The Asymptotic Cone of Teichmüller Space: Thickness and Divergence

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ABSTRACT

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Using the geometric model of the pants complex, we study the Asymptotic Cone of Teichmüller space equipped with the Weil-Petersson metric. In particular, we provide a characterization of the canonical finest pieces in the tree-graded structure of the asymptotic cone of Teichmüller space along the same lines as a similar characterization for right angled Artin groups in [4] and for mapping class groups in [8]. As a corollary of the characterization, we complete the thickness classification of Teichmüller spaces for all surfaces of finite type, thereby answering questions of Behrstock-Druţu [5], Behrstock-Druţu-Mosher [6], and Brock-Masur [21]. In particular, we prove that Teichmüller space of the genus two surface with one boundary component (or puncture) can be uniquely characterized in the following two senses: it is thick of order two, and it has superquadratic yet at most cubic divergence. In addition, we characterize strongly contracting quasi-geodesics in Teichmüller space, generalizing results of Brock-Masur-Minsky [23]. As a tool in the thesis, we develop a natural relative of the curve complex called the complex of separating multicurves, $\mathbb{S}(S)$, which may be of independent interest.

The final chapter includes various related and independent results including, under mild hypotheses, a proof of the equivalence of wideness and unconstrictedness in the CAT(0) setting, as well as adapted versions of three preprints, [63, 64, 65]; the last was recently published in the New York Journal of Mathematics. Specifically, in the three preprints we characterize hyperbolic type quasi-geodesics in CAT(0) spaces, we prove that $\mathcal{C}_{sep}(S_{2,0})$ satisfies a quasi-distance formula and is δ-hyperbolic, and we study the net of separating pants decompositions in the pants complex.

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Chapter 1

Introduction

"One geometry cannot be more true than another; it can only be more convenient."

-Henri Poincaré

1.1 Overview and context

For S a surface of finite type, *Teichmüller space*, denoted $\mathcal{T}(S)$, with origins in the work of Fricke, Fenchel, and Nielsen is a classical space which parameterizes isotopy classes of hyperbolic structures on S. In the literature there are various natural metrics with which Teichmüller space can be equipped. Hereinafter, we always consider $\mathcal{T}(S)$ with the Weil-Petersson metric. The Weil-Petersson metric on $\mathcal{T}(S)$ is a complex analytically defined Riemannian metric of variable non-positive curvature. While the space is not complete, its completion, $\overline{\mathcal{T}}(S)$, obtained by augmenting Teichmüller spaces of lower complexity surfaces corresponding to limit points in the space with pinched curves, is a CAT(0) metric space [66, 68].

As we will see, the spectrum of non-positively curved geometries of $\mathcal{T}(S)$ for various surfaces S is extremely broad. In particular, the geometry of Teichmüller spaces includes on the one hand examples of hyperbolic and strongly relatively hyperbolic metric spaces, and on the other hand thick of order one and thick of order two metric spaces. The example of a

Teichmüller space which is thick of order two is a novelty of this thesis. In a similar vein, we will see that the divergence function of Teichmüller spaces includes examples of spaces with quadratic divergence, superquadratic yet at most cubic divergence, exponential divergence, and infinite divergence. Again, the example of a Teichmüller space which has superquadratic yet at most cubic divergence is a novelty of this thesis.

The above referenced notion of thickness, developed in [6] and further explored in [5], is aptly named as it stands in stark contrast to relative hyperbolicity. In fact, if a space is thick of any finite order than it is not strongly relatively hyperbolic, [6]. Informally the *order of thickness of a space* should thought as a precise means of interpolating between product spaces, which are thick of order zero, and (relatively)-hyperbolic spaces, which are not thick of any finite order, or are thick of order infinity. More specifically, thickness is defined inductively. A space is *thick of order zero* if none of its asymptotic cones have cut-points. More generally, a space is *thick of order at most* n if the entire space is coarsely the union of a collection of quasi-convex subsets which are thick of order $n-1$, and such that given any two thick of order $n-1$ subsets in the collection, there is a finite chain of thick of order $n-1$ subsets between the two given subsets, such that each of the subsets in the chain coarsely intersects its neighboring subsets in an infinite diameter set. Finally, a space is *thick of order* n, if n is the smallest integer such that it is thick order at most n.

The large scale geometry of Teichmüller space has been an object of recent interest, especially within the circles of ideas surrounding Thurston's Ending Lamination Conjecture. In this context, the pants complex, $\mathcal{P}(S)$, a combinatorial complex associated to a hyperbolic surface S, becomes relevant. Specifically, by a groundbreaking theorem of Brock [19], $\mathcal{P}(S)$ is quasi-isometric to $\mathcal{T}(S)$. Accordingly, in order to study large scale geometric properties of Teichmüller space, it suffices to study the pants complex of a surface. For instance, significant recent results of Behrstock [3], Behrstock-Minsky [10], Brock-Farb [20], Brock-Masur [21], and Brock-Masur-Minsky [22, 23], among others, can be viewed from this perspective. Similarly, all of the results of this thesis regarding the coarse structure of the pants complex should be interpreted as coarse results regarding Teichmüller space.

The pants complex is closely related to two central objects in the field of geometric group theory: the curve complex, $\mathcal{C}(S)$, and the mapping class group, $\mathcal{MCG}(S)$. By definition, vertices of the pants complex are in correspondence with maximal simplices of the curve complex. Moreover, a necessary condition for two vertices in the pants complex to be connected by an edge is that their corresponding maximal simplices in the curve complex share a co-dimension one face. In fact, for low complexity surfaces the pants complex is precisely the curve complex. On the other hand, the pants complex also shares similarity with the mapping class group. In fact, the mapping class group acts co-finitely and by isometries on the pants complex. However, this action is not properly discontinuous. Nonetheless, the *marking complex*, a quasi-isometric combinatorial model for the mapping class group, is strikingly similar to the pants complex.

We will exploit the similarity between the marking complex and the pants complex. Specifically, we apply numerous tools developed primarily in the course of studying the marking complex (or equivalently, the mapping class group) by Behrstock [3], Behrstock-Kleiner-Minsky-Mosher [8], Behrstock-Minsky [10], and Masur-Minsky [48], to the pants complex (or equivalently, Teichmüller space).

The asymptotic cone of a metric space, a notion invented by Gromov and further developed by van den Dries and Wilkie, is an important tool for understanding the large scale geometry of a metric space, [36, 27]. In recent years, study of asymptotic cones has proven extremely fruitful in considering the coarse geometry of groups and spaces. See for instance [7, 28, 30]. One aspect in common to the aforementioned studies of asymptotic cones is interest in *cut-points*, namely single points whose removal disconnects the asymptotic cone. The general theme is that cut-points in asymptotic cones correspond to a weak form of hyperbolicity in the underlying space. One of the highlights of the thesis is a characterization of when two points in the asymptotic cone of Teichmüller space are separated by a cut-point, see Theorem 4.2.3.

On the one hand, it is shown in [3] that in the asymptotic cone of Teichmüller space, every point is a global cut-point. On the other hand, for high enough complexity surfaces, Teichmüller space has natural quasi-isometrically embedded flats, or *quasi-flats*, [10, 20, 48]. In turn, this implies the existence of naturally embedded flats in the asymptotic cone and hence the existence of nontrivial subsets of the asymptotic cone without cut-points. Putting things together, for high enough complexity surfaces, the asymptotic cone of Teichmüller space is a *tree-graded space*. In such a setting, there are canonically defined *finest pieces* of the tree-graded structure, which are defined to be maximal subsets of the asymptotic cone subject to the condition that no two points in a finest piece can be separated by the removal of a point. A highlight of this thesis is the following theorem that characterizes the finest pieces in tree-graded structure of the asymptotic cone of Teichmüller space.

Theorem 4.2.3. *Let* $S = S_{g,n}$*, and let* $\mathcal{P}_\omega(S)$ *be any asymptotic cone of* $\mathcal{P}(S)$*. Then* $\forall a_\omega, b_\omega \in \mathcal{P}\omega(S)$*, the following are equivalent:*

1. *No point separates* a_ω *and* b_ω*, or equivalently* a_ω *and* b_ω *are in the same canonical finest piece, and*

2. *In any neighborhood of* a_ω, b_ω*, respectively, there exists* a'_ω, b'_ω*, with representative sequences* (a'_n)*,*(b'_n)*, such that* $\lim_\omega d_{\mathbb{S}(S)}(a'_n, b'_n) < \infty$*.*

The characterization of finest pieces in Theorem 4.2.3 is given in terms of the complex of separating multicurves $\mathbb{S}(S)$ which encodes information about the natural product structures in the pants complex. The complex of separating multicurves will be defined and explored in Chapter 3. The proof of Theorem 4.2.3 relies heavily on a notion of *structurally integral corners* to be developed in Section 4.1. Roughly speaking, a structurally integral corner is a point in the asymptotic cone whose removal disconnects particular natural product regions. Structurally integral corners only exist for low complexity surfaces. Theorem 4.2.3 should be compared with Theorem 4.6 of [4] and Theorem 7.9 of [8] where similar characterizations of the finest pieces are proven for right angled Artin groups and mapping class groups, respectively.

The following two celebrated theorems can be recovered as special cases of Theorem 4.2.3.

Corollary 4.2.6. ([3, 20] Theorem 5.1, Theorem 1.1). $\mathcal{T}(S_{1,2})$ *and* $\mathcal{T}(S_{0,5})$ *are* δ*-hyperbolic.*

Corollary 4.2.7. ([21] Theorem 1). *For $S \in \{S_{0,6}, S_{1,3}, S_{2,0}\}$, $\mathcal{T}(S)$ is relatively hyperbolic with respect to natural quasi-convex product regions consisting of all pairs of pants with a fixed separating curve.*

More generally, in the course of studying non-positively curved metric spaces, such as $\mathcal{T}(S)$, one is frequently interested in families of geodesics which admit *hyperbolic type* properties, or properties exhibited by geodesics in hyperbolic space which are not exhibited by geodesics in Euclidean space. In the geometric group theory literature there are various well studied examples of such hyperbolic type properties including being Morse, being contracting, and having cut-points in the asymptotic cone. Such studies have proven fruitful in analyzing right angled Artin groups [4], Teichmüller space [3, 20, 21, 23], the mapping class group [3], CAT(0) spaces [5, 13, 26], and Out(F_n) [1] among others (for instance [29, 30, 41, 55, 47]).

A *morse geodesic* γ is defined by the property that all quasi-geodesics σ with endpoints on γ remain within a bounded distance from γ. A *strongly contracting geodesic* has the property that metric balls disjoint from the geodesic have nearest point projections onto the geodesic with uniformly bounded diameter. It is an elementary fact that in hyperbolic space all geodesics are Morse and strongly contracting. On the other end of the spectrum, in product spaces such as Euclidean spaces of dimension two and above, there are no Morse or strongly contracting geodesics. Relatedly, there are no cut-points in any asymptotic cones of product spaces, whereas all asymptotic cones of a δ-hyperbolic spaces are $\mathbb{R}$-trees, and hence any two distinct points are separated by a cut-point.

In [63], which is reproduced in Section 6.2, we prove that in CAT(0) spaces the aforementioned hyperbolic type properties of quasi-geodesics are closely related. Specifically, we have the following theorem building on similar theorems in [13, 26, 29, 41].

Theorem 6.2.5. *Let X be a CAT(0) space and $\gamma \subset X$ a quasi-geodesic. Then, the following are equivalent:*

1. *γ is (b,c)–contracting,*
2. *γ is strongly contracting,*

3. *γ is Morse, and*

4. *In every asymptotic cone X_ω, any two distinct points in the ultralimit γ_ω are separated by a cut-point.*

In particular, any of the properties listed above implies that γ has at least quadratic divergence.

Combining Theorems 4.2.3 and 6.2.5, in the following Theorem we characterize all strongly contracting (or equivalently Morse) quasi-geodesics in $\mathcal{T}(S)$. This family of strongly contracting quasi-geodesics represents a generalization of quasi-geodesics with *bounded combinatorics* studied in [23] and similarly in [3]. In the aforementioned papers it is shown that quasi-geodesics in $\mathcal{P}(S)$ which have uniformly bounded subsurface projections to all connected proper essential subsurfaces, or equivalently stay in the thick part of Teichmüller space, are necessarily contracting. Generalizing this result, we prove the following:

Theorem 4.3.3. *Let γ be a quasi-geodesic in $\overline{\mathcal{T}}(S)$, and let γ' be a corresponding quasi-geodesic in $\mathcal{P}(S)$. Then γ is strongly contracting if and only if there exists a constant C such that for all $Y \in \mathcal{SE}(S)$, the subsurface projection $\pi_Y(\gamma')$ has diameter bounded above by C.*

In particular, as an immediate consequence of Theorem 4.3.3, we have the following corollary highlighting a distinction between $\mathcal{M}CG(S)$ and $\mathcal{T}(S)$.

Corollary 4.3.4. *Let γ be any partial pseudo-Anosov axis in $\overline{\mathcal{T}(S)}$ supported on a connected nonseparating essential subsurface $Y \in \mathcal{NE}(S)$, then γ is strongly contracting.*

Later in the thesis we focus in particular on the Teichmüller space of the surface $S_{2,1}$ which in the literature has previously proven to be difficult to analyze. As noted, for "small" complexity surfaces which don't admit any nontrivial separating curves, Brock-Farb [20] prove that $\mathcal{T}(S)$ is hyperbolic. A new proof was later provided by Behrstock in [3]. Similarly, for "medium" complexity surfaces, which admit nontrivial separating curves, yet have the property that any two separating curves intersect, Brock-Masur prove that $\mathcal{T}(S)$ is relatively hyperbolic, [21]. Finally, for all the remaining "large" complexity surfaces excluding $S_{2,1}$, whose complexes of separating multicurves only have a single infinite diameter connected component, the combined work of [3, 21], implies that the Teichmüller spaces of these surfaces are not relatively hyperbolic and in fact are thick of order one. However, unlike all other surfaces of finite type, the surface $S_{2,1}$ has the peculiar property that it is "large enough" such that it admits disjoint separating curves, although "too small" such that the complex of separating multicurves has infinitely many infinite diameter connected components. As we will see, this phenomenon makes the study of the Teichmüller space of $S_{2,1}$ quite rich.

Using Theorem 4.2.3 in conjunction with a careful analysis of the Brock-Masur construction for showing that $\mathcal{T}(S_{2,1})$ is thick of order at most two [21], we prove the following theorem answering question 12.8 of [6].

Theorem 5.2.7. $\mathcal{T}(S_{2,1})$ *is thick of order two.*

Notably, Theorem 5.2.7 completes the thickness classification of the Teichmüller spaces of all surfaces of finite type. Moreover, among all surfaces of finite type, $S_{2,1}$ is the only surface that is thick of order two.

The *divergence* of a metric space measures the inefficiency of detour paths. More formally, divergence along a geodesic is defined as the growth rate of the length of detour paths connecting sequences of pairs of points on a geodesic, where the distance between the pairs of points is growing linearly while the detour path is forced to avoid linearly sized metric balls centered along the geodesic between the pairs of points. It is an elementary fact of Euclidean geometry that Euclidean space has linear divergence. On the other end of the spectrum, hyperbolic space has exponential divergence.

Given this gap between the linear divergence in Euclidean space and the exponential divergence in hyperbolic space, the exploration of spaces with "intermediate divergence" provides a means of understanding a rich spectrum of non-positively curved geometries which interpolate between flat and negatively curved geometries. The history of this exploration goes back to Gromov, who noticed that δ-hyperbolic spaces, like $\mathbb{H}^n$, have at least exponential divergence, [37]. Gromov then asked if there were non-positively curved spaces whose divergence functions were superlinear yet subexponential, [38]. Soon afterward, Gersten answered this question in the affirmative by constructing CAT(0) groups with quadratic divergence, [35]. In short order Gersten proved that in fact the family of fundamental groups of graph manifolds provided natural examples of spaces with quadratic divergence [34]. Moreover, in recent years it has been shown that various other well studied groups such as mapping class groups, right angled Artin groups, and Teichmüller spaces with the Teichmüller metric also have quadratic divergence, [3, 4, 31].

After identifying spaces with quadratic divergence, Gersten went on to reformulate Gromov's question and asked if there existed CAT(0) spaces with superquadratic yet subexponential divergence. This latter question of Gersten was recently answered in the affirmative by independent papers of Behrstock-Druţu and Macura who each constructed CAT(0) groups with polynomial of degree n divergence functions for every natural number n, [5, 45]. In Section 5.3 we show that a naturally occurring Teichmüller space, $\mathcal{T}(S_{2,1})$, which is CAT(0), also provides an example answering Gersten's question in the affirmative. In fact, we prove the following theorem answering question 4.19 in [5]:

Theorem 5.3.7. *$\mathcal{T}(S_{2,1})$ has superquadratic yet at most cubic divergence. Moreover, it is the unique Teichmüller space with this property.*

A common approach to proving that a geodesic has at least quadratic divergence is to show that a geodesic is contracting. Contraction implies that in order for a connected subsegment of a detour path avoiding a ball of radius R centered on the geodesic to have nearest point projection onto the geodesic of more than a uniformly bounded diameter, the length of the subsegment must be linear in R. In turn, it follows that a detour path must travel at least a linear amount of linear distances, and hence at least a quadratic distance. See [3] for such an approach in proving that $\mathcal{MCG}$ has (at least) quadratic divergence. In the proof of Theorem 5.3.7 we follow the previously sketched outline, although we pick a careful example of a quasi-geodesic such that we can show that a detour path must in fact travel a linear amount of superlinear distances, thereby ensuring superquadratic divergence. Since cut-points in asymptotic cones correspond to instances of superlinear divergence, Theorem 4.2.3 has a role in the proof of Theorem 5.3.7. It should be noted that conjecturally we believe that $\mathcal{T}(S_{2,1})$ has cubic divergence. An approach toward proving this is presented in Section 5.4.

In the literature there are a couple of closely related notions of thickness, see [5, 6], whose differences stem from the following distinction between wide and unconstricted spaces. Specifically, a metric space X is called *wide* if all asymptotic cones X_ω are without cut-points. On the other hand, a metric space X is called *unconstricted* if there exists some ultrafilter ω and some sequence of scalars s_i such that any asymptotic cone $Cone_\omega(X, \cdot, (s_i))$ does not have cut-points. In this thesis, as in [5], we adopt a strong form of thickness which uses wide spaces as the base case of its inductive definition. Nonetheless, while in general being unconstricted is strictly weaker than being wide, in Section 6.1 we prove the following theorem which may be of independent interest. The proof is based on Lemma 6.1.2, which ensures a minimal uniformity of nearest point projection maps in CAT(0) spaces onto convex subspaces, and may also be of independent interest.

Theorem 6.1.1. *For X a CAT(0) space with extendable geodesics, X is wide if and only if it is unconstricted. Moreover, if in addition X is coarsely homogeneous, then either every asymptotic cone of X has a cut-point, or no asymptotic cone of X has a cut-point.*

The final three sections of Chapter 6 represent adapted versions of the following papers, [63, 64, 65]. In Section 6.2 we use methods of CAT(0) geometry and asymptotic cones to study hyperbolic type quasi-geodesics in CAT(0) spaces. The proof of Theorem 6.2.5 cited above is the highlight of the section.

In Section 6.3, using some nice properties of Farey graphs we prove that the separating curve complex $\mathbb{S}(S_{2,0})$ is δ-hyperbolic, answering a question of Schleimer [61]. More specifically, we prove the following quasi-distance formula for $\mathbb{S}(S_{2,0})$ which is similar to as well as motivated by quasi-distance formulas for $\mathcal{M}(S)$ and $\mathcal{P}(S)$ in [48]:

Theorem 6.4.4. *There is constant K_0 such that for all $k \geq K_0$ there exist quasi-isometry constants such that* $\forall \alpha, \beta \in \mathbb{S}(S_{2,0})$,

$$d_{\mathcal{S}(S_{2,0})}(\alpha, \beta) \approx \sum_{Y \subset S_{2,0} | \partial Y \notin \mathbb{S}(S_{2,0})} \{d_{\mathcal{C}(Y)}(\alpha, \beta)\}_k$$

where the threshold function $\{f(x)\}_k := f(x)$ *if* $f(x) \geq k$, *and* 0 *otherwise.*

In Section [65], using graph theoretic methods, we provide the following asymptotically sharp bounds on the maximal distance in the pants complex from any pants decomposition to the set of pants decompositions containing a separating curve, $\mathcal{P}_{sep}(S)$.

Theorem 6.4.1. *Let* $S = S_{g,n}$ *and set* $D_n(g) = \max_{P \in \mathcal{P}(S)}(d_{\mathcal{P}(S)}(P, \mathcal{P}_{sep}(S)))$. *Then, for any fixed number of boundary components (or punctures)* n*, the function* $D_n(g)$ *grows proportionally to* $\log(g)$. *On the other hand, for any fixed genus* $g \geq 2$, $\forall n \geq 6g - 5$, $D_{g,n} = 2$.

The lower bounds in Theorem 6.4.1 follow from an original and explicit constructive algorithm for an infinite family of high girth at most cubic graphs with the property that the minimum cardinality of connected cut-sets is a logarithmic function with respect to the vertex size of the graphs.

1.2 Outline of subsequent chapters

Chapter 2 provides background material. Section 2.1 reviews background material, including but not limited to essential subsurfaces, the curve and pants complexes, the asymptotic cone of a space, and thickness. Section 2.2 reviews coarse geometric tools of the curve and pants complex, most prominently subsurface projections and hierarchy paths. In addition, Section 2.2 also reviews tools of Behrstock-Kleiner-Minsky-Mosher, and Behrstock-Minsky, [8, 10] including convex regions, regions of sublinear growth, the consistency theorem, and jets.

Chapter 3 introduces and analyzes the complex of separating multicurves, $\mathbb{S}(S)$. This natural combinatorial complex encodes the network of natural quasi-convex product regions in Teichmüller space. Properties of the complex including connectivity and the existence of a quasi-distance formula are proven. In addition, as an example which is relevant to some of the analysis of $\mathcal{P}(S_{2,1})$ in Chapter 5, the complex $\mathbb{S}(S_{2,1})$ is considered at length and is related to the point pushing subgroup of the mapping class group. Finally, the relationship between the complex and asymptotic cones are considered.

In Chapter 4, we study the asymptotic cone of Teichmüller space. In particular, we characterize the canonical finest pieces in the tree-graded structure of the asymptotic cone. In Section

4.1 the notion of a structurally integral corner in the asymptotic cone of the pants complex is developed. Informally, a structurally integral corner entails the joining of two particular natural convex product regions in the asymptotic cone at a corner such that the removal of the corner separates the two product regions. This separation property in the asymptotic cone of the pants complex provided by a structurally integral corner will be a major ingredient in the proof of the characterization of the finest pieces of the asymptotic cone of the pants complex. Structurally integral corners are motivated by an attempt to generalize the theory of jets developed in [8] in the context of the curve complex to the separating complex.

Section 4.2 contains the characterization of the finest pieces in the tree-graded structure of the asymptotic cone of the pants complex. Applications of this characterization include Corollaries 4.2.6 and 4.2.7, above. Next, Section 4.3 uses the analysis of the previous section to characterize strongly contracting quasi-geodesics in Teichmüller space.

In Chapter 5 we prove that $\mathcal{T}(S_{2,1})$ is thick of order two. We begin in Section 5.1 by carefully considering the construction of Brock-Masur [21], which proves that $\mathcal{T}(S_{2,1})$ is thick of order at most two. Then, in Section 5.2, using the characterization of the finest pieces in the tree-graded structure of the asymptotic cone of the pants complex, we show that $\mathcal{T}(S_{2,1})$ cannot be thick of order one, by showing that a maximal thick of order one subset in the pants complex has infinite Hausdorff distance from the entire space. Next, in Section 5.3, we prove that $\mathcal{T}(S_{2,1})$ has superquadratic divergence by constructing an explicit example. Finally, Section 5.4 concludes with progress toward proving that $\mathcal{T}(S_{2,1})$ has cubic divergence as well as some related open questions.

Chapter 6 consists of four independent sections. In Section 6.1, under mild conditions we prove the equivalence of wideness and unconstrictedness in the CAT(0) setting. In Section 6.2 we characterize hyperbolic type quasi-geodesics in CAT(0) spaces. In Section 6.3 we prove that $\mathcal{C}_{sep}(S_{2,0})$ is δ-hyperbolic. Finally, in Section 6.4 we study the net of pants decompositions containing a separating curve in the pants complex.

Chapter 2

Preliminaries

The chapter is broken up into two sections. Section 2.1 focuses on background material and establishes some notations. Section 2.2 focuses on certain tools developed primarily in the course of studying the large scale geometry of the mapping class group via the marking complex. These tools are described via the curve complex.

2.1 Background

2.1.1 Quasi-Isometries and Coarse intersections

In studying the large scale properties of a space, in place of the usual notions of intersections between subsets and continuous maps between spaces, it is useful to consider the notions of coarse intersection and quasi-isometries. The latter notions are natural generalizations to the large scale setting of the former notions.

Definition 2.1.1 (coarse intersection)**.** Given a metric space X, and subsets $A, B \subset X$, the subsets *coarsely intersect*, denoted $A \hat{\cap} B \neq \emptyset$, if there exists a positive constant r such that any two elements in the collection of subsets $\{N_R(A) \cap N_R(B) | R \geq r\}$ have finite Hausdorff distance. Moreover, if $C \subset X$ has finite Hausdorff distance from any set $N_R(A) \cap N_R(B)$, then C is the *coarse intersection* of the subsets A and B. In particular, we will be interested in the situations where C has bounded diameter, in which case we say the subsets A and B have *bounded coarse intersection*.

Note that two subsets may fail to coarsely intersect, although if they do, then their coarse intersection as defined in Definition 2.1.1 is well defined, [54].

Definition 2.1.2 (quasi-isometry)**.** Given metric spaces $(X, d_X), (Y, d_Y)$, a map $f\colon\ (X, d_X) \to (Y, d_Y)$ is called a (K, L) *quasi-isometric embedding of* X *into* Y if there exist constants $K \geq 1, L \geq 0$ such that for all $x, x' \in X$ the following inequality holds:

$$K^{-1} d_X(x, x') - L \leq d_Y(f(x), f(x')) \leq K d_X(x, x') + L$$

If in addition, the map f is *roughly onto*, i.e. a fixed neighborhood of the image must be the entire codomain, f is called a *quasi-isometry*. Two metric spaces are called *quasi-isometric* if and only if there exists a quasi-isometry between them. The special case of a quasi-isometric embedding with domain a line (segment, ray, or bi-infinite) is a *quasi-geodesic*.

Remark 2.1.3. To simplify notation, we sometimes write:

$$d_X(x, x') \approx_{K,L} d_Y(y, y') \text{ to imply } K^{-1} d_X(x, x') - L \leq d_Y(y, y') \leq K d_X(x, x') + L$$

for some K, L. Similarly, we write $d_X(x, x') \lesssim_{K,L} d_Y(y, y')$ to imply $d_X(x, x') \leq K d_Y(y, y') + L$. When the constants K, L are not important, they will be omitted from the notation.

2.1.2 Curves and Essential Subsurfaces

Let $S = S_{g,n}$, by any surface of finite type. That is, S is a genus g surface with n boundary components (or punctures). The *complexity* of S, denoted $\xi(S)$, is defined to be $3g - 3 + n$. While in terms of the mapping class group there is a distinction between boundary components of a surface and punctures on a surface, as elements of the mapping class group must fix the former, yet can permute the latter, for the purposes of this thesis such a distinction will not be relevant. Accordingly, throughout this thesis while we will always refer to surfaces with boundary components, the same results hold mutatis mutandis for surfaces with punctures.

A simple closed curve γ on a surface S is *peripheral* if it bounds a disk, once punctured disk, or annulus; a non-peripheral curve is *essential*. Throughout the thesis we only consider essential simple closed curves up to isotopy and by abuse of notation will refer to the isotopy classes simply as curves. Since we consider curves up to isotopy, we can always assume that their intersections are transverse and cannot be removed. Equivalently, $S \setminus (\gamma_1 \cup \gamma_2)$ does not contain any bigons. We say that two curves are *disjoint*, denoted $\gamma_1 \cap \gamma_2 = \emptyset$, if they can be drawn disjointly on the surface. Otherwise, we say that the curves *intersect*, denoted $\gamma_1 \cap \gamma_2 \neq \emptyset$. A *multicurve* is a set of disjoint non parallel curves.

An *essential subsurface* Y of a surface S is a subsurface $Y \subseteq S$ such that Y is a union of (not necessarily all) complementary components of a multicurve. Throughout the thesis we always consider essential subsurfaces and by abuse of notation will refer to the isotopy classes of essential subsurfaces simply as essential subsurfaces. Furthermore, we always assume every connected component of every essential subsurface $Y \subset S$ has complexity at least one. In particular, unless otherwise noted annuli or pairs of pants are not considered essential subsurfaces and do not appear as connected components of essential subsurfaces. For a fixed surface S, let $\mathcal{E}(S)$ denote the set of all connected essential subsurfaces of S.

Given any essential subsurface Y we define the *essential complement of* Y, denoted Y^c, to be the maximal (in terms of containment) essential subsurface in the complement $S \setminus Y$ if such an essential subsurface exists, and to be the empty set otherwise. An essential subsurface Y is called a *separating essential subsurface* if the complement $S \setminus Y$ contains an essential subsurface, or equivalently Y^c is nontrivial. The reason for the name separating essential subsurface is due to that the fact that Y is a separating essential subsurface if and only if the boundary ∂Y is a *separating multicurve*, an object we will consider at length in Chapter 3. All other essential subsurfaces which are not separating essential subsurfaces, are defined to be *nonseparating essential subsurfaces*. For example, if Y is an essential subsurface such that the complement $S \setminus Y$ consists of a disjoint union of annuli and pairs of pants, then Y is a nonseparating essential subsurface. Let the subsets $\mathcal{SE}(S), \mathcal{NE}(S) \subset \mathcal{E}(S)$ denote the sets of all connected separating, nonseparating essential subsurfaces of S, respectively.

An essential subsurface Y is *proper* if it is not all of S. If two essential subsurfaces W, V have representatives which can be drawn disjointly on a surface they are said to be *disjoint*. On the other hand, we say W is *nested* in V, denoted $W \subset V$, if W has a representative which can be realized as an essential subsurface inside a representative of the essential subsurface V. If W and V are not disjoint, yet neither essential subsurface is nested in the other, we say that W *overlaps* V, denoted $W \pitchfork V$. In general, if two essential subsurfaces W, V either are nested or overlap, we say that the surfaces *intersect* each other. In such a setting we define the *essential intersection*, denoted $W \cap V$, to be the maximal essential subsurface which is nested in both W and V, if such an essential subsurface exists, and the emptyset otherwise. Note that $W \cap V$ may be trivial even if the essential subsurfaces W, V are not disjoint, as the intersection $W \cap V$ may be supported in a subsurface which is not essential. For instance, see Figure 1. Similarly, the *essential complement of V in W*, denoted $W \setminus V$, is defined to be the maximal essential subsurface in $(S \cap W) \setminus Y$ if such an essential subsurface exists, and to be the empty set otherwise.

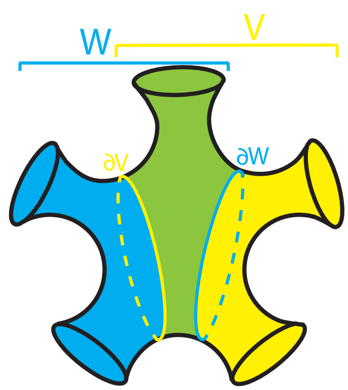

Figure 1: $W, V \in \mathcal{E}(S)$, $W \pitchfork V$. W is drawn in blue, V is drawn in yellow. Note that in this case the essential intersection $W \cap V$ is the emptyset.

A multicurve C is *disjoint* from an essential subsurface Y, denoted $C \cap Y = \emptyset$, if the multicurve and essential subsurface have representatives which can be drawn disjointly on the surface. Otherwise, the multicurve C and the essential subsurface Y are said to *intersect*. In particular, given a proper essential subsurface $Y \subsetneq S$, the boundary parallel curve(s) ∂Y are disjoint from Y.

2.1.3 Curve and Pants Complex

For any surface S with positive complexity, the *curve complex* of S, denoted $\mathcal{C}(S)$, is the simplicial complex obtained by associating a 0-cell to each curve, and more generally a k-cell to each multicurve with $k+1$ elements. In the special case of low complexity surfaces which do not admit disjoint curves, we relax the notion of adjacency to allow edges between vertices corresponding to curves which intersect minimally on the surface. $\mathcal{C}(S)$ was first defined by Harvey [39] and is a central object in the field of geometric group theory. The curve complex endowed with the graph metric is a locally infinite, infinite diameter, δ-hyperbolic metric space, see [47] (as well as [16] for an independent proof).

We will be particularly interested in maximal multicurves, or *pants decompositions*. Equivalently, a pants decomposition is a multicurve $\{\gamma_1, ..., \gamma_m\}$ such that $S - \{\gamma_1, ..., \gamma_m\}$ consists of a disjoint union of *pairs of pants*, or $S_{0,3}$'s. For example, in Figure 1 the multicurve $\{\partial W, \partial V\}$ is a pants decomposition of $S_{0,5}$.

Related to the curve complex, $\mathcal{C}(S)$, there is another natural complex associated to any surface of finite type with positive complexity: the *pants complex*, $\mathcal{P}(S)$. To be sure, the pants complex is a 2-complex, although for purposes of this thesis, since we are only interested in the quasi-isometry type of the pants complex, it will suffice to consider the 1-skeleton of the pants complex, the *pants graph*. By abuse of notation, we often refer the pants graph as the pants complex. The pants graph has vertices corresponding to different pants decompositions of the surface up to isotopy, and edges between two vertices when the two corresponding pants decompositions differ by a so-called *elementary pants move*. Specifically, two pants decompositions of a surface differ by an elementary pants move if the two decompositions differ in exactly one curve and inside the unique connected complexity one essential subsurface in the complement of all the other agreeing curves of the pants decompositions (topologically either an $S_{1,1}$ or an $S_{0,4}$) the differing curves intersect minimally (namely, once if the connected complexity one essential subsurface is $S_{1,1}$ and twice if the connected complexity one essential subsurface is $S_{0,4}$). The pants graph is connected [40], and hence it makes sense to endow $\mathcal{P}(S)$ with the graph metric which we denote $d_{\mathcal{P}}(P_1, P_2)$.

2.1.4 The Pants complex and Teichmüller space

Definition 2.1.4 (Teichmüller space)**.** For S a surface of finite type with $\chi(S) < 0$, the ***Teichmüller space of S*** is the set of isotopy classes of hyperbolic structures on S. Formally,

$$\mathcal{T}(S) = \{(f, X) | f\colon S \to X\} / \sim,$$

where S is a *model surface* (a topological surface without a metric), X is a surface with a hyperbolic metric, the map f is a homeomorphism called a *marking*, and the equivalence relation is given by:

$$(g, Y) \sim (f, X) \iff gf^{-1} \text{ is isotopic to an isometry.}$$

Often we omit the marking from the notation.

As Teichmüller space can be equivalently defined in terms of complex structures on S, Teichmüller space is a classical deformation space which is also of interest to complex analysts. It is a standard result that as a topological space, $\mathcal{T}(S)$ homeomorphic to $\mathcal{R}^{6g-6+2b+3p}$, where g is the genus, b is the number of boundary components, and n is the number of punctures; for instance, see [32] for a proof. On the other hand, a more interesting and active area of research is to study $\mathcal{T}(S)$ as a metric space. To be sure, there are various natural metrics throughout the literature with which Teichmüller space can be equipped. Among these, one of the most important metrics is the Weil-Petersson (WP) metric which is defined in terms of the complex analytic framework. Throughout, we always assume implicitly that $\mathcal{T}(S)$ is equipped with the WP metric. The WP metric is a Riemannian metric of variable non-positive curvature. While $\mathcal{T}(S)$ with the WP metric is not complete, as in finite time a curve in the surface can be degenerated to a point, its completion $\overline{\mathcal{T}}(S)$ obtained by augmenting the Teichmüller spaces of lower complexity nodded surfaces is CAT(0), [66, 68]. See [67] for a survey on the WP metric and its completion.

For purposes of this thesis, since we seek to explore the large scale geometric properties of Teichmüller space, we will not need to use the actual integral form definition of the WP metric but in its place will use the pants complex as a combinatorial model for studying $\mathcal{T}(S)$. Specifically, as justified by the conjunction of the following two theorems, in order to study quasi-isometry invariant properties of $\mathcal{T}(S)$, such as for instance thickness and divergence, it suffices to study the quasi-isometric model of Teichmüller space given by the pants complex.

Theorem 2.1.5 ([11, 12] Bers constant)**.** *$\exists$ a* Bers constant $B(S)$, *such that* $\forall X \in \mathcal{T}(S)$, *there exists a* Bers pants decomposition $X_B \in \mathcal{P}^0(S)$ *such that* $\forall \alpha \in X_B$, *the length* $l_X(\alpha) \leq B$. *In other words, every point in Teichmüller space has a pants decomposition consisting of all short curves, where short is measured relative to a uniform constant depending only on the topology of the surface.*

Intuitively, the proof of Theorem 2.1.5 is based on the standard fact that in hyperbolic geometry one can make certain curves on a surface long, but since the total area of the surface is bounded in terms of the topology of the surface, doing so is perforce at the expense of making other curves short. For instance, see [24] for a proof. Using the mapping suggested by Theorem 2.1.5, the following groundbreaking theorem of Brock proves that $\mathcal{T}(S)$ and $\mathcal{P}(S)$ are quasi-isometric.

Theorem 2.1.6 ([19] Theorem 3.2)**.** *The mapping* $\Psi\colon\ (\mathcal{T}(S), WP) \to (\mathcal{P}(S), \textit{graph metric})$ *given by*

$$X \mapsto B_X$$

where $B_X \in \mathcal{P}(S)$ *is a Bers pants decomposition of* X *as in Theorem 2.1.5, is coarsely well-defined, and moreover, is a quasi-isometry.*

2.1.5 Marking Complex

The marking complex can be thought of as a slight generalization of the pants complex. Specifically, a *complete marking* μ on S is a collection of *base curves* $\{\gamma_i\}$ and *transverse curves* $\{t_i\}$ subject to the following conditions:

1. The base curves $\{\gamma_1, ..., \gamma_n\}$ are a pants decomposition, i.e. $n = \xi(S)$.

2. Each base curve γ_i has a corresponding transverse curve t_i transversely intersecting γ_i such that t_i intersects γ_i exactly once if γ_i has nontrivial homology, and twice if γ_i is null-homologous.

A complete marking μ is said to be *clean* if in addition each transverse curve t_i is disjoint from all other base curves γ_j.

Let $\mu = \{(\gamma_i, t_i)\}$ be a complete clean marking, then we define an *elementary move* to be one of the following two operations applied to the marking μ :

1. *Twist*: For some i, we replace (γ_i, t_i) with (γ_i, t_i') where t_i' is the result of one full or half twist (when possible) of t_i around γ_i.

2. *Flip*: For some i we interchange the base and transverse curves. After a flip move, it is possible that the resulting complete marking is no longer clean, in which case as part of the flip move we then replace the non-clean complete marking with a *compatible* clean complete marking. Specifically, two complete markings μ, μ' are compatible if they have the same base curves and moreover for all i the annular distance $d_{\gamma_i}(t_i, t_i')$ is minimal over all choices of t_i'. See [48] for technical details regarding the annular complex. For our purposes it suffices to use the fact that traveling in the annular complex is accomplished by taking an arc in a regular neighborhood of the annulus and Dehn twisting it around the core curve of the annulus. In [48] it is shown that there is a uniform bound, depending only on the topological type of S, on the number of clean markings which are compatible with any other given marking. Hence, a flip move is coarsely well-defined by choosing any compatible complete clean marking.

The *marking complex*, $\mathcal{M}(S)$, is defined to be the graph formed by taking complete clean markings of S to be vertices and connecting two vertices by an edge if they differ by an elementary move. Notice that by construction, $\mathcal{M}(S)$ is a locally finite graph on which $\mathcal{MCG}(S)$ acts by isometries. Since there are only finitely many topological types of complete clean markings, the action is cocompact. Moreover, it can be checked that the action is properly discontinuous and hence by the Milnor-Svarc Lemma (see for instance [18, 60] for explicit statements and proofs of the Milnor-Svarc Lemma which is based on results in [51, 62]), it follows that $\mathcal{M}(S)$ is quasi-isometric to $\mathcal{M}CG(S)$.

2.1.6 Ultrapowers and Asymptotic Cones

The asymptotic cone of a space captures the large scale geometric properties of a space. Informally, an asymptotic cone of a metric space (X, d), denoted $\text{Cone}_\omega(X)$, can be described as looking at a space from far away. After introducing necessary concepts, we will give a more formal definition of the asymptotic cone.

We begin by defining a *non principal ultrafilter* ω, which is a tool of logic and is useful for example to ensure the convergence of sequences. Specifically, a non-principal ultrafilter is a subset $\omega \subset 2^{\mathbb{N}}$, satisfying the following properties:

1. ω is non empty; ω does not contain the empty set (filter),
2. $X, Y \in \omega \implies X \cap Y \in \omega$ (filter),
3. $X \subset Y, X \in \omega \implies Y \in \omega$ (filter),
4. $X \notin \omega \implies (\mathbb{N} \setminus X) \in \omega$ (ultrafilter), and
5. $|X| < \infty \implies X \notin \omega$ (non-principal).

Given a sequence of points (x_i) and an ultrafilter ω, the *ultralimit of* (x_i), denoted $\lim_\omega x_i$, is defined to be x if for any neighborhood U of x, the set $\{i : x_i \in U\} \in \omega$. That is, ω *almost surely (or ω–a.s.)* $x_i \in U$. Ultralimits are unique when they exist.

Remark 2.1.7 (Ultrafilter Lemma). Non principal ultrafilters exist by *Zorn's Lemma*: "Every partially ordered set, in which every totally ordered subset has an upper bound, contains at least one maximal element." In fact, the following argument proves that there exists an (non-principal) ultrafilter containing an arbitrary (non-principal) filter of $P(\mathbb{N})$.

The set of (non-principal) filters is a partially ordered set under containment. Moreover, by taking unions, every totally ordered subset has an upper bound. Hence, Zorn's Lemma implies there must exist a maximal (non-principal) filter. We can take such a maximal (non-principal) filter to be our (non-principal) ultrafilter ω. It suffices to show that the maximal (non-principal) filter ω is an ultrafilter, or equivalently that ω contains every set or its complement. Assume ω contains neither (an infinite order set) X nor $\mathbb{N} \setminus X$. Since $X \notin \omega$, maximality of ω implies $\exists$ a (infinite order) set $Y \in \omega$ such that $|X \cap Y| = 0$, (is finite). If not, $\omega \cup \{X\}$ generates a (non-principal) filter larger than $\omega \Rightarrow\Leftarrow$. Similarly, since $\mathbb{N} \setminus X \notin \omega$, there $\exists$ a (infinite order) set $Z \in \omega$ such that $|Z \cap (\mathbb{N} \setminus X)| = 0$, (is finite). By the properties of a filter $Y \cap Z \in \omega$, however, $(Y \cap Z) \cap X$ and $(Y \cap Z) \cap (\mathbb{N} \setminus X)$ each have empty intersection (are finite sets), implying that $Y \cap Z$ is the empty set (a finite set) and hence cannot be an element of ω. This is a contradiction to the assumption that ω is a (non-principal) filter. □

Given any set S and an ultrafilter ω, we define the *ultrapower* of S, denoted S^{ω}, as sequences $\overline{s}$ or (s_i) under the equivalence relation $\overline{s} \sim \overline{s'} \iff \omega$–a.s. $s_i = s'_i$. Elements of the ultrapower will be denoted s^{ω} and their representative sequences will be denoted by $\overline{s}$ or (s_i). By abuse of notation we will sometimes denote elements of the ultrapower and similarly elements of the asymptotic cone by their representative sequences.

For a metric space (X, d), we define the *asymptotic cone of* X, relative to a fixed choice of ultrafilter ω, a sequence of base points in the space (x_i), and an unbounded sequence of positive scaling constants (s_i), as follows:

$$Cone_{\omega}(X, (x_i), (s_i)) \equiv \lim_{\omega}(X, x_i, d_i = \frac{d}{s_i})$$

When the choice of scaling constants and base points are not relevant we denote the asymptotic cone of a metric space X by X_{ω}. Elements of asymptotic cones will be denoted x_{ω} with representatives denoted by $\overline{x}$ or (x_i). For $\mathcal{P}(S)$ we denote $Cone_{\omega}(\mathcal{P}(S), (P_0^i), (s_i)) = \mathcal{P}_{\omega}(S)$. In particular, we assume a fixed base point of our asymptotic cone with representative given by (P_i^0). Furthermore, unless otherwise specified always assume a fixed ultrafilter ω.

More generally, given a subset $Y \subset X$, and a choice of asymptotic cone X_{ω}, throughout we will often consider the *ultralimit* of Y, denoted Y_ω, defined as follows:

$$Y_\omega =: \{y_\omega \in X_\omega | y_\omega \text{ has a representative sequence } (y_i') \text{ with } y_i' \in Y \ \omega\text{-a.s}\}$$

In particular, when dealing with ultralimits we will always be considering the ultralimits as subsets contained inside an understood asymptotic cone. Furthermore, given a sequence of subspaces $Y_i \subset X$, we can similarly define the ultralimit, Y_ω. Based on the context it will be clear which type of ultralimit is being considered.

Consider the following two elementary examples of asymptotic cones: For K a compact metric space, K_ω is a singleton, whereas the asymptotic cone of $\mathbb{Z}^n$ is isometric to $\mathbb{R}^n$ equipped with the so-called *Manhattan distance*, or L^1 metric. The following theorem organizes some well known elementary facts about asymptotic cones:

Theorem 2.1.8. *For metric spaces X, Y and any asymptotic cones X_ω, Y_ω,*

1. $(X \times Y)_\omega = X_\omega \times Y_\omega$.

2. *For X a geodesic metric space, X_ω is a geodesic metric space and in particular is locally path connected.*

3. *$X \approx Y$ implies X_ω and Y_ω are bi-Lipschitz equivalent.*

Proof. The first fact follows from the observation that the construction of the asymptotic cone commutes with a product structure. The second fact follows from the consideration that the ultralimit of a sequence of geodesics gives rise to a geodesic in the asymptotic cone.

For the third fact, consider $f\colon (X, d) \to (Y, d')$ a (K, L) quasi-isometry. The map f induces maps $f^m\colon (X, \frac{d}{s_m}) \to (Y, \frac{d'}{s_m})$ which are $(K, \frac{L}{s_m})$ quasi-isometries. In the limit we have an induced map $f^\infty\colon Cone_\omega(X) \to Cone_\omega(Y)$ a $(K, \lim_\omega \frac{L}{s_i}) = (K, 0)$ quasi-isometry, or a bi-Lipschitz map. □

A couple of points regarding the relationship between elements of asymptotic cones and elements of ultrapowers are in order. First, the equivalence relation of ultrapowers is strictly weaker than the equivalence relation of asymptotic cones. In the case of the ultrapowers, representative sequences are identified precisely when they agree on a subset which is an element of the ultrafilter; whereas in the case of the asymptotic cones, even representative sequences which have sublinear distance, with respect to the scaling sequence, on a set in the ultrafilter are identified. Second, ultrapowers are more general than asymptotic cones, as the construction of an ultrapower can be applied to arbitrary sets as opposed to the construction of an asymptotic cone which can only be applied to metric spaces. In fact, we will often be interested in ultrapowers of objects such as $\mathcal{E}^{\omega}(S)$, or the ultrapower of connected essential subsurfaces of S. Similarly, we will consider $\mathcal{SE}^{\omega}(S), \mathcal{NE}^{\omega}(S))$, or the ultrapowers of separating, nonseparating connected essential subsurfaces of S, respectively.

The next elementary lemma will be useful on a couple of occasions.

Lemma 2.1.9 ([2] Lemma 2.2.6)**.** *If A is a finite set, then any $\overline{\alpha} \in A^{\omega}$ is ω–*a.s. constant. *That is, $\exists! a_0 \in A$ such that $\{i | \alpha_i = a_0\} \in \omega$. In particular, $|A^{\omega}| = |A|$.*

As an application of Lemma 2.1.9, since any essential subsurface is either separating or nonseparating, for any ultrapower of essential subsurfaces $\overline{Y}$, ω–a.s. Y_i is either always separating or always nonseparating. In particular, any $\overline{Y} \in \mathcal{E}^{\omega}(S)$ is either in $\mathcal{SE}^{\omega}(S)$ or $\mathcal{NE}^{\omega}(S)$, and the two options are mutually exclusive. Additionally, since are only a finite number of topological types of essential subsurfaces in any fixed surface of finite type, it follows that any ultrapower of essential subsurfaces $\overline{Y}$ ω–a.s. has constant topological type. Accordingly, just as we can talk about the complexity of an essential subsurface Y we can likewise talk about the complexity of $\overline{Y}$, denoted $\xi(\overline{Y})$, and define it to be the complexity of the ω–a.s. constant topological type of $\overline{Y}$. Similarly, since the number of connected components in any essential subsurface is bounded above by the complexity of the fixed ambient surface of finite type, $|\overline{Y}|$ is well-defined. Finally, along similar lines, given any pairs of ultrapowers of essential subsurfaces, $\overline{W}, \overline{V} \in \mathcal{E}^{\omega}(S)$ since W_i and V_i are either disjoint, nested, or overlapping, it follows that ω–a.s. one and only one of the relationships holds and hence we can say $\overline{W}$ and $\overline{V}$ are disjoint, nested, or overlapping as the case may be.

The next lemma implies that $P_{\omega}(S)$ is a homogeneous space:

Lemma 2.1.10. *$MCG^{\omega}(S)$ acts transitively by isometries on $\mathcal{P}_{\omega}(S)$ thereby making $\mathcal{P}_{\omega}(S)$ a homogeneous space.*

Proof. Since $\mathcal{M}CG(S)$ acts co-finitely by isometries on $\mathcal{P}(S)$, $MCG^{\omega}(S)$ similarly acts co-finitely by isometries on $\mathcal{P}^{\omega}(S)$. Since $\mathcal{P}_{\omega}(S)$ consists of equivalence classes of $\mathcal{P}^{\omega}(S)$ which in particular identifies uniformly bounded sequences, the desired result follows. □

2.1.7 CAT(0) geometry

The terminology "CAT(0)" was coined by Gromov and is an acronym for Cartan, Aleksandrov and Toponogov, all three of whom are considered pioneers in the study of non-positive curvature. By definition, CAT(0) spaces are geodesic metric spaces defined by the property that all geodesic triangles are no fatter than the corresponding comparison triangles in Euclidean space, where the *comparison Euclidean triangle* is the unique (up to isometry) Euclidean triangle with the prescribed side lengths. Specifically, any cordal length in any geodesic triangle is bounded above by the length of the corresponding cordal length of the comparison triangle in Euclidean space. Using this defining property one can prove the following lemma, see [18, Section II.2] for details.

Lemma 2.1.11. *Let X be a CAT(0) space.*

C1: (Projections onto convex subsets). Let C be a convex subset, complete in the induced metric, then there is a well-defined distance non-increasing nearest point projection map $\pi_C\colon X \to C$. In particular, π_C is continuous. We will be interested in the special case where C is a geodesic.

C2: (Convexity). Let $c_1\colon [0,1] \to X$ and $c_2\colon [0,1] \to X$ be any pair of geodesics parameterized proportional to arc length. Then the following inequality holds for all $t \in [0,1]$:

$$d(c_1(t), c_2(t)) \leq (1-t)d(c_1(0), c_2(0)) + td(c_1(1), c_2(1))$$

C3: (Unique geodesic space). $\forall x, y \in X$ there is a unique geodesic connecting x and y.

2.1.8 (Relative) Hyperbolicity and Thickness

The following notions of hyperbolicity and relative hyperbolicity introduced by Gromov, are fundamental in the field of geometric group theory, [37]. For points x_1, x_2 in any geodesic metric space X, we use the notation $[x_1, x_2]$ to denote a geodesic between the points x_1 and x_2.

Definition 2.1.12 (δ-hyperbolic)**.** A geodesic metric space X is said to be δ*-hyperbolic* if it satisfies the δ*-thin triangles inequality*. Specifically, there exists some constant $\delta \geq 0$ such that for any three points in the space x_1, x_2, x_3 and $[x_i, x_j]$ any geodesic connecting x_i and x_j, then

$$[x_1, x_3] \subset N_\delta([x_1, x_2]) \bigcup N_\delta([x_2, x_3]).$$

A metric space is called *hyperbolic* if it is δ-hyperbolic for some δ.

Example 2.1.13 ($\mathbb{R}$-Tree). Let T be an $\mathbb{R}$-*Tree*, that is, a metric space with the property that between any two points there is a unique embedded arc having them as endpoints. By definition, triangles in T are either lines or tripods. In either case it is immediate that T is 0-hyperbolic.

An important generalization of hyperbolicity is the notion of relative hyperbolicity. Informally, a metric space X is relatively hyperbolic with respect to a collection of subsets $\mathcal{A}$, if when all of the subsets in $\mathcal{A}$ are collapsed to finite diameter sets, the resulting "electric space," $X/\mathcal{A}$, is hyperbolic. To exclude trivialities we can assume no set $A \in \mathcal{A}$ has finite Hausdorff distance from X. More specifically, spaces satisfying the above are said to be *weakly relatively hyperbolic*. If, in addition, a weakly relatively hyperbolic space X has the *bounded coset penetration property*, namely quasi-geodesics with the same endpoints travel roughly through the same subsets in $\mathcal{A}$ both entering and exiting the same subsets near each other, then X is said to be *strongly relatively hyperbolic*. We will use the following equivalent definition of strong relative hyperbolicity of a metric space due to [30] formulated in terms of asymptotic cones:

Definition 2.1.14 (Relatively Hyperbolic)**.** A metric space (X, d) is said to be *hyperbolic relative* to a collection of *peripheral subsets* $\mathcal{A}$ if X is *asymptotically tree-graded*, with respect to $\mathcal{A}$. That is,

1. Every asymptotic cone X_ω is *tree-graded* with respect to the *pieces* A_ω for $A \in \mathcal{A}$. More specifically, the intersection of each pair of distinct pieces, A_ω, A'_ω, has at most one point and every simple geodesic triangle (a simple loop composed of three geodesics) in X_ω lies in one piece A_ω.

2. X is not contained in a finite radius neighborhood of any of the subsets in $\mathcal{A}$.

In contrast to earlier concepts of hyperbolicity or relatively hyperbolicity, we have the a notion of thickness developed in [6] and explored further in [5]. We will use the following definition of thickness of a metric space defined inductively.

Definition 2.1.15 (Thickness)**.**

1. A space X is said to be *thick of order zero* if none of its asymptotic cones X_ω have cut-points, or equivalently X is *wide*, and moreover it satisfies the following nontriviality condition: there is a constant c such that every $x \in X$ is distance at most c from a bi-infinite quasi-geodesic in X.

2. A space X is said to be *thick of order at most* $n+1$ if there exist subsets $P_\alpha \subset X$, satisfying the following conditions:

   (i) The subsets P_α are *quasi-convex* (namely, there exist constants (K, L, C) such that any two points in P_α can be connected by a (K,L)-quasi-geodesic remaining inside $N_C(P_\alpha)$) and are thick of order at most n when endowed with the restriction metric from the space X,

   (ii) The subsets are *almost everything*. Namely, $\exists$ a fixed constant R_1 such that $\bigcup_\alpha N_{R_1}(P_\alpha) = X$,

   (iii) The subsets can be *chained together thickly*. Specifically, for any subsets P_α, P_β, there exists a sequence of subsets $P_\alpha = P_{\gamma_1}, ..., P_{\gamma_n} = P_\beta$ such that for some fixed constant $R_2 \geq 0$, $diam(N_{R_2}(P_{\gamma_i}) \bigcap N_{R_2}(P_{\gamma_{i+1}})) = \infty$. In particular, due to the quasi-convexity assumption in (i), it follows that the coarse intersection between consecutive subsets being chained together is coarsely connected.

3. A space X is *thick of order n* if n is the lowest integer such that X is thick of order at most n.

In Chapter 5 we will often be interested in subspaces $Y \subset X$ which are *thick of order zero*. Namely, we say that a subspace Y is thick of order zero if in every asymptotic cone X_ω the subset corresponding to the ultralimit Y_ω has the property that any two distinct points in Y_ω are not separated by a cut-point (notice that this can be satisfied vacuously if Y_ω is trivial). Additionally, we require that Y satisfies the nontriviality condition of every point being distance at most c from a bi-infinite quasi-geodesic in Y.

Remark 2.1.16. It should be mentioned that Definition 2.1.15 of thickness is what is in fact called *strongly thick* in [5], as opposed to the slightly more general version of thickness considered in [6]. As in [5], for our purposes the notion of strong thickness is more natural as it proves to be more conducive to proving results regarding divergence, such as we will do in Chapter 5. There are two differences between the different definitions of thickness which we explain presently.

First, as opposed to requirement in Definition 2.1.15 (or equivalently in the definition of strong thickness in [5]) that thick of order zero subsets be wide, in [6] a thick of order zero subset is only required to be *unconstricted*. Namely, there exists some ultrafilter ω and some sequence of scalars s_i such that any asymptotic cone $Cone_\omega(X, \cdot, (s_i))$ does not have cut-points. Nonetheless, as noted in [6] for the special case of finitely generated groups, the definition of thick of order zero in Definition 2.1.15 (or being wide) is equivalent to the definition considered in [6] (or being unconstricted). Moreover, in Section 6.1, and in particular in Theorem 6.1.1, we will prove that for CAT(0) spaces with extendable quasi-geodesics, the notions of wide and unconstricted are similarly equivalent.

Second, the requirement for quasi-convexity in condition (i) of Definition 2.1.15 is omitted in the definition of thickness in [6].

The following theorem of [6], which in fact inspired the development of the notion of thickness, captures the contrasting relationship between hyperbolicity and thickness:

Theorem 2.1.17 ([6] Corollary 7.9)**.** *A metric space X which is thick of any finite order is not strongly relatively hyperbolic with respect to any subsets, i.e.* non relatively hyperbolic (NRH).

Another perspective is to understand thickness as a means of interpolating between two ends of the spectrum of non-positively curved spaces: product spaces and hyperbolic spaces. On the one hand, nontrivial product spaces are thick of order zero (this follows from Theorem 2.1.8 statement (2) as nontrivial products do not contain cut-points). On the other hand, Theorem 2.1.17 says that strongly relatively hyperbolic and hyperbolic spaces are not thick of any order, or equivalently can be thought of as thick of order infinity. Then, in this sense the higher the order of thickness of a metric space the closer the space resembles hyperbolic space and shares features of negative curvature. From this point of view, the close connections between thickness and divergence explored in [5] as well as in Chapter 5 are very natural.

2.1.9 Hyperbolicity/thickness of Teichmüller spaces

Excluding the genus two surface with one boundary component, $S_{2,1}$, the thickness of Teichmüller space for all surfaces of finite type was previously known through the work of Behrstock-Druţu-Mosher [6] and Brock-Masur [21], and in fact is determined by complexity. Specifically, the previously known results regarding the hyperbolicity/thickness of Teichmüller spaces of surfaces of finite type in the literature will be classified presently. Table 1 summarizes the results.

Surfaces with $\xi(S) \in \{1, 2\}$: The Teichmüller space of all of these surfaces are hyperbolic. For $S_{0,4}, S_{1,1}$, any pants decomposition of the respective surfaces consists of a single curve. Hence, the pants complexes are isomorphic to their curve complexes. The curve complexes of both $S_{0,4}$ and $S_{1,1}$ are both isometric to the classical Farey graph and in particular is well known to be δ-hyperbolic. For an explicit proof see [52].

The Teichmüller spaces of $S_{0,5}, S_{1,2}$, were proven to be δ-hyperbolic by Brock-Farb, [20].

Later a new proof of this result was proven by Behrstock in [3]. Using the hyperbolicity of the curve complex, the former authors showed that the pants complexes were strongly hyperbolic relative to natural hyperbolic subsets. The latter author, using hierarchies, showed that the spaces have a transitive family of quasi-geodesics with (b, c)-contraction. Specifically, it is shown in [3] that the asymptotic cones of the spaces are $\mathbb{R}$-trees, a known equivalence of hyperbolicity, [28, 37].

Surfaces with $\xi(S) = 3$: The Teichmüller spaces of all of these surfaces are all strongly relatively hyperbolic with respect to natural product regions consisting of all hyperbolic structures on S in which there is a fixed short separating curve, [21]. In fact, any quasi-flat is coarsely contained in exactly one of these natural product regions, and any two distinct natural product regions have bounded coarse intersection.

Surfaces with $\xi(S) \in \{4, 5\}$: The Teichmüller spaces of all of these surfaces are thick and hence are all NRH. In fact, all surfaces of mid range complexity, excluding the surface $S_{2,1}$, are explicitly shown to be thick of order exactly one, [6, 21]. For the case of $S_{2,1}$, as will be explained in Section 5.1, Behrstock proved that the Teichmüller space is thick of order at least one [3], while Brock-Masur explicitly show that the Teichmüller spaces is thick of order at most two, [21]. In Theorem 5.2.7 we bridge the gap between their results by proving $\mathcal{T}(S_{2,1})$ is thick of order exactly two.

Surfaces with $\xi(S) \geq 6$: In [6] it is shown that the Teichmüller spaces of all of these surfaces are thick of order exactly one and hence are all NRH. For any fixed surface there is a bound on the maximal distance of any pants decomposition from a pants decomposition containing a separating curve. It follows that any pants decomposition is bounded by a constant, depending only on the topological type of surface, from a natural Z^2 quasi-flat in the pants complex. Then, the fact that $\mathcal{T}(S)$ is thick of order one for high complexity surfaces follows from the connectivity of the separating curve complexes. Specifically, the connectivity of the separating curve complex ensures that any two natural quasi-flats corresponding to regions with short fixed separating curves can be thickly chained together.

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
7	T1	T1	T1	T1	T1	T1	...
6	RH	T1	T1	T1	T1	T1	...
5	H	T1	T1	T1	T1	T1	...
4	H	T1	T1	T1	T1	T1	...
3		RH	T1	T1	T1	T1	...
2		H	T1	T1	T1	T1	...
1		H	**T2**	T1	T1	T1	...
0			RH	T1	T1	T1	...
$n \uparrow \ g \rightarrow$	0	1	2	3	4	5	...

Table 1: Hyperbolicity/Thickness classification of Teichmüller spaces for all surfaces. H=hyperbolic, RH=relatively hyperbolic, T1=thick of order one, and T2=thick of order two.

2.2 Tools from mapping class groups

In this section we review some tools developed by Behrstock [3], Behrstock-Kleiner-Minsky-Mosher [8], Behrstock-Minsky [10], and Masur-Minsky [48] in their geometric analyses of the curve complex, $\mathcal{C}(S)$, and the marking complex, $M(S)$. If fact, in the aforementioned papers, many of these tools developed for the marking complex have simplifications which immediately apply to the pants complex.

In analogy with hyperbolic space, $\mathbb{H}^n$, the curve complex $\mathcal{C}(S)$ is δ-hyperbolic and admits a notion of subsurface projection which coarsely exhibits properties similar to those of nearest point retractions onto totally geodesic subspaces in hyperbolic space. Recall that in $\mathbb{H}^n$, given a totally geodesic subspace $L \subset \mathbb{H}^n$ we have a well-defined nearest point projection map $\pi_L \colon \mathbb{H}^n \rightarrow L$. Moreover, we have the property that given any geodesic in the space γ (possibly bi-infinite) such that $\gamma \cap L \neq \emptyset$, $\pi_L(\gamma)$ has uniformly bounded diameter. Presently, we develop a similar coarse subsurface projection in the curve complex.

2.2.1 Subsurface projections

Given a curve $\alpha \in \mathcal{C}(S)$ and a connected essential subsurface $Y \in \mathcal{E}(S)$ such that α intersects Y, we can define the projection of α to $2^{\mathcal{C}(Y)}$, denoted $\pi_{\mathcal{C}(Y)}(\alpha)$, to be the collection of vertices in $\mathcal{C}(Y)$ obtained in the following surgical manner. Specifically, the intersection $\alpha \cap Y$ consists of either the curve α, if $\alpha \subset Y$, or a non-empty disjoint union of arc subsegments of α with the endpoints of the arcs on boundary components of Y. In the former case we define the projection $\pi_{\mathcal{C}(Y)}(\alpha) = \alpha$. In the latter case, $\pi_{\mathcal{C}(Y)}(\alpha)$ consists of all curves obtained by the following process. If an arc in $\alpha \cap Y$ has both endpoints on the same boundary component of ∂Y, then $\pi_{\mathcal{C}(Y)}(\alpha)$ includes the curves obtained by taking the union of the arc and the boundary component containing the endpoints of the arc. Note that this yields at most two curves, at least one of which is essential. On the other hand, if an arc in $\alpha \cap Y$ has endpoints on different boundary components of ∂Y, then $\pi_{\mathcal{C}(Y)}(\alpha)$ includes the curve on the boundary of a regular neighborhood of the union of the arc and the different boundary components containing the end points of the arc. See Figure 2 for an example. Note that above we have only defined the projection $\pi_{\mathcal{C}(Y)}$ for curves intersecting Y, for all curves γ disjoint from Y, the projection $\pi_{\mathcal{C}(Y)}(\gamma) = \emptyset$.

In any context concerning the curve complex of an essential subsurface, $\mathcal{C}(Y)$ in order to avoid distractions we alway assume that $Y \in \mathcal{E}(Y)$, i.e. the essential subsurface Y is connected. If not, then by definition $\mathcal{C}(Y)$ is a nontrivial join and hence has diameter two.

To simplify notation, we write $d_{\mathcal{C}(Y)}(\alpha_1, \alpha_2)$ as shorthand for $d_{\mathcal{C}(Y)}(\pi_{\mathcal{C}(Y)}(\alpha_1), \pi_{\mathcal{C}(Y)}(\alpha_2))$. In particular, this distance is only well-defined if α_1, α_2 intersect Y. Similarly, for $A \subset \mathcal{C}(S)$, we write $diam_{\mathcal{C}(Y)}(A)$ as shorthand for $diam_{\mathcal{C}(Y)}(\pi_{\mathcal{C}(Y)}(A))$.

The following lemma ensures that the subsurface projection $\pi_{\mathcal{C}(Y)}$ defined above gives a coarsely well-defined projection $\pi_{\mathcal{C}(Y)} \colon \mathcal{C}(S) \to \mathcal{C}(Y) \cup \emptyset$.

Lemma 2.2.1 ([48], Lemma 2.2). *For α any curve and any $Y \in \mathcal{E}(Y)$ the set of curves $\pi_{\mathcal{C}(Y)}(\alpha)$ has diameter bounded above by three. Hence, we have a coarsely well-defined subsurface projection map which by abuse of notation we refer to as $\pi_{\mathcal{C}(Y)} \colon \mathcal{C}(S) \to \mathcal{C}(Y) \cup \emptyset$. In particular, if σ is any connected path in $\mathcal{C}(S)$ of length n, and Y is any connected subsurface such that*

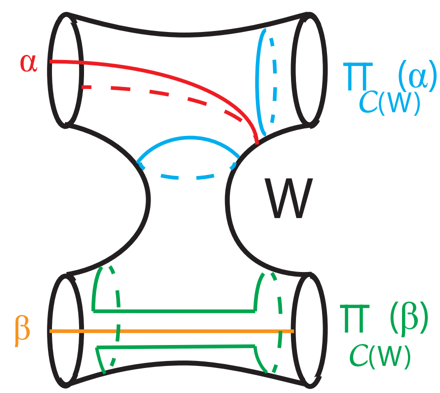

Figure 2: Performing s surgery on arcs in the connected proper essential subsurface $W \subsetneq S$ which makes them into curves in $\mathcal{C}(W)$. The arc α has both endpoints on the same boundary component of W, whereas β has endpoints on different boundary components of W.

every curve in the path σ intersects Y, then $diam_{\mathcal{C}(Y)}(\sigma) \leq 3n$.

The next theorem describes a situation in which subsurface projection maps geodesics in the curve complex to uniformly bounded diameter subsets in the curve complex of a connected essential subsurface.

Theorem 2.2.2 ([48], Theorem 3.1; Bounded Geodesic Image)**.** *Let $Y \in \mathcal{E}(S)$ be a connected proper essential subsurface of S, and let g be a geodesic (segment, ray, or bi-infinite) in $\mathcal{C}(S)$ such that every curve corresponding to a vertex of g intersects Y, then $diam_{\mathcal{C}(Y)}(g)$ is uniformly bounded by a constant $K(S)$ depending only on the topological type of S.*

In addition to projecting curves, we can similarly project multicurves. In particular, we can project pants decompositions of surfaces to essential subsurfaces. Specifically, for any essential subsurface Y we have an induced coarsely well-defined projection map:

$$\pi_{\mathcal{P}(Y)} \colon \mathcal{P}(S) \to \mathcal{P}(Y)$$

The induced map is defined as follows. Beginning with any pair of pants $P \in \mathcal{P}(S)$ there is at least one curve $\alpha_1 \in P$ intersecting Y. We then proceed to construct a pants decomposition of Y inductively. As our first curve we simply pick any curve $\beta_1 \in \pi_{\mathcal{C}(Y)}(\alpha_1)$. Then, we consider the surface $Y \setminus \beta_1$ and notice that $\xi(Y \setminus \beta_1) = \xi(Y) - 1$. Replace Y by $Y \setminus \beta_1$ and repeat this process until the complexity is reduced to zero. At this point, the curves $\{\beta_i\}$ are a pants decomposition of the essential subsurface Y. Due to all the choice, the above process does not produce a unique pants decomposition. Nonetheless, as in Lemma 2.2.1 the map is coarsely well-defined and in fact is coarsely Lipschitz with uniform constants [48, 3].

The next lemma makes precise a sense in which distances under projections to curve complexes of overlapping surfaces are related to each other. Intuitively, the point is that the distance in one subsurface projection can be large only at the expense of the distance in all overlapping essential subsurfaces being controlled.

Lemma 2.2.3 ([3, 46] Theorem 4.3, Lemma 2.5; Behrstock Inequality)**.** *For* $S = S_{g,n}$, *let* $W, V \in \mathcal{E}(S)$ *be such that* $W \pitchfork V$. *Then,* $\forall P \in \mathcal{P}(S)$:

$$\min\left(d_{\mathcal{C}(W)}(\mu, \partial V), d_{\mathcal{C}(V)}(P, \partial W)\right) \leq 10$$

Utilizing the projection $\pi_{\mathcal{P}(Y)} \colon \mathcal{P}(S) \to \mathcal{P}(Y)$, for $\overline{Y} \in \mathcal{E}^{\omega}(S)$ we can define $\mathcal{P}_{\omega}(\overline{Y})$ to be the ultralimit of $\mathcal{P}(Y_i)$. It is clear that $\mathcal{P}_{\omega}(\overline{Y})$ is isomorphic to $\mathcal{P}_{\omega}(Y)$ for Y an essential subsurface ω–a.s. isotopic to Y_i. Moreover, extending the coarsely well-defined Lipschitz projection $\pi_{\mathcal{P}(Y)} \colon \mathcal{P}(S) \to \mathcal{P}(Y)$ to the asymptotic cone, we have a Lipschitz projection

$$\pi_{\mathcal{P}_{\omega}(\overline{Y})} : \mathcal{P}_{\omega}(S) \to \mathcal{P}_{\omega}(\overline{Y}).$$

2.2.2 Tight Geodesics and Hierarchies

A fundamental obstacle in studying geodesics in the curve complex stems from the fact that the 1-skeleton is locally infinite. In an effort to navigate this problem, in [48] Masur-Minsky introduced a notion of *tight multigeodesics*, or simply *tight geodesics*, in $\mathcal{C}(S)$. Specifically, for S a surface of finite type with $\xi(S) \geq 2$, a tight geodesic in $\mathcal{C}(S)$ is a sequence of simplices $\sigma = (w_0, ..., w_n)$ such that the selection of any curves $v_i \in w_i$ yields a geodesic in $\mathcal{C}(S)$ and moreover, for $1 \leq i \leq n-1$, the simplex w_i is the boundary of the essential subsurface filled by the curves $w_{i-1} \cup w_{i+1}$. In the case of a surface S with $\xi(S) = 1$ every geodesic is considered tight. For σ a tight geodesic as above, we use the notation $[w_i, w_j] = (w_i, ..., w_j)$ to refer to a subsegment of the tight geodesic. In [48] it is shown that any two curves in $\mathcal{C}(S)$ can be joined by a tight geodesic (and in fact there are only finitely many).

Using tight geodesics, in [48] a 2-transitive family of quasi-geodesics, with constants depending on the topological type of S, in $\mathcal{P}(S)$ called *hierarchies*, are developed. Since we are interested in paths in the pants complex as opposed to the marking complex, unless specified otherwise we use the term "hierarchies" to refer to what are in fact called "resolutions of hierarchies without annuli" in [48]. The construction of hierarchies which are defined inductively as a union of tight geodesics in the curve complexes of connected essential subsurfaces of S is technical. For our purposes, it will suffice to record some of their properties in the following theorem. See [21] Definition 9 for a similar statement.

Theorem 2.2.4 ([48] Section 4; Hierarchies)**.** *For S any surface of finite type, given $P, Q \in \mathcal{P}(S)$, there exists a hierarchy path $\rho = \rho(P,Q)\colon\ [0,n] \to \mathcal{P}(S)$ with $\rho(0) = P$, $\rho(n) = Q$. Moreover, ρ is a quasi-isometric embedding with uniformly bounded constants depending only on the topological type of S, which has the following properties:*

- *H1: The hierarchy ρ* shadows *a tight $\mathcal{C}(S)$ geodesic g_S from a multicurve $p \in P$ to a multicurve $q \in Q$, called the* main geodesic of the hierarchy. *That is, there is a monotonic map $\nu\colon\ \rho \to g_S$ such that $\forall i,\ \nu_i = \nu(\rho(i)) \in g_S$ is a curve in the pants decomposition $\rho(i)$.*

- *H2: There is a constant M_1 such that if $Y \in \mathcal{E}(S)$ satisfies $d_{\mathcal{C}(Y)}(P,Q) > M_1$, then there is a maximal connected interval $I_Y = [t_1, t_2]$ and a tight geodesic g_Y in $\mathcal{C}(Y)$ from a multicurve in $\rho(t_1)$ to a multicurve in $\rho(t_2)$ such that for all $t_1 \leq t \leq t_2$, ∂Y is a multicurve in $\rho(t)$, and $\rho|_{I_Y}$ shadows the geodesic g_Y. Such a connected essential subsurface Y is called an M_1-*component domain *or simply a* component domain *of ρ. By convention the entire surface S is always considered a component domain.*

- *H3: If $Y_1 \pitchfork Y_2$ are two component domains of ρ, then there is a notion of time ordering $<_t$ of the domains with the property that $Y_1 <_t Y_2$, implies $d_{Y_2}(P, \partial Y_1) < M_1$ and*

$d_{Y_1}(Q, \partial Y_2) < M_1$. *Moreover, the time ordering is independent of the choice of the hierarchy* ρ *from* P *to* Q.

H4: For Y *a component domain with* $I_Y = [t_1, t_2]$, *let* $0 \leq s \leq t_1, t_2 \leq u \leq n$. *Then,*

$$d_{\mathcal{C}(Y)}(\rho(s), \rho(t_1)), d_{\mathcal{C}(Y)}(\rho(u), \rho(t_2)) \leq M_1.$$

As a corollary of Theorem 2.2.4, we have the following quasi-distance formula for computing distances in $\mathcal{P}(S)$ in terms of a sum of subsurface projection distances, where the sum is over all connected essential subsurfaces above a certain threshold.

Theorem 2.2.5 ([48] Theorem 6.12; Quasi-Distance Formula)**.** *For* $S = S_{g,n}$ *there exists a* minimal threshold M_2 *depending only on the surface* S *and quasi-isometry constants depending only on the surface* S *and the threshold* $M \geq M_2$ *such that:*

$$d_{\mathcal{P}(S)}(P, Q) \approx \sum_{Y \in \mathcal{E}(S)} \{d_{\mathcal{C}(Y)}(P, Q)\}_M$$

where the threshold function $\{f(x)\}_M := f(x)$ *if* $f(x) \geq M$, *and* 0 *otherwise.*

Note that by setting $M' = \max\{10, K, M_1, M_2\}$ we have a single constant M', depending only on the topology of the surface S, which simultaneously satisfies Lemmas 2.2.1 and 2.2.3, and Theorems 2.2.2, 2.2.4, and 2.2.5. Throughout we will use this constant M'.

Sequences of hierarchies in the pants complex give rise to ultralimits of hierarchies in the asymptotic cone of the pants complex. Specifically, given $x_\omega, y_\omega \in \mathcal{P}_\omega(S)$ with representatives $(x_i), (y_i)$, respectively, let ρ_ω be the ultralimit of the sequence of hierarchy paths ρ_i from x_i to y_i. Note that by construction, since ρ_i are quasi-geodesics with uniform constants, as in Theorem 2.1.8 it follows that ρ_ω is a (K,0)-quasi-geodesic path in the asymptotic cone from x_ω to y_ω.

2.2.2.1 Bowditch Tight Geodesics

In [17] Bowditch introduces a slightly weaker definition of tight geodesics. Specifically, a geodesic sequence $\{\gamma_i\}_{i=0}^n \subset \mathcal{C}(S)$ is said to be *Bowditch tight* if for all $0 < i < n$ and all geodesic segments $\gamma_{i-1}, \gamma_i', \gamma_{i+1}$, we have $d_{\mathcal{C}(S)}(\gamma_i, \gamma_i') \leq 1$. While tight geodesics always exist, the following combinatorial question regarding Bowditch tight geodesics remains open:

Question 2.2.6 (Bowditch). Given any two vertices in $\mathcal{C}(S)$ of distance at least three, does there exist a Bowditch tight geodesic between them?

The following lemma, which in fact is closely related to Lemma 4.5 in [48], provides partial progress toward Question 2.2.6.

Lemma 2.2.7. *For S with $\xi(S) \leq 3$ between any two vertices in $\mathcal{C}(S)$ there exists a Bowditch tight geodesic.*

Proof. Let $\{\gamma_i\}$ be any $\mathcal{C}(S)$ geodesic. Observe that if the geodesic is not Bowditch tight at some vertex γ_i, then the connected essential subsurface $F(i-1, i+1)$ filled by γ_{i-1} and γ_{i+1} must be a separating essential subsurface. In this case, replace γ_i with $\gamma_i' \in \partial F(i-1, i+1)$. Observe that for all surfaces covered by the statement of the lemma, separating multicurves are in fact separating curves, and hence γ_i' is a separating curve.

For surfaces with $\xi(S) \leq 3$, it follows that the connected components of $S \setminus \gamma'$ have complexity one. In particular, after replacement, the geodesic is automatically Bowditch tight at γ_{i-1} and γ_{i+1}. For example, consider the geodesic segment $\{\gamma_i', \gamma_{i+1}, \gamma_{i+2}\}$. Since γ_{i+2} and γ_i' intersect, it follows that the connected essential subsurface $F(i', i+2)$ is nonseparating. Equivalently, Bowditch tightness at γ_{i+1} is guaranteed. We have shown that failure of Bowditch tightness at any vertex can be fixed via replacement of the non-Bowditch tight vertex, and after replacement adjacent vertices are automatically ensured to be Bowditch tight. This completes the proof of the theorem. □

2.2.3 Consistency Theorem

We have already seen in Subsection 2.2.1 that a pants decomposition can be projected to the curve complexes of connected essential subsurfaces. In this section, we consider when this process can be reversed. The answer is provided by the following theorem:

Theorem 2.2.8 ([8] Theorem 4.3; Consistency Theorem)**.** *Given a tuple* $x_W \in \prod_{W \subseteq S} \mathcal{C}(W)$*, such that* $\exists$ *constants* c_1, c_2 *satisfying the following consistency conditions* $\forall V, W \in \mathcal{E}(S)$*:*

C1: $V \pitchfork W \implies \min\left(d_{\mathcal{C}(W)}(x_W, \partial V), d_{\mathcal{C}(V)}(x_V, \partial W)\right) < c_1$

C2: $V \subsetneq W$ *and* $d_{\mathcal{C}(W)}(x_W, \partial V) > c_2 \implies d_{\mathcal{C}(V)}(x_V, x_W) < c_1$

Then $\exists$ *a constant* c_3 *and* $P \in \mathcal{P}(S)$ *such that* $\forall W \in \mathcal{E}(S)$, $d_{\mathcal{C}(W)}(P, x_W) < c_3$.

The following application of Theorem 2.2.8, which is closely related to Lemma 5.3 in [8], will be used in the proof of Lemma 4.1.7.

Lemma 2.2.9. *Let* $P, Q, R \in \mathcal{P}(S)$*, and for* $W \in \mathcal{E}(S)$ *let* σ_W *be a* $\mathcal{C}(W)$ *geodesic from* P *to* Q*. Let* $\pi_{\sigma_W}(R)$ *be the nearest point projection of* $\pi_{\mathcal{C}(W)}(R)$ *onto the geodesic* σ_W*. Then* $\prod_W \pi_{\sigma_W}(R)$ *satisfies the consistency conditions of Theorem 2.2.8 for* $c_1 = 3M'$ *and* $c_2 = M'$*. In particular, there is a constant* c_3 *and a pants decomposition* $X \in \mathcal{P}(S)$ *such that for all* $W \in \mathcal{E}(S)$ *we have* $d_{\mathcal{C}(W)}(X, \pi_{\sigma_W}(R)) < c_3$*.*

Proof. First we show consistency condition [C1] holds. That is, assuming $V \pitchfork W$ and $d_{\mathcal{C}(W)}(\pi_{\sigma_W}(R), \partial V) > 3M'$, we will show $d_{\mathcal{C}(V)}(\pi_{\sigma_V}(R), \partial W) < 3M'$. Notice that if

$$d_{\mathcal{C}(W)}(\{P, Q\}, \partial V) > M',$$

then, Lemma 2.2.3 implies

$$d_{\mathcal{C}(V)}(P, \partial W), d_{\mathcal{C}(V)}(Q, \partial W) < M'$$

It follows that $d_{\mathcal{C}(V)}(\pi_{\sigma_W}(R), \partial W) < M' < 3M'$. Hence, without loss of generality we can assume $d_{\mathcal{C}(W)}(P, \partial V) < M'$. Since we are assuming $d_{\mathcal{C}(W)}(\pi_{\sigma_W}(R), \partial V) > 3M'$, in particular, $d_{\mathcal{C}(W)}(Q, \partial V) > M'$. Similarly, it follows that $d_{\mathcal{C}(W)}(R, \partial V) > M'$. If not, since π_{σ_W} is a nearest point projection we would have $d_{\mathcal{C}(W)}(R, \pi_{\sigma_W}(R)) < 2M'$ which leads to a contradiction when considering edge lengths of triangle $\triangle(\partial V, R, \pi_{\sigma_W}(R))$ in $\mathcal{C}(W)$. Namely, $d_{\mathcal{C}(W)}(R, \pi_{\sigma_W}(R)) < 2M'$ and $d_{\mathcal{C}(W)}(R, \partial V) < M'$, however this contradicts the fact that $d_{\mathcal{C}(W)}(\pi_{\sigma_W}(R), \partial V) > 3M'$. Thus we can assume $d_{\mathcal{C}(W)}(Q, \partial V) > M'$ and $d_{\mathcal{C}(W)}(R, \partial V) >$

M'. In this case, Lemma 2.2.3 implies that $d_{\mathcal{C}(V)}(R, \partial W) < M'$ and $d_{\mathcal{C}(V)}(Q, R) < 2M'$. Again, since π_{σ_W} is a nearest point projection we have $d_{\mathcal{C}(V)}(R, \pi_{\sigma_W}(R)) < 2M'$. The following inequality completes the proof of [C1]:

$$\begin{aligned} d_{\mathcal{C}(V)}(\pi_{\sigma_V}(R), \partial W) &\leq d_{\mathcal{C}(V)}(\pi_{\sigma_V}(R), R) + d_{\mathcal{C}(V)}(R, \partial W) \\ &< 2M' + M' = 3M'. \end{aligned}$$

Next, we will show that consistency condition [C2] holds. Namely, assuming $V \subsetneq W$ and $d_{\mathcal{C}(W)}(\pi_{\sigma_W}(R), \partial V) > M'$ we will show $d_{\mathcal{C}(V)}(\pi_{\sigma_W}(R), \pi_{\sigma_V}(R)) < 3M'$. First assume that $d_{\mathcal{C}(W)}(\sigma_W, \partial V) > 1$. In this case, since every curve in the $\mathcal{C}(W)$ geodesic σ_W intersects the connected essential subsurface V, by Theorem 2.2.2 $diam_{\mathcal{C}(V)}(\sigma_W) < M'$. In particular,

$$d_{\mathcal{C}(V)}(P, Q), d_{\mathcal{C}(V)}(P, \pi_{\sigma_W}(R)) < M'.$$

However, $d_{\mathcal{C}(V)}(P, Q) < M'$ implies that $d_{\mathcal{C}(V)}(P, \pi_{\sigma_V}(R)) < M'$. Then, by the triangle inequality we are done:

$$\begin{aligned} d_{\mathcal{C}(V)}(\pi_{\sigma_W}(R), \pi_{\sigma_V}(R)) &\leq d_{\mathcal{C}(V)}(\pi_{\sigma_W}(R), P) + d_{\mathcal{C}(V)}(P, \pi_{\sigma_V}(R)) \\ &< M' + M' < 3M'. \end{aligned}$$

Accordingly, we can assume that $d_{\mathcal{C}(W)}(\sigma_W, \partial V) \leq 1$. Since $d_{\mathcal{C}(W)}(\pi_{\sigma_W}(R), \partial V) > M'$, it follows that either the segment $\sigma_W|_{[P, \pi_{\sigma_W}(R)]}$ or the segment $\sigma_W|_{[\pi_{\sigma_W}(R), Q]}$ has all of its curves disjoint from ∂V. Without loss of generality we can assume the former, namely $\sigma_W|_{[P, \pi_{\sigma_W}(R)]}$ has all of its curves disjoint from ∂V. Similarly, the $\mathcal{C}(W)$ geodesic between R and its projection $\pi_{\sigma_W}(R)$ also has all of its curves disjoint from ∂V. Applying Theorem 2.2.2 it follows that $d_{\mathcal{C}(V)}(\pi_{\sigma_W}(R), R) < M'$ and $d_{\mathcal{C}(V)}(R, P) < 2M'$. Since π_{σ_V} is a nearest point projection, in particular $d_{\mathcal{C}(V)}(R, \pi_{\sigma_V}(R)) < 2M'$. The following inequality completes the proof:

$$d_{\mathcal{C}(V)}(\pi_{\sigma_W}(R), \pi_{\sigma_V}(R)) \leq d_{\mathcal{C}(V)}(\pi_{\sigma_W}(R), R) + d_{\mathcal{C}(V)}(R, \pi_{\sigma_V}(R)) < M' + 2M' < 3M'.$$

□

2.2.4 Convex Regions, Extensions of Multicurves, and Regions of Sublinear Growth

Given a multicurve $C \subset \mathcal{C}(S)$, by Theorem 2.2.5 we have a *natural quasi-convex region*:

$$\mathcal{Q}(C) \equiv \{P \in \mathcal{P}(S) | C \subset P\}. \tag{2.2.1}$$

Consider that an element $Q \in \mathcal{Q}(C)$ is determined by a choice of a pants decomposition of $S \setminus C$. Hence, $\mathcal{Q}(C)$ can be naturally identified with $\mathcal{P}(S \setminus C)$, which has nontrivial product structure in the event that $S \setminus C$ is a disjoint union of two or more connected essential subsurfaces. For example, given $W \in \mathcal{SE}(S)$, $\mathcal{Q}(\partial W) \approx \mathcal{P}(W) \times \mathcal{P}(W^c)$.

After taking ultralimits, quasi-convex regions give rise to convex regions in the asymptotic cone. Specifically, given an asymptotic cone $\mathcal{P}_\omega(S)$ and element of the ultrapower of multicurves $\overline{C}$ we have an ultralimit

$$\mathcal{Q}_\omega(\overline{C}) =: \{x_\omega \in \mathcal{P}_\omega(S) | x_\omega \text{ has a representative } (x_i') \text{ with } x_i' \in \mathcal{Q}(C_i) \ \omega\text{-a.s}\}.$$

Note that unless $\lim_\omega \frac{1}{s_i} d_{\mathcal{P}(S)}(P_i^0, \mathcal{Q}(C_i)) < \infty$, the ultralimit $\mathcal{Q}_\omega(\overline{C})$ is trivial. On the other hand, if $\lim_\omega \frac{1}{s_i} d_{\mathcal{P}(S)}(P_i^0, \mathcal{Q}(C_i)) < \infty$, then $\mathcal{Q}_\omega(\overline{C})$ can be naturally identified with $\mathcal{P}_\omega(S \setminus \overline{C})$, which has a nontrivial product structure in the event that the multicurves C_i ω–a.s. separate the surface S into at least two disjoint connected essential subsurfaces. Recall that we always assume essential subsurfaces have complexity at least one.

Given a multicurve C on a surface S and a pants decomposition $X \in \mathcal{P}(S)$, we define the coarsely well-defined *extension of C by X*, denoted $C \lrcorner X$, by:

$$C \lrcorner X \equiv C \cup \pi_{\mathcal{P}(S \setminus C)}(X).$$

More generally, for $\overline{C}$ an element of the ultrapower of multicurves satisfying

$$\lim_\omega \frac{1}{s_i} d_{\mathcal{P}(S)}(P_i^0, \mathcal{Q}(C_i)) < \infty,$$

and $x_\omega \in \mathcal{P}_\omega(S)$ we can define the *extension of* $\overline{C}$ *by* x_ω, denoted $\overline{C} \lrcorner x_\omega$, by:

$$\overline{C} \lrcorner x_\omega \equiv \lim_\omega (C_i \lrcorner X_i) \in \mathcal{P}_\omega(S),$$

where (X_i) is any representative of x_ω.

In [8] the set of natural quasi-convex regions $\mathcal{Q}(C)$ and their generalization to the asymptotic cone is studied at length. In particular, the following theorem is proven:

Theorem 2.2.10 ([8] Lemma 3.3, Section 3.4)**.** *Given two quasi-convex regions $\mathcal{Q}(C)$, $\mathcal{Q}(D)$ for C, D isotopy classes of multicurves, the closest point set in $\mathcal{Q}(C)$ to $\mathcal{Q}(D)$ is coarsely $\mathcal{Q}(C \lrcorner D)$. In particular,*

$$\mathcal{Q}(C) \hat{\cap} \mathcal{Q}(D) = \mathcal{Q}(C \lrcorner D) \cap \mathcal{Q}(D \lrcorner C).$$

For convex regions $\mathcal{Q}_\omega(\overline{C})$, $\mathcal{Q}_\omega(\overline{D})$ in the asymptotic cone $\mathcal{P}_\omega(S)$, the closest point set in $\mathcal{Q}_\omega(\overline{C})$ to $\mathcal{Q}_\omega(\overline{D})$ is $\mathcal{Q}_\omega(\overline{C \lrcorner D})$. In fact, the intersection $\mathcal{Q}_\omega(\overline{C}) \cap \mathcal{Q}_\omega(\overline{D})$ is nonempty if and only if $\mathcal{Q}_\omega(\overline{C \lrcorner D}) = \mathcal{Q}_\omega(\overline{D \lrcorner C})$. Moreover, in this case the intersection is equal to $\mathcal{Q}_\omega(\overline{C \lrcorner D})$.

With the result of Theorem 2.2.5 in mind, [3] and later [10] developed a stratification of $\mathcal{P}_\omega(S)$ by considering regions of so-called *sublinear growth*. Specifically, given $\overline{W} \in \mathcal{E}^\omega(S)$ and $x_\omega \in \mathcal{P}_\omega(\overline{W})$, we define the subset of $\mathcal{P}_\omega(\overline{W})$ with *sublinear growth from* x_ω, denoted $F_{\overline{W},x_\omega}$, as follows:

$$F_{\overline{W},x_\omega} = \{y_\omega \in \mathcal{P}_\omega(\overline{W}) \mid \forall \overline{U} \subsetneq \overline{W},\ d_{\mathcal{P}_\omega(\overline{U})}(x_\omega, y_\omega) = 0\}.$$

See [3] for an example showing that the sublinear growth regions $F_{\overline{W},x_\omega}$ are nontrivial.

The following theorem organizes some properties of subsets of sublinear growth.

Theorem 2.2.11 ([10] Theorem 3.1)**.** *With the same notation as above,*

S1: $z_\omega \neq z'_\omega \in F_{\overline{W},x_\omega} \implies \lim_\omega d_{\mathcal{C}(W_i)}(z_i, z'_i) \to \infty$ *for* $(z_i), (z'_i)$ *any representatives of* z_ω, z'_ω*, respectively. In particular, if* γ_i *is a hierarchy between* z_i *and* z'_i *shadowing a tight main geodesic* β_i *in* $\mathcal{C}(W_i)$ *connecting any curves in the simplices* z_i *and* z'_i*, then* $\lim_\omega |\beta_i|$ *is unbounded.*

S2: $F_{\overline{W},x_\omega} \subset \mathcal{P}_\omega(\overline{W})$ *is a convex* $\mathbb{R}$*-tree.*

S3: There is a continuous nearest point projection

$$\rho_{\overline{W},x_\omega}\colon\ \mathcal{P}_\omega(\overline{W}) \to F_{\overline{W},x_\omega}$$

where $\rho_{\overline{W},x_\omega}$ is the identity on $F_{\overline{W},x_\omega}$ and locally constant on $\mathcal{P}_\omega(\overline{W}) \setminus F_{\overline{W},x_\omega}$.

We record a proof of property [S1] as ideas therein will be used later in the proof of Theorem 4.2.3. For a proof of the rest of the theorem see [10].

Proof. Proof of [S1]: Assume not. That is, assume $\exists$ a constant $K \geq 0$ such that ω–a.s. $\lim_\omega d_{\mathcal{C}(S)}(z_i, z_i') \leq K$. Since $\{0, ..., K\}$ is a finite set, by Lemma 2.1.9 there is a $k \leq K$ such that ω–a.s. $\lim_\omega d_{\mathcal{C}(S)}(z_i, z_i') = k$. In particular, ω–a.s. there is a tight geodesic β_i in $\mathcal{C}(S)$, with simplices $b_{i0}, ..., b_{ik}$ such that $b_{i0} \subset z_i$, $b_{ik} \subset z_i'$. Thus ω–a.s. we can construct a quasi-geodesic hierarchy path γ_i between z_i and z_i' with main geodesic β_i of length k.

At the level of the asymptotic cone we have a quasi-geodesic γ_ω from z_ω to z_ω' which travels through a finite list of regions $\mathcal{Q}_\omega(\overline{b_j})$ where $\overline{b_j} = (b_{i,j})_i \in \mathcal{C}(S)^\omega$ for $j \in \{0, ..., k\}$. Moreover, γ_ω enters each region $\mathcal{Q}_\omega(\overline{b_j})$ at the point $\overline{b_j} \lrcorner z_\omega$ and exits each region at the point $\overline{b_j} \lrcorner z_\omega'$. Since $z_\omega, z_\omega' \in F_{\overline{W},x_\omega}$, by definition for any $\overline{Y} \subsetneq \overline{W}$ $\pi_{\mathcal{P}_\omega(\overline{Y})}(z_\omega) = \pi_{\mathcal{P}_\omega(\overline{Y})}(z_\omega')$. In particular, this holds for $\overline{Y^j}$ with $Y_i^j = W_i \setminus b_{i,j}$ for any j. It follows that the ultralimit of the hierarchy paths γ_ω enters and exits each region $\mathcal{Q}_\omega(\overline{b_j})$ at the same point. Since the regions $\mathcal{Q}_\omega(\overline{b_j})$ are convex, we can assume the quasi-geodesic γ_ω intersects each region in a single point. This leads to a contradiction since by assumption $z_\omega \neq z_\omega'$, yet there is a quasi-geodesic path γ_ω of length zero connecting the two points. □

In [10], regions of sublinear growth are used to stratify product regions in the asymptotic cone. Specifically, for $\overline{W} \in \mathcal{E}^\omega(S)$ such that $\lim_\omega \frac{1}{s_i} d_{\mathcal{P}(S)}(P^0, \mathcal{Q}(\partial W_i)) < \infty$, and $x_\omega \in \mathcal{P}_\omega(\overline{W})$, we define the set $P_{\overline{W},x_\omega} \subset \mathcal{Q}_\omega(\overline{\partial W})$ as follows:

$$P_{\overline{W},x_\omega} = \{y_\omega \in \mathcal{Q}_\omega(\overline{\partial W}) \mid \pi_{\mathcal{P}_\omega(\overline{W})}(y_\omega) \in F_{\overline{W},x_\omega}\} \cong \mathcal{P}_\omega(\overline{W^c}) \times F_{\overline{W},x_\omega}.$$

By precomposition with the projection $\pi_{\mathcal{P}_\omega(\overline{W})}\colon\ \mathcal{P}_\omega(S) \to \mathcal{P}_\omega(\overline{W})$, the continuous nearest point projection of property [S3] gives rise to a continuous map:

$$\Phi_{\overline{W},x_\omega} = \rho_{\overline{W},x_\omega} \circ \pi_{\mathcal{P}_\omega(\overline{W})}\colon\ \mathcal{P}_\omega(S) \to F_{\overline{W},x_\omega}. \tag{2.2.2}$$

The following theorem regarding the above projection is an extension of Theorem 2.2.11.

Theorem 2.2.12 ([10] Theorem 3.5). *$\Phi_{\overline{W},x_\omega}$ restricted to $P_{\overline{W},x_\omega}$ is a projection onto the $F_{\overline{W},x_\omega}$ factor in its natural product structure, and $\Phi_{\overline{W},x_\omega}$ is locally constant on $\mathcal{P}_\omega(S) \setminus P_{\overline{W},x_\omega}$.*

The following lemma shows that the sets $F_{\overline{W},x_\omega}$ can be used to study distance in $\mathcal{P}_\omega(S)$.

Lemma 2.2.13 ([10] Theorem 3.6). *$\forall x_\omega \neq y_\omega \in \mathcal{P}_\omega(S), \exists \overline{W} \in \mathcal{E}^\omega(S)$ such that*

$$\lim_\omega \frac{1}{s_i} d_{\mathcal{P}(S)}(P_i^0, \mathcal{Q}(\partial W_i)) < \infty,$$

with the property that $\pi_{\mathcal{P}_\omega(\overline{W})}(x_\omega) \neq \pi_{\mathcal{P}_\omega(\overline{W})}(y_\omega) \in F_{\overline{W},x_\omega}$.

Proof. Since $x_\omega \neq y_\omega$, by definition $d_{\mathcal{P}_\omega(S)}(x_\omega, y_\omega) \neq 0$. If there is no element $\overline{U} \in \mathcal{E}^\omega(S)$, such that

$$\lim_\omega \frac{1}{s_i} d_{\mathcal{P}(S)}(P_i^0, \mathcal{Q}(\partial U_i)) < \infty$$

with the property that the projection $d_{\mathcal{P}_\omega(\overline{U})}(x_\omega, y_\omega) \neq 0$, then set $\overline{W} = \overline{S}$ and we are done. If not, we iterate the above with $\overline{S}$ replaced by $\overline{U}$. Since the complexity of the original surface is finite, and at each stage the complexity decreases, the proof follows by induction. □

The following corollary provides sufficient condition for identifying when two sequences representing points in the asymptotic cone, actually represent the same point in the asymptotic cone. The proof follows immediately from Lemma 2.2.13 and property [S1] of Theorem 2.2.11.

Corollary 2.2.14. *Let $(x_i), (y_i)$ be sequences representing the points $x_\omega, y_\omega \in \mathcal{P}(S)$, and assume for all $\overline{W} \in \mathcal{E}^\omega(S)$ that $\lim_\omega d_{\mathcal{C}(W_i)}(x_i, y_i)$ is bounded. Then $x_\omega = y_\omega$.*

2.2.5 Jets

In [8], subsets of $\mathcal{P}_\omega(S)$ called jets are developed. Jets are particular subsets of the asymptotic cone corresponding to sequences of geodesics in the curve complexes of connected essential subsurfaces which give rise to separation properties in $\mathcal{P}_\omega(S)$.

Fix $P, Q \in \mathcal{P}(S)$, $Y \in \mathcal{E}(S)$ a connected essential subsurface, and σ a tight geodesic in $\mathcal{C}(Y)$ from an element of $\pi_{\mathcal{C}(Y)}(P)$ to an element of $\pi_{\mathcal{C}(Y)}(Q)$. If $g = [\alpha, \beta]$ is a subsegment of

σ, (g, P, Q) is called a *tight triple* supported in Y with *ambient geodesic* σ. For (g, P, Q) a tight triple as above, we define the *initial pants* of the triple, denoted $\iota(g, P, Q) \equiv \alpha \cup \pi_{\mathcal{P}(S\setminus\alpha)}(P)$. Similarly, we define the *terminal pants* of the triple, denoted $\tau(g, P, Q) \equiv \beta \cup \pi_{\mathcal{P}(S\setminus\beta)}(Q)$. Then, we define the *length* of a tight triple supported in Y by

$$\|g\| = \|(g, P, Q)\|_Y \equiv d_{\mathcal{P}(Y)}(\iota(g, P, Q), \tau(g, P, Q)).$$

For $\overline{P}, \overline{Q} \in \mathcal{P}^\omega(S)$ which have nontrivial ultralimits in $\mathcal{P}_\omega(S)$, a ***Jet*** J, is a quadruple of ultrapowers $(\overline{g}, \overline{Y}, \overline{P}, \overline{Q})$, where (g_i, P, Q) are tight triples supported in Y_i. Associated to our jet J with support $\overline{Y}$ we have an *initial point* or *basepoint* of our jet $\iota(J) = \iota_\omega(\overline{g}, \overline{P}, \overline{Q}) \in \mathcal{P}_\omega(S)$ with a representative ultrapower $\iota(g_i, P, Q)$. Similarly, we a terminal point of our jet $\tau(J) = \tau_\omega(\overline{g}, \overline{P}, \overline{Q}) \in \mathcal{P}_\omega(S)$ with a representative ultrapower $\tau(g_i, P, Q)$. A jet is called *macroscopic* if $\iota(J) \neq \tau(J)$ and *microscopic* otherwise. To simplify notation, we set $\|(g_i, P, Q)\|_{Y_i} = \|g_i\|_J$. We will only consider microscopic jets.

Let J be a microscopic jet with support $\overline{Y}$ and tight geodesics g_i. Then we can consider the ultralimit $\mathcal{Q}_\omega(\overline{\iota} \cup \overline{\partial Y})$ which can be though of as $\iota(J) \times \mathcal{P}_\omega(\overline{Y^c}) \subset \mathcal{P}_\omega(S)$. Then we can define an equivalence relation on $\mathcal{P}_\omega(S) \setminus \left(\iota(J) \times \mathcal{P}_\omega(\overline{Y^c})\right)$ given by:

$$x_\omega \sim_J x'_\omega \iff \lim_\omega d_{\mathcal{C}(Y_i)}(\pi_{g_i}(x_i), \pi_{g_i}(x'_i)) < \infty.$$

The following theorems regarding the existence and separation properties of microscopic jets will have application in Chapter 4.

Theorem 2.2.15 ([8] Lemma 7.5)**.** *Let $a_\omega, b_\omega \in \mathcal{P}_\omega(S)$ with representatives $(a_i), (b_i)$ respectively. Assume that $\overline{W} \in \mathcal{E}^\omega(S)$ is such that $\lim_\omega d_{\mathcal{C}(W)}(a_i, b_i) \to \infty$. Then there exists a microscopic jet $J = (\overline{g}, \overline{W}, \overline{a}, \overline{b})$ such that $a_\omega \not\sim_J b_\omega$. Moreover, the subsegments g_i can be constructed to be contained in tight $\mathcal{C}(W_i)$ geodesic of a hierarchy between a_i and b_i.*

Theorem 2.2.16 ([8] Theorem 7.2)**.** *For J a microscopic jet, each equivalence class under the relation $\sim_J$ is open. In particular, $x_\omega, x'_\omega \in \mathcal{P}_\omega(S) \setminus \left(\iota(J) \times \mathcal{P}_\omega(\overline{Y^c})\right)$, $x_\omega \not\sim_J x'_\omega \implies x_\omega$ and x'_ω are separated by $\iota(J) \times \mathcal{P}_\omega(\overline{Y^c})$.*

Chapter 3

Complex of separating multicurves

Along the lines of the curve complex and the pants complex, in this section we introduce and analyze another natural complex associated to a surface, namely the *complex of separating multicurves*, or simply the *separating complex*. The separating complex, denoted $\mathbb{S}(S)$, can be thought of as a generalizations of the separating curve complex and the Torelli Complex. Formally, we have the following definition:

Definition 3.0.17 (Separating complex)**.** Given a surface S of finite type, define the *separating complex*, denoted $\mathbb{S}(S)$, to have vertices corresponding to isotopy classes of *separating multicurves* $C \subset \mathcal{C}(S)$, that is multicurves C such that at least two connected components of $S \setminus C$ are essential subsurfaces. More generally, the separating complex has k-cells corresponding to a sets of $(k+1)$ isotopy classes of separating multicurves the complement of whose union in the surface S contains an essential subsurface. As usual, we will be interested in the one skeleton of $\mathbb{S}(S)$ equipped with the graph metric. See Figure 3 for an example of separating multicurves in $\mathbb{S}(S_{3,0})$.

Notice that a vertex in the separating complex representing a separating multicurve C, corresponds to a natural quasi-convex product regions in the pants complex, $\mathcal{Q}(C)$, defined in Equation 2.2.1. More generally, k-cells in the separating complex correspond to a set of $(k+1)$ quasi-convex product regions $\mathcal{Q}(C_0)$,..., $\mathcal{Q}(C_k)$ such that the coarse intersection between the $k+1$ regions has infinite diameter. Specifically, consider the multicurve $D =$

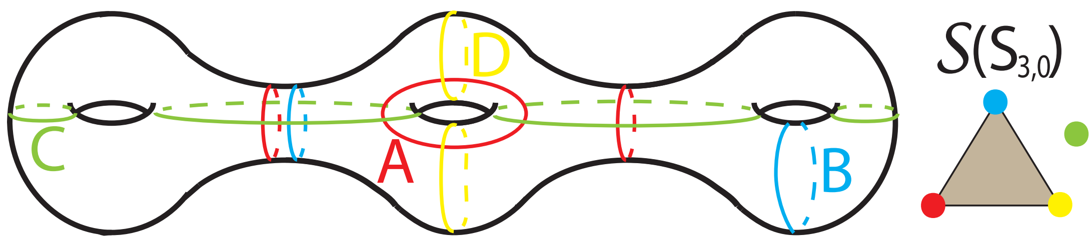

Figure 3: The red separating multicurve A, the blue separating multicurve B, and the yellow separating multicurve D, form a 2-simplex in $\mathbb{S}(S_{3,0})$. On the other hand, the green separating multicurve C is not connected to any of the other separating multicurves. In fact, C is in its own connected component of $\mathbb{S}(S_{3,0})$.

$C_0 \sqcup C_1 \sqcup \ldots \sqcup C_k$, and note that by Definition 3.0.17 there is an essential subsurface Y contained in the complement $S \setminus D$. By Theorem 2.2.10, the coarse intersection between the product regions $\hat{\bigcap}_{i=0}^{k} \mathcal{Q}(C_i) = \mathcal{Q}(D)$, which in particular has infinite diameter as the complement $S \setminus D$ contains an essential subsurface. This latter point of view motivates the definition of $\mathbb{S}(S)$.

Remark 3.0.18. Note that in Definition 3.0.17, in the definition of higher dimensional simplices in $\mathbb{S}(S)$ we did not require disjointness between separating multicurves corresponding to adjacent vertices. If we let $\mathbb{S}'(S)$ denote a natural relative of our separating complex defined identically to $\mathbb{S}(S)$ in conjunction with an additional assumption of disjointness between representatives of adjacent vertices, then we have the following bi-Lipschitz relation:

$$\forall C, D \in \mathbb{S}(S), \quad d_{\mathbb{S}(S)}(C, D) \leq d_{\mathbb{S}'(S)}(C, D) \leq 2 d_{\mathbb{S}(S)}(C, D). \tag{3.0.1}$$

The point is that while adjacent vertices $C, D \in \mathbb{S}(S)$ need not have disjoint separating multicurve representatives, by definition in the complement $S \setminus \{C, D\}$ there must exist a separating multicurve, E. Then in $\mathbb{S}'(S)$ we have the connected sequence of vertices C, E, D. As we will see, the complex $\mathbb{S}(S)$ is more natural from the point of view of Teichmüller space and in particular from the point of view of the asymptotic cones. Nonetheless, there are situations in this section where for the sake of simplifying the exposition we will prove certain results using $\mathbb{S}'(S)$, and then note that the bi-Lipschitz Equation 3.0.1 implies related results for $\mathbb{S}(S)$.

Example 3.0.19 ($\mathbb{S}(S)$ for $\xi(S) \leq 3$). By definition, for surfaces with $\xi(S) \leq 2$, $\mathbb{S}(S)$ is the empty set as for these surfaces there are no (nontrivially) separating multicurves. For complexity three surfaces, separating multicurves are precisely separating curves. Moreover, by topological considerations any two distinct separating curves on complexity three surfaces perforce have trivial complement in the surface. Hence, for complexity three surfaces, $\mathbb{S}(S)$ or equivalently $\mathcal{C}_{sep}(S)$, consists of infinitely many isolated points.

A couple of remarks relating the curve complex and its relative the separating complex are in order. First, notice that for $C, D \in \mathbb{S}(S)$, as an immediate consequence of the definition of $\mathbb{S}'(S)$ in conjunction with Equation 3.0.1 we have the following inequality:

$$d_{\mathcal{C}(S)}(C, D) \leq d_{\mathbb{S}'(S)}(C, D) \leq 2d_{\mathbb{S}(S)}(C, D). \tag{3.0.2}$$

On the other hand, it is possible to have separating curves which are distance one in the curve complex, yet are not even in the same connected component of the separating complex. For example, see multicurves C and D in Figure 3. Second, recall that in $\mathcal{C}(S)$, two curves are distance three or more if and only if they fill the surface. Similarly, the following elementary lemma describes the implications of having $\mathbb{S}(S)$ distance bounded below by four.

Lemma 3.0.20. *Let $C, D \in \mathbb{S}(S)$. $d_{\mathbb{S}(S)}(C, D) \geq 4$ implies that any connected essential subsurface of $S \setminus C$ overlaps any connected essential subsurface of $S \setminus D$.*

Proof. Assume not, then there are connected essential subsurfaces $Z \subseteq S \setminus C$, $Z' \subseteq S \setminus D$ such that Z and Z' are identical, nested, or disjoint. If $Z \subseteq Z'$ (or equivalently $Z' \subseteq Z$) then by definition, $d_{\mathcal{S}(S)}(C, D) \leq 1$. Finally, if $Z \cap Z' = \emptyset$ then $d_{\mathcal{S}(S)}(C, D) \leq 3$, as in $\mathcal{S}(S)$ we have a connected path: $C, \partial Z, \partial Z', D \Rightarrow\Leftarrow$. □

In light of our definitions, the following lemma which will have application in Chapter 5.

Lemma 3.0.21. *Let $\overline{W}, \overline{V} \in \mathcal{SE}^{\omega}(S)$ such that ω–a.s. $d_{\mathbb{S}(S)}(\partial W_i, \partial V_i) \geq 2$. Then*

$$\Phi_{\overline{W},x_\omega}(\mathcal{Q}_\omega(\overline{\partial V})) = \{pt\}, \;\; \Phi_{\overline{V},y_\omega}(\mathcal{Q}_\omega(\overline{\partial W})) = \{pt\},$$

where $\Phi_{\overline{W},x_\omega}$ is the projection defined in Equation 2.2.2.

Proof. Recall the definition of $\Phi_{\overline{W},x_\omega} = \rho_{\overline{W},x_\omega} \circ \pi_{\mathcal{P}(\overline{W})}$. By assumption, the complement in the surface S of $\partial W_i \cup \partial V_i$ ω–a.s. does not contain an essential subsurface. Hence, it follows that $\pi_{\mathcal{P}(\overline{W})}(\mathcal{Q}_\omega(\overline{\partial V})) = \{pt\}$, as for any $a_\omega \in \mathcal{Q}_\omega(\overline{\partial V})$ we can choose a representative (a_i) of a_ω which ω–a.s. contains ∂V_i. Thus, the projection to $\mathcal{P}(W_i)$ is coarsely entirely determined by the projection of the curve ∂V_i. □

Remark 3.0.22 (The diameter of $\mathbb{S}(S)$ is infinite). For surfaces S such that $\mathbb{S}(S)$ is disconnected by definition the diameter of $\mathbb{S}(S)$ is infinte. More generally, since $\mathcal{C}(S)$ has infinite diameter, see [47], and because $N_1^{\mathcal{C}(S)}(\mathbb{S}(S)) = \mathcal{C}(S)$, by Equation 3.0.2 it follows that $\mathbb{S}(S)$ has infinite diameter. To see that $N_1^{\mathcal{C}(S)}(\mathbb{S}(S)) = \mathcal{C}(S)$, by the "change of coordinates principle" of [32] it is easy to see that any curve which is not a nontrivially separating curve, is disjoint from a nontrivially separating curve.

3.1 Properties: connectivity and quasi-distance formula

3.1.1 Separating Complex is connected

In this subsection we prove that for high enough complexity surfaces $\mathbb{S}(S)$ is connected and in fact satisfies a quasi-distance formula.

Theorem 3.1.1. *Let $S = S_{g,n}$, then $\mathbb{S}(S)$ is connected if and only if $|\chi(S)| \geq 5$.*

The proof of Theorem 3.1.1 will follow from a couple of lemmas. The first lemma says that for almost all surfaces with $\chi(S) \leq -5$, one can find certain separating multicurves which are subsets of any pants decompositions and moreover, the distance between any two such separating multicurves is uniformly bounded. The author would like to acknowledge Lee Mosher for his help in providing the current version of the proof of the lemma.

Lemma 3.1.2. *Let $S = S_{g,n}$ be a surface of finite type with $\chi(S) \leq -5$, excluding $S = S_{0,7}, S_{0,8}, S_{1,5}$. Let $P \in \mathcal{P}(S)$ and for all $\gamma \in P$, let $X_\gamma \subset S$ denote the unique connected complexity one essential subsurface of $S \setminus (P - \gamma)$. Then $\forall \gamma \in P$, $\partial X_\gamma \in \mathbb{S}(S)$. Moreover, $diam_{\mathbb{S}(S)}(\bigcup_{\gamma \in P} \partial X_\gamma)$ is uniformly bounded.*

Remark 3.1.3. The statement of Lemma 3.1.2 is sharp in the sense that for all surfaces S excluded by the lemma, there exists pants decompositions P containing a curve $\gamma \in P$ such that $\partial X_\gamma \notin \mathbb{S}(S)$.

Proof. **Step One:** $\partial X_\gamma \in \mathbb{S}(S)$ or equivalently $X_\gamma \in \mathcal{SE}(S)$.

Notice that if $|P \setminus (\gamma \cup \partial X_\gamma)| \geq 1$, then we are done as it follows that the complement $S \setminus X_\gamma$ contains an essential subsurface. However, since by topological considerations $|\partial X_\gamma| \leq 4$, and because in all cases considered $\xi(S) \geq 6$, step one follows.

Step Two: $diam_{\mathbb{S}(S)}(\bigcup_{\gamma \in P} \partial X_\gamma)$ is uniformly bounded.

Fix some ∂X_γ, and consider any $\partial X_{\gamma'}$ for some $\gamma' \neq \gamma \in P$. It follows that $|P \cap (\partial X_\gamma \cup \partial X_{\gamma'})| \leq 8$. Hence, if $\xi(S) \geq 9$ then, as in step one, the complement $S \setminus (X_\gamma \cup X_{\gamma'})$ contains an essential subsurface thus implying that ∂X_γ and $\partial X_{\gamma'}$ are adjacent in $\mathcal{S}(S)$. Since $\gamma' \in P$ was arbitrary, in this case we are done with step two. Without loss of generality we can assume that $\xi(S) \leq 8$. Note that for the remaining cases, the surfaces $S = S_{g,n}$ covered by the theorem all have $n \geq 1$. Proceeding as above we can now choose our starting fixed γ such that $|\partial X_\gamma \cap \partial S| \geq 1$. In this case, then $|P \cap (\partial X_\gamma \cup \partial X_{\gamma'})| \leq 7$. Hence, as above if $\xi(S) \geq 8$ we are also done.

For the case of $\xi(S) = 7$ we will use the same argument as in the $\xi(S) = 8$ case, although with a little more care. Specifically, as in the $\xi(S) = 8$ case, fix some γ such that $|\partial X_\gamma \cap \partial S| \geq 1$. Without loss of generality we can assume $P = \partial X_\gamma \cup \partial X_{\gamma'}$. In particular, $\gamma \in \partial X_{\gamma'}$. Then, since γ and γ' then lie in a common pair of pants of the pants decomposition $S \setminus P$, we now have that $|P \cap (\partial X_\gamma \cup \partial X_{\gamma'})| \leq 6$, and thus we are done for $\xi(S) \geq 7$.

The three remaining $\xi(S) = 6$ cases are $S_{0,9}, S_{1,6}$, and $S_{2,3}$. Assume some pair of pants in $S \setminus P$ contains two boundary components of the ambient surface S, then fix our starting γ to be the third curve in the pair of pants with two boundary components of the ambient surface. As in the $\xi(S) = 7$ case without loss of generality we can assume $P = \partial X_\gamma \cup \partial X_{\gamma'}$ and hence in this case, we now have that $|P \cap (\partial X_\gamma \cup \partial X_{\gamma'})| \leq 5$, and thus we are done with step two under the assumption that some pair of pants in $S \setminus P$ contains two boundary components of the ambient surface S. Since any pants decomposition of $S_{0,9}$ consists of 7 pairs of pants, by pigeon hole considerations the assumption that some pair of pants in $S_{0,9} \setminus P$ contains two boundary components of the ambient surface $S_{0,9}$ must be true thus completing the proof of step two for $S_{0,9}$.

Similarly, for $S_{1,6}$ since any pants decomposition of $S_{1,6}$ consists of 6 pairs of pants, without loss of generality we can assume that every pair of pants in $S_{1,6} \setminus P$ contains exactly one boundary components of the ambient surface $S_{1,6}$. In particular, As usual, without loss of generality we can assume $P = \partial X_\gamma \cup \partial X_{\gamma'}$ and hence in this case, we similarly have that $|P \cap (\partial X_\gamma \cup \partial X_{\gamma'})| \leq 5$, thus completing the proof of step two for $S_{1,6}$.

Finally, for $S_{2,3}$ as usual without loss of generality we can assume ∂X_γ contains one boundary component of the ambient surface and that $P = \partial X_\gamma \cup \partial X_{\gamma'}$. By topological considerations, $X_\gamma \cup X_{\gamma'}$ is a connected separating essential subsurface of topological type $S_{1,4}$ whose complement in the surface S consists of two disjoint pairs of pants Q_1, Q_2 each of which contains one boundary component of the ambient surface and has its other two boundary components in $\partial X_\gamma \cup \partial X_{\gamma'}$. There are two topological types of situations which arise as presented in Figure 4. As noted in the caption, the statement of step two is easily verified in both cases, thus completing the proof.

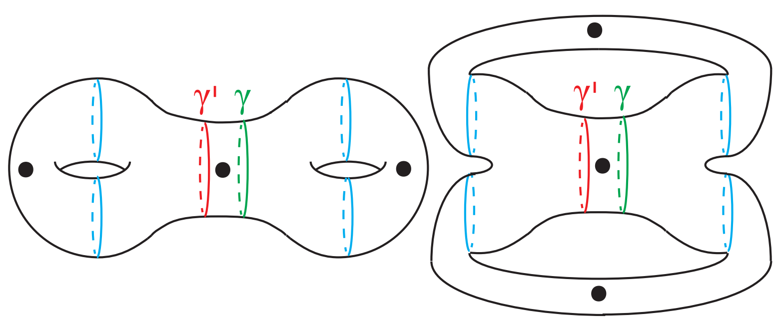

Figure 4: For the pants decomposition on the left $diam_{\mathbb{S}(S)}(\bigcup_{\gamma \in P} \partial X_\gamma) = 2$, while for the pants decomposition on the right, $diam_{\mathbb{S}(S)}(\bigcup_{\gamma \in P} \partial X_\gamma) = 3$.

□

Lemma 3.1.2 holds for almost all surfaces with $\chi(S) \leq -5$. The following lemma contains a slightly weakened statement which applies for all surfaces with $\chi(S) \leq -5$.

Lemma 3.1.4. *For S any surface of finite type with $\chi(S) \leq -5$, and any $P \in \mathcal{P}(S)$, there exists a separating multicurve $C \in \mathbb{S}(S)$, such that $C \subset P$. Moreover, for $P, Q \in \mathcal{P}(S)$ such that $d_{\mathcal{P}(S)}(P, Q) \leq 1$, and for any $C, D \in \mathbb{S}(S)$ such that $C \subset P, D \subset Q$, $d_{\mathbb{S}(S)}(C, D)$ is uniformly bounded. In particular, any two separating multicurves which are subsets of the same pants decompositions have uniformly bounded distance in the separating complex.*

Proof. Using Lemma 3.1.2 it is not hard to see that the statement of the lemma follows for all surfaces with $\chi(S) \leq -5$ excluding $S = S_{0,7}, S_{0,8}, S_{1,5}$.

Let $S = S_{0,7}, S_{0,8}$. Observe that for these surfaces, all pants decompositions contain a separating multicurve and furthermore all separating multicurves are in fact multicurves of separating curves. Hence, for these cases it suffices to notice that two separating curves which intersect in at most two points have uniformly bounded distance in $\mathbb{S}(S)$.

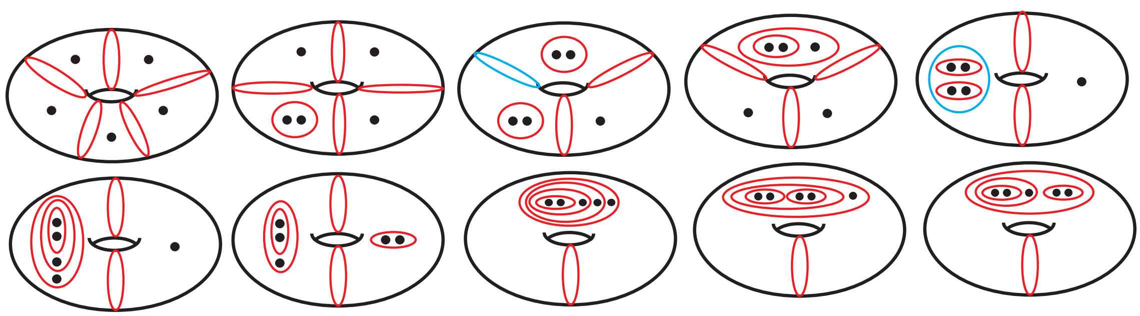

Figure 5: The ten topological types of pants decompositions of $S_{1,5}$. In the figure, for α a curve in red, ∂X_α is a separating multicurve, while for β a curve in blue, ∂X_α is not a separating multicurve.

Finally, the case $S = S_{1,5}$ follows from direct consideration. Specifically, up to the action of the mapping class group there are exactly ten types of pants decompositions of S as presented in Figure 5. It can be verified directly that (i) any pants decomposition contains a separating multicurve, (ii) any two separating multicurves contained in a common pants decomposition have uniformly bounded distance in $\mathbb{S}(S)$, and (iii) any pants decompositions differing by an elementary move contain separating multicurves with uniformly bounded distance in $\mathbb{S}(S)$. □

Proof of Theorem 3.1.1. The statement of Lemma 3.1.4 in conjunction with the connectivity of the pants complex, [40], implies the theorem. □

Remark 3.1.5. It should be noted that the lower bound cases of surfaces considered in Theorem 3.1.1 is strict. Specifically, for $S = S_{0,6}, S_{1,4}, S_{2,2}, S_{3,0}$, the complex $\mathbb{S}(S)$ has infinitely many connected components. The problem here is that for each of these surfaces there are separating multicurves which decompose the surface into two $S'_{0,4}s$. Each such topological type of separating multicurve, of which there are infinitely many distinct isotopy classes, corresponds to an isolated point in $\mathbb{S}(S)$. See Figure 6 for examples.

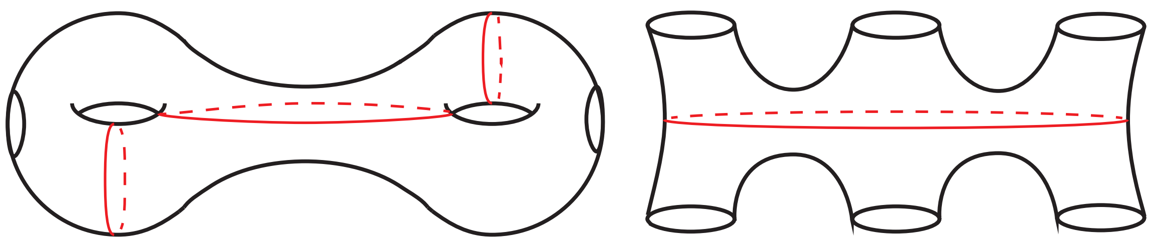

Figure 6: Singleton connected components in $\mathbb{S}(S_{0,6})$ and $\mathbb{S}(S_{2,2})$.

3.1.2 $\mathbb{S}(S)$ satisfies a quasi-distance formula.

In this subsection we use machinery from Masur-Schleimer, [50], to provide a quasi-distance formula for the separating complex, akin to the quasi-distance formula in Theorem 2.2.5 for the pants complex.

The main object in [50] is the notion of a *hole* which in the context of the separating complex is defined to be any connected essential subsurface Y such that all of $\mathbb{S}(S)$ has nontrivial subsurface projection into it. Equivalently, holes for the separating complex are precisely the set $\mathcal{NE}(S)$. Notice that no two holes of the separating complex are disjoint. The general philosophy in [50] is that distances in combinatorial complexes can be approximated by summing up distances in the curve complex projections to all holes. In this section we prove that this philosophy holds for $\mathbb{S}(S)$.

We begin by recalling a Theorem of [50] which in particular ensures a quasi-lower bound for a quasi-distance formula for $\mathbb{S}(S)$. As noted by Masur-Schleimer, the proof of the following lemma follows almost verbatim from similar arguments in [48]:

Lemma 3.1.6 ([50] Theorem 5.10)**.** *Let S be a surface of finite type, then there is a constant C_0 such that $\forall c \geq C_0$ there exists quasi-isometry constants such that $\forall \alpha, \beta \in \mathbb{S}(S)$:*

$$\sum_{Y \text{ a hole for } \mathbb{S}(S)} \{d_{\mathcal{C}(Y)}(\alpha, \beta)\}_c \lesssim d_{\mathbb{S}(S)}(\alpha, \beta)$$

In light of Lemma 3.1.6, in order to obtain a quasi-distance formula for $\mathbb{S}(S)$, it suffices to obtain a quasi-upper bound on $\mathbb{S}(S)$ distance in terms of the sum of subsurface projections to holes. As motivated by [50], our approach for doing so will be by relating pants decompositions to separating multicurves and more generally hierarchy paths in the pants complex to paths in the separating complex.

In fact, notice that Lemma 3.1.4 provides a coarsely well-defined mapping $\phi\colon \mathcal{P}(S) \to \mathbb{S}(S)$ which is natural with respect to elementary pants moves in the sense that if $d_{\mathcal{P}(S)}(P, Q) \leq 1$ then $d_{\mathbb{S}(S)}(\phi(P), \phi(Q))$ is uniformly bounded. As exploited in the proof of Theorem 3.1.1 we have the following procedure for finding a path between any two separating multicurves. Given $\alpha, \beta \in \mathbb{S}(S)$, complete the separating multicurves into pants decompositions μ and ν. Then construct a hierarchy path ρ in $\mathcal{P}(S)$ between μ and ν. Applying the mapping ϕ to our hierarchy path ρ, and interpolating as necessary, yields a path in $\mathbb{S}(S)$ between the separating multicurves α and β with length quasi-bounded above by the length of the hierarchy path ρ. In the following theorem we show that if we are careful, the above approach gives rise to a quasi-upper bound on $\mathbb{S}(S)$ distance in terms of the sum of subsurface projections to all holes with sufficiently large projections.

Theorem 3.1.7. *Let S be a surface with $\chi(S) \leq -5$. Then there is a constant K_0 such that $\forall k \geq K_0$ there exists quasi-isometry constants such that $\forall \alpha, \beta \in \mathbb{S}(S)$:*

$$d_{\mathbb{S}(S)}(\alpha, \beta) \lesssim \sum_{Y \in \mathcal{NE}(S)} \{d_{\mathcal{C}(Y)}(\alpha, \beta)\}_k$$

Proof. As noted, we have a quasi-upper bound on $\mathbb{S}(S)$ distance given by the length of any hierarchy path ρ connecting pants decompositions containing the given separating curves. In other words, by the quasi-distance formula of Theorem 2.2.5 we have already have a quasi-upper bound of the form:

$$d_{\mathbb{S}(S)}(\alpha,\beta) \lesssim \sum_{Y \in \mathcal{E}(S)} \{d_{\mathcal{C}(Y)}(\alpha,\beta)\}_k$$

It suffices to show that $\forall Y \in \mathcal{E}(S) \setminus \mathcal{NE}(S)$ in the above sum we can choose can choose our mapping ϕ such that the $\mathbb{S}(S)$ diameter of $\phi(I_Y)$ is uniformly bounded, where I_Y is as in property [H2] of Theorem 2.2.4. However, for any $Y \in \mathcal{E}(S) \setminus \mathcal{NE}(S) = \mathcal{SE}(S)$ we have that $\partial Y \in \mathbb{S}(S)$. Hence we can choose ∂Y as a constant representative for all $\phi(I_Y)$, thus implying that the $\mathbb{S}(S)$ diameter of $\phi(I_Y)$ is uniformly bounded. This completes the proof. □

Lemma 3.1.6 and Theorem 3.1.7 imply a quasi-distance formula for $\mathbb{S}(S)$:

Corollary 3.1.8. *There is constant K_0 such that for all $k \geq K_0$ there exists quasi-isometry constants such that $\forall \alpha,\beta \in \mathbb{S}(S)$:*

$$d_{\mathcal{S}(S_{2,0})}(\alpha,\beta) \approx \sum_{Y \in \mathcal{NE}(S)} \{d_{\mathcal{C}(Y)}(\alpha,\beta)\}_k$$

Remark 3.1.9. Since $\mathbb{S}(S)$ satisfies a quasi-distance formula as in Corollary 3.1.8 and because no two holes for $\mathbb{S}(S)$ are disjoint, it follows that for $|\chi(S)| \geq 5$, the complex $\mathbb{S}(S)$ is δ-hyperbolic. Specifically, in Section 20 of [50] it is shown that if distance in a combinatorial complex is approximated by a quasi-distance formula and it is known that no two holes overlap, then δ-hyperbolicity follows. The same ideas are implicit in [3]. To be sure, however, an explicit theorem as described above providing sufficient conditions for δ-hyperbolicity of a combinatorial complex, is not present in the current version of [50]. Accordingly as a consequence of this fact, in conjunction with the fact that the hyperbolicity of $\mathbb{S}(S)$ is unnecessary for any results in this thesis, we relegate the fact that $\mathbb{S}(S)$ is δ-hyperbolic to this remark.

3.2 Separating complex of $S_{2,1}$

3.2.1 Connected components of $\mathbb{S}(S_{2,1})$ and Point Pushing

In this subsection, we consider the connected components of $\mathbb{S}(S_{2,1})$, which will be of interest later in Chapter 5. By Remark 3.0.18 the connected components of $\mathbb{S}'(S)$ and $\mathbb{S}(S)$ are equivalent, and hence for the sake of simplifying the exposition, in this section we will in fact consider the connected components of $\mathbb{S}'(S_{2,1})$. By topological considerations, $\mathbb{S}'(S_{2,1})$ consists of separating curves or disjoint pairs thereof. Hence, vertices of $\mathbb{S}'(S_{2,1})$ and simplices of $\mathcal{C}_{sep}(S_{2,1})$ are in correspondence. Moreover, vertices in $\mathbb{S}'(S_{2,1})$ are adjacent if and only if the corresponding simplices are adjacent in $\mathcal{C}_{sep}(S_{2,1})$. Thus, the connected components of $\mathbb{S}'(S_{2,1})$, or equivalently $\mathbb{S}(S_{2,1})$, are precisely the connected components of $\mathcal{C}_{sep}(S_{2,1})$.

To study the connected components of $\mathcal{C}_{sep}(S_{2,1})$, we begin by considering the projection $\pi_{\mathcal{C}(S_{2,0})} = \pi_{\mathcal{C}(S_{2,0})}\colon \mathcal{C}(S_{2,1}) \to \mathcal{C}(S_{2,0})$ given by forgetting about the boundary component. Up to homeomorphism there is only one separating curve on the surfaces $S_{2,1}$ and $S_{2,0}$. In fact under the projection $\pi_{\mathcal{C}(S_{2,0})}$ the image of a separating curve is a separating curve, and similarly the preimage of a separating curve is a union of separating curves.

Lemma 3.2.1. *The map $\pi_{\mathcal{C}(S_{2,0})} = \pi_{\mathcal{C}}$ has a natural well-defined surjective restriction*

$$\pi_{\mathcal{C}_{sep}(S_{2,0})} = \pi_{\mathcal{C}_{sep}}\colon \mathcal{C}_{sep}(S_{2,1}) \to \mathcal{C}_{sep}(S_{2,0}).$$

Lemma 3.2.2. *The fibers of $\pi_{\mathcal{C}_{sep}}$ are connected.*

Proof. Consider two separating curves $\alpha \neq \beta \in \pi_{\mathcal{C}_{sep}}^{-1}(\gamma)$. If α and β are disjoint, we are done. If not, we will complete the proof by induction on the number of intersections between the curves α and β. Look for an innermost bigon B formed by the union of α and β, namely a bigon with two vertices given by intersection points of the curves and such that neither of the curves enters the interior of the bigon. By topological considerations such a bigon must exist. We can assume that the boundary component of the surface is included in the bigon B. If not, up to a choice of representatives of our curves α and β we reduce the intersection number.

Then we can perform a surgery on α along the bigon B to create the curve α', as in Figure 7. We can assume that α' is nontrivial, for if not then our original curve $\gamma \in \mathcal{C}_{sep}(S_{2,0})$ would be trivial $\Rightarrow\Leftarrow$. Moreover, it is also clear that $\alpha' \in \pi^{-1}_{\mathcal{C}_{sep}}(\gamma)$. Replacing our original curve α with α' reduces the intersection number by two, thereby completing the proof by induction.

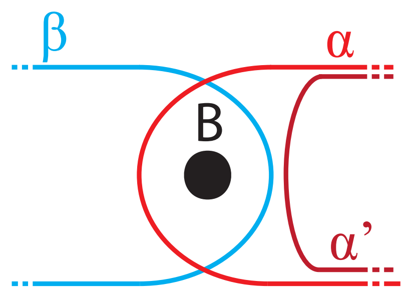

Figure 7: Performing surgery to a curve along a bigon to reduce intersection numbers.

□

Lemma 3.2.3. *The fibers of $\pi_{\mathcal{C}_{sep}}$ coincide with the connected components of $\mathcal{C}_{sep}(S_{2,1})$. In particular, since there are infinitely many curves in the range, $\mathcal{C}_{sep}(S_{2,0})$, it follows that there are infinitely many fibers, and hence infinitely many connected components of $\mathcal{C}_{sep}(S_{2,1})$.*

Proof. Since Lemma 3.2.2 ensures that any fiber of $\pi_{\mathcal{C}_{sep}}$ is connected, to prove the lemma it suffices to show that any two curves α, β which can be connected in $\mathcal{C}_{sep}(S_{2,1})$ must satisfy $\pi_{\mathcal{C}}(\alpha) = \pi_{\mathcal{C}}(\beta)$. Without loss of generality we can assume that $\alpha \cap \beta = \emptyset$. Ignoring the boundary component, we have disjoint representatives of $\pi_{\mathcal{C}}(\alpha)$, and $\pi_{\mathcal{C}}(\beta)$. However, there are no distinct isotopy classes of separating curves in $S_{2,0} \implies \pi_{\mathcal{C}}(\alpha) = \pi_{\mathcal{C}}(\beta)$. □

The *point pushing subgroup* is an important subgroup of the mapping class group of a surface with boundary first considered by Birman, [15]. Specifically, for $S_{g,n+1}$ with a fixed boundary component labeled x, such that if we fill in the boundary component x we obtain a topological $S_{g,n}$ with a marked base point x, we have the following short exact sequence:

$$1 \to \pi_1(S_{g,n}, x) \rightarrowtail \mathcal{MCG}(S_{g,n+1}) \twoheadrightarrow \mathcal{MCG}(S_{g,n}) \to 1.$$

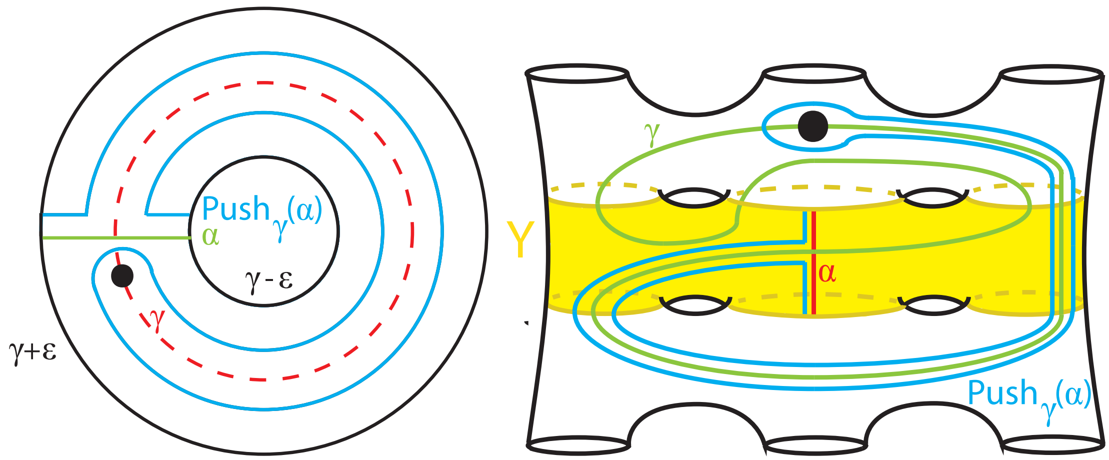

Figure 8: The point pushing map applied to an arcs $\alpha \subset S$.

The second map is defined by taking a homeomorphism of $S_{g,n+1}$ and viewing it as a homeomorphism of the surface $S_{g,n}$ obtained by filling in the boundary component x. On the other hand, the first map is give by "point pushing." Specifically, given a loop $\gamma \in \pi_1(S_{g,n}, x)$, the image of the point pushing map of γ, denoted $Push_\gamma$, is defined to be $T_{\gamma+\epsilon} \circ T^{-1}_{\gamma-\epsilon} \in \mathcal{M}CG(S_{g,n+1})$ where $\gamma + \epsilon$ and $\gamma - \epsilon$ are the two homotopically distinct push-offs of γ in $S_{g,n+1}$. The point pushing subgroup of the mapping class group is defined to be the group generated by point pushing maps for all loops $\gamma \in \pi_1(S_{g,n}, x)$. See Figure 8 for examples.

By construction, the image of this point pushing map is in the kernel of the projection $p : \mathcal{M}CG(S_{g,n+1}) \twoheadrightarrow MCG(S_{g,n})$ as the curves $\gamma + \epsilon$ and $\gamma - \epsilon$ viewed in the surface $S_{g,n}$ are the same up to homotopy. Specifically, since p is a homomorphism we have $p(T_{\gamma+\epsilon} \circ T_{\gamma-\epsilon}) = p(T_{\gamma+\epsilon}) \circ p(T^{-1}_{\gamma-\epsilon}) = T_\gamma T^{-1}_\gamma = Id \in \mathcal{M}CG(S_{g,n})$. We have just shown the following:

Lemma 3.2.4. *The point pushing subgroup $Push \subset \mathcal{M}CG(S_{2,1})$ preserves the connected components of $\mathcal{C}_{sep}(S_{2,1})$. Similarly, $Push \subset \mathcal{M}CG(S_{2,1})$ preserves the fibers of the projection $\pi_{\mathcal{P}} \colon \mathcal{P}(S_{2,1}) \to \mathcal{P}(S_{2,0})$.*

Since there exist pseudo-Anosov point pushing maps, [42], and because pseudo-Anosov axes have infinite diameter in $\mathcal{C}(S)$, [47], which in particular ensures that the axes have infinite diameter in $\mathcal{C}_{sep}(S)$, by Lemma 3.2.4 it follows that the connected components of $\mathcal{C}_{sep}(S_{2,1})$

have infinite diameter. Putting together Lemmas 3.2.3 and 3.2.4, we have the following corollary which uniquely characterizes the surface $S_{2,1}$ and which is the underlying reason for the unique phenomenon regarding the thickness and divergence of $\mathcal{T}(S_{2,1})$ studied in Chapter 5.

Corollary 3.2.5. *$\mathcal{C}_{sep}(S_{2,1})$, and similarly $\mathbb{S}(S_{2,1})$, has infinitely many connected connected components, each with infinite diameter.*

3.3 $\mathbb{S}_\omega(S)$, the ultralimit of $\mathbb{S}(S)$.

3.3.1 Asymptotic Separating Complex, $\mathbb{S}_\omega(S)$

Throughout this section we assume a fixed asymptotic cone $\mathcal{P}_\omega(S)$, and consider the ultralimit of $\mathbb{S}(S)$, which we denote $\mathbb{S}_\omega(S)$. Formally,

Definition 3.3.1 ($\mathbb{S}_\omega(S)$)**.** Given a surface S of finite type, define $\mathbb{S}_\omega(S)$ to have vertices corresponding to $\overline{C} \in \mathbb{S}(S)^\omega$ such that $\lim_\omega \frac{1}{s_i} d_{\mathcal{P}(S)}(P_i^0, \mathcal{Q}(C_i)) < \infty$. Equivalently, vertices in $\mathbb{S}_\omega(S)$ correspond to natural convex nontrivial product regions $\mathcal{Q}_\omega(\overline{C}) \subset \mathcal{P}_\omega(S)$. By abuse of notation, we will sometimes interchange between these two equivalent descriptions of vertices in $\mathbb{S}_\omega(S)$. Furthermore, define $\mathbb{S}_\omega(S)$ to have an edge between vertices $\mathcal{Q}_\omega(\overline{C})$ and $\mathcal{Q}_\omega(\overline{D})$ if in the asymptotic cone $\mathcal{Q}_\omega(\overline{C \lrcorner D}) = \mathcal{Q}_\omega(\overline{D \lrcorner C})$, and moreover ω–a.s. the complement $S \setminus \{C_i, D_i\}$ contains an essential subsurface Y_i. By Theorem 2.2.10 this is equivalent to the statement that the intersection between the convex product regions, $\mathcal{Q}_\omega(\overline{C}) \cap \mathcal{Q}_\omega(\overline{D})$, has nontrivial (in fact infinite) diameter in the asymptotic cone. We can define higher dimensional simplices similarly, although they will not be necessary as we will only be interested in the one skeleton of $\mathbb{S}_\omega(S)$ equipped with the graph metric.

Given our definition of $\mathbb{S}_\omega(S)$, we can define a related $[0, \infty]$–valued pseudometric on the asymptotic cone which gives information about the natural product structures connecting points in the asymptotic cone. Specifically, define

$$d_{\mathbb{S}_\omega(S)}(a_\omega, b_\omega) \equiv \inf_{\overline{A},\overline{B}} d_{\mathbb{S}_\omega(S)}(\overline{A}, \overline{B})$$

where the infimum is taken over all pairs $\overline{A}, \overline{B}$ in the vertex set of $\mathbb{S}_\omega(S)$ having the property that $a_\omega \in \mathcal{Q}_\omega(\overline{A})$ and $b_\omega \in \mathcal{Q}_\omega(\overline{B})$.

This definition is well-defined, as given any pants decompositions $P \in \mathcal{P}(S)$ there is a bound $D(S)$ depending only on the topological type of the surface S, such that there exists a pants decomposition $P' \in \mathcal{P}(S)$ containing a separating curve and $d_{\mathcal{P}(S)}(P, P') \leq D(S)$. In fact, in Section 6.4 we compute the asymptotics of $D(S)$. In particular, given any element of the asymptotic cone a_ω with any representative (A_i) there exists an alternative representative, (A'_i), with A'_i containing a separating curve, thus making it clear that a_ω lies in some natural convex product region of the asymptotic cone. The following theorem ensures appropriate compatibility of $\mathbb{S}(S)$ and $\mathbb{S}_\omega(S)$.

Theorem 3.3.2. *Let $\overline{C}, \overline{D}$ be vertices in $\mathbb{S}_\omega(S)$. Then we have the following inequality:*

$$d_{\mathbb{S}_\omega(S)}(\overline{C}, \overline{D}) \leq 2 \lim_\omega d_{\mathbb{S}(S)}(C_i, D_i) \leq 2 d_{\mathbb{S}_\omega(S)}(\overline{C}, \overline{D}).$$

Moreover, when $d_{\mathbb{S}_\omega(S)}(\overline{C}, \overline{D})$ is finite yet nontrivial, for each of the finite number of natural convex product regions $\mathcal{Q}_\omega(\overline{A}) \subset \mathcal{P}_\omega(S)$ traveled through in the path between $\mathcal{Q}_\omega(\overline{C})$ and $\mathcal{Q}_\omega(\overline{D})$, the separating curve A_i is ω–a.s. in the same connected components as the finite $\mathbb{S}(S)$ geodesic from C_i to D_i.

Remark 3.3.3. The multiplicative term of 2 in the bi-Lipschitz inequality of Theorem 3.3.2 is not believed to be necessary, although is used for technical aspects of the proof recorded below.

Proof of Theorem 3.3.2. First we will prove $\lim_\omega d_{\mathbb{S}(S)}(C_i, D_i) \leq d_{\mathbb{S}_\omega(S)}(\overline{C}, \overline{D})$. It suffices to assume that $d_{\mathbb{S}_\omega(S)}(\overline{C}, \overline{D}) = 1$ and show that $\lim_\omega d_{\mathbb{S}(S)}(C_i, D_i) \leq 1$. Since

$$d_{\mathbb{S}_\omega(S)}(\overline{C}, \overline{D}) = 1$$

it follows that in the asymptotic cone, the natural convex product regions $\mathcal{Q}_\omega(\overline{C})$, $\mathcal{Q}_\omega(\overline{D})$ whose intersection is $\mathcal{Q}_\omega(\overline{C \lrcorner D}) = \mathcal{Q}_\omega(\overline{D \lrcorner C})$ is an infinite diameter set. In particular, $S \setminus (C_i \cup D_i)$ ω–a.s. contains an essential subsurface, Y_i. Accordingly, in $\mathbb{S}(S)$ ω–a.s. we have a connected chain C_i, D_i thus proving $\lim_\omega d_{\mathbb{S}(S)}(C_i, D_i) \leq 1$ as desired.

In order to complete the proof we will show $d_{\mathbb{S}_\omega(S)}(\overline{C},\overline{D}) \leq 2\lim_\omega d_{\mathbb{S}(S)}(C_i,D_i)$. Considering the first part of the proof we can assume $\lim_\omega d_{\mathbb{S}(S)}(C_i,D_i)$ is finite, which by Lemma 2.1.9 implies that ω–a.s. $d_{\mathbb{S}(S)}(C_i,D_i) = n$ for some non-negative constant n. By Remark 3.0.18, it follows that ω–a.s. $d_{\mathbb{S}'(S)}(C_i,D_i) = n' \leq 2n$. Hence, ω–a.s. we have a finite $\mathbb{S}'(S)$ geodesic $\overline{C} = \overline{C^0}, ..., \overline{C^{n'}} = \overline{D}$. By assumption ω–a.s. $C_i^j \cap C_i^{j+1}$ are disjoint and it follows that ω–a.s. $C_i^j \lrcorner C_i^{j+1} = C_i^{j+1} \lrcorner C_i^j$. Hence, $\lim_\omega \frac{1}{s_i} d_{\mathcal{P}(S)}(C_i^j \lrcorner C_i^{j+1}, C_i^{j+1} \lrcorner C_i^j) = 0$. Moreover, since by assumption C_i^j, C_i^{j+1} are ω–a.s. connected in $\mathbb{S}'(S)$ by definition it follows that the complement in the surface of the two multicurves ω–a.s. contains an essential subsurface. Putting things together, in order to prove that $d_{\mathbb{S}_\omega(S)}(\overline{C},\overline{D}) \leq 2n$, and hence complete the proof of the lemma, it suffices to show that there are natural convex product regions $\mathcal{Q}_\omega(\overline{C^j}) \subset \mathcal{P}_\omega(S)$ in the asymptotic cone for $j \in \{1, ..., n'-1\}$ corresponding to the terms in the sequence of $\mathbb{S}(S)$ geodesics $C_i^0, ..., C_i^{n'}$. Equivalently, it suffices to show that $\lim_\omega \frac{1}{s_i} d_{\mathcal{P}(S)}(P_i^0, \mathcal{Q}(C_i^j)) < \infty$ for all $j \in \{1, ..., n'-1\}$ (by the assumptions of our lemma we already have this for $j = 0, n'$). Once we show this, we will have the following chain of natural convex product regions in the asymptotic cone with each product region intersecting its neighbor in an infinite diameter set:

$$\mathcal{Q}_\omega(\overline{C}) = \mathcal{Q}_\omega(\overline{C^0}), ..., \mathcal{Q}_\omega(\overline{C^{n'}}) = \mathcal{Q}_\omega(\overline{D}).$$

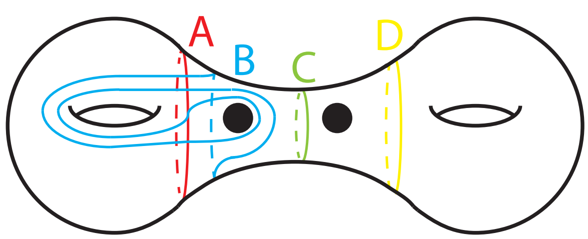

Figure 9: $d_{\mathbb{S}'(S_{2,2})}(A,B) = 2$ and in fact the sequence A, D, B is a $\mathbb{S}'(S)$ geodesic. Note that $D \not\subset \mathcal{N}(A \cup B)$. However, replacing D by C, we have a new $\mathbb{S}'(S)$ geodesic A, C, B with $C \subset \mathcal{N}(A \cup B)$. This replacement process is akin to tightening in Subsection 2.2.2.

Fix some $\overline{C^j}$, for $j \in \{1, ..., n'-1\}$. By replacement if necessary, we can assume C_i^j is contained in a regular neighborhood of C_i^{j-1} and C_i^{j+1}. We denote this latter condition by $C_i^j \subset \mathcal{N}(C_i^{j-1} \cup C_i^{j+1})$. See Figure 9 for an example of such a replacement. We will show that $\lim_\omega \frac{1}{s_i} d_{\mathcal{P}(S)}(P_i^0, \mathcal{Q}(C_i^j)) < \infty$. Then, iteratively repeating the same argument for each of two the resulting shorter sequences $\overline{C^0}, ..., \overline{C^j}$ and $\overline{C^j}, ..., \overline{C^{n'}}$, we eventually obtain an entire chain of length n' with the desired property, namely $\lim_\omega \frac{1}{s_i} d_{\mathcal{P}(S)}(P_i^0, \mathcal{Q}(C_i^j)) < \infty$ for all $j \in \{1, ..., n'-1\}$.

In order to show that $\lim_\omega \frac{1}{s_i} d_{\mathcal{P}(S)}(P_i^0, \mathcal{Q}(C_i^j)) < \infty$ we will show:

$$d_{\mathcal{P}(S)}(P_i^0, \mathcal{Q}(C_i^j)) \lesssim d_{\mathcal{P}(S)}(P_i^0, \mathcal{Q}(C_i^0)) + d_{\mathcal{P}(S)}(P_i^0, \mathcal{Q}(C_i^{n'})). \tag{3.3.1}$$

Then assumption $\lim_\omega \frac{1}{s_i} \left(d_{\mathcal{P}(S)}(P_i^0, C_i^0) + d_{\mathcal{P}(S)}(P_i^0, C_i^{n'}) \right) < \infty$, in conjunction with Equation 3.3.1 completes the proof of the theorem.

In order to prove equation 3.3.1, by Lemma 2.2 of [10] it suffices to show that for any connected essential subsurface $Y \in \mathcal{E}(S)$ such that $Y \cap C_i^j$,

$$d_{\mathcal{C}(Y)}(P_i^0, C_i^j) \leq d_{\mathcal{C}(Y)}(P_i^0, \{C_i^0, C_i^{n'}\}) + r'n$$

where r' is some constant. First assume that Y intersects C_i^m ω–a.s. for all $m \in \{0, .., j-1\}$. In this case we are done as by Lemma 2.2.1 it follows that $d_{\mathcal{C}(Y)}(C_i^0, C_i^j) \leq n'r'$. Similarly, we are done if Y intersects C_i^m ω–a.s. for all $m \in \{j+1, ..., n'\}$. Since $\{C_i^j\}_{j_0}^{n'}$ is a geodesic in $\mathbb{S}'(S)$, it follows that if Y is ω–a.s. disjoint from C_i^k then Y intersects all C_i^l for all l such that $|l-k| \geq 3$. Since by assumption $Y \cap C^j \neq \emptyset$ and because $C_i^j \subset \mathcal{N}(C_i^{j-1} \cup C_i^{j+1})$, it follows that either $Y \cap C^{j-1} \neq \emptyset$ or $Y \cap C^{j+1} \neq \emptyset$. In other words, any connected essential subsurface Y which intersects C_i^j actually intersects two consecutive separating multicurves: either C_i^{j-1}, C_i^j or C_i^j, C_i^{j+1}. In either case, it follows that Y must ω–a.s. intersect C_i^m either for all $m \in \{0, ..., j-1\}$ or for all $m \in \{j+1, ..., n'\}$, thereby completing the proof. □

The bi-Lipschitz relation in Theorem 3.3.2 guarantees that one of the terms is infinite if and only if the other term is infinite. It should be stressed that the term $\lim_\omega d_{\mathbb{S}(S)}(C_i, D_i)$ can be infinite due to two different reasons. On the one hand, it is possible that ω–a.s. C_i and D_i are connected in $\mathbb{S}(S)$ however their distances are unbounded. On the other hand, it is possible that ω–a.s. C_i and D_i are in different connected components of $\mathbb{S}(S)$. This distinction will be crucial in Chapter 4. Incidentally, by Theorem 3.1.1, the latter possibility cannot occur for surfaces with $\chi(S) \leq -5$.

Chapter 4

The asymptotic cone of Teichmüller space: cut-points, finest pieces, and applications thereof

In this chapter we explore the asymptotic cone of Teichmüller space by way of studying the asymptotic cone of the pants complex. In particular, we characterize cut-points in the asymptotic cone of the pants complex. By Theorem 2.1.8, since the pants complex is quasi-isometric to Teichmüller space [19], and because cut-points are preserved under a bi-Lipschitz map, our characterization of cut-points in the asymptotic cone of the pants complex immediately applies to Teichmüller space. The chapter is divided into three sections which we outline presently.

In Section 4.1 we introduce a notion called structurally integral corners which only arise in certain low complexity surfaces and will provide a desired separation property in the asymptotic cone. The motivation for the construction of structurally integral corners is based on the concept of microscopic jets developed in [8]. As an overly simplistic although conceptually accurate analogy, structurally integral corners are to the separating complex what microscopic jets are to the curve complex. The highlight of Section 4.2 is the proof of Theorem 4.2.3 in which we characterize when two points in the asymptotic cone of Teichmüller space are separated by a cut-point. Finally, in Section 4.3 we characterize the family of strongly contracting quasi-geodesics in Teichmüller space. This family of hyperbolic type quasi-geodesics generalizes the family of quasi-geodesics with *bounded combinatorics* studied in [23].

4.1 Structurally integral corners

4.1.1 Structurally integral corners are well-defined

Informally, a structurally integral corner entails the joining of two particular natural convex product regions in the asymptotic cone of the pants complex at a "corner" such that the removal of the corner joining the regions separates the two product regions from each other. More formally, fixing some ultrafilter ω, we have the following definition:

Definition 4.1.1 (structurally integral corner)**.** Let $\overline{\alpha} \neq \overline{\beta} \in \mathbb{S}^\omega$ be such that the following conditions hold:

1. ω–a.s. α_i and β_i are in different connected components of $\mathbb{S}(S)$. In particular, it follows that $\lim_\omega d_{\mathbb{S}(S)}(\alpha_i, \beta_i) \to \infty$ and $\alpha_i \lrcorner \beta_i, \beta_i \lrcorner \alpha_i \in \mathcal{P}(S)$. And,

2. $\lim_\omega d_{\mathcal{P}(S)}(\alpha_i \lrcorner \beta_i, \beta_i \lrcorner \alpha_i)$ is bounded. In particular, for any $\overline{Y} \in \mathcal{E}^\omega(S)$, the limit

$$\lim_\omega d_{\mathcal{C}(Y_i)}(\alpha_i \lrcorner \beta_i, \beta_i \lrcorner \alpha_i) \text{ is bounded.}$$

In this setting we call the point $(\alpha \lrcorner \beta)^\omega$ (or equivalently the point $(\beta \lrcorner \alpha)^\omega$) a *structurally integral corner*, and denote it by ${}_{\overline{\alpha}}C_{\overline{\beta}}$.

Remark 4.1.2. It should be stressed that due to condition (1) in Definition 4.1.1, in light of Theorem 3.1.1, structurally integral corners can only exist for surfaces S with $|\chi(S)| \leq 4$. In fact, for surfaces with $|\chi(S)| \geq 5$, if $\lim_\omega d_{\mathcal{P}(S)}(\alpha_i \lrcorner \beta_i, \beta_i \lrcorner \alpha_i)$ is bounded then comparing the quasi-distance formulas of $\mathcal{P}(S), \mathbb{S}(S)$ in Theorem 2.2.5, Corollary 3.1.8, respectively, it follows that $\lim_\omega d_{\mathbb{S}(S)}(\alpha_i, \beta_i)$ is also bounded. In other words, for surfaces with $|\chi(S)| \geq 5$, contradicting condition (2) in Definition 4.1.1 contradicts condition (1).

After descending from elements of ultrapowers to elements of the asymptotic cone, the structurally integral corners $(\alpha \lrcorner \beta)_\omega$ and $(\beta \lrcorner \alpha)_\omega$ will be identified and moreover, this point will serve as a cut-point between the quasi-convex product regions $\mathcal{Q}_\omega(\overline{\alpha})$ and $\mathcal{Q}_\omega(\overline{\beta})$. Note that we must assume that our cone $\mathcal{P}_\omega(S)$ contains the corner $(\alpha \lrcorner \beta)_\omega$, or equivalently we must assume the sequence $(\alpha_i \lrcorner \beta_i)$ satisfies $\lim_\omega \frac{1}{s_i} d_{\mathcal{P}(S)}(P_i^0, \alpha_i \lrcorner \beta_i) < \infty$.

Example 4.1.3 (A structurally integral corner in $\mathcal{P}_\omega(S_{2,1})$). Let $\alpha_i, \beta_i \in \mathcal{C}_{sep}(S_{2,1})$ be such that $\lim_\omega \frac{1}{s_i} d_{\mathcal{P}(S)}(P_i^0, \mathcal{Q}(\alpha_i) < \infty$, $\lim_\omega \frac{1}{s_i} d_{\mathcal{P}(S)}(P_i^0, \mathcal{Q}(\beta_i) < \infty$. Moreover, assume that ω–a.s. (i) the intersection number $i(\alpha_i, \beta_i)$ is bounded, and (ii) α_i, β_i are in different connected components of $\mathcal{C}_{sep}(S)$. In this case ${}_{\overline{\alpha}}C_{\overline{\beta}}$ is a structurally integral corner in $\mathcal{P}_\omega(S_{2,1})$. The only non-trivial point to note is that the bound on the intersection number between α_i and β_i guarantees condition (2) of Definition 4.1.1.

Given the notion of a structurally integral corner, we will now introduce a relation $\sim_{\overline{\alpha},\overline{\beta}}$ on $\mathcal{P}^\omega(S)$ which descends to an equivalence relation on $\mathcal{P}\omega(S) \setminus {}_{\overline{\alpha}}C_{\overline{\beta}}$. Moreover, each equivalence class is open. In particular, it will follow that in the asymptotic cone, $\mathcal{P}_\omega(S)$, the corner ${}_{\overline{\alpha}}C_{\overline{\beta}}$ is a cut-point between points of $\mathcal{P}_\omega(S) \setminus_{\overline{\alpha}} C_{\overline{\beta}}$ which are in different equivalence classes under the relation $\sim_{\overline{\alpha},\overline{\beta}}$. We begin with the following definition of a relation $\sim_{\overline{\alpha},\overline{\beta}}$ on $\mathcal{P}^\omega(S)$.

Definition 4.1.4. Let ${}_{\overline{\alpha}}C_{\overline{\beta}}$ be a structurally integral corner. Then we have relation $\sim_{\overline{\alpha},\overline{\beta}}$ on $\mathcal{P}^\omega(S)$ given by saying $\overline{P} \sim_{\overline{\alpha},\overline{\beta}} \overline{Q}$ if and only if $\overline{P}$ and $\overline{Q}$ fall into the same case under the following trichotomy. Namely, given $\overline{P}$,

1. $\overline{P}$ is in case one if $\exists \overline{W_\alpha} \in \mathcal{SE}^\omega(S)$ such that the following two conditions hold:

   (i) $\lim_\omega d_{\mathbb{S}(S)}(\alpha_i, \partial W_{\alpha,i})$ is bounded, and

   (ii) $\lim_\omega d_{\mathcal{C}(W_{\alpha,i})}(P_i, \beta_i) \to \infty$.

2. $\overline{P}$ is in case two if $\exists \overline{W_\beta} \in \mathcal{SE}^\omega(S)$ such that the following two conditions hold:

   (i) $\lim_\omega d_{\mathbb{S}(S)}(\beta_i, \partial W_{\beta,i})$ is bounded, and

   (ii) $\lim_\omega d_{\mathcal{C}(W_{\beta,i})}(P_i, \alpha_i) \to \infty$.

3. $\overline{P}$ is in case three if neither the conditions of case one nor case two apply to $\overline{P}$

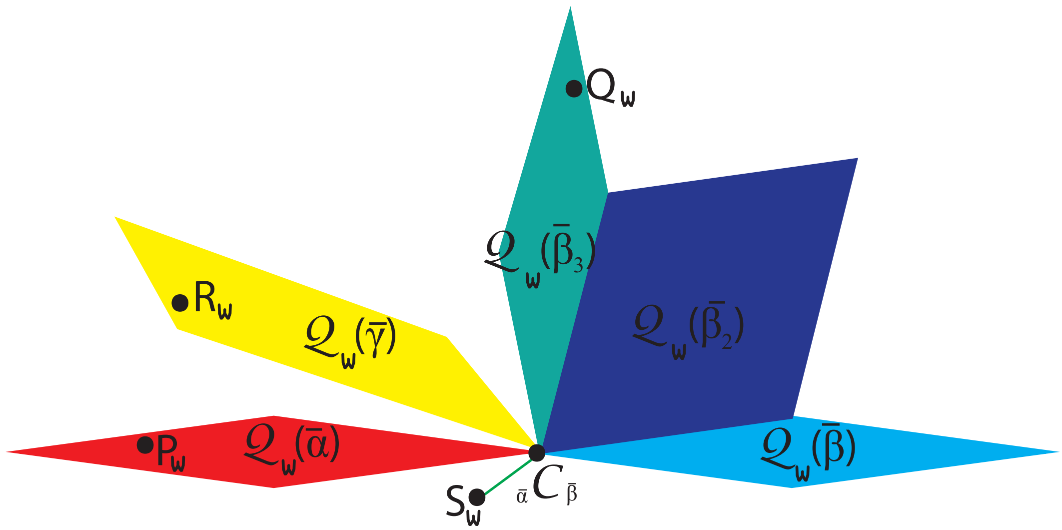

Figure 10: A structurally integral corner $_{\overline{\alpha}}C_{\overline{\beta}} \in \mathcal{P}_\omega(S)$. P_ω is in case one of the equivalence relation $\sim_{\overline{\alpha},\overline{\beta}}$, Q_ω is in case two, and the points R_ω, S_ω are in case three. In the picture we are assuming $d_{\mathbb{S}_\omega(S)}(\overline{\alpha}, \{\overline{\beta}, \overline{\beta_2}, \overline{\beta_3}\}) = \infty$, $d_{\mathbb{S}_\omega(S)}(\{\overline{\alpha}, \overline{\beta}, \overline{\beta_1}, \overline{\beta_2}\}, \overline{\gamma}) = \infty$.

See Figure 10 for an illustration of a structurally integral corner. As a first order of business, the following lemma guarantees the mutual exclusivity of the three cases in the definition of $\sim_{\overline{\alpha},\overline{\beta}}$, thus ensuring that the equivalence relation of Definition 4.1.4 is well-defined.

Lemma 4.1.5. *Let $\overline{P} \in \mathcal{P}^\omega(S)$. Then $\overline{P}$ falls into one and only one of the three cases in the trichotomy of Definition 4.1.4.*

Proof. It suffices to show that $\overline{P}$ cannot simultaneously be in cases one and two. Assume not, that is, assume $\exists$ elements $\overline{W_\alpha}, \overline{W_\beta} \in \mathcal{SE}^\omega(S)$ such that

$$\lim_\omega d_{\mathbb{S}(S)}(\alpha_i, \partial W_{\alpha,i}) \text{ and } \lim_\omega d_{\mathbb{S}(S)}(\beta_i, \partial W_{\beta,i})$$

are bounded (and by Remark 3.0.18 similarly for $\mathbb{S}'(S)$), while

$$\lim_\omega d_{\mathcal{C}(W_{\alpha,i})}(P_i, \beta_i) \text{ and } \lim_\omega d_{\mathcal{C}(W_{\beta,i})}(P_i, \alpha_i)$$

are unbounded.

Since ${}_{\overline{\alpha}}C_{\overline{\beta}}$ is a structurally integral corner, in particular, we have that $\lim_\omega d_{\mathbb{S}(S)}(\alpha_i, \beta_i)$ is unbounded, and consequently by our assumptions, $\lim_\omega d_{\mathbb{S}(S)}(\partial W_{\alpha,i}, \partial W_{\beta,i})$ is unbounded as well. Lemma 3.0.20 then guarantees that $\overline{W_\alpha} \pitchfork \overline{W_\beta}$.

By Lemma 2.2.1 if $Y_i \in \mathcal{E}(S)$ ω–a.s. intersects every separating multicurve in the bounded path of disjoint separating multicurves in $\mathbb{S}'(S)$ connecting β_i and $\partial W_{\beta,i}$, then

$$\lim_\omega d_{\mathcal{C}(Y_i)}(\beta_i, \partial W_{\beta,i})$$

is bounded as well. In particular, since the distance in $\mathbb{S}'(S)$ between $\partial W_{\alpha,i}$ and the bounded path connecting β_i and $\partial W_{\beta,i}$, is unbounded, Lemma 3.0.20 implies that ω–a.s. $\partial W_{\alpha,i}$ intersects every separating multicurve in the bounded path of separating multicurves in $\mathbb{S}'(S)$ connecting β_i and $\partial W_{\beta,i}$. Hence, $\lim_\omega d_{\mathcal{C}(W_{\alpha,i})}(\beta_i, \partial W_{\beta,i})$ is bounded. Similarly, $\lim_\omega d_{\mathcal{C}(W_{\beta,i})}(\alpha_i, \partial W_{\alpha,i})$ is bounded. In conjunction with our assumptions, it follows that $\lim_\omega d_{\mathcal{C}(W_{\alpha,i})}(P_i, \partial W_{\beta,i})$ and $\lim_\omega d_{\mathcal{C}(W_{\beta,i})}(P_i, \partial W_{\alpha,i})$ are unbounded. Since $\overline{W_\alpha} \pitchfork \overline{W_\beta}$, this contradicts Lemma 2.2.3. □

4.1.2 Equivalence relation induced by structurally integral corners

Having proven that the relation $\sim_{\overline{\alpha},\overline{\beta}}$ is well-defined, in this subsection we will prove that the relation in fact descends to an equivalence relation on $\mathcal{P}_\omega(S) \setminus_{\overline{\alpha}} C_{\overline{\beta}}$.

Theorem 4.1.6. *The relation $\sim_{\overline{\alpha},\overline{\beta}}$ descends to an equivalence relation on $\mathcal{P}_\omega(S) \setminus {}_{\overline{\alpha}}C_{\overline{\beta}}$. Moreover, each equivalence class is open.*

The proof of Theorem 4.1.6 will follow from the following technical lemma.

Lemma 4.1.7. *There exists a constant $C \geq 0$ such that for ${}_{\overline{\alpha}}C_{\overline{\beta}}$ a structurally integral corner if $\overline{P}, \overline{Q}$ are sequences representing points $P_\omega, Q_\omega \in \mathcal{P}_\omega(S)$, and if $\overline{P} \not\sim_{\overline{\alpha},\overline{\beta}} \overline{Q}$. Then,*

$$d_{\mathcal{P}_\omega(S)}(P_\omega, Q_\omega) \geq C d_{\mathcal{P}_\omega(S)}(P_\omega, {}_{\overline{\alpha}}C_{\overline{\beta}}).$$

Proof of Theorem 4.1.6. Assume that $\overline{P}$ and $\overline{Q}$ are representatives of the same point of the asymptotic cone. Then by Lemma 4.1.7 either $\overline{P} \sim_{\overline{\alpha},\overline{\beta}} \overline{Q}$ or in the asymptotic cone, $P_\omega = {}_{\overline{\alpha}}C_{\overline{\beta}}$. Hence, the relation $\sim_{\overline{\alpha},\overline{\beta}}$ descends to a relation on $\mathcal{P}_\omega(S) \setminus_{\overline{\alpha}} C_{\overline{\beta}}$ which is reflexive. Furthermore, since by definition it is immediate that $\sim_{\overline{\alpha},\overline{\beta}}$ is symmetric and transitive, it follows that $\sim_{\overline{\alpha},\overline{\beta}}$ descends to an equivalence relation on $\mathcal{P}_\omega(S) \setminus_{\overline{\alpha}} C_{\overline{\beta}}$. To see that each equivalence class is open, notice that Lemma 4.1.7 implies that any point $P_\omega \in \mathcal{P}_\omega(S) \setminus_{\overline{\alpha}} C_{\overline{\beta}}$ has an open neighborhood consisting entirely of points which are in the same equivalence class.

□

Proof of Lemma 4.1.7. $P_i, Q_i, \alpha_i \lrcorner \beta_i$ are pants decompositions of a surface and hence have non-trivial subsurface projection to any essential subsurface. For any $W \in \mathcal{E}(S)$, let σ_i^W be a $\mathcal{C}(W)$ geodesic from P_i to Q_i. Moreover, let $\pi_{\sigma_i^W}(\alpha_i \lrcorner \beta_i)$ be the nearest point projection of $\pi_{\mathcal{C}(W)}(\alpha_i \lrcorner \beta_i)$ onto the geodesic σ_i^W. Notice that by definition, $\forall W \in \mathcal{E}(S)$ we have

$$d_{\mathcal{C}(W)}(P_i, Q_i) \geq d_{\mathcal{C}(W)}(P_i, \pi_{\sigma_i^W}(\alpha_i \lrcorner \beta_i)). \tag{4.1.1}$$

By Lemma 2.2.9, there is a pants decomposition X_i with subsurface projections within a uniformly bounded distance of $\pi_{\sigma_i^W}(\alpha_i \lrcorner \beta_i)$ for all $W \in \mathcal{E}(S)$. That is, there is a uniform constant k such that $\forall W \in \mathcal{E}(S)$, we have

$$d_{\mathcal{C}(W)}(X_i, \pi_{\sigma_i^W}(\alpha_i \lrcorner \beta_i)) < k. \tag{4.1.2}$$

We will see that the sequence (X_i) represents an element $X_\omega \in \mathcal{P}_\omega(S)$. This can be shown directly, although such an argument is not necessary as later in the proof we will in fact show that $X_\omega = (\alpha \lrcorner \beta)_\omega = (\beta \lrcorner \alpha)_\omega$.

Combining Equations 4.1.1 and 4.1.2, for all $Y \in \mathcal{E}(S)$, we have:

$$d_{\mathcal{C}(Y)}(P_i, Q_i) \geq d_{\mathcal{C}(Y)}(P_i, X_i) - k, \tag{4.1.3}$$

where k is any constant. In particular, by Theorem 2.2.5 we have the following inequality:

$$d_{\mathcal{P}_\omega(S)}(P_\omega, Q_\omega) \geq C d_{\mathcal{P}_\omega(S)}(P_\omega, X_\omega). \tag{4.1.4}$$

Using inequality 4.1.4, in order to complete the proof of the lemma it suffices to show that X_ω is in fact equal to the corner point ${}_{\overline{\alpha}}C_{\overline{\beta}}$. The remainder of the proof will deal with proving this equality.

In order to show $X_\omega = {}_{\overline{\alpha}}C_{\overline{\beta}}$, by Corollary 2.2.14 it suffices to show that for any $\overline{Y} \in \mathcal{E}^\omega(S)$ we have that $\lim_\omega d_{\mathcal{C}(Y_i)}(X_i, \alpha_i \lrcorner \beta_i)$ is bounded. In fact, by the definition of X_i, it suffices to show that $\lim_\omega d_{\mathcal{C}(Y_i)}(\sigma_i^{Y_i}, \alpha_i \lrcorner \beta_i)$ is bounded. Moreover, by condition (2) in the definition of a structurally integral corner ${}_{\overline{\alpha}}C_{\overline{\beta}}$ it follows that $\lim_\omega \text{diam}_{\mathcal{C}(Y)}(\{\alpha_i, \beta_i, \alpha_i \lrcorner \beta_i, \beta_i \lrcorner \alpha_i\})$ is bounded, and hence, it suffices to show that $\lim_\omega d_{\mathcal{C}(Y_i)}(\sigma_i^{Y_i}, \{\alpha_i, \beta_i\})$ is bounded.

By assumption $\overline{P}$ and $\overline{Q}$ are in different equivalence classes, and hence by definition $\overline{P}$ and $\overline{Q}$ fall into different cases in Definition 4.1.4. By symmetry of the cases, without loss of generality we can assume that $\overline{P}$ is in case one of Definition 4.1.4, while $\overline{Q}$ is not. Namely, $\exists \overline{W_\alpha} \in \mathcal{SE}^\omega(S)$ such that $\lim_\omega d_{\mathbb{S}(S)}(\alpha_i, \partial W_{\alpha,i})$ is bounded, while $\lim_\omega d_{\mathcal{C}(W_{\alpha,i})}(P_i, \beta_i) \to \infty$. Furthermore, for any element $\overline{U} \in \mathcal{SE}^\omega(S)$ such that $\lim_\omega d_{\mathbb{S}(S)}(\alpha_i, \partial U_i)$ is bounded, perforce $\lim_\omega d_{\mathcal{C}(U_i)}(Q_i, \beta_i)$ is also bounded. By Remark 3.0.18 the same statements hold for $\mathbb{S}'(S)$.

We proceed by considering cases for the relationship between $\overline{Y}$ and $\overline{W_\alpha}$ where $\overline{Y}$ is an arbitrary element of the ultrapower of connected essential subsurfaces. By Lemma 2.1.9 since there are only a finite number of possibilities for the relationship between two essential subsurfaces - identical, nested, overlapping, and disjoint - it follows that there are similarly the same finitely many possibilities for the relationship between $\overline{Y}$ and $\overline{W_\alpha}$. In each case we will show $\lim_\omega d_{\mathcal{C}(Y_i)}(\sigma_i^{Y_i}, \{\alpha_i, \beta_i\})$ is bounded, thus completing the proof of the lemma.

Case 1: Either $\overline{Y} \subset \overline{W_\alpha}$ or $\overline{Y} \cap \overline{W_\alpha} = \emptyset$. In either case, ω–a.s. $d_{\mathbb{S}(S)}(\partial W_{\alpha,i}, \partial Y_i) \leq 1$ and hence by our assumptions $\lim_\omega d_{\mathbb{S}(S)}(\alpha_i, \partial Y_i)$ is bounded. Since $\overline{Q}$ is not in case one of the equivalence relation $\sim_{\overline{\alpha},\overline{\beta}}$, it follows that $\lim_\omega d_{\mathcal{C}(Y_i)}(Q_i, \beta_i)$ is bounded. In particular, this implies that $\lim_\omega d_{\mathcal{C}(Y_i)}(\sigma_i^{Y_i}, \{\alpha_i, \beta_i\})$ is bounded, completing this case.

Case 2: $\overline{W_\alpha} \subset \overline{Y}$ and $\lim_\omega d_{\mathcal{C}(Y_i)}(\partial W_{\alpha,i}, \{\alpha_i, \beta_i\})$ is bounded. By our assumptions,

$$\lim_\omega d_{\mathcal{C}(W_{\alpha,i})}(P_i, \beta_i) \to \infty,$$

while $\lim_\omega d_{\mathcal{C}(W_{\alpha,i})}(Q_i, \beta_i)$ is bounded. In particular, $\lim_\omega d_{\mathcal{C}(W_{\alpha,i})}(P_i, Q_i) \to \infty$. Then ω–a.s. $d_{\mathcal{C}(Y_i)}(\partial W_{\alpha,i}, \sigma_i^{Y_i}) \leq 1$. If not, then Theorem 2.2.2 would imply that ω–a.s. $d_{\mathcal{C}(W_{\alpha,i})}(P_i, Q_i)$ is uniformly bounded which is a contradiction. However, the assumption of the case that $\lim_\omega d_{\mathcal{C}(Y_i)}(\partial W_{\alpha,i}, \{\alpha_i, \beta_i\})$ is bounded then implies that $\lim_\omega d_{\mathcal{C}(Y_i)}(\{\alpha_i, \beta_i\}, \sigma_i^{Y_i})$ is bounded, thus completing this case.

Note that the special case of $\overline{Y} = \overline{S}$ must fall into this case. Specifically, recall we have assumed that $\lim_\omega d_{\mathbb{S}(S)}(\alpha_i, \partial W_{\alpha,i})$ is bounded which in particular ensures the same result for $\mathcal{C}(S)$ as distance in $\mathbb{S}(S)$ is coarsely bounded from below by $\mathcal{C}(S)$ distance, see Equation 3.0.2.

Case 3: $\overline{Y} \pitchfork \overline{W_\alpha}$ and $\lim_\omega d_{\mathcal{C}(Y_i)}(\partial W_{\alpha,i}, \{\alpha_i, \beta_i\})$ is bounded. As in Case 2, by our assumptions $\lim_\omega d_{\mathcal{C}(W_{\alpha,i})}(P_i, \beta_i) \to \infty$, while $\lim_\omega d_{\mathcal{C}(W_{\alpha,i})}(Q_i, \beta_i)$ is bounded. In particular,

$$\lim_\omega d_{\mathcal{C}(W_{\alpha,i})}(P_i, Q_i) \to \infty.$$

Since ω–a.s. $W_{\alpha,i} \pitchfork Y_i$, it follows that $\lim_\omega d_{\mathcal{C}(Y_i)}(\partial W_{\alpha,i}, \{P_i, Q_i\})$ is uniformly bounded. If not, then Lemma 2.2.3 implies that $d_{\mathcal{C}(W_{\alpha,i})}(P_i, Q_i)$ is uniformly bounded which is a contradiction. However, the assumption of the case that

$$\lim_\omega d_{\mathcal{C}(Y_i)}(\partial W_{\alpha,i}, \{\alpha_i, \beta_i\})$$

is bounded then implies that $\lim_\omega d_{\mathcal{C}(Y_i)}(\{\alpha_i, \beta_i\}, \{P_i, Q_i\})$ is bounded. Since $\sigma_i^{Y_i}$ is $\mathcal{C}(Y_i)$ geodesic between P_i and Q_i, it follows that $\lim_\omega d_{\mathcal{C}(Y_i)}(\{\alpha_i, \beta_i\}, \sigma_i^{Y_i})$ is bounded, thus completing this case.

Case 4: Either $\overline{W_\alpha} \subset \overline{Y}$ or $\overline{Y} \pitchfork \overline{W_\alpha}$, and in both cases, $\lim_\omega d_{\mathcal{C}(Y_i)}(\partial W_{\alpha,i}, \{\alpha_i, \beta_i\})$ is unbounded. Since $\lim_\omega d_{\mathbb{S}'(S)}(\alpha_i, \partial W_{\alpha,i})$ is bounded, it follows that there is a bounded path of connected multicurves in the curve complex $\mathcal{C}(S)$ from α_i to $\partial W_{\alpha,i}$ such that each multicurve is a separating multicurve. Call this path ρ_i. On the other hand, the assumption of the case is that $\lim_\omega d_{\mathcal{C}(Y_i)}(\partial W_{\alpha,i}, \{\alpha_i, \beta_i\}) \to \infty$. Putting things together, by Lemma 2.2.1 it follows ω–a.s. Y_i is disjoint from some vertex in ρ_i. By construction, it follows that $\partial Y_i \in \mathbb{S}(S)$, and in fact $\lim_\omega d_{\mathbb{S}(S)}(\alpha_i, \partial Y_i)$ is bounded. Since $\overline{Q}$ is not in case one of the equivalence relation $\sim_{\overline{\alpha},\overline{\beta}}$, it follows that $\lim_\omega d_{\mathcal{C}(Y_i)}(Q_i, \beta_i)$ is bounded. It follows that $\lim_\omega d_{\mathcal{C}(Y_i)}(\{\alpha_i, \beta_i\}, \sigma_i^{Y_i})$ is bounded. This completes the proof of the final case thereby completing the proof of the lemma. □

4.1.3 Separation property of structurally integral corners

As an immediate corollary of Theorem 4.1.6 we have the following useful separation property of structurally integral corners in the asymptotic cone. This separation property should be compared with the separation property of microscopic jets recorded in Theorem 2.2.16.

Corollary 4.1.8. *Let ${}_{\overline{\alpha}}C_{\overline{\beta}}$ be a structurally integral corner, and let $x_\omega, x'_\omega \in \mathcal{P}_\omega(S) \setminus_{\overline{\alpha}} C_{\overline{\beta}}$ be points in the asymptotic cone such that $x_\omega \not\sim_{\overline{\alpha},\overline{\beta}} x'_\omega$. Then x_ω and x'_ω are separated by the corner ${}_{\overline{\alpha}}C_{\overline{\beta}}$.*

Example 4.1.9. (Structurally integral corners in surfaces S with $\xi(S) = 3$) If the surface S has $\xi(S) = 3$, then $\mathbb{S}(S)$ has connected components consisting of singletons. In this case, the definition of a structurally integral corner and moreover the corresponding equivalence relation become much simpler. Specifically, Definition 4.1.1 simplifies to α_i and β_i being distinct elements in $\mathbb{S}^\omega(S)$ such that $\lim_\omega d_{\mathcal{P}(S)}(\alpha_i \lrcorner \beta_i, \beta_i \lrcorner \alpha_i)$ is bounded. Moreover, in Definition 4.1.4 the only possibilities for $\overline{W_\alpha}, \overline{W_\beta}$ are the two complexity one connected components of $\overline{S \setminus \alpha}$, $\overline{S \setminus \beta}$ respectively.

4.2 Finest pieces

Behrstock showed that every point in the asymptotic cone of both the mapping class group and Teichmüller space is a global cut-point, [3]. On the other hand, it is well established that for surfaces S with $\xi(S) \geq 2$, the mapping class group admits quasi-isometric embeddings of $\mathbb{Z}'$ flats, while for surfaces with $\xi(S) \geq 3$ Teichmüller space admits quasi-isometric embeddings of $\mathbb{Z}'$ flats, [10, 20, 48]. Hence, for high enough complexity surfaces the mapping class group and Teichmüller space are not δ-hyperbolic and in particular, their asymptotic cones are not $\mathbb{R}$-trees. Putting things together, for high enough complexity surfaces, the asymptotic cones of the mapping class group and Teichmüller space are nontrivial *tree-graded* spaces with the property that every point is a cut-point globally, but not locally for some nontrivial local regions. In such settings, we have canonically defined *finest pieces of the tree-graded structure* which are maximal subsets of the asymptotic cone subject to the condition that no two points in a finest piece can be separated by the removal of a point. In this subsection, we will characterize of the canonically defined finest pieces in the tree-graded structure of $\mathcal{T}_\omega(S)$. Our theorem is motivated by and should be compared with the following theorem of [8]:

Theorem 4.2.1 ([8] Theorem 7.9)**.** *Let $S = S_{g,n}$ and let $\mathcal{M}CG_\omega(S)$ be any asymptotic cone of $\mathcal{MCG}(S)$. Then for all $a_\omega, b_\omega \in \mathcal{MCG}_\omega(S)$, the following are equivalent:*

1. *No point separates a_ω and b_ω, and*

2. *In any neighborhood of a_ω, b_ω there exists a'_ω, b'_ω, with representatives $(a'_i), (b'_i)$ respectively, such that:*
$$\lim_\omega d_{\mathcal{C}(S)}(a'_i, b'_i) < \infty.$$

Example 4.2.2 ($\mathcal{M}CG(S)$ vs $\mathcal{P}(S)$; a partial pseudo-Anosov axis). The following example demonstrates that Theorem 4.2.1 cannot be applied without modification to $\mathcal{P}(S)$. Consider a representative (P_i^0) of the basepoint of our asymptotic cone $\mathcal{P}_\omega(S)$, and let $\gamma_i \in P_i^0$ be a non-separating curve. Let $g_i \in \mathcal{M}CG(S \setminus \gamma_i)$ be a pseudo-Anosov map. Then consider the following two points in the asymptotic cone:

$$a_\omega = (P_i^0), \quad b_\omega = (g_i^{s_i} P_i^0).$$

By construction, $a_\omega \neq b_\omega$ lie on a partial psuedo-Anosov axis in the asymptotic cone. Furthermore, by construction, using notation from Subsection 2.2.4 we have:

$$a_\omega, b_\omega \in P_{\overline{S\setminus\gamma}, a_\omega} = F_{\overline{S\setminus\gamma}, a_\omega} \times \{\overline{\gamma}\} = \mathbb{R}\text{-tree} \times \{pt\} \subset \mathcal{P}_\omega(S).$$

Hence, a_ω, and b_ω can be separated by a cut-point. Nonetheless, a_ω and b_ω have representatives (P_i^0), $(g_i^{s_i} P_i^0)$, respectively, each containing γ_i. In particular, $\forall i \in \mathbb{N}$, $d_{\mathcal{C}(S)}(P_i^0, g_i^{s_i} P_i^0) = 0$. Hence in $\mathcal{P}(S)$, statement (1) of Theorem 4.2.1 can fail even though statement (2) holds.

Despite the fact that Theorem 4.2.1 does not apply verbatim to $\mathcal{P}(S)$, the following slightly modified theorem with condition (2) strengthened does apply to $\mathcal{P}(S)$.

Theorem 4.2.3. *Let $S = S_{g,n}$ and let $\mathcal{P}_\omega(S)$ be any asymptotic cone of $\mathcal{P}(S)$. Then for all $a_\omega, b_\omega \in \mathcal{P}_\omega(S)$, the following are equivalent:*

1. *No point separates a_ω and b_ω, or equivalently a_ω and b_ω are in the same canonical finest piece, and*

2. *In any neighborhood of a_ω, b_ω, respectively, there exists a'_ω, b'_ω, with representative sequences (a'_i),(b'_i), such that* $\lim_\omega d_{\mathbb{S}(S)}(a'_i, b'_i) < \infty$.

A couple of remarks are in order.

Remark 4.2.4. Note that condition (2) of Theorem 4.2.3 implies condition (2) of Theorem 4.2.1 as distance in $\mathbb{C}(S)$ is coarsely bounded above by distance in $\mathbb{S}(S)$, see Equation 3.0.2. Moreover, note that by Theorem 3.3.2, condition (2) of Theorem 4.2.3 can be replaced by the following statement: In any neighborhood of a_ω, b_ω, respectively, there exist points a'_ω, b'_ω, such that $d_{\mathbb{S}_\omega(S)}(a'_\omega, b'_\omega) < \infty$.

Proof of Theorem 4.2.3. (2) $\implies$ (1): As noted in Remark 4.2.4, Property (2) implies that a_ω, b_ω are limit points of sequences in the asymptotic cone which have finite $\mathbb{S}_\omega(S)$ distance. Since the canonically defined finest pieces are closed sets [30], it suffices to show that points in the asymptotic cone with finite $\mathbb{S}_\omega(S)$ distance cannot be separated by a point. Specifically, assume we have a chain of natural convex nontrivial product regions $\mathcal{Q}_\omega(\overline{\gamma_0})$, ..., $\mathcal{Q}_\omega(\overline{\gamma_K})$ in the asymptotic cone $\mathcal{P}_\omega(S)$ such that $a'_\omega \in \mathcal{Q}_\omega(\overline{\gamma_0})$, $b'_\omega \in \mathcal{Q}_\omega(\overline{\gamma_K})$, and for all $j \in \{0, ..., K-1\}$ $\mathcal{Q}_\omega(\overline{\gamma_j}) \bigcap \mathcal{Q}_\omega(\overline{\gamma_{j+1}})$ has infinite diameter intersection. Clearly, each product region cannot be separated by a point. Furthermore, by assumption each product region cannot be separated from its neighbor by a point. It follows that a'_ω and b'_ω cannot be separated by a point, thus completing the proof of $(2) \implies (1)$.

$(1) \implies (2)$: We will prove the contrapositive, namely $\sim (2) \implies \sim (1)$. The negation of property (2) implies that there exists an $r_1 > 0$ such that all points in r_1 open neighborhoods of a_ω and b_ω respectively have infinite or undefined $\mathbb{S}_\omega(S)$ distance. By Theorem 2.1.8, $\mathcal{P}_\omega(S)$ is locally path connected. Let $r_2 > 0$ be a constant such that the r_2 open neighborhoods of a_ω and b_ω are path connected. Set $3r = \min(r_1, r_2)$. By choosing r_1 to be sufficiently small, we can assume that $d_{\mathcal{P}_\omega(S)}(a_\omega, b_\omega) > 6r$.

Let the sequences $(a'_i), (b'_i)$ represent any points a'_ω, b'_ω in r neighborhoods of a_ω, b_ω respectively, let γ_i be a hierarchy path between a'_i and b'_i, and let γ_ω represent its ultralimit. By construction γ_ω is a $(K, 0)$-quasi-geodesic. Let a''_ω denote a point on γ_ω of distance r from a'_ω, and let a'''_ω denote a point on γ_ω of distance $2r$ from a'_ω. Similarly, let b''_ω denote a point on γ_ω of distance r from b'_ω, and let b'''_ω denote a point on γ_ω of distance $2r$ from b'_ω. See Figure 11. We will show that the quasi-geodesic γ_ω contains a cut-point between the points a''_ω and b''_ω. Then, local path connectedness implies that the cut-point also separates a_ω and b_ω, thus completing the proof of the negation of (1) and hence the proof of the Theorem. Specifically, since by assumption a_ω and a'_ω (and similarly b_ω and b'_ω) are within distance r of each other, and because the cut-point between a''_ω and b''_ω is at least distance r from a'_ω, (and similarly from b'_ω) it follows that a geodesic path between a_ω and a'_ω (and similarly between b_ω and b'_ω) does not contain the cut-point.

We will proceed by considering two cases. In the first case we will obtain a cut-point using the machinery of microscopic jets and in the second case we will obtain a cut-point using the machinery of structurally integral corners.

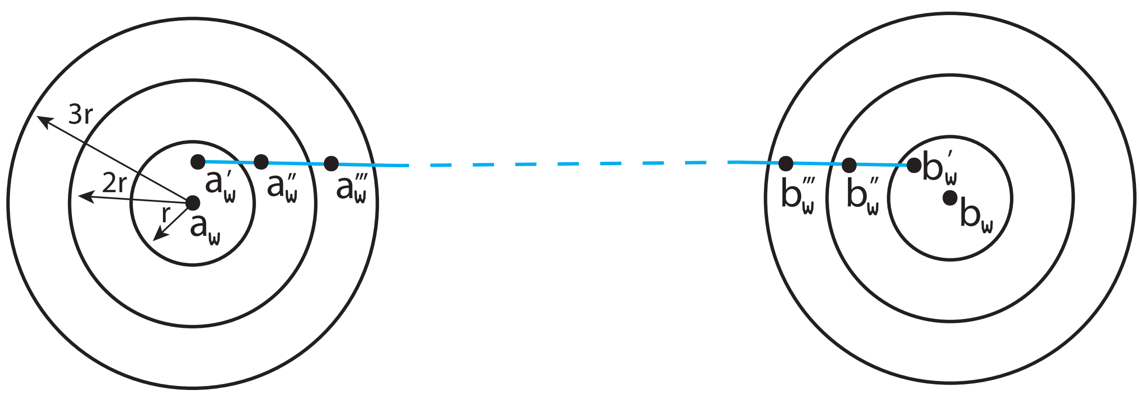

Figure 11: The dotted line is a quasi-geodesic γ_ω from a'_ω to b'_ω.

Case One: $\exists r'$ such that for all a^0_ω, b^0_ω in $3r'$ neighborhoods of a_ω, b_ω, with (a^0_i),(b^0_i) any representatives thereof, respectively, $\exists \overline{Y} \in \mathcal{NE}^\omega(S)$ with $\lim_\omega d_{\mathcal{C}(Y_i)}(a^0_i, b^0_i) \to \infty$.

By abuse of notation assume that we have replaced r described above by $r = \min\{r, r'\}$. In particular, since a'''_ω, b'''_ω are contained in $3r'$ neighborhoods of a_ω, b_ω, respectively, the assumption of the case ensures that for some $\overline{Y} \in \mathcal{NE}^\omega(S)$, we have $\lim_\omega d_{\mathcal{C}(Y_i)}(a'''_i, b'''_i) \to \infty$. Then, by Theorem 2.2.15 there exists a microscopic jet $J = (\overline{g}, \overline{Y}, \overline{a'''}, \overline{b'''})$ with $\overline{g} \subset \gamma_\omega|_{[a'''_\omega, b'''_\omega]}$ and such that $a'''_\omega \not\sim_J b'''_\omega$. By definition, $\lim_\omega d_{\mathcal{C}(Y_i)}(\pi_{g_i}(a'''_i), \pi_{g_i}(b'''_i)) \to \infty$. By the properties of hierarchies in Theorem 2.2.4 it follows that $\lim_\omega d_{\mathcal{C}(Y_i)}(\pi_{g_i}(a''_i), \pi_{g_i}(b''_i)) \to \infty$, and hence $a''_\omega \not\sim_J b''_\omega$.

Since the complement $\overline{Y^c}$ is the emptyset, $\iota(J) \times \mathcal{P}_\omega(\overline{Y^c})$ is a single point in the asymptotic cone. Moreover, by construction it is not equal to either a''_ω or b''_ω. Theorem 2.2.16 implies that the initial point of the jet is a cut-point between a''_ω and b''_ω. This completes the proof of case one. It should be noted that the proof of case one follows closely the proof of Theorem 4.2.1 in [8]. In fact, for the special case of $\overline{Y} = S$ the proofs are identical.

Case Two: The negation of case one. Namely, in any neighborhoods of a_ω, b_ω there exists a^0_ω, b^0_ω with representatives (a^0_i),(b^0_i), such that $\forall \overline{Y} \in \mathcal{NE}^\omega(S), \lim_\omega d_{\mathcal{C}(Y_i)}(a^0_i, b^0_i) < \infty$.

For r neighborhoods of a_ω, b_ω set the points a^0_ω, b^0_ω with representatives (a^0_i),(b^0_i), guaranteed to exist by the hypothesis of the case to be equal to a'_ω, b'_ω, with representatives (a'_i),(b'_i), respectively. Then as above, let γ_i be a hierarchy path between a'_i and b'_i, and similarly define the points $a''_i, a'''_i, b''_i, b'''_i$. By the assumptions of the case the hierarchies γ_i have the property that for all $Y \in \mathcal{NE}(S)$, the projection of γ_i to $\mathcal{C}(Y)$ is uniformly bounded. In particular, the hierarchies γ_i have uniformly bounded main geodesic length and travels for uniformly bounded distances in all connected nonseparating essential subsurfaces Y. By Lemma 2.1.9 there is a k such that ω–a.s. the main geodesic in γ_i has length exactly k. Specifically, ω–a.s. there is a tight main geodesic in $\mathcal{C}(S)$, with simplices $g_{0i}, ..., g_{ki}$ such that $g_{0i} \subset a'_i$, $g_{ki} \subset b'_i$. By construction, the hierarchy γ_i travels through the finite set of quasi-convex regions, $\mathcal{Q}(g_{0i})$, ..., $\mathcal{Q}(g_{ki})$. See Figure 12.

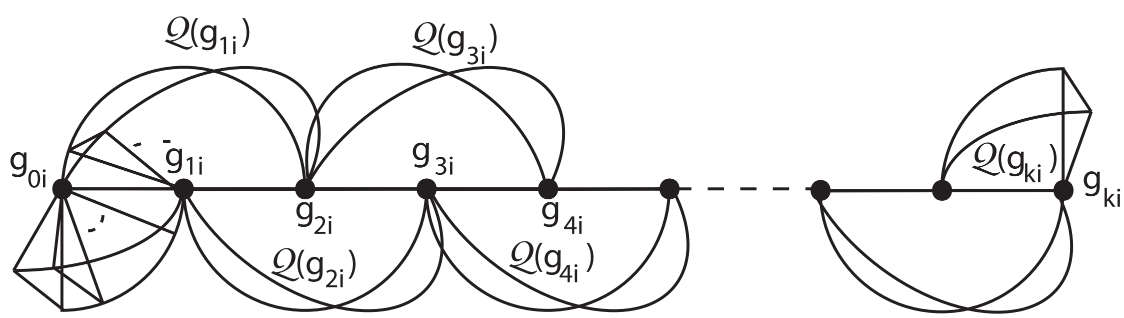

Figure 12: The ultralimit of hierarchy paths with a uniformly bounded main geodesics. Notice that each of the vertices along the finite length main geodesic are separating multicurves.

Without loss of generality we can assume that for all j, either $\gamma_{ji} \in \mathcal{P}(S)$, i.e γ_{ji} is an entire pants decomposition of a surface, or for any (W_i) a sequence of connected essential subsurfaces in the complement $S \setminus g_{ji}$, we have $\lim_\omega d_{\mathcal{C}(W_i)}(a_i', b_i') \to \infty$. If not, by iterating the argument we used above for a finite length $\mathcal{C}(S)$ main geodesic we can ω–a.s. replace the multicurve g_{ji} by a finite list of connected simplices in $\mathcal{C}(S)$ each containing g_{ji} as a proper multicurve. This iteration process of replacing a multicurve g_{ji} from our our finite list $\{g_{0i}, ..., g_{ki}\}$ with finite sequences of multicurves each containing the original multicurve as a proper multicurve must terminate due to the finite complexity of the surface S. Accordingly, we have a finite list of nontrivial quasi-convex regions and singletons through which our hierarchy path γ_i from a_i' to b_i' ω–a.s. travels. Since the list of nontrivial quasi-convex regions and singletons is bounded ω–a.s. , coarsely we can ignore the singletons. That is, coarsely our hierarchy path γ_i from a_i' to b_i' ω–a.s. travels through only a finite list of nontrivial quasi-convex regions, $\mathcal{Q}(g_{0i})$, ..., $\mathcal{Q}(g_{k'i})$ such that for any (W_i) a sequence of connected component of $S \setminus g_{ji}$, we have $\lim_\omega d_{\mathcal{C}(W_i)}(a_i', b_i') \to \infty$. By the assumptions of our case, for each j, ω-a.s $g_{i,j}$ is a separating multicurve, or equivalently for each j the region $\mathcal{Q}(g_{ji})$ is a nontrivial quasi-convex product region. Moreover, by construction for all j, $\lim_\omega d_{\mathcal{P}(S)}(g_{ij} \lrcorner g_{(i+1)j}, g_{(+1)j} \lrcorner g_{ij},)$ is bounded. Notice that all of the above analysis holds after restricting to the subquasi-geodesic $\gamma_i|_{a_i''', b_i'''}$. Assume we have done so.

However, by the negation of condition (2) of the theorem, it follows that there exist consecutive separating multicurves, $g_{ji}, g_{(j+1)i}$ in our list such that:

$$\lim_\omega d_{\mathbb{S}(S)}(g_{ji}, g_{(j+1),i}) \to \infty.$$

In particular, in conjunction with the analysis of the previous paragraph, we have a structurally integral corner ${}_{\overline{g_j}}C_{\overline{g_{j+1}}}$. Moreover, by construction $a_\omega'', b_\omega'' \neq {}_{\overline{g_j'}}C_{\overline{g_{j+1}'}}$ as the corner is on the quasi-geodesic $\gamma_\omega|_{[a_\omega''', b_\omega''']}$. Furthermore, $a_\omega'' \not\sim_{\overline{g_j'}, \overline{g_{j+1}'}} b_\omega''$, as by our assumptions a_ω'' is in case one of the equivalence relation $\sim_{\overline{g_j'}, \overline{g_{j+1}'}}$ while b_ω'' is in case two of the equivalence relation $\sim_{\overline{g_j'}, \overline{g_{j+1}'}}$. Corollary 4.1.8 implies that the structurally integral corner ${}_{\overline{g_j'}}C_{\overline{g_{j+1}'}}$ is a cut-point between the points a_ω'', b_ω''. This completes the proof of the theorem. $\square$

Remark 4.2.5. Precisely as in Remark 4.1.2, it should be stressed that Case Two in the proof of Theorem 4.2.3 cannot occur for surfaces with $|\chi(S)| \geq 5$.

4.2.1 Applications of the classification of finest pieces

Special cases of Theorem 4.2.3 include the following celebrated theorems of others.

Corollary 4.2.6 ([3, 20] Theorem 5.1, Theorem 1.1)**.** *Let $S = S_{1,2}$ or $S_{0,5}$. Then $\mathcal{P}(S)$ is δ-hyperbolic.*

Proof. It suffices to show that for all choices of asymptotic cones, $\mathcal{P}_\omega(S)$ is an $\mathbb{R}$-tree, see [28, 37]. Equivalently, it suffices to show that the finest pieces in any asymptotic cone are trivial, or equivalently, any two points $a_\omega \neq b_\omega \in \mathcal{P}_\omega(S)$ can be separated by a point. However, by Theorem 4.2.3 this is immediate as $\mathbb{S}(S) = \emptyset$. □

Corollary 4.2.7 ([21] Theorem 1)**.** *Let $\xi(S) = 3$, then $\mathcal{P}(S)$ is relatively hyperbolic with respect to natural quasi-convex product regions consisting of all pairs of pants with a fixed separating curve.*

Proof. It suffices to show that $\mathcal{P}(S)$ is asymptotically tree-graded with respect to peripheral subsets consisting of all natural quasi-convex product regions $\mathcal{Q}(\gamma)$ for any $\gamma \in \mathcal{C}_{sep}(S)$.

By topological considerations any two separating curves $\gamma \neq \delta \in \mathcal{C}_{sep}(S)$, $S \setminus (\gamma \cup \delta)$ does not contain an essential subsurface. Consequently, $d_{\mathbb{S}(S)} \in \{0, \infty\}$, and similarly for all $\overline{C}, \overline{D} \in \mathbb{S}^\omega(S)$, the expression $\lim_\omega d_{\mathbb{S}(S)}(C_i, D_i)$ takes values in $\{0, \infty\}$. Accordingly, Theorem 4.2.3 implies any two points a_ω, b_ω are either in a common natural convex product region (such regions are closed) or are separated by a cut-point. In particular, any simple nontrivial geodesic triangle in $\mathcal{P}_\omega(S)$ must be contained entirely inside a single piece $\mathcal{Q}_\omega(\overline{\gamma})$. □

While stated for $\mathcal{P}(S)$, Corollaries 4.2.6 and 4.2.7 immediately apply to $\mathcal{T}(S)$ as hyperbolicity and strong relative hyperbolicity are quasi-isometry invariant properties.

4.3 Hyperbolic type quasi-geodesics

In this section, after some definitions of the various types of hyperbolic type geodesics, we will characterize hyperbolic type quasi-geodesics in Teichmüller space. See [43] for a similar analysis of strongly contracting quasi-geodesics in Teichmüller space equipped with the Lipschitz metric. We begin with the notion of a Morse (quasi-)geodesic which has roots in the classical paper [53]:

Definition 4.3.1 (Morse)**.** A (quasi-)geodesic γ is called a *Morse (quasi-)geodesic* if every (K, L)-quasi-geodesic with endpoints on γ is within a bounded distance from γ, with the bound depending only on the constants K, L. Similarly, the definition of Morse can be associated to a sequence of (quasi-)geodesic segments with uniform quasi-isometry constants.

The following generalized notion of contracting quasi-geodesics can be found for example in [3, 21], and is based on a slightly more general notion of (a,b,c)–contraction found in [47] where it serves as a key ingredient in the proof of the hyperbolicity of the curve complex.

Definition 4.3.2 (contracting quasi-geodesic)**.** A quasi-geodesic γ is said to be *(b,c)–contracting* if $\exists$ constants $0 < b \leq 1$ and $0 < c$ such that $\forall x, y \in X$:

$$d_X(x, y) < bd_X(x, \pi_\gamma(x)) \implies d_X(\pi_\gamma(x), \pi_\gamma(y)) < c.$$

For the special case of a (b,c)–contracting quasi-geodesic where b can be chosen to be 1, the quasi-geodesic γ is called *strongly contracting.*

In [63], which is reproduced in Section 6.2, hyperbolic type quasi-geodesics in CAT(0) spaces are analyzed. In particular, the following result is proven:

Theorem 6.2.5. *Let X be a CAT(0) space and $\gamma \subset X$ a quasi-geodesic. Then, the following are equivalent: (1) γ is (b,c)–contracting, (2) γ is strongly contracting, (iii) γ is Morse, and (iv) In every asymptotic cone X_ω, any two distinct points in the ultralimit γ_ω are separated by a cut-point.*

Recall that $\overline{\mathcal{T}}(S)$ is CAT(0). Combining Theorems 4.2.3 and 6.2.5, the following corollary characterizes all strongly contracting quasi-geodesics in $\overline{\mathcal{T}}(S)$. Equivalently, in light of Theorem 6.2.5 the theorem also characterizes Morse quasi-geodesics in $\overline{\mathcal{T}}(S)$. The characterization represents a generalization of quasi-geodesics with *bounded combinatorics* studied in [3, 23]. Specifically, in [3, 23] it is shown that quasi-geodesics in $\mathcal{P}(S)$ which have uniformly bounded subsurface projections to all connected proper essential subsurfaces. More generally, we show:

Theorem 4.3.3. *Let γ be a quasi-geodesic in $\overline{\mathcal{T}}(S)$, and using Theorem 2.1.6 let γ' be a corresponding quasi-geodesic in $\mathcal{P}(S)$. Then γ is strongly contracting if and only if there exists a constant C such that for all $Y \in \mathcal{SE}(S)$ the subsurface projection $\pi_Y(\gamma')$ has diameter bounded above by C.*

Proof. Assume there is no uniform bound C on the subsurface projection $\pi_Y(\gamma')$, where Y ranges over $\mathcal{SE}(S)$. Then we can construct $\overline{Y} \in \mathcal{SE}^\omega(S)$ such that $\lim_i diam(\pi_{Y_i}(\gamma')) \to \infty$. By the properties of hierarchies in Theorem 2.2.4, it follows that there is a sequence of hierarchy quasi-geodesic segments $\{\gamma'_r\}_r$ with endpoints on γ' traveling through product regions $\mathcal{Q}(\partial Y_r)$ for unbounded connected subsegments. In particular, the sequence of quasi-geodesics $\{\gamma'_r\}_r$ are not Morse, and furthermore since the hierarchy segments γ'_r are all quasi-geodesics with uniform constants which have endpoints on γ', the quasi-geodesic γ' is also not Morse. Moreover, considering the quasi-isometry taking γ' to γ, it similarly follows that γ is not Morse. By Theorem 6.2.5, γ is not strongly contracting.

On the other hand, assume $\forall Y \in \mathcal{SE}(S)$ that the subsurface projection $\pi_Y(\gamma')$ is uniformly bounded. Let $\mathcal{P}_\omega(S)$ be any asymptotic cone with a_ω, b_ω any two distinct points on γ'_ω with representatives sequences $(a_i), (b_i) \in \gamma'$, respectively. Proceeding as in Case One of the proof of Theorem 4.2.3, consider a sequence of hierarchy quasi-geodesic segments $\rho(a_i, b_i)$, between the points a_i and b_i on γ', and define distinct points $a''_\omega, a'''_\omega, b''_\omega, b'''_\omega$ with representatives $(a''_i), (a'''_i), (b''_i), (b'''_i)$ along the sequence of hierarchy quasi-geodesic segments $\rho(a_i, b_i)$. By assumption, $\forall \overline{Y} \in \mathcal{SE}^\omega(S), \lim_\omega d_{\mathcal{C}(Y_i)}(a'''_i, b'''_i)$ is bounded. On the other hand, since $a'''_\omega \neq b'''_\omega$ by Corollary 2.2.14 there is some $\overline{W} \in \mathcal{E}^\omega(S)$ such that $\lim_\omega d_{\mathcal{C}(W_i)}(a_i, b_i)$ is unbounded. Perforce, $\overline{W} \in \mathcal{NE}^\omega(S)$. Then, as in Case One of the proof of Theorem 4.2.3, there exists a microscopic jet which gives rise to a cut-point between a_ω and b_ω. Since a_ω and b_ω are arbitrary and because cut-points in asymptotic cones are preserved by quasi-isometries, by Theorem 6.2.5 γ is strongly contracting. □

As a corollary of Theorem 4.3.3, we have the following result highlighting a strong distinction between $\mathcal{MCG}(S)$ and $\mathcal{T}(S)$. Let $Y \in \mathcal{NE}(S)$ be a connected proper essential subsurface, and let f be a *partial pseudo-Anosov* mapping class group supported on Y. That is, f is a reducible mapping class group which restricts to a pseudo-Anosov homeomorphism on the non-separating essential subsurface Y, and to the identity in the complement of Y. In terms of $\mathcal{MCGS}$, it is not hard to see that the axis of f in the mapping class group is not a contracting element. In fact, the entire axis of f, $< f >$, is contained in a nontrivial quasi-convex product subspace of $\mathcal{MCG}(S)$, as there is an infinte order Dehn twist subgroup with support the annulus ∂Y which commutes with the subgroup $< f > \subset \mathcal{MCG}(S)$. On the other hand, in contrast to the mapping class group setting, in $\mathcal{P}(S)$, this same partial pseudo-Anosov axis is a contracting quasi-geodesic.

Corollary 4.3.4. *Let γ be any partial pseudo-Anosov axis in $\overline{\mathcal{T}(S)}$ supported on a connected nonseparating essential subsurface $Y \in \mathcal{NE}(S)$, then γ is strongly contracting.*

Proof. By Theorem 4.3.3 we must show that the corresponding quasi-geodesic γ' in $\mathcal{P}(S)$ has uniformly bounded subsurface projection for all $W \in \mathcal{SE}(S)$. If $W \cap \partial Y \neq \emptyset$ then we are done as the curve ∂Y is fixed along the partial pseudo-Anosov axis, and hence the subsurface projection of γ' into the curve complex of the essential subsurface W remains in a uniform diameter of $\pi_{\mathcal{C}(W)}(\partial Y)$. So without loss of generality W and ∂Y are disjoint and hence W is nested in either Y or Y^c. However, since $Y \in \mathcal{NE}(S)$, Y^c is not an essential subsurface and hence cannot contain W. So $W \subset Y$. Moreover, since $Y \in \mathcal{NE}(S)$ and $W \in \mathcal{SE}(S)$, the essential subsurface W is properly nested in Y. Then, pseudo-Anosov axes have uniformly bounded subsurface projections to all proper essential subsurfaces, [3, 48], thus completing the proof. □

Chapter 5

Thickness and Divergence of Teichmüller Spaces

In this chapter we focus our analysis on the surface $S_{2,1}$ which has previously proven to be difficult to understand, as is apparent from the surrounding literature. In particular, in this chapter we complete the thickness classification of Teichmüller space of all surfaces of finite type described in Section 2.1 and presented in Table 1. Specifically, we prove that the Teichmüller space of the surface $S_{2,1}$ is thick of order two and has superquadratic divergence, thereby answering questions of [5, 6, 21]. The proof in this chapter is broken up into three sections. In Section 5.1 we carefully analyze from a geometric viewpoint the construction in [21] where it is shown that $\mathcal{T}(S_{2,1})$ is thick of order at least one and at most two. Then, in Section 5.2 we prove that $\mathcal{T}(S_{2,1})$ cannot be thick of order one. In Section 5.3 using our understanding from the previous sections we prove that $\mathcal{T}(S_{2,1})$ can be uniquely characterized among all Teichmüller spaces as it has a divergence function which is superquadratic yet subexponential. Finally, we conclude with some open questions in Section 5.4. Throughout this section we will use the pants complex as a quasi-isometric model for Teichmüller space, often making statements and theorems about Teichmüller space with proofs obtained from considering the pants complex.

5.1 $\mathcal{T}(S_{2,1})$ is thick of order one or two

In this section we recall results of Behrstock in [3] and Brock-Masur in [21]. Specifically, we first recall a result of Behrstock that shows that for all surfaces $\mathcal{T}(S)$ is never wide. By definition, it follows that $\mathcal{T}(S)$ is never thick of order zero. Then, we record a slightly adapted version of a proof in [21] that $\mathcal{T}(S_{2,1})$ is thick of order at most two. Putting things together, this section implies that $\mathcal{T}(S_{2,1})$ is thick of order one or two. The reason for the necessary slight adaptation in this section of the proof in [21] is due to the various versions of thickness in the literature, as noted in Remark 2.1.16.

We begin by recalling the following theorem of Behrstock:

Theorem 5.1.1 ([3] Theorem 7.1)**.** *Let γ be any pseudo-Anosov axis in $\mathcal{P}(S)$, and let γ_ω be its ultralimit in any asymptotic cone $\mathcal{P}_\omega(S)$. Then any distinct points on γ_ω are separated by a cut-point.*

Since all mapping class groups of surfaces with positive complexity contain pseudo-Anosov elements, and given any pseudo-Anosov axis, one can choose an asymptotic cone in which its ultralimit is nontrivial, by Theorem 5.1.1 it follows that $\mathcal{T}(S)$ is never wide, and hence never thick of order zero.

Next, we consider the proof in [21] proving that $\mathcal{T}(S_{2,1})$ is thick of order at most two. Given $\alpha \in \mathcal{C}_{sep}(S_{2,0})$, let $\tilde{\alpha} \in \mathcal{C}_{sep}(S_{2,1})$ denote any lift of α with respect to the projection $\pi = \pi_{\mathcal{C}(S_{2,0})} \colon \mathcal{C}_{sep}(S_{2,1}) \to \mathcal{C}_{sep}(S_{2,0})$ which forgets about the boundary component. By topological considerations $S \setminus \tilde{\alpha} = Y_1 \sqcup Y_2 = S_{1,1} \sqcup S_{1,2}$. Since $diam(\mathcal{P}(Y_i)) = \infty$, we can choose bi-infinite geodesics $\rho_i \in \mathcal{P}(Y_i)$, and in fact, by Theorem 2.2.5, the span of any two such bi-infinite geodesics in the different connected components Y_1, Y_2 comprise a quasi-flat. In particular, it follows that the sets $\mathcal{Q}(\tilde{\alpha})$ are nontrivial product regions, and in particular are wide. Again, using Theorem 2.2.5, it is also immediate that subsets $\mathcal{Q}(\tilde{\alpha})$ are quasi-convex. Moreover, using the property of hierarchies in Theorem 2.2.4, it follows that these subsets $\mathcal{Q}(\tilde{\alpha})$ satisfy the non triviality property of every point having a bi-infinite quasi-geodesic through it. Hence, the subsets $\mathcal{Q}(\tilde{\alpha})$ are thick of order zero.

With the notation as above, set

$$\mathcal{X}(\alpha) = \{Q \in \mathcal{P}(S_{2,1}) \mid \alpha \in \pi(Q)\} = \bigcup_{\tilde{\alpha}} \mathcal{Q}(\tilde{\alpha}) \tag{5.1.1}$$

Presently we will prove the following theorem:

Theorem 5.1.2 ([21] Theorem 18). *$\mathcal{T}(S_{2,1})$ is thick of order at most two.*

To prove Theorem 5.1.2, Brock-Masur show that the subsets $\mathcal{X}(\alpha)$ are thick of order at most one, any two subsets $\mathcal{X}(\alpha), \mathcal{X}(\alpha')$ can be thickly chained together, and the union of uniform neighborhoods of all subsets $\mathcal{X}(\alpha)$ is all of $\mathcal{P}(S_{2,1})$. Each of these steps will be worked out.

1. $\mathcal{X}(\alpha)$ is thick of order one: For a given separating curve $\alpha \in \mathcal{C}_{sep}(S_{2,0})$, consider the set of all thick of order zero subsets $\mathcal{Q}(\tilde{\alpha})$, with $\pi(\tilde{\alpha}) = \alpha$. By definition, the union of all the thick of order zero subsets $\mathcal{Q}(\tilde{\alpha})$ is precisely all of $\mathcal{X}(\alpha)$. Furthermore, since by Lemma 3.2.2 the fiber of α under the projection map π is connected in $\mathcal{C}_{sep}(S_{2,1})$, in order to prove thick connectivity of elements in the set of all thick of order zero subsets $\mathcal{Q}(\tilde{\alpha})$, it suffices to notice that for $\tilde{\alpha}$ and $\tilde{\alpha'}$ disjoint separating curves, the quasi-convex product regions $\mathcal{Q}(\tilde{\alpha})$ and $\mathcal{Q}(\tilde{\alpha'})$ thickly intersect. However, this is immediate as $\mathcal{Q}(\tilde{\alpha}) \cap \mathcal{Q}(\tilde{\alpha'}) = \mathcal{Q}(\tilde{\alpha} \cup \tilde{\alpha'})$ is itself a natural quasi-convex nontrivial product regions and in particular has infinite diameter.

2. Subsets $\mathcal{X}(\alpha)$ and $\mathcal{X}(\alpha')$ can be thickly chained together: Given any separating curves $\alpha, \alpha' \in \mathcal{C}_{sep}(S_{2,0})$ there is a sequence of separating curves between them such that each separating curve intersects its neighboring curves in the sequence minimally. Specifically, there is a sequence of separating curves

$$\alpha = a_0, a_1, ..., a_n = \alpha'$$

with $|a_i \cap a_{i+1}| = 4$. Hence, we can assume that α, α' intersect four times. Up to homeomorphism there are only a finite number of such similar situations, one of which is presented in Figure 13.

As in Figure 13, we then have pants decompositions, $P_1 \in \mathcal{X}(\alpha), P_1' \in \mathcal{X}(\alpha')$ such that $d_{\mathcal{P}(S_{2,1})}(P_1, P_1') = D$, for some uniform constant D. Then for any (partial) pseudo-Anosov

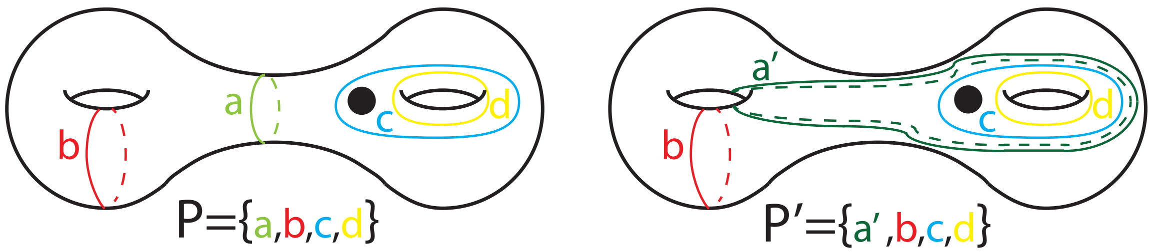

Figure 13: Pants decompositions with minimally intersecting separating curves that are distance two in $\mathcal{P}(S_{2,0})$. In fact, the curves $\{b,c,d\}$ are in common to both pants decompositions P, P' while the curves a, a' are distance two (they intersect four times) in the connected essential subsurface $S \setminus \{b,c,d\}$.

element $g \in Push \subset \mathcal{MCG}(S_{2,1})$, set $P_n = g^n P_1, \quad P'_n = g^n P'_1$. By Lemma 3.2.4, $\forall\ n \in \mathbb{Z}$ $P_n \in \mathcal{X}(\alpha), P'_n \in \mathcal{X}(\alpha')$, and moreover,

$$d_{\mathcal{P}(S_{2,1})}(P_n, P'_n) = d_{\mathcal{P}(S_{2,1})}(g^n P_1, g^n P'_1) = d_{\mathcal{P}(S_{2,1})}(P_1, P'_1) = D.$$

It follows that $diam(N_D(\mathcal{X}(\alpha)) \cap N_D(\mathcal{X}(\alpha'))) = \infty$ as it contains the axes of (partial) pseudo-Anosov elements.

3. $N_1(\bigcup_\alpha \mathcal{X}(\alpha)) = \mathcal{P}(S_{2,1})$: This follows immediately from the observation that any pair of pants in $\mathcal{P}(S_{2,1})$ is distance at most one from a pair of pants decomposition containing a separating curve. For further considerations regarding the net of the separating pants complex in the entire pants complex for a general surface of finite type, see [65] which is reproduced in Section 6.4.

In the course of proving that $\mathcal{P}(S_{2,1})$ is thick of order two we will consider ultralimits of the subsets $\mathcal{X}(\alpha)$ in the asymptotic cone. Specifically, for a given asymptotic cone $\mathcal{P}_\omega(S)$, and for $\overline{\alpha} \in \mathcal{C}^\omega_{sep}(S_{2,0})$, denote

$$\mathcal{X}_\omega(\overline{\alpha}) =: \{x_\omega \in \mathcal{P}_\omega(S) | x_\omega \text{has a representative } (x'_i) \text{ with } x'_i \in \mathcal{X}(\alpha_i)\ \omega\text{-a.s}\}. \qquad (5.1.2)$$

Unfortunately, the above argument for proving that $\mathcal{P}(S_{2,1})$ is thick of order at most two is using a version of thickness which is weaker than the version of thickness in Definition 2.1.15, and hence we must adapt their proof slightly. Specifically, recall that in our definition of thickness to show that a space is thick of order at most two it is required that the space have a collection of subsets that are quasi-convex, thick of order one, coarsely make up the entire space, and thickly intersect. However, notice that in the argument above from [21] proving that $\mathcal{P}(S_{2,1})$ is thick of order at most two we satisfied all the requirements with the exception of quasi-convexity. In fact, Example 5.1.3 suggests that the thick of order one subsets $\mathcal{X}(\alpha)$ may not be quasi-convex. Nonetheless, we will see that we can modify the above argument such that the conclusion that $\mathcal{P}(S_{2,1})$ is thick of order at most two remains true even with the stronger definition of thickness as in Definition 2.1.15. The idea will be to consider particular quasi-convex subsets of the sets $\mathcal{X}(\alpha)$.

Example 5.1.3. Fix $\alpha \in \mathcal{C}_{sep}(S_{2,0})$ and let β be another separating curve of $S_{2,0}$ which is arbitrarily far from α in $\mathcal{C}(S_{2,0})$. Then let $\hat{\alpha}, \hat{\beta}$ be any separating curves of $S_{2,1}$ which project to α, β under the map π which forgets about the boundary component, respectively. Furthermore, let $P \in \mathcal{P}(S_{2,1})$ be any pants decomposition containing $\hat{\alpha}$. Next, let f be a reducible point pushing mapping class which restricts to a partial pseudo-Anosov with support on one of the connected components of $S_{2,1} \setminus \hat{\beta}$. Then consider a hierarchy path between the pants decompositions P and $f^N P$. For large N, a large component domain of the hierarchy will be a connected component of $S \setminus \hat{\beta}$. In other words the hierarchy quasi-geodesic between P and $f^N P$ - with endpoints in the same thick of order zero subset $\mathcal{X}(\alpha)$ as f is point pushing, see Lemma 3.2.4 - will travel for an arbitrary long amount of time (at the expense of increasing N) in the region $\mathcal{X}(\beta)$. However, since α and β can be chosen to be arbitrarily far in $\mathcal{C}(S_{2,0})$, it follows that the subsets $\mathcal{X}(\alpha)$ and $\mathcal{X}(\beta)$ in $\mathcal{P}(S_{2,1})$ are arbitrarily far apart. In other words we have just shown that the quasi-geodesic hierarchy connecting two points in the same subset $\mathcal{X}(\alpha)$ travels through a subset $\mathcal{X}(\beta)$ where the subset $\mathcal{X}(\beta)$ is as far as desired from $\mathcal{X}(\alpha)$. To be sure, this does not necessarily imply that the subsets $\mathcal{X}(\alpha)$ are not quasi-convex as there may exist some other (non hierarchy) quasi-geodesic connecting P and $f^N P$ while remaining coarsely in $\mathcal{X}(\alpha)$.

Let $\tilde{\alpha} \in \mathcal{C}_{sep}(S_{2,1})$ with $\pi(\tilde{\alpha}) = \alpha \in \mathcal{C}_{sep}(S_{2,0})$, and let f be any point pushing pseudo-Anosov mapping class of $S_{2,1}$, such that $d_{\mathcal{C}_{sep}(S_{2,1})}(\tilde{\alpha}, f(\tilde{\alpha}))$ is less than some uniform bound.

Let $\rho = \rho(f, \tilde{\alpha}, Q)$ be any quasi-geodesic axis of f in the pants complex which goes through some point Q in $\mathcal{Q}(\tilde{\alpha})$. Then consider the set

$$\mathcal{X}(f, \tilde{\alpha}, Q) =: \rho \bigcup_n \mathcal{Q}(f^n(\tilde{\alpha})).$$

Intuitively, this set $\mathcal{X}(f, \tilde{\alpha}, Q)$ should be thought of as a point pushing pseudo-Anosov axis thickened up by product regions which it crosses through. Note that by construction the sets $\mathcal{X}(f, \tilde{\alpha}, Q)$ are coarsely contained in $\mathcal{X}(\alpha)$ and moreover, the Brock-Masur proof recorded earlier that $\mathcal{X}(\alpha)$ is thick of order one, (part (1) of the Brock-Masur proof) carries through to show that the subsets $\mathcal{X}(f, \tilde{\alpha}, Q)$ are similarly thick of order one in the induced metric from the pants complex. In fact, all of the thick of order zero subsets $\mathcal{Q}(\alpha)$ are either coarsely contained or coarsely disjoint from the set $\mathcal{X}(f, \tilde{\alpha}, Q)$. Given any thick of order zero subsets $f^j(\tilde{\alpha})$ and $f^k(\tilde{\alpha})$ which are contained in a set $\mathcal{X}(f, \tilde{\alpha}, Q)$, since f is a point pushing pseudo-Anosov map, by Lemma 3.2.4 it follows that the thick of order zero subsets $f^j(\tilde{\alpha})$ and $f^k(\tilde{\alpha})$ are contained in a common set $\mathcal{X}(\alpha)$. Precisely these types of thick of order zero subsets were shown to be possible to be thickly connected in part (1) of the Brock-Masur proof. In fact, by our assumption on f, there is a uniform bound on the number of thick of order zero quasi-convex product regions $\mathcal{Q}(C)$ traveled through in connecting any $f^j(\tilde{\alpha})$ and $f^{j+1}(\tilde{\alpha})$. Moreover, the following lemma shows that the subsets $\mathcal{X}(f, \tilde{\alpha}, Q)$ are quasi-convex.

Lemma 5.1.4. *The sets $\mathcal{X}(f, \tilde{\alpha}, Q)$ are quasi-convex.*

Proof. Pick any elements $A, B \in \mathcal{X}(f, \tilde{\alpha}, Q)$. We will see that they can be connected by a hierarchy quasi-geodesic $\sigma(A, B)$ that remains in a uniform neighborhood of $\mathcal{X}(f, \tilde{\alpha}, Q)$. Without loss of generality we can assume that A and B are contained in natural product regions $\mathcal{Q}(f^j(\tilde{\alpha})), \mathcal{Q}(f^k(\tilde{\alpha}))$, respectively. But then, remaining in the natural product regions $\mathcal{Q}(f^j(\tilde{\alpha})), \mathcal{Q}(f^k(\tilde{\alpha}))$, the points A, B can be connected to points $f^j(Q), f^k(Q)$, respectively, both of which lie on the pseudo-Anosov axis ρ.

Since pseudo-Anosov axes have uniformly bounded subsurface projections to all connected proper essential subsurfaces [3, 48], it follows that there is a hierarchy quasi-geodesic path connecting $f^j(Q)$ and $f^k(Q)$ in which the only component domain, for some sufficiently large threshold, is the entire surface S. Accordingly, in the hierarchy quasi-geodesic $\sigma(A, B)$ the only component domains, for some sufficiently large threshold, are the entire surface S and possibly connected essential subsurfaces Y with $Y \subset S \setminus f^j(\tilde{\alpha})$ or with $Y \subset S \setminus f^k(\tilde{\alpha})$. By definition, the portion of the σ traveling through the component domains of connected essential subsurfaces Y with $Y \subset S \setminus f^j(\tilde{\alpha})$ or with $Y \subset S \setminus f^k(\tilde{\alpha})$ is coarsely contained in the set $\mathcal{X}(f, \tilde{\alpha}, Q)$.

Furthermore, as a special case of Corollary 4.3.4, ρ the axis of a pseudo-Anosov element in the pants complex is Morse. In fact, the special case is actually already worked out in [3, 23]. It follows that any pants decompositions that the hierarchy path σ travels through along the component domain corresponding to the whole surface, or equivalently the main geodesic, are uniformly close to ρ. This completes the proof of the Theorem. □

Proof of Theorem 5.1.2. Let $\{P\}_\Gamma$ be the set consisting of all thick of order zero subsets $\mathcal{Q}(\gamma)$ for γ any separating curve in $\mathcal{C}_{sep}(S_{2,1})$ as well as all quasi-convex thick of order one subsets of the form $\mathcal{X}(f, \tilde{\gamma}, Q)$. It is immediate that the union of the sets is coarsely the entire space. In fact, this is true for just the union of the thick of order zero subsets in $\{P\}_\Gamma$. Finally, to complete our argument we will show that any two subsets $P_a, P_b \in \{P\}_\Gamma$ can be thickly chained together. Without loss of generality we can assume that P_a and P_b are thick of order zero subsets $\mathcal{Q}(\alpha)$, $\mathcal{Q}(\beta)$ for α, β in different connected components of $\mathcal{C}_{sep}(S_{2,1})$. But then we can construct a sequence of separating curves $\alpha = \gamma_1, ..., \gamma_n = \beta$ such that each of the consecutive curves are either disjoint or intersect minimally (four times), [61]. Hence, we can reduce the situation to showing that we can thickly connect $\mathcal{Q}(\alpha)$ and $\mathcal{Q}(\beta)$ where α, β are separating curves in different connected components of $\mathcal{C}_{sep}(S_{2,1})$ which intersect four times. Fix any thick of order one sets $P_c = \mathcal{X}(f, \alpha, \alpha \lrcorner \beta), P_d = \mathcal{X}(f, \beta, \beta \lrcorner \alpha)$. By construction, we have the following chain of thickly intersecting subsets: P_a, P_c, P_d, P_b. Note that the fact that P_c and P_d have infinite diameter coarse intersection was precisely what was in fact shown in part (2) of the Brock-Masur proof recorded earlier in this section. □

5.2 $\mathcal{T}(S_{2,1})$ is thick of order two

In this section we will use our characterization of the finest pieces in the asymptotic cone of the pants complex, Theorem 4.2.3, in order to prove that $\mathcal{T}(S_{2,1})$ cannot be thick of order one and hence by the conclusion of Section 5.1, must be thick of order exactly two.

Recall the definition of the sets $\mathcal{X}(\alpha), \mathcal{X}_\omega(\overline{\alpha})$ defined in Equations 5.1.1, 5.1.2, respectively. In the following lemma we prove that the ultralimit $\mathcal{X}_\omega(\overline{\alpha})$ is a closed set in the asymptotic cone.

Lemma 5.2.1. *For $\overline{\alpha} \in \mathcal{C}^\omega_{sep}(S_{2,0})$, $\mathcal{X}_\omega(\overline{\alpha}) \subset \mathcal{P}(S_{2,1})$ is a closed set.*

Proof. Consider the continuous projection $\pi_{\mathcal{P}_\omega(S_{2,0})} \colon \mathcal{P}_\omega(S_{2,1}) \to \mathcal{P}_\omega(S_{2,0})$ which takes a representative sequence (a_i) for a_ω and maps it to a representative sequence of $(\pi_{\mathcal{P}(S_{2,0})}(a_i))$ where the map $\pi_{\mathcal{P}(S_{2,0})} \colon \mathcal{P}(S_{2,1}) \to \mathcal{P}(S_{2,0})$ is the natural projection which forgets about the boundary component. Continuity of the projection map $\pi_{\mathcal{P}_\omega(S_{2,0})}$ follows from continuity of the 1-Lipschitz map $\pi_{\mathcal{P}(S_{2,0})}$. By definition $\left(\pi_{\mathcal{P}_\omega(S_{2,0})}\right)^{-1}(\mathcal{Q}_\omega(\overline{\alpha})) = \mathcal{X}_\omega(\overline{\alpha})$. By continuity, the result of the lemma follows from the fact that $\mathcal{Q}_\omega(\overline{\alpha}) \subset \mathcal{P}_\omega(S_{2,0})$ is closed. (In fact, $\mathcal{Q}_\omega(\overline{\alpha})$ is a finest piece in the tree-graded structure of $\mathcal{P}_\omega(S_{2,0})$.) □

Recall Lemma 3.0.21. In light of the notation developed in this section, as a special case we have the following corollary:

Corollary 5.2.2. *Assume $\overline{\alpha} \neq \overline{\beta} \in \mathcal{C}^\omega_{sep}(S_{2,1})$, and let $\mathcal{X}_\omega(\overline{\alpha}) = \bigcup \mathcal{Q}_\omega(\overline{\tilde{\alpha}})$ and $\mathcal{X}_\omega(\overline{\beta}) = \cup \mathcal{Q}_\omega(\overline{\tilde{\beta}})$. Then $|\mathcal{Q}_\omega(\overline{\tilde{\alpha}}) \cap \mathcal{Q}_\omega(\overline{\tilde{\beta}})| \leq 1$ and moreover, for $\overline{W}, \overline{V} \in \mathcal{E}^\omega(S)$ with $\overline{\partial W} = \overline{\tilde{\alpha}}$, $\overline{\partial V} = \overline{\tilde{\beta}}$ we have:*

$$\Phi_{\overline{W},x_\omega}(\mathcal{Q}_\omega(\overline{\tilde{\beta}})) = \{pt\}, \quad \Phi_{\overline{V},y_\omega}(\mathcal{Q}_\omega(\overline{\tilde{\alpha}})) = \{pt\},$$

where $\Phi_{\overline{W},x_\omega}$ is the projection defined in Equation 2.2.2.

The next theorem will be used to prove that the ultralimit of any thick of order zero subset Z in $\mathcal{P}(S_{2,1})$ must be contained entirely inside a particular single closed set of the form $\mathcal{X}_\omega(\overline{\alpha})$. Recall that by definition, a quasi-convex subspace Z is thick of order zero if (i) it is wide, namely in every asymptotic cone $P_\omega(S_{2,1})$, the subset corresponding to the ultralimit

$$Z_\omega =: \{x_\omega \in \mathcal{P}_\omega(S_{2,1}) | x_\omega \text{ has a representative sequence } (x'_i) \text{ with } x'_i \in Z \ \omega\text{-a.s}\}$$

has the property that any two distinct points in Z_ω are not separated by a cut-point, and moreover (ii) Z satisfies the nontriviality condition of every point being distance at most c from a bi-infinite quasi-geodesic in Z.

Theorem 5.2.3. *Let $(Z_i) \subset \mathcal{P}(S_{2,1})$ be any sequence of subsets, and let $\mathcal{P}_\omega(S_{2,1})$ be any asymptotic cone such that the ultralimit Z_ω does not have cut-points. Then $Z_\omega \subset \mathcal{X}_\omega(\overline{\alpha})$, for some $\overline{\alpha} \in \mathcal{C}^\omega_{sep}(S_{2,0})$. Moreover, if in any asymptotic cone $\mathcal{P}_\omega(S_{2,1})$, the ultralimit Z_ω contains at least two points, then there exists a unique such $\overline{\alpha}$ satisfying the following condition: in any neighborhoods of $a_\omega \neq b_\omega \in Z_\omega$ there are points a'_ω, b'_ω with $d_{\mathbb{S}_\omega(S_{2,1})}(a'_\omega, b'_\omega)$ bounded, and such that each of the natural quasi-convex product regions $\mathcal{Q}_\omega(\overline{C}) \in \mathcal{P}_\omega(S)$ in a finite $\mathbb{S}_\omega(S_{2,1})$ chain from a'_ω to b'_ω are entirely contained in $\mathcal{X}_\omega(\overline{\alpha})$.*

Before proving Theorem 5.2.3 we first prove the following lemma.

Lemma 5.2.4. *Let $(Z_i) \subset \mathcal{P}(S_{2,1})$ be any sequence of subsets, and let $\mathcal{P}_\omega(S_{2,1})$ be any asymptotic cone such that the ultralimit Z_ω is nontrivial and does not have cut-points. Then $\forall a_\omega \neq b_\omega \in Z_\omega$, it follows that $a_\omega, b_\omega \subset \mathcal{X}_\omega(\overline{\alpha})$, for some $\overline{\alpha} \in \mathcal{C}^\omega_{sep}(S_{2,0})$. In fact, $\overline{\alpha}$ can be uniquely identified by the following condition: in any neighborhoods of $a_\omega \neq b_\omega \in Z_\omega$ there are points a'_ω, b'_ω with $d_{\mathbb{S}_\omega(S_{2,1})}(a'_\omega, b'_\omega)$ bounded, and such that each of the natural quasi-convex product regions $\mathcal{Q}_\omega(\overline{C}) \in \mathcal{P}_\omega(S)$ in a finite $\mathbb{S}_\omega(S_{2,1})$ chain from a'_ω to b'_ω are entirely contained in $\mathcal{X}_\omega(\overline{\alpha})$.*

Remark 5.2.5. Alternatively, as in the proof of Theorem 3.3.2 the unique characterization of the element $\overline{\alpha} \in \mathcal{C}^\omega_{sep}(S_{2,0})$ in Theorem 5.2.3 and Lemma 5.2.4 can be described as follows: in any neighborhoods of $a_\omega \neq b_\omega \in Z_\omega$ there are points a'_ω, b'_ω with representatives $(a'_i), (b'_i)$ with $\lim_\omega d_{\mathcal{C}_{sep}(S_{2,1})}(a'_i, b'_i)$ bounded, and such that ω–a.s. a finite $\mathcal{C}_{sep}(S_{2,1})$ geodesic between (a'_i) and (b'_i) is contained in the connected components of $\mathcal{C}_{sep}(S_{2,1})$ corresponding to $\overline{\alpha}$.

Proof. Since Z_ω does not have any cut points, by Theorem 4.2.3 and Remark 4.2.4, in any neighborhoods of a_ω, b_ω there exist points a'_ω, b'_ω with $d_{\mathbb{S}_\omega(S_{2,1})}(a'_\omega, b'_\omega)$ bounded. That is, there is a finite chain of convex nontrivial product regions $\mathcal{Q}_\omega(\overline{\widetilde{\alpha}_1})$,...,$\mathcal{Q}_\omega(\overline{\widetilde{\alpha}_K})$ such that $a'_\omega \in \mathcal{Q}_\omega(\overline{\widetilde{\alpha}_1})$,

$b'_\omega \in \mathcal{Q}_\omega(\overline{\tilde{\alpha}_K})$, and $|\mathcal{Q}_\omega(\overline{\tilde{\alpha}_j}) \cap \mathcal{Q}_\omega(\overline{\tilde{\alpha}_{j+1}})| \geq 2$. As suggested by the notation, for all $j \in \{1, ..., K\}$, $\pi_{\mathcal{C}^\omega(S_{2,0})}(\overline{\tilde{\alpha}_j}) = \overline{\alpha}$ for some fixed $\overline{\alpha} \in \mathcal{C}^\omega(S_{2,0})$ where the projection

$$\pi_{\mathcal{C}^\omega(S_{2,0})} \colon \mathcal{C}^\omega(S_{2,1}) \to \mathcal{C}^\omega(S_{2,0})$$

is the extension to the ultrapower of the natural projection map which forgets about the boundary component. In particular, all the natural convex product regions $\mathcal{Q}_\omega(\overline{\tilde{\alpha}_j})$ in the chain connecting a'_ω, b'_ω are contained in the set $\mathcal{X}_\omega(\overline{\alpha})$.

Since by Lemma 5.2.1 the sets $\mathcal{X}_\omega(\overline{\alpha})$ are closed, in order to complete the proof of the lemma it suffices to show that for all a'_ω, b'_ω in small enough neighborhoods of a_ω, b_ω, respectively, such that $d_{\mathbb{S}_\omega(S_{2,1})}(a'_\omega, b'_\omega)$ is bounded, we have that a'_ω and b'_ω are all always contained in the same set $\mathcal{X}_\omega(\overline{\alpha})$ as above. Assume not, that is, assume that in any neighborhoods of a_ω, b_ω there exist points a^1_ω, b^1_ω and a^2_ω, b^2_ω such that $d_{\mathbb{S}_\omega(S_{2,1})}(a^1_\omega, b^1_\omega) < \infty$ and $d_{\mathbb{S}_\omega(S_{2,1})}(a^2_\omega, b^2_\omega) < \infty$, yet $a^1_\omega, b^1_\omega \in \mathcal{X}_\omega(\overline{\alpha})$ while $a^2_\omega, b^2_\omega \in \mathcal{X}_\omega(\overline{\beta})$ where $\overline{\alpha} \neq \overline{\beta}$. In particular, we can assume that a^1_ω, b^1_ω lie in an r-neighborhood of a_ω and a^2_ω, b^2_ω lie in an r-neighborhood of b_ω where $r \geq 0$ is a constant such that open r-neighborhoods of a_ω, b_ω are path connected. In addition, we can assume that $2r < d_{\mathcal{P}_\omega(S)}(a_\omega, b_\omega)$. See Figure 14 for an illustration of this.

Let $\mathcal{Q}_\omega(\overline{\tilde{\alpha}_1})$,...,$\mathcal{Q}_\omega(\overline{\tilde{\alpha}_m})$ be a finite chain of convex nontrivial product regions in $\mathcal{X}_\omega(\overline{\alpha})$ connecting a^1_ω and b^1_ω. Moreover, as in Theorem 4.2.3 there is a quasi-geodesic path ρ^1_ω, the ultralimit of hierarchy paths, through the product regions connecting a^1_ω and b^1_ω. Similarly, let $\mathcal{Q}_\omega(\overline{\tilde{\beta}_1})$,...,$\mathcal{Q}_\omega(\overline{\tilde{\beta}_n})$ be a finite chain of convex nontrivial product regions in $\mathcal{X}_\omega(\overline{\beta})$ connecting a^2_ω and b^2_ω, and let ρ^2_ω be a quasi-geodesic path through the product regions connecting a^2_ω and b^2_ω. By omitting product regions as necessary and using properties of hierarchies in Theorem 2.2.4 we can assume that initial product region $\mathcal{Q}_\omega(\overline{\tilde{\alpha}_1})$ of the path ρ^1_ω has the property that ρ^1_ω exits the product region $\mathcal{Q}_\omega(\overline{\tilde{\alpha}_1})$ once at a point $e_\omega \neq a^1_\omega$. By Lemma 2.2.13, there is some $\overline{W} \in \mathcal{SE}^\omega(S)$ which is ω–a.s. a connected component of $\overline{S \setminus \tilde{\alpha}_1}$, such that $\pi_{\mathcal{P}_\omega(\overline{W})}(a^1_\omega) \neq \pi_{\mathcal{P}_\omega(\overline{W})}(e_\omega) \in F_{\overline{W},a^1_\omega}$.

By our assumptions, a^1_ω and a^2_ω are connected by a path that remains entirely inside an r-neighborhood of a_ω. Let $[a^1_\omega, a^2_\omega]$ denote such a path. Similarly, let $[b^1_\omega, b^2_\omega]$ denote a path between the points b^1_ω and b^2_ω. We can assume that $(a^1_\omega, a^2_\omega]$ and $(b^1_\omega, b^2_\omega]$ are contained in $\mathcal{P}_\omega(S_{2,1}) \setminus$

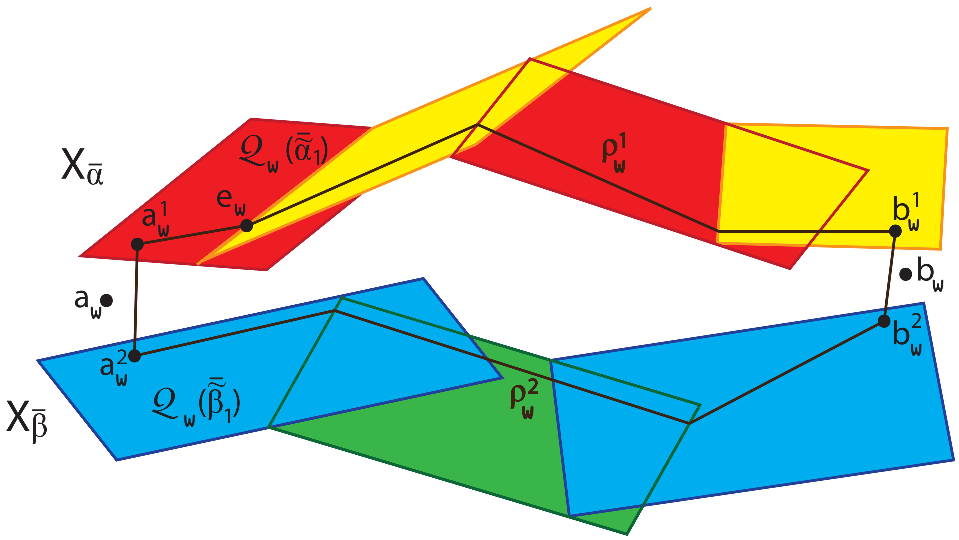

Figure 14: In neighborhoods of a_ω, b_ω there exist points a^1_ω, b^1_ω and a^2_ω, b^2_ω, respectively, such that $d_{\mathbb{S}_\omega(S)}(a^1_\omega, b^1_\omega) < \infty$, $d_{\mathbb{S}_\omega(S)}(a^2_\omega, b^2_\omega) < \infty$, yet $a^1_\omega, b^1_\omega \in \mathcal{X}_\omega(\overline{\alpha})$ while $a^2_\omega, b^2_\omega \in \mathcal{X}_\omega(\overline{\beta})$ where $\overline{\alpha} \neq \overline{\beta}$. This situation cannot occur in $\mathcal{P}_\omega(S_{2,1})$.

$\mathcal{Q}_\omega(\overline{\tilde{\alpha}_1})$. If not, we can replace a^1_ω and/or b^1_ω with points closer to a^2_ω and/or b^2_ω respectively such that this is the case.

Consider the closed pentagon P with vertices $\{a^1_\omega, e_\omega, b^1_\omega, b^2_\omega, a^2_\omega\}$ and edges

$$\rho^1_\omega|_{[a^1_\omega, e_\omega]}, \rho^1_\omega|_{[e_\omega, b^1_\omega]}, [b^1_\omega, b^2_\omega], \rho^2_\omega, [a^1_\omega, a^2_\omega]$$

It should be noted that some sides of the pentagon may be trivial, although this does not affect the argument. Applying the continuous projection $\Phi_{\overline{W}, x_\omega}$ of Theorem 2.2.12 to the pentagon P, we have $\Phi_{\overline{W}, x_\omega}(e_\omega) = \Phi_{\overline{W}, x_\omega}(b^1_\omega) = \Phi_{\overline{W}, x_\omega}(b^2_\omega)$. Similarly, $\Phi_{\overline{W}, x_\omega}(a^1_\omega) = \Phi_{\overline{W}, x_\omega}(a^2_\omega)$ as by construction the edges $\rho^1_\omega|_{[e_\omega, b^1_\omega]}, [b^1_\omega, b^2_\omega]$ and $[a^1_\omega, a^2_\omega]$ are contained in $\mathcal{P}_\omega(S_{2,1}) \setminus P_{\overline{W}, x_\omega}$. Furthermore, by Corollary 5.2.2 and continuity of the projection, $\Phi_{\overline{W}, x_\omega}(\rho^2_\omega)$ is a single point and is in fact equal to $\Phi_{\overline{W}, x_\omega}(a^2_\omega) = \Phi_{\overline{W}, x_\omega}(b^2_\omega)$. Putting things together we have

$$\Phi_{\overline{W}, x_\omega}(e_\omega) = \Phi_{\overline{W}, x_\omega}(b^1_\omega) = \Phi_{\overline{W}, x_\omega}(b^2_\omega) = \Phi_{\overline{W}, x_\omega}(a^2_\omega) = \Phi_{\overline{W}, x_\omega}(a^1_\omega)$$

However, this is a contradiction to our assumption that $\Phi_{\overline{W}, x_\omega}(a^1_\omega) \neq \Phi_{\overline{W}, x_\omega}(e_\omega)$, thus completing the proof. □

Using the proof of Lemma 5.2.4, presently we prove Theorem 5.2.3.

Proof of Theorem 5.2.3. By Lemma 5.2.4 and its proof, we know that given any two distinct points $a_\omega, b_\omega \in Z_\omega$, the points a_ω, b_ω are contained in a common subset $\mathcal{X}_\omega(\overline{\alpha})$ where $\overline{\alpha} \in \mathcal{C}^\omega_{sep}(S_{2,0})$ is such that in any neighborhoods of $a_\omega \neq b_\omega \in Z_\omega$ there are points a'_ω, b'_ω with $d_{\mathbb{S}_\omega(S_{2,1})}(a'_\omega, b'_\omega)$ bounded, and such that each of the natural quasi-convex product regions $\mathcal{Q}_\omega(\overline{C})$ in a finite $\mathbb{S}_\omega(S_{2,1})$ chain from a'_ω to b'_ω are entirely contained in $\mathcal{X}_\omega(\overline{\alpha})$.

Let $c_\omega \in Z_\omega$ be any third point in Z_ω, (possibly the same as a_ω or b_ω). Similarly, it follows that the points a_ω, c_ω (b_ω, c_ω) are contained in a common subset $\mathcal{X}_\omega(\overline{\beta})$ $(\mathcal{X}_\omega(\overline{\gamma}))$ where $\overline{\beta}$ $(\overline{\gamma})$ is an element of $\mathcal{C}^\omega_{sep}(S_{2,0})$ such that in any neighborhoods of a_ω and c_ω (b_ω and c_ω) there are points a'_ω, c'_ω (b'_ω, c'_ω) with $d_{\mathbb{S}_\omega(S_{2,1})}(a'_\omega, c'_\omega)$ bounded ($d_{\mathbb{S}_\omega(S_{2,1})}(b'_\omega, c'_\omega)$ bounded), and such that each of the natural quasi-convex product regions $\mathcal{Q}_\omega(\overline{C}) \in \mathcal{P}_\omega(S)$ in a finite $\mathbb{S}_\omega(S_{2,1})$ chain from a'_ω to c'_ω (b'_ω to c'_ω) are entirely contained in $\overline{\beta}$ $(\overline{\gamma})$. But then, considering the triangle between the points $a'_\omega, b'_\omega, c'_\omega$ and using the same projection arguments in Lemma 5.2.4 to generalize the contradiction argument with the pentagon, it follows that $\overline{\alpha} = \overline{\beta} = \overline{\gamma}$. Notice if c_ω is the same as a_ω or b_ω, the proof is identical to the proof in Lemma 5.2.4.

Since c_ω is arbitrary, it follows that $Z_\omega \subset \mathcal{X}_\omega(\overline{\alpha})$ where $\overline{\alpha}$ is uniquely determined by the property described in the statement of the theorem. □

As a corollary of the proof of Lemma 5.2.4, we have the following corollary:

Corollary 5.2.6. *Let $(Z_i), (Z'_i) \subset \mathcal{P}(S_{2,1})$ be any sequences subsets, and let $\mathcal{P}_\omega(S_{2,1})$ be an asymptotic cone such that $Z_\omega, Z'_\omega \subset \mathcal{P}_\omega(S_{2,1})$ each one contains at least two points, and each one has no cut-points. As in Theorem 5.2.3 assume that $Z_\omega \subset \mathcal{X}_\omega(\overline{\alpha})$ and $Z'_\omega \subset \mathcal{X}_\omega(\overline{\beta})$ for some $\overline{\alpha}, \overline{\beta} \in \mathcal{C}^\omega_{sep}(S_{2,0})$, such that ω–a.s. $\alpha_i \neq \beta_i$, then:*

$$|Z_\omega \cap Z'_\omega| \leq 1.$$

In particular, if the asymptotic cone $\mathcal{P}_\omega(S_{2,1})$ has a constant base point, and the sequences of subsets $(Z_i) = \overline{Z}$ and $(Z'_i) = \overline{Z'}$ are constant and quasi-convex, then the subsets Z and Z' have bounded coarse intersection.

Proof. We will show $|Z_\omega \cap Z'_\omega| \leq 1$ by contradiction. That is, assume $a_\omega \neq b_\omega \in (Z_\omega \cap Z'_\omega)$. By Theorem 4.2.3, in any neighborhoods of a_ω, b_ω there exist points a^1_ω, b^1_ω and a^2_ω, b^2_ω, such that $d_{\mathbb{S}_\omega(S_{2,1})}(a^1_\omega, b^1_\omega) < \infty$ and $d_{\mathbb{S}_\omega(S_{2,1})}(a^2_\omega, b^2_\omega) < \infty$, yet $a^1_\omega, b^1_\omega \in \mathcal{X}_\omega(\overline{\alpha})$ while $a^2_\omega, b^2_\omega \in \mathcal{X}_\omega(\overline{\beta})$ where $\overline{\alpha} \neq \overline{\beta}$. Precisely this situation was shown to be impossible in the proof of Lemma 5.2.4.

Next, consider the special case of the first part of the Corollary in which the asymptotic cone $\mathcal{P}_\omega(S_{2,1})$ has a constant base point, and the sequences of subsets $(Z_i) = \overline{Z}$ and $(Z'_i) = \overline{Z'}$ are constant and quasi-convex. Then the coarse intersection $\overline{Z}\hat{\cap}\overline{Z'}$ is the constant quasi-convex, and hence connected, sequence of subsets $\overline{Z\hat{\cap}Z'}$. Since our asymptotic cone has a constant base point, assuming $Z\hat{\cap}Z'$ is nontrivial (if not then we are done), its ultralimit $\overline{Z\hat{\cap}Z'} = (Z\hat{\cap}Z')_\omega$ in the asymptotic cone is similarly nontrivial. That is, in the asymptotic cone $(Z\hat{\cap}Z')_\omega$ contains at least - and hence by the first part exactly- one point, namely the point in the cone with constant representative sequence. It follows that the diameter of the connected coarse intersection $Z\hat{\cap}Z'$ is sublinear in s_i. On the other hand, since the diameter of the coarse intersection $Z\hat{\cap}Z'$, is not only sublinear but also constant, it follows that Z and Z' have bounded coarse intersection. □

Using Theorem 5.2.3 and Corollary 5.2.6, we are now prepared to prove the following highlight of the thesis.

Theorem 5.2.7. *$\mathcal{T}(S_{2,1})$ is thick of order two.*

Proof. Since thickness is a quasi-isometry invariant property, [6], it suffices to prove that $\mathcal{P}(S_{2,1})$ is thick of order two. In Section 5.1 we showed that $\mathcal{P}(S_{2,1})$ is thick of order at most two and at least one. Hence, it suffices to show that $\mathcal{P}(S_{2,1})$ is not thick of order one. In fact, we will show that any thick of order one subset is entirely contained inside a *nontrivially proper subset* of the entire pants complex (that is, a subset which has infinite Hausdorff from the entire pants complex).

Fix an asymptotic cone $\mathcal{P}_\omega(S_{2,1})$ with a constant base point and scaling sequence s_i. Note that since $\mathcal{P}(S_{2,1})$ is connected, for any $q \in \mathcal{P}(S_{2,1})$, the constant sequence $\overline{q}$ all represent the same base point of the asymptotic cone $\mathcal{P}_\omega(S_{2,1})$.

Let Z be any thick of order zero subset in $\mathcal{P}(S_{2,1})$. By hypothesis, Z coarsely contains a bi-infinite quasi-geodesic through any point. Fix some point $z \in Z$, and some quasi-geodesic ray

γ beginning near z and remaining in Z. Then for every s_i, set $y_i = \gamma(s_i) \in Z$. By construction, in the asymptotic cone the sequences $\overline{z}$ and (y_i) represent distinct points contained in $Z_\omega \subset \mathcal{P}_\omega(S_{2,1})$. In particular, we have just shown that every thick of order zero subset $Z \subset \mathcal{P}(S_{2,1})$ has ultralimit Z_ω containing at least two distinct points in the asymptotic cone $\mathcal{P}(S_{2,1})$. By Theorem 5.2.3 it follows that every thick of order zero subset Z in $\mathcal{P}(S_{2,1})$ can be assigned a unique element $\overline{\alpha} \in \mathcal{C}^\omega_{sep}(S_{2,0})$. Moreover, Corollary 5.2.6 implies that a necessary condition for any two thick of order zero subsets Z, Z' to be thickly chained together, as in condition (ii) of 2.1.15, is that the two thick of order zero subsets Z, Z' are assigned the same element $\overline{\alpha} \in \mathcal{C}^\omega_{sep}(S_{2,1})$.

It follows that any thick of order one subset Y of the space $\mathcal{P}(S_{2,1})$ can consist of at most the union of thick of order zero subsets with the same labels $\overline{\alpha} \in \mathcal{C}^\omega_{sep}(S_{2,0})$. Hence, the ultralimit Y_ω in the asymptotic cone $\mathcal{P}_\omega(S_{2,1})$ is entirely contained inside the subset $\mathcal{X}_\omega(\overline{\alpha})$ which we will see is a proper subset of $\mathcal{P}_\omega(S_{2,1})$. The proof of the Theorem then follows from the observation that if a subset $Y \subset X$ has finite Hausdorff distance from X, then in any asymptotic cone the ultralimit $Y_\omega = X_\omega$.

To see that $\mathcal{X}_\omega(\overline{\alpha})$ is a proper subset of $\mathcal{P}_\omega(S_{2,1})$, notice that under under the surjective projection $\pi\colon\ \mathcal{P}_\omega(S_{2,1}) \twoheadrightarrow \mathcal{P}_\omega(S_{2,0})$, the subset $\mathcal{X}_\omega(\overline{\alpha})$ is mapped into the natural quasi-convex product region $\mathcal{Q}_\omega(\overline{\alpha})$, a proper subset of $\mathcal{P}_\omega(S_{2,0})$. □

Remark 5.2.8. Theorem 5.2.7 completes the thickness classification of the pants complexes of all surfaces of finite type as described in Section 2.1. Moreover, among all surfaces of finite type of equal or higher complexity, $S_{2,1}$ is the only surface such that its pants complex is not thick of order one, thus making its pants complex particularly rich.

5.3 $\mathcal{T}(S_{2,1})$ has superquadratic divergence

Informally the *divergence* of a metric space, a notion introduced by Gromov, is a measure of inefficiency of detours paths. More specifically, divergence quantifies the cost of going from a point x to a point y in a (typically one-ended geodesic) metric space X while avoiding a metric ball based at a point z. Throughout the literature there are a couple of closely related definitions of divergence that emerge based on stipulations regarding the points x, y, z. See [29] for a comparison of various definitions and criterion for when the different definitions agree. We will consider the following definition of divergence which is a lower bound on all other definitions of divergence in the literature. In particular, it follows that the novel result in this section regarding the superquadratic divergence of $\mathcal{T}(S_{2,1})$ remains true for any definition of divergence.

Definition 5.3.1 (Divergence)**.** Let γ be a coarsely arc length parameterized bi-infinite quasi-geodesic in a one-ended geodesic metric space. Then the *divergence along* γ, denoted $div(\gamma, \epsilon)$ is defined to be the growth rate of the function

$$d_{X \setminus B_{\epsilon r}(\gamma(0))}(\gamma(-r), \gamma(r))$$

with respect to r where the scalar $\epsilon > 0$ is chosen so that $\gamma(\pm r) \notin B_{\epsilon r}(\gamma(0))$. As divergence is independent of the choice of a small ϵ, we will often omit ϵ from the notation. Divergence can be similarly associated to a sequence of quasi-geodesic segments γ_i. The *divergence of* X denoted $div(X)$ is defined to be $\max_{\gamma,\epsilon} div(\gamma, \epsilon)$.

Example 5.3.2. In $\mathbb{R}^n$ for $n \geq 2$, it is an elementary fact that divergence is linear. On the other hand as we will demonstrate presently, $div(\mathbb{H}^n)$ is exponential for $n \geq 2$. Due to homogeneity, it suffices to consider any standard geodesic in $\mathbb{H}^2$ and show that this geodesic has exponential divergence. In particular, consider the unit disk model of $\mathbb{H}^2$ and let γ be the equatorial geodesic with $y = 0$. Give γ an arc length parameterization by setting $\gamma(0) = (0, 0)$, and more generally $\gamma(r) = (0, \frac{e^r-1}{e^r+1})$. Then, using the fact that the element of hyperbolic arc length in the disk model is $\frac{2|dx|}{1-|x|^2}$, and considering the parameterized semicircle, $\sigma(\theta) = (\frac{e^r-1}{e^r+1}\cos(\theta), \frac{e^r-1}{e^r+1}\sin(\theta))$ with $\theta \in [0, \pi]$, explicit computation shows that

$$d_{\mathbb{H}^2 \setminus B_r(0,0)}(\gamma(-r), \gamma(r)) = \int_0^\pi \frac{2\frac{e^r-1}{e^r+1}}{1 - (\frac{e^r-1}{e^r+1})^2} d\theta = \pi \frac{e^{2r}-1}{2e^r}$$

In particular, the growth rate of the length of a detour path is an exponential function in r, thus showing that $\mathbb{H}^2$ has exponential divergence.

There is a relationship between the divergence of a metric space and the existence of cut-points in the asymptotic cone of a metric space. Specifically, we have the following straightforward lemma.

Lemma 5.3.3 ([29] Lemma 3.15)**.** *Let X be a geodesic metric space, X_ω any asymptotic cone, and assume $a_\omega \neq b_\omega \in X_\omega$ have representative sequences $(a_i), (b_i)$, respectively. Then, the following are equivalent:*

1. *X_ω has a global cut-point separating a_ω and b_ω.*
2. *ω–a.s. the sequence of geodesics $[a_i, b_i]$ has superlinear divergence.*

The plan for the rest of the section is to show that $\mathcal{T}(S_{2,1})$ has at least superquadratic and at most cubic divergence. First we prove the lower bound, and then see that the upper bound follows from Theorem 5.2.7 in conjunction with results in [5].

5.3.1 $\mathcal{T}(S_{2,1})$ has at least superquadratic divergence

Recall Theorem 6.2.5 which characterizes contracting quasi-geodesics in CAT(0) spaces. The proof of Theorem 6.2.5 appears in Section 6.2. Presently, we will provide a standard argument for the following small ingredient of the theorem as it serves as motivation for ideas in this section.

Lemma 5.3.4. *A (b,c)–contracting quasi-geodesic γ in a geodesic metric space X has at least quadratic divergence.*

Proof. To streamline the exposition we will assume γ is a strongly contracting geodesic, although the same argument carries through for γ a (b,c)–contracting quasi-geodesic. Recall that by Definition 4.3.2 since γ is strongly contracting geodesic there exists a constant c such that $\forall x, y \in X$ if $d(x, y) < d(x, \gamma)$ then $d(\pi_\gamma(x), \pi_\gamma(y)) < c$, where the map $\pi_\gamma \colon X \to 2^\gamma$ is a nearest point projection. To prove the lemma we will consider an arbitrary detour path α_r connecting $\gamma(-r)$ and $\gamma(r)$ while avoiding the metric ball $B_r(\gamma(0))$, and show that the length of α_r is at least a quadratic function in r.

Presently we will discretize the detour path in terms of nearest point projections on to the subgeodesic $[\gamma(-r/2), \gamma(r/2)]$. Specifically, for each

$$j \in \{-\lfloor \frac{r}{2c} \rfloor, ..., -1, 0, 1, ..., \lfloor \frac{r}{2c} \rfloor\},$$

fix $z_r^{jc} \in \alpha_r$ such that $z_r^{jc} \in \pi_\gamma^{-1}(\gamma(jc))$. Notice that by construction $d(z_r^{jc}, \gamma) \geq \frac{r}{2}$. Furthermore, since $d(\pi_\gamma(z_r^{jc}), \pi_\gamma(z_r^{(j+1)c})) = c$, by the strongly contracting property it follows that

$$d(z_r^{jc}, z_r^{(j+1)c}) \geq d(z_r^{jc}, \gamma) \geq \frac{r}{2}.$$

Putting things together, the following inequality gives the desired lower bound on the length of the detour path α_r :

$$|\alpha_r| \geq \sum_{j=1}^{\lfloor \frac{r}{c} \rfloor} d(z_r^{(j-1)c}, z_r^{jc}) \geq \sum_{j=1}^{\lfloor \frac{r}{c} \rfloor} \frac{r}{2} \geq \frac{r^2}{2c} - 1.$$

Since c is a uniform constant, the statement of the lemma follows. □

The following lemma is closely related to ideas in [13] regarding the thinness of polygons with edges along a contracting contracting geodesic.

Lemma 5.3.5. *Using the notation from Lemma 5.3.4, let $\sigma = \sigma_r^{jc}$ be the concatenated path*

$$[z_r^{jc}, \gamma(jc)] \cup [\gamma(jc), \gamma((j+1)c)] \cup [\gamma((j+1)c), z_r^{(j+1)c}],$$

then σ is a (2,c)-quasi-geodesic.

Proof. Let x, y be any points along σ. If $x, y \in [z_r^{jc}, \gamma(jc)]$, then it is immediate that

$$d(x, y) = d_\sigma(x, y),$$

where $d_\sigma(x, y)$ represents the distance along σ from x to y. In particular, for any points $x, y \in [z_r^{jc}, \gamma(jc)]$, the $(2, c)$ quasi-isometric inequality is trivially satisfied. Similarly, the same conclusion holds for $x, y \in [z_r^{(j+1)c}, \gamma((j+1)c)]$ or $x, y \in [\gamma(jc), \gamma((j+1)c)]$. Moreover, since $|[\gamma(jc), \gamma((j+1)c)]| = c$, for the cases $x \in [z_r^{jc}, \gamma(jc)] \cup [\gamma((j+1)c), z_r^{(j+1)c}]$ and $y \in [\gamma(jc), \gamma((j+1)c)]$ (or vice versa) the $(2, c)$ quasi-isometric inequality is similarly satisfied.

Hence, we can assume $x \in [z_r^{jc}, \gamma(jc)]$ and $y \in [\gamma((j+1)c), z_r^{(j+1)c}]$. Since x and y have nearest point projections onto γ which are distance c apart, by (1,c)-contraction of γ we have:

$$\max\{d(x,\gamma(jc)), d(y,\gamma((j+1)c))\} = D \leq d(x,y).$$

Specifically, since $d(\gamma(jc),\gamma((j+1)c)) = d(\gamma((j+1)c),\gamma(jc)) = c$ the definition of (1,c)-contraction (Definition 4.3.2) implies that:

$$d(x,y) \geq d(x,\gamma(jc)) \text{ and similarly } d(y,x) \geq d(y,\gamma((j+1)c)).$$

But then, we have the following inequality completing the proof:

$$d(x,y) \leq d_\sigma(x,y) = d(x,\gamma(jc)) + c + d(y,\gamma((j+1)c)) \leq 2D + c \leq 2d(x,y) + c.$$

□

Remark 5.3.6. Note that in the special case of γ a strongly contracting quasi-geodesic in Lemma 5.3.5 we showed that the piecewise geodesic paths σ_r^{jc} are (2,c)-quasi-geodesics. More generally, for γ a (b,c)-contracting quasi-geodesic it is not hard to see that the piecewise geodesic paths σ_r^{jc} are $(\frac{2}{b}, \frac{c}{b})$-quasi-geodesics. In particular, all the quasi-geodesics σ_r^{jc} have uniformly bounded quasi-isometry constants.

We will now aim toward proving the following main theorem of this subsection.

Theorem 5.3.7. $\mathcal{T}(S_{2,1})$ *has at least superquadratic divergence.*

Recall in the proof of Lemma 5.3.4 we showed a contracting quasi-geodesic has at least quadratic divergence by showing that in order for a detour path to have more than a uniformly bounded "shadow" (i.e. nearest point projection set) onto γ the detour path must travel at least a linear distance. In other words, the at least quadratic divergence was a consequence of the fact that the detour path had to travel a linear amount of at least linear distances. To prove Theorem 5.3.7 we will construct a quasi-geodesic such that a detour path must travel a linear amount of at least superlinear distances.

More specifically, recall the sequence of quasi-geodesic segments $\{\sigma_r^{jc}\}_r$ which coincide with γ along the segment $[\gamma(jc), \gamma((j+1)c)]$ in the proof of Lemma 5.3.4. By definition, the portion of the detour path α_r which connects the endpoints of σ_r^{jc} cannot fellow travel with σ_r^{jc}. In fact, by construction α_r lies outside of the ball $N_{r/2}([\gamma(jc), \gamma((j+1)c)])$. In particular, in order to prove that γ has at least superquadratic divergence, we will show that the sequence of quasi-geodesic segments $\{\sigma_r^{jc}\}_r$ for almost all j have superlinear divergence.

To be sure, showing that a detour path must travel a linear amount of superlinear distances without controlling the degree of superlinearity does not ensure superquadratic divergence. Specifically, consider the following example.

Example 5.3.8. Let α_r be a sequence of paths with each α_r partitioned into r subsegments $\{\tau_{r,j}\}_{j=1}^r$ such that for any fixed j, the length of the sequence of segments $\{\tau_{r,j}\}_r$ is superlinear in r. Then,

$$\begin{aligned} |\alpha_r| &\geq \sum_{j=1}^r |\tau_{r,j}| \\ &= \sum_{j=1}^r r\epsilon_j(r) \quad \text{where for any fixed } j,\ \lim_r \epsilon_j(r) \to \infty \\ &\geq r^2 \min_{j=1}^r(\epsilon_j(r)). \end{aligned}$$

Taking the limit, it does not necessarily follow that $|\alpha_r|$ is superquadratic in r. For example, if we define the functions ϵ_j as follows,

$$\begin{aligned} \epsilon_j(r) &= 1 \ \text{ if } r \leq j \\ &= r \ \text{ otherwise.} \end{aligned}$$

Notice that $\min_{j=1}^r(\epsilon_j(r)) = 1$ and by our approach above, it follows that $|\alpha_r|$ can only be bounded below by r^2.

Nonetheless, the potential problem highlighted in Example 5.3.8 will be avoided by using the periodicity of γ in conjunction with a contradiction argument. Specifically, presently we will prove a lemma which provides a sufficient criterion for proving superquadratic divergence. Before stating the lemma, we fix some notation.

As in the proof of Lemma 5.3.4 let γ be a contracting quasi-geodesic, let α_r be a sequence of detour paths avoiding balls $B_r(\gamma(0))$, and let z_r^{jc} denote fixed points on α_r which have nearest point projections to $\gamma(jc)$. Then, for all $jc \in \mathbb{Z}c$ we obtain sequences of points $\overline{z^{jc}} = \{z_r^{jc}\}_{r=2c|j|}^{\infty}$, and similarly sequences of quasi-geodesic paths $\overline{\sigma^{jc}} = \{\sigma_r^{jc}\}_{r=2c|j|}^{\infty}$. Let τ_r^{jc} denote the restriction of the quasi-geodesics σ_r^{jc} to the intersection $\sigma_r^{jc} \cap B_{r/2}(\gamma(jc))$, and by abuse of notation refer to the endpoints of τ_r^{jc} by z_r^{jc} and $z_r^{(j+1)c}$. In fact, by even further abuse of notation, let $\{z_r^{jc}\}_r$ represent any sequence of points of distance $r/2$ from γ such that the nearest point projection of z_r^{jc} onto γ is $\gamma(jc)$, and similarly, let τ_r^{jc} denote the quasi-geodesic between consecutive points z_r^{jc} and $z_r^{(j+1)c}$, given by the concatenation:

$$\tau_r^{jc} =: [z_r^{jc}, \gamma(jc)] \cup [\gamma(jc), \gamma((j+1)c)] \cup [\gamma((j+1)c), z_r^{(j+1)c}].$$

Lemma 5.3.9. *With the notation from above, assume in addition that γ is a periodic quasi-geodesic such that for all fixed j, the sequence of quasi-geodesic segments $\{\tau_r^{jc}\}_r$ has divergence which is superlinear in r, the natural numbers. Then, γ has superquadratic divergence. Similarly, the same conclusion holds if γ is a periodic quasi-geodesic such that there is a constant C such that for any fixed j, and any consecutive sequence of sequences of quasi-geodesic segments*

$$\{\tau_r^{jc}\}_r, \{\tau_r^{(j+1)c}\}_r, ..., \{\tau_r^{(j+C)c}\}_r$$

with each one beginning from the terminal point of the previous one, at least one of the sequences of quasi-geodesic segments $\{\tau_r^{(j+m)c}\}_r$ in the list has divergence which is superlinear in r.

Proof. To simplify the exposition we will prove the first case, although the proof of the similar statement follows almost identically. Fix a sequence of detour paths α_r and corresponding quasi-geodesics τ_r^{jc}. By assumption, for any fixed j the divergence of the sequence τ_r^{jc} is superlinear, say $r\epsilon_j(r)$ where $\lim_r \epsilon_j(r) \to \infty$. We will prove the lemma by contradiction. That is, assume there is a constant N such that $\lim_r |\alpha_r| < r^2 N$. Since γ is contracting, as in Lemma 5.3.4 we have:

$$|\alpha_r| \geq \sum_{j=1}^{\lfloor \frac{r}{c} \rfloor} d_{X \setminus B_r(\gamma(0))}(z_r^{(j-1)c}, z_r^{jc}) \gtrsim \sum_{j=1}^{\lfloor \frac{r}{c} \rfloor} r\epsilon_j(r) \geq \frac{r^2}{c} \min_{j=1}^{r}(\epsilon_j(r))$$

Putting things together, it follows that

$$\lim_r \min_{j=1}^r (\epsilon_j(r))$$

is uniformly bounded. (In the situation of Example 5.3.8 the uniform bound was one). Set $\min_{j=1}^r(\epsilon_j(r)) = \epsilon_{j_{min}}(r)$. Then for all values of $r \in \mathbb{N}$ we can use the periodicity of γ to translate the points $z_r^{j_{min}c}$ to points z_r^0, and correspondingly the quasi-geodesics $\tau_r^{j_{min}c}$ to quasi-geodesics τ_r^0. After translation, we have a sequence of quasi-geodesic segments $\{\tau_r^0\}_r$ with linear divergence. This is a contradiction to the hypotheses of the theorem and hence completes the proof by contradiction. □

Next, consider the following Lemma of [57], which we will use in the construction of a quasi-geodesic with superquadratic divergence in $\mathcal{P}(S_{2,1})$:

Lemma 5.3.10 ([57] Theorem 2.1)**.** *For any surface $S_{g,n}$ there exists an isometric embedding $i : \mathcal{C}(S_{g,n}) \to \mathcal{C}(S_{g,n+1})$ such that $\pi \circ i$ is the identity map, where $\pi\colon\ \mathcal{C}(S_{g,n+1}) \to \mathcal{C}(S_{g,n})$ is given by forgetting about the puncture.*

Fix $\bar{\alpha}_0 \in \mathcal{C}_{sep}(S_{2,0})$, and let $\bar{f}$ be a pseudo-Anosov axis in $\mathcal{C}(S_{2,0})$ through the curve $\bar{\alpha}_0$. Furthermore, assume that $|\bar{f}(\bar{\alpha}_0) \cap \bar{\alpha}_0| = 4$. See Figure 15 for an example.

Denote the separating curve $\bar{f}^i(\bar{\alpha}_0)$ by $\bar{\alpha}_i$ for all $i \in \mathbb{Z}$. Since $\forall i \neq j$, $\bar{\alpha}_i, \bar{\alpha}_j$ are in different separating curves of $\mathcal{C}_{sep}(S_{2,0})$, by topological considerations it follows that $\bar{\alpha}_i \lrcorner \bar{\alpha}_j$ can be coarsely identified with a pants decomposition of $S_{2,0}$. In particular, for all $i \in \mathbb{Z}$, let $\bar{P}_i$ denote a fixed pants decomposition of the form $\bar{\alpha}_i \lrcorner \bar{\alpha}_{i+1}$. Let $\bar{\gamma}_n$ denote a piecewise geodesic path in the pants complex traveling through the pairs of pants $\bar{P}_{-n}, ..., \bar{P}_0, ..., \bar{P}_n$. Moreover, let $\bar{\gamma}$ denote the limit of the paths $\bar{\gamma}_n$. Note that we can assume $\bar{f}^i(\bar{P}_j) = \bar{P}_{i+j}$, and hence $\bar{f}$ acts by translations on the path $\bar{\gamma}$. It follows that $\bar{\gamma} \in \mathcal{P}(S_{2,0})$ is a contracting quasi-geodesic as it is the axis of a pseudo-Anosov mapping class and has bounded combinatorics, [3, 23]. Moreover, it is clear that by construction in every asymptotic cone $P_\omega(S_{2,0})$ any two distinct points on $\bar{\gamma}_\omega$ are not contained in the ultralimit of a natural product region of the form $\mathcal{Q}_\omega(\overline{\alpha})$ for any $\bar{\alpha} \in \mathcal{C}^\omega_{sep}(S_{2,0})$. In particular, by Theorem 4.2.3 for the special case of $S_{2,0}$, it follows that any two points on $\bar{\gamma}_\omega$ are separated by a cut-point.

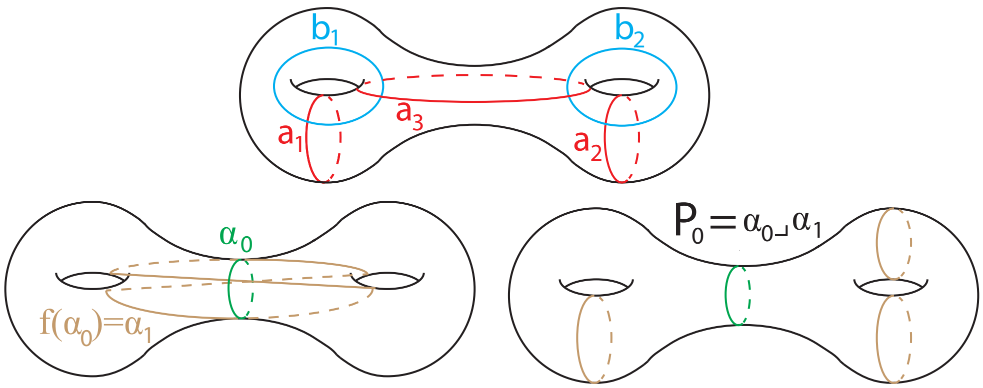

Figure 15: $\bar{f} = T_{a_3}T_{b_2}^{-1}T_{b_1}^{-1}T_{a_2}T_{a_1}$ is a non-point pushing pseudo-Anosov mapping class. Note that in the lower left figure $|\bar{f}(\bar{\alpha}_0) \cap \bar{\alpha}_0| = 4$. Moreover, in the lower right figure note that since $\bar{\alpha}_0$ and $\bar{f}(\bar{\alpha}_0)$ are different separating curves, by topological considerations $\bar{\alpha}_0 \lrcorner \bar{f}(\bar{\alpha}_0)$ is a pants decomposition.

Then, using the isometric embedding $i\colon\ \mathcal{C}(S_{2,0}) \to \mathcal{C}(S_{2,1})$ of Lemma 5.3.10, we can lift all the aforementioned structure from $S_{2,0}$ to $S_{2,0}$. Specifically, we can lift the separating curves $\bar{\alpha}_i$ to separating curves $\alpha_i \in \mathcal{C}_{sep}(S_{2,1})$, the pants decompositions $\bar{P}_i$ to pants decompositions $P_i \in \mathcal{P}(S_{2,1})$, and the periodic quasi-geodesic $\bar{\gamma}$ with bounded combinatorics to a periodic geodesic $\gamma \subset \mathcal{P}(S_{2,1})$ which also has bounded combinatorics as it too is the axis of a pseudo-Anosov map f which is a lift of $\bar{f}$. Then, by construction it follows that in every asymptotic cone $P_\omega(S_{2,1})$ any two distinct points on γ_ω are not contained in the ultralimit of a common subset of the form $\mathcal{Q}_\omega(\overline{\alpha})$ for $\bar{\alpha}$ for any $\bar{\alpha} \in \mathcal{C}^{\omega}_{sep}(S_{2,0})$. Moreover, considering Corollary 4.2.7 it follows that any region of the form $\mathcal{Q}_\omega(\overline{\alpha})$ has a unique nearest point on $\bar{\gamma}_\omega$ whose removal separates the region $\mathcal{Q}_\omega(\overline{\alpha})$ from the two resulting components of $\bar{\gamma}_\omega$.

Presently we will prove Theorem 5.3.7 by showing that this periodic and contracting quasi-geodesic $\gamma \subset \mathcal{P}(S_{2,1})$ has superquadratic divergence.

Proof of Theorem 5.3.7. In light of Lemma 5.3.9 in order to prove the theorem it suffices to show that the above constructed periodic and contracting quasi-geodesic $\gamma \subset \mathcal{P}(S_{2,1})$ satisfies the hypothesis of Lemma 5.3.9. Assume γ does not satisfy the hypothesis of Lemma 5.3.9. Specifically, for any positive integer k there exists some consecutive sequence of sequences of

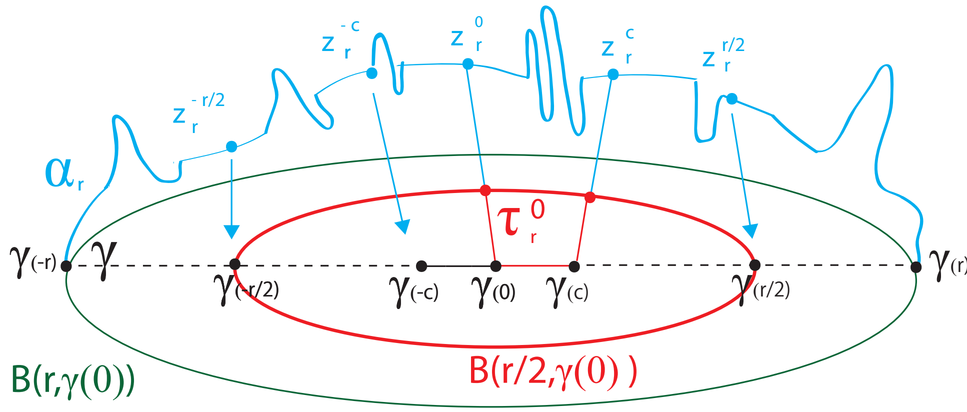

Figure 16: The detour path α_r connects γ_{-r} and γ_r and avoids the metric ball $B(r, \gamma(0))$. The points $z_r^{jc} \in \alpha_r$ project to $\gamma(jc)$ under the nearest point projection onto γ. By Lemma 5.3.5, $\tau_r^{jc} = [z_r^{jc}, \gamma(jc)] \cup [\gamma(jc), \gamma((j+1)c)] \cup [\gamma((j+1)c), z_r^{(j+1)c}]$ is a (2,c)-quasi-geodesic. Furthermore, by our assumptions on γ, the sequence of quasi-geodesics $\{\tau_r^j\}_r$ almost always has superlinear divergence.

quasi-geodesic segments

$$\{\tau_r^{j_k c}\}_r, \{\tau_r^{(j_k+1)c}\}_r, .., \{\tau_r^{(j_k+k)c}\}_r$$

each one beginning from the terminal point of the previous one, such that for each fixed $m \in \{0, ..., k\}$, the sequence of quasi-geodesic segments $\{\tau_r^{(j_k+m)c}\}_r$ in the list has divergence linear in r, the natural numbers.

Since the sequence of geodesics $[z_r^{(j_k+m)c}, \gamma((j_k + m)c)]$ are contained as subsegments of $\tau_r^{(j_k+m)c}$ with roughly half the total length, and because $\tau_r^{(j_k+m)c}$ have linear divergence, it follows that $[z_r^{(j_k+m)c}, \gamma((j_k + m)c)]$ also have linear divergence. By Lemma 5.3.3, in the asymptotic cone $Cone_\omega(\mathcal{P}(S_{2,1}), \gamma(j_k c), (rc))$ the ultralimit of $[z_r^{(j_k+m)c}, \gamma((j_k + m)c)]$ is non-trivial and does not have any cut-points. By Theorem 5.2.3 it follows that the ultralimit of the form $[z_r^{(j_k+m)c}, \gamma((j_k + m)c)]$ is completely contained in subset of the form $\mathcal{X}_\omega(\overline{\alpha})$ for a unique $\overline{\alpha}$ an element of $\mathcal{C}_{sep}(S_{2,0})^\omega$.

Considering the sequence of geodesic quadrilaterals with vertices given by

$$\{z_r^{(j_k+m)c}, z_r^{(j_k+m+1)c}, \gamma((j_k + m + 1)c), \gamma((j_k + m)c)\}.$$

The sequence of edges $[\gamma((j_k+m)c), \gamma((j_k+m+1)c)]$ have bounded (constant) length. On the other hand, the three remaining sequence of edges all have lengths growing linearly in r and have linear divergence. As in Theorem 5.2.3 it follows that in same asymptotic cone

$$Cone_\omega(\mathcal{P}(S_{2,1}), \gamma(j_k c), (rc)),$$

the ultralimits of the sequences of quadrilaterals and in particular the edges of them

$$[z_r^{(j_k+m)c}, \gamma((j_k+m)c)], [z_r^{(j_k+m+1)c}, \gamma((j_k+m+1)c)]$$

are completely contained in a common subset of the form $\mathcal{X}_\omega(\overline{\alpha})$. Repeating this argument and using the fact that adjacent pairs of ultralimits

$$\begin{aligned} &[z_r^{(j_k+m)c}, \gamma((j_k+m)c)], [z_r^{(j_k+m+1)c}, \gamma((j_k+m+1)c)] \\ \text{and} \quad &[z_r^{(j_k+m+1)c}, \gamma((j_k+m+1)c)], [z_r^{(j_k+m+2)c}, \gamma((j_k+m+2)c)] \end{aligned}$$

have nontrivial intersection in the asymptotic cone, by Corollary 5.2.6 it follows that the consecutive string of sequences

$$[z_r^{j_k c}, \gamma(j_k c)], ..., [z_r^{(j_k+k)c}, \gamma((j_k+k)c)]$$

have ultralimit in the asymptotic cone $Cone_\omega(\mathcal{P}(S_{2,1}), \gamma(j_k c), (rc))$, completely contained in a common subset of the form $\mathcal{X}_\omega(\overline{\alpha})$.

Now consider the asymptotic cone $Cone_\omega(\mathcal{P}(S_{2,1}), \gamma(j_{3r}), (rc))$. In particular, consider the distinct points in the asymptotic cone with representative sequences $\{z_{rc}^{j_{3r}c}\}_r$and$\{z_{rc}^{(j_{3r}+3r)c}\}_r$. We have seen that these points in the cone have representative sequences that identify them as being contained in a common subset of the form $\mathcal{X}_\omega(\overline{\alpha})$. Furthermore, the points

$$\{z_{rc}^{(j_{3r}c)}\}_r, \text{ and } \{z_{rc}^{((j_{3r}+3r)c)}\}_r$$

are of distance at most (in fact exactly) one from the distinct points with representatives

$$\{\gamma(j_{3r}c)\}_r, \text{ and } \{\gamma((j_{3r}+3r)c)\}_r$$

on the ultralimit γ_ω, respectively. Projecting this situation from $S_{2,1}$ to $S_{2,0}$, we obtain points $\{\bar{z}_{rc}^{(j_{3r}c)}\}_r, \{\bar{z}_{rc}^{((j_{3r}+3r)c)}\}_r$ which are of distance at most one (the projection is Lipschitz) from the distinct points with representatives $\{\bar{\gamma}(j_{3r}c)\}_r, \{\bar{\gamma}((j_{3r}+3r)c)\}_r$ on the ultralimit $\bar{\gamma}_\omega$, respectively. On the other hand, by assumption the points $\{\bar{z}_{rc}^{(j_{3r}c)}\}_r, \{\bar{z}_{rc}^{((j_{3r}+3r)c)}\}_r$ are in a common subset of the form $\mathcal{Q}_\omega(\overline{\alpha})$. It follows that there is a path ρ_ω connecting the points $\{\bar{\gamma}(j_{3r}c)\}_r, \{\bar{\gamma}((j_{3r}+3r)c)\}_r$ which travels for distance at most two (namely $\{[\bar{\gamma}(j_{3r}c), \bar{z}_{rc}^{(j_{3r}c)}]\}_r$ and $\{[\bar{\gamma}((j_{3r}+3r)c), \bar{z}_{rc}^{(j_{3r}+3r)c}]\}_r$ each of which has length at most one) outside of the region $\mathcal{Q}_\omega(\overline{\alpha})$. However, since in the asymptotic cone $Cone_\omega(\mathcal{P}(S_{2,1}), \gamma(j_{3r}c), (rc))$, the points $\{\bar{\gamma}(j_{3r}c)\}_r$, and $\{\bar{\gamma}((j_{3r}+3r)c)\}_r$ are distance three apart and because any region of the form $\mathcal{Q}_\omega(\overline{\alpha})$ has a unique nearest point on $\bar{\gamma}_\omega$ whose removal separates the region $\mathcal{Q}_\omega(\overline{\alpha})$ from the two resulting components of $\bar{\gamma}_\omega$, this is a contradiction, thus completing the proof. □

5.3.2 $\mathcal{T}(S_{2,1})$ has at most cubic divergence

In addition to the relationship between divergence and cut-points in the asymptotic cone as in Lemma 5.3.3, there is a strong relationship between the divergence of a metric space and its thickness. Preliminarily, as a consequence of Lemma 5.3.3 it follows that a geodesic metric space is thick of order zero if and only if the divergence of the space is linear. More generally, considering the inductive nature of the definition of degree of thickness of a space, a natural conjecture is that the polynomial order of divergence of a sufficiently nice metric space - such as the pants complex - is equal one plus the degree of thickness of the space, [5]. Presently we record a theorem providing partial progress toward this conjecture.

Theorem 5.3.11 ([5] Corollary 4.17)**.** *Let X be a geodesic metric space which is thick of order n, then the divergence of along any geodesic in X is at most polynomial of order $n+1$. In particular, by Theorem 5.2.7 it follows that $\mathcal{T}(S_{2,1})$ has at most cubic divergence.*

Note that in light of Theorem 5.3.7, Theorem 5.3.11 provides an alternative proof of the fact that $\mathcal{T}(S_{2,1})$ is thick of order at least two, as proven in Theorem 5.2.7.

5.3.3 Divergence of Teichmüller spaces

Just as the proof of Theorem 5.2.7 uniquely characterizes $\mathcal{T}(S_{2,1})$ among all Teichmüller spaces and completes the thickness classification of Teichmüller spaces, so too Theorems 5.3.7 and 5.3.11 also uniquely characterize $\mathcal{T}(S_{2,1})$ among all Teichmüller spaces and (almost) complete the divergence classification of all Teichmüller spaces. See Table 2.

Notice that the Teichmüller spaces of low complexity surfaces that are either hyperbolic or relatively hyperbolic, perforce have at least exponential divergence. It is immediate by observation that for complexity one surfaces the pants complex, or equivalently the Farey graph, has infinitely many ends. On the other hand, it follows from recent work of [33, 58] that for complexity at least two surfaces, the pants complex is one ended and hence the divergence is in fact exponential. Specifically, building off of work of Gabai in [33]. Rafi-Schleimer in Proposition 4.1 of [58] show that the curve complex is one ended for complexity at least two surfaces. In particular, it follows that the same result holds for the corresponding pants complexes.

5.4 An approach toward cubic divergence

In this section we present an approach toward proving that $\mathcal{T}(S_{2,1})$ has cubic divergence. In particular, considering the proof of the previous section that $\mathcal{T}(S_{2,1})$ has superquadratic divergence, in order to prove cubic divergence, our approach will be to consider the quasi-geodesics $\{\sigma_r^{jc}\}_r$ and show that they not only have superlinear divergence, but in fact have quadratic divergence. With this goal in mind, we will we wish to prove particular sequences of quasi-geodesics have at least quadratic divergence.

Recalling the quasi-distance formula of Theorem 2.2.5, presently we will consider the divergence of various types of sequences of quasi-geodesics in terms of the component domains through which the hierarchy paths travel. Recall the set $\mathcal{NE}(S_{2,1})$ consists of all connected nonseparating essential subsurfaces of $S_{2,1}$. In fact for $S_{2,1}$ all nonseparating essential subsurfaces are connected. On the other hand for a fixed curve $\alpha \in \mathcal{C}_{sep}(S_{2,0})$ set $\mathcal{S}_\alpha =: \{W \in \mathcal{SE}(S_{2,1}) | \pi_{\mathcal{C}(S_{2,0})}(\partial W) = \alpha\}$. That is, $\mathcal{S}_\alpha \subset \mathcal{SE}(S_{2,1})$ consists of all connected *α-type es-*

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
7	quadratic	quadratic	quadratic	quadratic	$\ldots$
6	exponential	quadratic	quadratic	quadratic	$\ldots$
5	exponential	quadratic	quadratic	quadratic	$\ldots$
4	infinite	quadratic	quadratic	quadratic	$\ldots$
3		exponential	quadratic	quadratic	$\ldots$
2		exponential	quadratic	quadratic	$\ldots$
1		infinite	superquadratic yet at most cubic	quadratic	$\ldots$
0			exponential	quadratic	$\ldots$
$n \uparrow \;\; g \rightarrow$	0	1	2	3	$\ldots$

Table 2: Divergence of Teichmüller spaces for all surfaces of finite type, a summary.

sential subsurfaces or connected essential subsurfaces W which have quasi-convex product regions $\mathcal{Q}(\partial W) \subset \mathcal{X}(\alpha)$.

In light of Example 5.1.3, we will use the following refinement of our subset of connected α-type essential subsurfaces, $\mathcal{S}_\alpha$. Given any fixed separating curve $\alpha' \in \mathcal{C}_{sep}(S_{2,1})$, let $\pi_{\mathcal{C}(S_{2,0})}(\alpha') = \alpha$. Then for any $z \in \mathcal{P}(S)$ and constant $M \geq M'$, define

$$\begin{aligned} \mathcal{S}_\alpha^{M,z} \;\; = \;\; & \{W \subset \mathcal{S}_\alpha | \text{ there exists a hierarchy path } \rho \text{ from } \mathcal{Q}(\partial W) \text{ to } z \\ & \text{such that all at least M-component domains of } \rho \text{ are in } \mathcal{S}_\alpha.\} \end{aligned}$$

By Theorem 2.2.4 for sufficiently large values of M we can equivalently define the sets $\mathcal{S}_\alpha^{M,z}$ to consist of all connected essential subsurfaces $W \subset \mathcal{S}_\alpha$ such that $\forall Y \notin \mathcal{S}_\alpha$, the projection $d_{\mathcal{C}(Y)}(\partial W, z) < M$. Note that since $Y \notin \mathcal{S}_\alpha$, by topological considerations, the subsurface projection $\pi_{\mathcal{C}(Y)}(\partial W)$ is well-defined. In light of this reformulation it can been seen that for large enough values of M, $\mathcal{S}_\alpha^{M,z}$ is not empty. Specifically, for $W \in \mathcal{S}_\alpha$, the intersection number $|\partial W \cap \alpha'|$ gives an upper bound on the subsurface projection distances between α and α' into any connected essential subsurface $Y \notin \mathcal{S}_\alpha$. Additionally note that by definition if $N \geq M$, then $\mathcal{S}_\alpha^{M,z} \subset \mathcal{S}_\alpha^{N,z}$.

Next, we will prove a couple of lemmas to be used later in the section. We begin with a lemma describing the relationship between connected essential subsurfaces in $\mathcal{S}_\alpha^{M,z}$ and $\mathcal{S}_\alpha^{2M,z}$.

Lemma 5.4.1. *Assume $W \in \mathcal{S}_\alpha^{M,z}$ and $Y \notin \mathcal{S}_\alpha^{2M,z}$ then either $W \pitchfork Y$ or $W \subsetneq Y$.*

Proof. The only other options for the relationship between W, Y are $W \cap Y = \emptyset$ or $Y \subseteq W$. In both cases, it follows that $\partial W, \partial Y$ are disjoint separating multicurves. We will see that using this condition in conjunction with the assumption of the lemma that $W \in \mathcal{S}_\alpha^{M,z}$, implies that $Y \in \mathcal{S}_\alpha^{2M,z}$ which is a contradiction. This will complete the proof.

First, if W is an α-type connected essential subsurface, and ∂W and ∂Y are disjoint, then Y must also be an α-type essential subsurface. We can assume Y is connected; if not we can replace it by a connected component. Then similarly an elementary application of Theorem 2.2.2 to the geodesic segment $\partial W, \partial Y$ it is clear that for any essential subsurface $Z \notin \mathcal{S}_\alpha$ (which in particular implies Z intersects both ∂W and ∂Y), we have $d_{\mathcal{C}(Z)}(\partial W, \partial Y) < K$. Then since by definition there is a hierarchy path connecting $\mathcal{Q}(W)$ to z whose component domains are all α-type connected essential subsurfaces, the same condition holds for $\mathcal{Q}(Y)$ at the expense of possibly increasing the constant M to $M + K \leq 2M$. $\square$

Next we prove the following *generalized contraction property*:

Lemma 5.4.2 (generalized contraction property)**.** *There is a constant C such that for all $x, y \in \mathcal{P}(S_{2,1})$ with*

$$\sum_{Y \in \mathcal{S}_\alpha^{M,z}} \{d_{\mathcal{C}(Y)}(x, y)\} > C,$$

then

$$\sum_{Y \notin \mathcal{S}_\alpha^{2M,z}} \{d_{\mathcal{C}(Y)}([x, y], z)\} \text{ is uniformly bounded.}$$

where $[x, y]$ is a hierarchy in $\mathcal{P}(S_{2,1})$ between x and y.

In other words, Lemma 5.4.2 says that if x, y have sufficiently far apart projections into the subsurface projections in the set of connected essential subsurfaces $\mathcal{S}_{\alpha}^{M,z}$ then a quasi-geodesic connecting them goes close to z in every subsurface projection that is not in the set of connected essential subsurfaces $\mathcal{S}_{\alpha}^{2M,z}$.

Proof. Since by assumption $\sum_{Y \in \mathcal{S}_{\alpha}^{M,z}} \{d_{\mathcal{C}(Y)}(x, y)\} > C$, it follows that there is a connected essential subsurface $W \in \mathcal{S}_{\alpha}^{M,z}$ such that $d_{\mathcal{C}(W)}(x, y) > M$. Fix any $Y \notin \mathcal{S}_{\alpha}^{2M,z}$. By Lemma 5.4.1, either $W \pitchfork Y$ or $W \subsetneq Y$. Using Theorem 2.2.2 and Lemma 2.2.3 it follows that $d_{\mathcal{C}(Y)}([x, y], \partial W)$ is uniformly bounded. Hence, if $d_{\mathcal{C}(Y)}(\partial W, z)$ is uniformly bounded, by the triangle inequality we are done. Accordingly without loss of generality we can assume $d_{\mathcal{C}(Y)}(\partial W, z) > M$. Since $W \in \mathcal{S}_{\alpha}^{M,z}$, and the hierarchy ρ from $\mathcal{Q}(\partial W)$ to z contains Y as an M-component domain, by definition it follows that $Y \in \mathcal{S}_{\alpha}$. On the other hand, using the properties of hierarchies from Theorem 2.2.4 in conjunction with the fact that all M-component domains of ρ are in $\mathcal{S}_{\alpha}$, it follows that for any $Z \notin \mathcal{S}_{\alpha}$, the diameter $diam_{\mathcal{C}(Z)}(\rho) < M$. Putting things together, it follows that $Y \in \mathcal{S}_{\alpha}^{2M,z}$, which contradicting our hypotheses. □

We are now prepared to analyze the divergence of particular hierarchy paths in terms of the types of component domains traveled through. Let $\{\gamma_n\}_n$ be a fixed sequence of hierarchy quasi-geodesics between the points $P_i, Q_i \in \mathcal{P}(S_{2,1})$ with lengths increasing as a linear function in n. As a first case, assume a nontrivial ratio of the distance occurs in nonseparating component domains. Specifically, assume the fraction

$$\frac{\sum_{Y \in \mathcal{NE}(S_{2,1})} \{d_{\mathcal{C}(Y)}(P_n, Q_n)\}}{\sum_{Y \subset S} \{d_{\mathcal{C}(Y)}(P_n, Q_n)\}}$$

is a linear function in n then the sequence of hierarchy paths $\{\gamma_n\}_n$ have at least quadratic divergence. The reason for this is that by Theorem 4.3.3 the sequence of quasi-geodesics satisfy strong contraction for a nontrivial ratio of their total length, and hence the argument used in Lemma 5.3.4 applies to the nontrivial portion of the total length.

Since we wish to prove that $\{\gamma_n\}_n$ have at least quadratic divergence, without loss of generality we can assume that roughly all of the distance in the sequence of quasi-geodesics $\{\gamma_n\}_n$

occurs in separating component domains. Then for the quasi-geodesics $\{\gamma_n\}_n$, consider the possibilities for the relationships between the separating component domains. It is possible that all the separating component domains are all coarsely contained in a common subset of the form $\mathcal{X}_\omega(\overline{\alpha})$, or alternatively, the ultralimit γ_ω goes through a structurally integral corner in the asymptotic cone. In the former case, we have little control over the divergence, and in fact the divergence can easily seen to be linear, quadratic, or possibly in between. On the other hand, regarding the latter case, which in particular we are assured of for our sequence of quasi-geodesic segments $\{\sigma_n^{jc}\}_n$, we conjecture that the quasi-geodesic must have at least quadratic divergence. In particular, the generalized contraction lemma, Lemma 5.4.2, provides direction toward proving this. We close with a couple of additional related open questions.

Question 5.4.3 ([5] Question 4.21). Let X be a coarsely homogenous CAT(0) space. If X is thick of order n, does it follow that X has the divergence of polynomial of degree $n + 1$? Equivalently, by Theorem 5.3.11 is the divergence is at least polynomial of degree $n + 1$.

In particular, by Theorem 5.2.7 an affirmative answer to Question 5.4.3 guarantees cubic divergence of $\mathcal{T}(S_{2,1})$. Another more general conjecture which may guarantee cubic divergence of $\mathcal{T}(S_{2,1})$ is the following:

Question 5.4.4 ([5] Question 1.5). Let X be a coarsely homogenous CAT(0) space. Can X have a divergence which is strictly superpolynomial of degree n but strictly subpolynomial of degree $n + 1$?

Chapter 6

Odds and Ends

This chapter contains four independent and self contained although related results. In Section 6.1 we compare the notions of wideness and unconstrictedness in the CAT(0) setting. In the three remaining sections we present adapted versions of the following papers, [63, 64, 65]. In Section 6.2 we prove the equivalence of various hyperbolic type properties for quasi-geodesics in CAT(0) spaces. As a corollary, we provide a converse to the usual Morse stability lemma in the CAT(0) setting. In addition, as a warmup we include an alternative proof of the fact, originally proven in Behrstock-Druţu [5], that in CAT(0) spaces Morse quasi-geodesics have at least quadratic divergence. In Section 6.3 using some nice properties of Farey graphs we prove that the separating curve complex $\mathbb{S}(S_{2,0})$ is δ-hyperbolic, answering a question in [61]. More specifically, we prove the following quasi-distance formula for $\mathbb{S}(S_{2,0})$ which is similar to as well as motivated by quasi-distance formulas for $\mathcal{P}(S)$ in Theorem 2.2.5. Finally, in Section 6.4 we study the topological types of pants decompositions of a surface by associating to any pants decomposition P, its *pants decomposition graph*, $\Gamma(P)$. This perspective provides a convenient way to analyze the maximum distance in the pants complex of any pants decomposition to a pants decomposition containing a nontrivial separating curve for all surfaces of finite type. We provide an asymptotically sharp approximation of this nontrivial distance in terms of the topology of the surface. In particular, for closed surfaces of genus g we show the maximum distance in the pants complex of any pants decomposition to a pants decomposition containing a separating curve grows asymptotically like the function $\log(g)$.

6.1 Wide versus unconstricted in CAT(0) spaces.

Recall the definitions of wide and unconstricted metric spaces. Specifically, a metric space X is wide if all asymptotic cones X_ω are without cut-points. On the other hand, a metric space X is unconstricted if there exists some ultrafilter ω and some sequence of scalars s_i such that any asymptotic cone $Cone_\omega(X, \cdot, (s_i))$ does not have cut-points. Presently, we will show that under mild hypotheses in the CAT(0) setting the notions of wide and unconstricted are equivalent.

Theorem 6.1.1. *For X a CAT(0) space with extendable geodesics, X is wide if and only if it is unconstricted. Moreover, if in addition X is coarsely homogeneous (it admits a coarsely transitive group action by isometries), then either every asymptotic cone of X has a cut-point, or no asymptotic cone of X has a cut-point (i.e. X is wide).*

In order to prove Theorem 6.1.1, we will first prove a Lemma which represents a strengthened version of Property [C1] in Lemma 2.1.11. The author would like to acknowledge Igor Belegradek for help formulating and proving the precise form of the lemma.

Lemma 6.1.2. *Let X be a CAT(0) space, and $\gamma_R \subset X$ a geodesic segment of length $2R$ and center $\gamma(0)$. For all $r \leq R$, let α_r denote a minimal length detour path connecting $\gamma(-r)$ and $\gamma(r)$ which has interior disjoint from the metric ball $B_{\gamma(0)}(r)$. Then, we have the following inequality:*

$$\frac{|\alpha_r|}{r} \leq \frac{|\alpha_R|}{R}.$$

Proof. For any small value of $\epsilon > 0$, such that $\frac{|\alpha_R|}{\epsilon}$ is an integer, discretize the detour path α_R into $\frac{\alpha_R}{\epsilon}$ subsegments such that all but one have length at most ϵ, and the final one has length at most 2ϵ, where the ith subsegment connects the points $x_i, x_{i+1} \in \alpha_R$. For each pair of consecutive points x_i, x_{i+1} consider the geodesic triangle $\triangle(x_i, x_{i+1}, \gamma(0))$ in X. By assumption, $[x_i, x_{i+1}]$ has length at most ϵ. Let y_i denote the point on the geodesic $[\gamma(0), x_i]$ with distance r from $\gamma(0)$, and similarly let y_{i+1} denote the point on the geodesic $[\gamma(0), x_{i+1}]$ with distance r

from $\gamma(0)$. Using the defining CAT(0) triangle comparison property, it follows that the length of the cord $[y_i, y_{i+1}]$ is bounded above by $\frac{r\epsilon}{R}$. Performing this process for all pairs of consecutive points x_i, x_{i+1}, we obtain the following concatenated detour path

$$[y_0, y_1] \cup .... \cup [y_{\frac{\alpha_R}{\epsilon}-1}, y_{\frac{\alpha_R}{\epsilon}}]$$

which by construction remains outside the ball of radius $B_{r-\epsilon}(0)$ and has total length bounded above by $\frac{|\alpha_R|}{\epsilon}\frac{r\epsilon}{R}$. In particular, it follows that

$$|\alpha_{r-\epsilon}| \leq \frac{|\alpha_R|}{\epsilon}\frac{r\epsilon}{R} = \frac{r|\alpha_R|}{R}.$$

Letting ϵ limit to 0 completes the proof of the lemma. □

Proof. (Proof of Theorem 6.1.1) By definition wide implies unconstricted. Hence, to prove the first statement of the theorem it suffices to show that assuming X is not wide yet is unconstricted yields a contradiction.

Assuming X is not wide, there is an asymptotic cone $Cone_{\omega'}(X, (x_i), (s_i))$ with a cut-point. In particular, in light of Lemma 5.3.3 there exist sequences of points (a_i) and (b_i) with lengths linear in s_i such that the divergence of the sequence of geodesics $[a_i, b_i]$ is superlinear in s_i. Let c_i, be the centers of the geodesic $[a_i, b_i]$. Note that if α_i are detour paths of $[a_i, b_i]$ avoiding metric balls centered at c_i with radii growing linearly in s_i, then

$$\lim_{\omega'} \frac{|\alpha_i|}{s_i} \to \infty.$$

Assume that X is unconstricted. Then $\exists$ an ultrafilter ω and scaling sequence (t_i) such that any asymptotic cone $Cone_\omega(X, \cdot, (t_i))$ has no cut-points. Fix some $A \in \omega'$, and for each t_i, let $s_{j(i)}$ be the largest term in the sequence (s_i) such that $j \in A$ and moreover $s_j < t_i$. By unconstrictedness, the asymptotic cone $Cone_\omega(X, (c_{j(i)}), (t_i))$ has no cut-points.

Using the fact that geodesics in X can be extended, extend the geodesics $[a_{j(i)}, b_{j(i)}]$ in both directions to geodesics $[a'_{j(i)}, b'_{j(i)}]$ which are still centered at $c_{j(i)}$ however now have total length t_i. Let α'_i be detour paths of $[a'_{j(i)}, b'_{j(i)}]$ avoiding metric balls centered at c_i with radii

t_i. By our assumption that $Cone_\omega(X,(c_{j(i)}),(t_i))$ has no cut-points, in particular it follows that for any infinite set $B \subset \mathbb{N}$

$$\lim_B \frac{|\alpha_i'|}{t_i} \text{ is bounded.}$$

However, using Lemma 6.1.2,

$$\lim_{\omega'} \frac{|\alpha_i|}{s_i} \leq \lim_{\omega'} \frac{|\alpha_i'|}{t_i}.$$

Putting things together, we obtain a contradiction, thus completing the proof of the first statement of the theorem.

For the "moreover" statement of the theorem, assume X is not wide, then as we have seen from the first part, X is not unconstricted. Namely, if X has some asymptotic cone with a cut-point, then for any choices of ω and (s_i), there exists a sequence of base points (x_i), such that the asymptotic cone $Cone_\omega(X,(x_i),(s_i))$ has a cut-point. However, since we have the additional hypothesis that X is coarsely homogeneous, it follows that for all choices of basepoints (y_i) the asymptotic cone $Cone_\omega(X,(y_i),(s_i))$ also has a cut-point. The completes the proof of the theorem. □

Remark 6.1.3. Considering the proof of Theorem 6.1.1 it follows that conclusion of the theorem that wide is equivalent to unconstricted, actually holds in greater generality than stated. Specifically, it enough to assume that X a CAT(0) space with the *extendable quasi-geodesic*, namely there exist uniform constants such that any geodesic segment can be extended to a bi-infinite quasi-geodesic with the given quasi-isometry constants.

It is worth pointing out that Theorem 6.1.1 has applications related to a the following of open problems of [5]:

Question 6.1.4 ([5] Questions 6.9-10). Let X be a CAT(0) space with the property that all asymptotic cones have cut-points. Then, must X at least quadratic divergence? Must X have a Morse geodesic? In light of Theorem 6.1.1 we have the following equivalent reformulations for the special cases of CAT(0) spaces: Let X be a coarsely homogeneous CAT(0) space with extendable geodesic. If X is not wide, must X at least quadratic divergence? Must X have a Morse geodesic?

6.2 Hyperbolic quasi-geodesics in CAT(0) spaces

In this section we will explore the close relationship between various hyperbolic type properties of quasi-geodesics in CAT(0) spaces. In fact, a highlight of this section is the proof of Theorem 6.2.5 which was has already been recorded in both Chapters 1 and 4. Specifically, in Theorem 6.2.5 we prove that for X a CAT(0) space and $\gamma \subset X$ a quasi-geodesic, the following four statements are equivalent: (i) γ is Morse, (ii) γ is (b,c)–contracting, (iii) γ is strongly contracting, and (iv) in every asymptotic cone X_ω, any two distinct points in the ultralimit γ_ω are separated by a cut-point.

Theorem 6.2.5 should be considered in the context of related theorems in [13, 26, 29, 41]. Specifically, in [41] it is shown that periodic geodesics with superlinear divergence have at least quadratic divergence. In [29] it is shown that properties (3) and (4) in Theorem 6.2.5 are equivalent for arbitrary metric spaces. In [13] it is shown that in proper CAT(0) spaces a geodesic which is the axis of a hyperbolic isometry is strongly contracting if and only if the geodesic fails to bound a half plane. In [26] it is shown that geodesics with superlinear lower divergence are equivalent to strongly contracting geodesics and are Morse. The proof of Theorem 6.2.5 relies on careful applications of CAT(0) geometry and asymptotic cones.

Generalizing results of [41, 13], in [5] it is shown that in CAT(0) spaces Morse quasi-geodesics have at least quadratic divergence. As a warmup for Theorem 6.2.5, we provide an alternative proof of this result.

Theorem 6.2.3. ([5] Theorem 6.4). *A Morse quasi-geodesic in a CAT(0) space has at least quadratic divergence.*

The plan for this section is as follows. Subsection 6.2.1 provides background. Subsection 6.2.2 includes the proof of Lemma 6.2.4 and Theorems 6.2.3 and 6.2.5.

6.2.1 Background

Recall that a *(K,L) quasi-geodesic* $\gamma \subset X$ is the image of a map $\gamma \colon I \to X$ where I is a connected interval in $\mathbb{R}$ (possibly all of $\mathbb{R}$) such that $\forall s, t \in I$ we have the following quasi-isometric inequality:

$$\frac{|s-t|}{K} - L \leq d_X(\gamma(s), \gamma(t)) \leq K|s-t| + L.$$

We will refer to the quasi-geodesic $\gamma(I)$ by γ, and when the constants (K, L) are not relevant omit them.

An arbitrary quasi-geodesic in any geodesic metric space can be replaced by a continuous rectifiable quasi-geodesic by replacing the quasi-geodesic with a piecewise geodesic path connecting consecutive integer valued parameter points of the original quasi-geodesic. It is clear that this replacement process yields a continuous rectifiable quasi-geodesic which is in a bounded Hausdorff neighborhood of the original quasi-geodesic. When doing so will not affect an argument, by replacement if necessary we will assume quasi-geodesics are continuous and rectifiable. One upshot of the assumption of continuous quasi-geodesics is that the for γ, σ quasi-geodesics, the distance function $\psi(t) = d(\gamma(t), \sigma)$ is continuous. More generally, for non-continuous quasi-geodesics this distance function can have jump discontinuities controlled by the constants of the quasi-geodesics. Throughout, for γ any continuous and rectifiable path, we will denote its length by $|\gamma|$.

The following theorem of [29] characterizing Morse geodesics in terms of the asymptotic cone has application:

Theorem 6.2.1 ([29] Proposition 3.24)**.** *γ is a Morse quasi-geodesic if and only if in every asymptotic cone X_ω, every pair of distinct points in the ultralimit γ_ω are separated by a cut-point.*

6.2.2 Proof of Theorems

As a warmup for Theorem 6.2.5, we begin this subsection by giving an alternative proof of the fact that Morse quasi-geodesics in CAT(0) spaces have at least quadratic divergence. This result was originally proven in [5]. The present alternative proof is inspired by similar methods in [41] and follows immediately from the following lemma. For the sake of simplifying the exposition, in Lemma 6.2.2 we consider the special case of γ a geodesic rather than a quasi-geodesic. Hence, properties in Lemma 2.1.11 can be applied. Nonetheless, below we will show that the current form of the lemma suffices to prove Theorem 6.2.3 concerning quasi-geodesics.

Lemma 6.2.2. *Let X be a CAT(0) space, and γ a geodesic. If for every asymptotic cone X_ω, any two distinct points in the ultralimit γ_ω are separated by a cut-point, then γ has at least quadratic divergence. Similarly, the same result holds for the case of $\{\gamma_n\}$ a sequence of geodesic segments in X with lengths growing proportionally to a linear function.*

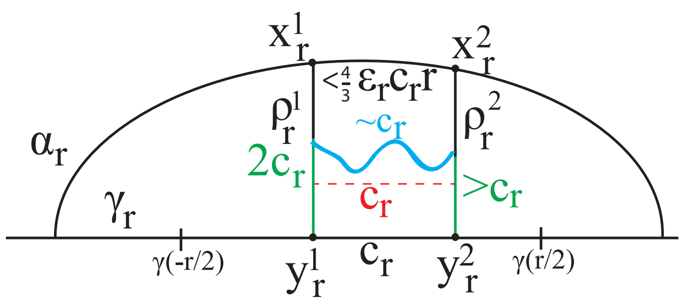

Figure 17: In CAT(0) spaces subquadratic divergence implies the existence of an asymptotic cone X_ω in which distinct points in the ultralimit of the geodesic are not separated by a cut-point.

Proof. We will prove the first statement in the Lemma. The "similarly" statement follows by the same argument.

By contradiction. That is, assume γ has subquadratic divergence. By definition, for each $r \in \mathbb{N}$, there is a continuous rectifiable detour path α_r connecting $\gamma(-r)$ and $\gamma(r)$ while remaining outside the ball $B_r(\gamma(0))$, such that $|\alpha_r| \leq \epsilon_r r^2$ where the function ϵ_r satisfies $\lim_{r\to\infty} \epsilon_r = 0$. Fix a sequence $\{c_r\}_{r\in\mathbb{N}}$ such that:

1. $4c_r \leq r$,

2. $\lim_{r\to\infty} c_r \to \infty$, and

3. $\lim_{r\to\infty} c_r^2 \epsilon_r = 0$.

For example, set $c_r = \min\{\epsilon_r^{-1/3}, \frac{r}{4}\}$.

For each r, let $n \in \{0, 1, ..., \lfloor \frac{r}{c_r} \rfloor\}$, and fix $z_r^n \in \alpha_r$ such that $z_r^n \in \pi_\gamma^{-1}(\gamma(-r/2 + nc_r))$. Since the total length of α_r is at most $\epsilon_r r^2$, it follows that for some m, the distance on α_r between z_r^m and z_r^{m+1} is at most

$$\frac{\epsilon_r r^2}{\lfloor \frac{r}{c_r} \rfloor} \leq \frac{\epsilon_r r^2}{\frac{r-c_r}{c_r}} = \frac{\epsilon_r c_r r^2}{r - c_r} \leq \frac{\epsilon_r c_r r^2}{r - \frac{r}{4}} = \frac{4\epsilon_r c_r r}{3}.$$

Set $x_r^1 = z_r^m$, $x_r^2 = z_r^{m+1}$, and $y_r^i = \pi_\gamma(x_r^i)$. By construction, $d(x_r^1, x_r^2) \leq \frac{4}{3}\epsilon_r c_r r$ while $d(y_r^1, y_r^2) = c_r$. Let $\rho_r^i \colon [0,1] \to X$ be a geodesic parameterized proportional to arc length joining $y_r^i = \rho_r^i(0)$ and $x_r^i = \rho_r^i(1)$. See Figure 17. Note that by construction since $y_r^i \in \gamma[-r/2, r/2]$ and $x_r^i \in \alpha_r$, it follows that

$$|\rho_r^i| \geq \frac{r}{2} \geq 2c_r.$$

Consider the function $\psi_r(t) = d(\rho_r^1(t), \rho_r^2(t))$. Note that $\psi_r(0) = c_r$ and $\psi_r(1) \leq \frac{4}{3}\epsilon_r c_r r$. CAT(0) convexity (Lemma 2.1.11 property C2) implies that

$$\psi_r\left(\frac{2c_r}{|\rho_r^1|}\right) \leq \left(1 - \frac{2c_r}{|\rho_r^1|}\right)c_r + \frac{8c_r}{3|\rho_r^1|}\epsilon_r c_r r \leq c_r + \frac{16}{3}c_r^2\epsilon_r.$$

Since $\lim_{r\to\infty} c_r^2\epsilon_r = 0$, for large enough r we can assume $d(\rho_r^1\left(\frac{2c_r}{|\rho_r^1|}\right), \rho_r^2\left(\frac{2c_r}{|\rho_r^1|}\right))$ is arbitrarily close to c_r.

Since y_r^2 is a nearest point projection of x_r^2 onto γ, it follows that $|\rho_r^2| \leq |\rho_r^1| + \frac{4}{3}\epsilon_r c_r r$. Since $\lim_{r\to\infty} c_r^2\epsilon_r = 0$ and $\lim_{r\to\infty} c_r \to \infty$, in particular $\lim_{r\to\infty} c_r\epsilon_r = 0$. Hence, for sufficiently large r we can assume $c_r\epsilon_r \leq \frac{3}{8}$. Then we have the following inequality:

$$|\rho_r^2| \leq |\rho_r^1| + \frac{4}{3}\epsilon_r c_r r \leq |\rho_r^1| + \frac{1}{2}r \leq |\rho_r^1| + |\rho_r^1| = 2|\rho_r^1|.$$

Running the same argument with the roles of ρ_r^1 and ρ_r^2 reversed, it follows that

$$\frac{1}{2}|\rho_r^1| \leq |\rho_r^2| \leq 2|\rho_r^1|.$$

In particular, $d(y_r^2, \rho_r^2\left(\frac{2c_r}{|\rho_r^1|}\right))$ is at most $4c_r$ and at least c_r.

Putting things together, on the one hand we have a geodesic segment $[y_r^1, y_r^2] \subset \gamma$ of length c_r. While on the other hand we have a piecewise geodesic path

$$\sigma_r = [y_r^1, \rho_r^1\left(\frac{2c_r}{|\rho_r^1|}\right)] \bigcup [\rho_r^1\left(\frac{2c_r}{|\rho_r^1|}\right), \rho^2\left(\frac{2c_r}{|\rho_r^1|}\right)] \bigcup [\left(\frac{2c_r}{|\rho_r^1|}\right), y_r^2],$$

of total length arbitrarily close to at most $7c_r$. Moreover, note that by construction we can bound from below the distance between the geodesics $[y_r^1, y_r^2]$ and $[\rho_r^1\left(\frac{2c_r}{|\rho_r^1|}\right), \rho_r^2\left(\frac{2c_r}{|\rho_r^1|}\right)]$. Specifically, it follows that the distance

$$d([\rho_r^1\left(\frac{2c_r}{|\rho_r^1|}\right), \rho_r^2\left(\frac{2c_r}{|\rho_r^1|}\right)], [y_r^1, y_r^2])$$

is at least arbitrarily close to c_r. Consider the asymptotic cone $Cone_\omega(X, (y_r^1), (c_r))$. In this asymptotic cone, the distinct points y_ω^1, y_ω^2 in the ultralimit γ_ω are not separated by a cut-point due to the path σ_ω connecting them. This completes the proof. □

Using Lemma 6.2.2 in conjunction with Theorem 6.2.1, proven in [29], we provide an alternative proof of the following Theorem, originally proven in [5]:

Theorem 6.2.3 ([5] Theorem 6.4)**.** *Let γ be a Morse quasi-geodesic in a CAT(0) space X, then γ has at least quadratic divergence.*

Proof. Given a Morse quasi-geodesic γ, construct a sequence of geodesic segments γ'_n connecting the points $\gamma(-n)$ and $\gamma(n)$. By the Morse property, all the geodesic segments γ'_n are contained in a uniformly bounded Hausdorff neighborhood of γ. By Theorem 6.2.1, in any asymptotic cone X_ω, any distinct points in γ_ω are separated by a cut-point. However, since the sequence of geodesics γ'_n are in a uniformly bounded Hausdorff neighborhood of γ it follows that in any asymptotic cone X_ω, any distinct points in γ'_ω are similarly separated by a cut-point. Applying Lemma 6.2.2 to the sequence of geodesic segments γ'_n, it follows that the sequence of geodesic segments has quadratic divergence. However, since the quasi-geodesic γ and sequence of geodesic segments γ'_n are in a bounded Hausdorff neighborhood of each other they have the same order of divergence. □

With the end goal of proving Theorem 6.4.4, presently we write down a proof of the following generalized Morse stability lemma. While versions of Morse stability lemmas are explicit in [1, 26] as well as implicit in [3, 29], there does not seem to be a recorded proof for the following version of the lemma in the literature. Accordingly, presently we include an explicit proof, closely based on a similar proofs in [1, 26].

Lemma 6.2.4. *Let X be a geodesic metric space and $\gamma \subset X$ a (b,c)–contracting quasi-geodesic. Then γ is Morse. Specifically, if σ is a (K,L) quasi-geodesic with endpoints on γ, then $d_{Haus}(\gamma, \sigma)$ is uniformly bounded in terms of only the constants b, c, K, L.*

Proof. Since γ is (b,c)–contracting, in particular the nearest point projection π_γ is coarsely well-defined. Set $D = \max\{K, L, 1\}$, $A = \frac{2(1+cD)}{b}$, and $R = \max\{d(\gamma, \sigma) \mid t \in \mathbb{R}\}$. Without loss of generality we can assume $R > A$. Since we wish to show that σ is in a bounded neighborhood of γ, by replacement if necessary we can assume σ is a continuous rectifiable quasi-geodesic.

Let $[s_1, s_2]$ be any maximal connected subinterval in the domain of σ such that $\forall s \in [s_1, s_2]$, we have $d(\sigma(s), \gamma) \geq A$. Since σ is continuous, we can subdivide the interval $[s_1, s_2]$ such that $s_1 = r_1, ..., r_m, r_{m+1} = s_2$ where $|\sigma(r_i, r_{i+1})| = \frac{Ab}{2}$ for $i \leq m$ and $|\sigma(r_m, r_{m+1})| \leq \frac{Ab}{2}$. Hence,

$$|\sigma(s1, s2)| \geq \frac{mAb}{2}. \tag{6.2.1}$$

Fix $P_i \in \pi_\gamma(\sigma(r_i))$. Then since $d(\sigma(r_i), P_i) \geq A$ and $d(\sigma(r_i), \sigma(r_{i+1})) \leq \frac{Ab}{2} < Ab$, by (b,c)–contraction, $d(P_i, P_{i+1}) < c$. Therefore $d(P_1, P_{m+1}) < c(m+1)$. It follows that

$$d(\sigma(s_1), \sigma(s_2)) < 2(A + L) + c(m + 1).$$

Note that since we are not assuming γ is a continuous quasi-geodesic, the distance function $d(\sigma(t), \gamma)$ can have jump discontinuities of L. Using the fact that σ is a quasi-geodesic, it follows that

$$|\sigma(s1, s2)| \leq D(d(\sigma(s_1), \sigma(s_2))) + D \leq D(2A + 2L + cm + c + 1). \tag{6.2.2}$$

Combining inequalities 6.2.1 and 6.2.2, after some manipulation we obtain

$$m < \frac{D(2A + 2L + c + 1)}{\frac{Ab}{2} - cD} = D(2A + 2L + c + 1).$$

Thus, $\forall s \in [s_1, s_2]$ we have the following inequality:

$$\begin{aligned} d(\sigma(s), \gamma) &\leq d(\sigma(s), \sigma(s_2)) + d(\sigma(s_2), \gamma) \\ &\leq |\sigma[s_1, s_2]| + A + L \\ &\leq D(2A + 2L + cm + c + 1) + A + L \\ &< D(2A + 2L + c\,(D(2A + 2L + c + 1)) + c + 1) + A + L \end{aligned}$$

Since the constants A, D are defined in terms of the constants b, c, K, L, the lemma follows. □

Using Lemma 6.2.4, we will prove the following theorem:

Theorem 6.2.5. *Let X be a CAT(0) space and $\gamma \subset X$ a (K,L)–quasi-geodesic. Then the following are equivalent:*

1. *γ is (b,c)–contracting*
2. *γ is (1,c)–contracting, (or strongly contracting)*
3. *γ is Morse, and*
4. *In every asymptotic cone X_ω, any two distinct points in the ultralimit γ_ω are separated by a cut-point.*

In particular, any of the properties listed above implies that γ has at least quadratic divergence.

Proof. $(2) \implies (1)$: This follows immediately from the definitions. $(1) \implies (3)$: This is precisely Lemma 6.2.4. $(3) \implies (4)$: This is precisely Theorem 6.2.1, proven in [29].

In the remainder of the proof we will prove $(4) \implies (2)$: By contradiction. That is, assuming γ is not (1,c)–contracting we will show that there is an asymptotic cone X_ω such

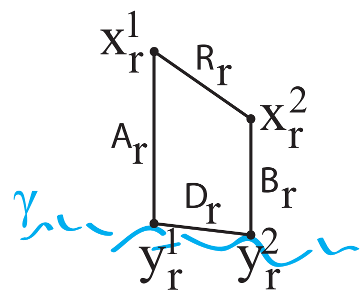

Figure 18: In a CAT(0) space, assuming a quasi-geodesic γ is not (1,c)–contracting implies it is not Morse.

that distinct points in the ultralimit γ_ω are not separated by a cut-point. Since γ is not (1,c)–contracting, it follows that for all $r \in \mathbb{N}$, we can make the following choices satisfying the stated conditions:

(i) Fix points $x_r^1 \in X \setminus \gamma$, and $y_r^1 \in \pi_\gamma(x_r^1)$ such that $d(x_r^1, y_r^1) = A_r$ and

(ii) Fix points $x_r^2 \in X \setminus \gamma$, and a point $y_r^2 \in \pi_\gamma(x_r^2)$ such that $d(x_r^1, x_r^2) = R_r < A_r$, and $d(y_r^1, y_r^2) = D_r$, for some $D_r \geq r$. Set $B_r = d(x_r^2, y_r^2)$.

Let $\rho_r^i \colon [0,1] \to X$ be a geodesic parameterized proportional to arc length joining $y_r^i = \rho_r^i(0)$ and $x_r^i = \rho_r^i(1)$. See Figure 18 for an illustration of the situation. Note we are not assuming the nearest point projection maps π_γ are even coarsely well-defined, but instead are simply picking elements of the set of nearest points subject to certain restrictions guaranteed by the negation of (1,c)–contraction. In fact, we cannot have assumed that y_r^2 could have been chosen such that $d(y_r^1, y_r^2) = r$, as the nearest point projection map onto quasi-geodesics need not be continuous. Moreover, it is possible that x_r^1 and x_r^2 are even the same point.

Since $d(y_r^1, y_r^2) = D_r$, it follows that $A_r + R_r + B_r \geq D_r$. Moreover, since $R_r < A_r$ and $B_r \leq R_r + A_r$, it follows that $A_r > \frac{D_r}{4}$. Fix $t = \frac{D_r}{4A_r} \in (0,1)$. Additionally, since $B_r < 2A_r$ it follows that $|[y_r^2, \rho_r^2(t)]| < \frac{D_r}{2}$.

Since $A_r > D_r/4$, the ratio $\frac{D_r}{A_r} \in (0,4)$, and hence there exists some subsequence such that $\frac{D_r}{A_r}$ converges.

Case 1: There exists some subsequence such that $\frac{D_r}{A_r} \to \epsilon \neq 0$.

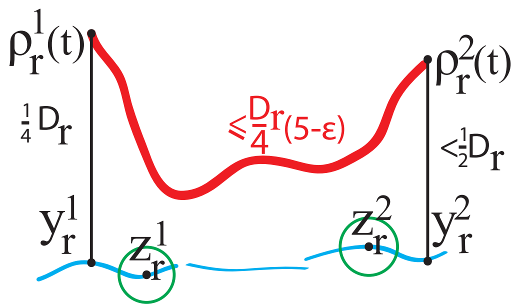

Figure 19: Case (1) of the proof of Theorem 6.2.5.

CAT(0) convexity (Lemma 2.1.11 property C2) applied to the geodesics ρ_r^i implies:

$$\begin{aligned} d(\rho_r^1(t), \rho_r^2(t)) &\leq \left(1 - \frac{D_r}{4A_r}\right) D_r + \frac{D_r R_r}{4A_r} \\ &\leq D_r - \frac{D_r^2}{4A_r} D_r + \frac{D_r}{4} \leq \frac{D_r}{4}\left(5 - \frac{D_r}{A_r}\right). \end{aligned}$$

For large enough values of r in the convergent subsequence, it follows that $d(\rho_r^1(t), \rho_r^2(t))$ is arbitrarily close to $\frac{D_r}{4}\left(5-\epsilon\right)$.

Let z_r^1 be a point on γ between y_r^1 and y_r^2 such that $d(y_r^1, z_r^1)$ is in the range $[\frac{\epsilon D_r}{28}, \frac{\epsilon D_r}{28} + L]$. Similarly, let z_r^2 be a point on γ between y_r^1 and y_r^2 such that $d(y_r^2, z_r^2)$ is in the range $[\frac{\epsilon D_r}{28}, \frac{\epsilon D_r}{28} + L]$. Since ρ_r^i are geodesics minimizing the distance from a fixed point to γ, it follows that ρ_r^i are disjoint from the interiors of the metric balls $B(z_r^i, \frac{\epsilon D_r}{56})$.

Moreover, by construction, for large enough values of r in the convergence subsequence, the geodesic $[\rho_r^1(t), \rho_r^2(t)]$ is disjoint from either the metric ball $B(z_r^1, \frac{\epsilon D_r}{56})$ or the metric ball $B(z_r^2, \frac{\epsilon D_r}{56})$. For if not, then

$$\begin{aligned} |[\rho_r^1(t), \rho_r^2(t)]| &\geq d(\rho_r^1(t), \{B(z_r^1, \frac{\epsilon D_r}{56}), B(z_r^2, \frac{\epsilon D_r}{56})\}) + d(B(z_r^1, \frac{\epsilon D_r}{56}), B(z_r^2, \frac{\epsilon D_r}{56})) \\ &\geq \left(\frac{D_r}{4} - \frac{\epsilon D_r}{56}\right) + \left(D_r - 6\frac{\epsilon D_r}{56}\right) = \frac{D_r}{4}\left(5 - \frac{\epsilon}{2}\right). \end{aligned}$$

However, this contradicts the fact that $d(\rho_r^1(t), \rho_r^2(t))$ is arbitrarily close to $\frac{D_r}{4}\left(5-\epsilon\right)$. On the other hand, if for large enough values of r in the convergence subsequence, the geodesic $[\rho_r^1(t), \rho_r^2(t)]$ is disjoint from the metric ball $B(z_r^i, \frac{\epsilon D_r}{56})$, then we will construct an asymptotic cone in which distinct points on γ_ω are not separated by a cut-point, thus completing the proof in this case.

Specifically, let ω be a non-principal ultrafilter such that the set of values of r in the convergence subsequence are an element of ω. Consider the asymptotic cone $Cone_\omega(X, (y_r^1), (D_r))$. In this asymptotic cone, the points $v_\omega^\pm$ in the intersection of γ_ω and the metric ball $B(z_\omega^i, \frac{\epsilon D_r}{56})$ are not separated by a cut-point due to the existence of a path $[v_\omega^+, z_\omega^i] \cup [z_\omega^i, v_\omega^-]$ connecting them in the interior of the ball $B(z_r^i, \frac{\epsilon D_r}{56})$, as well as the path connecting them outside the ball $B(z_r^i, \frac{\epsilon D_r}{56})$ given by the union of paths

$$[v_\omega^-, y_\omega^1] \cup [y_\omega^1, \rho_\omega^1(t)] \cup [\rho_\omega^1(t), \rho_\omega^2(t)] \cup [\rho_\omega^2(t), y_\omega^2] \cup [y_\omega^2, v_\omega^+].$$

See figure 19 for an illustration of the proof in Case (1).

Case 2: There exists some subsequence such that $\frac{B_r}{D_r} = 0$.

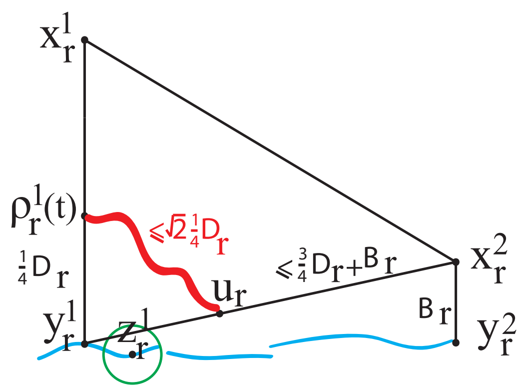

Figure 20: Case (2) of the proof of Theorem 6.2.5.

Let $\sigma_r \colon [0,1] \to X$ be a geodesic parameterized proportional to arc length joining $y_r^1 = \sigma_r(0)$ and $x_r^2 = \sigma_r(1)$. By the triangle inequality, $|\sigma_r|$ is in the range $[D_r - B_r, D_r + B_r]$.

Consider the triangle in X with vertices (y_r^1, x_r^2, x_r^1), and let the comparison triangle in Euclidean space have vertices $(\overline{y_r^1}, \overline{x_r^2}, \overline{x_r^1})$, Since $R_r < A_r$, it follows that the angle between the sides $[\overline{x_r^2}, \overline{y_r^1}]$ and $[\overline{x_r^1}, \overline{y_r^1}]$, is less than $\frac{\pi}{2}$. Let $\overline{u}_r$ denote the point in $[\overline{y_r^1}, \overline{x_r^2}]$, such that $d(\overline{y_r^1}, \overline{u_r}) = \frac{D_r}{4}$. Elementary Euclidean trigonometry implies that $d(\overline{\rho_r^1(t)}, \overline{u_r}) < \frac{\sqrt{2}D_r}{4}$. Hence, by the CAT(0) property, it follows that $d(\rho_r^1(t), u_r) < \sqrt{2}\epsilon D_r$.

Note that $d(u_r, x_r^2) \leq \frac{3D_r}{4} + B_r$, and hence $d(u_r, y_r^2) \leq \frac{3D_r}{4} + 2B_r$. Putting things together, it follows that $d(\rho_r^1(t), y_r^2) < \frac{\sqrt{2}D_r}{4} + \frac{3D_r}{4} + 2B_r$.

As in Case (1), let z_r^1 be a point on γ between y_r^1 and y_r^2 such that $d(y_r^1, z_r^1)$ is in the range $[\frac{(2-\sqrt{2})D_r}{16}, \frac{(2-\sqrt{2})D_r}{16} + L]$. Again as in Case (1), note that ρ_r^1 is disjoint from the interior of the metric balls $B(z_r^1, \frac{(2-\sqrt{2})D_r}{32})$.

Furthermore, for large enough values of r in the convergence subsequence, the geodesic $[\rho_r^1(t), y_r^2]$ is also disjoint from the metric ball $B(z_r^1, \frac{(2-\sqrt{2})D_r}{32})$. For if not, then

$$\begin{aligned} |[\rho_r^1(t), y_r^2]| &\geq d(\rho_r^1(t), B(z_r^1, \frac{(2-\sqrt{2})D_r}{32})) + d(B(z_r^1, \frac{(2-\sqrt{2})D_r}{32}), y_r^2) \\ &\geq \left(\frac{D_r}{4} - \frac{(2-\sqrt{2})D_r}{32}\right) + \left(D_r - 3\frac{(2-\sqrt{2})D_r}{32}\right) \geq D_r + \frac{\sqrt{2}D_r}{8}. \end{aligned}$$

However, in conjunction with the assumption of the case, this contradicts the fact that $d(\rho_r^1(t), y_r^2)$ is at most $\frac{3D_r}{4} + \frac{\sqrt{2}D_r}{4} + 2B_r$. On the other hand, if for large enough values of r the geodesic $[\rho_r^1(t), y_r^2]$ is disjoint from the metric ball $B(z_r^1, \frac{(2-\sqrt{2})\epsilon D_r}{32})$, then as in Case (1), in the asymptotic cone $Cone_\omega(X, (y_r^1), (D_r))$ we can find distinct points on γ_ω that are not separated by a cut-point. This completes the proof in Case (2). See figure 20 for an illustration of the proof in Case (2).

Case 3: We are not in Cases (1) or (2):

Since we are not in Case (2), by passing to a subsequence if necessary we can assume that the ratio $\frac{B_r}{D_r}$ either converges to $\epsilon' > 0$ or diverges to infinity. In the former case, set $\epsilon = \min(\frac{1}{4}, \epsilon')$, and in the latter case set $\epsilon = \frac{1}{4}$. Set $s = \frac{\epsilon D_r}{A_r}$. By construction $s \in (0, 1)$.

Let $\tau_r\colon [0,1] \to X$ be a geodesic parameterized proportional to arc length joining $x_r^2 = \tau_r(0)$ and $x_r^1 = \tau_r(1)$. Similarly, let $\sigma_r\colon [0,1] \to X$ be a geodesic parameterized proportional to arc length joining $y_r^2 = \sigma_r(0)$ and $x_r^1 = \sigma_r(1)$. By construction, $|\sigma_r|$ is in the range $[A_r, A_r + D_r]$. Since we are not in Case (1), it follows that $|[\sigma_r(0), \sigma_r(s)]|$ is arbitrarily close to ϵD_r. Moreover, CAT(0) convexity (Lemma 2.1.11 property C2) applied to the geodesics ρ_r^1 and σ_r immediately implies $d(\rho_r^1(s), \sigma_r(s))$ is bounded above by D_r.

Consider the triangle in X with vertices (x_r^1, x_r^2, y_r^2), and let the comparison triangle in Euclidean space have vertices $(\overline{x_r^1}, \overline{x_r^2}, \overline{y_r^2})$, As in Case (1), since $R_r < A_r$, it follows that the angle between the sides $[\overline{x_r^1}, \overline{y_r^2}]$ and $[\overline{x_r^2}, \overline{y_r^2}]$, is less than $\frac{\pi}{2}$. Let $\overline{w_r}$ denote the point in $[\overline{y_r^2}, \overline{x_r^2}]$, such that $d(\overline{y_r^2}, \overline{w_r}) = \epsilon D_r$. Note that since $|[\sigma_r(0), \sigma_r(s)]|$ is arbitrarily close to ϵD_r, elementary Euclidean trigonometry implies that $d(\overline{\sigma_r(s)}, \overline{w_r})$ is at most arbitrarily close to $\sqrt{2}\epsilon D_r$. Hence, by the CAT(0) property, it follows that $d(\sigma_r(s), w_r)$ is at most arbitrarily close to $\sqrt{2}\epsilon D_r$. Putting things together, it follows that $d(\rho_r^1(s), w_r)$ is at most arbitrarily close to $D_r + \sqrt{2}\epsilon D_r$.

As in Case (1), let z_r^1 be a point on γ between y_r^1 and y_r^2 such that $d(y_r^1, z_r^1)$ is in the range $[\frac{(2-\sqrt{2})\epsilon D_r}{16}, \frac{(2-\sqrt{2})\epsilon D_r}{16} + L]$. Similarly, let z_r^2 be a point on γ between y_r^1 and y_r^2 such that $d(y_r^2, z_r^2)$ is in the range $[\frac{(2-\sqrt{2})\epsilon D_r}{16}, \frac{(2-\sqrt{2})\epsilon D_r}{16} + L]$. For large enough values of r in the convergence subsequence, the geodesic $[\rho_r^1(s), w_r]$ is disjoint from either the metric ball $B(z_r^1, \frac{(2-\sqrt{2})\epsilon D_r}{32})$ or the metric ball $B(z_r^2, \frac{(2-\sqrt{2})\epsilon D_r}{32})$. For if not, then

$$
\begin{aligned}
|[\rho_r^1(s), w_r]| &\geq d(\rho_r^1(t), \{B(z_r^1, \frac{(2-\sqrt{2})\epsilon D_r}{32}), B(z_r^2, \frac{(2-\sqrt{2})\epsilon D_r}{32})\}) \\
&+ d(B(z_r^1, \frac{(2-\sqrt{2})\epsilon D_r}{32}), B(z_r^2, \frac{(2-\sqrt{2})\epsilon D_r}{32})) \\
&+ d(w_r, \{B(z_r^1, \frac{(2-\sqrt{2})\epsilon D_r}{32}), B(z_r^2, \frac{(2-\sqrt{2})\epsilon D_r}{32})\}) \\
&\geq \left(\epsilon D_r - \frac{(2-\sqrt{2})\epsilon D_r}{32}\right) + \left(D_r - 6\frac{(2-\sqrt{2})\epsilon D_r}{32}\right) \\
&+ \left(\epsilon D_r - \frac{(2-\sqrt{2})\epsilon D_r}{32}\right) \\
&> D_r + \frac{3\epsilon D_r}{2}.
\end{aligned}
$$

However, this is a contradiction to the fact that $d(\rho_r^1(s), w_r)$ is at most arbitrarily close to $D_r + \sqrt{2}\epsilon D_r$. On the other hand, if for large enough values of r in the convergence subsequence, the geodesic $[\rho_r^1(s), w_r]$ is disjoint from the metric ball $B(z_r^i, \frac{(2-\sqrt{2})\epsilon D_r}{32})$, then as in Case (1), the asymptotic cone $Cone_\omega(X, (y_r^1), (D_r))$ contains distinct points of γ_ω not separated by a cut-point, thereby completing the proof in the final case and hence completing the proof of $(4) \implies (2)$. See figure 21 for an illustration of the proof in Case (3).

Finally, the "in particular" clause of the theorem follows from Theorem 6.2.3. □

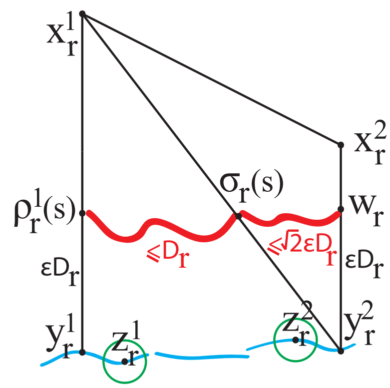

Figure 21: Case (3) of the proof of Theorem 6.2.5.

6.3 A proof of the hyperbolicity of $\mathcal{C}_{sep}(S_{2,0})$

It is well known that the curve complex $\mathcal{C}(S)$ is δ-hyperbolic for all surfaces of positive complexity, see [47]. On the other hand, the separating curve complex $\mathcal{C}_{sep}(S)$ in general is not δ-hyperbolic. In particular, for all closed surfaces $S = S_{g,0}$ with genus $g \geq 3$, as noted in [61], $\mathcal{C}_{sep}(S)$ contains natural nontrivial *quasi-flats*, or quasi-isometric embeddings of Euclidean flats; an obstruction to hyperbolicity. For $S_{2,0}$ however, unlike closed surfaces of higher genus, there are no natural nontrivial quasi-flats. Given this context, Schleimer conjectures that $\mathcal{C}_{sep}(S_{2,0})$ is δ-hyperbolic; see [61] Conjecture 2.48. In this section, we prove this conjecture in the affirmative. Note that the natural embedding $i\colon \mathcal{C}_{sep}(S) \to \mathcal{C}(S)$ is known not to be a quasi-isometric embedding for all surfaces, and hence the proof of the conjecture does not follow from the hyperbolicity of the curve complex, [47].

Remark 6.3.1. While a proof that $\mathcal{C}_{sep}(S_{2,0})$ is δ-hyperbolic is implicit in the work of Brock-Masur, [21], it is somewhat hidden, and so in this section we present an alternative proof of this fact which is independent of their results. In fact, since writing up this result, I have been informed that a recent paper of Ma, [44], proved the δ-hyperbolicity of $\mathcal{C}_{sep}(S_{2,0})$ using the aforementioned work of Brock-Masur.

The ideas in this section are similar to, as well as motivated by, work of Masur-Schleimer in [50]. Specifically, in [50], using ideas implicit in [3], axioms are established for when a combinatorial complex has a quasi-distance formula and is δ-hyperbolic. In particular, Masur and Schleimer use these axioms to prove that the disk complex and the arc complex are δ-hyperbolic. While due to a technicality, the Masur-Schleimer axioms do not all hold in the case of $\mathcal{C}_{sep}(S_{2,0})$, nonetheless, with enough care we are able to show by a direct argument that $\mathcal{C}_{sep}(S_{2,0})$ has a quasi-distance formula. Furthermore, careful consideration of the Masur-Schleimer proof of δ-hyperbolicity for a complex satisfying their axioms reveals that their proof in fact holds in the case of $\mathcal{C}_{sep}(S_{2,0})$.

The outline of the section is as follows. In Subsection 6.3.1 relevant background material is introduced. Subsection 6.3.2 contains the core content of the section including a proof of the quasi-distance formula for $\mathcal{C}_{sep}(S_{2,0})$ as well as a proof of δ-hyperbolicity.

6.3.1 Background

6.3.1.1 Combinatorial Complexes and Holes

In this section, a *combinatorial complex*, $\mathcal{G}(S)$, will be a graph with vertices defined in terms of multicurves on the surface and edge relations defined in terms of upper bounds on intersections between the multicurves. In addition, we will assume that combinatorial complexes are invariant under an isometric action of the mapping class group, $\mathcal{MCG}$. Examples of combinatorial complexes include the separating curve complex, $\mathcal{C}_{sep}(S)$, the arc complex, $\mathcal{A}(S)$, the pants complex, $\mathcal{P}(S)$, the marking complex, $\mathcal{M}(S)$, as well as many others in the literature.

A *hole* for $\mathcal{G}(S)$ is defined to be any connected essential subsurface (here unlike in the rest of the thesis, essential subsurfaces need not have non-trivial complexity) such that the entire combinatorial complex has nontrivial subsurface projection into it. For example, it is not hard to see that holes for the arc complex $\mathcal{A}(S)$, are precisely all connected subsurfaces Y such that $\partial S \subset \partial Y$.

The central idea in [50], which is also implicit in [3], is that distance in a combinatorial complex is approximated by summing over the distances in the subsurface projections to the curve complexes of holes. In particular, due to the action by $\mathcal{MCG}$, if a complex has disjoint holes then the complex admits nontrivial quasi-flats, and hence cannot be δ-hyperbolic. Conversely, if a combinatorial complex has the property that no two holes are disjoint, then assuming a couple of additional Masur-Schleimer axioms, see [50], the complex is δ-hyperbolic.

6.3.1.2 Farey Graph

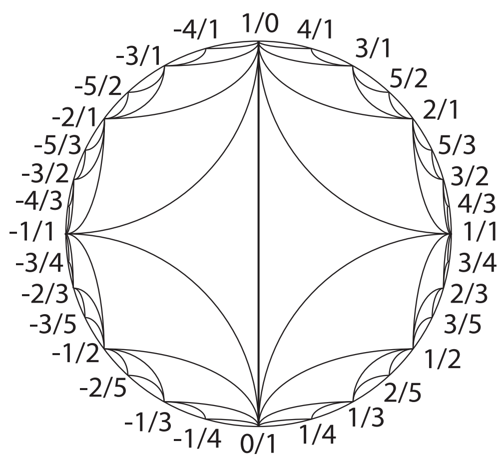

Figure 22: A finite portion of the Farey Graph with labeled vertices.

The Farey graph is a classical graph with direct application to the study of the curve complex. Vertices of the Farey graph corresponding to elements of $\mathbb{Q} \cup \{\infty = \frac{1}{0}\}$, with edges between two rational numbers in lowest terms $\frac{p}{q}$ and $\frac{r}{s}$ if $|ps - qr| = 1$. The Farey graph can be drawn as an ideal triangulation of the unit disk as in Figure 22. A nice feature of the Farey graph is the so-called *Farey addition property* which ensures that if rational number $\frac{p}{q}$ and $\frac{r}{s}$ are connected in the Farey graph, then there is an ideal triangle in the Farey graph with vertices $\frac{p}{q}$, $\frac{r}{s}$, and $\frac{p+r}{q+s}$.

The curve complexes $\mathcal{C}(S_{0,4})$ and $\mathcal{C}(S_{1,1})$ are isomorphic to the Farey graph. The isomorphism is given by sending the positively oriented meridional curve of the surfaces to $\frac{1}{0}$, the positively oriented longitudinal curve of the surfaces to $\frac{0}{1}$, and more generally sending the (p, q) curve to $\frac{p}{q}$.

6.3.2 Separating curve complex of the closed genus two surface is hyperbolic: proof

Theorem 6.3.2. *$\mathcal{C}_{sep}(S_{2,0})$ is δ-hyperbolic.*

The proof of Theorem 6.4.4 is broken down into two steps. In the first step we show by a direct argument that $\mathcal{C}_{sep}(S_{2,0})$ has a quasi-distance formula. In the second step, using step one, we show that the Masur-Schleimer proof for δ-hyperbolicity of a combinatorial complex found in Section 20 of [50] applies to $\mathcal{C}_{sep}(S_{2,0})$ despite the fact that not all the Masur-Schleimer axioms hold.

6.3.2.1 Step One: $\mathcal{C}_{sep}(S_{2,0})$ has a quasi-distance formula.

We begin by recalling a lemma of [50] which ensures a quasi-lower bound for a quasi-distance formula for $\mathcal{C}_{sep}(S_{2,0})$. As noted by Masur-Schleimer, the proof of the following lemma follows almost verbatim from similar arguments in [48] regarding the marking complex:

Lemma 6.3.3 ([50] Theorem 5.10)**.** *Let S be a surface of finite type, and let $\mathcal{G}(S)$ be a combinatorial complex. There is a constant C_0 such that $\forall c \geq C_0$ there exists quasi-isometry constants such that $\forall \alpha, \beta \in \mathcal{G}(S)$:*

$$\sum_{Y \text{ a hole for } \mathcal{G}(S)} \{d_{\mathcal{C}(Y)}(\alpha,\beta)\}_c \lesssim d_{\mathcal{G}(S)}(\alpha,\beta)$$

In light of Lemma 6.3.3, in order to obtain a quasi-distance formula for $\mathcal{C}_{sep}(S_{2,0})$, it suffices to obtain a quasi-upper bound on $\mathcal{C}_{sep}(S_{2,0})$ distance in terms of the sum of subsurface projections to holes. As motivated by [50], our approach for doing so will be by relating markings to separating curves and more generally marking paths to separating paths. In the rest of this subsection let $S = S_{2,0}$.

Let $\mu \in \mathcal{M}(S)$. Presently we will define a coarsely well-defined mapping $\phi\colon \mathcal{M}(S) \to \mathcal{C}_{sep}(S)$. If $base(\mu)$ contains a separating curve γ_i, then we define $\phi(\mu) = \gamma_i$. On the other hand, if all three base curves of μ, $\gamma_1, \gamma_2, \gamma_3$, are non-separating curves, then for any $i, j, k \in \{1, 2, 3\}$,

$i \neq j \neq k \neq i$, denote the essential subsurface $S_{i,j} := S \setminus \gamma_i, \gamma_j \simeq S_{0,4}$. Note that $\mathcal{C}(S_{i,j})$ is a Farey graph containing the adjacent curves γ_k and t_k. Let o_k be a curve in $S_{i,j}$ such that γ_k, t_k, o_k form a triangle in $\mathcal{C}(S_{i,j})$. Note that o_k is not uniquely determined by this condition; in fact, there are exactly two possibilities for o_k. Nonetheless, the Farey addition property implies that the two possible curves for o_k intersect four times and are distance two in $\mathcal{C}(S_{i,j})$. In this case, assuming none of the base curves are separating curves, we claim that exactly one of o_k or t_k is a separating curve of S, and define $\phi(\mu)$ to be either t_k or o_k, depending on which one is a separating curve.

Claim 6.3.4. *With the notation from above, let γ_k, t_k, o_k form a triangle in the Farey graph $\mathcal{C}(S_{i,j})$. Then one (and only one) of the curves t_k and o_k are separating curves of S.*

Proof. $S_{i,j}$ has four boundary components which glue up in pairs inside the ambient surface S. Moreover, any curve $\alpha \in \mathcal{C}(S_{i,j})$ gives rise to a partition of the four boundary components of $S_{i,j}$ into pairs given by pairing boundary components in the same connected component of $S_{i,j} \setminus \alpha$.

In total there are $\binom{4}{2} = 3$ different ways to partition the four boundary components of $S_{i,j}$ into pairs, and in fact it is not hard to see that the partition of a boundary components determined by a curve $\frac{p}{q} \in \mathcal{C}(S_{i,j})$ is entirely determined by the parity of p and q. Specifically, the three partitions correspond to the cases (i) p and q are both odd, (ii) p is odd and q is even, and (iii) p is even and q is odd. By topological considerations, since we are assuming none of the base curves of the marking are separating curves, it follows that all curves in $\mathcal{C}(S_{i,j})$ corresponding to exactly one of the three cases, (i),(ii) or (iii), are separating curves of the ambient surface S.

Hence, in order to prove the claim it suffices to show that any triangle in the Farey graph has exactly one vertex from each of the three cases (i), (ii) and (iii). This follows from basic arithmetic computation: First note that no two vertices from a single case are adjacent in the Farey graph. For example a vertex of type (odd/odd) cannot be adjacent to another vertex of type (odd/odd) as the adjacency condition fails, namely

$$|odd^2 - odd^2| = |odd' - odd'| = even \neq 1.$$

Similar calculations show that two vertices of type (odd/even) or two vertices of type (even/odd) cannot be adjacent to each other. Moreover, the Farey addition property implies that if a triangle contains vertices of two of the different cases, then the third vertex in any such triangle perforce corresponds to the third case. For example if a triangle has vertices of type (odd/odd) and (odd/even), the Farey addition property implies that the third vertex in any such triangle will be of type (even/odd). The claim follows. □

The following theorem ensures that the mapping $\phi\colon \mathcal{M}(S) \to \mathcal{C}_{sep}(S)$ is coarsely well-defined.

Theorem 6.3.5. *Using the notation from above, let μ be a marking with no separating base curves, and let t_i, t_j be transversals which are separating curves. Then t_i and t_j are connected in the separating curve complex $\mathcal{C}_{sep}(S)$. Similarly, if t_i and o_j, or o_i and o_j are separating curves the same result holds.*

Proof. We will prove the first case; the "similarly" statement follows from the same proof. Specifically, we will show that the separating curves t_i, t_j intersect four times. Up to action of $\mathcal{M}CG$, there is only one picture for a marking μ which does not contain a separating base curve, as presented in Figure 23. Without loss of generality we can assume $t_i = t_1$ and $t_j = t_2$. Notice that in the essential subsurface $S_{2,3}$, as in Figure 23, the base curve γ_1 corresponds to the meridional curve $\frac{1}{0}$, and similarly in the essential subsurface $S_{1,3}$ the base curve γ_2 also corresponds to the meridional curve $\frac{1}{0}$. Since t_1 is connected to γ_1 in the Farey graph $\mathcal{C}(S_{2,3})$ it follows that $t_1 \in \mathcal{C}(S_{2,3})$ is a curve of the form $\frac{n}{1}$ for some integer n. Similarly, $t_2 \in \mathcal{C}(S_{1,3})$ is a curve of the form $\frac{m}{1}$ for some integer m. As in the examples in Figure 23 it is easy to draw representatives of the two curves which intersect four times. □

The following lemma says that our coarsely well-defined mapping ϕ which associates a separating curve to a complete clean marking is natural with respect to elementary moves in the marking complex.

Lemma 6.3.6. *If $d_{\mathcal{M}(S)}(\mu, \mu') \leq 1$ then $d_{\mathcal{C}_{sep}(S)}(\phi(\mu), \phi(\mu')) \leq 2$.*

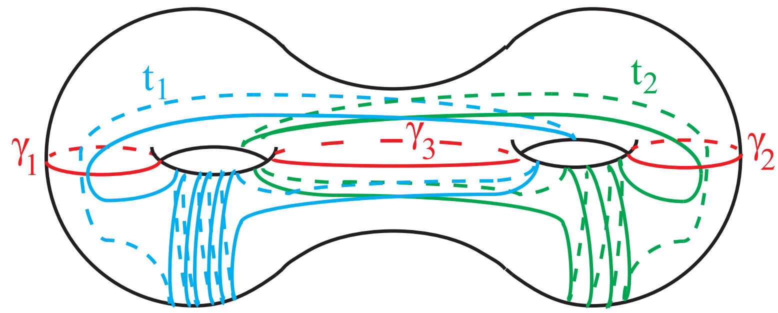

Figure 23: A marking μ on $S_{2,0}$ with no separating curves, with transversal curves t_1, t_2 separating curves. Notice that $d_{\mathcal{C}_{sep}(S)}(t_1, t_2) = 1$.

Remark 6.3.7. To be sure, as will be evident in the proof of the Lemma 6.3.6, up to choosing appropriate representatives of $\phi(\mu)$ and $\phi(\mu')$ it is in fact true that $d_{\mathcal{C}_{sep}(S)}(\phi(\mu), \phi(\mu')) \leq 1$. However, the statement of the lemma holds for any representatives of $\phi(\mu)$ and $\phi(\mu')$.

Proof. The proof will proceed by considering cases. First assume μ and μ' differ by a twist move applied to the pair (γ_i, t_i). If μ has a separating base curve, and hence so does μ' as twists do not affect base curves, then we are done as ϕ associates to both markings this common separating base curve. On the other hand, if μ has no separating base curves, and hence neither does μ', we can let ϕ assign to both markings the same separating curve either t_j or o_j, for $i \neq j$, depending on which one is a separating curve. In either case we are done.

Next assume μ and μ' differ by a flip move applied to the pair (γ_i, t_i). Recall that after the flip move is performed one must pass to a compatible clean marking. Let us consider the situation more carefully. Specifically, assume $\mu = \{(\gamma_i, t_i), (\gamma_j, t_j), (\gamma_k, t_k)\}$. Then $\mu' = \{(t_i, \gamma_i), (\gamma_j, t_j'), (\gamma_k, t_k')\}$, where the transversals t_j', t_k' are obtained by passing to a compatible clean marking if necessary. If γ_j or γ_k is a separating base curve we are done. If not, then if γ_i is a separating curve we are similarly done as ϕ can be chosen to assign to both markings the separating curve γ_i. Finally, if none of the base curves are separating curves, then we also done as we can choose ϕ to assign to both markings the same separating curve either t_j or o_j, depending on which one is a separating curve. □

Combining the existence of well-defined mapping $\phi\colon\ \mathcal{M}(S) \to \mathcal{C}_{sep}(S)$ with the result of

Lemma 6.3.6, we have the following procedure for finding a path between any two separating curves. Given $\alpha, \beta \in \mathcal{C}_{sep}(S)$, complete the separating curves into complete clean markings μ and ν such that $\alpha \in base(\mu)$ and $\beta \in base(\nu)$. Then construct a hierarchy path ρ in $\mathcal{M}(S)$ between μ and ν. Applying the mapping ϕ to our hierarchy path ρ, and interpolating as necessary, yields a path in $\mathcal{C}_{sep}(S)$ between the separating curves α and β with length quasi-bounded above by the length of the marking path ρ. In fact, if we are careful we can obtain the following corollary which provides a quasi-upper bound on $\mathcal{C}_{sep}(S_{2,0})$ distance in terms of the sum of subsurface projection to holes. Note that together with Lemma 6.3.3, the corollary gives a quasi-distance formula for $\mathcal{C}_{sep}(S)$, thus completing step one.

Corollary 6.3.8. *For $S = S_{2,0}$, there is a constant K_0 such that $\forall k \geq K_0$ there exists quasi-isometry constants such that $\forall \alpha, \beta \in \mathcal{C}_{sep}(S)$:*

$$d_{\mathcal{C}_{sep}(S)}(\alpha, \beta) \lesssim \sum_{Y \textit{ a hole for } \mathcal{C}_{sep}(S)} \{d_{\mathcal{C}(Y)}(\alpha, \beta)\}_k$$

Proof. As suggested above we have a quasi-upper bound on $\mathcal{C}_{sep}(S)$ distance given by the length any hierarchy path ρ connecting markings containing the given separating curves as base curves. In conjunction with the quasi-distance formula for $\mathcal{M}(S)$ in [48], we have already have a quasi-upper bound of the form:

$$d_{\mathcal{C}_{sep}(S)}(\alpha, \beta) \lesssim \sum_{\xi(Y) \geq 1, \text{ or } Y \text{ an annulus}} \{d_{\mathcal{C}(Y)}(\alpha, \beta)\}_k$$

Hence, it suffices to show that for all components domains Y in the above sum which are not holes of $\mathcal{C}_{sep}(S)$, we can choose can choose our mapping ϕ such that the $\mathcal{C}_{sep}(S)$ diameter of $\phi(I_Y)$ is uniformly bounded, where I_Y is as in property [H2] of Theorem 2.2.4.

Holes for $\mathcal{C}_{sep}(S)$ consist of all connected essential subsurfaces excluding essential subsurfaces whose boundary is a separating curve of the surface. Hence, we must show that for all component domains Y which are either annuli or proper connected essential subsurfaces with boundary component a separating curve of the surface, that the $\mathcal{C}_{sep}(S)$ diameter of $\phi(I_Y)$ is uniformly bounded. First consider the case of Y an annulus. In this case, the subpath of ρ in the marking complex corresponding to I_Y is acting by twist moves on transversal curves of a fixed base curve γ_i. As in the proof of Lemma 6.3.6, if there is a separating base curve in the marking, then we are done as the base curves are fixed by the twisting and we can pick the fixed base curve as our separating curve for all of $\phi(I_Y)$. Otherwise, if none of the base curves are separating then for $i \neq j$ we can pick t_j or o_j, depending on which is a separating curve, as our constant representative for all of $\phi(I_Y)$, as this transversal is unaffected by the twist moves applied to the base curve γ_i. Next consider the case of Y a proper connected essential subsurface with boundary a separating curve of the surface. Since every marking in I_Y contains the separating curve ∂Y, the desired result follows as we set all of $\phi(I_Y)$ to be equal to the fixed separating curve ∂Y. □

6.3.2.2 Step Two: $\mathcal{C}_{sep}(S_{2,0})$ is δ-hyperbolic.

In Section 13 of [50], sufficient axioms are established for implying a combinatorial complex admits a quasi-distance formula and furthermore is δ-hyperbolic. The first axiom is that no two holes for the combinatorial complex are disjoint. This is easily verified for $\mathcal{C}_{sep}(S_{2,0})$. The rest of the axioms are related to the existence of an appropriate marking path $\{\mu_i\}_{i=0}^{N} \subset \mathcal{M}(S)$ and a corresponding well suited combinatorial path $\{\gamma_i\}_{i=0}^{K} \subset \mathcal{G}(S)$. In particular, there is a strictly increasing reindexing function $r\colon [0,K] \to [0,N]$ with $r(0) = 0$ and $r(K) = N$. In the event that one uses a hierarchy as a marking path, the rest of the axioms can be simplified to the following:

1. (Combinatorial:) There is a constant C_2 such that for all i, $d_{\mathcal{C}(Y)}(\gamma_i, \mu_{r(i)}) < C_2$ for every hole Y, and moreover $d_{\mathcal{G}(S)}(\gamma_i, \gamma_{i+1}) < C_2$.

2. (Replacement:) There is a constant C_4 such that:

   [R1] If Y is a hole and $r(i) \in I_Y$, then there is a vertex $\gamma' \in \mathcal{G}(S)$ with $\gamma' \subset Y$ and $d_{\mathcal{G}(S)}(\gamma, \gamma') < C_4$.

   [R2] If Y is a non-hole and $r(i) \in I_Y$, then there is a vertex $\gamma' \in \mathcal{G}(S)$ with $\gamma' \subset Y$ or $\gamma' \subset S \setminus Y$ and $d_{\mathcal{G}(S)}(\gamma, \gamma') < C_4$.

3. (Straight:) For any subinterval $[p,q] \subset [0,K]$ with $d_{\mathcal{C}(Y)}(\mu_{r(p)}, \mu_{r(q)})$ uniformly bounded, where Y ranges over all non-holes, then $d_{\mathcal{G}(S)}(\gamma_p, \gamma_q) \lesssim d_{\mathcal{C}(S)}(\gamma_p, \gamma_q)$.

Presently we will show that in the case of the separating curve complex $\mathcal{C}_{sep}(S_{2,0})$ all of above axioms with the exception of axiom [R2] hold. Let $\rho = \{\mu_i\}_{i=0}^{N}$ be a hierarchy path between two complete clean markings each containing a separating base curve. Then define the combinatorial path $\{\gamma_i\}_{i=0}^{K} \subset \mathcal{C}_{sep}(S)$ by interpolating between the elements of $\phi(\rho)$ subject to making choices for images of the coarsely well-defined mapping ϕ such that for component domains of ρ which are not holes of $\mathcal{C}_{sep}(S_{2,0})$, the $\mathcal{C}_{sep}(S)$ diameter of $\phi(I_Y)$ is uniformly bounded. This is precisely what was proven to be possible in Corollary 6.3.8. In other words, we can assume the combinatorial path is a quasi-geodesic in the separating curve complex obtained from considering the mapping ϕ applied to a hierarchy path ρ and with representative chosen in a manner such that as the hierarchy path potentially travels for an arbitrary distance in a non-hole component domain, the combinatorial path in the separating curve complex only travels a uniformly bounded distance. Let the reindexing function r be given by sending an element γ_i of the combinatorial path to any marking μ_j such that $\phi(\mu_j) = \gamma_i$.

Given this setting, the combinatorial axiom is immediate from the definition of ϕ in conjunction with Lemma 6.3.6. Similarly, the straight axiom follows from the properties of hierarchy paths of Theorem 2.2.4 in conjunction with the construction of the combinatorial path. Replacement axiom [R1] also holds for if Y is a hole, then ∂Y contains at most two non-separating curves. Then for all markings $\mu \in I_Y$, $base(\mu)$ contains the at most two non-separating curves ∂Y. Let γ_i be a base curve of μ not in ∂Y. Then we can choose $\phi(\mu)$ to be either γ_i, t_i, or o_i, depending on which is a separating curve, all of which are properly contained in the connected essential subsurface Y. Claim 6.3.4 ensures that exactly one of the three curves γ_i, t_i, and o_i is a separating curve. On the other hand, axiom [R2] fails as if Y is an essential subsurface which is a non-hole then it is possible that $\partial Y \in \mathcal{C}_{sep}(S)$. In this case, by elementary topological considerations there cannot exist any separating curve properly contained in either Y or $S \setminus Y$.

Nonetheless, while the Masur-Schleimer axioms fail due to the failure of axiom [R2], the Masur-Schleimer proof that a combinatorial complex satisfying the axioms is δ-hyperbolic carries through in the case of $\mathcal{C}_{sep}(S_{2,0})$. Specifically, the Masur-Schleimer proof has two distinct parts. First they show that a combinatorial complex satisfying their axioms satisfies a quasi-distance formula, and then they show that the complex is δ-hyperbolic. Moreover, the replacement axiom [R2] is only used in the first step of the Masur-Schleimer argument, namely the proof of the existence of a quasi-distance formula. However, replacement is not used in the second step which uses the quasi-distance formula to obtain hyperbolicity. Accordingly, since we have provided an independent proof of a quasi-distance formula for $\mathcal{C}_{sep}(S_{2,0})$, the δ-hyperbolicity of $\mathcal{C}_{sep}(S_{2,0})$ follows from the second part of the Masur-Schleimer argument. In fact, the idea that a complex satisfying a quasi-distance formula and has no holes is δ-hyperbolic is in fact implicit in [3] where such methods are used to prove the hyperbolicity of various low complexity marking and pants complexes.

6.4 Separating pants decompositions in the pants complex

As noted, the large scale geometry of Teichmüller space has been an object of interest in recent years, and in this context, the pants complex, $\mathcal{P}(S)$, becomes relevant, as by Theorem 2.1.6 of Brock $\mathcal{P}(S)$ is quasi-isometric to the Teichmüller space. Accordingly, in order to study large scale geometric properties of Teichmüller space, it suffices to study the pants complex of a surface. One feature of the coarse geometry of the pants complex in common to many analyses of the subject is the existence of natural quasi-isometrically embedded product regions in the thin part of Teichmüller space. These product regions, which are obstructions to δ-hyperbolicity, correspond to pants decompositions of the surface containing a fixed nontrivially separating (multi)curve. In fact, often in the course of studying the coarse geometry of the pants complex it proves advantageous to pass to the net of pants decompositions that contain a nontrivially separating curve. See for instance [21, 6] in which such methods are used to prove that the certain pants complexes are relatively hyperbolic or thick, respectively. Similarly, work of [49], uses similar methods to prove the pants complex is one ended.

In this section, we study the net of pants decompositions of a surface that contain a non-

8	0	2	2	3	4	4	4	5	5
7	0	2	2	3	4	4	4	5	6
6	1	2	3	4	4	4	5	5	6
5		2	3	4	4	4	5	5	6
4		2	3	4	4	4	5	5	6
3		2	3	3	4	4	4	5	5
2			2	3	3	3	4	5	5
1			1	2	3	3	4	4	5
0			1	2	3	3	4	4	5
$n \uparrow \; g \rightarrow$	0	1	2	3	4	5	6	7	8

Table 3: The maximum distance in the pants complex of any pants decomposition to a pants decomposition containing a nontrivial separating curve for some low complexity surfaces.

trivially separating curve within the entire pants complex. Specifically, for all surfaces of finite type we approximate the maximum distance in the pants complex of any pants decomposition to a pants decomposition containing a nontrivially separating curve, thereby proving the following theorem:

Theorem 6.4.1. *Let* $S = S_{g,n}$ *and set* $D_{g,n} = \max_{P \in \mathcal{P}(S)}(d_{\mathcal{P}(S)}(P, \mathcal{P}_{sep}(S)))$. *Then, for any fixed number of boundary components (or punctures)* n, $D_{g,n}$ *grows asymptotically like the function* $\log(g)$, *that is* $D_{g,n} = \Theta(\log(g))$. *On the other hand, for any fixed genus* $g \geq 2$, $\forall n \geq 6g-5$, $D_{g,n} = 2$.

Table 3 computes $D_{g,n}$ for some low complexity examples.

There is a sharp contrast between the nets provided by the subcomplexes $\mathcal{C}_{sep}(S) \subset \mathcal{C}(S)$ and $\mathcal{P}_{sep}(S) \subset \mathcal{P}(S)$. It is easy to see that $N_1(\mathcal{C}_{sep}(S)) = \mathcal{C}(S)$. On the other hand, Theorem 6.4.1 says that the maximal distance from an arbitrary pants decomposition to any pants decompositions containing a nontrivial separating curve is a nontrivial function depending on the topology of the surface. The lower bounds in Theorem 6.4.1 follow from an original and explicit constructive algorithm for an infinite family of high girth at most cubic graphs with the following expander like property, namely the minimum cardinality of connected cut-sets is a logarithmic function with respect to the vertex size of the graphs. This family of graphs may be of independent interest.

The following lemma used in the course of proving the lower bounds in Theorem 6.4.1 may also be of independent interest. Its proof brings together ideas related to the topology of the surfaces and graph theory in a simple yet elegant manner.

Lemma 6.4.7. *For $P \in \mathcal{P}(S)$ and $\Gamma(P)$ its pants decomposition graph, let d be the cardinality of a minimal nontrivial connected cut-set $C \subset \Gamma(P)$. Then*

$$d_{\mathcal{P}(S)}(P, P') \geq \min\{\mathit{girth}(\Gamma(P)), d\} - 1,$$

for P' any pants decomposition containing a separating curve cutting off genus.

The results of this section have some overlap with recent results in [25, 59]. Nonetheless, the results presented are in fact distinct from the aforementioned articles. Specifically, due to the fact that the quasi-isometry constants of Theorem 2.1.6 between the pants complex and Teichmüller space equipped with the Weil-Petersson metric are dependent on the topology of the surface, the results of this section are more properly related to complex of cubic graphs than to Moduli Space. Conversely, while methods in [25] do contain lower bounds on the diameter of entire complex of cubic graphs, this section focuses on the finer question of the density of a natural subset inside the entire space. On the other hand, while methods in [59] provide an independent and alternative (albeit nonconstructive) proof of the lower bounds achieved in Subection 6.4.4 of this section by considering pants decompositions whose pants decomposition graphs are expanders. The explicit and constructive nature of the family of graphs in Subsection 6.4.4 is a novelty of this section.

The outline of the section is as follows. In Subsection 6.4.1 we review relevant background concepts. In Subsection 6.4.2 we introduce a pants decomposition graph. In Subsection 6.4.3

we prove Theorem 6.4.1 modulo a construction of an infinite family of high girth, log length connected, at most cubic graphs, which is explicitly described in Subsection 6.4.4.

6.4.1 Preliminaries

6.4.1.1 Graph Theory

Let $\Gamma = \Gamma(V, E)$ be an undirected graph with vertex set V and edge set E. The *degree of a vertex* $v \in V$, is the number of times that the vertex v arises as an endpoint in E. The *degree of a graph* is the maximal degree over all vertices. Γ is called *at most cubic* if the degree of Γ is at most three, and *cubic* if every vertex has degree exactly three. A simple closed path in a graph is called a *cycle*. A cycle of length one is a *loop*. The *girth* of a graph Γ is defined to be the length of a shortest cycle in Γ, unless Γ is acyclic, in which case the girth is infinity.

Given a graph, $\Gamma(V, E)$ for any subset $S \subset V(\Gamma)$, the *complete subgraph of* S *in* Γ, denoted $\Gamma[S]$, is the subgraph of Γ with vertex set S and edges between any pair of vertices $x, y \in S$ if and only if there is a corresponding edge $e \in E(\Gamma)$. A graph Γ is said to be *connected* if there is a path between any two vertices of the graph, and disconnected otherwise. If a subset of vertices, $C \subset V$, has the property that the *deletion subgraph*, $\Gamma[V \setminus C]$, is disconnected, then C is called a *cut-set* of a graph. If the deletion subgraph $\Gamma[V \setminus C]$, is disconnected and moreover it has at least two connected components each consisting of at least two vertices or a single vertex with a loop, C is said to be a *nontrivial cut-set*. A (nontrivial) [connected] cut-set C is called a *minimal sized (nontrivial) [connected] cut-set* if $|C|$ is minimal over all (nontrivial) [connected] cut-sets of Γ.

We will be interested in families of graphs that are robust with regard to nontrivial disconnection by the removal of connected cut-sets. More formally, we define an infinite family of graphs, $\Gamma_i(V_i, E_i)$, with increasing vertex size to be *log length connected* if they have the property that the size of minimal nontrivial connected cut-sets of the graphs, asymptotically grows logarithmically in the vertex size of the graphs. Specifically, if we set the function $f(i)$ to be equal to the cardinality of a minimal nontrivial connected cut-set of the graph Γ_i, then

$f(i) = \Theta(\log(|V_i|))$. The robust connectivity property of log length connected graphs is quite different than the connectivity property enjoyed by expander graphs. Informally, a family of graphs are expanders if the graphs are strongly connected in the sense the deletion of small number of arbitrary vertices will not separate the graph. On the other hand, a family of graphs are log length connected if the graphs are strongly connected in the sense that the deletion of a small locally connected subgraph will not separate the graph. This seems to be a novel type of connectivity property for graphs and may be of independent interest.

6.4.1.2 Curves and Pants

Among simple closed curves on a surface of finite type we differentiate between two types of curves. Specifically, a simple closed curve $\gamma \subset S$ is called a *nontrivially separating curve*, or simply a *separating curve*, if $S \setminus \gamma$ consists of two connected components Y_1 and Y_2 such that $\xi(Y_i) \geq 1$. Any other simple closed curve is *nonseparating*. It should be stressed that a *trivially separating curve*, that is a simple closed curve that cuts off two boundary components of the surface for our purposes is not considered a separating curve. While counterintuitive, this point of view is in fact quite natural in the context of Teichmüller space. Restricting $\mathcal{C}(S)$ to the set of separating curves one obtains the *complex of separating curves*, $\mathcal{C}_{sep}(S)$. Similarly, restricting $\mathcal{P}(S)$ to the set of pants decompositions containing a separating curve we have the *pants complex of separating curves*, $\mathcal{P}_{sep}(S)$. This section analyzes the net of $\mathcal{P}_{sep}(S)$ in $\mathcal{P}(S)$.

6.4.2 Pants Decomposition Graph

By topological considerations, for $P \in P(S_{g,n})$, $|P| = \xi(S) = 3g - 3 + n$, while the number of connected components, or "pairs of pants," in the complement $S \setminus P$ is equal to $|\chi(S)| = 2(g-1)+n$. Given $P \in \mathcal{P}(S)$ we define its *pants decomposition graph*, $\Gamma(P)$, as follows: $\Gamma(P)$ is a graph with vertices corresponding the connected components of $S \setminus P$, and edges between vertices corresponding to connected components that share a common boundary curve. See Figure 24 for an example.

Remark 6.4.2. The notion of pants decomposition graphs is considered in [24] as well as in

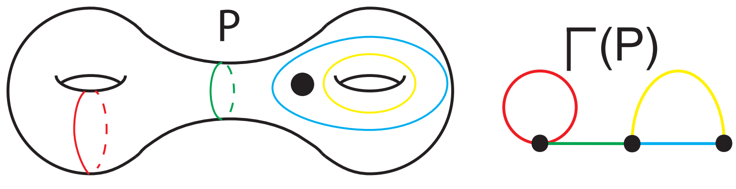

Figure 24: $\Gamma(P)$ for $P \in P(S_{2,1})$.

[56]. Moreover, replacing the vertices in $\Gamma(P)$ with edges and vice versa yields the *adjacency graph* in [9].

The following self evident lemma organizes elementary properties of $\Gamma(P)$ and gives a one to one correspondence between certain graphs and pants decomposition graphs:

Lemma 6.4.3. *For $P \in \mathcal{P}(S_{g,n})$, and $\Gamma(P)$ its pants decomposition graph:*

1. *$\Gamma(P)$ is a connected graph with $2(g-1)+n$ vertices and $3(g-1)+n$ edges. In particular, $\pi_1(\Gamma(P))$ is the free group of rank g.*

2. *$\Gamma(P)$ is at most cubic*

Moreover, for all $q, p \in \mathbb{N}$, given any connected, at most cubic graph $\Gamma = \Gamma(V, E)$ with $|V| = 2(p-1)+q$ and $|E| = 3(p-1)+q$, there exists a pants decomposition $P \in \mathcal{P}(S_{p,q})$ with $\Gamma(P) \cong \Gamma$.

6.4.2.1 Elementary moves and pants decomposition graphs.

Recall the two types of elementary moves:

E1 Inside a $S_{1,1}$ component of the surface in the complement of all of the pants curves except α, the curve α is replaced with β where α and β intersect once.

E2 Inside a $S_{0,4}$ component of the surface in the complement of all of the pants curves except α, the curve α is replaced with β where α and β intersect twice.

Elementary move E1 has a trivial action on $\Gamma(P)$, while the impact of the elementary move E2 can be described as follows: identify any two adjacent vertices, v_1, v_2 in the pants decomposition graph connected by an edge e, then the action of an elementary move E2 on the pants decomposition graph has the effect of interchanging any edge other than e impacting v_1, or possibly the empty set, with any edge other than e, impacting v_2, or possibly the empty set. The one stipulation is that in the event that the empty set is being interchanged with an edge, the result of the action must yield a connected at most cubic graph.

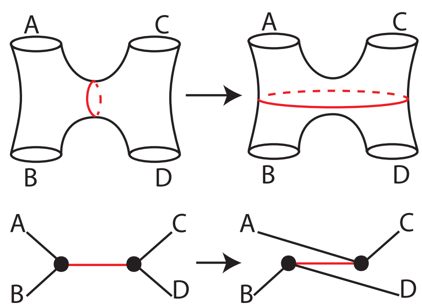

Figure 25: An example of an elementary pants move action on $\Gamma(P)$

6.4.2.2 Adding boundary components

Any pants decomposition of $S_{g,n+1}$ can be obtained by beginning with a suitable pants decomposition of $S_{g,n}$, adding a boundary component appropriately, and then appropriately completing the resulting multicurve into a pants decomposition of $S_{g,n+1}$. The effect that this process of adding a boundary component has on the pants decomposition graph has two forms as depicted in Figure 26.

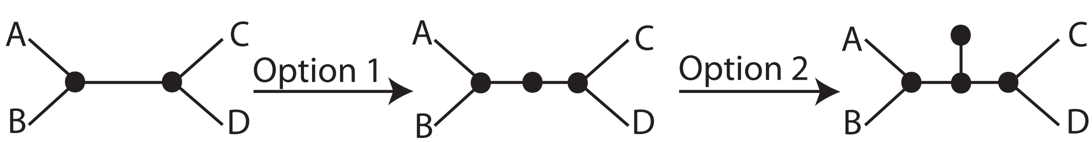

Figure 26: Adding a boundary component to a pants decomposition graph.

6.4.2.3 Separating curves and pants decomposition graphs.

Notice that a curve in a pants decomposition is a separating curve if and only if the effect of removing the corresponding edge in $\Gamma(P)$ nontrivially separates the graph into two connected components. Recall that a nontrivial separation of a graph is a separation such that there are at least two connected components each consisting of at least two vertices or a single vertex and a loop.

We differentiate two categories of separating curves, separating curves that *cut off genus* and separating curves that *cut off boundary components*. By the former, we refer to separating curves whose removal separates that surface into two nontrivial essential subsurfaces each with genus at least one. By the latter, we refer to separating curves whose removal separates that surface into two nontrivial essential subsurfaces at least one of which is a topological sphere with boundary components. Equivalently, a separating curve cuts off genus if the removal of the edge in $\Gamma(P)$ corresponding to the curve disconnects the graph into two cyclic components, otherwise if at least one of the connected components is acyclic, then the separating curve cuts off boundary components. Separating curves that cut off genus only exist on surfaces $S_{g,n}$ with $g \geq 2$, while separating curves that cut off boundary components only exist on surfaces with $n \geq 3$.

6.4.3 Proof of Theorem 6.4.1

In this section we will prove the following theorem which in particular implies Theorem 6.4.1. The proof will from directly from the combination of the Lemmas and Corollaries. To simplify the exposition we will first deal with the case of closed surfaces, and then we will explain how boundary components affect the arguments.

Theorem 6.4.4. *Let* $S = S_{g,n}$ *and set* $D_{g,n} = \max_{P \in \mathcal{P}(S)}(d_{\mathcal{P}(S)}(P, \mathcal{P}_{sep}(S)))$. *Then, for any fixed number of boundary components (or punctures)* n*,* $D_{g,n}$ *grows asymptotically like the function* $\log(g)$. *that is* $D_{g,n} = \Theta(\log(g))$. *In particular, for closed surfaces of sufficiently large* *genus,*

$$\frac{\log_2(2g+2)}{2} - 2 \leq D_{g,0} \leq \lfloor 2\log_2(g-1)+1 \rfloor$$

On the other hand, for any fixed genus $g \geq 2,\ \forall n \geq 6g-5,\ D_{g,n} = 2.$

Note 6.4.5. It is not hard to see by direct consideration that $D_{0,6} = 1$. More generally, for $n \geq 7$, $D_{0,n} = 0$, and $\forall n \geq 3$, $D_{1,n} = 2$. The exact terms in the upper and lower bounds on $D_{g,0}$ while necessary for the technical details in the proofs are not believed to be sharp.

6.4.3.1 Upper bounds for closed surfaces using girth

Lemma 6.4.6. *For* $P \in \mathcal{P}(S)$ *and* $\Gamma(P)$ *its pants decomposition graph,*

$$d_{\mathcal{P}}(P, \mathcal{P}_{sep}) \leq \textit{girth}(\Gamma(P)) - 1.$$

In particular, $D_{g,0} \leq \lfloor 2\log_2(g-1)+1 \rfloor$.

Proof. By valence considerations, a loop in $\Gamma(P)$ implies P contains a separating curve. Hence, to prove the first statement it suffices to show that given any cycle of length $n \geq 2$, there exists an elementary move decreasing the length of the cycle by one. See Figure 27.

Regarding the second statement, it is known that a girth h cubic graph must have at least $2^{h/2}$ vertices, [14]. It follows that any cubic graph Γ with $2(g-1)$ vertices, has girth$(\Gamma) \leq 2\log_2(g-1)+2\rfloor$. The second statement now follows from the first one. □

6.4.3.2 Lower bounds for closed surfaces

Recall that a separating curve $\gamma \in \mathcal{C}_{sep}(S)$ is said to *cut off genus* if $S \setminus \gamma$ consists of two connected essential subsurfaces neither of which is topologically a sphere with boundary components. Also recall that for a graph $\Gamma(V,E)$, a subset $C \subset V$ is called a *nontrivial connected cut-set* of Γ if $\Gamma[C]$ is a connected graph and $\Gamma[V \setminus C]$ has at least two connected components each consisting of at least two vertices or a vertex and a loop. The following lemma gives a lower bound on the distance of a pants decomposition to a pants decomposition which cuts off genus.

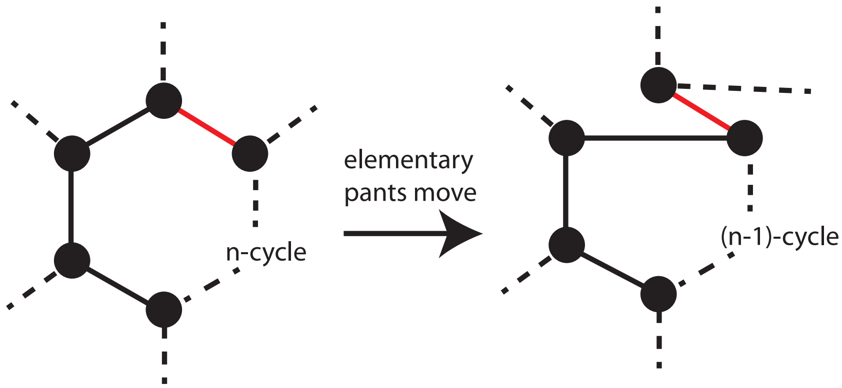

Figure 27: Elementary pants move decreasing the length of a cycle in Γ.

Lemma 6.4.7. *For $P \in \mathcal{P}(S)$ and $\Gamma(P)$ its pants decomposition graph, let d be the cardinality of a minimal nontrivial connected cut-set $C \subset \Gamma(P)$. Then*

$$d_{\mathcal{P}(S)}(P, P') \geq \min\{\textit{girth}(\Gamma(P)) - 1, d - 1\}$$

for P' any pants decomposition containing a separating curve cutting off genus.

Proof. Let γ be any curve in the pants decomposition P, and let α be any separating curve of the surface S that cuts off genus. It suffices to show that the number of elementary moves needed to take the curve γ to α is at least $\min\{\text{girth}(\Gamma(P)) - 1, d - 1\}$. In fact, considering the effect of an elementary move, it suffices to show that α nontrivially intersects at least $\min\{\text{girth}(\Gamma(P)), d\}$ different connected components of $S \setminus P$.

Corresponding to α consider the subgraph $[\alpha] \subset \Gamma(P)$ consisting of all vertices in $\Gamma(P)$ corresponding to connected components of $S \setminus P$ nontrivially intersected by α, as well as all edges in $\Gamma(P)$ corresponding to curves of the pants decomposition P nontrivially intersected by α. By construction, the subgraph $[\alpha]$ is connected. Note that the subgraph $[\alpha]$ need not be equal to the induced subgraph $\Gamma[\alpha]$, but may be a proper subgraph of it. Nonetheless, $V(\Gamma[\alpha]) = V([\alpha])$. (See Figure 28 for an example of a subgraph [a] $\subset \Gamma(P)$.)

As noted, it suffices to show $|V(\Gamma[\alpha])| \geq \min\{\text{girth}(\Gamma(P)),\ d\}$. Assuming not, by the girth condition it follows that $\Gamma[\alpha]$ is acyclic. However, this implies that α is entirely contained in a union of connected components of $S \setminus P$ such that in the ambient surface S, the connected components glue together to yield a subsurface Y, which is topologically a sphere with boundary components. Moreover, by the cardinality of the minimal nontrivial connected cut-set condition, it follows that the removal of the subsurface Y, or any subsurface thereof, from the ambient surface S does not, nontrivially separate S. In particular, for all $U \subset Y$, $S \setminus U$ consists of a disjoint union of at most one nontrivial subsurface as well as some number of pairs of pants. It follows that α cannot be a separating curve cutting off genus. □

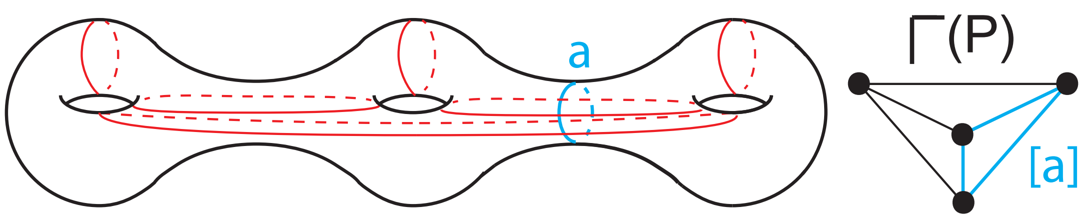

Figure 28: An example of a subgraph $[a] \subset \Gamma(P)$ corresponding to a separating curve $a \subset S_{3,0}$, cutting off genus. In this example, the girth of $\Gamma(P)$ is three and there are no nontrivial connected cut-sets of $\Gamma(P)$. By Lemma 6.4.7, the distance from P to any pants decomposition with a separating curve cutting off genus is at least (and in fact exactly) two.

In Subsection 6.4.4 for any even number $2m$, such that h is the largest integer satisfying $\left(\lceil\frac{2^h-4}{h-4}\rceil\right)\cdot h \leq 2m$, we construct a graph, Γ_{2m}, such that $|V(\Gamma_{2m})| = 2m$, girth$(\Gamma_{2m}) = h$, and any connected cut-set of the graph contains at least $\lfloor\frac{h}{2}\rfloor$ vertices. By Lemma 6.4.7, the pants decomposition corresponding to Γ_{2m} is distance at least $\frac{h}{2} - 2$ from a pants decomposition containing a separating curve. Because the pants decomposition graph Γ_{2m} corresponds to a pants decomposition of a closed surfaces of genus $m-1$, it follows that $\frac{h}{2} - 2 < D_{m-1,0}$. Since for large enough values of h,

$$2m < \left(\lceil\frac{2^{h+1}-4}{h-3}\rceil\right)\cdot(h+1) < 2^{h+2},$$

after algebraic manipulation one obtains $\frac{\log_2(2(m-1)+2)}{2} - 2 < \frac{h}{2} - 2$. In conjunction with Lemma 6.4.6 we have proven the following:

Corollary 6.4.8. *For large enough values of g, we have the bounds on $D_{g,0}$ recorded in Theorem 6.4.4. In particular, $D_{g,0} = \Theta(\log(g))$*

6.4.3.3 Adding boundary components

In this section we modify the previously described arguments to allow for the case that our surface S has boundary components. We begin with a lemma describing a local situation in $\Gamma(P)$ ensuring that a pants decomposition is close to a pants decomposition containing a separating curve.

Lemma 6.4.9. *For $P \in \mathcal{P}(S)$ and $\Gamma(P)$ its pants decomposition graph. If $\Gamma(P)$ has three consecutive vertices of degree at most two, then $d_{\mathcal{P}}(P, \mathcal{P}_{sep}) \leq 2$.*

Proof. See Figure 29. □

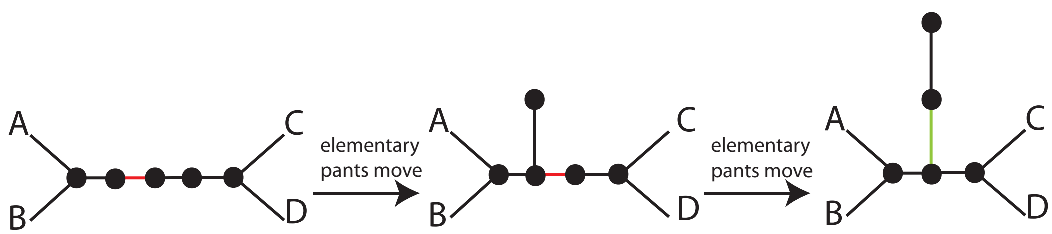

Figure 29: Two elementary moves creating a separating curve that cuts off boundary components in Γ beginning from a pants decomposition graph with three consecutive valence at most two vertices.

Using Lemma 6.4.9, presently we generalize Lemma 6.4.6 to surfaces with boundary.

Corollary 6.4.10. $\forall g \geq 2,\ D_{g,n} \leq \lfloor 2\log_2(g-1)+3 \rfloor$.

Proof. Recall that in Lemma 6.4.6 we obtained an upper bound of $\lfloor 2\log_2(g-1)+1 \rfloor$ on $D_{g,0}$. Specifically, this upper bound was obtained by taking the smallest cycle C in the graph $\Gamma(P)$ which had length at most $\lfloor 2\log_2(g-1)+2 \rfloor$ and then successively decreasing the length of cycle C by elementary pants moves as in the proof of Lemma 6.4.6. Consider what can happen to this cyclic subgraph C as we add boundary components to our surface as in subsection 6.4.2.2. If the added boundary components do not affect the length of cycle C, the upper bound is unaffected. On the other hand, if the added boundary components increase the length of the cycle C by adding one (two) degree two vertex (vertices) to the cycle C, then the distance to a separating curve increases by at most one (two). However, once at least three degree two vertices have been added to the cycle C, instead of reducing the cycle to a loop, we can use elementary moves to gather together three consecutive vertices of degree two and then create a separating curve locally, as in Lemma 6.4.9. The statement of the corollary follows. □

Again using Lemma 6.4.9 we have the following corollary, also proving a special case of Theorem 6.4.4.

Corollary 6.4.11. *For all* $g \geq 2, n \geq 6g-5 \implies D_{g,n} = 2$.

Proof. By Lemma 6.4.3 for $P \in \mathcal{P}(S_{g,n})$, $\Gamma(P)$ is a connected at most cubic graph with $2(g-1)+n$ vertices and $3(g-1)+n$ edges. Since $n \geq 6g-5$, by pigeon hole considerations it follows that $\Gamma(P)$ has three consecutive vertices of degree at most two. By Lemma 6.4.9, $D_{g,n} \leq 2$. Then to see that $D_{g,n} = 2$ it suffices to explicitly exhibit connected at most cubic graphs with $2(g-1)+n$ vertices and $3(g-1)+n$ edges for all $g \geq 2, n \geq 6g-5$ such that the graphs neither contain nontrivial cut edges nor are one elementary move away from a graph with a nontrivial cut edge. See Figure 30 for an explicit construction of such a family of graphs. □

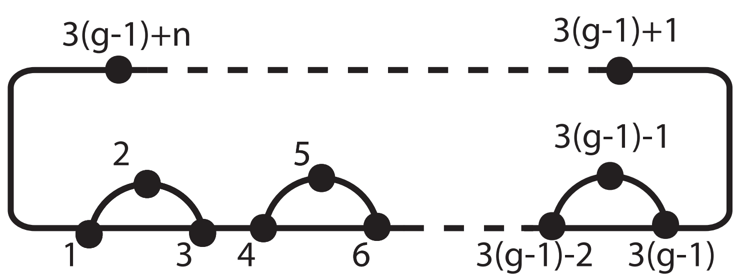

Figure 30: Pants decompositions graphs of pants decompositions which are distance two from a pants decomposition containing a separating curve.

Generalizing the aforementioned family of cubic graphs, Γ_{2m}, in Subsection 6.4.4 we show that for any fixed $n \in \mathbb{N}$ we can add n boundary components to the our graphs, Γ_{2m}, creating a family of pants decomposition graphs Γ_{2m}^n, whose corresponding pants decompositions have girth, minimum nontrivial cut-set size, and distance between valence less than three vertices growing logarithmically in the vertex size of the graph. By Lemma 6.4.7, the fact that girth and minimum nontrivial connected cut-set size grow logarithmically in the vertex size of the graph implies that the distance between pants decompositions corresponding to the constructed graphs to any pants decompositions containing a separating curve cutting off genus grows logarithmically in the vertex size of the graph. Moreover, the fact that the distance between valence less than three vertices grows logarithmically in the vertex size of the graphs, implies that the distance between pants decompositions corresponding to the constructed graphs to any pants decompositions containing a separating curve cutting off boundary components also grows logarithmically in the vertex size of the graphs. As a corollary, we have:

Corollary 6.4.12. *For any fixed* $n \in \mathbb{N}$, $D_{g,n} = \Theta(\log(g))$.

6.4.4 Construction of Large Girth, Log Length Connected Graphs

We first describe a construction for a family, Γ_h, of cubic girth $h \geq 5$ graphs with

$$\left(\lceil \frac{2^h - 4}{h - 4} \rceil\right) \cdot h + \{0, 1\}$$

vertices (where the final term is simply to ensure the total number of vertices is even), which have the property that any connected cut-set of Γ_h contains at least $\lfloor \frac{h}{2} \rfloor$ vertices. Afterward, we generalize our construction, interpolating between the family of graphs Γ_h. Specifically, for all $m \in \mathbb{N}$, such that $h \geq 5$ is the largest integer satisfying $2m \geq \left(\lceil \frac{2^h-4}{h-4} \rceil\right) \cdot h$, there exists a cubic girth h graph Γ_{2m} with $2m$ vertices and the property that any connected cut-set of the graph contains at least $\lfloor \frac{h}{2} \rfloor$ vertices. Finally, we demonstrate that for any fixed number of boundary components n, we can add n boundary components to our graphs Γ_{2m} yielding a family of graphs Γ_{2m}^n with the same desired properties.

6.4.4.1 Construction of Γ_h

Begin with $\left(\lceil\frac{2^h-4}{h-4}\rceil\right)$ disjoint cycles each of length h (possibly one of length $h+1$ if necessary to make the total number of vertices even). Then, chain together these disjoint cycles into an at most cubic connected tower T_h, connecting each cycle to its neighboring cycle(s) by adding two edges between pairs of vertices, one from each cycle, such that each of the two vertices from the same cycle, to which edges are being attached, are of distance at least $\lfloor\frac{h}{2}\rfloor$.

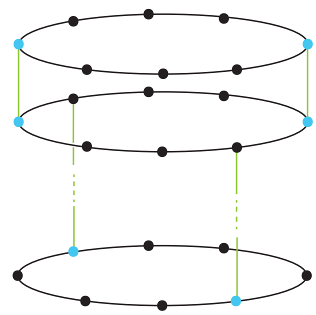

Figure 31: T_8, an at most cubic girth eight tower graph.

By construction, the tower graphs, T_h, have the following properties:

T1 T_h has $\left(\lceil\frac{2^h-4}{h-4}\rceil\right)\cdot h+\{0,1\}$ vertices.

T2 T_h is an at most 3-regular and at least 2-regular graph with girth h.

T3 If we denote the subset of vertices of T_h of valence two by $V_2^{T_h}$, then $|V_2^{T_h}|\geq 2^h$.

T4 Any connected cut-set of T_h has at least $\lfloor\frac{h}{2}\rfloor$ vertices.

6.4.4.2 Algorithm completing T_h to a 3-regular graph Γ_h

Motivated by ideas in [14], presently we describe a constructive algorithm to add edges to the tower T_h completing it to a 3-regular graph $\Gamma=\Gamma_h$ which also has girth h, and retains the property that any connected cut-set of Γ_h has at least $\lfloor\frac{h}{2}\rfloor$ vertices. By abuse of notation, we will always refer to the graph that has been constructed up to the current point as Γ. In terms of ensuring the girth condition, the main observation being used implicitly throughout is that removing edges from a graph never decreases girth, while adding an edge connecting vertices which were previously at least distance $h-1$ apart, in a girth at least h graph, yields a girth at least h graph.

Step One An Easy Opportunity to Add an Edge

If Γ is 3-regular, we're done. If not, fix a vertex $v \in V_2^{T_h}$ of valence two. If there exists another vertex $x \in V_2^{T_h}$ with $d_\Gamma(v, x) \geq h-1$, add an edge between x and v.

Step Two Exhaust Easy Opportunities

Iterate step one until all possibilities to add edges to Γ are exhausted.

Step Three One Step Backward, Two Steps Forward

If Γ is 3-regular, we're done. If not, since the total number of vertices is even, there must exist at least two vertices, x and y, of valence two. Consider the sets $U = N_{h-2}^\Gamma(x) \cup N_{h-2}^\Gamma(y)$ and $I = N_{h-2}^\Gamma(x) \cap N_{h-2}^\Gamma(y)$. Due to the valence considerations, since x, y are valence TWO vertices in an at most cubic graph it follows that $|N_{h-2}^\Gamma(x)| \leq 1+2+\ldots+2^{h-2} = 2^{h-1}-1$, and similarly for $N_{h-2}^\Gamma(y)$. Note that $|U| = |N_{h-2}^\Gamma(x)| + |N_{h-2}^\Gamma(y)| - |I| \leq 2^h - 2 - |I|$. Then consider the set $W = V_2^{T_h} \setminus U$. Since $|V_2^{T_h}| \geq 2^h$, it follows that $|W| \geq 2 + |I|$. In particular, the set W is nonempty. Furthermore, considering that step two was completed to exhaustion, it follows that $\forall w \in W$, w is of valence three in Γ. Moreover, by definition, the vertex w is of valence two in T_h. Denote the vertex that is connected to w in Γ but not in T_h by w'. Perforce, w' is distance at least $h-2$ from both x and y. In fact, we can assume that w' is not exactly distance $h-2$ from both x and y because $|W| > |I|$. For concreteness, we can assume that $d_\Gamma(x, w') \geq h-1$.

Remove from Γ the edge e connecting w to w', and in its place include two edges: e_1 between x and w', and e_2 between w and y. Adding the two edges e_1 and e_2 does not decrease girth to less than h as they each connect vertices that were distance at least $h-1$ apart: After removing e, the vertices w and w' are distance at least $h-1$ because Γ was girth at least h. Hence, even after adding edge e_1 we can still be sure that the vertices

y and w remain distance at least $h-1$ apart, thereby allowing us to add edge e_2 without decreasing girth to less than h.

Step Four Repeat

If Γ is not yet 3-regular, return to step three.

The algorithm terminates as if the graph is not yet 3-regular Step three can be performed, and net effect of Step three increases the number of edges in the at most 3-regular graph. By construction the graph Γ_h has girth h. Moreover, as that the algorithm never removes edges from the tower T_h, and hence the resulting graph Γ_h includes the tower T_h as a subgraph. Using the girth condition in conjunction with the fact that any connected cut-set of Γ_h is a cut-set of T_h, it is not hard to see that any connected cut-set of Γ_h has at least $\lfloor \frac{h}{2} \rfloor$ vertices.

6.4.4.3 Construction of Γ_{2m}

For any even number of vertices $2m$ such that $2m \geq |V(\Gamma_h)|$, for some h, we can construct a 3-regular girth h graph on $2m$ vertices, which we denote Γ_{2m}, with the property that any connected cut-set of Γ_{2m} contains at least $\lfloor \frac{h}{2} \rfloor$ vertices. In fact, we can construct Γ_{2m} using the exact same process as in the construction of Γ_h with the exception that we now start with $\lfloor \frac{2m}{h} \rfloor$ cycles of length h, and $(h+1)$ as necessary, in the building our initial tower which is subsequently completed to a cubic graph.

6.4.4.4 Adding a fixed number n of boundary components to Γ_{2m}

For any fixed number $n \in \mathbb{N}$, we can add n boundary components to the graphs Γ_{2m}, to obtain graphs Γ^n_{2m}. Moreover, we can ensure that no two added boundary components are within distance $\lfloor \frac{h}{2} \rfloor$ from each other, past some minimal threshold for $2m$. This is because for x, an added boundary component in Γ_{2m}, $|N_{\lfloor \frac{h}{2} \rfloor}(x)| \leq 2^{\lfloor \frac{h}{2} \rfloor + 1}$, while $|V(\Gamma_{2m})| \geq 2^h$. It follows that that for any fixed number of boundary components n, we have a family of graphs Γ^n_{2m} with girth, nontrivial minimum cut-set size, and the distance between valence less than three vertices all growing logarithmically in the vertex size of the graph.

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