, closed under substitution. Assume that $\mathcal{G}$ is $\chi$-bounded by a non-decreasing function $f : \mathbb{N}_0 \to \mathbb{R}$. Then $\mathcal{G}^{\#}$ is $\chi$-bounded by $f$.*

In view of the results of section 7.1, 7.2.8 easily implies 7.2.7 and 7.2.11 (see the proof of these two theorems at the end of this section). The idea of the proof of 7.2.8 is as follows. We first prove a certain structural result for graphs in the class $\mathcal{G}^{\#}$, where $\mathcal{G}$ is a hereditary class, closed under substitution (see 7.2.9). We then use 7.2.9 to show that if $\mathcal{G}$ is a hereditary class, closed under substitution, then for every graph $G \in \mathcal{G}^{\#}$, there exists a graph $G' \in \mathcal{G}$ such that $G'$ is an induced subgraph of $G$ and $\chi(G') = \chi(G)$ (see 7.2.10). Finally, 7.2.10 easily implies 7.2.8.

We begin with some definitions. Let $\mathcal{G}$ be a hereditary class. Given non-empty graphs $G, G_0 \in \mathcal{G}^\#$ with $V_{G_0} = \{v_1, ..., v_t\}$, we say that $G$ is an *expansion* of $G_0$ provided that there exist non-empty graphs $G_1, ..., G_t \in \mathcal{G}^\#$ with pairwise disjoint vertex-sets such that $G$ is obtained by substituting $G_1, ..., G_t$ for $v_1, .., v_t$ in $G_0$. We observe that every non-empty graph in $\mathcal{G}^\#$ is an expansion of itself. We say that a non-empty graph $G \in \mathcal{G}^\#$ is *decomposable* provided that there exists a non-empty graph $G' \in \mathcal{G}^\#$ such that $G$ is an expansion of $G'$, and there exist non-empty graphs $H, K \in \mathcal{G}^\#$ with inclusion-wise incomparable vertex-sets such that $G'$ can be obtained by gluing $H$ and $K$ along a clique. We now prove a structural result for graphs in $\mathcal{G}^\#$, when $\mathcal{G}$ is a hereditary class, closed under substitution.

**7.2.9.** *Let $\mathcal{G}$ be a hereditary class, closed under substitution. Then for every graph $G \in \mathcal{G}^\#$, either $G \in \mathcal{G}$, or there exists a non-empty set $S \subseteq V_G$ such that $S$ is a (not necessarily proper) homogeneous set in $G$ and $G[S]$ is decomposable.*

*Proof.* Let $G \in \mathcal{G}^\#$, and assume inductively that the claim holds for every graph in $\mathcal{G}^\#$ with fewer than $|V_G|$ vertices. If $G \in \mathcal{G}$, then we are done. So assume that $G \in \mathcal{G}^\# \setminus \mathcal{G}$. If $G$ can be obtained from two graphs in $\mathcal{G}^\#$, each with fewer than $|V_G|$ vertices, by gluing along a clique, then $G$ is decomposable, and we are done. So assume that this is not the case. Then there exist non-empty graphs $G_1, G_2 \in \mathcal{G}^\#$ such that $V_{G_1} \cap V_{G_2} = \emptyset$ and $|V_{G_i}| < |V_G|$ for each $i \in \{1, 2\}$, and a vertex $u \in V_{G_1}$, such that $G$ is obtained by substituting $G_2$ for $u$ in $G_1$.

By the induction hypothesis, either $G_2 \in \mathcal{G}$ or there exists a homogeneous set $S_2 \subseteq V_{G_2}$ in $G_2$ such that $G_2[S_2]$ is decomposable. In the latter case, it easy to see that the set $S_2$ is a homogeneous set in $G$ as well, and that $G[S_2]$ is decomposable. So from now on, we assume that $G_2 \in \mathcal{G}$. Now, if $G_1 \in \mathcal{G}$, then since $G_2 \in \mathcal{G}$ and $\mathcal{G}$ is closed under substitution, we get that $G \in \mathcal{G}$, which is a contradiction. Thus, $G_1 \notin \mathcal{G}$. By the induction hypothesis then, there exists a non-empty set $S_1 \subseteq V_{G_1}$ such that $S_1$ is a homogeneous set in $G_1$ and

$G_1[S_1]$ is decomposable. If $u \notin S_1$, then it is easy to see that $S_1$ is a homogeneous set in $G$ and that $G[S_1]$ is decomposable. So assume that $u \in S_1$. Set $S = (S_1 \smallsetminus \{u\}) \cup V_{G_2}$. Clearly, $S$ is a homogeneous set in $G$ (as $S_1$ is a homogeneous set in $G_1$). Further, $G[S]$ is obtained by substituting $G_2$ for $u$ in the decomposable graph $G_1[S_1]$, and so it is easy to see that $G[S]$ is decomposable. This completes the argument. $\square$

**7.2.10.** *Let $\mathcal{G}$ be a hereditary class, closed under substitution. Then for all $G \in \mathcal{G}^{\#}$, there exists a graph $G' \in \mathcal{G}$ such that $G'$ is an induced subgraph of $G$ and $\chi(G') = \chi(G)$.*

*Proof.* Fix a graph $G \in \mathcal{G}^{\#}$, and assume inductively that the claim holds for every graph in $\mathcal{G}^{\#}$ that has fewer than $|V_G|$ vertices. If $G \in \mathcal{G}$, then the result is immediate; so assume that $G \notin \mathcal{G}$. Then by 7.2.9, there exists a non-empty set $S \subseteq V_G$ such that $S$ is a homogeneous set in $G$ and $G[S]$ is decomposable.

Since $G[S]$ is decomposable, there exist graphs $G_0, H_0, K_0 \in \mathcal{G}^{\#}$ such that $H_0$ and $K_0$ have inclusion-wise incomparable vertex-sets, such that $G_0$ can be obtained by gluing $H_0$ and $K_0$ along a clique, and such that $G[S]$ is an expansion of $G_0$. Set $C = V_{H_0} \cap V_{K_0}$; then $C$ is a clique in both $H_0$ and $K_0$, and $G_0$ is obtained by gluing $H_0$ and $K_0$ along $C$. Set $C = \{c_1, ..., c_r\}$, $V_{H_0} \smallsetminus C = \{h_1, ..., h_s\}$, and $V_{K_0} \smallsetminus C = \{k_1, ..., k_t\}$. Let $C_1, ..., C_r, H_1, ..., H_s, K_1, ..., K_t$ be non-empty graphs with pairwise disjoint vertex-sets such that $G[S]$ is obtained by substituting $C_1, ..., C_r, H_1, ..., H_s, K_1, ..., K_s$ for $c_1, ..., c_r$, $h_1, ..., h_s, k_1, ..., k_t$ in $G_0$. Set $\tilde{C} = \bigcup_{i=1}^{r} V_{C_i}$. Let $H$ be the graph obtained by substituting $C_1, ..., C_r, H_1, ..., H_s$ for $c_1, ..., c_r, h_1, ..., h_s$ in $H_0$; and let $K$ be the graph obtained by substituting $C_1, ..., C_r, K_1, ..., K_t$ for $c_1, ..., c_r, k_1, ..., k_t$ in $K_0$. Clearly, both $H$ and $K$ are proper induced subgraphs of $G[S]$. Our goal is to show that $\chi(G[S]) = \max\{\chi(H), \chi(K)\}$. Since $H$ and $K$ are induced subgraphs of $G[S]$, it suffices to show that $\chi(G[S]) \leq \max\{\chi(H), \chi(K)\}$.

Let $b'_H : V_H \to \{1, ..., \chi(H)\}$ be an optimal coloring of $H$. Since $V_{C_1}, ..., V_{C_r}$ are pairwise disjoint and complete to each other, we know that $b'_H$ uses pairwise disjoint color sets on these sets. Now, let $b_H : V_H \to \{1, ..., \chi(H)\}$ be defined as follows: for all $v \in V_H \smallsetminus \tilde{C}$, set $b_H(v) = b'_H(v)$, and for all $i \in \{1, ..., r\}$, assume that $b_H \upharpoonright V_{C_i}$ is an optimal coloring of $C_i$ using only the colors from $b'_H[C_i]$. As $V_{C_1}, ..., V_{C_r}$ are homogeneous sets in $H$, and $b_H[C_i] \subseteq b'_H[C_i]$ for all $i \in \{1, ..., r\}$, it easily follows that $b_H$ is a proper coloring of $H$. Now, note that $b_H : V_H \to \{1, ..., \chi(H)\}$ is an optimal coloring of $H$, $b_H[V_{C_1}], ..., b_H[V_{C_r}]$ are pairwise disjoint, and for each $i \in \{1, ..., r\}$, $|b_H[V_{C_i}]| = \chi(C_i)$. Similarly, there exists an optimal coloring $b_K : V_K \to \{1, ..., \chi(K)\}$ of $K$ such that $b_K[V_{C_1}], ..., b_K[V_{C_r}]$ are pairwise disjoint, and for each $i \in \{1, ..., r\}$, $|b_K[V_{C_i}]| = \chi(C_i)$. Relabeling if necessary, we may assume that $b_H \upharpoonright \tilde{C} = b_K \upharpoonright \tilde{C}$; as $V_H \cap V_K = \tilde{C}$, we can define $b_S : S \to \{1, ..., \max\{\chi(H), \chi(K)\}\}$ by setting

$$b_S(v) \quad = \quad \begin{cases} b_H(v) & \text{if} \quad v \in V_H \\ b_K(v) & \text{if} \quad v \in V_K \end{cases}$$

Since $V_H \smallsetminus \tilde{C}$ is anti-complete to $V_K \smallsetminus \tilde{C}$ in $G[S]$, this is a proper coloring of $G[S]$. It follows that $\chi(G[S]) = \max\{\chi(H), \chi(K)\}$. By symmetry, we may assume that $\chi(K) \leq \chi(H)$, so that $\chi(G[S]) = \chi(H)$.

Now, since $S$ is a homogeneous set in $G$, there exists a graph $\tilde{G} \in \mathcal{G}^\#$ such that $V_{\tilde{G}} \cap S = \emptyset$, and a vertex $u \in V_{\tilde{G}}$ such that $G$ is obtained by substituting $G[S]$ for $u$ in $\tilde{G}$. Let $G_H$ be the graph obtained by substituting $H$ for $u$ in $\tilde{G}$. Since $\chi(G[S]) = \chi(H)$, it is easy to see that $\chi(G[S]) = \chi(G_H)$. Since $H$ is a proper induced subgraph of $G[S]$, we have that $G_H$ is a proper induced subgraph of $G$. By the induction hypothesis then, there exists a graph $G' \in \mathcal{G}$ such that $G'$ is an induced subgraph of $G_H$ and $\chi(G') = \chi(G_H)$. But then $G' \in \mathcal{G}$ is an induced subgraph of $G$ and $\chi(G') = \chi(G)$. This completes the argument. $\square$

We are now ready to prove 7.2.8, restated below.

**7.2.8.** *Let $\mathcal{G}$ be a hereditary class, closed under substitution. Assume that $\mathcal{G}$ is $\chi$-bounded* *by a non-decreasing function $f : \mathbb{N}_0 \to \mathbb{R}$. Then $\mathcal{G}^{\#}$ is $\chi$-bounded by $f$.*

*Proof.* Fix $G \in \mathcal{G}^{\#}$. By 7.2.10, there exists a graph $G' \in \mathcal{G}$ such that $G'$ is an induced subgraph of $G$ and $\chi(G') = \chi(G)$. Since $G'$ is an induced subgraph of $G$, we know that $\omega(G') \leq \omega(G)$. Since $G' \in \mathcal{G}$ and $\mathcal{G}$ is $\chi$-bounded by $f$, we have that $\chi(G') \leq f(\omega(G'))$. Finally, since $\omega(G') \leq \omega(G)$ and $f$ is non-decreasing, $f(\omega(G')) \leq f(\omega(G))$. Now we have the following:

$$\chi(G) \quad = \quad \chi(G') \quad \leq \quad f(\omega(G')) \quad \leq \quad f(\omega(G)).$$

It follows that $\mathcal{G}^{\#}$ is $\chi$-bounded by $f$. □

We now use 7.2.8 and the results of section 7.1, in order to prove 7.2.7 (restated below), as well as 7.2.11, which is a strengthening of 7.2.7 in some special cases.

**7.2.7.** *Let $\mathcal{G}$ be a class of graphs, $\chi$-bounded by a non-decreasing function $f : \mathbb{N}_0 \to \mathbb{R}$. Then $\mathcal{G}^{\#}$ is $\chi$-bounded by the function $g(k) = f(k)^k$.*

*Proof.* We may assume that $\mathcal{G}$ is hereditary (otherwise, instead of $\mathcal{G}$, we consider the closure of $\mathcal{G}$ under isomorphism and taking induced subgraphs). Now, if $\mathcal{G}$ contains no non-empty graphs, then neither does $\mathcal{G}^{\#}$, and then $\mathcal{G}^{\#}$ is $\chi$-bounded by $g$ because $g(0) = 1$ and $\chi(H) = \omega(H) = 0$ for the empty graph $H$. So assume that $\mathcal{G}$ contains at least one non-empty graph; this implies that $f(0) \geq 0$ and that $f(n) \geq 1$ for all $n \in \mathbb{N}$; since $f$ is non-decreasing, this implies that $g$ is non-decreasing. Now, by 7.1.2, $\mathcal{G}^*$ is $\chi$-bounded by $g$. Next, note that $\mathcal{G}^{\#}$ is the closure of $\mathcal{G}^*$ under substitution and gluing along a clique. Since $\mathcal{G}^*$ closed under substitution and is $\chi$-bounded by the non-decreasing function $g$, 7.2.8 implies that $\mathcal{G}^{\#}$ is $\chi$-bounded by $g$. □

**7.2.11.** *Let $\mathcal{G}$ be a class of graphs, $\chi$-bounded by a polynomial (respectively: exponential) function $f : \mathbb{N}_0 \to \mathbb{R}$. Then $\mathcal{G}^{\#}$ is $\chi$-bounded by some polynomial (respectively: exponential) function $g : \mathbb{N}_0 \to \mathbb{R}$.*

*Proof.* The proof is analogous to the proof of 7.2.7, with 7.1.3 and 7.1.6 being used instead of 7.1.2. □

## 7.3 Open Questions

Let us say that an operation $O$ defined on the class of graphs *preserves $\chi$-boundedness* (respectively: *preserves hereditariness*) if for for every $\chi$-bounded (respectively: hereditary) class $\mathcal{G}$, the closure of $\mathcal{G}$ under the operation $O$ is again $\chi$-bounded (respectively: hereditary). This work raises the following natural question. Suppose that some $\chi$-boundedness preserving operations are given. Is the closure of a $\chi$-bounded class with respect to all the operations together $\chi$-bounded? In general, the answer is no. The *Mycielskian* $M(G)$ of a graph $G$ on $\{v_1, \dots, v_n\}$ is defined as follows: start from $G$ and for all $i \in \{1, ..., n\}$, add a vertex $w_i$ complete to $N_G(v_i)$ (note that $\{w_1, \dots, w_n\}$ is a stable set in $M(G)$); then add a vertex $w$ complete to $\{w_1, \dots, w_n\}$. It is well known (see [49]) that $\omega(M(G)) = \omega(G)$ and $\chi(M(G)) = \chi(G) + 1$ for every graph $G$ that has at least one edge. Now define two operations on graphs: $O_1(G)$ (respectively: $O_2(G)$) is defined to be $M(G)$ if $\chi(G)$ is odd (respectively: even), and to be $G$ otherwise. Clearly, $O_1$ (respectively: $O_2$) preserves $\chi$-boundedness; this follows from the fact that applying $O_1$ (respectively: $O_2$) repeatedly can increase the chromatic number of a graph at most by 1. But taken together, $O_1$ and $O_2$ may build triangle-free graphs of arbitrarily large chromatic number: by applying them alternately to the complete graph on two vertices for instance. However, this example may look artificial; perhaps some more "natural" kinds of operations, to be defined, have better behavior? Note that, unlike the three operations discussed in this chapter (substitution, gluing along a clique, and gluing along a bounded number of vertices), the Mycielskian does not preserve hereditariness. This suggests a candidate for which we have no counterexample:

**Question.** *If $O_1$ and $O_2$ are operations that (individually) preserve hereditariness and $\chi$-boundedness, do $O_1$ and $O_2$ together preserve $\chi$-boundedness?*

Note that we do not know the answer in the following particular case:

**Question.** *Is the closure of a $\chi$-bounded class under substitution and gluing along a bounded number of vertices $\chi$-bounded?*

Are there other operations that in some sense preserve $\chi$-boundedness? A *star* in a graph $G$ is a set $S \subseteq V_G$ such that some vertex $v \in S$ is complete to $S \setminus \{v\}$. A *star cutset* of a graph is star whose deletion yields a disconnected graph. Star cutsets are interesting in our context because their introduction by Chvátal [23] was the first step in a series of theorems that culminated in the proof of the Strong Perfect Graph Conjecture 1.1.2. Also, several classes of graphs that are notoriously difficult to decompose are decomposed with star cutsets or variations on star cutsets: star cutsets are used to decompose even-hole-free graphs (see [58]); skew partitions are used to decompose Berge graphs (see [16]); double star cutsets are used to decompose odd-hole-free graphs (see [26]). Could it be that some of these decompositions preserve $\chi$-boundedness? If so, the following open question could be a good starting point (and should have a positive answer):

**Question.** *Is there a constant $c$ such that if a graph $G$ is triangle-free and all induced subgraphs of $G$ either are 3-colorable or admit a star cutset, then $G$ is $c$-colorable?*

# Bibliography

[1] N. Alon, D.J. Kleitman, M. Saks, P. Seymour, and C. Thomassen, "Subgraphs of large connectivity and chromatic number in graphs of large chromatic number", *J. Graph Theory* 11 (1987), 367–371.

[2] C. Berge, "Les problèmes de coloration en théorie des graphes", Publ. Inst. Statist. Univ. Paris, 9 (1960), 123–160.

[3] C. Berge and V. Chvátal (Eds.), *Topics on perfect graphs*, volume 21 of *Annals of Discrete Mathematics*, North Holland, Amsterdam, 1984.

[4] M. Burlet and J. Fonlupt, "A polynomial time algorithm to recognize a Meyniel graph", In *Berge and Chvátal [3]*, 225–252.

[5] K. Cameron, "Antichain sequences", *Order*, 2 (1985), 249–255.

[6] M. Chudnovsky, "Berge trigraphs and their applications", PhD thesis, Princeton University, 2003.

[7] M. Chudnovsky, "The structure of bull-free graphs I - three-edge-paths with centers and anticenters", *Journal of Combinatorial Theory. Ser B,* 102 (2012), 233–251.

[8] M. Chudnovsky, "The structure of bull-free graphs II - elementary bull-free graphs", *Available online in the Supplementary material at doi:10.1016/j.jctb.2011.07.002.*

[9] M. Chudnovsky, "The structure of bull-free graphs III - global structure", *Available online in the Supplementary material at doi:10.1016/j.jctb.2011.07.002.*

[10] M. Chudnovsky, "The structure of bull-free graphs II and III - a summary", *Journal of Combinatorial Theory. Ser B,* 102 (2012), 252–282.

[11] M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour, and K. Vušković, "Recognizing Berge graphs", *Combinatorica*, 25 (2005), 143-186.

[12] M. Chudnovsky and I. Penev, "The structure of bull-free perfect graphs", *submitted for publication.*

[13] M. Chudnovsky, I. Penev, A.D. Scott, and N. Trotignon, "Excluding induced subdivisions of the bull and related graphs", *to appear in Journal of Graph Theory.*

[14] M. Chudnovsky, I. Penev, A.D. Scott, and N. Trotignon, "Substitution and $\chi$-Boundedness", *submitted for publication.*

[15] M. Chudnovsky and M. Plumettaz, "The structure of claw-free perfect graphs", *submitted for publication.*

[16] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, "The strong perfect graph theorem", *Annals of Mathematics* 164 (2006), 51–229.

[17] M. Chudnovsky and S. Safra, "The Erdős-Hajnal conjecture for bull-free graphs", *Journal of Combinatorial Theory, Ser. B*, 98(6) (2008), 1301–1310.

[18] M. Chudnovsky and P. Seymour, "Claw-free Graphs I. Orientable prismatic graphs", *Journal of Combinatorial Theory. Ser B*, 97 (2007), 867–901.

[19] M. Chudnovsky and P. Seymour, "Claw-free Graphs II. Non-orientable prismatic graphs", *Journal of Combinatorial Theory. Ser B*, 98 (2008), 249–290.

[20] M. Chudnovsky and P. Seymour, "Claw-free Graphs III. Circular Interval Graphs", *Journal of Combinatorial Theory. Ser B*, 98 (2008), 812–834.

[21] M. Chudnovsky and P. Seymour, "Claw-free Graphs IV. Decomposition theorem", *Journal of Combinatorial Theory. Ser B*, 98 (2008), 839–938.

[22] M. Chudnovsky and P. Seymour, "Claw-free Graphs V. Global structure", *Journal of Combinatorial Theory. Ser B*, 98 (2008), 1373–1410.

[23] V. Chvátal, "Star-cutsets and perfect graphs", *Journal of Combinatorial Theory, Ser. B*, 39 (1985), 189–199.

[24] V. Chvátal and N. Sbihi, "Bull-free Berge graphs are perfect", *Graphs and Combin.*, 3 (1987), 127–139.

[25] F. Cicalese and M. Milanič, "Graphs of separability at most 2", *to appear in Discrete Applied Mathematics.*

[26] M. Conforti, G. Cornuéjols, and K. Vušković, "Decomposition of odd-hole-free graphs by double star cutsets and 2-joins", *Discrete Applied Mathematics*, 141 (2004), 41–91.

[27] G. Cornuéjols and W.H. Cunningham, "Composition for perfect graphs", *Discrete Mathematics*, 55 (1985), 245–254.

[28] A. Cournier and M. Habib, "A new linear algorithm for modular decomposition", in *Trees in Algebra and Programming - CAAP*, 1994 (Edinburgh), Lecture Notes in Comput. Sci. 787, Springer-Verlag, Berlin, 1994, 68–84.

[29] E. Dahlhaus, J. Gustedt, and R.M. McConnell, "Efficient and practical modular decomposition", in *Proceedings of the 8th Annual ACM-SIAM Symposium on Discrete Algorithms*, ACM, New York, SIAM, Philadelphia, 1997, 26–35.

[30] P. Erdős, "Graph theory and probability. II", *Canadian J. Math.*, 13 (1961), 346–352.

[31] P. Erdős and A. Hajnal, "On chromatic number of graphs and set systems", *Acta Math. Sci. Hung.* 17 (1966), 61–69.

[32] U. Faigle and G. Frahling, "A combinatorial algorithm for weighted stable sets in bipartite graphs", *Discrete Appl. Math.*, 154 (2006), 1380–1391.

[33] C.M.H. de Figueiredo and F. Maffray, "Optimizing bull-free perfect graphs", *SIAM J. Discrete Math*, 18 (2004), 226-240.

[34] C.M.H. de Figueiredo, F. Maffray, and O. Porto, "On the structure of bull-free perfect graphs", *Graphs and Combin.*, 13 (1997), 31-55.

[35] T. Gallai. "Graphen mit triangulierbaren ungeraden Vielecken." *A Magyar Tudományos Akadémia — Matematikai Kutató Intézetének Közleményei*, 7 (1962), 3–36.

[36] G. Gasparyan, "Minimal imperfect graphs: A simple approach", *Combinatorica* 16 (2) (1996), 209–212.

[37] M.C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Second edition, Annals of Discrete Mathematics 57, Elsevier, 2004.

[38] M.C. Golumbic, "The complexity of comparability graph recognition and coloring", *Computing*, 18 (1977), 199–208.

[39] M. Grötschel, L. Lovász, and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization*, Springer-Verlag, New York, 1988.

[40] A. Gyárfás, "On Ramsey covering-numbers", *Coll. Math. Soc. János Bolyai* In *Infinite and Finite Sets*, North Holland/American Elsevier, New York, 1975, 10.

[41] A. Gyárfás, "Problems from the world surrounding perfect graphs", *Zastowania Matematyki Applicationes Mathematicae*, 19 (1987), 413–441.

[42] A. Gyárfás, E. Szemerédi, and Zs. Tuza, "Induced subtrees in graphs of large chromatic number", *Discrete Mathematics*, 30 (1980), 235-244.

[43] C.T. Hoàng, "Efficient algorithms for minimum weighted colouring of some classes of perfect graphs", *Discrete Appl. Math.*, 55 (1994), 133-143.

[44] H.A. Kierstead and S.G. Penrice, "Radius two trees specify $\chi$-bounded classes", *Journal of Graph Theory*, 18 (1994), 119-129.

[45] H.A. Kierstead and Y. Zhu, "Radius three trees in graphs with large chromatic number", *SIAM Journal of Discrete Mathematics*, 17 (2004), 571-581.

[46] J.H. Kim, "The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$", *Random Structures and Algorithms*, 7(3) (1995), 173–208.

[47] B. Lévêque, F. Maffray, and N. Trotignon, "On graphs with no induced subdivision of $K_4$", *submitted for publication.*

[48] L. Lovász, "Normal hypergraphs and the perfect graph conjecture", *Discrete Mathematics* 2 (1972) 253–267.

[49] J. Mycielski, "Sur le coloriage des graphs", *Colloquium Mathematicum*, 3 (1955), 161–162.

[50] S. Olariu, "Paw-free graphs", *Information Processing Letters*, 28 (1988), 53–54.

[51] A. Pawlik, J. Kozik, T. Krawczyk, M. Lasoń, P. Micek, W. Trotter, and B. Walczak, "Triangle-free intersection graphs of line segments with large chromatic number", *manuscript.*

[52] I. Penev, "Coloring bull-free perfect graphs", *submitted for publication.*

[53] J.L. Ramírez Alfonsín and B.A. Reed (Eds.), *Perfect graphs*, Series in Discrete Mathematics and Optimization, Wiley-Interscience, 2001.

[54] B. Reed and N. Sbihi, "Recognizing bull-free perfect graphs", *Graphs and Combin.*, 11 (1995), 171–178.

[55] A. Schrijver, *Combinatorial Optimization: Polyhedra and Efficiency*, Springer-Verlag, New York, 2003.

[56] A.D. Scott. "Induced cycles and chromatic number", *Journal of Combinatorial Theory, Ser. B*, 76 (1999), 70–75.

[57] A.D. Scott, "Induced trees in graphs of large chromatic number", *Journal of Graph Theory*, 24 (1997), 297–311.

[58] M.V.G. da Silva and K. Vušković, "Decomposition of even-hole-free graphs with star cutsets and 2-joins", *preprint*, (2008).

[59] J. Spinrad, "On comparability and permutation graphs", *SIAM J. Comput.*, 14 (1985), 658–670.

[60] D.P. Sumner, "Subtrees of a graph and chromatic number", In *The Theory and Applications of Graphs*, G. Chartrand (Ed.), John Wiley and Sons, New York (1981), 557–576.

[61] N. Trotignon and K. Vušković, "A structure theorem for graphs with no cycle with a unique chord and its consequences", *Journal of Graph Theory*, 63 (1) (2010), 31–67.