# Genus Distributions of Graphs Constructed Through Amalgamations 

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Submitted in partial fulfillment of the requirements for the degree
of Doctor of Philosophy
in the Graduate School of Arts and Sciences

## COLUMBIA UNIVERSITY

## ABSTRACT

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Graphs are commonly represented as points in space connected by lines. The points in space are the vertices of the graph, and the lines joining them are the edges of the graph. A general definition of a graph is considered here, where multiple edges are allowed between two vertices and an edge is permitted to connect a vertex to itself. It is assumed that graphs are connected, i.e., any vertex in the graph is reachable from another distinct vertex either directly through an edge connecting them or by a path consisting of intermediate vertices and connecting edges. Under this visual representation, graphs can be drawn on various surfaces. The focus of my research is restricted to a class of surfaces that are characterized as compact connected orientable 2-manifolds. The drawings of graphs on surfaces that are of primary interest follow certain prescribed rules. These are called 2 -cellular graph embeddings, or simply embeddings.

A well-known closed formula makes it easy to enumerate the total number of 2-cellular embeddings for a given graph over all surfaces. A much harder task is to give a surface-wise breakdown of this number as a sequence of numbers that count the number of 2-cellular embeddings of a graph for each orientable surface. This sequence of numbers for a graph is known as the genus distribution of a graph. Prior research on genus distributions of graphs has primarily focused on making calculations of genus distributions for specific families of graphs. These families of graphs have often been contrived, and the methods used for finding their genus distributions have not been general enough to extend to other graph families. The research I have undertaken aims at developing and using a general method that frames the problem of calculating genus distributions of large graphs in terms of a partitioning of the genus distributions of smaller graphs. To this end, I use various operations such as edge-
amalgamation, self-edge-amalgamation, and vertex-amalgamation to construct large graphs out of smaller graphs, by coupling their vertices and edges together in certain consistent ways. This method assumes that the partitioned genus distribution of the smaller graphs is known or is easily calculable by computer, for instance, by using the famous HeffterEdmonds algorithm. As an outcome of the techniques used, I obtain general recurrences and closed-formulas that give genus distributions for infinitely many recursively specifiable graph families. I also give an easily understood method for finding non-trivial examples of distinct graphs having the same genus distribution. In addition to this, I describe an algorithm that computes the genus distributions for a family of graphs known as the $4^{-}$ regular outerplanar graphs.

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## Acknowledgments

I owe a depth of gratitude to my research advisor, Professor Jonathan L. Gross, without whose guidance this research would not be possible. I have benefited immensely from his counsel, encouragement and enthusiasm, and I would like to convey my sincerest thanks to him for his input at every stage of this work. I would especially like to thank him for a rich and a thoroughly enjoyable doctoral research experience and not least of all for his regard for my professional and general well-being.

This research was made possible with a grant from the Fulbright/HEC/USAID program. The efforts of the staff at the Institute of International Education, the US Educational Foundation for Pakistan and the Higher Education Commission of Pakistan are much appreciated and I regret not being able to individually thank the many anonymous faces at these institutions who have been striving to make higher education a reality for many aspiring students.

I am deeply grateful and indebted to Professor Tal G. Malkin for her overwhelming support over the years and for the great deal of her time and energy spent on my behalf.

I would also like to thank my undergraduate mentors Irfan ullah Chaudhary and Omar Khan for their extraordinary efforts and for shining a light through a dark tunnel.

I am immensely grateful to my family for their love, patience, unwavering support and prayers, especially my mother Tauseef Irfan, my father Irfan Rehman, and my parents-inlaw Mujahidda Khanum and Farid-uz-Zaman, all of whom I cannot appreciate enough for a million little things language cannot express. Last but not the least I want to thank Imran F. Khan, who has been an anchor of support as a spouse and truly a pleasure to work with as a co-researcher.

Dedicated to my grandmother Razia Irshad.

## Chapter 1

## Introduction

For over a century, drawings of graphs on various surfaces have been a source of many interesting questions that have captivated the imaginations of mathematicians. These drawings of graphs, called graph embeddings, have spurred interest not only in a characterization of the embeddings themselves but also in the properties of the underlying surfaces that contain those embeddings. The research herein falls in the former category and aims at finding genus distributions for many graph families. A genus distribution is an inventory that catalogues the number of embeddings of a graph into each surface.

While the study of genus distributions spans only a few decades, its origins can be attributed to the classical Heawood map-coloring problem that arose in the nineteenth century and was an open problem for the better part of the subsequent 78 years. The problem was eventually resolved in 1968 by Ringel and Youngs, and came to be known as the Heawood map-coloring theorem. The Heawood map-coloring theorem characterizes for each surface, other than the sphere, the minimum number of colors that are needed to color any possible map on that surface, such that mutually adjacent countries of the map are assigned distinct colors. The solution of the Heawood map-coloring problem hinged upon establishing a closed formula for a graph invariant known as the minimum genus, for the complete graph $K_{n}$, for all $n \geq 3$. In fact, a surface is uniquely characterized by a number known as the genus of that surface. The minimum genus of a graph, as the name suggests, refers to the surface of smallest genus in which the graph can be embedded. It was in this context that the problem of minimum genus gained impetus and created interest
in other graph invariants such as maximum genus, average genus, genus distribution, etc. The research that constitutes the core of this document involves determination of genus distributions of graphs that are produced from smaller graphs through specific operations.

A few fundamental concepts, including a formal definition of genus distribution, are reviewed here, followed by a description of the undertaken research. We refer the reader to Gross and Tucker, 2001, Bonnington and Little, 1995], White, 2001, and Mohar and Thomassen, 2001 for developing greater familiarity with the basics in topological graph theory, and we recommend Beineke et al., 2009 as a compendium of historical and recent trends in the area.

### 1.1 Preliminaries

## Graphs as Topological Spaces

Graphs are well understood as abstract combinatorial structures comprising a set of vertices and a set of edges, each of which is incident on one or two vertices. We use a general definition of graphs that permits multi-edges and self-loops. Since our primary concern is with drawing graphs on surfaces, it is useful to regard them as topological spaces, with the vertices of the graphs corresponding to points in the Euclidean space $\mathbb{R}^{3}$ and edges corresponding to curves that can be parameterized by the $[0,1]$ unit interval. Thus every edge, including self-loops, has two edge-ends. Under this parameterization, the parts of an edge $e$ that correspond to the intervals $[0, \epsilon)$ and $(1-\epsilon, 1]$, for $\epsilon \ll 1$, are known as the 0 -end and the 1 -end of the edge $e$, respectively. They are also known collectively as the edge-ends of the edge $e$. As long as we are consistent when considering a particular edge, it is not important which end we consider the 0 -end and which end the 1 -end. For visualization, we give artificial directions to the edges under which the 0 -end corresponds to the tail of the directed-edge and the 1 -end corresponds to its head. We denote the 0 -end of edge $e$ by $e^{+}$and the 1 -end by $e^{-}$.

## Surfaces

A 2-manifold is a subspace of $\mathbb{R}^{3}$ in which each point has a neighborhood homeomorphic to an open disc. A 2-manifold that cannot be represented as a disjoint union of other 2manifolds is said to be connected, and it is said to be compact if (i) there is a real number $N$ such that the distance between any point on it and a fixed point of origin is less than $N$ and (ii) the endpoints of an open arc lying in the surface also lie in the surface. We refer to a connected compact 2 -manifold as a surface.

The connected sum of two surfaces is an operation on the surfaces, where an open disc is excised from each of the two surfaces followed by gluing the surfaces together along their boundaries.

Surfaces are of two types: orientable and non-orientable. Orientable surfaces are those which do not contain a Möbius band as a subspace. These are completely classified, by a well-known result in topology, as the sphere $S_{0}$, the torus $S_{1}$, and the $k$-torus $S_{k}$, for integer $k \geq 2$, where $S_{k}$ is defined recursively by taking a connected sum of $S_{k-1}$ and $S_{1}$. Orientable surfaces of genus three or less are shown in Figure 1.1. The subscript $k$ in $S_{k}$ is known as the genus of the orientable surface $S_{k}$, for $k \geq 0$.


Figure 1.1: Orientable surfaces $S_{0}, S_{1}, S_{2}$ and $S_{3}$.

Amongst the non-orientable surfaces, the projective plane $N_{1}$ is the simplest surface; it is obtained by removing an open disc from a sphere and gluing it to a Möbius band along its boundary. Non-orientable surfaces are completely classified as $N_{k}$, for $k \geq 1$, where $N_{k}$ is the surface obtained by a connected sum of $k$ projective planes. These non-orientable surfaces are embeddable in 4 -space, but not in 3 -space. The subscript $k$ in $N_{k}$ is known as the crosscap number of the non-orientable surface $N_{k}$, for $k \geq 1$.

Other than surfaces, graph embeddings have been studied on an $n$-page book space, consisting of a collection of $n$ half-planes that share a common boundary, as well as on generalized pseudo-surfaces, which are surfaces containing finitely many points that do
not have neighborhoods homeomorphic to open discs. While embeddings on these other topological spaces have been a subject of prior investigations, we confine our focus only to embeddings on orientable surfaces.

## Embeddings and Rotation Systems

A crossing-free drawing of a graph $G$ on an oriented surface is referred to as an embedding. More formally, if $G$ is regarded as a topological space, then it is the $1-1$ function $\iota: G \rightarrow S$ that maps the graph $G$ to its image $\iota(G)$ on the surface $S$. Given an embedding $\iota: G \rightarrow S$, the connected components of $S-\iota(G)$ are known as the regions of that embedding. A region with its boundary is known as a face of the embedding. Embeddings that belong to the same equivalence class under an orientation-preserving homeomorphism of the underlying surface are regarded as equivalent.

A 2-cellular embedding is an embedding in which every region is homeomorphic to an open disc. The regions of a 2 -cellular embedding can be made non-cellular by attaching handles to them, that is, by excising one or more disjoint open discs from the interior of each region to be made non-cellular, followed by excising an equal number of disjoint open discs from another surface $S$ and subsequently gluing the boundaries of $S$ to the boundaries of the surface of embedding. The constraint of 2-cellularity on the regions of an embedding renders the class of 2-cellular embeddings finite, and consequently makes the notion of embedding graphs on surfaces more structured and meaningful. Moreover, a 2-cellular embedding is related to the genus of the underlying surface through an algebraic relation known as the Euler polyhedral formula. We assume, therefore, that all graph embeddings under discussion are 2-cellular embeddings unless indicated otherwise. This implies that graphs are connected, since embedding a disconnected graph on a surface would always induce one or more regions that are not homeomorphic to open discs.

Let $G=(V, E)$ be a connected graph with $V$ as the set of vertices and $E$ as the set of edges. Given a 2-cellular embedding $\iota: G \rightarrow S_{k}$ on an oriented surface, we refer to the subscript $k$ as the genus of the embedding. The Euler polyhedral formula for the orientable surfaces specifies that all embeddings of a graph into a fixed surface $S_{k}$ also have the same number of faces. It relates the genus $k$ of an oriented embedding to the cardinality
$|F|$ of its set of faces as:

$$
|V|-|E|+|F|=2-2 k
$$

An embedding on a sphere can also be regarded as a planar embedding under the Riemann projection of the sphere, with the point at infinity lying in the "outer" face. For this reason, it is not unusual to use the expression "planar embeddings" to indicate embeddings on the sphere.

A rotation at a vertex $v$ is the cyclic permutation of the edge-ends incident on it. A vertex of degree $d$ has $(d-1)$ ! rotations. A rotation-system is a set of rotations, one for each vertex. Thus, a graph that has vertices $v_{1}, \cdots, v_{n}$ having degrees $d_{1}, \cdots, d_{n}$ has $\prod_{i}\left(d_{i}-1\right)$ ! rotation systems. A well-known result pioneered by Heffter and rediscovered by Edmonds states that there is a $1-1$ correspondence between rotation systems and 2cellular graph embeddings. In fact, a rotation system is a combinatorial representation of an embedding. Figure 1.2 shows a toroidal embedding of the dipole graph $D_{3}$ to the left and the corresponding rotation system to the right, where the dipole $D_{n}$ consists of two vertices connected by $n$ multi-edges, for $n \geq 1$.


$$
\begin{aligned}
& v_{1} \cdot a^{+} b^{-} c^{-} \\
& v_{2} \cdot a^{-} b^{+} c^{+}
\end{aligned}
$$

Figure 1.2: Toroidal embedding of $D_{3}$ and corresponding rotation system.

## Genus Distribution

The genus distribution of a connected graph $G$ is a graph invariant. It is defined as a sequence of numbers $g_{0}(G), g_{1}(G), g_{2}(G), \cdots$, where $g_{k}(G)$ is the number of 2-cellular embeddings of the graph $G$ in the oriented surface $S_{k}$. It is usually encoded as a finite univariate polynomial:

$$
g[G](x)=g_{0}+g_{1} x+g_{2} x^{2}+\cdots
$$

For example, the dipole $D_{3}$ has a total of 2 embeddings on a sphere and 2 embeddings on a torus and, therefore, has genus distribution $2+2 x^{1}$. The finiteness of the genus distribution polynomial follows from the finiteness of the number of distinct embeddings of a graph.

### 1.2 Problem Statement

A survey of the research efforts invested into the problem of calculating the genus distributions of graphs reveals that much of that research has been geared towards graphs that are highly symmetrical. Typically, such research is carried out on families of graphs that have bounded degree. In a few cases, the graphs have an arbitrary degree, as is the case for bouquets Gross et al., 1989, dipoles Rieper, 1990, and the more recent result on generalized fan graphs Chen et al., 2011a, but there, too, the scope is usually limited to specific graph families. The methods that have been used in such instances are also exploitative of this symmetry and for this reason it is not practicable to extend the techniques used on one graph family to another graph family.

This thesis focuses on genus distributions of graphs built from smaller graphs using various kinds of amalgamations. It is possible to define amalgamation operations on graphs where a graph can be pasted to another graph on a vertex, an edge or even on subgraphs. Whereas amalgamating on arbitrary subgraphs is an ambitious future goal, work on genus distributions of graphs produced by amalgamating vertices and edges has met with considerable progress under the aegis of an umbrella project by Gross, Khan and the author. The research included here fits under this large project. A large portion of the research discussed in $\$ 2-\$ 4$ has also been published elsewhere (see Poshni et al., 2010, Poshni et al., 2011 and (Poshni et al., 2012]).

In this document, we describe a general method that frames the problem of calculating genus distribution of large graphs in terms of known partitioned genus distributions of smaller graphs. This method is employed to calculate the genus distributions of those infinite families of graphs that are obtained by iteratively amalgamating copies of smaller graphs, called base graphs, along their root-edges. It is presumed here that the partitioned genus distributions of the base graphs are known and that their root-edges have two 2 -valent
endpoints. I augment this general method to enable calculations of genus distributions for graphs produced by identifying together the two root-edges of the same graph. The afore-mentioned techniques involve analyzing face-boundary walks of graph embeddings and modeling the information collectively in what we refer to as partials and productions. These partials and productions are designed and adapted depending on the context, and they are used for deriving simultaneous recurrences or formulas for genus distributions. I demonstrate the power of these techniques by describing an easily understood method for generating examples of non-homeomorphic graphs having identical genus distributions.

In contrast to historical trends where genus distributions have been calculated for graph families that have mostly been artificially constructed and have not been a source of prior interest, outerplanar graphs have been of great interest to graph theorists working in other areas (see Brehaut, 1977], Syslo, 1979], Heath, 1986], and Bienstock and Dean, 1992]). I describe an $O\left(n^{2}\right)$-time algorithm for calculating the genus distribution of 4-regular outerplanar graphs. This is a significant improvement over the $O\left(6^{n}\right)$ time-complexity of the naive Heffter-Edmonds algorithm. The new algorithm breaks down a given instance of a 4 -regular outerplanar graph into an auxiliary graph with multiple components, and then applies amalgamations to those components, while finding the genus distribution of the original graph, with active use of the contextually developed notions of partials, productions and partitioned genus distribution.

Remark 1. It should be pointed out that Gross, Khan and the author are the first to conceptualize the useful notions of "partitioned genus distribution" and "production" in [Gross et al., 2010] in the context of vertex-amalgamation on 2-valent vertices.

### 1.3 Related Literature

Minimum genus and maximum genus of a graph are graph invariants that refer to the smallest and largest genus of an orientable surface on which the graph can be embedded 2-cellularly. Prior to the 1980's much of the work on oriented graph embeddings was focused on characterizing the minimum or the maximum genus of graphs (see Ringel, 1955, Battle et al., 1962, Ringel, 1965, and Ringel and Youngs, 1968 for some classical results on
minimum genus, and see Nordhaus et al., 1971, Xuong, 1979a, and Ringeisen, 1979 for results on maximum genus), or was concerned with the related problem of finding triangular or other highly symmetric embeddings of important or interesting graphs Goddyn et al., 2007, Grannell et al., 2002.

Youngs published an algorithm, now known as the Heffter-Edmonds algorithm, to find the minimum genus of a surface on which a graph can be embedded Youngs, 1963. The algorithm combinatorially generates all rotation systems and uses these to specify the faces of the corresponding graph embeddings. The number of faces obtained in this manner for each embedding can then be used with the Euler-polyhedral equation to obtain the genus of the embedding. Since the algorithm requires enumeration of all $\prod_{i}\left(d_{i}-1\right)$ ! rotation systems, its time-complexity is superexponential in the size of the graph. It is now known that the genus problem, i.e., the problem of deciding if the minimum genus of a given graph is bounded by a given integer, is NP-complete [Thomassen, 1989]. In fact, the genus problem is also NP-complete for cubic graphs Thomassen, 1997 as well as for apex graphs Mohar, 2001. Other works on minimum genus are of an enumerative nature that involve counting graph embeddings for interesting graphs in a minimum-genus surface. These include Bonnington et al., 2000], Grannell and Griggs, 2008], Goddyn et al., 2007, and Korzhik and Voss, 2002.

A better understood topological invariant for graphs is the maximum genus of a graph. An oft-cited result by Duke, now known as the interpolation theorem, establishes that all the numbers that fall in the interval between the minimum and the maximum genus of a graph, are valid genus values for the embeddings of that graph Duke, 1966. Duke's interpolation theorem generated interest in maximum genus. Pioneering work on the maximum genus came by Nordhaus, Stewart and White in Nordhaus et al., 1971, where they established many general results, including an easily provable upper bound of $\left\lfloor\frac{\beta(G)}{2}\right\rfloor$ on the maximum genus of a graph $G$ in terms of its Betti number $\beta(G)$. Graphs that achieve this upper bound are said to be upper-embeddable (see Jungerman, 1978, Xuong, 1979b, and Cai et al., 2010). In 1979, Xuong characterized maximum genus by giving a closed formula Xuong, 1979a. However, Xuong's formula used a graph invariant that required consideration of all spanning trees for its computation and is, therefore, not computationally feasible. In 1988,

Furst, Gross and McGeoch designed a polynomial-time algorithm for finding maximum genus Furst et al., 1988]. Maximum genus is now considered quite well-understood. Recent research has focused on obtaining lower bounds on maximum genus in terms of connectivity, Betti number, independence number, girth and minimum degree Skoviera, 1991, Huang and Liu, 2000, Huang and Zhao, 2005 and Ouyang et al., 2010.

In 1987, Gross and Furst laid out a program of research in which they introduced several new graph invariants for connected graphs Gross and Furst, 1987. Genus distribution is one such invariant. Other invariants introduced by them include average genus, region-size distribution and embedding distribution.

The average genus $\gamma_{a v g}(G)$ of a graph $G$ is the average value of the genus of $G$ over all of its oriented embeddings. For example, $\gamma_{\text {avg }}\left(D_{3}\right)=(2 \times 0+2 \times 1) / 4=1 / 2$. This invariant has received considerable attention Chen and Gross, 1992a, Chen and Gross, 1992b, Chen and Gross, 1993, Gross et al., 1993, Chen et al., 1995, Stahl, 1995a, Stahl, 1995b.

The region-size or face-size distribution is another invariant expressed as a fintite polynomial, $s[G](x)=f_{0}+f_{1} x+f_{2} x^{2}+\cdots$, where $f_{j}(G)$ denotes the number of $j$-sided faces of $G$ taken over all oriented embeddings. For example, $s\left[D_{3}\right](x)=6 x^{2}+2 x^{6}$.

The embedding distribution $i[G]\left(z_{j}\right)$ of a graph $G$ is a multi-variate polynomial. Each multi-variate monomial corresponds to a type of embedding of the graph $G$ and consists of factors of the form $z_{j}^{k}$, signifying that the corresponding embedding has $k$ regions that are $j$-sided. For example, $i\left[D_{3}\right]\left(z_{j}\right)=2 z_{6}^{1}+2 z_{2}^{3}$ is the embedding distribution of the dipole $D_{3}$. It indicates that $D_{3}$ has two embeddings with one 6 -sided face and two embeddings with three 2-sided faces.

Gross and Furst, 1987] pointed out that these invariants (and the previously known invariants of minimum and maximum genus) are in a hierarchical relationship shown in Figure 1.3 with respect to the amount of information contained within each of these. Thus, for instance, if the number of vertices in a graph are known then one can calculate the genus distribution polynomial from the embedding polynomial. Similarly, given the embeddingdistribution polynomial, one can calculate the region-size distribution with recourse to no other information. This can be seen readily in the examples of the embedding and region-
size distributions calculated above for $D_{3}$. For example, the terms $2 z_{2}^{3}$ in the embedding polynomial for $D_{3}$ are the only terms that encode embeddings with a 2 -sided face. Therefore, these terms contribute $2 \times 3=62$-sided regions over all oriented embeddings of $D_{3}$. This accounts for the term $6 x^{2}$ in the region-size polynomial for $D_{3}$.


Figure 1.3: Hierarchy of invariants.

Some of the invariants introduced by Gross and Furst garnered more attention than others. Of particular interest amongst these is the genus distribution of a graph or, its counterpart for non-orientable surfaces, the crosscap distribution of a graph. Perhaps they are at the right level of "granularity" with regard to the information embodied. They represent a birds-eye view of the trends pertaining to embeddings of a graph, but at the same time they don't require drilling down to the much finer structural level details, as is the case with the invariants of embedding distribution or region-size distribution. They are also sufficiently high in the hierarchy to make them easy to use in finding other important topological invariants of minimum, maximum and average genus or crosscap.

The first calculation of genus distribution was made for two infinite graph families called the closed-end ladders and cobblestone paths by Furst et al., 1989 in 1989. The genus distribution sequences for both of these families was derived in the form of nice closed formulas. This was followed by calculations of genus distributions (and crosscap distributions) for many other graph families. For instance, McGeoch as part of his dissertation calculated the genus distribution of circular and Möbius ladders McGeoch, 1987. Rieper Rieper, 1990 and later Kwak and Lee Kwak and Lee, 1993 independently calculated the genus distribution of dipoles. Other examples include but are not limited to calculations for ( $r, s$ )-type necklaces Gross et al., 1993, Ringel ladders by Tesar Tesar, 2000 etc. Some of these graph families are shown in Figure 1.4 .


Figure 1.4: Graph families left to right, top to bottom: closed-end ladders $L_{n}$, cobblestone paths $J_{n}$, Ringel ladders $R_{n}$, circular ladders $C L_{n}$, Möbius ladders $M L_{n}$, bouquets $B_{n}$, dipoles $D_{n}$, and generalized fan graphs $F_{(1, n)}$ and $F_{t_{1}, \cdots, t_{n}}$.

Genus distributions for these and other graph families have been calculated in many different forms such as closed formulas, generating functions, recurrences, enumerative tables and algorithms. A variety of combinatorial and topological techniques have been employed to obtain such results. For example, a result published in 1989 marks the first use of representation theory for calculating the genus distribution for the infinite graph family of bouquets of circles Gross et al., 1989. Kwak and Shim, 2002. uses edge-attaching surgery for calculating both genus and crosscap distributions for dipoles and bouquets. Chen et al., 1994, Chen et al., 2006, Chen et al., 2011b use Mohar's overlap matrices to calculate both the genus and crosscap distributions for many graph families for which genus distribution had been computed before. Wan and Liu in Wan and Liu, 2008 calculate the genus distributions for three different types of cubic graphs, using a surface generating method on the basis of joint trees of a graph which were introduced by Liu, 2003. A noticeable trend amongst these calculations are that they are carried out on graph families that are highly symmetric. Stahl considers, what he refers to as, the $H$-linear graph families Stahl, 1991a. By an $H$-linear family of graphs $G_{n}$, he means graphs constructed from $n$ copies of
the graph $H$, chained together in some consistent manner. He describes how to construct recursively defined matrices that can be used for finding generating functions for the region distributions of these graph families. The region distributions of graphs can, in turn, be used for finding their genus distributions.

As for more general results pertaining to genus distributions, not many are known. The seminal paper by Gross and Furst Gross and Furst, 1987] established a general result that specifies the genus distribution of the bar-amalgamation of two graphs. Bar-amalgamation refers to obtaining a new graph from two distinct graphs by adding an edge (called a bar) between any of their vertices. Gross and Furst established that the genus distribution of the graph obtained by running an edge $(u, v)$ between the vertex $u$ of a graph $G$ and the vertex $v$ of a graph $H$ is the convolution of the genus distributions of $G$ and $H$ times a constant, that is the product of degrees of $u$ and $v$ in $G$ and $H$, respectively.

In 1989, Gross et al. conjectured that the genus sequence for any graph is a (strongly) unimodal sequence Gross et al., 1989. This is now known as the (strong) unimodality conjecture. Pioneering calculations proving unimodality of genus distributions for certain families of graphs were made by Furst et al., 1989 and Gross et al., 1989. In Stahl, 1990, Stahl used group characters theory to show that the genus distribution of every bouquet is proportional to the unsigned Stirling numbers of the first kind, which are well known to form a unimodal sequence of numbers. Rieper around the same time showed that the genus distribution of every dipole is proportional to the Stirling numbers Rieper, 1990]. In 1991 Stahl conjectured that the Stirling-like nature of the genus distribution is true for almost all graphs and verified the conjecture for wheels and for other graphs obtained by joining some of the vertices of a forest to an exterior vertex with an arbitrary number of edges or multi-edges Stahl, 1991b]. In 1997 Stahl proved that the genus distribution of several infinite families of graphs are log concave, and are therefore unimodal Stahl, 1997. Stahl also conjectured that the zeros of the genus polynomial are all real and negative, which if true would imply strong-unimodality of the genus polynomial. However, this conjecture was proved false by Chen and Liu in Chen and Liu, 2010. There are no known graphs for which the (strong) unimodality conjecture fails, and it is still considered an open problem.

More recently, Gross et al., 2010 and Gross, 2011a calculate genus distributions for graphs produced as a result of amalgamating 2-valent vertices of distinct smaller graphs or graphs produced as a result of identifying two 2 -valent vertices of the same graph. Khan et al., 2010 generalizes this to calculate genus distributions for graphs produced by amalgamating vertices of distinct graphs, where one of the vertices is 2 -valent and the other can have an arbitrarily high degree. Khan et al., 2011 determines the genus distribution of graphs produced by edge-addition and self-amalgamation on vertices of the same graph, one of which is 2 -valent and the other $n$-valent, for $n \geq 2$. Gross in Gross, 2010 examines the effect on genus distribution of operations like adding an edge, contracting an edge, and splitting a vertex (which is the inverse operation of contracting an edge). In this context, he proves that given a graph $G$ with a 4 -valent vertex $w$,

$$
2 g d(G)=g d\left(H_{1}\right)+g d\left(H_{2}\right)+g d\left(H_{3}\right)
$$

where the function $g d$, applied to a graph, represents the genus polynomial of that graph, and where $H_{1}, H_{2}$, and $H_{3}$ are the three graphs produced by splitting the vertex $w$ of $G$ into two new vertices, with an edge running between them, so as to render the new vertices 3 -valent.

A predecessor to the algorithm on genus distribution of 4-regular outerplanar graphs is the quadratic-time algorithm for calculating genus distribution of cubic outerplanar graphs Gross, 2011b. Two recent results that focus on hybrid operations include a 3 -way $\pi$-merge for calculating genus distributions of cubic Halin graphs and a simultaneous edge-addition for calculating genus distributions of the graphs $P_{3} \times P_{n}$ (see Gross, 2011c and Khan et $a l ., 2012)$. These results provide hope that perhaps characterization of genus distributions is more tractable when its scope is limited to graphs of bounded degree and bounded treewidth.

### 1.4 Content Organization

The rest of this document is organized along the following lines:

1. 2 highlights how genus distributions can be calculated for graphs that are produced from the edge-amalgamation operation. In doing so, attention is paid to some foun-
dational ideas that will be useful in conceptualizing the theoretical underpinnings of other results in this document. An easy method is given for producing examples of non-homeomorphic graphs having the same genus distribution.
2. $\$ 3$ further develops the ideas in $\$ 2$ to derive closed formulas for other families of infinite graphs that are called closed chains.
3. $\$ 4$ describes an algorithm that calculates the genus distribution of any given 4-regular outerplanar graph.
4. $\$ 5$ gives some concluding remarks. In addition, it suggests future directions for research by discussing my work in the larger context of some significant problems.
5. Appendix A includes the complete set of productions for edge-amalgamations, as well as all the recurrences for genus distributions of graphs produced from edgeamalgamation.

## Part I

## Main Corpus

## Chapter 2

## Genus Distributions of

## Edge-Amalgamated Graphs

This section discusses techniques that enable us to formulate recurrences that specify genus distributions for the arbitrarily large graphs known as open chains. An open chain can be constructed by iteratively amalgamating smaller graph units of known partitioned genus distributions on root-edges that have two 2 -valent endpoints. These smaller graphs, called base graphs, may have arbitrarily large degrees at vertices on which the root-edges are not incident. In this manner, one can construct open chains consisting of copies of the same graph or, alternatively, one can interleave copies of many distinct graphs. In the first half of this section, we discuss how to obtain recurrences for single-edge-rooted graphs. While in the second half, these ideas are extended to obtain recurrences for double-edge-rooted graphs. Both discussions are followed by a few applications to demonstrate the use of these recurrences in computing genus distributions for different graph families. Also given in 82.6 is some insight into a simple technique for generating examples of non-homeomorphic graphs having identical genus distributions. The discussion is preceded with a note on terminology.

A double-edge-rooted graph is a graph that has two distinct edges designated as rootedges, or more simply, as roots. Each root-edge is required to have two 2-valent endpoints. The notation $(G, e, f)$ signifies that the graph $G$ is double-edge-rooted, with edges $e$ and $f$ serving as root-edges. The graph $(G, e, f)$ is abbreviated as $G$ where it is clear from context that a double-edge-rooted graph is intended.

The edge-amalgamation of a pair of double-rooted graphs $(G, e, d)$ and $(H, g, f)$ is the graph obtained by merging the roots $d$ and $g$. This operation is denoted by an asterisk:

$$
(G, e, d) *(H, g, f)=(W, e, f)
$$

where $W$ is the merged graph and $e$ and $f$ are its roots. There are two different ways of amalgamating edges $d$ and $g$, depending on how the endpoints of $d$ are paired up with the endpoints of $g$. This information is not captured in the above notation, and it is usually obvious from context what is intended for a particular scenario. Insofar as the genus distributions are concerned, it will later be established that graphs resulting from either way of edge-amalgamation have identical genus distributions.

The definition of edge-amalgamation for graphs carries over naturally to the edgeamalgamation of graph embeddings. The embeddings of the graph $W=G * H$ are obtained by combining the rotation systems for the graphs $G$ and $H$ in all possible ways. Thus, each embedding $\iota_{W}$ of the graph $W$ induces unique embeddings $\iota_{G}$ and $\iota_{H}$ of the graphs $G$ and $H$, respectively, such that the rotation system corresponding to $\iota_{W}$ is consistent with the rotation systems corresponding to $\iota_{G}$ and $\iota_{H}$.

A closed walk traced just inside the boundary of a face of an embedding is referred to as a face-boundary walk. The abbreviation $\boldsymbol{f b}$-walk is used for face-boundary walk. A related concept is that of a strand with respect to a root-edge $e$ (or an e-strand for short), which is defined to be an open subwalk of an fb-walk that runs between any two occurrences of the endpoints of $e$, such that it has in its interior neither an occurrence of $e$ nor of the endpoints of $e$.

The effects of amalgamating two graph embeddings are analyzed by using schematic representations called productions, which will be described later in this section.

### 2.1 Partitioned Genus Distributions

In order to explain what a production is, we first describe ways to categorize an embedding of a double-rooted graph. We are primarily interested here in the fb-walks incident on the root-edges, as the crux of the work in this section focuses on how these fb-walks change in response to the amalgamation operation on the graphs. Each root-edge has two 2-valent
endpoints, which implies that either each root has two distinct face-boundaries incident on it, or the same fb-walk is incident on both sides of it. Accordingly, the mnemonic $d$ is used for double and the mnemonic $s$ for single in defining the double-root partials in Table 2.1 . Note that the subscript $i$ in the definitions refers to the genus of the surface $S_{i}$.

Table 2.1: Double-root partials of $(G, e, f)$.

| Partial | Counts these embeddings in $S_{i}$ |
| :---: | :---: |
| $d d_{i}(G, e, f)$ | $e$ and $f$ both occur on two fb-walks |
| $d s_{i}(G, e, f)$ | $e$ occurs on two fb-walks and $f$ on one fb-walk |
| $s d_{i}(G, e, f)$ | $e$ occurs on one fb-walk and $f$ on two fb-walks |
| $s s_{i}(G, e, f)$ | $e$ occurs on one fb-walk and $f$ on one fb-walk |

Moreover, the fb-walk incident once or twice on one root-edge might also be incident on the other root-edge. Thereby arises the need for refinement of these partials into subpartials. It will be seen later in this section and in $\$ 3$ that this abstraction may necessitate an additional level of refinement to facilitate the calculation of genus distributions of doublerooted open chains as well as other graph families known as closed chains. For this reason, the sub-partials at the first level of abstraction are termed as first-order sub-partials. These sub-partials are defined as follows:

## First-order Sub-partials of $(G, e, f)$

The following three numbers are the sub-partials of $d d_{i}(G, e, f)$ :

$$
\begin{aligned}
d d_{i}^{0}(G, e, f)= & \text { the number of embeddings of type } d d_{i} \text { such that } \\
& \text { neither fb-walk at } e \text { is incident on } f . \\
d d_{i}^{\prime}(G, e, f)= & \text { the number of embeddings of type } d d_{i} \text { such that } \\
& \text { exactly one fb-walk at } e \text { is incident on } f . \\
d d_{i}^{\prime \prime}(G, e, f)= & \text { the number of embeddings of type } d d_{i} \text { such that } \\
& \text { both fb-walks at } e \text { are incident on } f .
\end{aligned}
$$

We observe, by definition, that

$$
d d_{i}(G)=d d_{i}^{0}(G)+d d_{i}^{\prime}(G)+d d_{i}^{\prime \prime}(G)
$$

Similarly, the sub-partials of $d s_{i}(G, e, f)$ and $s d_{i}(G, e, f)$ are as follows:

$$
\begin{aligned}
d s_{i}^{0}(G, e, f)= & \text { the number of embeddings of type } d s_{i} \text { such that } \\
& \text { neither fb-walk at } e \text { is incident on } f . \\
d s_{i}^{\prime}(G, e, f)= & \text { the number of embeddings of type } d s_{i} \text { such that } \\
& \text { exactly one fb-walk at } e \text { is incident on } f . \\
s d_{i}^{0}(G, e, f)= & \text { the number of embeddings of type } s d_{i} \text { such that } \\
& \text { the fb-walk at } e \text { is not incident on } f . \\
s d_{i}^{\prime}(G, e, f)= & \text { the number of embeddings of type } s d_{i} \text { such that } \\
& \text { the fb-walk at } e \text { is incident on } f .
\end{aligned}
$$

Thus,

$$
d s_{i}(G)=d s_{i}^{0}(G)+d s_{i}^{\prime}(G) \quad \text { and } \quad s d_{i}(G)=s d_{i}^{0}(G)+s d_{i}^{\prime}(G)
$$

Finally, the partial $s s_{i}(G, e, f)$ has these sub-partials:

$$
\begin{aligned}
s s_{i}^{0}(G, e, f)= & \text { the number of embeddings of type } s s_{i} \text { such that } \\
& \text { the fb-walk at } e \text { is not incident on } f . \\
s s_{i}^{1}(G, e, f)= & \text { the number of embeddings of type } s s_{i} \text { such that } \\
& \text { removing the two occurrences of the edge } e \text { from } \\
& \text { the fb-walk breaks it into two strands, exactly } \\
& \text { one of which contains both occurrences of } f . \\
s s_{i}^{2}(G, e, f)= & \text { the number of embeddings of type } s s_{i} \text { such that } \\
& \text { removing the two occurrences of the edge } e \text { from } \\
& \text { the fb-walk breaks it into two strands, each } \\
& \text { containing an occurrence of } f .
\end{aligned}
$$

Clearly,

$$
s s_{i}(G)=s s_{i}^{0}(G)+s s_{i}^{1}(G)+s s_{i}^{2}(G)
$$

The partitioning of genus distribution of a graph into partials and sub-partials constitutes a partitioned genus distribution of the graph. It follows from these definitions that

$$
g_{i}(G)=d d_{i}(G)+d s_{i}(G)+s d_{i}(G)+s s_{i}(G)
$$

## Single-root partials of $(G, e)$

Similarly, the embeddings of single-rooted graphs can be differentiated into two distinct types depending on whether the two occurrences of the root-edge are in the same or in different fb-walks of an embedding. Thus, the number $g_{i}(G, e)$ is the sum of the following single-root partials:

$$
\begin{aligned}
s_{i}(G, e)= & \text { The number of embeddings of } G \text { such that } \\
& e \text { occurs twice on the same fb-walk. } \\
d_{i}(G, e)= & \text { The number of embeddings of } G \text { such that } \\
& e \text { occurs on two different fb-walks. }
\end{aligned}
$$

### 2.2 Modeling Edge-Amalgamation

A construct known as production models the effect of applying an operation to one or more graph embeddings. Accordingly, the definition of production needs to be tailored to the operation for which it is being conceived. In this section, only the productions for edge-amalgamation are considered.

Let $p$ and $q$ be any of the partials such as those discussed above. Then a production for edge-amalgamation expresses how an embedding of the single-rooted graph ( $G, e$ ) of type $p$ on surface $S_{i}$ and an embedding of the double-rooted graph $(H, g, f)$ of type $q$ on surface $S_{j}$ amalgamate on root-edges $e$ and $g$ to give certain types of embeddings of the single-rooted graph $(W, f)$. This is represented as

$$
p_{i}(G) * q_{j}(H) \longrightarrow c_{1} u_{k_{1}}(W)+c_{2} v_{k_{2}}(W)+c_{3} w_{k_{3}}(W)+c_{4} z_{k_{4}}(W)
$$

where $c_{1}, c_{2}, c_{3}, c_{4}$ are integer constants and $k_{1}, k_{2}, k_{3}, k_{4}$ are integer-valued functions of $i$ and $j$. Such a production can be read as follows:

An embedding of the graph $(G, e)$ of type $p$ on surface $S_{i}$ and an embedding of the graph $(H, g, f)$ of type $q$ on surface $S_{j}$ amalgamate on edges $e$ and $g$ to give $c_{1}, c_{2}, c_{3}$ and $c_{4}$ embeddings of the graph $(W, f)$ having types $u, v, w$ and $z$, respectively, on surface $S_{k_{1}}, S_{k_{2}}, S_{k_{3}}$ and $S_{k_{4}}$.

The left-hand side of the production is known as the production head and the righthand side of the production as the production body. The number of terms in the production body could be larger if the degrees of the endpoints of root-edges were larger. This is also true for productions defined for other graph operations, such as those encountered in subsequent sections.

Remark 2. In 2.8, we see a variation in the definition of a production for edge-amalgamation where the amalgamand graph $G$ is taken to be a double-edge-rooted graph, as is the graph $W$ resulting from the edge-amalgamation.

### 2.3 Productions for Amalgamands: $(G, e)$ and $(H, g, f)$

Since there are two single-root partials for $G$ and ten first-order double-root sub-partials for $H$, there are a total of twenty productions. While these are not so many in number, their derivations are fairly routine; thus, the only productions derived are those that are necessary for developing our examples.

Theorem 1. Let $(G, e)$ be a single-edge-rooted graph and ( $H, g, f)$ a double-edge-rooted graph, where each of the root-edges has two 2-valent endpoints. Then the following two productions, which cover all possible cases of edge-amalgamation where the embedding of $H$ is of type dd ${ }^{\prime \prime}$, hold true:

$$
\begin{align*}
& d_{i}(G) * d d_{j}^{\prime \prime}(H) \longrightarrow 2 d_{i+j}(W)+2 s_{i+j+1}(W)  \tag{2.1}\\
& s_{i}(G) * d d_{j}^{\prime \prime}(H) \longrightarrow 4 d_{i+j}(W) \tag{2.2}
\end{align*}
$$

Proof. When an embedding of $(G, e)$ is amalgamated with an embedding of $(H, g, f)$, the fb-walks on edges $e$ and $g$ are broken into strands that recombine into new fb-walks in the resulting embedding of $(W, f)$, i.e., the embedding whose rotations at all vertices are
consistent with those of the embeddings of $G$ and $H$. On the amalgamated edge there are two possibilities for the rotations at each of its two endpoints. This is illustrated in Figure 2.1, which gives a pictorial representation of the Production (2.1). The production head shows the types of embeddings to undergo amalgamation on their root-edges, where the three root-edges of the single-rooted and the double-rooted amalgamands are shown parallel to each other. The production body shows the four types of embeddings of the single-rooted graph that is produced as result of the edge-amalgamation. The figure also demonstrates the changes in the fb-walks resulting from recombining the strands. In all four cases there is a decrease of 2 vertices and 1 edge after the amalgamation.


Figure 2.1: Production $d_{i}(G) * d d_{j}^{\prime \prime}(H) \longrightarrow 2 d_{i+j}(W)+2 s_{i+j+1}(W)$.

The first and the last embedding of $W$ show a decrease of 1 face, as only one fb-walk at edge $e$ combines with only one fb-walk at edge $g$. These are $d$-type embeddings of $W$. By using the Euler polyhedral equation, it can be established that the genus of the resulting embedding of $W$ is the sum of the genera of the embeddings of $G$ and $H$.

The second and the third embedding of $W$ show a decrease of 3 faces as the 2 fb -walks at $e$ and the 2 at $g$ are merged into a single fb-walk. Both of these embeddings are $s$-type embeddings of $W$. By the Euler polyhedral equation, one can see that the genus of the resulting embedding of $W$ is the sum of the genera of the embeddings of $G$ and $H$ with an additional increment of one. This proves Production (2.1).

Production (2.2) similarly follows from the Euler polyhedral equation and yields embeddings of type $d$ in all four cases for embeddings of $W$ as evident from Figure 2.2 .


Figure 2.2: Production $s_{i}(G) * d d_{j}^{\prime \prime}(H) \longrightarrow 4 d_{i+j}(W)$.

Theorem 2. Let $(G, e)$ be a single-edge-rooted graph and ( $H, g, f$ ) a double-edge-rooted graph, where each of the root-edges has two 2-valent endpoints. Then the following productions, which cover all possible cases of edge-amalgamation where the embedding of $H$ is of type ss ${ }^{0}$ or ss ${ }^{1}$, hold true:

$$
\begin{align*}
d_{i}(G) * s s_{j}^{0}(H) & \longrightarrow 4 s_{i+j}(W)  \tag{2.3}\\
s_{i}(G) * s s_{j}^{0}(H) & \longrightarrow 4 s_{i+j}(W)  \tag{2.4}\\
d_{i}(G) * s s_{j}^{1}(H) & \longrightarrow 4 s_{i+j}(W)  \tag{2.5}\\
s_{i}(G) * s s_{j}^{1}(H) & \longrightarrow 4 s_{i+j}(W) \tag{2.6}
\end{align*}
$$

Proof. For Productions $(2.3)$ and $(2.4)$, the fb-walk at edge $f$ remains unaffected by the amalgamation. Thus, all four embeddings of $W$ induced by the amalgamation of an embedding of $G$ with an embedding of $H$ are $s$-type embeddings. An examination of the recombinant strands tells us that the amalgamation merges two faces incident at the rootedges. This is shown for Production (2.3) in Figure 2.3 .


Figure 2.3: Production $d_{i}(G) * s s_{j}^{0}(H) \longrightarrow 4 s_{i+j}(W)$.

The same is also true for the Productions (2.4), (2.5) and (2.6). The proofs for Production (2.4) and (2.5) are omitted and the proof of Production (2.6) is demonstrated in Figure 2.4 .


Figure 2.4: Production $s_{i}(G) * s s_{j}^{1}(H) \longrightarrow 4 s_{i+j}(W)$.

Theorem 3. Let $(G, e)$ be a single-edge-rooted graph and $(H, g, f)$ be a double-edge-rooted graph, where all roots have two 2-valent endpoints. Then the following productions hold true:

$$
\begin{aligned}
& d_{i}(G) * d d_{j}^{0}(H) \longrightarrow 2 d_{i+j}(W)+2 d_{i+j+1}(W) \\
& s_{i}(G) * d d_{j}^{0}(H) \longrightarrow 4 d_{i+j}(W) \\
& d_{i}(G) * d d_{j}^{\prime}(H) \longrightarrow 2 d_{i+j}(W)+2 d_{i+j+1}(W) \\
& s_{i}(G) * d d_{j}^{\prime}(H) \longrightarrow 4 d_{i+j}(W) \\
& d_{i}(G) * d s_{j}^{0}(H) \longrightarrow 2 s_{i+j}(W)+2 s_{i+j+1}(W) \\
& s_{i}(G) * d s_{j}^{0}(H) \longrightarrow 4 s_{i+j}(W) \\
& d_{i}(G) * d s_{j}^{\prime}(H) \longrightarrow 2 s_{i+j}(W)+2 s_{i+j+1}(W) \\
& s_{i}(G) * d s_{j}^{\prime}(H) \longrightarrow 4 s_{i+j}(W) \\
& d_{i}(G) * s d_{j}^{0}(H) \longrightarrow 4 d_{i+j}(W) \\
& s_{i}(G) * s d_{j}^{0}(H) \longrightarrow 4 d_{i+j}(W) \\
& d_{i}(G) * s d_{j}^{\prime}(H) \longrightarrow 4 d_{i+j}(W) \\
& s_{i}(G) * s d_{j}^{\prime}(H) \longrightarrow 4 d_{i+j}(W)
\end{aligned}
$$

$$
\begin{aligned}
d_{i}(G) * s s_{j}^{2}(H) & \longrightarrow 2 d_{i+j}(W)+2 s_{i+j}(W) \\
s_{i}(G) * s s_{j}^{2}(H) & \longrightarrow 4 s_{i+j}(W)
\end{aligned}
$$

Proof. The proof is omitted for the sake of brevity.

To illustrate a technique that uses productions, a derivation of the genus distribution of the historically significant family of closed-end ladders is presented here (Furst et al., 1989. In revisiting the closed-end ladders, the intent is to bring to attention how, in some cases, it may be possible to solve the recurrences and obtain closed formulas. As another application of this technique, a new family formed from open chains of copies of the closed-end ladder $\ddot{L}_{2}$ is also examined.

### 2.4 Application: Closed-End Ladders

Let $L_{0}$ be the closed-end ladder with end-rungs but no middle-rung. It is equivalent under barycentric sub-division to the 4 -cycle $C_{4}$, with two non-adjacent edges serving as the rootedges. Let $L_{n}$ be the closed-end ladder with $n$ middle rungs; one end-rung is trisected, and the middle third serves as a single root-edge. Thus, $L_{n}=L_{n-1} * L_{0}$ (for $n \geq 1$ ). See Figure 2.5


Figure 2.5: Closed-end ladders.

Remark 3. For $L_{1}=L_{0} * L_{0}$, it is understood here that the first amalgamand is singlerooted, whereas the second is double-rooted.

Applying the face-tracing algorithm Gross and Tucker, 2001 on $L_{0}$ reveals that $d d_{0}^{\prime \prime}$ is the only non-zero partial of $L_{0}$. Theorem 1 lists the productions necessary for edgeamalgamation when the second amalgamand is a $d d^{\prime \prime}$-type embedding, and it has the following implications:

Theorem 4. Let $\left(L_{n}, f\right)=\left(L_{n-1}, e\right) *\left(L_{0}, g, f\right)$, where each of the root-edges $e, g, f$ has two 2-valent endpoints. Then

$$
\begin{align*}
& d_{k}\left(L_{n}\right)=\sum_{i=0}^{k}\left(2 d_{i}\left(L_{n-1}\right)+4 s_{i}\left(L_{n-1}\right)\right) \times d d_{k-i}^{\prime \prime}\left(L_{0}\right)  \tag{2.7}\\
& s_{k}\left(L_{n}\right)=\sum_{i=0}^{k-1} 2 d_{i}\left(L_{n-1}\right) \times d d_{k-1-i}^{\prime \prime}\left(L_{0}\right) \tag{2.8}
\end{align*}
$$

Proof. Production (2.1) indicates that amalgamating a $d$-type embedding of the singlerooted graph $L_{n-1}$ on $S_{i}$ with a $d d^{\prime \prime}$-type embedding of $L_{0}$ on surface $S_{j}$ induces four embeddings of the single-rooted graph $L_{n}$, two on the surface $S_{i+j}$ and two on the surface $S_{i+j+1}$. This explains the terms $\sum_{i=0}^{k} 2 d_{i}\left(L_{n-1}\right) \times d d_{k-i}^{\prime \prime}\left(L_{0}\right)$ of Equation 2.7 and accounts for the Equation 2.8. The terms $\sum_{i=0}^{k} 4 s_{i}\left(L_{n-1}\right) \times d d_{k-i}^{\prime \prime}\left(L_{0}\right)$ of Equation 2.7 follow from the Production (2.2).

Since $d d_{i}^{\prime \prime}\left(L_{0}\right)=1$ for $i=0$ and 0 otherwise, we obtain the recurrences:

$$
\begin{align*}
& d_{k}\left(L_{n}\right)=\left(2 d_{k}\left(L_{n-1}\right)+4 s_{k}\left(L_{n-1}\right)\right) \times d d_{0}^{\prime \prime}\left(L_{1}\right)=2 d_{k}\left(L_{n-1}\right)+4 s_{k}\left(L_{n-1}\right)  \tag{2.9}\\
& s_{k}\left(L_{n}\right)=2 d_{k-1}\left(L_{n-1}\right) \times d d_{0}^{\prime \prime}\left(L_{1}\right)=2 d_{k-1}\left(L_{n-1}\right) \tag{2.10}
\end{align*}
$$

which are analogous to the forms of recurrences obtained for cobblestone paths in Furst $e t$ al., 1989, and which can be solved identically to produce this formula, which was also first computed by Furst et al., 1989. This can be done as follows by substituting the value of $s_{k}\left(L_{n-1}\right)$ from Equation 2.10 into Equation 2.9.

$$
\begin{align*}
d_{k}\left(L_{n}\right) & =2 d_{k}\left(L_{n-1}\right)+4 s_{k}\left(L_{n-1}\right)=2 d_{k}\left(L_{n-1}\right)+4 \times 2 d_{k-1}\left(L_{n-2}\right) \\
& =2 d_{k}\left(L_{n-1}\right)+8 d_{k-1}\left(L_{n-2}\right) \tag{2.11}
\end{align*}
$$

The notation $d_{k, n}$ is used to denote $d_{k}\left(L_{n}\right)$ in the following algebraic manipulations. Multiplying both sides of the Equation 2.11 by $x^{n}$ and summing over all $n \geq 2$, we have:

$$
\sum_{n=2}^{\infty} d_{k, n} x^{n}=2 \sum_{n=2}^{\infty} d_{k, n-1} x^{n}+8 \sum_{n=2}^{\infty} d_{k-1, n-2} x^{n}
$$

Let $D_{k}(x)$ be the generating function for $d_{k, n}$. Then $D_{k}(x)=\sum_{n=0}^{\infty} d_{k, n} x^{n}$. We know that $d_{k, 0}=0$ and $d_{k, 1}=0$ for all $k>1$. Therefore, for $k>1$,

$$
\begin{aligned}
D_{k}(x)-d_{k, 0}-d_{k, 1} & =2 \sum_{n=2}^{\infty} d_{k, n-1} x^{n}+8 \sum_{n=2}^{\infty} d_{k-1, n-2} x^{n} \\
D_{k}(x) & =2 x D_{k}(x)+8 x^{2} D_{k-1}(x) \\
& =\frac{8 x^{2}}{1-2 x} D_{k-1}(x) \\
& =\left(\frac{8 x^{2}}{1-2 x}\right)^{k-1} D_{1}(x) \quad \text { for } \quad k>1
\end{aligned}
$$

Equation 2.11 implies that $d_{0, n}=2^{n}$ and consequently $D_{0}(x)=(1-2 x)^{-1}$. This fact and Equation 2.11 is used to conclude that

$$
\begin{aligned}
d_{1, n} & =2 d_{1, n-1}+8 d_{0, n-2}=2 d_{1, n-1}+8 \times 2^{n-2} \\
& =(n-1) 2^{n+1} \text { for } n \geq 1
\end{aligned}
$$

This expression enables us to find the generating function for $D_{1}(x)$ as follows:

$$
\begin{align*}
D_{1}(x) & =\sum_{n=0}^{\infty} d_{1, n} x^{n} \\
& =\sum_{n=1}^{\infty}(n-1) 2^{n+1} x^{n} \quad \because d_{1,0}=0 \\
& =8 x^{2} \sum_{n=1}^{\infty}(n-1) 2^{n-2} x^{n-2} \\
& =8 x^{2} \sum_{n=0}^{\infty}(n+1) 2^{n} x^{n} \\
\Longrightarrow & \\
D_{1}(x) & =\frac{8 x^{2}}{(1-2 x)^{2}} \\
\Longrightarrow & \\
D_{k}(x) & =\left(\frac{8 x^{2}}{1-2 x}\right)^{k-1} D_{1}(x) \\
& =\left(\frac{8 x^{2}}{1-2 x}\right)^{k-1} \frac{8 x^{2}}{(1-2 x)^{2}} \\
& =\frac{\left(8 x^{2}\right)^{k}}{(1-2 x)^{k+1}} \\
& =8^{k} x^{2 k}(1-2 x)^{-(k+1)} \quad \text { for } \quad k>1 \tag{2.12}
\end{align*}
$$

From the expressions for $D_{0}(x)$ and $D_{1}(X)$, it is clear that Equation 2.12 holds not only for $k>1$ but for all values of $k$. The coefficient of $x^{n-2 k}$ in the power series expansion of $(1-2 x)^{-(k+1)}$ is

$$
\binom{(n-2 k)+(k-1)+1}{n-2 k} 2^{n-2 k}=\binom{n-k}{k} 2^{n-2 k}
$$

Therefore, the coefficient of $x^{n}$ in $D_{k}(x)$ is

$$
d_{k, n}=8^{k}\binom{n-k}{k} 2^{n-2 k}=\binom{n-k}{k} 2^{n+k}
$$

This along with Equation 2.10 implies that

$$
\begin{aligned}
g_{k}\left(L_{n}\right) & =d_{k}\left(L_{n}\right)+s_{k}\left(L_{n}\right) \\
& =\binom{n-k}{k} 2^{n+k}+2\binom{n-k}{k-1} 2^{n+k-2} \\
& =2^{n+k-1}\left[2\binom{n-k}{k}+\binom{n-k}{k-1}\right] \\
& =2^{n+k-1}\left[\binom{n-k}{k}+\binom{n-k+1}{k}\right] \\
& =2^{n+k-1}\binom{n-k+1}{k}\left[\frac{n-2 k+1}{n-k+1}+1\right] \\
& =2^{n+k-1}\binom{n-k+1}{k} \frac{2 n-3 k+2}{n-k+1} \\
g_{k}\left(L_{n}\right) & =\left\{\begin{array}{l}
2^{n-1+k}\binom{n+1-k}{k} \frac{2 n+2-3 k}{n+1-k} \text { for } k \leq \frac{n+1}{2}, \\
0 \text { otherwise }
\end{array}\right.
\end{aligned}
$$

### 2.5 Application: Open Chains of Copies of $\ddot{L}_{2}$

Let $\ddot{L}_{2}$ be the graph obtained from the ladder $L_{2}$ by trisecting the two side-rungs and designating the middle third of these trisected edges as root-edges. Let $G_{0}$ be a single-rooted graph homeomorphic to $\ddot{L}_{2}$, with the middle third of the only trisected side-rung serving as a root-edge. An open chain $G_{n}$ of copies of $\ddot{L}_{2}$ can be formed by taking $G_{n}=G_{n-1} * \ddot{L}_{2}$ (for $n \geq 1$ ) as shown in Figure 2.6.

Face-tracing of $\ddot{L}_{2}$ demonstrates that its only non-zero-valued double-root first-order sub-partials are $d d_{0}^{\prime \prime}\left(\ddot{L}_{2}\right), s s_{1}^{0}\left(\ddot{L}_{2}\right)$ and $s s_{1}^{1}\left(\ddot{L}_{2}\right)$. Thus, the only productions needed for


Figure 2.6: Open chains of copies of $\ddot{L}_{2}$.
calculating the genus distribution of an open chain of copies of $\ddot{L}_{2}$ are those listed in Theorems 1 and 2. These productions make contributions to $d_{k}\left(G_{n}\right)$ or $s_{k}\left(G_{n}\right)$ as captured in the following equations:

$$
\begin{aligned}
& d_{k}\left(G_{n}\right)= \sum_{i=0}^{k}\left[2 d_{i}\left(G_{n-1}\right) * d d_{k-i}^{\prime \prime}\left(\ddot{L}_{2}\right)+4 s_{i}\left(G_{n-1}\right) * d d_{k-i}^{\prime \prime}\left(\ddot{L}_{2}\right)\right] \\
& s_{k}\left(G_{n}\right)=\sum_{i=0}^{k}\left[4 d_{i}\left(G_{n-1}\right) * s s_{k-i}^{0}\left(\ddot{L}_{2}\right)+4 s_{i}\left(G_{n-1}\right) * s s_{k-i}^{0}\left(\ddot{L}_{2}\right)+4 d_{i}\left(G_{n-1}\right) * s s_{k-i}^{1}\left(\ddot{L}_{2}\right)\right. \\
&\left.+4 s_{i}\left(G_{n-1}\right) * s s_{k-i}^{1}\left(\ddot{L}_{2}\right)\right]+\sum_{i=0}^{k-1}\left[2 d_{i}\left(G_{n-1}\right) * d d_{k-1-i}^{\prime \prime}\left(\ddot{L}_{2}\right)\right] \\
&=\sum_{i=0}^{k}\left[4 g_{i}\left(G_{n-1}\right) * s s_{k-i}^{0}\left(\ddot{L}_{2}\right)+4 g_{i}\left(G_{n-1}\right) * s s_{k-i}^{1}\left(\ddot{L}_{2}\right)\right]+\sum_{i=0}^{k-1}\left[2 d_{i}\left(G_{n-1}\right)\right. \\
&\left.\quad * d d_{k-1-i}^{\prime \prime}\left(\ddot{L}_{2}\right)\right]
\end{aligned}
$$

## Genus distribution of $G_{n}$

Since $d d_{0}^{\prime \prime}\left(\ddot{L}_{2}\right)=4, s s_{1}^{0}\left(\ddot{L}_{2}\right)=4, s s_{1}^{1}\left(\ddot{L}_{2}\right)=8$, it follows that

$$
\begin{align*}
& d_{k}\left(G_{n}\right)=2 d_{k}\left(G_{n-1}\right) * d d_{0}^{\prime \prime}\left(\ddot{L}_{2}\right)+4 s_{k}\left(G_{n-1}\right) * d d_{0}^{\prime \prime}\left(\ddot{L}_{2}\right) \\
& s_{k}\left(G_{n}\right)=4 g_{k-1}\left(G_{n-1}\right) * s s_{1}^{0}\left(\ddot{L}_{2}\right)+4 g_{k-1}\left(G_{n-1}\right) * s s_{1}^{1}\left(\ddot{L}_{2}\right)+2 d_{k-1}\left(G_{n-1}\right) * d d_{0}^{\prime \prime}\left(\ddot{L}_{2}\right) \\
& \quad \Longrightarrow \\
& d_{k}\left(G_{n}\right)=8 g_{k}\left(G_{n-1}\right)+8 s_{k}\left(G_{n-1}\right)  \tag{2.13}\\
& s_{k}\left(G_{n}\right)=48 g_{k-1}\left(G_{n-1}\right)+8 d_{k-1}\left(G_{n-1}\right) \tag{2.14}
\end{align*}
$$

As $\ddot{L}_{2} \cong G_{0}$, the partitioned genus distribution of $\ddot{L}_{2}$ implies that $d_{0}\left(G_{0}\right)=4$ and $s_{1}\left(G_{0}\right)=12$. Therefore, we can iteratively plug values into Equations 2.13) and 2.14, and calculate the genus distributions of open chains $G_{n}$, for $n \geq 1$. The genus distributions of $G_{1}, G_{2}$, and $G_{3}$ are given in Tables 2.2-2.4.

Table 2.2: Genus distribution of $G_{1}$.

| $k$ | $k=0$ | $k=1$ | $k=2$ | $k \geq 3$ |
| :---: | :---: | :---: | :---: | :---: |
| $d_{k}\left(G_{1}\right)$ | 32 | 192 | 0 | 0 |
| $s_{k}\left(G_{1}\right)$ | 0 | 224 | 576 | 0 |
| $g_{k}\left(G_{1}\right)$ | 32 | 416 | 576 | 0 |

Table 2.3: Genus distribution of $G_{2}$.

| $k$ | $k=0$ | $k=1$ | $k=2$ | $k=3$ | $k \geq 4$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $d_{k}\left(G_{2}\right)$ | 256 | 5120 | 9216 | 0 | 0 |
| $s_{k}\left(G_{2}\right)$ | 0 | 1792 | 21504 | 27648 | 0 |
| $g_{k}\left(G_{2}\right)$ | 256 | 6912 | 30720 | 27648 | 0 |

Table 2.4: Genus distribution of $G_{3}$.

| $k$ | $k=0$ | $k=1$ | $k=2$ | $k=3$ | $k=4$ | $k \geq 5$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $d_{k}\left(G_{3}\right)$ | 2048 | 69632 | 417792 | 442368 | 0 | 0 |
| $s_{k}\left(G_{3}\right)$ | 0 | 14336 | 372736 | 1548288 | 1327104 | 0 |
| $g_{k}\left(G_{3}\right)$ | 2048 | 83968 | 790528 | 1990656 | 1327104 | 0 |

Remark 4. From Tables 2.2 2.4, the genus distributions for open chains of copies of $\ddot{L}_{2}$ appear to support the unimodality conjecture that all graphs have unimodal genus distributions. The amalgamation approach is likely to be useful in such contexts either by producing counterexamples to the conjecture or by providing recurrences like Equations (2.13) and 2.14) which may be instrumental in proving unimodality for certain families of graphs.

### 2.6 Non-Homeomorphic Graphs with Identical Genus Distributions

The earliest published example for non-homeomorphic graphs with identical genus distributions is given in Gross et al., 1993. [McGeoch, 1987] provides a more general method
for generating such pairs. A simple method for constructing such examples is also given here. Whereas all previously known pairs of graphs exhibiting this property have been homeomorphic to non-simple graphs, pairs of graphs generated through this method are not subject to this restriction.

There are two ways of edge-amalgamating the graphs $(G, e)$ and $(H, f)$, depending on how the endpoints of the root-edges $e$ and $f$ are paired. It can be observed that all the productions for edge-amalgamation in Theorems 1 - 3 are independent of how the endpoints of the respective root-edges are paired, that is, they are true for both possible pairings. Thus, for both ways of pasting, we get the same genus distribution.

One can exploit this fact to construct pairs of non-homeomorphic graphs having the same genus distribution. For instance, Figure 2.7 shows two non-homeomorphic graphs resulting from the two ways of edge-amalgamating the same graphs. They have the same genus distributions. To prove that they are non-isomorphic, consider the set of distances between the two double adjacencies. Since these two graphs are 3-regular, they are also non-homeomorphic.


Figure 2.7: Non-homeomorphic graphs with the same genus distribution: $32+928 x+$ $6720 x^{2}+7680 x^{3}+1024 x^{4}$.

### 2.7 Second-Order Sub-Partials

The first-order sub-partials that can be further partitioned into second-order sub-partials are characterized by having an fb-walk incident on both roots, but not on all four occurrences of these roots. In particular, these are $d d^{\prime}, d d^{\prime \prime}, d s^{\prime}$ and $s d^{\prime}$. These four first-order sub-partials are further refined into second-order sub-partials. To define these, we assign arbitrary orientations to root-edges $e$ and $f$ of the graph $(G, e, f)$. These assigned orientations are referred to as pasting-orientations of root-edges. Given an oriented embedding
of ( $G, e, f$ ), as we walk along the oriented root-edge $e$ towards its head, the left side of the edge is labeled 1 and the right side is labeled 2 . Whereas when we walk along root-edge $f$ towards its head, the left side is labeled 3 and the right side is labeled 4 . This is illustrated in Figure 2.8.

$$
{ }_{1}^{e}{ }_{2}
$$

Figure 2.8: Labeling edge-sides of root-edges for second-order sub-partials.

Distinguishing which of these labeled sides come together in an fb-walk is an important piece of information, which we would like to capture in our second-order sub-partial, as it is essential for double-rooted edge-amalgamation. Thus, for example, a $d d^{\prime}$-type embedding may combine the faces 1 and 3 , faces 1 and 4 , faces 2 and 3 , or faces 2 and 4 . Accordingly, the second-order sub-partials for $d d^{\prime}$ are defined as illustrated in the top half of Figure 2.9 . The remaining sub-partials are shown in the bottom half of the figure.


Figure 2.9: Models for second-order sub-partials.

The second-order sub-partials are thus defined as follows:
$\overline{d d^{\prime}}{ }_{i}(G, e, f)=$ the number of embeddings of type $d d_{i}^{\prime}$ such that the sides 1 and 4 occur in the same fb-walk.
$\widetilde{d d^{\prime}}{ }_{i}(G, e, f)=$ the number of embeddings of type $d d_{i}^{\prime}$ such that the sides 2 and 3 occur in the same fb-walk.
$\overrightarrow{d d^{\prime}}{ }_{i}(G, e, f)=$ the number of embeddings of type $d d_{i}^{\prime}$ such that the sides 1 and 3 occur in the same fb-walk.
$\overleftarrow{d d^{\prime}}{ }_{i}(G, e, f)=$ the number of embeddings of type $d d_{i}^{\prime}$ such that the sides 2 and 4 occur in the same fb-walk.

Similarly,
${\overrightarrow{d d^{\prime \prime}}}_{i}(G, e, f)=$ the number of embeddings of type $d d_{i}^{\prime \prime}$ such that the sides 1 and 4 occur in the same fb-walk and the sides 2 and 3 in another.
$\overleftarrow{d d^{\prime \prime}}{ }_{i}(G, e, f)=$ the number of embeddings of type $d d_{i}^{\prime \prime}$ such that the sides 1 and 3 occur in the same fb-walk and the sides 2 and 4 in another.

And finally,

$$
\begin{aligned}
{\overrightarrow{d s^{\prime}}}_{i}(G, e, f)= & \text { the number of embeddings of type } d s_{i}^{\prime} \text { such that } \\
& \text { the sides } 1,3,4 \text { occur in the same fb-walk. } \\
\overleftarrow{\mathrm{st}^{\prime}}{ }_{i}(G, e, f)= & \text { the number of embeddings of type } d s_{i}^{\prime} \text { such that } \\
& \text { the sides } 2,3,4 \text { occur in the same fb-walk. } \\
{\overrightarrow{s d^{\prime}}}^{\prime}(G, e, f)= & \text { the number of embeddings of type } s d_{i}^{\prime} \text { such that } \\
& \text { the sides } 1,2,4 \text { occur in the same fb-walk. } \\
\overleftarrow{s d^{\prime}}{ }_{i}(G, e, f)= & \text { the number of embeddings of type } s d_{i}^{\prime} \text { such that } \\
& \text { the sides } 1,2,3 \text { occur in the same fb-walk. }
\end{aligned}
$$

### 2.8 Productions for Double-Edge-Rooted Amalgamands

A complete list of productions for edge-amalgamation using only double-root partials can be derived in a manner akin to the method in $\S 2.2$. One could work out all $16 \times 16=256$ productions by using only the double-root first-order sub-partials and substituting the use of $d d^{\prime}, d d^{\prime \prime}, d s^{\prime}$ and $s d^{\prime}$ by their respective second-order sub-partials defined in 2.7 . We proceed to derive the few productions needed for the first of our target applications in 82.9 . The complete set of productions for edge-amalgamation of double-edge-rooted graphs are listed in Appendix A.

Theorem 5. Let $(G, e, d)$ and $(H, g, f)$ be double-edge-rooted graphs, where all four roots have two 2-valent endpoints. Then the following productions apply when the fb-walks on both roots of the embedding of $G$ are distinct from each other and the embedding of $H$ is of type $\overrightarrow{d d^{\prime \prime}}$ :

$$
\begin{align*}
& d d_{i}^{0}(G) *{\overrightarrow{d d^{\prime \prime}}}_{j}(H) \longrightarrow 2 d d_{i+j}^{0}(W)+2 d s_{i+j+1}^{0}(W)  \tag{2.15}\\
& d s_{i}^{0}(G) *{\overrightarrow{d d^{\prime \prime}}}^{\prime}(H) \longrightarrow 4 d d_{i+j}^{0}(W)  \tag{2.16}\\
& s d_{i}^{0}(G) *{\overrightarrow{d d^{\prime \prime}}}_{j}(H) \longrightarrow 2 s d_{i+j}^{0}(W)+2 s s_{i+j+1}^{0}(W)  \tag{2.17}\\
& s s_{i}^{0}(G) *{\overrightarrow{d d^{\prime \prime}}}^{\prime}(H) \longrightarrow 4 s d_{i+j}^{0}(W) \tag{2.18}
\end{align*}
$$

Proof. Productions (2.15) and 2.17) are both of form

$$
x d_{i}^{0}(G, e, d) *{\overrightarrow{d d^{\prime \prime}}}^{\prime}(H, g, f) \longrightarrow 2 x d_{i+j}^{0}(W, e, f)+2 x s_{i+j+1}^{0}(W, e, f)
$$

where $x$ is $d$ in the former case and $s$ in the latter case. Figure 2.10 shows how the fb-walks change in response to the breaking of fb-walks incident on the root-edges and recombining of the resulting strands. The first and last embeddings show one less face as a result of amalgamation, while the middle two embeddings show a decrease of three faces. The result follows from the Euler polyhedral equation.

In all cases, the fb-walks at edge $e$ remain unaffected. Thus, the resulting embedding for graph $W$ has $d$ or $s$ for $x$, depending on whether there are two distinct fb-walks incident on edge $e$ or only one in the graph $G$. The proofs of Productions 2.16) and 2.18) are very similar, and are omitted.


Figure 2.10: Production $x d_{i}^{0}(G) * d d_{j}^{\prime \prime}(H) \longrightarrow 2 x d_{i+j}^{0}(W)+2 x s_{i+j+1}^{0}(W)$.

Theorem 6. Let $(G, e, d)$ and $(H, g, f)$ be double-edge-rooted graphs, where all four roots have two 2-valent endpoints. Then the following productions apply to the remaining cases where the fb-walks on each of the two roots of the embedding of $G$ are distinct and the embedding of $H$ is of type $\overrightarrow{d d^{\prime \prime}}$.

$$
\begin{align*}
& {\overline{d d^{\prime}}}_{i}(G) *{\overrightarrow{d d^{\prime \prime}}}^{j}(H) \longrightarrow d d_{i+j}^{0}(W)+{\overline{d d^{\prime}}}_{i+j}(W)+2{\overrightarrow{d s^{\prime}}}_{i+j+1}(W)  \tag{2.19}\\
& {\widetilde{d d^{\prime}}}_{i}(G) *{\overrightarrow{d d^{\prime \prime}}}_{j}(H) \longrightarrow d d_{i+j}^{0}(W)+{\widetilde{d d^{\prime}}}_{i+j}(W)+2{\overleftarrow{d s^{\prime}}}^{i+j+1}{ }^{\prime}(W)  \tag{2.20}\\
& {\overrightarrow{d d^{\prime}}}_{i}(G) *{\overrightarrow{d d^{\prime \prime}}}^{j}(H) \longrightarrow d d_{i+j}^{0}(W)+{\overrightarrow{d d^{\prime}}}_{i+j}(W)+2{\overrightarrow{d s^{\prime}}}_{i+j+1}(W)  \tag{2.21}\\
& {\overleftarrow{d d^{\prime}}}_{i}(G) *{\overrightarrow{d d^{\prime \prime}}}_{j}(H) \longrightarrow d d_{i+j}^{0}(W)+{\overleftarrow{d d^{\prime}}}_{i+j}(W)+2{\overleftarrow{d s^{\prime}}}^{\prime}{ }_{i+j+1}(W)  \tag{2.22}\\
& \overrightarrow{d d^{\prime \prime}}{ }_{i}(G) *{\overrightarrow{d d^{\prime \prime}}}^{\prime}{ }^{\prime}(H) \longrightarrow{\overline{d d^{\prime}}}_{i+j}(W)+{\widetilde{d d^{\prime}}}_{i+j}(W)+2 s s_{i+j+1}^{2}(W)  \tag{2.23}\\
& \overleftarrow{d d^{\prime \prime}}{ }_{i}(G) *{\overrightarrow{d d^{\prime \prime}}}^{j}(H) \longrightarrow{\overrightarrow{d d^{\prime}}}_{i+j}(W)+{\overleftarrow{d d^{\prime}}}_{i+j}(W)+2 s s_{i+j+1}^{2}(W) \tag{2.24}
\end{align*}
$$

Proof. As before, we consider the way amalgamation on the root-edges in embeddings of graphs $G$ and $H$ generates new fb-walks by recombining strands in the embedding of the graph $W$. For the proof of Production (2.19), Figure 2.11 shows the new fb-walks of $W$ as they arise from fb-walks in embeddings of $G$ and $H$.

Productions $2.20-2.22$ also deal with amalgamation of a $d d^{\prime}$-type embedding of $G$ with a $\overrightarrow{d d^{\prime \prime}}$-type embedding of $H$. However, in each case the particular second-order partial of $d d^{\prime}$ causes different types of embeddings to be generated. For example, Figure 2.12 highlights this contrast by providing the proof for Production (2.21).


Figure 2.11: Production ${\overline{d d^{\prime}}}_{i}(G) *{\overrightarrow{d d^{\prime \prime}}}^{j}(H) \longrightarrow d d_{i+j}^{0}(W)+{\overline{d d^{\prime}}}^{i+j}(W)+2{\overrightarrow{d s^{\prime}}}^{i+j+1}{ }^{\prime}(W)$.


Figure 2.12: Production ${\overrightarrow{d d^{\prime}}}_{i}(G) *{\overrightarrow{d d^{\prime \prime}}}^{\prime}(H) \longrightarrow d d_{i+j}^{0}(W)+{\overrightarrow{d d^{\prime}}}^{i+j}(W)+2{\overrightarrow{d s^{\prime}}}^{i+j+1}{ }^{\prime}(W)$.

Similarly, the picture proof of the Production (2.24) is given in Figure 2.13. The first and last embedding of the graph $W$ in the production body show one less face, while the second and the third embedding of $W$ show a decrease of 3 faces as all the faces at rootedges merge into a single face. The result follows. The proofs of the remaining productions are omitted.


Figure 2.13: Production $\overleftarrow{d d^{\prime \prime}}{ }_{i}(G) *{\overrightarrow{d d^{\prime \prime}}}^{j}(H) \longrightarrow{\overrightarrow{d d^{\prime}}}^{i+j}(W)+\overleftarrow{d d^{\prime}}{ }_{i+j}(W)+2 s s_{i+j+1}^{2}(W)$.

Theorem 7. Let $(G, e, d)$ and $(H, g, f)$ be double-edge-rooted graphs, where all four roots have two 2-valent endpoints. Then the following productions apply when the embedding of $G$ is of type $d s^{\prime}$ or $s d^{\prime}$ and the embedding of $H$ is of type $\overrightarrow{d d^{\prime \prime}}$.

$$
\begin{align*}
& {\overrightarrow{d s^{\prime}}}_{i}(G) *{\overrightarrow{d d^{\prime \prime}}}_{j}(H) \longrightarrow 2{\overline{d d^{\prime}}}_{i+j}(W)+2{\overrightarrow{d d^{\prime}}}_{i+j}(W)  \tag{2.25}\\
& {\overleftarrow{d s^{\prime}}}_{i}(G) *{\overrightarrow{d d^{\prime \prime}}}^{j}(H) \longrightarrow 2{\widetilde{d d^{\prime}}}^{i+j}(W)+2{\overleftarrow{d d^{\prime}}}^{\prime}(W+j(W)  \tag{2.26}\\
& {\overrightarrow{s d^{\prime}}}_{i}(G) *{\overrightarrow{d d^{\prime \prime}}}^{\prime}(H) \longrightarrow s d_{i+j}^{0}(W)+{\overrightarrow{s d^{\prime}}}_{i+j}(W)+2 s s_{i+j+1}^{1}(W)  \tag{2.27}\\
& \overleftarrow{s d^{\prime}}{ }_{i}(G) *{\overrightarrow{d d^{\prime \prime}}}^{\prime}(H) \longrightarrow s d_{i+j}^{0}(W)+{\overleftarrow{s d^{\prime}}}_{i+j}(W)+2 s s_{i+j+1}^{1}(W) \tag{2.28}
\end{align*}
$$

Proof. The proof for Production (2.25) follows from Figure 2.14. In all four cases that can arise as a consequence of amalgamation, the fb-walks incident at the root-edge $d$ of graph $G$ and the root-edge $g$ of graph $H$ break into strands that merge to yield one less face. Thus, the resulting genus of the embedding of graph $W$ is precisely the sum of the genera of embeddings of $G$ and $H$.

The proof of Production (2.27) is similar. It follows by face-tracing, using as a model for $\overrightarrow{s d^{\prime}}$ a $180^{\circ}$ rotation of the model for $\overrightarrow{d s^{\prime}}$ that we used in Figure 2.14.

Production (2.28) is illustrated by Figure 2.15. It is easy to use a $180^{\circ}$ rotation of the model used for $\overleftarrow{s d^{\prime}}$ and to use face-tracing to establish the proof of Production 2.26.


Figure 2.14: Production ${\overrightarrow{d s^{\prime}}}_{i}(G) *{\overrightarrow{d d^{\prime \prime}}}_{j}(H) \longrightarrow 2{\overline{d d^{\prime}}}_{i+j}(W)+2{\overrightarrow{d d^{\prime}}}_{i+j}(W)$.


Figure 2.15: Production $\overleftarrow{s d^{\prime}}{ }_{i}(G) * \overrightarrow{d d^{\prime \prime}}{ }_{j}(H) \longrightarrow s d_{i+j}^{0}(W)+\overleftarrow{s d^{\prime}}{ }_{i+j}(W)+2 s s_{i+j+1}^{1}(W)$.

Theorem 8. Let $(G, e, d)$ and $(H, g, f)$ be double-edge-rooted graphs, where all four roots have two 2-valent endpoints. Then the following productions apply to all the remaining cases where the embedding of $G$ is of type ss and the embedding of $H$ is of type $\overrightarrow{d d^{\prime \prime}}$.

$$
\begin{align*}
& s s_{i}^{1}(G) *{\overrightarrow{d d^{\prime}}}^{j}(H) \longrightarrow 2{\overrightarrow{s d^{\prime}}}_{i+j}(W)+2{\overleftarrow{s d^{\prime}}}_{i+j}(W)  \tag{2.29}\\
& s s_{i}^{2}(G) *{\overrightarrow{d d^{\prime}}}^{\prime}(H) \longrightarrow{\overrightarrow{d d^{\prime \prime}}}_{i+j}(W)+{\overleftarrow{d d^{\prime \prime}}}_{i+j}(W)+{\overrightarrow{s d^{\prime}}}_{i+j}(W)+{\overleftarrow{s d^{\prime}}}_{i+j}(W) \tag{2.30}
\end{align*}
$$

Proof. The proofs of Productions (2.29) and 2.30) are clear from Figures 2.16 and 2.17 , respectively. For both productions, in all four cases, the genus of the induced embedding surface of graph $W$ is equal to the sum of the genera of the embedding surfaces of the graphs $G$ and $H$. However, the embedding types of the graph $W$ yielded by both productions are different.


Figure 2.16: Production $s s_{i}^{1}(G) *{\overrightarrow{d d^{\prime \prime}}}_{j}(H) \longrightarrow 2{\overrightarrow{s d^{\prime}}}_{i+j}(W)+2 \overleftarrow{s d^{\prime}}{ }_{i+j}(W)$.


Figure 2.17: Production $s s_{i}^{2}(G) *{\overrightarrow{d d^{\prime \prime}}}^{j}(H) \longrightarrow{\overrightarrow{d d^{\prime \prime}}}^{i+j}{ }^{\prime}(W)+\overleftarrow{d d^{\prime \prime}}{ }_{i+j}(W)+{\overrightarrow{s d^{\prime}}}_{i+j}(W)+$ $\overleftarrow{s d^{\prime}}{ }_{i+j}(W)$

The results of Theorems 58 are summarized in Table 2.5 , where the partials are abbreviated through omission of the graphs $G, H$ and $W$.

Table 2.5: A subset of the productions for the edge-amalgamation $(G, e, d) *(H, g, f)$.

| Productions |
| :---: |
|  |

In general, when amalgamating copies of a base graph, some of the partials of the base graph may be zero-valued. Accordingly, one can eliminate a lot of unnecessary work by using a smaller subset of productions relevant to a particular application. The productions in Table 2.5 lead to Theorem 9 .

Theorem 9. Let $(W, e, f)=(G, e, d) *(H, g, f)$, where each of the root-edges e, $d, g, f$ has two 2-valent endpoints and the embeddings of the graph $H$ are of type $\overrightarrow{d d^{\prime \prime}}$. Then

$$
\begin{align*}
d d_{k}^{0}(W) & =\sum_{i=0}^{k}\left(2 d d_{i}^{0}(G)+d d_{i}^{\prime}(G)+4 d s_{i}^{0}(G)\right) \times{\overrightarrow{d d^{\prime \prime}}}_{k-i}(H)  \tag{2.31}\\
{\overline{d d^{\prime}}}^{k}(W) & =\sum_{i=0}^{k}\left({\overline{d d^{\prime}}}_{i}(G)+{\overrightarrow{d d^{\prime \prime}}}_{i}(G)+2{\overrightarrow{d s^{\prime}}}_{i}(G)\right) \times{\overrightarrow{d d^{\prime \prime}}}_{k-i}(H)  \tag{2.32}\\
{\widetilde{d d^{\prime}}}^{\prime}(W) & =\sum_{i=0}^{k}\left({\widetilde{d d^{\prime}}}_{i}(G)+{\overrightarrow{d d^{\prime \prime}}}_{i}(G)+2 \overleftarrow{{d s^{\prime}}^{\prime}}(G)\right) \times{\overrightarrow{d d^{\prime \prime}}}_{k-i}(H) \tag{2.33}
\end{align*}
$$

$$
\begin{align*}
& {\overrightarrow{d d^{\prime}}}^{k}(W)=\sum_{i=0}^{k}\left({\overrightarrow{d d^{\prime}}}_{i}(G)+\overleftarrow{d d^{\prime \prime}}{ }_{i}(G)+2{\overrightarrow{d s^{\prime}}}_{i}(G)\right) \times{\overrightarrow{d d^{\prime \prime}}}_{k-i}(H)  \tag{2.34}\\
& {\overleftarrow{d d^{\prime}}}^{k}(W)=\sum_{i=0}^{k}\left({\overleftarrow{d d^{\prime}}}_{i}(G)+{\overleftarrow{d d^{\prime \prime}}}_{i}(G)+2{\overleftarrow{d s^{\prime}}}_{i}(G)\right) \times{\overrightarrow{d d^{\prime \prime}}}_{k-i}(H)  \tag{2.35}\\
& \overrightarrow{d d^{\prime \prime}}{ }_{k}(W)=\sum_{i=0}^{k} s s_{i}^{2}(G) \times{\overrightarrow{d d^{\prime \prime}}}^{k-i}(H)  \tag{2.36}\\
& \overleftarrow{d d^{\prime \prime}}{ }_{k}(W)=\sum_{i=0}^{k} s s_{i}^{2}(G) \times{\overrightarrow{d d^{\prime \prime}}}_{k-i}(H)  \tag{2.37}\\
& d s_{k}^{0}(W)=\sum_{i=0}^{k-1} 2 d d_{i}^{0}(G) \times{\overrightarrow{d d^{\prime \prime}}}_{k-1-i}(H)  \tag{2.38}\\
& {\overrightarrow{d s^{\prime}}}^{\prime}(W)=\sum_{i=0}^{k-1} 2\left({\overline{d d^{\prime}}}_{i}(G)+{\overrightarrow{d d^{\prime}}}_{i}(G)\right) \times{\overrightarrow{d d^{\prime \prime}}}_{k-1-i}(H)  \tag{2.39}\\
& {\overleftarrow{d s^{\prime}}}_{k}(W)=\sum_{i=0}^{k-1} 2\left({\widetilde{d d^{\prime}}}_{i}(G)+{\overleftarrow{d d^{\prime}}}_{i}(G)\right) \times{\overrightarrow{d d^{\prime \prime}}}_{k-1-i}(H)  \tag{2.40}\\
& s d_{k}^{0}(W)=\sum_{i=0}^{k}\left(2 s d_{i}^{0}(G)+s d_{i}^{\prime}(G)+4 s s_{i}^{0}(G)\right) \times \overrightarrow{d d^{\prime \prime}}{ }_{k-i}(H)  \tag{2.41}\\
& {\overrightarrow{s d^{\prime}}}_{k}(W)=\sum_{i=0}^{k}\left({\overrightarrow{s d^{\prime}}}_{i}(G)+2 s s_{i}^{1}(G)+s s_{i}^{2}(G)\right) \times{\overrightarrow{d d^{\prime \prime}}}_{k-i}(H)  \tag{2.42}\\
& \overleftarrow{s d^{\prime}}{ }_{k}(W)=\sum_{i=0}^{k}\left(\overleftarrow{s d^{\prime}}{ }_{i}(G)+2 s s_{i}^{1}(G)+s s_{i}^{2}(G)\right) \times{\overrightarrow{d d^{\prime}}}_{k-i}(H)  \tag{2.43}\\
& s s_{k}^{0}(W)=\sum_{i=0}^{k-1} 2 s d_{i}^{0}(G) \times \overrightarrow{d d^{\prime \prime}}{ }_{k-1-i}(H)  \tag{2.44}\\
& s s_{k}^{1}(W)=\sum_{i=0}^{k-1} 2 s d_{i}^{\prime}(G) \times{\overrightarrow{d d^{\prime \prime}}}^{k-1-i}(H)  \tag{2.45}\\
& s s_{k}^{2}(W)=\sum_{i=0}^{k-1} 2 d d_{i}^{\prime \prime}(G) \times \overrightarrow{d d^{\prime \prime}}{ }_{k-1-i}(H) \tag{2.46}
\end{align*}
$$

Proof. Consider the production:

$$
d d_{i}^{0}(G) *{\overrightarrow{d d^{\prime \prime}}}_{j}(H) \longrightarrow 2 d d_{i+j}^{0}(W)+2 d s_{i+j+1}^{0}(W)
$$

It indicates that each $d d^{0}$-type embedding of $G$ on $S_{i}$ when amalgamated with a $\overrightarrow{d d^{\prime \prime}}$-type embedding of $H$ on surface $S_{j}$, induces two embeddings of $W$ having type $d d^{0}$ on surface $S_{i+j}$ and two of type $d s^{0}$ on surface $S_{i+j+1}$.

These contributions account for the terms $\sum_{i=0}^{k} 2 d d_{i}^{0} \times \overrightarrow{d d^{\prime \prime}}{ }_{k-i}$ in Equation 2.31) and for the Equation (2.38). Taking into account all contributions made by the productions in Table 2.5, the result follows.

Remark 5. The complete sets of productions for edge-amalgamations and the recurrences obtained from them are given in Appendix $A$.

### 2.9 Application: Double-rooted Closed-End Ladders

It was shown in $\$ 2.4$ how the single-root partials for the genus distributions of closed-end ladders can be computed. The same can be accomplished for double-root partials of closedend ladders. It is known by face-tracing that all partials for $L_{0}$ are zero-valued except for $\overrightarrow{d d^{\prime \prime}}{ }_{0}\left(L_{0}\right)$, whose value is 1 . Using the value of this partial, Theorem 9 can be applied iteratively to obtain the partitioned genus distribution for the closed-end ladders. The partitioned genus distribution for some small closed-end ladders is derived in this manner in Table 2.6.

Table 2.6: Double-root partials of $L_{n}$.

| $L_{n}$ | $L_{0}$ | $L_{1}$ |  | $L_{2}$ |  | $L_{3}$ |  |  | $L_{4}$ |  | $L_{5}$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $k$ | 0 | $0 \quad 1$ | 0 | 1 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 3 |
| $d d_{k}^{0}$ | 0 | $0 \quad 0$ | 2 | 0 | 6 | 0 | 0 | 14 | 40 | 0 | 30 | 168 | 0 | 0 |
| $\overline{d d^{\prime}}{ }_{k}$ | 0 | 10 | 1 | 0 | 1 | 6 | 0 | 1 | 10 | 0 | 1 | 14 | 56 | 0 |
| $\widetilde{d d^{\prime}}{ }_{k}$ | 0 | 10 | 1 | 0 | 1 | 6 | 0 | 1 | 10 | 0 | 1 | 14 | 56 | 0 |
| ${\overrightarrow{d d^{\prime}}}^{\prime}$ | 0 | $0 \quad 0$ | 0 | 0 | 0 | 6 | 0 | 0 | 10 | 0 | 0 | 14 | 56 | 0 |
| ${\overleftarrow{d d^{\prime}}}^{\prime}{ }_{k}$ | 0 | $0 \quad 0$ | 0 | 0 | 0 | 6 | 0 | 0 | 10 | 0 | 0 | 14 | 56 | 0 |
| $\overrightarrow{d d^{\prime \prime}}{ }_{k}$ | 1 | $0 \quad 0$ | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | 0 |
| ${\overleftarrow{d d^{\prime \prime}}}_{k}$ | 0 | $0 \quad 0$ | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | 0 |
| $d s_{k}^{0}$ | 0 | $0 \quad 0$ | 0 | 0 | 0 | 4 | 0 | 0 | 12 | 0 | 0 | 28 | 80 | 0 |
| $\overrightarrow{d s^{\prime}}{ }_{k}$ | 0 | $0 \quad 0$ | 0 | 2 | 0 | 2 | 0 | 0 | 2 | 24 | 0 | 2 | 40 | 0 |
| ${\overleftarrow{d s^{\prime}}}^{\prime}{ }^{2}$ | 0 | $0 \quad 0$ | 0 | 2 | 0 | 2 | 0 | 0 | 2 | 24 | 0 | 2 | 40 | 0 |
| $s d_{k}^{0}$ | 0 | $0 \quad 0$ | 0 | 0 | 0 | 4 | 0 | 0 | 12 | 0 | 0 | 28 | 80 | 0 |
| $\overrightarrow{s d^{\prime}}{ }_{k}$ | 0 | $0 \quad 0$ | 0 | 2 | 0 | 2 | 0 | 0 | 2 | 24 | 0 | 2 | 40 | 0 |
| $\overleftarrow{s d^{\prime}}{ }_{k}$ | 0 | $0 \quad 0$ | 0 | 2 | 0 | 2 | 0 | 0 | 2 | 24 | 0 | 2 | 40 | 0 |
| $s s_{k}^{0}$ | 0 | $0 \quad 0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 24 | 0 |
| $s s_{k}^{1}$ | 0 | $0 \quad 0$ | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 8 | 0 | 0 | 8 | 96 |
| $s s_{k}^{2}$ | 0 | $0 \quad 2$ | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 32 |
| $g_{k}$ | 1 | 22 | 4 | 12 | 8 | 40 | 16 | 16 | 112 | 128 | 32 | 288 | 576 | 128 |

Remark 6. The fact that $\overrightarrow{d d^{\prime \prime}}{ }_{0}\left(L_{0}\right)=1$ is the only non-zero-valued partial for $L_{0}$ is the vital piece of information utilized by us in selecting the sixteen productions listed in Table 2.5. from amongst a total of two hundred and fifty six productions listed in Appendix A.

Remark 7. In §3, these double-root partials are used for calculating genus distributions of closed chains which are "cycles" of copies of a given base graph. The two closed chains corresponding to closed-end ladders are circular ladders and Möbius ladders.

One can observe that the values for $g_{k}\left(L_{n}\right)$ agree with the values first obtained by Furst et al., 1989. It may also be observed that the same results could have also been achieved using first-order sub-partials and may question the need for using second-order subpartials for amalgamating double-rooted graphs. However, in general, with more complex applications requiring amalgamations of double-rooted graphs having higher degrees, one is likely to need the additional information captured in second-order sub-partials to obtain closed chains from open chains. One such application is calculating the genus distribution of an open chain of copies of the prism graph given in 2.10 . Another is genus distribution calculation of an open chain of copies of $K_{3,3}$ in $\$ 2.11$.

### 2.10 Application: Open Chains of Copies of a Triangular Prism Graph

A triangular prism graph is illustrated in Figure 2.18 at the left, where two of its edges are trisected and their middle-thirds are designated as root-edges. The root-edges are shown darker by convention. Let $\Delta_{G}$ denote the double-edge-rooted triangular prism graph. Figure 2.18 shows some small double-edge-rooted open chains of copies of the graph $\Delta_{G}$. An open chain consisting of $n$ copies of $\Delta_{G}$ is denoted by $\mathcal{P} r_{n}$.


Figure 2.18: Open chains of copies of a triangular prism graph.

The detailed calculations for genus distributions for $\mathcal{P} r_{1}, \mathcal{P} r_{2}$ and $\mathcal{P} r_{3}$ are omitted and only the results computed by using Theorems $23-38$ from Appendix A are listed in Tables 2.7-2.8.

Table 2.7: Genus distributions of the open chains $\mathcal{P} r_{1}$ and $\mathcal{P} r_{2}$ of 1 and 2 copies of $\Delta_{G}$, respectively.

| $\mathcal{P} r_{n}$ | $\mathcal{P} r_{1}$ |  |  | $\mathcal{P} r_{2}$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $k$ | 0 | 1 | 2 | 0 | 1 | 2 | 3 | 4 |
| $d d_{k}^{0}$ | 0 | 0 | 0 | 6 | 176 | 704 | 0 | 0 |
| $\overline{d d^{\prime}}{ }_{k}$ | 1 | 0 | 0 | 1 | 46 | 400 | 0 | 0 |
| $\widetilde{d d^{\prime}}{ }_{k}$ | 1 | 0 | 0 | 1 | 46 | 400 | 0 | 0 |
| ${\overrightarrow{d d^{\prime}}}_{k}$ | 0 | 0 | 0 | 0 | 22 | 256 | 0 | 0 |
| $\overleftarrow{d d^{\prime}}{ }_{k}$ | 0 | 0 | 0 | 0 | 22 | 256 | 0 | 0 |
| $\overrightarrow{d d^{\prime \prime}}{ }_{k}$ | 0 | 12 | 0 | 0 | 0 | 48 | 0 | 0 |
| $\overleftarrow{d d^{\prime \prime}}{ }_{k}$ | 0 | 0 | 0 | 0 | 0 | 48 | 0 | 0 |
| $d s_{k}^{0}$ | 0 | 4 | 0 | 0 | 44 | 800 | 384 | 0 |
| $\overrightarrow{d s^{\prime}}{ }_{k}$ | 0 | 4 | 0 | 0 | 6 | 272 | 960 | 0 |
| $\overleftarrow{4 s^{\prime}}{ }_{k}$ | 0 | 4 | 0 | 0 | 6 | 272 | 960 | 0 |
| $s d_{k}^{0}$ | 0 | 4 | 0 | 0 | 44 | 800 | 384 | 0 |
| $\overrightarrow{s d^{\prime}}{ }_{k}$ | 0 | 4 | 0 | 0 | 6 | 272 | 960 | 0 |
| ${\overleftarrow{s d \prime^{\prime}}}^{\prime}$ | 0 | 4 | 0 | 0 | 6 | 272 | 960 | 0 |
| $s s_{k}^{0}$ | 0 | 0 | 0 | 0 | 0 | 320 | 1888 | 0 |
| $s s_{k}^{1}$ | 0 | 0 | 24 | 0 | 0 | 72 | 1664 | 2304 |
| $s s_{k}^{2}$ | 0 | 2 | 0 | 0 | 0 | 8 | 288 | 0 |
| $g_{k}$ | 2 | 38 | 24 | 8 | 424 | 5200 | 8448 | 2304 |

Table 2.8: Genus distributions of the open chain $\mathcal{P r} r_{3}$ of 3 copies of $\Delta_{G}$.

| $\mathcal{P} r_{n}$ | $\mathcal{P r}{ }_{3}$ |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $d d_{k}^{0}$ | 30 | 2080 | 40000 | 211840 | 67584 | 0 | 0 |
| $\overline{d d^{\prime}}{ }_{k}$ | 1 | 82 | 2608 | 27104 | 38400 | 0 | 0 |
| $\widetilde{d d^{\prime}}{ }_{k}$ | 1 | 82 | 2608 | 27104 | 38400 | 0 | 0 |
| $\overrightarrow{d d^{\prime}}{ }_{k}$ | 0 | 46 | 2176 | 25376 | 38400 | 0 | 0 |
| $\overleftarrow{d d^{\prime}}{ }_{k}$ | 0 | 46 | 2176 | 25376 | 38400 | 0 | 0 |
| $\overrightarrow{d d^{\prime \prime}}{ }_{k}$ | 0 | 0 | 0 | 288 | 3456 | 0 | 0 |
| $\overleftarrow{d d^{\prime \prime}}{ }_{k}$ | 0 | 0 | 0 | 288 | 3456 | 0 | 0 |
| $d s_{k}^{0}$ | 0 | 212 | 11744 | 149568 | 336896 | 36864 | 0 |
| ${\overrightarrow{d s^{\prime}}}{ }_{k}$ | 0 | 6 | 640 | 17664 | 113792 | 92160 | 0 |
| $\overleftarrow{d s^{\prime}}{ }_{k}$ | 0 | 6 | 640 | 17664 | 113792 | 92160 | 0 |
| $s d_{k}^{0}$ | 0 | 212 | 11744 | 149568 | 336896 | 36864 | 0 |
| $\overrightarrow{s d^{\prime}}{ }_{k}$ | 0 | 6 | 640 | 17664 | 113792 | 92160 | 0 |
| $\overleftarrow{s d^{\prime}}{ }_{k}$ | 0 | 6 | 640 | 17664 | 113792 | 92160 | 0 |
| $s s_{k}^{0}$ | 0 | 0 | 1496 | 61728 | 393728 | 417792 | 0 |
| $s s_{k}^{1}$ | 0 | 0 | 72 | 6144 | 112000 | 411648 | 221184 |
| $s s_{k}^{2}$ | 0 | 0 | 0 | 32 | 3456 | 0 | 0 |
| $g_{k}$ | 32 | 2784 | 77184 | 755072 | 1866240 | 1271808 | 221184 |

### 2.11 Application: Open Chains of Copies of $K_{3,3}$

Trisect two edges of $K_{3,3}$ as shown in Figure 2.19 and use the middle-thirds as root-edges.


Figure 2.19: Open chains $K_{3,3}^{1}, K_{3,3}^{2}, K_{3,3}^{3}$.

This open chain is denoted by $K_{3,3}^{1}$. An open chain of $n$ copies of $K_{3,3}$ is denoted by $K_{3,3}^{n}$ and it consists of $n$ edge-amalgamated copies of $K_{3,3}^{1}$. We use Theorems $23-38$ from Appendix A to compute the results listed in Table 2.9. In particular, we list only the non-zero columns.

Table 2.9: Genus distributions of open chains of copies of $K_{3,3}$.

| $K_{3,3}^{n}$ | $K_{3,3}^{1}$ | $K_{3,3}^{2}$ |  |  | $K_{3,3}^{3}$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $k$ | 12 | 2 | 3 | 4 | 3 | 4 | 5 | 6 |
| $d d_{k}^{0}$ | 0 0 | 1656 | 0 | 0 | 262976 | 436224 | 0 | 0 |
| $\overline{d d^{\prime}}{ }_{k}$ | 40 | 344 | 440 | 0 | 13296 | 78064 | 31040 | 0 |
| $\widetilde{d d^{\prime}}{ }_{k}$ | 40 | 344 | 440 | 0 | 13296 | 78064 | 31040 | 0 |
| $\overrightarrow{d d^{\prime}}{ }_{k}$ | 60 | 280 | 440 | 0 | 13808 | 78064 | 31040 | 0 |
| $\overleftarrow{d d^{\prime}}{ }_{k}$ | 60 | 280 | 440 | 0 | 13808 | 78064 | 31040 | 0 |
| $\overrightarrow{d d^{\prime \prime}}{ }_{k}$ |  | 24 | 144 | 0 | 144 | 2160 | 5184 | 0 |
| $\overleftarrow{d d^{\prime \prime}}{ }_{k}$ | 60 | 24 | 144 | 0 | 144 | 2160 | 5184 | 0 |
| $d s_{k}^{0}$ | 20 | 424 | 1040 | 0 | 58784 | 339488 | 171392 | 0 |
| $\overrightarrow{d s^{\prime}}{ }_{k}$ | 20 | 104 | 1016 | 0 | 4384 | 68256 | 158336 | 0 |
| $\overleftarrow{d s^{\prime}}{ }_{k}$ | 20 | 104 | 1016 | 0 | 4384 | 68256 | 158336 | 0 |
| $s d_{k}^{0}$ | 20 | 424 | 1040 | 0 | 58784 | 339488 | 171392 | 0 |
| $\overrightarrow{s d^{\prime}}{ }_{k}$ | 20 | 116 | 1088 | 0 | 4336 | 68688 | 160064 | 0 |
| $\overleftarrow{s d^{\prime}}{ }_{k}$ | 20 | 92 | 944 | 0 | 4432 | 67824 | 156608 | 0 |
| $s s_{k}^{0}$ | 0 0 | 96 | 664 | 0 | 12928 | 133312 | 280576 | 0 |
| $s s_{k}^{1}$ | $0 \quad 12$ | 32 | 704 | 2016 | 1408 | 36416 | 230336 | 214272 |
| $s s_{k}^{2}$ | 212 | 8 | 168 | 288 | 32 | 1440 | 8640 | 6912 |
| $g_{k}$ | $40 \quad 24$ | 4352 | 9728 | 2304 | 466944 | 1875968 | 1630208 | 221184 |

### 2.12 Application: Open Chains of Alternating Copies of Two Distinct Graphs

Genus distribution calculations are also streamlined for open chains consisting of different base graphs. Here, the base graphs $\Delta_{G}$ and $K_{3,3}$ from our previous two applications are
used to form open chains using alternating copies of these two graphs. The open chains $\mathcal{A} l_{2}$ and $\mathcal{A} l_{3}$, shown in Figure 2.20, consist of two and three base graphs, respectively. The partitioned genus distributions for these open chains is given in Table 2.10.


Figure 2.20: Open chains $\mathcal{A l} l_{2}$ and $\mathcal{A} l_{3}$.

Table 2.10: Genus distributions of open chains $\mathcal{A l} l_{2}$ and $\mathcal{A} l_{3}$.

| $\mathcal{A} l_{n}$ | $\mathcal{A l}{ }_{2}$ |  |  |  | $\mathcal{A l}{ }_{3}$ |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $k$ | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 | 5 | 6 |
| $d d_{k}^{0}$ | 104 | 1232 | 0 | 0 | 808 | 27488 | 220416 | 67584 | 0 | 0 |
| $\overline{d d^{\prime}}{ }_{k}$ | 10 | 308 | 0 | 0 | 18 | 1320 | 22256 | 36672 | 0 | 0 |
| $\widetilde{d d^{\prime}}{ }_{k}$ | 10 | 308 | 0 | 0 | 18 | 1320 | 22256 | 36672 | 0 | 0 |
| $\overrightarrow{d d^{\prime}}{ }_{k}$ | 18 | 404 | 0 | 0 | 26 | 1512 | 23408 | 36672 | 0 | 0 |
| $\overleftarrow{d d^{\prime}}{ }_{k}$ | 18 | 404 | 0 | 0 | 26 | 1512 | 23408 | 36672 | 0 | 0 |
| $\overrightarrow{d d^{\prime \prime}}{ }_{k}$ | 0 | 36 | 144 | 0 | 0 | 0 | 240 | 2880 | 0 | 0 |
| $\overleftarrow{d d^{\prime \prime}}{ }_{k}$ | 0 | 36 | 144 | 0 | 0 | 0 | 240 | 2880 | 0 | 0 |
| $d s_{k}^{0}$ | 24 | 440 | 384 | 0 | 0 | 5744 | 123360 | 379264 | 36864 | 0 |
| $\overrightarrow{d s^{\prime}}{ }_{k}$ | 4 | 176 | 816 | 0 | 0 | 264 | 11840 | 105856 | 92160 | 0 |
| $\overleftarrow{d s^{\prime}}{ }_{k}$ | 4 | 176 | 816 | 0 | 0 | 264 | 11840 | 105856 | 92160 | 0 |
| $s d_{k}^{0}$ | 0 | 792 | 1408 | 0 | 0 | 5744 | 123360 | 379264 | 36864 | 0 |
| $\overrightarrow{s d^{\prime}}{ }_{k}$ | 0 | 172 | 1456 | 0 | 0 | 276 | 11984 | 105856 | 92160 | 0 |
| $\overleftarrow{s d^{\prime}}{ }_{k}$ | 0 | 148 | 1456 | 0 | 0 | 252 | 11696 | 105856 | 92160 | 0 |
| $s s_{k}^{0}$ | 0 | 176 | 1136 | 0 | 0 | 0 | 40736 | 353920 | 550912 | 0 |
| $s s_{k}^{1}$ | 0 | 48 | 1072 | 2304 | 0 | 0 | 3168 | 80192 | 437504 | 221184 |
| $s s_{k}^{2}$ | 0 | 8 | 192 | 0 | 0 | 0 | 32 | 2496 | 6912 | 0 |
| $g_{k}$ | 192 | 4864 | 9024 | 2304 | 896 | 45696 | 650240 | 1838592 | 1437696 | 221184 |

## Chapter 3

## Genus Distributions of

## Self-Edge-Amalgamated Graphs

In this section, two closed formulas are developed that use the partitioned genus distribution of a double-edge-rooted graph to calculate the genus distribution of the graph obtained from it by pasting together the two root-edges. Combined with the results derived in the previous section, one can first obtain a recursion for the genus distributions of an infinite family of open chains of edge-amalgamated copies of a base graph, and then apply the two formulas derived in this section to obtain genus distributions of the corresponding one or two infinite families of closed chains. In this manner, it is possible to calculate the genus distribution for arbitrarily large graphs. While the results in this section are predominantly self-contained, there is heavy reliance on the concepts of partials and productions defined in $\$ 2$.

The self-edge-amalgamation of a graph is an operation whereby two root-edges of a double-edge-rooted graph are pasted together. Informally, the expression pasting refers to any kind of amalgamation. Where it is clear from context that a self-edge-amalgamation is intended, the terminology self-amalgamation may be used. This operation is denoted by a monadic operator acting on a double-edge-rooted graph operand $(G, e, f)$ :

$$
W=*_{e f}(G, e, f)
$$

The self-edge-amalgamation of a double-edge-rooted graph $(G, e, f)$ produces a graph $W$ obtained from $G$ by identifying edges $e$ and $f$. The pasting-orientations on the root-edges $e$
and $f$ are fixed arbitrarily. Accordingly, the edge-ends of $e$ and $f$ at the tail are $e^{-}$and $f^{-}$, while the ones at the head are $e^{+}$and $f^{+}$. The edges $e$ and $f$ can then be pasted in two different ways. One way of pasting, called co-self-amalgamation, identifies the edge-end $e^{-}$with $f^{-}$and the edge-end $e^{+}$with $f^{+}$. The other way of pasting, called contra-selfamalgamation, pairs the edge-end $e^{-}$with $f^{+}$and the edge-end $e^{+}$with $f^{-}$. These two ways of self-edge-amalgamating a graph produces graphs which may be non-isomorphic, as seen later in this paper.

Remark 8. When introducing second-order partials in 2.7, we required that the assignment of labels 1 through 4 to the edge-sides of root-edges be relative to these same pastingorientations.

Graphs that are obtained from a self-amalgamation of double-edge-rooted open chains are referred to as closed chains. Depending on which type of pasting is used, these may be classified as co-pasted or contra-pasted closed chains. For example, circular ladders and Möbius ladders are co- and contra-pasted closed chains obtained by self-edge-amalgamating closed-end ladders.

We work under the assumption that we already have the partitioned genus distribution of the graph that we wish to self-edge-amalgamate. For smaller graphs, this can be done easily by using the Heffter-Edmonds algorithm Gross and Tucker, 2001. For large open chains, one can rely on the recurrences presented in Appendix A for finding partitioned genus distributions.

### 3.1 Productions for Self-Edge-Amalgamation

In 82 , productions were used to highlight the behavior of fb-walks of two embeddings as they underwent edge-amalgamation. The concept of production is now adapted for the self-edge-amalgamation operation.

Let $x_{i}$ be any double-root sub-partial. Then a production for self-edge-amalgamation is used to represent the ways in which an embedding of a double-edge-rooted graph ( $G, e, f$ ) of type $x$ on surface $S_{i}$ self-edge-amalgamates on the root-edges $e$ and $f$ to give various types
of embeddings of the resulting graph $W$. Formally, we write

$$
x_{i}(G, e, f) \longrightarrow g_{k_{1}}(W)+g_{k_{2}}(W)+g_{k_{3}}(W)+g_{k_{4}}(W)
$$

where $k_{1}, k_{2}, k_{3}, k_{4}$ are (not necessarily distinct) integer-valued functions of $i$. This can be read as follows:

An embedding of $(G, e, f)$ of type $x$ on surface $S_{i}$ self-amalgamates on the root-edges $e$ and $f$ to give four embeddings of the graph $W$ on the surfaces $S_{k_{1}}, S_{k_{2}}, S_{k_{3}}$ and $S_{k_{4}}$.

The production, as defined above, does not specify whether the self-amalgamation is a co-self-paste or a contra-self-paste. However, as we shall see, for an application that seeks to find the genus distribution of a self-amalgamated graph, the system of productions will consistently refer to only one of the two types of self-pasting. While considering the self-edge-amalgamation for an embedding on an oriented surface and modeling it using a production, it is important to maintain a sense of orientation of the strands. Each embedding of a self-edge-amalgamated graph $W=*_{e f}(G, e, f)$ induces a unique embedding of the graph $G$, such that the rotation system of $W$ is consistent with the rotation system of $G$.

Theorem 10. Let $(G, e, f)$ be a double-edge-rooted graph, where both root-edges have two 2-valent endpoints. Then the following productions apply to all scenarios of co-self-paste and contra-self-paste in which no fb-walk of the embedding of $G$ is incident on both root-edges e and $f$ :

$$
\begin{align*}
d d_{i}^{0}(G) & \longrightarrow 2 g_{i+1}(W)+2 g_{i+2}(W)  \tag{3.1}\\
d s_{i}^{0}(G) & \longrightarrow 4 g_{i+1}(W)  \tag{3.2}\\
s d_{i}^{0}(G) & \longrightarrow 4 g_{i+1}(W)  \tag{3.3}\\
s s_{i}^{0}(G) & \longrightarrow 4 g_{i+1}(W) \tag{3.4}
\end{align*}
$$

Proof. The proof of Production (3.1) follows by face-tracing of the fb-walks incident on both root-edges of the graph $G$. The recombination of strands caused by a self-edgeamalgamation is now examined where the embedding of graph $G$ is of type $d d^{0}$. The
production shown in the upper half of Figure 3.1 depicts the case of a co-self-paste while the drawing in the lower half shows a contra-self-paste on the same root-edges. Both cases yield the same results.


Figure 3.1: Productions for co-self-pasting and contra-self-pasting a $d d^{0}$-type embedding of $(G, e, f)$.

The self-amalgamation produces two fewer vertices and one less edge in each of the four resultant graph embeddings. The first and last embeddings shown for each production have two fb-walks merging as a consequence of self-amalgamation. In the second and third embeddings, all four fb-walks that are incident on the two root-edges merge into a single fb-walk. Applying the Euler polyhedral equation, it can be seen that in the former case the decrease of a single face results in a genus increment of 1 , while in the latter case the decrease of three fb-walks results in a genus increment of 2 . The proofs for the remaining productions are similar and also follow by face-tracing. For the sake of brevity, these are omitted.

For embeddings in which one or two fb-walks are incident on both root-edges, the productions for co-self-pasting and contra-self-pasting may differ. In particular, for $d d^{\prime}$ and $d d^{\prime \prime}$ we get different results for both ways of pasting.

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Theorem 11. Let $(G, e, f)$ be a double-edge-rooted graph, where both root-edges have two 2-valent endpoints. Then the following productions describe all cases of co-self-pasting for embeddings of $G$ of type $d d^{\prime}$ :

$$
\begin{align*}
& {\overline{d d^{\prime}}}_{i}(G) \longrightarrow g_{i}(W)+3 g_{i+1}(W)  \tag{3.5}\\
& {\widetilde{d d^{\prime}}}_{i}(G) \longrightarrow g_{i}(W)+3 g_{i+1}(W)  \tag{3.6}\\
& {\overrightarrow{d d^{\prime}}}_{i}(G) \longrightarrow 4 g_{i+1}(W)  \tag{3.7}\\
& {\overleftarrow{d d^{\prime}}}_{i}(G) \longrightarrow 4 g_{i+1}(W) \tag{3.8}
\end{align*}
$$

Furthermore, the following productions describe all cases of co-self-pasting for embeddings of $G$ of type $d d^{\prime \prime}$ :

$$
\begin{align*}
& {\overrightarrow{d d^{\prime \prime}}}_{i}(G) \longrightarrow 4 g_{i}(W)  \tag{3.9}\\
& {\overleftarrow{d d^{\prime \prime}}}_{i}(G) \longrightarrow 2 g_{i}(W)+2 g_{i+1}(W) \tag{3.10}
\end{align*}
$$

Proof. For illustration of Production (3.5), which describes the effects of co-self-pasting a $\overline{d d^{\prime}}$-type embedding of $G$, we refer to the upper half of Figure 3.2 . Observe that the first embedding of graph $W$ stands out from the other three, in that there is a net increase of one fb-walk as the fb-walk incident on both roots of the embedding of $G$ breaks into two fb-walks during self-pasting. This does not occur in the other three cases, where there is a net decrease of one fb-walk in the resulting embedding. The former results in an unchanged genus of the resultant graph embedding, while the latter results in a genus increment of one. This accounts for Production (3.5).

For Production (3.7), the illustration in the bottom half of Figure 3.2 shows that all four embeddings resulting from the self-pasting of $G$ end up with one less face, thereby warranting a genus increment of 1 for the resulting embeddings. Proofs for Productions (3.6) and (3.8) are similar to the proofs for (3.5) and (3.7), respectively, and are omitted for brevity.

Productions (3.9 and 3.10 for co-self-pasting a $\overrightarrow{d d^{\prime \prime}}$ - or a $\overleftarrow{d d^{\prime \prime}}$-type embedding of $G$ can also be derived by using the same technique. Figure 3.3 is included for aiding with the proof of Production (3.10), and the proof of Production (3.9) is omitted.


Figure 3.2: Productions for co-self-pasting $\overline{d d^{\prime}}$-type and $\overrightarrow{d d^{\prime}}$-type embeddings of $(G, e, f)$.


Figure 3.3: Production for co-self-pasting a $\overleftarrow{d d^{\prime \prime}}$-type embedding of $(G, e, f)$.

Theorem 12. Let $(G, e, f)$ be a double-edge-rooted graph, where both root-edges have two 2-valent endpoints. Then the following productions apply for contra-self-pasting all dd'-type embeddings of $G$ :

$$
\begin{align*}
& {\overline{d d^{\prime}}}_{i}(G) \longrightarrow 4 g_{i+1}(W)  \tag{3.11}\\
& {\widetilde{d d^{\prime}}}_{i}(G) \longrightarrow 4 g_{i+1}(W)  \tag{3.12}\\
& {\overrightarrow{d d^{\prime}}}_{i}(G) \longrightarrow g_{i}(W)+3 g_{i+1}(W)  \tag{3.13}\\
& {\overleftarrow{d d^{\prime}}}^{\prime}(G) \longrightarrow g_{i}(W)+3 g_{i+1}(W) \tag{3.14}
\end{align*}
$$

Furthermore, the following productions apply for contra-self-pasting all dd ${ }^{\prime \prime}$-type embeddings of $G$ :

$$
\begin{align*}
& {\overrightarrow{d d^{\prime \prime}}}_{i}(G) \longrightarrow 2 g_{i}(W)+2 g_{i+1}(W)  \tag{3.15}\\
& {\overleftarrow{d d^{\prime \prime}}}_{i}(G) \longrightarrow 4 g_{i}(W) \tag{3.16}
\end{align*}
$$

Proof. Figure 3.4 illustrates Production (3.12) in the upper half and Production (3.14) in the bottom half. For Production (3.12), in all four embeddings resulting from the contra-selfpasting, the three fb-walks incident on the root-edges of the graph $G$ break into strands that merge into two fb-walks, as shown in Figure 3.4. While, for Production (3.14), this happens for only three of the resulting embeddings of graph $W$. For the remaining embedding, the fb-walks break into strands that recombine to give four distinct fb-walks. Proofs of Productions (3.11) and (3.13) are similar.


Figure 3.4: Productions for contra-self-pasting $\widetilde{d d^{\prime}}$-type and $\overleftarrow{d d^{\prime}}$-type embeddings of $(G, e, f)$.

Similarly, applying contra-self-pasting to the root-edges of a $\overrightarrow{d d^{\prime \prime}}$-type embedding of $(G, e, f)$ results in two embeddings of $W$ where all the fb-walks incident on the roots merge into a single fb-walk, and two embeddings where they break into strands that recombine into three distinct fb-walks, as shown in Figure 3.5. This proves Production 3.15. Proof for Production (3.16) is similar and is omitted for brevity.

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 GRAPHS

Figure 3.5: Production for contra-self-pasting $\overrightarrow{d d^{\prime \prime}}$-type embeddings of $(G, e, f)$.

Theorem 13. Let $(G, e, f)$ be a double-edge-rooted graph, where both roots have two 2valent endpoints. Then the following productions cover all scenarios for co-self-pasting and contra-self-pasting, where the embedding of $G$ is of type $d s^{\prime}$ or $s d^{\prime}$ :

$$
\begin{align*}
d s_{i}^{\prime}(G) & \longrightarrow 2 g_{i}(W)+2 g_{i+1}(W)  \tag{3.17}\\
s d_{i}^{\prime}(G) & \longrightarrow 2 g_{i}(W)+2 g_{i+1}(W) \tag{3.18}
\end{align*}
$$

Proof. Due to the symmetry of the models $\overrightarrow{d s^{\prime}}$ and $\overrightarrow{s d^{\prime}}$, and of $\overleftarrow{d s^{\prime}}$ and $\overleftarrow{s d^{\prime}}$, we need only provide the proof for one of the two Productions (3.17) and (3.18).


Figure 3.6: Productions for co-pasting and contra-pasting a $\overrightarrow{d s^{\prime}}$-type embedding of $(G, e, d)$.

Figure 3.6 illustrates the proof for the co-self-pasted and contra-self-pasted $\overrightarrow{d s^{\prime}}$-type embedding of $G$. Both operations result in two embeddings with a genus increment of 0
and two with a genus increment of 1 . The proof for a $\overleftarrow{d s^{\prime}}$-type embedding of $G$ is similar and leads to the observation that the production body for the embedding type $\overleftarrow{d s^{\prime}}$ is identical to the production body for the embedding type $\overrightarrow{d s^{\prime}}$. This is true for both co-self-pasting and contra-self-pasting operations. This completes the proof of Production (3.17).

Theorem 14. Let $(G, e, f)$ be a double-edge-rooted graph, where both roots have two 2-valent endpoints. Then the following productions apply for co-self-pasting or contra-self-pasting an ss ${ }^{1}$ or $s s^{2}$-type embedding of $G$ :

$$
\begin{align*}
& s s_{i}^{1}(G) \longrightarrow 4 g_{i}(W)  \tag{3.19}\\
& s s_{i}^{2}(G) \longrightarrow 3 g_{i}(W)+g_{i-1}(W) \tag{3.20}
\end{align*}
$$

Proof. When the embedding of a double-edge-rooted graph $G$ is of type $s s^{1}$, a self-amalgamation on the root-edges breaks the single fb-walk incident on both roots into strands that recombine to give two fb-walks in all the corresponding embeddings of the self-amalgamated graph $W$. The additional face balances out the decrease of two vertices and one edge to retain the same genus in each of the four resulting embeddings of $W$. This holds for co-self-pasting as well as contra-self-pasting as evident from Figure 3.7. The proof of Production 3.20 for the self-amalgamation of an $s s^{2}$-type is also similar and is omitted.


Figure 3.7: Productions for co-self-pasting and contra-self-pasting an $s s^{1}$-type embedding of $(G, e, f)$.

A convenient table summarizing the results of Theorems 1014 appears in Table 3.1 .
Table 3.1: The productions for self-edge-amalgamation.

| co- and contra-paste productions | reference |
| :---: | :---: |
| $d d_{i}^{0}(G) \longrightarrow 2 g_{i+1}+2 g_{i+2}$ | 3.1) |
| $d s_{i}^{0}(G) \longrightarrow 4 g_{i+1}$ | (3.2) |
| $s d_{i}^{0}(G) \longrightarrow 4 g_{i+1}$ | (3.3) |
| $s s_{i}^{0}(G) \longrightarrow 4 g_{i+1}$ | (3.4) |
| $d s_{i}^{\prime}(G) \longrightarrow 2 g_{i}+2 g_{i+1}$ | 3.17 |
| $s d_{i}^{\prime}(G) \longrightarrow 2 g_{i}+2 g_{i+1}$ | (3.18) |
| $s s_{i}^{1}(G) \longrightarrow 4 g_{i}$ | (3.19) |
| $s s_{i}^{2}(G) \longrightarrow 3 g_{i}+g_{i-1}$ | 3.20) |
| co-paste productions | reference |
| $\overline{d d}^{\prime}{ }_{i}(G) \longrightarrow g_{i}+3 g_{i+1}$ | (3.5) |
| $\widetilde{d d^{\prime}}{ }^{\prime}(G) \longrightarrow g_{i}+3 g_{i+1}$ | (3.6) |
| ${\overrightarrow{d d^{\prime}}}^{\prime}(G) \longrightarrow 4 g_{i+1}$ | (3.7) |
| $\overleftarrow{d d^{\prime}}{ }_{i}(G) \longrightarrow 4 g_{i+1}$ | (3.8) |
| ${\overrightarrow{d d^{\prime \prime}}}_{i}(G) \longrightarrow 4 g_{i}$ | (3.9) |
| ${\overleftarrow{d d^{\prime \prime}}}^{\prime}(G) \longrightarrow 2 g_{i}+2 g_{i+1}$ | 3.10 |
| contra-paste productions | reference |
|  | 3.11) |
| ${\widetilde{d d^{\prime}}}^{\prime}(G) \longrightarrow 4 g_{i+1}$ | (3.12) |
| $\overrightarrow{d d}^{\prime}{ }_{i}(G) \longrightarrow g_{i}+3 g_{i+1}$ | (3.13) |
| $\overleftarrow{d d^{\prime}}{ }_{i}(G) \longrightarrow g_{i}+3 g_{i+1}$ | (3.14) |
| $\overrightarrow{d d \prime \prime}^{\prime}(G) \longrightarrow 2 g_{i}+2 g_{i+1}$ | 3.15) |
| ${\overleftarrow{d d^{\prime \prime}}}_{i}(G) \longrightarrow 4 g_{i}$ | 3.16 |

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Remark 9. As it turns out, the productions for second-order sub-partial types of dd' and $d d^{\prime \prime}$ are the only ones that disagree for a co-self-paste and a contra-self-paste. Moreover, the results for the co- and contra-self-amalgamation for both $d d^{\prime}$ - and $d d^{\prime \prime}$-type embeddings are symmetric in the sense that the production body for co-self-pasting a $\overrightarrow{d d^{\prime \prime}}$-type (or a $\overleftarrow{d d^{\prime \prime}}$-type) embedding of $G$ is the same as the production body for contra-self-pasting $\overleftarrow{d d^{\prime \prime}}$-type (or a $\overrightarrow{d d^{\prime \prime}}$ type) embedding of $G$. Likewise, for the pairs of $\overline{d d^{\prime}}$-, $\widetilde{d d^{\prime}}$-types of embeddings of $G$ which show a symmetry with the $\overrightarrow{d d^{\prime}}$-, $\overleftarrow{d d^{\prime}}$-types of embeddings of $G$.

Theorem 15. Let $W$ be the graph formed by co-self-pasting of $(G, e, f)$. Then

$$
\begin{align*}
g_{i}(W)= & 2 d d_{i-2}^{0}(G)+2 d d_{i-1}^{0}(G)+3{\overline{d d^{\prime}}}_{i-1}(G)+3{\widetilde{d d^{\prime}}}^{i-1} \\
& +4{\overleftarrow{d d^{\prime}}}_{i-1}(G)+4{\overrightarrow{d d^{\prime}}}_{i-1}(G) \\
& +2 s{\overleftarrow{d d^{\prime \prime}}}_{i-1}^{\prime}(G)+4 d s_{i-1}^{0}(G)+4 s s_{i-1}^{0}(G)+{\overline{d d^{\prime}}}_{i}(G)+{\widetilde{d d^{\prime}}}_{i}(G)+4{\overrightarrow{d d^{\prime \prime}}}^{\prime}{ }_{i}(G)+4 s d_{i-1}^{0}(G) \\
& +2 d s_{i}^{\prime}(G)+2 s d_{i}^{\prime}(G)+4 s s_{i}^{1}(G)+3 s s_{i}^{2}(G)+s s_{i+1}^{2}(G) \tag{3.21}
\end{align*}
$$

Proof. The Production (3.1):

$$
d d_{i}^{0}(G) \longrightarrow 2 g_{i+1}(W)+2 g_{i+2}(W)
$$

indicates that each $d d^{0}$-type embedding of $(G, e, f)$ on the surface $S_{i}$ when self-amalgamated, induces two embeddings of $W$ on $S_{i+1}$ and two on the surface $S_{i+2}$. These contributions account for the first two terms $2 d d_{i-2}^{0}(G)+2 d d_{i-1}^{0}(G)$ on the right-hand side of Equation (3.21). Taking into account all the contributions made by productions listed in Theorems 10, 14, the result follows.

Theorem 16. Let $W$ be the graph formed by contra-self-pasting of $(G, e, f)$. Then

$$
\begin{align*}
g_{i}(W)= & 2 d d_{i-2}^{0}(G)+2 d d_{i-1}^{0}(G)+3{\overrightarrow{d d^{\prime}}}_{i-1}(G)+3{\overleftarrow{d d^{\prime}}}_{i-1}(G)+4{\overline{d d^{\prime}}}_{i-1}(G) \\
& +4{\widetilde{d d^{\prime}}}_{i-1}(G)+2{\overrightarrow{d d^{\prime \prime}}}^{\prime}{ }_{i-1}(G)+4 d s_{i-1}^{0}(G)+2 d s_{i-1}^{\prime}(G)+4 s d_{i-1}^{0}(G) \\
& +2 s d_{i-1}^{\prime}(G)+4 s s_{i-1}^{0}(G)+{\overrightarrow{d d^{\prime}}}_{i}(G)+{\overleftarrow{d d^{\prime}}}_{i}(G)+4{\overleftarrow{d d^{\prime \prime}}}_{i}(G)+2{\overrightarrow{d d^{\prime \prime}}}_{i}(G) \\
& +2 d s_{i}^{\prime}(G)+2 s d_{i}^{\prime}(G)+4 s s_{i}^{1}(G)+3 s s_{i}^{2}(G)+s s_{i+1}^{2}(G) \tag{3.22}
\end{align*}
$$

Proof. The proof for Equation (3.22) is obvious from Equation (3.21) and our earlier remarks on the symmetries of the productions for $d d^{\prime}$ and $d d^{\prime \prime}$ second-order sub-partial types.

Thus, depending on whether one plans on forming a closed chain through a co-selfamalgamation or through a contra-self-amalgamation, the genus distribution of the closed chain is calculated by using Theorems 15 or 16, respectively. For instance, the partitioned genus distributions calculated for the closed-end ladders in $\$ 2.9$ can now be used with Theorems 15 and 16 to obtain the partitioned genus-distributions for both circular and Möbius ladders.

### 3.2 Application: Revisiting Circular Ladders and Möbius Ladders

The genus distributions of circular ladders and Möbius ladders were first derived by McGeoch, 1987. 2.9 shows how calculation of the double-root genus distributions of closed-end ladders is reducible to a routine recursion. This in turn reduces the derivation of the genus distributions of circular and Möbius ladders, in turn, to a routine substitution into an equation.

Let $L_{n}$ be the closed-end ladder with 2 end-rungs and $n$ interior rungs, as shown in Figure 3.8. Let $C L_{n}$ denote the circular ladder with $n$ rungs as illustrated in Figure 3.9.


-     - •

Figure 3.8: Closed-end ladders $L_{n}$.

Observe that co-self-pasting the closed-end ladder $L_{n}$ on the root-edges yields $C L_{n+1}$.


$\mathrm{CL}_{3}$

$\mathrm{CL}_{4}$

Figure 3.9: Circular ladders $C L_{n}$.

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 GRAPHSSimilarly, a Möbius ladder with $n$ rungs is denoted by $M L_{n}$, as shown in Figure 3.10. It can be observed that contra-self-pasting the closed-end ladder $L_{n}$ on the root-edges yields $M L_{n+1}$.


Figure 3.10: Möbius ladders $M L_{n}$.

A small example is given here to demonstrate how the genus distributions of $C L_{4}$ and $M L_{4}$ can be calculated from the partitioned genus distribution of $L_{3}$. We begin by reproducing in Table 3.2 the partitioned genus distribution of the closed-end ladder $L_{3}$ originally derived in $\$ 2.9$.

Table 3.2: Double-root partials of $L_{3}$.

| $L_{3}$ |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $k$ | 0 | 1 |  | $k$ |  | 1 | 1 |
| $d d_{k}^{0}$ |  | 0 | 0 | $\overrightarrow{d s^{\prime}}{ }_{k}$ |  |  | 2 |
| $\overline{d d^{\prime}}{ }_{k}$ |  | 6 | 0 | $\overleftarrow{d s^{\prime}}{ }_{k}$ |  |  | 2 |
| $\widetilde{d d^{\prime}}{ }_{k}$ | 1 | 6 | 0 | $s d_{k}^{0}$ |  | 4 | 4 |
| ${\overrightarrow{d d^{\prime}}}^{\prime}$ | 0 | 6 | 0 | $\overrightarrow{s d^{\prime}}{ }_{k}$ | 0 | 2 | 2 |
| $\overleftarrow{d d^{\prime}}{ }_{k}$ | 0 | 6 | 0 | $\overleftarrow{s d^{\prime}}{ }_{k}$ |  | 2 | 2 |
| $\overrightarrow{d d^{\prime \prime}}{ }_{k}$ |  | 0 | 0 | $s s_{k}^{0}$ |  | 0 | 0 |
| $\overleftarrow{d d^{\prime \prime}}{ }_{k}$ | 0 | 0 | 0 | ssk |  | 0 | 8 |
| $d s_{k}^{0}$ |  | 4 | 0 | $s s^{2}$ |  | 0 | 0 |

Simply plugging the values from Table 3.2 into Equation (3.21) yields the genus distributions of circular ladder $C L_{4}$.

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$$
\begin{aligned}
g_{0}\left(C L_{4}\right)= & {\overline{d d^{\prime}}}_{0}\left(L_{3}\right)+{\widetilde{d d^{\prime}}}_{0}\left(L_{3}\right)+4{\overrightarrow{d d^{\prime \prime}}}_{0}\left(L_{3}\right)+2 \overleftarrow{d d^{\prime \prime}}{ }_{0}\left(L_{3}\right)+2 d s_{0}^{\prime}\left(L_{3}\right)+2 s d_{0}^{\prime}\left(L_{3}\right) \\
& +4 s s_{0}^{1}\left(L_{3}\right)+3 s s_{0}^{2}\left(L_{3}\right)+s s_{1}^{2}\left(L_{3}\right) \\
= & 1+1+4 \times 0+2 \times 0+2 \times 0+2 \times 0+4 \times 0+3 \times 0+0=2 \\
g_{1}\left(C L_{4}\right)= & 2 d d_{0}^{0}\left(L_{3}\right)+3{\overline{d d^{\prime}}}^{0}\left(L_{3}\right)+3 \widetilde{d d^{\prime}}{ }_{0}\left(L_{3}\right)+4{\overrightarrow{d d^{\prime}}}_{0}\left(L_{3}\right)+4 \overleftarrow{d d^{\prime}}{ }_{0}\left(L_{3}\right)+2 \overleftarrow{d d^{\prime \prime}}{ }_{0}\left(L_{3}\right) \\
& +4 d s_{0}^{0}\left(L_{3}\right)+2 d s_{0}^{\prime}\left(L_{3}\right)+4 s d_{0}^{0}\left(L_{3}\right)+2 s d_{0}^{\prime}\left(L_{3}\right)+4 s s_{0}^{0}\left(L_{3}\right)+{\overline{d d^{\prime}}}_{1}\left(L_{3}\right) \\
& +\widetilde{d d^{\prime}}{ }_{1}\left(L_{3}\right)+4{\overrightarrow{d d^{\prime \prime}}}_{1}\left(L_{3}\right)+2{\overleftarrow{d d^{\prime \prime}}}_{1}\left(L_{3}\right)+2 d s_{1}^{\prime}\left(L_{3}\right)+2 s d_{1}^{\prime}\left(L_{3}\right)+4 s s_{1}^{1}\left(L_{3}\right) \\
& +3 s s_{1}^{2}\left(L_{3}\right)+s s_{2}^{2}\left(L_{3}\right) \\
= & 2 \times 6+3 \times 1+3 \times 1+4 \times 0+4 \times 0+2 \times 0+4 \times 0+2 \times 0+4 \times 0 \\
& +2 \times 0+4 \times 0+6+6+4 \times 0+2 \times 0+2 \times 4+2 \times 4+4 \times 0+3 \times 0 \\
& +8=54 \\
g_{2}\left(C L_{4}\right)= & 2 d d_{0}^{0}\left(L_{3}\right)+2 d d_{1}^{0}\left(L_{3}\right)+3 \overline{d d^{\prime}}{ }_{1}\left(L_{3}\right)+3 \widetilde{d d^{\prime}}{ }_{1}\left(L_{3}\right)+4 \overrightarrow{d d^{\prime}}{ }_{1}\left(L_{3}\right)+4 \overleftarrow{d d^{\prime}}{ }_{1}\left(L_{3}\right) \\
& +2 \overleftarrow{d d^{\prime \prime}}{ }_{1}\left(L_{3}\right)+4 d s_{1}^{0}\left(L_{3}\right)+2 d s_{1}^{\prime}\left(L_{3}\right)+4 s d_{1}^{0}\left(L_{3}\right)+2 s d_{1}^{\prime}\left(L_{3}\right)+4 s s_{1}^{0}\left(L_{3}\right) \\
& +{\overline{d d^{\prime}}}_{2}\left(L_{3}\right)+\widetilde{d d^{\prime}}{ }_{2}\left(L_{3}\right)+4{\overrightarrow{d d^{\prime \prime}}}^{2}\left(L_{3}\right)+2 \overleftarrow{d d^{\prime \prime}}{ }_{2}\left(L_{3}\right)+2 d s_{2}^{\prime}\left(L_{3}\right)+2 s d_{2}^{\prime}\left(L_{3}\right) \\
& +4 s s_{2}^{1}\left(L_{3}\right)+3 s s_{2}^{2}\left(L_{3}\right)+s s_{3}^{2}\left(L_{3}\right) \\
= & 2 \times 6+2 \times 0+3 \times 6+3 \times 6+4 \times 6+4 \times 6+2 \times 0+4 \times 4+2 \times 4 \\
& +4 \times 4+2 \times 4+4 \times 0+0+0+4 \times 0+2 \times 0+2 \times 0+2 \times 0+4 \times 8 \\
& +3 \times 8+0=200 \\
g_{3}\left(C L_{4}\right)= & 0
\end{aligned}
$$

Whereas, plugging the values from Table 3.2 into Equation (3.22) produces the genus distribution of the Möbius ladder $M L_{4}$ :

$$
\begin{aligned}
g_{0}\left(M L_{4}\right)= & {\overrightarrow{d d^{\prime}}}_{0}\left(L_{3}\right)+{\overleftarrow{d d^{\prime}}}_{0}\left(L_{3}\right)+4{\overleftarrow{d d^{\prime \prime}}}_{0}\left(L_{3}\right)+2{\overrightarrow{d d^{\prime \prime}}}_{0}\left(L_{3}\right)+2 d s_{0}^{\prime}\left(L_{3}\right)+2 s d_{0}^{\prime}\left(L_{3}\right) \\
& +4 s s_{0}^{1}\left(L_{3}\right)+3 s s_{0}^{2}\left(L_{3}\right)+s s_{1}^{2}\left(L_{3}\right) \\
= & 0+0+4 \times 0+2 \times 0+2 \times 0+2 \times 0+4 \times 0+3 \times 0+0=0 \\
g_{1}\left(M L_{4}\right)= & 2 d d_{0}^{0}\left(L_{3}\right)+3{\overrightarrow{d d^{\prime}}}_{0}\left(L_{3}\right)+3{\overleftarrow{d d^{\prime}}}_{0}\left(L_{3}\right)+4{\overline{d d^{\prime}}}_{0}\left(L_{3}\right)+4{\widetilde{d d^{\prime}}}_{0}\left(L_{3}\right)+2{\overrightarrow{d d^{\prime \prime}}}^{0}{ }_{0}\left(L_{3}\right) \\
& +4 d s_{0}^{0}\left(L_{3}\right)+2 d s_{0}^{\prime}\left(L_{3}\right)+4 s d_{0}^{0}\left(L_{3}\right)+2 s d_{0}^{\prime}\left(L_{3}\right)+4 s s_{0}^{0}\left(L_{3}\right)+{\overrightarrow{d d^{\prime}}}_{1}\left(L_{3}\right)
\end{aligned}
$$

$$
\begin{aligned}
& +\overleftarrow{d d^{\prime}}{ }_{1}\left(L_{3}\right)+4 \overleftarrow{d d^{\prime \prime}}{ }_{1}\left(L_{3}\right)+2{\overrightarrow{d d^{\prime \prime}}}_{1}\left(L_{3}\right)+2 d s_{1}^{\prime}\left(L_{3}\right)+2 s d_{1}^{\prime}\left(L_{3}\right)+4 s s_{1}^{1}\left(L_{3}\right) \\
& +3 s s_{1}^{2}\left(L_{3}\right)+s s_{2}^{2}\left(L_{3}\right) \\
= & 2 \times 6+3 \times 0+3 \times 0+4 \times 1+4 \times 1+2 \times 0+4 \times 0+2 \times 0+4 \times 0 \\
& +2 \times 0+4 \times 0+6+6+4 \times 0+2 \times 0+2 \times 4+2 \times 4+4 \times 0+3 \times 0 \\
& +8=56 \\
g_{2}\left(M L_{4}\right)= & 2 d d_{0}^{0}\left(L_{3}\right)+2 d d_{1}^{0}\left(L_{3}\right)+3{\overrightarrow{d d^{\prime}}}_{1}\left(L_{3}\right)+3 \overleftarrow{d d^{\prime}}{ }_{1}\left(L_{3}\right)+4{\overline{d d^{\prime}}}^{\prime}\left(L_{3}\right)+4 \widetilde{d d^{\prime}}{ }_{1}\left(L_{3}\right) \\
& +2{\overrightarrow{d d^{\prime \prime}}}^{\prime}\left(L_{3}\right)+4 d s_{1}^{0}\left(L_{3}\right)+2 d s_{1}^{\prime}\left(L_{3}\right)+4 s d_{1}^{0}\left(L_{3}\right)+2 s d_{1}^{\prime}\left(L_{3}\right)+4 s s_{1}^{0}\left(L_{3}\right) \\
& +{\overrightarrow{d d^{\prime}}}_{i}\left(L_{3}\right)+\overleftarrow{d d^{\prime}}{ }_{i}\left(L_{3}\right)+4 \overleftarrow{d d^{\prime \prime}}{ }_{i}\left(L_{3}\right)+2{\overrightarrow{d d^{\prime \prime}}}_{i}\left(L_{3}\right)+2 d s_{i}^{\prime}\left(L_{3}\right)+2 s d_{i}^{\prime}\left(L_{3}\right) \\
& +4 s s_{i}^{1}\left(L_{3}\right)+3 s s_{i}^{2}\left(L_{3}\right)+s s_{i+1}^{2}\left(L_{3}\right) \\
= & 2 \times 6+2 \times 0+3 \times 6+3 \times 6+4 \times 6+4 \times 6+2 \times 0+4 \times 4+2 \times 4 \\
& +4 \times 4+2 \times 4+4 \times 0+0+0+4 \times 0+2 \times 0+2 \times 0+2 \times 0+4 \times 8 \\
& +3 \times 8+0=200 \\
g_{3}\left(M L_{4}\right)= & 0
\end{aligned}
$$

### 3.3 Application: Closed Chains of Copies of a Triangular Prism Graph

Open chains of copies of the triangular prism graph were encountered in 2.10. As an example of two entirely new calculations of genus distributions of closed chains, consider the closed chains of copies of the triangular prism graph. As before, the double-edge-rooted triangular prism graph is denoted by $\Delta_{G}$ and the open chain consisting of $n$ copies of $\Delta_{G}$ by $\mathcal{P} r_{n}$. The partitioned genus distribution of the open chains $\mathcal{P} r_{1}, \mathcal{P} r_{2}$ and $\mathcal{P} r_{3}$ were produced in Tables 2.7-2.8.

Let $\mathcal{C P} r_{n}$ be the co-self-amalgamated closed chain of $n$ copies of $\Delta_{G}$, as shown in Figure 3.11. We plug the values from Table 2.7 into Equation (3.21) to calculate genus distributions for $\mathcal{C P} r_{1}$ and $\mathcal{C P} r_{2}$.
This is illustrated as follows:

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Figure 3.11: Co-pasted closed chains of copies of a triangular prism graph.

$$
\begin{aligned}
g_{0}\left(\mathcal{C P} r_{1}\right)= & {\overline{d d^{\prime}}}_{0}\left(\mathcal{P} r_{1}\right)+\widetilde{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{1}\right)+4{\widetilde{d d^{\prime \prime}}{ }_{0}\left(\mathcal{P} r_{1}\right)+2 \overleftarrow{d d^{\prime \prime}}{ }_{0}\left(\mathcal{P} r_{1}\right)+2 d s_{0}^{\prime}\left(\mathcal{P} r_{1}\right)}+2 s d_{0}^{\prime}\left(\mathcal{P} r_{1}\right)+4 s s_{0}^{1}\left(\mathcal{P} r_{1}\right)+3 s s_{0}^{2}\left(\mathcal{P} r_{1}\right)+s s_{1}^{2}\left(\mathcal{P} r_{1}\right) \\
= & 1+1+4 \times 0+2 \times 0+2 \times 0+2 \times 0+4 \times 0+3 \times 0+2 \\
= & 4 \\
g_{1}\left(\mathcal{C P} r_{1}\right)= & 2 d d_{0}^{0}\left(\mathcal{P} r_{1}\right)+3 \overline{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{1}\right)+3 \widetilde{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{1}\right)+4{\overrightarrow{d d^{\prime}}}^{\prime}\left(\mathcal{P} r_{1}\right)+4 \overleftarrow{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{1}\right) \\
& +2 \overleftarrow{d d^{\prime \prime}}{ }_{0}\left(\mathcal{P} r_{1}\right)+4 d s_{0}^{0}\left(\mathcal{P} r_{1}\right)+2 d s_{0}^{\prime}\left(\mathcal{P} r_{1}\right)+4 s d_{0}^{0}\left(\mathcal{P} r_{1}\right)+2 s d_{0}^{\prime}\left(\mathcal{P} r_{1}\right) \\
& +4 s s_{0}^{0}\left(\mathcal{P} r_{1}\right)+{\overline{d d^{\prime}}{ }_{1}\left(\mathcal{P} r_{1}\right)+\widetilde{d d^{\prime}}{ }_{1}\left(\mathcal{P} r_{1}\right)+4{\widehat{d d^{\prime \prime}}}^{1}\left(\mathcal{P} r_{1}\right)+2 \overleftarrow{d d^{\prime \prime}}{ }_{1}\left(\mathcal{P} r_{1}\right)}+22 d s_{1}^{\prime}\left(\mathcal{P} r_{1}\right)+2 s d_{1}^{\prime}\left(\mathcal{P} r_{1}\right)+4 s s_{1}^{1}\left(\mathcal{P} r_{1}\right)+3 s s_{1}^{2}\left(\mathcal{P} r_{1}\right)+s s_{2}^{2}\left(\mathcal{P} r_{1}\right) \\
= & 2 \times 0+3 \times 1+3 \times 1+4 \times 0+4 \times 0+2 \times 0+4 \times 0+2 \times 0 \\
& +4 \times 0+2 \times 0+4 \times 0+0+0+4 \times 12+2 \times 0+2 \times 8+2 \times 8 \\
& +4 \times 0+3 \times 2+0
\end{aligned}
$$

$=92$

$$
\begin{aligned}
g_{2}\left(\mathcal{C P} r_{1}\right)= & 2 d d_{0}^{0}\left(\mathcal{P} r_{1}\right)+2 d d_{1}^{0}\left(\mathcal{P} r_{1}\right)+3{\overline{d d^{\prime}}}_{1}\left(\mathcal{P} r_{1}\right)+3{\widetilde{d d^{\prime}}}_{1}\left(\mathcal{P} r_{1}\right)+4{\overrightarrow{d d^{\prime}}}^{\prime}\left(\mathcal{P} r_{1}\right) \\
& +4 \overleftarrow{d d^{\prime}}{ }_{1}\left(\mathcal{P} r_{1}\right)+2 \overleftarrow{d d^{\prime \prime}}{ }_{1}\left(\mathcal{P} r_{1}\right)+4 d s_{1}^{0}\left(\mathcal{P} r_{1}\right)+2 d s_{1}^{\prime}\left(\mathcal{P} r_{1}\right)+4 s d_{1}^{0}\left(\mathcal{P} r_{1}\right) \\
& +2 s d_{1}^{\prime}\left(\mathcal{P} r_{1}\right)+4 s s_{1}^{0}\left(\mathcal{P} r_{1}\right)+{\overline{d d^{\prime}}{ }_{2}\left(\mathcal{P} r_{1}\right)+{\widetilde{d d^{\prime}}}^{\prime}\left(\mathcal{P} r_{1}\right)+4 \overrightarrow{d d^{\prime \prime}}{ }_{2}\left(\mathcal{P} r_{1}\right)}+2 \overleftarrow{d d^{\prime \prime}}{ }_{2}\left(\mathcal{P} r_{1}\right)+2 d s_{2}^{\prime}\left(\mathcal{P} r_{1}\right)+2 s d_{2}^{\prime}\left(\mathcal{P} r_{1}\right)+4 s s_{2}^{1}\left(\mathcal{P} r_{1}\right)+3 s s_{2}^{2}\left(\mathcal{P} r_{1}\right) \\
& +s s_{3}^{2}\left(\mathcal{P} r_{1}\right) \\
= & 2 \times 0+2 \times 0+3 \times 0+3 \times 0+4 \times 0+4 \times 0+2 \times 0+4 \times 4+2 \times 8 \\
& +4 \times 4+2 \times 8+4 \times 0+0+0+4 \times 0+2 \times 0+2 \times 0+2 \times 0 \\
& +4 \times 24+3 \times 0+0
\end{aligned}
$$

$$
=160
$$

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$$
g_{2}\left(\mathcal{C P} r_{2}\right)=2 d d_{0}^{0}\left(\mathcal{P} r_{2}\right)+2 d d_{1}^{0}\left(\mathcal{P} r_{2}\right)+3{\overline{d d^{\prime}}}_{1}\left(\mathcal{P} r_{2}\right)+3{\widetilde{d d^{\prime}}}_{1}\left(\mathcal{P} r_{2}\right)+4 \overrightarrow{d d^{\prime}}{ }_{1}\left(\mathcal{P} r_{2}\right)
$$

$$
+4 \overleftarrow{d d^{\prime}}{ }_{1}\left(\mathcal{P} r_{2}\right)+2 \overleftarrow{d d^{\prime \prime}}{ }_{1}\left(\mathcal{P} r_{2}\right)+4 d s_{1}^{0}\left(\mathcal{P} r_{2}\right)+2 d s_{1}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s d_{1}^{0}\left(\mathcal{P} r_{2}\right)
$$

$$
+2 s d_{1}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s s_{1}^{0}\left(\mathcal{P} r_{2}\right)+{\overline{d d^{\prime}}}_{2}\left(\mathcal{P} r_{2}\right)+{\widetilde{d d^{\prime}}}_{2}\left(\mathcal{P} r_{2}\right)+4 \overrightarrow{d d^{\prime \prime}}{ }_{2}\left(\mathcal{P} r_{2}\right)
$$

$$
+2 \overleftarrow{d d^{\prime \prime}}{ }_{2}\left(\mathcal{P} r_{2}\right)+2 d s_{2}^{\prime}\left(\mathcal{P} r_{2}\right)+2 s d_{2}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s s_{2}^{1}\left(\mathcal{P} r_{2}\right)+3 s s_{2}^{2}\left(\mathcal{P} r_{2}\right)
$$

$$
+s s_{3}^{2}\left(\mathcal{P} r_{2}\right)
$$

$$
=2 \times 6+2 \times 176+3 \times 46+3 \times 46+4 \times 22+4 \times 22+2 \times 0+4 \times 44
$$

$$
+2 \times 12+4 \times 44+2 \times 12+4 \times 0+400+400+4 \times 48+2 \times 48
$$

$$
+2 \times 544+2 \times 544+4 \times 72+3 \times 8+288
$$

$$
=5080
$$

$$
g_{3}\left(\mathcal{C P} r_{2}\right)=2 d d_{1}^{0}\left(\mathcal{P} r_{2}\right)+2 d d_{2}^{0}\left(\mathcal{P} r_{2}\right)+3 \overline{d d^{\prime}}{ }_{2}\left(\mathcal{P} r_{2}\right)+3 \widetilde{d d^{\prime}}{ }_{2}\left(\mathcal{P} r_{2}\right)+4 \overrightarrow{d d^{\prime}}{ }_{2}\left(\mathcal{P} r_{2}\right)
$$

$$
+4 \overleftarrow{d d^{\prime}}{ }_{2}\left(\mathcal{P} r_{2}\right)+2 \overleftarrow{d d^{\prime \prime}}{ }_{2}\left(\mathcal{P} r_{2}\right)+4 d s_{2}^{0}\left(\mathcal{P} r_{2}\right)+2 d s_{2}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s d_{2}^{0}\left(\mathcal{P} r_{2}\right)
$$

$$
+2 \overleftarrow{d d^{\prime \prime}}{ }_{3}\left(\mathcal{P} r_{2}\right)+2 d s_{3}^{\prime}\left(\mathcal{P} r_{2}\right)+2 s d_{3}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s s_{3}^{1}\left(\mathcal{P} r_{2}\right)+3 s s_{3}^{2}\left(\mathcal{P} r_{2}\right)
$$

$$
+s s_{4}^{2}\left(\mathcal{P} r_{2}\right)
$$

$$
\begin{aligned}
& g_{0}\left(\mathcal{C P} r_{2}\right)=\overline{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{2}\right)+\widetilde{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{2}\right)+4 \overrightarrow{d d^{\prime \prime}}{ }_{0}\left(\mathcal{P} r_{2}\right)+2 \overleftarrow{d d^{\prime \prime}}{ }_{0}\left(\mathcal{P} r_{2}\right)+2 d s_{0}^{\prime}\left(\mathcal{P} r_{2}\right) \\
& +2 s d_{0}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s s_{0}^{1}\left(\mathcal{P} r_{2}\right)+3 s s_{0}^{2}\left(\mathcal{P} r_{2}\right)+s s_{1}^{2}\left(\mathcal{P} r_{2}\right) \\
& =1+1+4 \times 0+2 \times 0+2 \times 0+2 \times 0+4 \times 0+3 \times 0+0 \\
& =2 \\
& g_{1}\left(\mathcal{C P} r_{2}\right)=2 d d_{0}^{0}\left(\mathcal{P} r_{2}\right)+3 \overline{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{2}\right)+3 \widetilde{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{2}\right)+4 \overrightarrow{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{2}\right)+4 \overleftarrow{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{2}\right) \\
& +2 \overleftarrow{d d^{\prime \prime}}{ }_{0}\left(\mathcal{P} r_{2}\right)+4 d s_{0}^{0}\left(\mathcal{P} r_{2}\right)+2 d s_{0}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s d_{0}^{0}\left(\mathcal{P} r_{2}\right)+2 s d_{0}^{\prime}\left(\mathcal{P} r_{2}\right) \\
& +4 s s_{0}^{0}\left(\mathcal{P} r_{2}\right)+\overline{d d^{\prime}}{ }_{1}\left(\mathcal{P} r_{2}\right)+\widetilde{d d^{\prime}}{ }_{1}\left(\mathcal{P} r_{2}\right)+4 \overrightarrow{d d^{\prime \prime}}{ }_{1}\left(\mathcal{P} r_{2}\right)+2 \overleftarrow{d d^{\prime \prime}}{ }_{1}\left(\mathcal{P} r_{2}\right) \\
& +2 d s_{1}^{\prime}\left(\mathcal{P} r_{2}\right)+2 s d_{1}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s s_{1}^{1}\left(\mathcal{P} r_{2}\right)+3 s s_{1}^{2}\left(\mathcal{P} r_{2}\right)+s s_{2}^{2}\left(\mathcal{P} r_{2}\right) \\
& =2 \times 6+3 \times 1+3 \times 1+4 \times 0+4 \times 0+2 \times 0+4 \times 0+2 \times 0+4 \times 0 \\
& +2 \times 0+4 \times 0+46+46+4 \times 0+2 \times 0+2 \times 12+2 \times 12+4 \times 0 \\
& +3 \times 0+8 \\
& =166
\end{aligned}
$$

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$$
\begin{aligned}
= & 2 \times 176+2 \times 704+3 \times 400+3 \times 400+4 \times 256+4 \times 256+2 \times 48 \\
& +4 \times 800+2 \times 544+4 \times 800+2 \times 544+4 \times 320+0+0+4 \times 0 \\
& +2 \times 0+2 \times 1920+2 \times 1920+4 \times 1664+3 \times 288+0 \\
= & 31360 \\
g_{4}\left(\mathcal{C P} r_{2}\right)= & 2 d d_{2}^{0}\left(\mathcal{P} r_{2}\right)+2 d d_{3}^{0}\left(\mathcal{P} r_{2}\right)+3 \overline{d d^{\prime}}{ }_{3}\left(\mathcal{P} r_{2}\right)+3 \widetilde{d d^{\prime}}{ }_{3}\left(\mathcal{P} r_{2}\right)+4 \overrightarrow{d d^{\prime}}{ }_{3}\left(\mathcal{P} r_{2}\right) \\
& +4 \overleftarrow{d d^{\prime}}{ }_{3}\left(\mathcal{P} r_{2}\right)+2 \overleftarrow{d d^{\prime \prime}}{ }_{3}\left(\mathcal{P} r_{2}\right)+4 d s_{3}^{0}\left(\mathcal{P} r_{2}\right)+2 d s_{3}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s d_{3}^{0}\left(\mathcal{P} r_{2}\right) \\
& +2 s d_{3}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s s_{3}^{0}\left(\mathcal{P} r_{2}\right)+{\overline{d d^{\prime}}}_{4}\left(\mathcal{P} r_{2}\right)+\widetilde{d d^{\prime}}{ }_{4}\left(\mathcal{P} r_{2}\right)+4 \overrightarrow{d d^{\prime \prime}}{ }_{4}\left(\mathcal{P} r_{2}\right) \\
& +2 \overleftarrow{d d^{\prime \prime}}{ }_{4}\left(\mathcal{P} r_{2}\right)+2 d s_{4}^{\prime}\left(\mathcal{P} r_{2}\right)+2 s d_{4}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s s_{4}^{1}\left(\mathcal{P} r_{2}\right)+3 s s_{4}^{2}\left(\mathcal{P} r_{2}\right) \\
& +s s_{5}^{2}\left(\mathcal{P} r_{2}\right) \\
= & 2 \times 704+2 \times 0+3 \times 0+3 \times 0+4 \times 0+4 \times 0+2 \times 0+4 \times 384 \\
& +2 \times 1920+4 \times 384+2 \times 1920+4 \times 1888+0+0+4 \times 0+2 \times 0 \\
& +2 \times 0+2 \times 0+4 \times 2304+3 \times 0+0 \\
= & 28928
\end{aligned}
$$

In a similar manner, one can compute the genus distributions for $\mathcal{C P} r_{n}$ for higher values of $n$. We omit details but list the genus distributions of $\mathcal{C P} r_{3}$.

$$
\begin{array}{lll}
g_{0}\left(\mathcal{C P} r_{3}\right)=2 & g_{1}\left(\mathcal{C P} r_{3}\right)=278 & g_{2}\left(\mathcal{C P} r_{3}\right)=17480 \\
g_{3}\left(\mathcal{C P} r_{3}\right)=447648 & g_{4}\left(\mathcal{C P} r_{3}\right)=3920896 & g_{5}\left(\mathcal{C P} r_{3}\right)=8667648 \\
g_{6}\left(\mathcal{C P} r_{3}\right)=3723264 & &
\end{array}
$$

Let $\mathcal{K} \mathcal{P} r_{n}$ be the contra-self-amalgamated closed chain of $n$ copies of $\Delta_{G}$, as illustrated in Figure 3.12. The genus distributions for contra-self-amalgamated closed chains $\mathcal{K} \mathcal{P} r_{1}$ and $\mathcal{K P} r_{2}$ are calculated by substituting values from Table 2.7 into Equation (3.22) as follows:


Figure 3.12: Contra-pasted closed chains of copies of a triangular prism graph.

$$
\begin{aligned}
g_{0}\left(\mathcal{K P} r_{1}\right)= & \overrightarrow{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{1}\right)+\overleftarrow{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{1}\right)+4 \overleftarrow{{d d^{\prime \prime}}^{0}}{ }_{0}\left(\mathcal{P} r_{1}\right)+2 \overrightarrow{d d^{\prime \prime}}{ }_{0}\left(\mathcal{P} r_{1}\right)+2 d s_{0}^{\prime}\left(\mathcal{P} r_{1}\right) \\
& +2 s d_{0}^{\prime}\left(\mathcal{P} r_{1}\right)+4 s s_{0}^{1}\left(\mathcal{P} r_{1}\right)+3 s s_{0}^{2}\left(\mathcal{P} r_{1}\right)+s s_{1}^{2}\left(\mathcal{P} r_{1}\right) \\
= & 0+0+4 \times 0+2 \times 0+2 \times 0+2 \times 0+4 \times 0+3 \times 0+2 \\
= & 2 \\
g_{1}\left(\mathcal{K P} r_{1}\right)= & 2 d d_{0}^{0}\left(\mathcal{P} r_{1}\right)+3 \overrightarrow{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{1}\right)+3 \overleftarrow{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{1}\right)+4 \overrightarrow{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{1}\right)+4 \widetilde{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{1}\right) \\
& +2 \overrightarrow{d d^{\prime \prime}}{ }_{0}\left(\mathcal{P} r_{1}\right)+4 d s_{0}^{0}\left(\mathcal{P} r_{1}\right)+2 d s_{0}^{\prime}\left(\mathcal{P} r_{1}\right)+4 s d_{0}^{0}\left(\mathcal{P} r_{1}\right)+2 s d_{0}^{\prime}\left(\mathcal{P} r_{1}\right) \\
& +4 s s_{0}^{0}\left(\mathcal{P} r_{1}\right)+{\overrightarrow{d d^{\prime}}{ }_{1}\left(\mathcal{P} r_{1}\right)+\overleftarrow{d d^{\prime}}{ }_{1}\left(\mathcal{P} r_{1}\right)+4 \overleftarrow{d d^{\prime \prime}}{ }_{1}\left(\mathcal{P} r_{1}\right)+2 \overrightarrow{d d^{\prime \prime}}{ }_{1}\left(\mathcal{P} r_{1}\right)}+2 d s_{1}^{\prime}\left(\mathcal{P} r_{1}\right)+2 s d_{1}^{\prime}\left(\mathcal{P} r_{1}\right)+4 s s_{1}^{1}\left(\mathcal{P} r_{1}\right)+3 s s_{1}^{2}\left(\mathcal{P} r_{1}\right)+s s_{2}^{2}\left(\mathcal{P} r_{1}\right) \\
= & 2 \times 0+3 \times 0+3 \times 0+4 \times 1+4 \times 1+2 \times 0+4 \times 0+2 \times 0+4 \times 0 \\
& +2 \times 0+4 \times 0+0+0+4 \times 0+2 \times 12+2 \times 8+2 \times 8+4 \times 0+3 \times 2 \\
& +0 \\
= & 70 \\
g_{2}\left(\mathcal{K} \mathcal{P} r_{1}\right)= & 2 d d_{0}^{0}\left(\mathcal{P} r_{1}\right)+2 d d_{1}^{0}\left(\mathcal{P} r_{1}\right)+3 \overrightarrow{d d^{\prime}}{ }_{1}\left(\mathcal{P} r_{1}\right)+3 \overleftarrow{d d^{\prime}}{ }_{1}\left(\mathcal{P} r_{1}\right)+4 \overline{d d^{\prime}}{ }_{1}\left(\mathcal{P} r_{1}\right) \\
& +4 \widetilde{d d^{\prime}}{ }_{1}\left(\mathcal{P} r_{1}\right)+2 \overrightarrow{d d^{\prime \prime}}{ }_{1}\left(\mathcal{P} r_{1}\right)+4 d s_{1}^{0}\left(\mathcal{P} r_{1}\right)+2 d s_{1}^{\prime}\left(\mathcal{P} r_{1}\right)+4 s d_{1}^{0}\left(\mathcal{P} r_{1}\right) \\
& +2 s d_{1}^{\prime}\left(\mathcal{P} r_{1}\right)+4 s s_{1}^{0}\left(\mathcal{P} r_{1}\right)+{\overrightarrow{d d^{\prime}}{ }_{2}\left(\mathcal{P} r_{1}\right)+\overleftarrow{d d^{\prime}}{ }_{2}\left(\mathcal{P} r_{1}\right)+4 \overleftarrow{d d^{\prime \prime}}{ }_{2}\left(\mathcal{P} r_{1}\right)}+2 \overrightarrow{d d^{\prime \prime}}{ }_{2}\left(\mathcal{P} r_{1}\right)+2 d s_{2}^{\prime}\left(\mathcal{P} r_{1}\right)+2 s d_{2}^{\prime}\left(\mathcal{P} r_{1}\right)+4 s s_{2}^{1}\left(\mathcal{P} r_{1}\right)+3 s s_{2}^{2}\left(\mathcal{P} r_{1}\right) \\
& +s s_{3}^{2}\left(\mathcal{P} r_{1}\right) \\
= & 2 \times 0+2 \times 0+3 \times 0+3 \times 0+4 \times 0+4 \times 0+2 \times 12+4 \times 4+2 \times 8 \\
& +4 \times 4+2 \times 8+4 \times 0+0+0+4 \times 0+2 \times 0+2 \times 0+2 \times 0+4 \times 24 \\
& +3 \times 0+0 \\
184 &
\end{aligned}
$$

$$
\begin{aligned}
g_{0}\left(\mathcal{K P} r_{2}\right)= & {\overrightarrow{d d^{\prime}}}_{0}\left(\mathcal{P} r_{2}\right)+\overleftarrow{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{2}\right)+4 \overleftarrow{d d^{\prime \prime}}{ }_{0}\left(\mathcal{P} r_{2}\right)+2{\overrightarrow{d d^{\prime \prime}}}_{0}\left(\mathcal{P} r_{2}\right)+2 d s_{0}^{\prime}\left(\mathcal{P} r_{2}\right) \\
& +2 s d_{0}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s s_{0}^{1}\left(\mathcal{P} r_{2}\right)+3 s s_{0}^{2}\left(\mathcal{P} r_{2}\right)+s s_{1}^{2}\left(\mathcal{P} r_{2}\right) \\
= & 0+0+4 \times 0+2 \times 0+2 \times 0+2 \times 0+4 \times 0+3 \times 0+0 \\
= & 0
\end{aligned}
$$

$$
\begin{aligned}
g_{1}\left(\mathcal{K} \mathcal{P} r_{2}\right)= & 2 d d_{0}^{0}\left(\mathcal{P} r_{2}\right)+3 \overrightarrow{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{2}\right)+3 \overleftarrow{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{2}\right)+4 \overline{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{2}\right)+4 \widetilde{d d^{\prime}}{ }_{0}\left(\mathcal{P} r_{2}\right) \\
& +2 \overrightarrow{d d^{\prime \prime}}{ }_{0}\left(\mathcal{P} r_{2}\right)+4 d s_{0}^{0}\left(\mathcal{P} r_{2}\right)+2 d s_{0}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s d_{0}^{0}\left(\mathcal{P} r_{2}\right)+2 s d_{0}^{\prime}\left(\mathcal{P} r_{2}\right) \\
& +4 s s_{0}^{0}\left(\mathcal{P} r_{2}\right)+{\overrightarrow{d d^{\prime}}{ }_{1}\left(\mathcal{P} r_{2}\right)+\overleftarrow{d d^{\prime}}{ }_{1}\left(\mathcal{P} r_{2}\right)+4 \overleftarrow{d d^{\prime \prime}}{ }_{1}\left(\mathcal{P} r_{2}\right)+2 \overrightarrow{d d^{\prime \prime}}{ }_{1}\left(\mathcal{P} r_{2}\right)}+2 d s_{1}^{\prime}\left(\mathcal{P} r_{2}\right)+2 s d_{1}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s s_{1}^{1}\left(\mathcal{P} r_{2}\right)+3 s s_{1}^{2}\left(\mathcal{P} r_{2}\right)+s s_{2}^{2}\left(\mathcal{P} r_{2}\right) \\
= & 2 \times 6+3 \times 0+3 \times 0+4 \times 1+4 \times 1+2 \times 0+4 \times 0+2 \times 0+4 \times 0 \\
& +2 \times 0+4 \times 0+22+22+4 \times 0+2 \times 0+2 \times 12+2 \times 12+4 \times 0 \\
& +3 \times 0+8
\end{aligned}
$$

$$
=120
$$

$$
\begin{aligned}
g_{2}\left(\mathcal{K P} r_{2}\right)= & 2 d d_{0}^{0}\left(\mathcal{P} r_{2}\right)+2 d d_{1}^{0}\left(\mathcal{P} r_{2}\right)+3 \overrightarrow{d d^{\prime}}{ }_{1}\left(\mathcal{P} r_{2}\right)+3 \overleftarrow{d d^{\prime}}{ }_{1}\left(\mathcal{P} r_{2}\right)+4{\overline{d d^{\prime}}{ }_{1}\left(\mathcal{P} r_{2}\right)}+4{\widetilde{d d^{\prime}}{ }_{1}\left(\mathcal{P} r_{2}\right)+2 \overrightarrow{d d^{\prime \prime}}{ }_{1}\left(\mathcal{P} r_{2}\right)+4 d s_{1}^{0}\left(\mathcal{P} r_{2}\right)+2 d s_{1}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s d_{1}^{0}\left(\mathcal{P} r_{2}\right)}+2 s d_{1}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s s_{1}^{0}\left(\mathcal{P} r_{2}\right)+{\overrightarrow{d d^{\prime}}{ }_{2}\left(\mathcal{P} r_{2}\right)+\overleftarrow{d d^{\prime}}{ }_{2}\left(\mathcal{P} r_{2}\right)+4 \overleftarrow{d d^{\prime \prime}}{ }_{2}\left(\mathcal{P} r_{2}\right)}+2{\overrightarrow{d d^{\prime \prime}}{ }_{2}\left(\mathcal{P} r_{2}\right)+2 d s_{2}^{\prime}\left(\mathcal{P} r_{2}\right)+2 s d_{2}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s s_{2}^{1}\left(\mathcal{P} r_{2}\right)+3 s s_{2}^{2}\left(\mathcal{P} r_{2}\right)}+{\operatorname{ss} 3_{3}^{2}\left(\mathcal{P} r_{2}\right)}_{=} 2 \times 6+2 \times 176+3 \times 22+3 \times 22+4 \times 46+4 \times 46+2 \times 0+4 \times 44 \\
& +2 \times 12+4 \times 44+2 \times 12+4 \times 0+256+256+4 \times 48+2 \times 48 \\
& +2 \times 544+2 \times 544+4 \times 72+3 \times 8+288
\end{aligned}
$$

$=4840$

$$
\begin{aligned}
g_{3}\left(\mathcal{K P} r_{2}\right)= & 2 d d_{1}^{0}\left(\mathcal{P} r_{2}\right)+2 d d_{2}^{0}\left(\mathcal{P} r_{2}\right)+3 \overrightarrow{d d^{\prime}}{ }_{2}\left(\mathcal{P} r_{2}\right)+3 \overleftarrow{d d^{\prime}}{ }_{2}\left(\mathcal{P} r_{2}\right)+4 \overrightarrow{d d^{\prime}}{ }_{2}\left(\mathcal{P} r_{2}\right) \\
& +4 \widetilde{d d^{\prime}}{ }_{2}\left(\mathcal{P} r_{2}\right)+2 \overrightarrow{d d^{\prime \prime}}{ }_{2}\left(\mathcal{P} r_{2}\right)+4 d s_{2}^{0}\left(\mathcal{P} r_{2}\right)+2 d s_{2}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s d_{2}^{0}\left(\mathcal{P} r_{2}\right) \\
& +2 s d_{2}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s s_{2}^{0}\left(\mathcal{P} r_{2}\right)+{\overrightarrow{d d^{\prime}}{ }_{3}\left(\mathcal{P} r_{2}\right)+\overleftarrow{d d^{\prime}}{ }_{3}\left(\mathcal{P} r_{2}\right)+4 \overleftarrow{d d^{\prime \prime}}{ }_{3}\left(\mathcal{P} r_{2}\right)}+2 \overrightarrow{d d^{\prime \prime}}{ }_{3}\left(\mathcal{P} r_{2}\right)+2 d s_{3}^{\prime}\left(\mathcal{P} r_{2}\right)+2 s d_{3}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s s_{3}^{1}\left(\mathcal{P} r_{2}\right)+3 s s_{3}^{2}\left(\mathcal{P} r_{2}\right) \\
& +s s_{4}^{2}\left(\mathcal{P} r_{2}\right) \\
= & 2 \times 176+2 \times 704+3 \times 256+3 \times 256+4 \times 400+4 \times 400+2 \times 48 \\
& +4 \times 800+2 \times 544+4 \times 800+2 \times 544+4 \times 320+0+0+4 \times 0 \\
& +2 \times 0+2 \times 1920+2 \times 1920+4 \times 1664+3 \times 288+0
\end{aligned}
$$

$$
=31648
$$

$$
\begin{aligned}
g_{4}\left(\mathcal{K P} r_{2}\right)= & 2 d d_{2}^{0}\left(\mathcal{P} r_{2}\right)+2 d d_{3}^{0}\left(\mathcal{P} r_{2}\right)+3 \overrightarrow{d d^{\prime}}{ }_{3}\left(\mathcal{P} r_{2}\right)+3 \overleftarrow{d d^{\prime}}{ }_{3}\left(\mathcal{P} r_{2}\right)+4 \overrightarrow{d d^{\prime}}{ }_{3}\left(\mathcal{P} r_{2}\right) \\
& +4{\widetilde{d d^{\prime}}}_{3}\left(\mathcal{P} r_{2}\right)+2 \overrightarrow{d d^{\prime \prime}}{ }_{3}\left(\mathcal{P} r_{2}\right)+4 d s_{3}^{0}\left(\mathcal{P} r_{2}\right)+2 d s_{3}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s d_{3}^{0}\left(\mathcal{P} r_{2}\right) \\
& +2 s d_{3}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s s_{3}^{0}\left(\mathcal{P} r_{2}\right)+{\overrightarrow{d d^{\prime}}{ }_{4}\left(\mathcal{P} r_{2}\right)+\overleftarrow{d d^{\prime}}{ }_{4}\left(\mathcal{P} r_{2}\right)+4 \overleftarrow{d d^{\prime \prime}}{ }_{4}\left(\mathcal{P} r_{2}\right)}+2 \overrightarrow{d d^{\prime \prime}}{ }_{4}\left(\mathcal{P} r_{2}\right)+2 d s_{4}^{\prime}\left(\mathcal{P} r_{2}\right)+2 s d_{4}^{\prime}\left(\mathcal{P} r_{2}\right)+4 s s_{4}^{1}\left(\mathcal{P} r_{2}\right)+3 s s_{4}^{2}\left(\mathcal{P} r_{2}\right) \\
& +s s_{5}^{2}\left(\mathcal{P} r_{2}\right) \\
= & 2 \times 704+2 \times 0+3 \times 0+3 \times 0+4 \times 0+4 \times 0+2 \times 0+4 \times 384 \\
& +2 \times 1920+4 \times 384+2 \times 1920+4 \times 1888+0+0+4 \times 0 \\
& +2 \times 0+2 \times 0+2 \times 0+4 \times 2304+3 \times 0+0 \\
= & 28928
\end{aligned}
$$

Similarly, one can routinely compute the genus distributions for $\mathcal{K} \mathcal{P} r_{n}$ for higher values of $n$. We conclude this section by listing the genus distributions of $\mathcal{K} \mathcal{P} r_{3}$ obtained similarly.

$$
\begin{array}{lll}
g_{0}\left(\mathcal{K P} r_{3}\right)=0 & g_{1}\left(\mathcal{K P} r_{3}\right)=208 & g_{2}\left(\mathcal{K P} r_{3}\right)=16688 \\
g_{3}\left(\mathcal{K P} r_{3}\right)=445056 & g_{4}\left(\mathcal{K P} r_{3}\right)=3924352 & g_{5}\left(\mathcal{K P} r_{3}\right)=8667648 \\
g_{6}\left(\mathcal{K P P r _ { 3 } )}=3723264\right. & &
\end{array}
$$

### 3.4 Application: Closed Chains of Copies of $K_{3,3}$

Let $\mathcal{C} K_{3,3}^{n}$ be the co-self-amalgamated closed chain of $n$ copies of $K_{3,3}$, as shown in Figure 3.13 and let $\mathcal{K} K_{3,3}^{n}$ be the contra-self-amalgamated closed chain of $n$ copies of $K_{3,3}$, as illustrated in Figure 3.14 .


Figure 3.13: Co-self-amalgamating open chains of $n$ copies of $K_{3,3}$.

The genus distributions for $\mathcal{C} K_{3,3}^{n}$ and $\mathcal{K} K_{3,3}^{n}$ are calculated by substituting values from Table 2.9 into Equation (3.21) and Equation (3.22), respectively. The genus distribution

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Figure 3.14: Contra-self-amalgamating open chains of $n$ copies of $K_{3,3}$.
polynomials for some small co-pasted and contra-pasted closed chains of copies of $K_{3,3}$ are given as follows:

$$
\begin{aligned}
& g\left[\mathcal{C} K_{3,3}^{1}\right](x)=2+54 x+200 x^{2} \\
& g\left[\mathcal{C} K_{3,3}^{2}\right](x)=8 x+1984 x^{2}+25752 x^{3}+37792 x^{4} \\
& g\left[\mathcal{C} K_{3,3}^{3}\right](x)=32 x^{2}+69696 x^{3}+2147296 x^{4}+8604864 x^{5}+5955328 x^{6} \\
& g\left[\mathcal{K} K_{3,3}^{1}\right](x)=2+70 x+184 x^{2} \\
& g\left[\mathcal{K} K_{3,3}^{2}\right](x)=8 x+1856 x^{2} 25880 x^{3}+37792 x^{4} \\
& g\left[\mathcal{K} K_{3,3}^{3}\right](x)=32 x^{2}+70720 x^{3}+2146272 x^{4}+8604864 x^{5}+5955328 x^{6}
\end{aligned}
$$

### 3.5 Application: Closed Chains of Alternating Copies of Two Distinct Graphs

The last example of this section draws on the partitioned genus distribution calculated in $\$ 2.12$ for open chains $\mathcal{A} l_{2}$ and $\mathcal{A} l_{3}$ consisting respectively of two and three interleaved copies of the triangular prism graph $\Delta_{G}$ and the complete bipartite graph $K_{3,3}$. Their corresponding co-pasted chains $\mathcal{C A} l_{2}$ and $\mathcal{C} \mathcal{A} l_{3}$ are shown in Figure 3.15 and their genus distributions are given as follows:


Figure 3.15: Co-self-amalgamating open chains $\mathcal{A l} l_{2}$ and $\mathcal{A} l_{3}$.

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\begin{aligned}
& g\left[\mathcal{C A} l_{2}\right](x)=44 x+3108 x^{2}+29616 x^{3}+32768 x^{4} \\
& g\left[\mathcal{C A} l_{3}\right](x)=36 x+6716 x^{2}+280608 x^{3}+3334272 x^{4}+8899840 x^{5}+4255744 x^{6}
\end{aligned}
$$

The corresponding contra-pasted chains $\mathcal{K} \mathcal{A} l_{2}$ and $\mathcal{K} \mathcal{A} l_{3}$ are shown in Figure 3.16 and their genus distributions are given as follows:

$$
\begin{aligned}
& g\left[\mathcal{K} \mathcal{A} l_{2}\right](x)=60 x+3284 x^{2}+29424 x^{3}+32768 x^{4} \\
& g\left[\mathcal{K} \mathcal{A} l_{3}\right](x)=52 x+7084 x^{2}+282528 x^{3}+3331968 x^{4}+8899840 x^{5}+4255744 x^{6}
\end{aligned}
$$



Figure 3.16: Contra-self-amalgamating open chains $\mathcal{A} l_{2}$ and $\mathcal{A} l_{3}$.

## Chapter 4

## Genus Distributions of 4-Regular Outerplanar Graphs

This section describes an $O\left(n^{2}\right)$-time divide-and-conquer algorithm for calculating genus distribution of any 4 -regular outerplanar graph. One special importance of 4 -regular graphs is that they occur as projections of knots and links. Another is that the medial graph of any embedded graph is a 4 -regular graph. Outerplanar and outerembeddable graphs have been a subject of interest, especially in the area of graph minors and obstructions Brehaut, 1977, Syslo, 1979], Heath, 1986, and Bienstock and Dean, 1992].

A graph $G$ is called an outerplanar graph if it has a planar embedding in which some face-boundary walk contains every vertex of $G$. We refer to such an embedding as an outerplane embedding, and we denote the face containing all the vertices by $f_{\infty}$, to indicate that it contains the point at infinity. An outerplane embedding is said to be normalized if all self-loops of the graph lie on the face-boundary walk of the face $f_{\infty}$. We designate the edges that constitute the face-boundary walk of $f_{\infty}$ as exterior edges, in contrast to the usage of interior edges for the remaining edges. We refer to the edge-ends of exterior and interior edges as exterior edge-ends and interior edge-ends, respectively. Figure 4.1 shows a 4-regular outerplane embedding before normalization, with the exterior edges shown darker than interior edges.


Figure 4.1: An unnormalized outerplane embedding of a 4-regular outerplanar graph.

In the previous two sections, we were dealing with rooted graphs containing one or two root-edges. In this section, we use methods that employ a variation on the concepts developed in $\$ 2$ and $\S 3$. We deal strictly with vertices, instead of edges, as roots. In this context, we extend some old terminology to encompass more general ideas, as well as introduce new terminology.

Any vertex in a graph may be designated a root-vertex. Insofar as the exposition of this section is concerned, a root always refers to a root-vertex, and a graph with one or more root-vertices is known as a rooted graph. Here, we primarily deal with graphs having two roots. We refer to such a graph as a double-rooted graph. We assume that each rootvertex in a double-rooted graph is 2 -valent. If a 2 -valent root-vertex $u$ occurs twice in an fb-walk, it breaks the fb-walk into two strands, which are the maximal subwalks such that $u$ is not an interior point. We refer to these strands as $u$-strands. For a double-rooted graph $(G, u, v)$, the vertex $u$ is referred to as the first-root of the graph $G$ and the vertex $v$ is referred to as the second-root of the graph $G$.

The layout of the rest of this section is as follows: In 4.1 and $\$ 4.2$, we lay groundwork for exploiting the structure of 4-regular outerplanar graphs for our present purpose. In $\$ 4.3$, we discuss the algorithm, and we do a dry run on a small example. In $\$ \sqrt[4.4]{4}$ the complexity of the algorithm is discussed. The proof of correctness is given in $\$ 4.5$.

### 4.1 Split Graphs and Incidence Trees

Given a normalized outerplane embedding of a 4-regular outerplanar graph $G$, we classify its vertices into two types. A Type-I vertex has two exterior and two interior incident edge-ends, whereas a Type-II vertex has four exterior incident edge-ends. Thus, every
cut-vertex is a Type-II vertex. Moreover, by requiring that every self-loop lie on the fb-walk of the face $f_{\infty}$, we can make its single endpoint be a Type-II vertex. All other vertices are Type-I.

The general term splitting of a vertex $v_{i}$ is used to mean either of the following two operations on the vertex $v_{i}$ :

- Type-I Vertices: In the rotation at a Type-I vertex $v_{i}$ in the outerplane embedding, the exterior edge-ends $e_{1}$ and $e_{2}$ incident on $v_{i}$ are contiguous, as are the interior edge-ends $d_{1}$ and $d_{2}$. Let the cyclic counter-clockwise order of the edge-ends incident on $v_{i}$ be ( $e_{1}, e_{2}, d_{1}, d_{2}$ ) in the outerplane embedding of graph $G$. Then splitting the vertex $v_{i}$ consists of introducing two new vertices $v_{i}^{\prime}$ and $v_{i}^{\prime \prime}$, called single-primed and double-primed vertices, respectively, with the edge-ends $d_{2}$ and $e_{1}$ incident on $v_{i}^{\prime}$ instead of on $v_{i}$, and with the edge-ends $e_{2}$ and $d_{1}$ incident on $v_{i}^{\prime \prime}$ instead of on $v_{i}$. The vertex $v_{i}$ is deleted. This is illustrated in Figure 4.2.


Figure 4.2: Splitting a Type-I vertex $v_{i}$.

- Type-II Vertices: Let the exterior edge-ends of $v_{i}$ be cyclically ordered as ( $e_{1}, e_{2}$, $e_{3}, e_{4}$ ), where $e_{1}$ and $e_{4}$ belong to one block and $e_{2}$ and $e_{3}$ either belong to another block or are the two edge-ends of the same self-loop. Then splitting the vertex $v_{i}$ consists of introducing two new vertices $\dot{v}_{i}$ and $\ddot{v}_{i}$. We refer to either of these as a dotted vertex. The edge-ends $e_{1}$ and $e_{4}$ are made incident on $\dot{v}_{i}$, while the edge-ends $e_{2}$ and $e_{3}$ are made incident on $\ddot{v}_{i}$ instead of on $v_{i}$. The vertex $v_{i}$ is deleted.

In this manner, every vertex of the normalized outerplane embedding of a 4-regular outerplanar graph $G$ may be split, thereby obtaining a graph $G^{\prime}$ where each vertex is 2 -valent and, therefore, each component is a cycle $C_{n}$ for some $n$. We refer to $G^{\prime}$ as the split graph for the graph $G$, and we refer to each pair of vertices obtained from a split as coupled vertices. The two vertices in a coupled pair belong to different components. These two
components are called coupled components with respect to that pair of vertices. An example of a 4-regular outerplanar graph and its split graph is shown in Figure 4.3 .


Figure 4.3: A 4-regular outerplanar graph and the split graph obtained from its normalized outerplane embedding.

Remark 10. Each component of a split graph is the boundary of a 2-cell, which is regarded as having a counter-clockwise orientation induced from the orientation of the outerplane embedding.

It is easy to visualize how the original graph $G$ can be reassembled by amalgamating each pair of coupled vertices in $G^{\prime}$. The devised algorithm utilizes this reconstructability of a 4-regular outerplanar graph. It calculates genus distribution of the outerplanar graph by simulating its reconstruction, while calculating the genus distributions for the subgraphs assembled at each step of the algorithm.

Component graph of the split graph $C\left(G^{\prime}\right)$ refers to the graph whose nodes are the components of the split graph, and in which two nodes are adjacent if they are coupled. An algorithm for building an ordered tree that can be regarded as a depth-first spanning tree of $C\left(G^{\prime}\right)$ is described as follows:

1. Designate an arbitrary component in the component graph $C\left(G^{\prime}\right)$ as the root node of the tree. Represent the root node visually with a round-shaped vertex.
2. Construct a depth-first ordered tree rooted at the root node in the component graph $C\left(G^{\prime}\right)$, such that the child components for each tree node $\mathcal{C}$ correspond to the components coupled with it only with respect to its single-primed and dotted vertices.
3. By ordered tree, we mean that the counter-clockwise rotation at each tree node imposes a linear ordering on its children. The order prescribed for the children of each tree node $\mathcal{C}$ is that in which these coupled child components are encountered under the counter-clockwise orientation on $\mathcal{C}$ in $G^{\prime}$. The first child node of the root node is chosen arbitrarily since the root node has no parent node. In contrast, for any other tree node $\mathcal{C}$ coupled with parent node $\mathcal{P}$, the first child node of $\mathcal{C}$ is chosen to be the first component coupled with it after $\mathcal{P}$ under the counter-clockwise orientation on $\mathcal{C}$.
4. Each new node added to the tree in step 2 is represented visually by a square node if it corresponds to a component coupled to its parent with respect to dotted vertices, otherwise it is represented by a round node.

The ordered tree formed in this manner is unique for a fixed root and a fixed first child of the root, and is referred to as the incidence tree of the outerplanar graph $G$ with respect to the given outerplane embedding. Depending on the context, the tree nodes of an incidence tree may interchangeably be regarded as the components of $C\left(G^{\prime}\right)$ or as their more abstract round and square visual representations. For the split graph in Figure 4.3 and a particular choice of the root component and the first child component, the corresponding incidence tree is shown in Figure 4.4. The darker round node 18 shown in Figure 4.4 is the root node. The arbitrarily selected first child of the root is illustrated by the dark directed edge incident on it from the root node.

The post-order traversal of an incidence tree prescribes the order in which coupled vertices are amalgamated when simulating the reconstruction of the outerplanar graph. In this sense, the incidence tree for a 4-regular outerplanar graph fills the same role as the "inner tree" for a 3-regular outerplanar graph in Gross, 2011b. However, as we have just seen, its construction involves more subtleties.

Remark 11. The purpose in introducing component graphs is to facilitate the conceptualization of incidence trees. In practice, an incidence tree can be constructed directly from a split graph without recourse to construction of the component graph.


Figure 4.4: An incidence tree for the split graph from Figure 4.3 .

### 4.2 Vertex-Amalgamations and Self-Vertex-Amalgamations

In order to simulate the reconstruction of a 4-regular outerplanar graph, we require two graph operations that involve amalgamation of root-vertices. These are vertex-amalgamation and self-vertex-amalgamation. For simplicity, we refer to these as amalgamation and selfamalgamation, respectively. As before, we define productions for these operations that algebraically represent the embeddings of the resulting graph. We also specify the relevant set of double-root partials, albeit with respect to root-vertices. The set of double-root partials in turn enables a precise understanding of what is entailed by partitioned genus distribution.

## Double-Root Partials

Double-Root partials for double-rooted graphs are defined analogously to the first-order partials for double-edge-rooted graphs with the difference that the roots in question are vertices and not edges. Any given 2-valent vertex appears exactly twice in the set of fbwalks of an embedding. This enables a partitioning of the embeddings of a double-rooted graph $(G, u, v)$ on a surface $S_{i}$ into the four basic types: $d d_{i}, d s_{i}, s d_{i}$, and $s s_{i}$. The first letter of each type represents the first-root $u$, and the second letter represents the second-
root $v$. The letters $s$ and $d$ are mnemonics for "same" and "distinct", indicating whether the corresponding 2 -valent root vertex occurs twice on the same fb-walk or once on each of two distinct fb-walks. Each double-root partial counts the number of embeddings of one of these four basic types. The four double-root partials are further refined by Gross et al., 2010 to express the specific relationships of the fb-walks incident on both roots. These refinements are known as sub-partials and are as follows:

- $d d_{i}^{0}, d s_{i}^{0}, s d_{i}^{0}$ and $s s_{i}^{0}$ are the numbers of embeddings of $G$ on surface $S_{i}$ of types $d d$, $d s, s d$ and $s s$, respectively, such that no fb-walk incident on $u$ is incident on $v$.
- $d d_{i}^{\prime}, d s_{i}^{\prime}$, and $s d_{i}^{\prime}$ are the numbers of embeddings of $G$ on surface $S_{i}$ of types $d d, d s$ and $s d$, respectively, such that exactly one fb-walk incident on $u$ is also incident on $v$.
- $d d_{i}^{\prime \prime}$ is the number of embeddings of $G$ on surface $S_{i}$ of type $d d$ such that both fb-walks incident on $u$ are also incident on $v$.
- $s s_{i}^{1}$ and $s s_{i}^{2}$ partition the number of embeddings on surface $S_{i}$ of type $s s$ where $s s^{1}$ counts the cases where exactly one $u$-strand contains both occurrences of the root $v$, while $s s^{2}$ counts the cases where each $u$-strand contains an occurrence of $v$.

It follows from these definitions that each double-root partial is the sum of its sub-partials. Moreover,

$$
g_{i}(G)=d d_{i}(G)+d s_{i}(G)+s d_{i}(G)+s s_{i}(G)
$$

There are also additional sub-partials that are refinements for the sub-partials $s d^{\prime}$ and $s s^{1}$. We build the context in which these sub-partials are needed and provide their definitions in 84.2 .

The collection of values of the sub-partials, for all values of $i$, is the partitioned genus distribution of the graph $(G, u, v)$. This collection includes the values of sub-partials defined as refinements of $s d^{\prime}$ and $s s^{1}$ in $\$ 4.2$.

## Productions for Self-Vertex-Amalgamation

A self-vertex-amalgamation of a double-rooted graph is an operation $(G, u, v) \longrightarrow W$, where the two roots of the graph are merged together to produce a new graph. Here,
we use the simpler alternative self-amalgamation to mean the same. We know from Gross, 2011a that when both roots $u$ and $v$ are 2 -valent, an embedding $\iota_{G}$ of $G$ under self-amalgamation induces six unique embeddings of $W$ such that the rotations at vertices in $\iota_{G}$ are consistent with rotations at vertices in the six corresponding embeddings of $W$. We also know that the genus of each of these embeddings of $W$ is a function of the genus of the embedding surface of $\iota_{G}$ and of the configuration of fb-walks on which the roots of $G$ lie. This information can be represented in a form known as a production for self-amalgamation.

Let $p_{i}$ be a double-root sub-partial of the double-rooted graph $(G, u, v)$. Then the standard representation for a self-amalgamation production, as laid out in Gross, 2011a, is of the form:

$$
p_{i}(G, u, v) \longrightarrow \alpha_{1} g_{i+k_{1}}(W)+\alpha_{2} g_{i+k_{2}}(W)
$$

where $\alpha_{1}, \alpha_{2}$ are non-negative integers whose sum is 6 , and where $k_{1}, k_{2}$ are integers within the range of -1 to 2 . This can be interpreted as follows:

A type $p$ embedding of $(G, u, v)$ on surface $S_{i}$ self-amalgamates on the rootvertices $u$ and $v$ to give six embeddings of the graph $W$. Out of these six resulting embeddings, $\alpha_{1}$ embeddings are on surface $S_{i+k_{1}}$, and $\alpha_{2}$ are on surface $S_{i+k_{2}}$.

The complete set of productions for self-amalgamation on 2 -valent roots is given in Gross, 2011a. However, the form of the production defined above does not capture rootrelated information for the graph $W$ that is produced as a result of self-amalgamation. This is problematic since we need to be able to repeatedly apply self-amalgamations and vertex-amalgamations, in order to build a larger graph from many of the smaller subgraphs. For this reason, after self-amalgamation, we pop new root vertices on the exterior edge $e$ incident on the first-root $u$ of the graph $(G, u, v)$. This is illustrated in Figure 4.5, where the edges $e$ and $f$ are incident on the first-root $u$ before self-amalgamation. The edge $f$ is necessarily an interior edge. New roots are popped on the edge $e$ after self-amalgamation. Nota bene, the root popped closer to the amalgamated vertex is considered the second-root of the resulting graph.

This entails adapting the production body to reflect the new roots. In particular, we need to replace each occurrence of $g_{i}$ in the production body by the relevant double-root


Figure 4.5: A model representing self-amalgamation.
sub-partial that is consistent with the face-boundaries incident on these new roots. In light of this, a production for self-vertex-amalgamation is redefined as follows:

$$
p_{i}(G, u, v) \longrightarrow \sum_{\substack{x_{k} \text { ranges over all } \\ \text { sub-partial types } \\ \text { with } k \in\{i-1, i, i+1, i+2\}}} \alpha_{x_{k}} x_{k}(W, s, v)
$$

where each $x_{k}$ is a double-root sub-partial type for the graph $W$ and where the numbers $\alpha_{x_{k}}$ are non-negative integers whose sum is 6 .

Since both new roots are popped on the same edge, the same fb-walk that passes through one root also passes through the other. Thus, each sub-partial in the production body for a self-amalgamation production is either of type $d d^{\prime \prime}$ or of type $s s^{1}$.

Adaptation of productions in this manner is straightforward for all sub-partials $p_{i}$ in the production head, except for the sub-partials $s d_{i}^{\prime}$ and $s s_{i}^{1}$. To facilitate the adaptation of productions for these two sub-partials, we further refine them as follows:

$$
\begin{aligned}
\uparrow s d_{i}^{\prime}(G, u, v)= & \text { the number of type } s d_{i}^{\prime} \text { embeddings such that the } \\
& u \text {-strand that contains the occurrence of vertex } v \\
& \text { also contains both occurrences of exterior edge } e \\
& \text { in it (see Figure 4.6). } \\
\downarrow s d_{i}^{\prime}(G, u, v)= & \text { the number of type } s d_{i}^{\prime} \text { embeddings such that the } \\
& u \text {-strand that contains the occurrence of vertex } v \\
& \text { does not contain the two occurrences of exterior edge } e \\
& \text { in it (see Figure 4.6). }
\end{aligned}
$$

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Therefore,

$$
s d_{i}^{\prime}(G, u, v)=\uparrow s d_{i}^{\prime}(G, u, v)+\downarrow s d_{i}^{\prime}(G, u, v)
$$

Similarly,

$$
\begin{aligned}
\uparrow s s_{i}^{1}(G, u, v)= & \text { the number of type } s s_{i}^{1} \text { embeddings such that the } \\
& u \text {-strand that contains both occurrences of the vertex } v \\
& \text { also contains both occurrences of exterior edge } e \text { in it } \\
& \text { (see Figure 4.6). } \\
\downarrow s s_{i}^{1}(G, u, v)= & \text { the number of type } s s_{i}^{1} \text { embeddings such that the } \\
& u \text {-strand that contains both occurrences of the vertex } v \\
& \text { does not contain the two occurrences of exterior edge } e \\
& \text { in it (see Figure 4.6). }
\end{aligned}
$$

Thus,

$$
s s_{i}^{1}(G, u, v)=\uparrow s s_{i}{ }^{1}(G, u, v)+\downarrow s s_{i}{ }^{1}(G, u, v)
$$



Figure 4.6: Refined partials types of $s d^{\prime}$ and $s s^{1}$.

The proofs in Gross, 2011a can now be adapted by popping two new roots on edge $e$, as shown on the right side of Figure 4.5. This chosen edge $e$ corresponds to the exterior edge of the outerplane embedding that is incident on the first-root undergoing self-amalgamation.

The following theorem adapts the productions for self-amalgamation by making the modification above.

Theorem 17. When an embedding of a double-rooted graph ( $G, s, t$ ) with 2-valent roots is self-amalgamated, the following productions hold:

$$
\begin{equation*}
d d_{i}^{0}(G, u, v) \longrightarrow 4 d d_{i+1}^{\prime \prime}(W, s, t)+2 \downarrow s s_{i+2}^{1}(W, s, t) \tag{4.1}
\end{equation*}
$$

$$
\begin{align*}
d d_{i}^{\prime}(G, u, v) & \longrightarrow d d_{i}^{\prime \prime}(W, s, t)+3 d d_{i+1}^{\prime \prime}(W, s, t)+2 \downarrow s s_{i+1}^{1}(W, s, t)  \tag{4.2}\\
d d_{i}^{\prime \prime}(G, u, v) & \longrightarrow 4 d d_{i}^{\prime \prime}(W, s, t)+2 \downarrow s s_{i+1}^{1}(W, s, t)  \tag{4.3}\\
d s_{i}^{0}(G, u, v) & \longrightarrow 6 d d_{i+1}^{\prime \prime}(W, s, t)  \tag{4.4}\\
d s_{i}^{\prime}(G, u, v) & \longrightarrow 3 d d_{i}^{\prime \prime}(W, s, t)+3 \downarrow s s_{i+1}^{1}(W, s, t)  \tag{4.5}\\
s d_{i}^{0}(G, u, v) & \longrightarrow 6 \downarrow s s_{i+1}^{1}(W, s, t)  \tag{4.6}\\
\uparrow s d_{i}^{\prime}(G, u, v) & \longrightarrow 3 d d_{i}^{\prime \prime}(W, s, t)+3 \downarrow s s_{i+1}^{1}(W, s, t)  \tag{4.7}\\
\downarrow s d_{i}^{\prime}(G, u, v) & \longrightarrow 3 \downarrow s s_{i}^{1}(W, s, t)+3 \downarrow s s_{i+1}^{1}(W, s, t)  \tag{4.8}\\
s s_{i}^{0}(G, u, v) & \longrightarrow 6 \downarrow s s_{i+1}^{1}(W, s, t)  \tag{4.9}\\
\uparrow s s_{i}^{1}(G, u, v) & \longrightarrow 6 d d_{i}^{\prime \prime}(W, s, t)  \tag{4.10}\\
\downarrow s s_{i}^{1}(G, u, v) & \longrightarrow 6 \downarrow s s_{i}^{1}(W, s, t)  \tag{4.11}\\
s s_{i}^{2}(G, u, v) & \longrightarrow d d_{i-1}^{\prime \prime}(W, s, t)+3 d d_{i}^{\prime \prime}(W, s, t)+2 \downarrow s s_{i}^{1}(W, s, t) \tag{4.12}
\end{align*}
$$

Proof. An $s d^{\prime}$-type embedding of $(G, u, v)$ has one fb-walk incident on root $u$ and two on root $v$. Moreover, the fb-walk incident on $u$ is also incident on root $v$. When such an embedding is self-amalgamated, the resulting graph $W$ has six corresponding embeddings. This however results in two different scenarios based on whether the embedding of $G$ is of sub-type $\uparrow s d_{i}^{\prime}$ or $\downarrow s d_{i}^{\prime}$. The first scenario, corresponding to an $\uparrow s d_{i}^{\prime}$-type embedding of $(G, u, v)$, is portrayed in Figure 4.7.


Figure 4.7: Self-amalgamation of a $\uparrow s d_{i}$-type embedding of $G$.

The six embedding models shown at the right of the figure correspond to the embeddings of $W$ resulting from the self-amalgamation of an embedding of $G$. As a result of self-
amalgamation, the fb-walks incident on both root vertices of $(G, u, v)$ break into strands, that recombine to make new fb-walks. Two new roots are popped on the exterior edge $e$ after self-amalgamation, as shown in the figure. The root farther from the amalgamated vertex is the first-root, and the one closer to it is the second-root. One observes that half of the embeddings of $W$ resulting from self-amalgamation are of type $d d^{\prime \prime}$, while the remaining are of type $\downarrow s s^{1}$. This accounts for Production 4.7.

Contrast this with the second scenario illustrated in Figure 4.8. This constitutes the proof of Production 4.8.


Figure 4.8: Self-amalgamation of a $\downarrow s d_{i}$-type embedding of $G$.

Remark 12. Figure 4.5 makes it clear that the $s^{1}$-type partials resulting from the selfamalgamation are always $\downarrow$ ss $^{1}$-sub-type.

The proofs for other productions are identical in substance to the proofs given for the corresponding productions in Gross, 2011a. However, a fine-tuning of the classification of the embeddings resulting from self-amalgamation is necessitated, as in the proof of the productions above. For the sake of brevity, the remaining productions are left for the reader to verify.

## Productions for Vertex-Amalgamation

Let $(G, s, t)$ be a graph with the vertices $s$ and $t$ designated as roots, and let $(H, u, v)$ be a graph with the vertices $u$ and $v$ as roots. Then amalgamating the graph $G$ at root vertex $t$ with the graph $H$ at root vertex $u$ yields a new graph $(W, s, v)$ with the vertices $s$ and $v$
serving as roots. The vertex-amalgamation operation is denoted by an asterisk as follows:

$$
(W, s, v)=(G, s, t) *(H, u, v)
$$

It is assumed that the roots are 2 -valent. Thus, when an embedding $\iota_{G}$ of $G$ and an embedding $\iota_{H}$ of $H$ amalgamate, they induce six unique embeddings of $W$, in which the rotations at all vertices of $W$ are consistent with the rotations at the corresponding vertices in both $\iota_{G}$ and $\iota_{H}$. Moreover, the genus of each of these embeddings of $W$ is a function of the genera of $\iota_{G}$ and $\iota_{H}$ and of the fb-walks on which the roots of $G$ and $H$ lie as they undergo amalgamation.

Let $p_{i}$ and $q_{j}$ be double-root sub-partials. Then a production for vertex-amalgamation represents the ways in which a type $p_{i}$ embedding of $(G, s, t)$ and a type $q_{j}$ embedding of ( $H, u, v$ ) amalgamate on their root vertices $t$ and $u$, respectively, to give various types of embeddings of the resulting graph $(W, s, v)$. We write

$$
p_{i}(G, s, t) * q_{j}(H, u, v) \longrightarrow \sum_{\substack{x_{k} \text { ranges over all } \\ \text { subb-partial types } \\ \text { with } k \in\{i+j, i+j+1\}}} \alpha_{x_{k}} x_{k}(W, s, v)
$$

where the coefficients $\alpha_{x_{k}}$ are non-negative integers that sum to six, and where each term in the production body indicates that there are $\alpha_{x_{k}}$ embeddings of the graph produced by the amalgamation, that have genus $k$ and a sub-partial type $x_{k}$. This can be read as follows:

Amalgamating a $p$-type embedding of $(G, s, t)$ on surface $S_{i}$ with a $q$-type embedding of ( $H, u, v$ ) on surface $S_{j}$ on the root-vertices $t$ and $u$ yields six embeddings of the graph $(W, s, v)$. Each of these six embeddings corresponds to a partial type $x$ on the surface $S_{i+j}$ or $S_{i+j+1}$, as specified by the subscript of $x$.

A method for deriving productions for vertex-amalgamation was presented in Gross, 2011a, but no distinction was made between the $\uparrow s s^{1}$ and $\downarrow s s^{1}$ sub-partials, or between the $\uparrow s d^{\prime}$ and $\downarrow s d^{\prime}$ sub-partials. The method in Gross, 2011a works equally well for these new sub-partials. The complete list of productions needed for our algorithm is given in Table 4.1 . The productions not involving sub-partial types $\uparrow s d^{\prime}, \downarrow s d^{\prime}, \uparrow s s^{1}$ or $\downarrow s s^{1}$ in the production body are taken from Gross, 2011a and are listed here only for the sake of completion. For brevity, we abbreviate the double-root partials by omitting the double-rooted graphs.

Even though there are twelve sub-partials defined in this paper, the number of productions directly needed for our algorithm is $2 \times 12=24$. This is because the order in which the various graph components are amalgamated necessitates that the roots of the first amalgamand in any vertex-amalgamation be adjacent. This allows three possibilities for the sub-partial types of such a component: $d d^{\prime \prime}, \uparrow s s^{1}$, and $\downarrow s s^{1}$. It turns out that an embedding of the first amalgamand is never of type $\uparrow s s^{1}$. The first amalgamand has an $s s^{1}$-type embedding only as an outcome of a previous self-amalgamation or as an outcome of a step in our algorithm that involves vertex-amalgamating a pair of dotted vertices. In our earlier remark, it was mentioned that self-amalgamation produces only $\downarrow s s^{1}$-type embeddings. The same is also true for the latter scenario as will become evident in the next section. Therefore, the sub-partials of the first amalgamand are limited to only two valid types: $d d^{\prime \prime}$ and $\downarrow s s^{1}$.

Table 4.1: Productions for vertex-amalgamation $(G, s, t) *(H, u, v)$ where the embedding of graph $G$ has partial type $d d^{\prime \prime}$ or $\downarrow s s^{1}$.

| $d d_{i}^{\prime \prime}(G, s, t)$ productions | $\downarrow s s_{i}^{1}(G, s, t)$ productions |
| :--- | :--- |
| $d d_{i}^{\prime \prime} * d d_{j}^{0} \longrightarrow 4 d d_{i+j}^{0}+2 s d_{i+j+1}^{0}$ | $\downarrow s s_{i}^{1} * d d_{j}^{0} \longrightarrow 6 s d_{i+j}^{0}$ |
| $d d_{i}^{\prime \prime} * d d_{j}^{\prime} \longrightarrow 2 d d_{i+j}^{0}+2 d d_{i+j}^{\prime}+\downarrow s d_{i+j+1}^{\prime}+\uparrow s d_{i+j+1}^{\prime}$ | $\downarrow s s_{i}^{1} * d d_{j}^{\prime} \longrightarrow 3 \downarrow s d_{i+j}^{\prime}+3 s d_{i+j}^{0}$ |
| $d d_{i}^{\prime \prime} * d d_{j}^{\prime \prime} \longrightarrow 4 d d_{i+j}^{\prime}+2 s s_{i+j+1}^{2}$ | $\downarrow s s_{i}^{1} * d d_{j}^{\prime \prime} \longrightarrow 6 \downarrow s d_{i+j}^{\prime}$ |
| $d d_{i}^{\prime \prime} * d s_{j}^{0} \longrightarrow 4 d s_{i+j}^{0}+2 s s_{i+j+1}^{0}$ | $\downarrow s s_{i}^{1} * d s_{j}^{0} \longrightarrow 6 s s_{i+j+1}^{0}$ |
| $d d_{i}^{\prime \prime} * d s_{j}^{\prime} \longrightarrow 2 d s_{i+j}^{0}+2 d s_{i+j}^{\prime}+\downarrow s s_{i+j+1}^{1}+\uparrow s s_{i+j+1}^{1}$ | $\downarrow s s_{i}^{1} * d s_{j}^{\prime} \longrightarrow 3 s s_{i+j}^{0}+3 \downarrow s s_{i+j}^{1}$ |
| $d d_{i}^{\prime \prime} * s d_{j}^{0} \longrightarrow 6 d d_{i+j}^{0}$ | $\downarrow s s_{i}^{1} * s d_{j}^{0} \longrightarrow 6 \downarrow s d_{i+j}^{\prime}$ |
| $d d_{i}^{\prime \prime} * \downarrow s d_{j}^{\prime} \longrightarrow 6 d d_{i+j}^{\prime}$ | $\downarrow s s_{i}^{1} * \uparrow s d_{j}^{\prime} \longrightarrow 6 \downarrow s d_{i+j}^{\prime}$ |
| $d d_{i}^{\prime \prime} * \uparrow s d_{j}^{\prime} \longrightarrow 6 d d_{i+j}^{\prime}$ | $\downarrow s s_{i}^{1} * \downarrow s d_{j}^{\prime} \longrightarrow 6 \downarrow s d_{i+j}^{\prime}$ |
| $d d_{i}^{\prime \prime} * s s_{j}^{0} \longrightarrow 6 d s_{i+j}^{0}$ | $\downarrow s s_{i}^{1} * s s_{j}^{0} \longrightarrow 6 s s_{i+j}^{0}$ |
| $d d_{i}^{\prime \prime} * \uparrow s s_{j}^{1} \longrightarrow 6 d s_{i+j}^{\prime}$ | $\downarrow s s_{i}^{1} * \uparrow s s_{j}^{1} \longrightarrow 6 \downarrow s s_{i+j}^{1}$ |
| $d d_{i}^{\prime \prime} * \downarrow s s_{j}^{1} \longrightarrow 6 d s_{i+j}^{\prime}$ | $\downarrow s s_{i}^{1} * \downarrow s s_{j}^{1} \longrightarrow 6 \downarrow s s_{i+j}^{1}$ |
| $d d_{i}^{\prime \prime} * s s_{j}^{2} \longrightarrow 4 d s_{i+j}^{\prime}+2 d d_{i+j}^{\prime \prime}$ | $\downarrow s s_{i}^{1} * s s_{j}^{2} \longrightarrow 6 \downarrow s s_{i+j}^{1}$ |

### 4.3 Algorithm

This section describes the algorithm that calculates the genus distribution of a 4 -regular $n$-vertex outerplanar graph in $O\left(n^{2}\right)$ time. The later part of this section also demonstrates how the algorithm works by illustrating it for a simple example.

Input: A rotation system that specifies an outerplane embedding of a 4-regular outerplanar graph $G$.

## Algorithm:

1. Normalize the outerplane embedding by changing rotations of all vertices that have a self-loop incident on them and by making the self-loops lie on the boundary of the face $f_{\infty}$.
2. Obtain the split graph $G^{\prime}$ from the normalized outerplane embedding, and form an incidence tree $T$ with respect to an arbitrarily designated root component and an arbitrarily chosen first child of the root component. At the outset, the only non-zero double-root sub-partial for each component of the split graph $G^{\prime}$ is $d d_{0}^{\prime \prime}=1$. As we see in an example developed in this section, splitting the base vertex of a self-loop leads to a component of $G^{\prime}$ with only one vertex and one edge. However, we pop a new root vertex adjacent to that one vertex and regard that component as also having two roots and as having the double-root partial $d d_{0}^{\prime \prime}=1$, thereby avoiding exceptional handling of this case.
3. Perform a post-order traversal of the incidence tree $T$ and process all the nodes of $T$ in that order. Processing each node requires a vertex-amalgamation, a selfamalgamation, or both operations on its associated component, in addition to certain other actions. When performing a vertex-amalgamation or a self-amalgamation, one calculates the double-root sub-partials for the resulting subgraph by applying the relevant productions to the non-zero double-root sub-partials of the components involved in the operation.

We elaborate on how to process a node based on its type:
(a) Processing a round node of $T$ requires two steps:
i. First the component associated with the round node is vertex-amalgamated on its first-root to the component associated with its parent node in the incidence tree.
ii. After the vertex-amalgamation, check whether the vertex coupled with the second-root of the component belongs to a different component or to the same component. If it is the same component, perform a self-amalgamation.
(b) Processing a square node simulates the amalgamation of coupled vertices that were initially produced by splitting a Type-II vertex. Let $\mathcal{P}$ be the component associated with the parent node of a square node, and let $\mathcal{S}$ be the component associated with the square node. Then processing the square node involves the following steps:
i. First the component $\mathcal{P}$ is vertex-amalgamated on its second-root to the component $\mathcal{S}$. The resulting graph has the first-root on what was previously the component $\mathcal{P}$, while the second-root is on what was previously the component $\mathcal{S}$. There are no further amalgamations to be performed on the subgraph $\mathcal{S}$, whereas we still need two root vertices on the subgraph $\mathcal{P}$ in order to process the parent node of the square node in the post-order traversal of the incidence tree. This necessitates dropping the second-root and popping a new root vertex adjacent to the first-root. Depending on whether the first-root lies in a type $d$ or a type $s$ embedding, the two new roots will now be in a type $d d^{\prime \prime}$ or in a type $\downarrow s s^{1}$ embedding, respectively. This explains the next step.
ii. All $d d_{i}$ and $d s_{i}$ partials for the graph produced in the previous step are added and saved as $d d_{i}^{\prime \prime}$ for each $i$, and all $s d_{i}$ and $s s_{i}$ partials for each $i$ are added and saved as $\downarrow s s_{i}^{1}$. Other than these two sub-partials, all other sub-partials are made zero-valued.
4. Once the entire incidence tree has been processed, the values of sub-partials constitute the partitioned genus distribution of the given graph $G$. The genus distribution can

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now be calculated by summing all non-zero double-root sub-partials for each $i$, i.e.,

$$
g_{i}(G)=\sum_{\substack{x_{i} \text { ranges over all } \\ \text { sub-partials }}} x_{i}(G, u, v)
$$

## Working Out an Example

We simulate the algorithm on a simple example of a 4-regular outerplanar graph, shown in Figure 4.9. The split graph and its corresponding incidence tree for an arbitrarily chosen root component are also shown in Figure 4.9. For ease of referencing, the components of the split graph are labeled with letters of the alphabet.


Figure 4.9: Graph $G$, its split graph and incidence tree.

1. Processing tree node 1 involves a vertex-amalgamation of components $A$ and $B$, followed by a self-amalgamation. We refer to the subgraph obtained as a result of the vertex-amalgamation as $U_{1}$, and to the subgraph resulting from the self-amalgamation of $U_{1}$ as $U_{2}$.
(a) Since $d d_{0}^{\prime \prime}(A)=1$ and $d d_{0}^{\prime \prime}(B)=1$ are the only non-zero sub-partials of components $A$ and $B$, there is only one applicable production for vertex-amalgamation:

$$
\begin{gathered}
d d_{i}^{\prime \prime}(A) * d d_{j}^{\prime \prime}(B) \longrightarrow 4 d d_{i+j}^{\prime}\left(U_{1}\right)+2 s s_{i+j+1}^{2}\left(U_{1}\right) \\
\Longrightarrow \\
\\
\hline
\end{gathered}{d d_{k}^{\prime}\left(U_{1}\right)=4 d d_{k}^{\prime \prime}(A) \times d d_{0}^{\prime \prime}(B)=4 d d_{k}^{\prime \prime}(A) \times 1=4 d d_{k}^{\prime \prime}(A)}^{s s_{k}^{2}\left(U_{1}\right)=2 d d_{k-1}^{\prime \prime}(A) \times d d_{0}^{\prime \prime}(B)=2 d d_{k-1}^{\prime \prime}(A) \times 1=2 d d_{k-1}^{\prime \prime}(A)} \begin{gathered}
d d_{0}^{\prime}\left(U_{1}\right)=4 d d_{0}^{\prime \prime}(A)=4 \times 1=4 \\
s s_{1}^{2}\left(U_{1}\right)=2 d d_{0}^{\prime \prime}(A)=2 \times 1=2
\end{gathered}
$$

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(b) For self-amalgamation of $U_{1}$, we need Productions 4.2 and 4.12 ,

$$
\begin{gathered}
d d_{i}^{\prime}\left(U_{1}\right) \longrightarrow d d_{i}^{\prime \prime}\left(U_{2}\right)+3 d d_{i+1}^{\prime \prime}\left(U_{2}\right)+2 \downarrow s s_{i+1}^{1}\left(U_{2}\right) \\
s s_{i}^{2}\left(U_{1}\right) \longrightarrow d d_{i-1}^{\prime \prime}\left(U_{2}\right)+3 d d_{i}^{\prime \prime}\left(U_{2}\right)+2 \downarrow s s_{i}^{1}\left(U_{2}\right) \\
\Longrightarrow \quad \\
\\
\\
\\
\\
\downarrow s d_{k}^{\prime \prime}\left(U_{2}\right)=d d_{k}^{\prime}\left(U_{1}\right)+3 d d_{k-1}^{\prime}\left(U_{1}\right)+s s_{k+1}^{2}\left(U_{1}\right)+3 s s_{k}^{2}\left(U_{1}\right) \\
\\
\\
d d_{0}^{\prime \prime}\left(U_{2}\right)=d d_{0}^{\prime}\left(U_{1}\right)+0+s s_{1}^{2}\left(U_{1}\right)+2 s s_{k}^{2}\left(U_{1}\right) \\
d d_{1}^{\prime \prime}\left(U_{2}\right)=0+3 d d_{0}^{\prime}\left(U_{1}\right)+0+3 s s_{1}^{2}\left(U_{1}\right)=3 \times 4+3 \times 2=18 \\
\downarrow s s_{1}^{1}\left(U_{2}\right)=2 d d_{0}^{\prime}\left(U_{1}\right)+2 s s_{1}^{2}\left(U_{1}\right)=2 \times 4+2 \times 2=12
\end{gathered}
$$

2. Processing tree node 2 involves two steps, since it is a square vertex:
(a) The first step involves amalgamating the component $C$ to the component $D$.

Remark 13. Notice that even though $D$ has a single vertex, we can consider a second-root vertex adjacent to the single vertex and then work as before, using $d d_{0}^{\prime \prime}(D)=1$ as the only non-zero sub-partial.

Since $d d_{0}^{\prime \prime}(C)=1$ and $d d_{0}^{\prime \prime}(D)=1$, this case is similar to what occurred while processing tree node 1 , where components $A$ and $B$ were vertex-amalgamated. The resulting graph $U_{3}=C * D$ will have the same values for sub-partials as were produced for the subgraph $U_{1}=A * B$. Thus, before the second step, the partials for $U_{3}$ are $d d_{0}^{\prime}\left(U_{3}\right)=4$ and $s s_{1}^{2}\left(U_{3}\right)=2$.
(b) In the second step, we save all partials of $U_{3}$ as $d d_{i}^{\prime \prime}$ and $\downarrow s s_{i}^{1}$ in order to simulate dropping the second-root of $U_{3}$ and popping the new root on that part of $U_{3}$ which was previously the component $C$ :

$$
d d_{0}^{\prime \prime}\left(U_{3}\right)=4, \downarrow s s_{1}^{1}\left(U_{3}\right)=2
$$

## CHAPTER 4. GENUS DISTRIBUTIONS OF 4-REGULAR OUTERPLANAR GRAPHS

3. Processing tree node 3 means amalgamating the component $U_{2}$, that was produced while processing node 1 , to the component $U_{3}$ produced while processing node 2 . We refer to the component $U_{2} * U_{3}$ as $U_{4}$.

The non-zero sub-partials of $U_{2}$ are

$$
d d_{0}^{\prime \prime}\left(U_{2}\right)=6, \quad d d_{1}^{\prime \prime}\left(U_{2}\right)=18, \quad \downarrow s s_{1}^{1}\left(U_{2}\right)=12
$$

and the non-zero sub-partial of $U_{3}$ are

$$
d d_{0}^{\prime \prime}\left(U_{3}\right)=4, \quad \downarrow s s_{1}^{1}\left(U_{3}\right)=2
$$

The productions needed for vertex-amalgamation of $U_{2}$ and $U_{3}$ are

$$
\begin{aligned}
& d d_{i}^{\prime \prime}\left(U_{2}\right) * d d_{j}^{\prime \prime}\left(U_{3}\right) \longrightarrow 4 d d_{i+j}^{\prime}\left(U_{4}\right)+2 s s_{i+j+1}^{2}\left(U_{4}\right) \\
& \downarrow s s_{i}^{1}\left(U_{2}\right) * d d_{j}^{\prime \prime}\left(U_{3}\right) \longrightarrow 6 \downarrow s d_{i+j}^{\prime}\left(U_{4}\right) \\
& d d_{i}^{\prime \prime}\left(U_{2}\right) * \downarrow s s_{j}^{1}\left(U_{3}\right) \longrightarrow 6 d s_{i+j}^{\prime}\left(U_{4}\right) \\
& \downarrow s s_{i}^{1}\left(U_{2}\right) * \downarrow s s_{j}^{1}\left(U_{3}\right) \longrightarrow 6 \downarrow s s_{i+j}^{1}\left(U_{4}\right) \\
& \Longrightarrow \quad \\
& \\
& \quad d d_{k}^{\prime}\left(U_{4}\right)= 4 d d_{k}^{\prime \prime}\left(U_{2}\right) \times d d_{0}^{\prime \prime}\left(U_{3}\right)=4 d d_{k}^{\prime \prime}\left(U_{2}\right) \times 4=16 d d_{k}^{\prime \prime}\left(U_{2}\right) \\
& s s_{k}^{2}\left(U_{4}\right)= \\
& \downarrow s d_{k}^{\prime}\left(U_{4}\right)= 2 d d_{k-1}^{\prime \prime}\left(U_{2}\right) \times d d_{0}^{\prime \prime}\left(U_{3}\right)=2 d d_{k-1}^{\prime \prime}\left(U_{2}\right) \times 4 d_{0}^{\prime \prime}\left(U_{3}\right)=6 \downarrow s s_{k}^{1}\left(U_{2}\right) \times 4=24 \downarrow s d_{k-1}^{\prime \prime}\left(U_{2}\right) \\
& d s_{k}^{\prime}\left(U_{4}\right)= \\
& \downarrow s s_{k}^{1}\left(U_{4}\right)= 6 d d_{k-1}^{\prime \prime}\left(U_{2}\right) \times \downarrow s s_{k-1}^{1}\left(U_{2}\right) \times \downarrow s s_{1}^{1}\left(U_{3}\right)=6 \downarrow s s_{k-1}^{1}\left(U_{2}\right) \times 2=12 \downarrow s s_{k-1}^{1 \prime}\left(U_{2}\right) \\
& \Longrightarrow \quad
\end{aligned}
$$

$$
\begin{array}{ll}
d d_{0}^{\prime}\left(U_{4}\right)=16 d d_{0}^{\prime \prime}\left(U_{2}\right)=16 \times 6=96 & \downarrow s d_{1}^{\prime}\left(U_{4}\right)=24 \downarrow s s_{1}^{1}\left(U_{2}\right)=24 \times 12=288 \\
d d_{1}^{\prime}\left(U_{4}\right)=16 d d_{1}^{\prime \prime}\left(U_{2}\right)=16 \times 18=288 & d s_{1}^{\prime}\left(U_{4}\right)=12 d d_{0}^{\prime \prime}\left(U_{2}\right)=12 \times 6=72 \\
s s_{1}^{2}\left(U_{4}\right)=8 d d_{0}^{\prime \prime}\left(U_{2}\right)=8 \times 6=48 & d s_{2}^{\prime}\left(U_{4}\right)=12 d d_{1}^{\prime \prime}\left(U_{2}\right)=12 \times 18=216 \\
s s_{2}^{2}\left(U_{4}\right)=8 d d_{1}^{\prime \prime}\left(U_{2}\right)=8 \times 18=144 & \downarrow s s_{2}^{1}\left(U_{4}\right)=12 \downarrow s s_{1}^{1}\left(U_{2}\right)=12 \times 12=144
\end{array}
$$

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4. Processing tree node 4 involves amalgamating subgraphs $E$ and $U_{4}$, followed by a selfamalgamation. We refer to the subgraph $E * U_{4}$ as $U_{5}$, and we refer to the subgraph that results from self-amalgamating $U_{5}$ as $U_{6}$.
(a) For this purpose, five productions are needed for the cases $d d^{\prime \prime} * d d^{\prime}, d d^{\prime \prime} * s s^{2}$, $d d^{\prime \prime} * \downarrow s d^{\prime}, d d^{\prime \prime} * d s^{\prime}$, and $d d^{\prime \prime} * \downarrow s s^{1}$, since $d d_{0}^{\prime \prime}(E)=1$ is the only non-zero sub-partial of $E$. These are the relevant productions:

$$
\begin{aligned}
& d d_{i}^{\prime \prime}(E) * d d_{j}^{\prime}\left(U_{4}\right) \longrightarrow 2 d d_{i+j}^{0}\left(U_{5}\right)+2 d d_{i+j}^{\prime}\left(U_{5}\right)+\uparrow s d_{i+j+1}^{\prime}\left(U_{5}\right)+\downarrow s d_{i+j+1}^{\prime}\left(U_{5}\right) \\
& d d_{i}^{\prime \prime}(E) * s s_{j}^{2}\left(U_{4}\right) \longrightarrow 4 d s_{i+j}^{\prime}\left(U_{5}\right)+2 d d_{i+j}^{\prime \prime}\left(U_{5}\right) \\
& d d_{i}^{\prime \prime}(E) * \downarrow s d_{j}^{\prime}\left(U_{4}\right) \longrightarrow 6 d d_{i+j}^{\prime}\left(U_{5}\right) \\
& d d_{i}^{\prime \prime}(E) * d s_{j}^{\prime}\left(U_{4}\right) \longrightarrow 2 d s_{i+j}^{0}\left(U_{5}\right)+2 d s_{i+j}^{\prime}\left(U_{5}\right)+\downarrow s s_{i+j+1}^{1}\left(U_{5}\right)+\uparrow s s_{i+j+1}^{1}\left(U_{5}\right) \\
& d d_{i}^{\prime \prime}(E) * \downarrow s s_{j}^{1}\left(U_{4}\right) \longrightarrow 6 d s_{i+j}^{\prime}\left(U_{5}\right) \\
& \Longrightarrow \\
& \Longrightarrow \\
& d d_{k}^{0}\left(U_{5}\right)=2 d d_{0}^{\prime \prime}(E) \times d d_{k}^{\prime}\left(U_{4}\right)=2 d d_{k}^{\prime}\left(U_{4}\right) \\
& d d_{k}^{\prime}\left(U_{5}\right)=2 d d_{0}^{\prime \prime}(E) \times d d_{k}^{\prime}\left(U_{4}\right)+6 d d_{0}^{\prime \prime}(E) \times \downarrow s d_{k}^{\prime}\left(U_{4}\right)=2 d d_{k}^{\prime}\left(U_{4}\right)+6 \downarrow s d_{k}^{\prime}\left(U_{4}\right) \\
& \uparrow s d_{k}^{\prime}\left(U_{5}\right)=d d_{0}^{\prime \prime}(E) \times d d_{k-1}^{\prime}\left(U_{4}\right)=d d_{k-1}^{\prime}\left(U_{4}\right) \\
& \downarrow s d_{k}^{\prime}\left(U_{5}\right)=d d_{0}^{\prime \prime}(E) \times d d_{k-1}^{\prime}\left(U_{4}\right)=d d_{k-1}^{\prime}\left(U_{4}\right) \\
& d s_{k}^{\prime}\left(U_{5}\right)=4 d d_{0}^{\prime \prime}(E) \times s s_{k}^{2}\left(U_{4}\right)+2 d d_{0}^{\prime \prime}(E) \times d s_{k}^{\prime}\left(U_{4}\right)+6 d d_{0}^{\prime \prime}(E) \times \downarrow s s_{k}^{1}\left(U_{4}\right) \\
&=4 s s_{k}^{2}\left(U_{4}\right)+2 d s_{k}^{\prime}\left(U_{4}\right)+6 \downarrow s s_{k}^{1}\left(U_{4}\right) \\
& d d_{k}^{\prime \prime}\left(U_{5}\right)=2 d d_{0}^{\prime \prime}(E) \times s s_{k}^{2}\left(U_{4}\right)=2 s s_{k}^{2}\left(U_{4}\right) \\
& d s_{k}^{0}\left(U_{5}\right)=2 d d_{0}^{\prime \prime}(E) \times d s_{k}^{\prime}\left(U_{4}\right)=2 d s_{k}^{\prime}\left(U_{4}\right) \\
& \uparrow s s_{k}^{1}\left(U_{5}\right)=d d_{0}^{\prime \prime}(E) \times d s_{k-1}^{\prime}\left(U_{4}\right)=d s_{k-1}^{\prime}\left(U_{4}\right) \\
& \downarrow s s_{k}^{1}\left(U_{5}\right)=d d_{0}^{\prime \prime}(E) \times d s_{k-1}^{\prime}\left(U_{4}\right)=d s_{k-1}^{\prime}\left(U_{4}\right) \\
& \Longrightarrow
\end{aligned} \quad \begin{aligned}
\Longrightarrow
\end{aligned}
$$

$$
\begin{aligned}
d d_{0}^{\prime}\left(U_{5}\right) & =2 d d_{0}^{\prime}\left(U_{4}\right)+6 \downarrow s d_{0}^{\prime}\left(U_{4}\right)=2 \times 96+0=192 \\
d d_{1}^{\prime}\left(U_{5}\right) & =2 d d_{1}^{\prime}\left(U_{4}\right)+6 \downarrow s d_{1}^{\prime}\left(U_{4}\right)=2 \times 288+6 \times 288=2304 \\
\uparrow s d_{1}^{\prime}\left(U_{5}\right) & =d d_{0}^{\prime}\left(U_{4}\right)=96 \\
\uparrow s d_{2}^{\prime}\left(U_{5}\right) & =d d_{1}^{\prime}\left(U_{4}\right)=288 \\
\downarrow s d_{1}^{\prime}\left(U_{5}\right) & =d d_{0}^{\prime}\left(U_{4}\right)=96 \\
\downarrow s d_{2}^{\prime}\left(U_{5}\right) & =d d_{1}^{\prime}\left(U_{4}\right)=288 \\
d s_{1}^{\prime}\left(U_{5}\right) & =4 s s_{1}^{2}\left(U_{4}\right)+2 d s_{1}^{\prime}\left(U_{4}\right)+6 \downarrow s s_{1}^{1}\left(U_{4}\right) \\
& =4 \times 48+2 \times 72+0=336 \\
d s_{2}^{\prime}\left(U_{5}\right) & =4 s s_{2}^{2}\left(U_{4}\right)+2 d s_{2}^{\prime}\left(U_{4}\right)+6 \downarrow s s_{2}^{1}\left(U_{4}\right) \\
& =4 \times 144+2 \times 216+6 \times 144=1872 \\
d d_{1}^{\prime \prime}\left(U_{5}\right) & =2 s s_{1}^{2}\left(U_{4}\right)=2 \times 48=96 \\
d d_{2}^{\prime \prime}\left(U_{5}\right) & =2 s s_{2}^{2}\left(U_{4}\right)=2 \times 144=288 \\
d s_{1}^{0}\left(U_{5}\right) & =2 d s_{1}^{\prime}\left(U_{4}\right)=2 \times 72=144 \\
d s_{2}^{0}\left(U_{5}\right) & =2 d s_{2}^{\prime}\left(U_{4}\right)=2 \times 216=432 \\
\uparrow s s_{2}^{1}\left(U_{5}\right) & =d s_{1}^{\prime}\left(U_{4}\right)=72 \\
\uparrow s s_{3}^{1}\left(U_{5}\right) & =d s_{2}^{\prime}\left(U_{4}\right)=216 \\
\downarrow s s_{2}^{1}\left(U_{5}\right) & =d s_{1}^{\prime}\left(U_{4}\right)=72 \\
\downarrow s s_{3}^{1}\left(U_{5}\right) & =d s_{2}^{\prime}\left(U_{4}\right)=216
\end{aligned}
$$

(b) Productions 4.1-4.5, 4.7-4.8, and 4.10-4.11 are needed for self-amalgamation of $U_{5}$ :

$$
\begin{array}{lrl}
d d_{i}^{0}\left(U_{5}\right) \longrightarrow 4 d d_{i+1}^{\prime \prime}\left(U_{6}\right)+2 \downarrow s s_{i+2}^{1}\left(U_{6}\right) & d d_{i}^{\prime}\left(U_{5}\right) \longrightarrow & d d_{i}^{\prime \prime}\left(U_{6}\right)+3 d d_{i+1}^{\prime \prime}\left(U_{6}\right) \\
& & +2 \downarrow s s_{i+1}^{1}\left(U_{6}\right) \\
d d_{i}^{\prime \prime}\left(U_{5}\right) \longrightarrow 4 d d_{i}^{\prime \prime}\left(U_{6}\right)+2 \downarrow s s_{i+1}^{1}\left(U_{6}\right) & & d s_{i}^{\prime}\left(U_{5}\right) \longrightarrow \\
d s_{i}^{0}\left(U_{5}\right) \longrightarrow & 3 d d_{i}^{\prime \prime}\left(U_{6}\right)+3 \downarrow s d_{i+1}^{1 \prime}\left(U_{6}\right) \\
\uparrow s d_{i}^{\prime}\left(U_{5}\right) \longrightarrow & 3 d d_{i}^{\prime \prime}\left(U_{6}\right)+3 \downarrow s s_{i+1}^{1}\left(U_{6}\right) & \downarrow s d_{i}^{\prime}\left(U_{5}\right) \longrightarrow \\
\uparrow s \downarrow s s_{i}^{1}\left(U_{6}\right)+3 \downarrow s s_{i+1}^{1}\left(U_{6}\right) \\
& & \downarrow s s_{i}^{1}\left(U_{5}\right) \longrightarrow 6 d d_{i}^{\prime \prime}\left(U_{6}\right) \longrightarrow \\
& &
\end{array}
$$

$$
\begin{aligned}
& d d_{k}^{\prime \prime}\left(U_{6}\right)=4 d d_{k-1}^{0}\left(U_{5}\right)+d d_{k}^{\prime}\left(U_{5}\right)+3 d d_{k-1}^{\prime}\left(U_{5}\right)+4 d d_{k}^{\prime \prime}\left(U_{5}\right)+6 d s_{k-1}^{0}\left(U_{5}\right) \\
& +3 d s_{k}^{\prime}\left(U_{5}\right)+3 \uparrow s d_{k}^{\prime}\left(U_{5}\right)+6 \uparrow s s_{k}^{1}\left(U_{5}\right) \\
& \downarrow s s_{k}^{1}\left(U_{6}\right)=2 d d_{k-2}^{0}\left(U_{5}\right)+2 d d_{k-1}^{\prime}\left(U_{5}\right)+2 d d_{k-1}^{\prime \prime}\left(U_{5}\right)+3 d s_{k-1}^{\prime}\left(U_{5}\right) \\
& +3 \uparrow s d_{k-1}^{\prime}\left(U_{5}\right)+3 \downarrow s d_{k}^{\prime}\left(U_{5}\right)+3 \downarrow s d_{k-1}^{\prime}\left(U_{5}\right)+6 \downarrow s s_{k}^{1}\left(U_{5}\right) \\
& \Longrightarrow \\
& d d_{0}^{\prime \prime}\left(U_{6}\right)=0+d d_{0}^{\prime}\left(U_{5}\right)+0+0+0+0+0+0=192 \\
& d d_{1}^{\prime \prime}\left(U_{6}\right)=4 d d_{0}^{0}\left(U_{5}\right)+d d_{1}^{\prime}\left(U_{5}\right)+3 d d_{0}^{\prime}\left(U_{5}\right)+4 d d_{1}^{\prime \prime}\left(U_{5}\right)+0+3 d s_{1}^{\prime}\left(U_{5}\right) \\
& +3 \uparrow s d_{1}^{\prime}\left(U_{5}\right)+0 \\
& =4 \times 192+2304+3 \times 192+4 \times 96+3 \times 336+3 \times 96=5328 \\
& d d_{2}^{\prime \prime}\left(U_{6}\right)=4 d d_{1}^{0}\left(U_{5}\right)+0+3 d d_{1}^{\prime}\left(U_{5}\right)+4 d d_{2}^{\prime \prime}\left(U_{5}\right)+6 d s_{1}^{0}\left(U_{5}\right)+3 d s_{2}^{\prime}\left(U_{4}\right) \\
& +3 \uparrow s d_{2}^{\prime}\left(U_{5}\right)+6 \uparrow s s_{2}^{1}\left(U_{5}\right) \\
& =4 \times 576+3 \times 2304+4 \times 288+6 \times 144+3 \times 1872+3 \times 288 \\
& +6 \times 72=18144 \\
& d d_{3}^{\prime \prime}\left(U_{6}\right)=0+0+0+0+6 d s_{2}^{0}\left(U_{5}\right)+0+0+6 \uparrow s s_{3}^{1}\left(U_{5}\right)=6 \times 432 \\
& +6 \times 216=3888 \\
& \downarrow s s_{0}^{1}\left(U_{6}\right)=0 \\
& \downarrow s s_{1}^{1}\left(U_{6}\right)=0+2 d d_{0}^{\prime}\left(U_{5}\right)+0+0+0+3 \downarrow s d_{1}^{\prime}\left(U_{5}\right)+0+0 \\
& =2 \times 192+3 \times 96=672 \\
& \downarrow s s_{2}^{1}\left(U_{6}\right)=2 d d_{0}^{0}\left(U_{5}\right)+2 d d_{1}^{\prime}\left(U_{5}\right)+2 d d_{1}^{\prime \prime}\left(U_{5}\right)+3 d s_{1}^{\prime}\left(U_{5}\right)+3 \uparrow s d_{1}^{\prime}\left(U_{5}\right) \\
& +3 \downarrow s d_{2}^{\prime}\left(U_{5}\right)+3 \downarrow s d_{1}^{\prime}\left(U_{5}\right)+6 \downarrow s s_{2}^{1}\left(U_{5}\right) \\
& =2 \times 192+2 \times 2304+2 \times 96+3 \times 336+3 \times 96+3 \times 288 \\
& +3 \times 96+6 \times 72=8064 \\
& \downarrow s s_{3}^{1}\left(U_{6}\right)=2 d d_{1}^{0}\left(U_{5}\right)+0+2 d d_{2}^{\prime \prime}\left(U_{5}\right)+3 d s_{2}^{\prime}\left(U_{5}\right)+3 \uparrow s d_{2}^{\prime}\left(U_{5}\right)+0+0 \\
& +3 \downarrow s d_{2}^{\prime}\left(U_{5}\right)+6 \downarrow s s_{3}^{1}\left(U_{5}\right)
\end{aligned}
$$

$$
\begin{aligned}
& =2 \times 576+2 \times 288+3 \times 1872+3 \times 288+3 \times 288+6 \times 216 \\
& =10368
\end{aligned}
$$

5. Processing tree node 5 returns immediately, since it is the root node. Thus, the assembled graph $U_{6}$ is the outerplanar graph $G$.
6. The genus distribution for $G$ can be obtained by summing the sub-partials as follows:

$$
\begin{aligned}
& g_{0}(G)=d d_{0}^{\prime \prime}(G)+0=192 \\
& g_{1}(G)=d d_{1}^{\prime \prime}(G)+\downarrow s s_{1}^{1}(G)=5328+672=6000 \\
& g_{2}(G)=d d_{2}^{\prime \prime}(G)+\downarrow s s_{2}^{1}(G)=18144+8064=26208 \\
& g_{3}(G)=d d_{3}^{\prime \prime}(G)+\downarrow s s_{3}^{1}(G)=3888+10368=14256
\end{aligned}
$$

### 4.4 Time-Complexity Analysis

Normalizing the outerplane embedding and obtaining the split graph are $O(n)$ operations, where $n$ is the number of vertices of the given graph. Since the split graph has fewer than $n$ components, it follows that forming an incidence tree is also $O(n)$.

Theorem 18. A connected subgraph $H$ of a 4-regular outerplanar graph on $k$ vertices has $O(k)$ number of partials.

Proof. Let $k$ be the number of vertices in a subgraph $H$ assembled using the algorithm. A connected 4-regular graph with $k$ vertices has cycle $\operatorname{rank} \beta=k+1$. Since $H$ is a subgraph of a connected 4-regular graph, the maximum genus of $H$ is bounded by $\left\lfloor\frac{\beta(H)}{2}\right\rfloor \leq\left\lfloor\frac{k+1}{2}\right\rfloor$. As there are 12 sub-partial types, the number of sub-partials of $H$ is bounded from above by $12 \times\left\lfloor\frac{k+1}{2}\right\rfloor$.

1. Time-Complexity of an Amalgamation Operation: If the parent component has $p$ vertices and the child component has $q$ vertices, then by Theorem 18 their number of partials are $O(p)$ and $O(q)$, respectively. Applying a single production for an amalgamation step is $O(1)$. Consequently, the complexity of applying all productions for a single amalgamation is $O(p q)$. The number of vertices in the subgraph resulting from amalgamation is $O(p+q)$.
2. Time-Complexity of a Self-Amalgamation Operation: If the graph component undergoing self-amalgamation has $p$ vertices then the complexity of applying selfamalgamation productions is $O(p)$. The number of vertices in the resulting graph component is $O(p)$.

Let $n_{1}, n_{2}, \cdots, n_{r}$ be the number of vertices in the components of the split graph of a graph $G$. From the first point above, it follows that if a component of size $\sum_{i \in I} n_{i}$ amalgamates to a component of size $\sum_{j \in J} n_{j}$, where $I$ and $J$ are some disjoint sets, then the time-complexity of performing the operation is

$$
\sum_{i \in I} n_{i} \sum_{j \in J} n_{j}
$$

and the size of the resulting graph is

$$
\sum_{i \in I \cup J} n_{i}
$$

As each coupled pair of vertices is amalgamated only once, the complexity of reconstructing the original graph is $O\left(\sum_{i \in I, j \in J} n_{i} n_{j}\right)$ for some disjoint sets $I$ and $J$. Therefore, the complexity of the given algorithm is $O\left(\left(n_{1}+\cdots+n_{r}\right)\left(n_{1}+\cdots+n_{r}\right)\right)=O(n \cdot n)=O\left(n^{2}\right)$.

### 4.5 Correctness

In order to show that the algorithm given in $\$ 4.3$ correctly computes genus distribution of 4-regular outerplanar graphs, we need to address the question of whether root vertices will be available at the right time and the right place for amalgamations and self-amalgamations. As before, we regard the components represented by round and square nodes of an incidence tree as nodes of the tree themselves, and we use expressions like "parent component", "child component" etc. We argue inductively for a tree node that the graph constructed by processing each of the child nodes of that node contains two root vertices and that these roots are available for the next amalgamation operation on that graph.

Lemma 19. Coupled components of an incidence tree are always in an ancestor-descendant relationship.

Proof. When two components are not coupled, they are said to be separated. Since an incidence tree is created in a depth-first manner and since depth-first trees have no cross-edges, it follows that the components from sibling subtrees of an incidence tree are separated from each other. Thus, the vertices that recombine under amalgamation or self-amalgamation must initially belong to coupled components that are in an ancestor-descendant relationship.

Theorem 20. Let $\mathcal{P}$ be a component with one or more child components, none of which correspond to square nodes. Then every graph in the sequence of graphs produced by processing the children of $\mathcal{P}$ contains two root vertices, such that these root vertices are available for the next amalgamation.

Proof. Before any of its child nodes are processed, $\mathcal{P}$ is homeomorphic to a cycle graph and has two roots. When $\mathcal{P}$ has more than one child, processing its first child involves an amalgamation of the child with $\mathcal{P}$ and necessarily ends with a self-amalgamation that produces two consecutive roots on the resulting graph. This is illustrated in Figure 4.10.


Figure 4.10: An example of propagation of root vertices.

The first-root of the double-rooted graph produced as a result of the self-amalgamation corresponds to the vertex popped farther in the counter-clockwise direction, as shown. The first- and second-root of each amalgamand are labeled 1 and 2, respectively. Thus, the roots
are available at the right place for the next child to be processed. All but the last child will be eventually processed similarly during the post-order traversal. In case of the last child or the only child, if $\mathcal{P}$ has exactly one child component, its eventual amalgamation with $\mathcal{P}$ may or may not be immediately followed by a self-amalgamation. If there is an immediate self-amalgamation then as before, it will produce two adjacent roots, which can again be used for attaching the resulting subgraph to its parent. On the other hand, if there is no immediate self-amalgamation, then the second-root is preserved till a later time, when it undergoes a self-amalgamation while processing an ancestor of $\mathcal{P}$ or $\mathcal{P}$ itself.

Lemma 21. No self-amalgamation is required while processing a square node.
Proof. Processing a square node represents the need to amalgamate coupled vertices that arise either by splitting a cut vertex or by splitting the endpoint of a loop. A square node has descendant components only in the former case. The descendants of a square node are separated from its ancestor components since these two sets of components arise from splitting different blocks of the outerplanar graph embedding. In addition, a component corresponding to a square node has exactly one vertex coupled to its parent component. Therefore, no self-amalgamation is required when processing a square node.

Theorem 22. Let $\mathcal{P}$ be a component with at-least one child component corresponding to a square node. Then every graph in the sequence of graphs produced by processing the children of $\mathcal{P}$ contains two root vertices, such that these root vertices are available for the next amalgamation.

Proof. Let $\mathcal{S}$ be the first child component of $\mathcal{P}$ corresponding to a square node. Then $\mathcal{S}$ is necessarily separated from its siblings as well as from the ancestor of $\mathcal{P}$, by Lemma 19 and 21. The amalgamation of $\mathcal{S}$ to its parent component produces two roots on the resulting graph, only the first of which lies on its subgraph $\mathcal{P}$. The other root, that lies on the subgraph $\mathcal{S}$, is redundant by Lemma 21. This redundant root is dropped and only the information for the first-root is retained in the form of single-root partials $s_{i}$ and $d_{i}$. We then pop up a new second-root adjacent to our first-root on the subgraph $\mathcal{P}$ and re-adjust the numbers we had for $s_{i}$ and $d_{i}$ as $\downarrow s s_{i}^{1}$ and $d d_{i}^{\prime \prime}$, respectively. Thus, as we continue to

## CHAPTER 4. GENUS DISTRIBUTIONS OF 4-REGULAR OUTERPLANAR GRAPHS

process the remaining child components of $\mathcal{P}$ or $\mathcal{P}$ itself, both roots will be available for the next amalgamation.

By Theorem 20 and 22, each component is readily amalgamated to its parent node. If a self-amalgamation is required, it is performed as soon as the opportunity presents itself. In this bottom up fashion, eventually the entire graph is reconstructed.

## Part II

## Conclusions

## Chapter 5

## Conclusions

In approaching the genus distribution problem, I follow the divide-and-conquer approach of computing genus distributions of large graphs in terms of the genus distributions of their smaller constituent subgraphs. Under this guiding principle, I look at graphs constructed from small base graphs through various kinds of edge- and vertex- amalgamations. Accordingly, there are three main components of my research.

1. Recurrences are devised to specify genus distributions of chain-like graphs that are constructed from base graphs pasted together along their edges. It is assumed that the edges being pasted have 2 -valent endpoints. It is also assumed that the partitioned genus distributions of these base graphs are known, where the partitioned genus distribution of a graph is understood to be a breakdown of its total number of embeddings into an inventory that specifies the number of embeddings of the graph for each surface and each type under a classification system formulated here.
2. Two closed formulas are derived for computing the genus distribution of graphs obtained from other graphs, by an amalgamation operation that involves identification of edges of the same graph. Though there are two different ways of performing such an identification, this operation is broadly qualified as self-edge-amalgamation. It is assumed, as in the case of edge-amalgamation, that the partitioned genus distribution of the graph undergoing self-edge-amalgamation is known and that the endpoints of the edges undergoing the pasting are 2 -valent.
3. A quadratic-time algorithm is demonstrated for calculating the genus distribution of any 4 -regular outerplanar graph. The algorithm causes a 4 -regular outerplanar graph to split into its constituent subgraphs, and then simulates the synthesis of the original graph, by identifying the necessary vertices in a prescribed order. In the process, it utilizes the genus distributions of the constituent subgraphs to calculate the genus distribution for the reassembled outerplanar graph.

### 5.1 Contributions

The salient contributions of my dissertation are summarized as follows:

1. It provides a general method based on edge-amalgamation that enables us to calculate the genus distribution of recursively defined infinite families of graphs, a task that has been not possible hitherto without recourse to the brute-force Heffter-Edmonds algorithm. The methods discussed have theoretical importance in view of the scope of prior research on the genus distribution problem, which has, on one hand, tended to focus on highly symmetrical graphs and has, on the other hand, focused on techniques that are specific to a particular graph family without applicability to other families of graphs. Apart from theoretical importance, there is also practical value in making genus distribution calculations computationally viable for many families of graphs.
2. This general plan enables genus distribution calculations for various edge-pasted chains constructed by using copies of different types of graphs or by using multiple copies of the same graph. In this manner, genus distributions can be computed for various infinite families of 3 -regular graphs, apart from many other infinite classes.
3. This is taken a step further, to find the genus distribution of graphs produced as a result of self-edge-amalgamation. Thus, further expanding the set of graphs for which genus distributions can be computed.
4. The recurrences for edge-amalgamation can in some cases be solved, whereby to yield closed formulas. These recurrences and formulas may be analyzed for proving unimodality for some classes of graphs. More importantly, they may provide an op-
portunity for producing a counterexample to the unimodality conjecture, if such a counterexample exists. Such opportunities for insight into the unimodality conjecture also present themselves through the two closed-formulas given for graphs undergoing self-edge-amalgamation.
5. 2.6 gives an easily understood method for constructing pairs of non-homeomorphic graphs with the same genus distribution.
6. The results in $\$_{2}$ have been used by Gross, 2011b to construct a quadratic-time algorithm for calculating the genus distribution of any 3-regular outerplanar graph.
7. An $O\left(n^{2}\right)$-time algorithm is described for calculating the genus distributions of 4regular outerplanar graphs. Insofar as time-complexity is concerned, this is a significant improvement over the previous $O\left(6^{n}\right)$ time-complexity encountered when applying the Heffter-Edmonds algorithm to 4-regular outerplanar graphs. Outerplanar graphs and their embeddings have been of particular interest to mathematicians working in the branch of topological graph theory that deals with obstructions and minors. Thus, whereas many of the graph families for which genus distributions have been calculated in the past are felt to be somewhat contrived, the class of outerplanar graphs is certainly an interesting family in its own right.

### 5.2 Future Research

My research has possibly made progress towards some very important problems in the area. We discuss some of these distant goals and explore avenues of future research that may have the potential to contribute something of use to the larger body of knowledge pertaining to these problems. However, the feasibility of conducting future research to solve these problems based on my research is an unknown.

1. The Heawood map-coloring problem is a classical problem in topological graph theory. Map-coloring refers to a coloring of the regions of an embedding such that all adjacent regions use distinct colors. The Heawood map-coloring problem asks how many colors suffice for a map-coloring on a given surface? A landmark solution to the

Heawood map-coloring problem came in 1968 Ringel and Youngs, 1968. A major part of the solution consisted of finding the minimum genus of the complete graphs $K_{n}$, for all $n \geq 3$. The Ringel-Youngs proof for the Heawood map-coloring problem comprised over 300 pages. Gross's topological generalization of current graphs reduced the length of the proof to about half [Gross and Alpert, 1974]. It might be possible to build upon the methods discussed earlier to find genus distributions of complete graphs through a unified approach that perhaps sheds light on the behavior of genus distributions of covering graphs. If this is accomplished, it might potentially simplify the solution, as well as further reduce the length of the existing solution.
2. The Genus distribution of planar graphs is an important problem. A step in that direction would be to improve upon my algorithm for 4-regular outerplanar graphs to the class of 4-regular Hamiltonian planar graphs.

A special case of 4-regular outerplanar graphs are the 4-regular Hamiltonian outerplanar graphs, which can be embedded on the sphere in such a way that a Hamiltonian cycle forms the boundary for the face at infinity, and such that the remaining edges of the graph can be regarded as comprising of polygons inscribed inside the Hamiltonian cycle. One observes that all 4-regular Hamiltonian planar graphs can also be characterized in a similar manner, as a Hamiltonian cycle with "outer" polygons as well as "inner" polygons. Are the techniques covered here, therefore, extendible to 4-regular planar Hamiltonian graphs?
3. The Unimodality conjecture has been an open problem for more than 20 years. For the first time, there is a general method available for examining embedding trends for large classes of graphs. There is great potential for an in-depth analysis of why graph families generated in certain systematic ways, such as ours, are unimodal.

One finds oneself motivated to ask if open as well as closed chains of base graphs with strongly unimodal genus distributions are unimodal? We saw a formulation of the genus distribution of closed chains as a linear combination of sub-partials of the corresponding open chains. In turn, each of these sub-partials of an open chain is calculated as a linear combination of convolutions of sub-partials of its constituent
subgraphs. Although convolutions of strongly unimodal sequences are strongly unimodal, linear combinations of unimodal sequences need not be unimodal, unless the modes are sufficiently close together. The task at hand seems quite complicated, due to the large number of sub-partials. Nevertheless, the analysis of the recurrences and closed-formulas presented here may prove to be useful for establishing structural results related to genus distributions as well as for providing insights into the larger question of unimodality. The recurrences and formulas can also be of great computational assistance in attempting to uncover a counterexample, if one exists.
4. It has not been explored how this technique would work in combination with other techniques. This is an avenue ripe for further investigation and can greatly extend the classes of graphs for which genus distribution can be computed.
5. The impact of bounding tree-width on genus distribution should also be examined in the light of these and other related techniques.
6. Adapting these ideas to the non-orientable case appears to be fairly accessible.
7. Other problems include the following:

- Calculating the genus distributions of graphs produced by amalgamating other graphs on edges with higher-valent endpoints as well as of graphs produced by amalgamating on more general subgraphs than $K_{1}$ and $K_{2}$
- Calculating the genus distributions of $k$-regular graphs for higher values of $k$, and the generalization of such calculations to outer-embeddable graphs.


## Part III

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## Part IV

## Appendices

## Appendix A

## Productions and Recurrences for Edge-Amalgamations

Table A.1: Productions for edge-amalgamation $(G, e, d) *(H, g, f)$ where the embedding of graph $H$ has partial type $d d^{0}$.

$$
\begin{array}{|l|}
\hline \hline \text { production } \\
\hline d d_{i}^{0} * d d_{j}^{0} \longrightarrow 2 d d_{i+j}^{0}+2 d d_{i+j+1}^{0} \\
d d_{i}^{\prime} * d d_{j}^{0} \longrightarrow 2 d d_{i+j}^{0}+2 d d_{i+j+1}^{0} \\
d d_{i}^{\prime \prime} * d d_{j}^{0} \longrightarrow 2 d d_{i+j}^{0}+2 s d_{i+j+1}^{0} \\
d s_{i}^{0} * d d_{j}^{0} \longrightarrow 4 d d_{i+j}^{0} \\
d s_{i}^{\prime} * d d_{j}^{0} \longrightarrow 4 d d_{i+j}^{0} \\
s d_{i}^{0} * d d_{j}^{0} \longrightarrow 2 s d_{i+j}^{0}+2 s d_{i+j+1}^{0} \\
s d_{i}^{\prime} * d d_{j}^{0} \longrightarrow 2 s d_{i+j}^{0}+2 s d_{i+j+1}^{0} \\
s s_{i}^{0} * d d_{j}^{0} \longrightarrow 4 s d_{i+j}^{0} \\
s s_{i}^{1} * d d_{j}^{0} \longrightarrow 4 s d_{i+j}^{0} \\
s s_{i}^{2} * d d_{j}^{0} \longrightarrow 2 d d_{i+j}^{0}+2 s d_{i+j}^{0} \\
\hline \hline
\end{array}
$$

Table A.2: Set I of III: Productions for edge-amalgamation $(G, e, d) *(H, g, f)$ where the embedding of graph $H$ has partial type $d d^{\prime}$.

| production |
| :---: |
| $\begin{aligned} d d_{i}^{0} * d d_{j}^{\prime} & \longrightarrow 2 d d_{i+j}^{0}+2 d d_{i+j+1}^{0} \\ d s_{i}^{0} * d d_{j}^{\prime} & \longrightarrow 4 d d_{i+j}^{0} \\ s d_{i}^{0} * d d_{j}^{\prime} & \longrightarrow 2 s d_{i+j}^{0}+2 s d_{i+j+1}^{0} \\ s s_{i}^{0} * d d_{j}^{\prime} & \longrightarrow 4 s d_{i+j}^{0} \end{aligned}$ |
|  |

Table A.3: Set II of III: Productions for edge-amalgamation $(G, e, d) *(H, g, f)$ where the embedding of graph $H$ has partial type $d d^{\prime}$.

| production |
| :---: |
|  |

Table A.4: Set III of III: Productions for edge-amalgamation $(G, e, d) *(H, g, f)$
where the embedding of graph $H$ has partial type $d d^{\prime}$.

| production |
| :---: |
|  |
|  |

Table A.5: Set I of II: Productions for edge-amalgamation $(G, e, d) *(H, g, f)$ where the embedding of graph $H$ has partial type $d d^{\prime \prime}$.

|  | production |
| :---: | :---: |
|  | $\begin{aligned} & d d_{i}^{0} * d d_{j}^{\prime \prime} \longrightarrow 2 d d_{i+j}^{0}+2 d s_{i+j+1}^{0} \\ & d s_{i}^{0} * d d_{j}^{\prime \prime} \longrightarrow 4 d d_{i+j}^{0} \\ & s d_{i}^{0} * d d_{j}^{\prime \prime} \longrightarrow 2 s d_{i+j}^{0}+2 s s_{i+j+1}^{0} \\ & s s_{i}^{0} * d d_{j}^{\prime \prime} \longrightarrow 4 s d_{i+j}^{0} \end{aligned}$ |
|  |  |

Table A.6: Set II of II: Productions for edge-amalgamation $(G, e, d) *(H, g, f)$ where the embedding of graph $H$ has partial type $d d^{\prime \prime}$.


Table A.7: Productions for edge-amalgamation $(G, e, d) *(H, g, f)$ where the embedding of graph $H$ has partial type $d s^{0}$.

| production |
| :---: |
| $d d_{i}^{0} * d s_{j}^{0} \longrightarrow 2 d s_{i+j}^{0}+2 d s_{i+j+1}^{0}$ |
| $d d_{i}^{\prime} * d s_{j}^{0} \longrightarrow 2 d s_{i+j}^{0}+2 d s_{i+j+1}^{0}$ |
| $d d_{i}^{\prime \prime} * d s_{j}^{0} \longrightarrow 2 d s_{i+j}^{0}+2 s s_{i+j+1}^{0}$ |
| $d s_{i}^{0} * d s_{j}^{0} \longrightarrow 4 d s_{i+j}^{0}$ |
| $d s_{i}^{\prime} * d s_{j}^{0} \longrightarrow 4 d s_{i+j}^{0}$ |
| $s d_{i}^{0} * d s_{j}^{0} \longrightarrow 2 s s_{i+j}^{0}+2 s s_{i+j+1}^{0}$ |
| $s d_{i}^{\prime} * d s_{j}^{0} \longrightarrow 2 s s_{i+j}^{0}+2 s s_{i+j+1}^{0}$ |
| $s s_{i}^{0} * d s_{j}^{0} \longrightarrow 4 s s_{i+j}^{0}$ |
| $s s_{i}^{1} * d s_{j}^{0} \longrightarrow 4 s s_{i+j}^{0}$ |
| $s s_{i}^{2} * d s_{j}^{0} \longrightarrow 2 d s_{i+j}^{0}+2 s s_{i+j}^{0}$ |

Table A.8: Set I of II: Productions for edge-amalgamation $(G, e, d) *(H, g, f)$ where the embedding of graph $H$ has partial type $d s^{\prime}$.


Table A.9: Set II of II: Productions for edge-amalgamation $(G, e, d) *(H, g, f)$ where the embedding of graph $H$ has partial type $d s^{\prime}$.

| production |
| :---: |
|  |

Table A.10: Productions for edge-amalgamation $(G, e, d) *(H, g, f)$ where the embedding of graph $H$ has partial type $s d^{0}$.

$$
\begin{array}{|c|}
\hline \hline \text { production } \\
\hline d d_{i}^{0} * s d_{j}^{0} \longrightarrow 4 d d_{i+j}^{0} \\
d d_{i}^{\prime} * s d_{j}^{0} \longrightarrow 4 d d_{i+j}^{0} \\
d d_{i}^{\prime \prime} * s d_{j}^{0} \longrightarrow 4 d d_{i+j}^{0} \\
d s_{i}^{0} * s d_{j}^{0} \longrightarrow 4 d d_{i+j}^{0} \\
d s_{i}^{\prime} * s d_{j}^{0} \longrightarrow 4 d d_{i+j}^{0} \\
s d_{i}^{0} * s d_{j}^{0} \longrightarrow 4 s d_{i+j}^{0} \\
s d_{i}^{\prime} * s d_{j}^{0} \longrightarrow 4 s d_{i+j}^{0} \\
s s_{i}^{0} * s d_{j}^{0} \longrightarrow 4 s d_{i+j}^{0} \\
s s_{i}^{1} * s d_{j}^{0} \longrightarrow 4 s d_{i+j}^{0} \\
s s_{i}^{2} * s d_{j}^{0} \longrightarrow 4 s d_{i+j}^{0} \\
\hline \hline
\end{array}
$$

Table A.11: Set I of II: Productions for edge-amalgamation $(G, e, d) *(H, g, f)$ where the embedding of graph $H$ has partial type $s d^{\prime}$.

| production |
| :---: |
| $\begin{aligned} d d_{i}^{0} * s d_{j}^{\prime} & \longrightarrow 4 d d_{i+j}^{0} \\ d s_{i}^{0} * s d_{j}^{\prime} & \longrightarrow 4 d d_{i+j}^{0} \\ s d_{i}^{0} * s d_{j}^{\prime} & \longrightarrow 4 s d_{i+j}^{0} \\ s s_{i}^{0} * s d_{j}^{\prime} & \longrightarrow 4 s d_{i+j}^{0} \end{aligned}$ |
|  |

Table A.12: Set II of II: Productions for edge-amalgamation $(G, e, d) *(H, g, f)$ where the embedding of graph $H$ has partial type $s d^{\prime}$.

| production |
| :---: |
|  |

Table A.13: Productions for edge-amalgamation $(G, e, d) *(H, g, f)$ where the embedding of graph $H$ has partial type $s s^{0}$.

| production |
| :---: |
| $d d_{i}^{0} * s s_{j}^{0} \longrightarrow 4 d s_{i+j}^{0}$ |
| $d d_{i}^{\prime} * s s_{j}^{0} \longrightarrow 4 d s_{i+j}^{0}$ |
| $d d_{i}^{\prime} * s s_{j}^{0} \longrightarrow 4 d s_{i+j}^{0}$ |
| $d d_{i}^{\prime \prime} * s s_{j}^{0} \longrightarrow 4 d s_{i+j}^{0}$ |
| $d s_{i}^{0} * s s_{j}^{0} \longrightarrow 4 d s_{i+j}^{0}$ |
| $d s_{i}^{\prime} * s s_{j}^{0} \longrightarrow 4 d s_{i+j}^{0}$ |
| $s d_{i}^{0} * s s_{j}^{0} \longrightarrow 4 s s_{i+j}^{0}$ |
| $s d_{i}^{\prime} * s s_{j}^{0} \longrightarrow 4 s s_{i+j}^{0}$ |
| $s s_{i}^{0} * s s_{j}^{0} \longrightarrow 4 s s_{i+j}^{0}$ |
| $s s_{i}^{1} * s s_{j}^{0} \longrightarrow 4 s s_{i+j}^{0}$ |
| $s s_{i}^{2} * s s_{j}^{0} \longrightarrow 4 s s_{i+j}^{0}$ |

Table A.14: Productions for edge-amalgamation $(G, e, d) *(H, g, f)$ where the embedding of graph $H$ has partial type $s s^{1}$.

| production |
| :---: |
|  |

Table A.15: Productions for edge-amalgamation $(G, e, d) *(H, g, f)$ where the embedding of graph $H$ has partial type $s s^{2}$.

|  | production |
| :---: | :---: |
|  |  |

The productions in Tables A.1 A. 15 lead to Theorems 23-38, In stating the recurrences for sub-partials in Theorems 23-38, we abbreviate the double-edge-rooted graph $(X, e, f)$ as $X$, as before, and omit the graphs $G$ and $H$ altogether. We also omit the proofs and refer the reader to the proof of Theorem 9 as an aid for transposing the productions to obtain the following results:

Theorem 23. Let $(X, e, f)=(G, e, d) *(H, g, f)$, where $e, d, g, f$ all have two 2-valent endpoints. Then

$$
\begin{aligned}
& d d_{k}^{0}(X)=\sum_{i=0}^{k}\left[\left(2 d d_{i}^{0}+2 d d_{i}^{\prime}+2 d d_{i}^{\prime \prime}+4 d s_{i}^{0}+4 d s_{i}^{\prime}+2 s s_{i}^{2}\right) \times d d_{k-i}^{0}+\left(\overline{d d^{\prime}}{ }_{i}+2{\overrightarrow{d d^{\prime}}}_{i}+\overleftarrow{d d^{\prime}}{ }_{i}\right.\right. \\
& \left.+2{\widetilde{d d^{\prime}}}_{i}\right) \times{\overline{d d^{\prime}}}_{k-i}+\left(2{\overline{d d^{\prime}}}_{i}+{\widetilde{d d^{\prime}}}_{i}+{\overrightarrow{d d^{\prime}}}_{i}+2 \overleftarrow{\mathrm{dd}^{\prime}}{ }_{i}\right) \times{\widetilde{d d^{\prime}}}_{k-i}+\left(\overline{d d^{\prime}}{ }_{i}+2{\overrightarrow{d d^{\prime}}}_{i}+{\overleftarrow{d d^{\prime}}}_{i}\right. \\
& \left.+2{\widetilde{d d^{\prime}}}^{i}\right) \times{\overrightarrow{d d^{\prime}}}^{k-i}+\left(2{\overline{d d^{\prime}}}_{i}+{\overrightarrow{d d^{\prime}}}_{i}+2 \overleftarrow{d d^{\prime}}{ }_{i}+{\widetilde{d d^{\prime}}}^{\prime}\right) \times \overleftarrow{d d^{\prime}}{ }_{k-i}+\left(2 d d_{i}^{0}+d d_{i}^{\prime \prime}+4 d s_{i}^{0}\right. \\
& \left.+2 d s_{i}^{\prime}\right) \times d d_{k-i}^{\prime}+\left(2 d d^{0} i+d d_{i}^{\prime}+4 d s_{i}^{0}\right) \times d d_{k-i}^{\prime \prime}+4\left(d d_{i}^{0}+d d_{i}^{\prime}+d d_{i}^{\prime \prime}+d s_{i}^{0}\right. \\
& \left.\left.+d s_{i}^{\prime}\right) \times s d_{k-i}^{0}+\left(4 d d_{i}^{0}+2 d d_{i}^{\prime}+4 d s_{i}^{0}\right) \times s d_{k-i}^{\prime}+2 d d_{i}^{0} * s s_{k-i}^{2}\right]+\sum_{i=0}^{k-1}\left[\left(2 d d_{i}^{0}\right.\right. \\
& \left.\left.+2 d d_{i}^{\prime}\right) \times d d_{k-1-i}^{0}+2 d d_{i}^{0} \times d d_{k-1-i}^{\prime}\right]
\end{aligned}
$$

Theorem 24. Let $(X, e, f)=(G, e, d) *(H, g, f)$, where $e, d, g, f$ all have two 2-valent endpoints. Then

$$
\begin{aligned}
&{\overline{d d^{\prime}}}^{k}(X)=\sum_{i=0}^{k} {\left[\left({\overline{d d^{\prime}}}_{i}+{\overrightarrow{d d^{\prime \prime}}}_{i}+2{\overrightarrow{d s^{\prime}}}_{i}+s s_{i}^{2}\right) \times{\overline{d d^{\prime}}}_{k-i}+\left({\overrightarrow{d d^{\prime}}}_{i}+{\overleftarrow{d d^{\prime \prime}}}_{i}+2{\overrightarrow{d s^{\prime}}}_{i}+s s_{i}^{2}\right) \times{\overleftarrow{d d^{\prime}}}^{\prime}{ }_{k-i}\right.} \\
&+\left({\overline{d d^{\prime}}}_{i}+{\overrightarrow{d d^{\prime \prime}}}_{i}+2{\overrightarrow{d s^{\prime}}}_{i}\right) \times{\overrightarrow{d d^{\prime \prime}}}_{k-i}+\left({\overrightarrow{d d^{\prime}}}_{i}+{\overleftarrow{d d^{\prime \prime}}}_{i}+2{\overrightarrow{d s^{\prime}}}_{i}\right) \times{\overleftarrow{d d^{\prime \prime}}}_{k-i}+\left(2{\overline{d d^{\prime}}}_{i}\right. \\
&\left.\left.+2{\overrightarrow{d d^{\prime}}}_{i}+2 d{d d^{\prime \prime}}_{i}+4{\overrightarrow{d s^{\prime}}}_{i}\right) \times{\overrightarrow{s d^{\prime}}}_{k-i}+\left({\overline{d d^{\prime}}}_{i}+{\overrightarrow{d d^{\prime}}}_{i}\right) \times s s_{k-i}^{2}\right]+\sum_{i=0}^{k-1}\left[\left(2{\overline{d d^{\prime}}}_{i}\right.\right. \\
&\left.+2{\overrightarrow{d d^{\prime}}}_{i}\right) \times{\overline{d d^{\prime}}}_{k-1-i}+\left(2{\overline{d d^{\prime}}}_{i}+2{\overrightarrow{d d^{\prime}}}_{i}\right) \times \overleftarrow{d d^{\prime}} \\
& \\
&
\end{aligned}
$$

Theorem 25. Let $(X, e, f)=(G, e, d) *(H, g, f)$, where $e, d, g, f$ all have two 2-valent endpoints. Then

$$
\begin{aligned}
& {\widetilde{d d^{\prime}}} \text { ( }(X)=\sum_{i=0}^{k}\left[\left({\widetilde{d d^{\prime}}}_{i}+{\overrightarrow{d d^{\prime \prime}}}_{i}+2 \overleftarrow{d s^{\prime}}{ }_{i}+s s_{i}^{2}\right) \times{\widetilde{d d^{\prime}}}^{k-i}+\left(\overleftarrow{d d^{\prime}}{ }_{i}+{\overleftarrow{d d^{\prime \prime}}}_{i}+2 \overleftarrow{d s^{\prime}}{ }_{i}+s s_{i}^{2}\right) \times{\overrightarrow{d d^{\prime}}}^{\prime}{ }_{k-i}\right. \\
& +\left({\widetilde{d d^{\prime}}}_{i}+{\overrightarrow{d d^{\prime \prime}}}_{i}+2{\overleftarrow{d s^{\prime}}}_{i}\right) \times{\overrightarrow{d d^{\prime \prime}}}_{k-i}+\left({\overleftarrow{d d^{\prime}}}_{i}+{\overleftarrow{d d^{\prime \prime}}}_{i}+2{\overleftarrow{d s^{\prime}}}_{i}\right) \times{\overleftarrow{d d^{\prime \prime}}}_{k-i}+\left(2{\widetilde{d d^{\prime}}}_{i}\right. \\
& \left.\left.+2 \overleftarrow{d d^{\prime}}{ }_{i}+2{\overrightarrow{d d^{\prime \prime}}}_{i}+2 \overleftarrow{d d^{\prime \prime}}{ }_{i}+4{\overleftarrow{d s^{\prime}}}_{i}\right) \times{\overleftarrow{s d^{\prime}}}_{k-i}+\left({\widetilde{d d^{\prime}}}_{i}+{\overleftarrow{d d^{\prime}}}_{i}\right) \times s s_{k-i}^{2}\right] \\
& +\sum_{i=0}^{k-1}\left[\left(2{\widetilde{d d^{\prime}}}_{i}+2 \overleftarrow{\mathrm{dd}^{\prime}}{ }_{i}\right) \times{\widetilde{d d^{\prime}}}_{k-1-i}+\left(2{\widetilde{d d^{\prime}}}_{i}+2 \overleftarrow{{d d^{\prime}}^{\prime}}{ }_{i}\right) \times{\overrightarrow{d d^{\prime}}}_{k-1-i}\right]
\end{aligned}
$$

Theorem 26. Let $(X, e, f)=(G, e, d) *(H, g, f)$, where $e, d, g, f$ all have two 2-valent endpoints. Then

$$
\begin{aligned}
{\overrightarrow{d d^{\prime}}}_{k}(X)= & \sum_{i=0}^{k}\left[\left({\overrightarrow{d d^{\prime}}}_{i}+{\overleftarrow{d d^{\prime \prime}}}_{i}+2{\overrightarrow{d s^{\prime}}}_{i}+s s_{i}^{2}\right) \times{\widetilde{d d^{\prime}}}_{k-i}+\left({\overline{d d^{\prime}}}_{i}+{\overrightarrow{d d^{\prime \prime}}}_{i}+2{\overrightarrow{d s^{\prime}}}_{i}+s s_{i}^{2}\right) \times{\overrightarrow{d d^{\prime}}}_{k-i}\right. \\
& +\left({\overrightarrow{d d^{\prime}}}_{i}+{\overleftarrow{d d^{\prime \prime}}}_{i}+2{\overrightarrow{d s^{\prime}}}_{i}\right) \times{\overrightarrow{d d^{\prime \prime}}}_{k-i}+\left({\overrightarrow{d d^{\prime}}}_{i}+{\overrightarrow{d d^{\prime \prime}}}_{i}+2{\overrightarrow{d s^{\prime}}}_{i}\right) \times{\overleftarrow{d d^{\prime \prime}}}_{k-i}+\left(2{\overline{d d^{\prime}}}_{i}\right. \\
& \left.\left.+2{\overrightarrow{d d^{\prime}}}_{i}+2{\overrightarrow{d d^{\prime \prime}}}_{i}+2{\overleftarrow{d d^{\prime \prime}}}_{i}+4{\overrightarrow{d s^{\prime}}}_{i}\right) \times{\overleftarrow{s d^{\prime}}}^{\prime}{ }_{k-i}+\left({\overline{d d^{\prime}}}_{i}+{\overrightarrow{d d^{\prime}}}_{i}\right) \times s s_{k-i}^{2}\right] \\
+ & \sum_{i=0}^{k-1}\left[\left(2{\overline{d d^{\prime}}}_{i}+2{\overrightarrow{d d^{\prime}}}_{i}\right) \times{\widetilde{d d^{\prime}}}_{k-1-i}+\left(2{\overrightarrow{d d^{\prime}}}_{i}+2{\overrightarrow{d d^{\prime}}}_{i}\right) \times{\overrightarrow{d d^{\prime}}}_{k-1-i}\right]
\end{aligned}
$$

Theorem 27. Let $(X, e, f)=(G, e, d) *(H, g, f)$, where $e, d, g, f$ all have two 2-valent endpoints. Then

$$
\begin{aligned}
{\overleftarrow{d d^{\prime}}}_{k}(X)=\sum_{i=0}^{k} & {\left[\left({\overleftarrow{d d^{\prime}}}_{i}+{\overleftarrow{d d^{\prime \prime}}}_{i}+2{\overleftarrow{d s^{\prime}}}_{i}+s s_{i}^{2}\right) \times{\overline{d d^{\prime}}}_{k-i}+\left({\widetilde{d d^{\prime}}}_{i}+{\overrightarrow{d d^{\prime \prime}}}_{i}+2{\overleftarrow{d s^{\prime}}}_{i}+s s_{i}^{2}\right) \times{\overleftarrow{d d^{\prime}}}_{k-i}\right.} \\
& +\left({\overleftarrow{d d^{\prime}}}_{i}+{\overleftarrow{d d^{\prime \prime}}}_{i}+2{\overleftarrow{d s^{\prime}}}_{i}\right) \times{\widehat{d d^{\prime \prime}}}_{k-i}+\left({\widetilde{d d^{\prime}}}_{i}+{\widehat{d d^{\prime \prime}}}_{i}+2{\overleftarrow{d s^{\prime}}}_{i}\right) \times{\overleftarrow{d d^{\prime \prime}}}^{k-i}+\left(2{\widetilde{d d^{\prime}}}_{i}\right. \\
& +2{\overleftarrow{d d^{\prime}}}_{i}+2{\left.\left.{\overrightarrow{d d^{\prime \prime}}}_{i}+2{\overleftarrow{d d^{\prime \prime}}}_{i}+4{\overleftarrow{d s^{\prime}}}_{i}\right) \times{{\overrightarrow{s d^{\prime}}}^{\prime}}_{k-i}+\left({\widetilde{d d^{\prime}}}_{i}+{\overleftarrow{d d^{\prime}}}_{i}\right) \times s s_{k-i}^{2}\right]}^{+} \sum_{i=0}^{k-1}\left[\left(2{\overleftarrow{d d^{\prime}}}_{i}+2{\widetilde{d d^{\prime}}}_{i}\right) \times{\overline{d d^{\prime}}}_{k-1-i}+\left(2{\widetilde{d d^{\prime}}}_{i}+2{\overleftarrow{d d^{\prime}}}_{i}\right) \times{\overleftarrow{d d^{\prime}}}{ }_{k-1-i}\right]
\end{aligned}
$$

Theorem 28. Let $(X, e, f)=(G, e, d) *(H, g, f)$, where $e, d, g, f$ all have two 2-valent endpoints. Then

$$
\overrightarrow{d d}_{k}^{\prime \prime}(X)=\sum_{i=0}^{k}\left[s s_{i}^{2} \times d d_{k-i}^{\prime \prime}+d d_{i}^{\prime \prime} \times s s_{k-i}^{2}\right]
$$

Theorem 29. Let $(X, e, f)=(G, e, d) *(H, g, f)$, where $e, d, g, f$ all have two 2-valent endpoints. Then

$$
\overleftarrow{d d_{k}^{\prime \prime}}(X)=\sum_{i=0}^{k}\left[s s_{i}^{2} \times d d_{k-i}^{\prime \prime}+d d_{i}^{\prime \prime} \times s s_{k-i}^{2}\right]
$$

Theorem 30. Let $(X, e, f)=(G, e, d) *(H, g, f)$, where $e, d, g, f$ all have two 2-valent endpoints. Then

$$
\begin{aligned}
d s_{k}^{0}(X)=\sum_{i=0}^{k} & {\left[\left(2 d d_{i}^{0}+2 d d_{i}^{\prime}+2 d d_{i}^{\prime \prime}+4 d s_{i}^{0}+4 d s_{i}^{\prime}+2 s s_{i}^{2}\right) \times d s_{k-i}^{0}+\left(\overline{d d^{\prime}}{ }_{i}+2{\widetilde{d d^{\prime}}}_{i}+2{\overrightarrow{d d^{\prime}}}_{i}\right.\right.} \\
& \left.+{\overleftarrow{d d^{\prime}}}^{\prime}+d d_{i}^{\prime \prime}+2 d s_{i}^{\prime}\right) \times{\overrightarrow{d s^{\prime}}}_{k-i}+\left(2{\overline{d d^{\prime}}}_{i}+{\widetilde{d d^{\prime}}}_{i}+{\overrightarrow{d d^{\prime}}}_{i}+2 \overleftarrow{d d^{\prime}}{ }_{i}+d d_{i}^{\prime \prime}+2 d s_{i}^{\prime}\right) \\
& \times{\overleftarrow{d s^{\prime}}}_{k-i}+\left(2 d d_{i}^{0}+4 d s_{i}^{0}\right) \times d s_{k-i}^{\prime}+4\left(d d_{i}^{0}+d d_{i}^{\prime}+d d_{i}^{\prime \prime}+d s_{i}^{0}+d s_{i}^{\prime}\right) \times s s_{k-i}^{0} \\
& \left.+\left(4 d d_{i}^{0}+2 d d_{i}^{\prime}+4 d s_{i}^{0}\right) \times s s_{k-i}^{1}+\left(2 d d_{i}^{0}+d d_{i}^{\prime}+4 d s_{i}^{0}\right) \times s s_{k-i}^{2}\right] \\
+ & \sum_{i=0}^{k-1}\left[2 d d_{i}^{0} \times d d_{k-1-i}^{\prime \prime}+\left(2 d d_{i}^{0}+2 d d_{i}^{\prime}\right) \times d s_{k-1-i}^{0}+2 d d_{i}^{0} \times d s_{k-1-i}^{\prime}\right]
\end{aligned}
$$

Theorem 31. Let $(X, e, f)=(G, e, d) *(H, g, f)$, where $e, d, g, f$ all have two 2-valent endpoints. Then

$$
\begin{aligned}
{\overrightarrow{d s^{\prime}}}_{k}(X)=\sum_{i=0}^{k} & {\left[\left({\overline{d d^{\prime}}}_{i}+{\overrightarrow{d d^{\prime \prime}}}_{i}+2{\overrightarrow{d s^{\prime}}}_{i}+s s_{i}^{2}\right) \times{\overrightarrow{d s^{\prime}}}^{\prime}{ }_{k-i}+\left({\overrightarrow{d d^{\prime}}}_{i}+{\overleftarrow{d d^{\prime \prime}}}_{i}+2{\overrightarrow{d s^{\prime}}}_{i}+s s_{i}^{2}\right) *{\overleftarrow{d s^{\prime}}}_{k-i}\right.} \\
& \left.+\left(2{\overline{d d^{\prime}}}_{i}+2{\overrightarrow{d d^{\prime}}}_{i}+2 d d_{i}^{\prime \prime}+4{\overrightarrow{d s^{\prime}}}_{i}\right) \times s s_{k-i}^{1}+\left({\overrightarrow{d d^{\prime}}}_{i}+{\overrightarrow{d d^{\prime}}}_{i}+d{d d_{i}^{\prime \prime}}^{\prime}+4{\overrightarrow{d s^{\prime}}}_{i}\right) \times s s_{k-i}^{2}\right] \\
+\sum_{i=0}^{k-1} & {\left[\left(2{\overline{d d^{\prime}}}_{i}+2{\overrightarrow{d d^{\prime}}}_{i}\right) \times{\overrightarrow{d d^{\prime \prime}}}^{k-1-i}+\left(2{\overline{d d^{\prime}}}_{i}+2{\overrightarrow{d d^{\prime}}}_{i}\right) \times{\overleftarrow{d d^{\prime \prime}}}^{k-1-i}+\left(2{\overline{d d^{\prime}}}_{i}+2{\overrightarrow{d d^{\prime}}}_{i}\right)\right.} \\
& \left.\times{\overrightarrow{d s^{\prime}}}_{k-1-i}+\left(2{\overline{d d^{\prime}}}_{i}+2{\overrightarrow{d d^{\prime}}}_{i}\right) \times{\overleftarrow{d s^{\prime}}}_{k-1-i}\right]
\end{aligned}
$$

Theorem 32. Let $(X, e, f)=(G, e, d) *(H, g, f)$, where $e, d, g, f$ all have two 2-valent endpoints. Then

$$
\begin{aligned}
{\overleftarrow{d s^{\prime}}}_{k}(X)= & \sum_{i=0}^{k}\left[\left(\overleftarrow{d d^{\prime}}{ }_{i}+{\overleftarrow{d d^{\prime \prime}}}_{i}+2{\overleftarrow{d s^{\prime}}}_{i}+s s_{i}^{2}\right) \times{\overrightarrow{d s^{\prime}}}_{k-i}+\left({\widetilde{d d^{\prime}}}_{i}+{\overrightarrow{d d^{\prime \prime}}}_{i}+2 \overleftarrow{d s^{\prime}}{ }_{i}+s s_{i}^{2}\right) \times{\overleftarrow{d s^{\prime}}}_{k-i}\right. \\
& \left.+\left(2{\widetilde{d d^{\prime}}}_{i}+2{\overleftarrow{d d^{\prime}}}_{i}+2 d d_{i}^{\prime \prime}+4{\overleftarrow{d s^{\prime}}}_{i}\right) \times s s_{k-i}^{1}+\left({\widetilde{d d^{\prime}}}_{i}+{\overleftarrow{d d^{\prime}}}_{i}+d d_{i}^{\prime \prime}+4{\overleftarrow{d s^{\prime}}}_{i}\right) \times s s_{k-i}^{2}\right] \\
+ & \sum_{i=0}^{k-1}\left[\left(2{\widetilde{d d^{\prime}}}_{i}+2{\overleftarrow{d d^{\prime}}}_{i}\right) \times d d_{k-1-i}^{\prime \prime}+\left(2{\widetilde{d d^{\prime}}}_{i}+2{\overleftarrow{d d^{\prime}}}_{i}\right) \times d{s_{k-1-i}^{\prime}}\right]
\end{aligned}
$$

Theorem 33. Let $(X, e, f)=(G, e, d) *(H, g, f)$, where $e, d, g, f$ all have two 2-valent endpoints. Then

$$
\begin{aligned}
s d_{k}^{0}(X)=\sum_{i=0}^{k} & {\left[\left(2 s d_{i}^{0}+2 s d_{i}^{\prime}+4 s s_{i}^{0}+4 s s_{i}^{1}+2 s s_{i}^{2}\right) \times d d_{k-i}^{0}+\left(2 s d_{i}^{0}+4 s s_{i}^{0}\right) \times d d_{k-i}^{\prime}+\left({\overrightarrow{s d^{\prime}}}_{i}\right.\right.} \\
& \left.+2 \overleftarrow{s d^{\prime}}{ }_{i}+2 s s_{i}^{1}+s s_{i}^{2}\right) \times{\overline{d d^{\prime}}}_{k-i}+\left(2{\overrightarrow{s d^{\prime}}}_{i}+\overleftarrow{s d^{\prime}}{ }_{i}+2 s s_{i}^{1}+s s_{i}^{2}\right) \times{\widetilde{d d^{\prime}}}_{k-i}+\left({\overrightarrow{s d^{\prime}}}_{i}\right. \\
& \left.+2 \overleftarrow{s d^{\prime}}{ }_{i}+2 s s_{i}^{1}+s s_{i}^{2}\right) \times{\overrightarrow{d d^{\prime}}}_{k-i}+\left(2 s{\overrightarrow{d d^{\prime}}}_{i}+\overleftarrow{s d^{\prime}}{ }_{i}+2 s s_{i}^{1}+s s_{i}^{2}\right) \times \overleftarrow{d d^{\prime}}{ }_{k-i}+\left(2 s d_{i}^{0}\right. \\
& \left.+s d_{i}^{\prime}+4 s s_{i}^{0}\right) \times d d_{k-i}^{\prime \prime}+4\left(s d_{i}^{0}+s d_{i}^{\prime}+s s_{i}^{0}+s s_{i}^{1}+s s_{i}^{2}\right) \times s d_{k-i}^{0}+\left(4 s d_{i}^{0}\right. \\
& \left.\left.+2 s d_{i}^{\prime}+4 s s_{i}^{0}\right) \times s d_{k-i}^{\prime}+2 s d_{i}^{0} \times s s_{k-i}^{2}\right]+\sum_{i=0}^{k-1}\left[\left(2 d d_{i}^{\prime \prime}+2 s d_{i}^{0}+2 s d_{i}^{\prime}\right)\right. \\
& \left.\times d d_{k-1-i}^{0}+2 s d_{i}^{0} \times d d_{k-1-i}^{\prime}\right]
\end{aligned}
$$

Theorem 34. Let $(X, e, f)=(G, e, d) *(H, g, f)$, where $e, d, g, f$ all have two 2-valent endpoints. Then

$$
\left.\begin{array}{rl}
{\overrightarrow{s d^{\prime}}}_{k}(X)=\sum_{i=0}^{k} & {\left[\left({\overrightarrow{s d^{\prime}}}_{i}+2 s s_{i}^{1}+s s_{i}^{2}\right) \times{\overline{d d^{\prime}}}_{k-i}+\left({\overleftarrow{s d^{\prime}}}_{i}+2 s s_{i}^{1}+s s_{i}^{2}\right) \times{\overleftarrow{d d^{\prime}}}_{k-i}+\left({\overrightarrow{s d^{\prime}}}_{i}\right.\right.} \\
& \left.+2 s s_{i}^{1}+s s_{i}^{2}\right) \times{\overrightarrow{d d^{\prime \prime}}}_{k-i}+\left({\overleftarrow{s d^{\prime}}}_{i}+2 s s_{i}^{1}+2 s s_{i}^{2}\right) \times{\overleftarrow{d d^{\prime \prime}}}^{k-i}+\left(2 s d_{i}^{\prime}+4 s s_{i}^{1}\right. \\
& \left.\left.+4 s s_{i}^{2}\right) \times{\overrightarrow{s d^{\prime}}}_{k-i}+s d_{i}^{\prime} \times s s_{k-i}^{2}\right]+\sum_{i=0}^{k-1}\left[\left(2{\overrightarrow{d d^{\prime \prime}}}_{i}+2{\overleftarrow{d d^{\prime \prime}}}_{i}+2{\overrightarrow{s d^{\prime}}}_{i}+2{\overleftarrow{s d^{\prime}}}_{i}\right)\right. \\
& \times{\overline{d d^{\prime}}}_{k-1-i}+\left(2{\overrightarrow{d d^{\prime \prime}}}_{i}+2{\overleftarrow{d d^{\prime \prime}}}_{i}+2{\overrightarrow{s d^{\prime}}}_{i}+2 \overleftarrow{s d^{\prime}} i\right) \times \overleftarrow{d d^{\prime}} \\
k-1-i
\end{array}\right]
$$

Theorem 35. Let $(X, e, f)=(G, e, d) *(H, g, f)$, where $e, d, g, f$ all have two 2-valent endpoints. Then

$$
\begin{aligned}
\overleftarrow{s d^{\prime}} & k \\
(X)=\sum_{i=0}^{k} & {\left[\left(\overleftarrow{s d^{\prime}}{ }_{i}+2 s s_{i}^{1}+s s_{i}^{2}\right) \times{\widetilde{d d^{\prime}}}_{k-i}+\left({\overrightarrow{s d^{\prime}}}_{i}+2 s s_{i}^{1}+s s_{i}^{2}\right) \times{\overrightarrow{d d^{\prime}}}_{k-i}+\left(\overleftarrow{s d^{\prime}}{ }_{i}+2 s s_{i}^{1}\right.\right.} \\
& \left.+s s_{i}^{2}\right) \times{\overrightarrow{d d^{\prime \prime}}}_{k-i}+\left({\overrightarrow{s d^{\prime}}}_{i}+2 s s_{i}^{1}\right) \times{\overleftarrow{d d^{\prime \prime}}}_{k-i}+\left(2 s d_{i}^{\prime}+4 s s_{i}^{1}+4 s s_{i}^{2}\right) \times \overleftarrow{s d^{\prime}}{ }_{k-i}
\end{aligned}
$$

$$
\begin{aligned}
& \left.+s d_{i}^{\prime} \times s s_{k-i}^{2}\right]+\sum_{i=0}^{k-1}\left[\left(2{\overrightarrow{d d^{\prime \prime}}}_{i}+2{\overleftarrow{d d^{\prime \prime}}}_{i}+2{\overrightarrow{s d^{\prime}}}_{i}+2{\overleftarrow{s d^{\prime}}}_{i}\right) \times{\widetilde{d d^{\prime}}}_{k-1-i}+\left(2 d d_{i}^{\prime \prime}\right.\right. \\
& \left.\left.+2 s d_{i}^{\prime}\right) \times{\overrightarrow{d d^{\prime}}}_{k-1-i}\right]
\end{aligned}
$$

Theorem 36. Let $(X, e, f)=(G, e, d) *(H, g, f)$, where $e, d, g, f$ all have two 2-valent endpoints. Then

$$
\begin{aligned}
s s_{k}^{0}(X)=\sum_{i=0}^{k} & {\left[\left(2 s d_{i}^{0}+2 s d_{i}^{\prime}+4 s s_{i}^{0}+4 s s_{i}^{1}+2 s s_{i}^{2}\right) \times d s_{k-i}^{0}+\left({\overrightarrow{s d^{\prime}}}_{i}+2 \overleftarrow{s d^{\prime}}{ }_{i}\right) \times{\overrightarrow{d s^{\prime}}}_{k-i}+\left(2{\overrightarrow{s d^{\prime}}}_{i}\right.\right.} \\
& \left.+\overleftarrow{s d^{\prime}}{ }_{i}\right) \times \overleftarrow{d s^{\prime}}{ }_{k-i}+\left(2 s d_{i}^{0}+4 s s_{i}^{0}+2 s s_{i}^{1}+s s_{i}^{2}\right) \times d s_{k-i}^{\prime}+4\left(s d_{i}^{0}+s d_{i}^{\prime}+s s_{i}^{0}+s s_{i}^{1}\right. \\
& \left.\left.+s s_{i}^{2}\right) \times s s_{k-i}^{0}+\left(4 s d_{i}^{0}+2 s d_{i}^{\prime}+4 s s_{i}^{0}\right) \times s s_{k-i}^{1}+\left(2 s d_{i}^{0}+s d_{i}^{\prime}+4 s s_{i}^{0}\right) \times s s_{k-i}^{2}\right] \\
+\sum_{i=0}^{k-1} & {\left[2 s d_{i}^{0} \times d d_{k-1-i}^{\prime \prime}+\left(2 d d_{i}^{\prime \prime}+2 s d_{i}^{0}+2 s d_{i}^{\prime}\right) \times d s_{k-1-i}^{0}+2 s d_{i}^{0} \times d s_{k-1-i}^{\prime}\right] }
\end{aligned}
$$

Theorem 37. Let $(X, e, f)=(G, e, d) *(H, g, f)$, where $e, d, g, f$ all have two 2-valent endpoints. Then

$$
\begin{aligned}
s s_{k}^{1}(X)=\sum_{i=0}^{k} & {\left[\overrightarrow{s d^{\prime}}{ }_{i} \times{\overrightarrow{d s^{\prime}}}_{k-i}+\overleftarrow{s d^{\prime}}{ }_{i} \times \overleftarrow{d s^{\prime}}{ }_{k-i}+\left(2 s s_{i}^{1}+s s_{i}^{2}\right) \times d s_{k-i}^{\prime}+\left(2 s d_{i}^{\prime}+4 G \cdot s s_{i}^{1}+4 s s_{i}^{2}\right)\right.} \\
& \left.\times s s_{k-i}^{1}+\left(s d_{i}^{\prime}+4 s s_{i}^{1}+2 s s_{i}^{2}\right) \times s s_{k-i}^{2}\right]+\sum_{i=0}^{k-1}\left[2 s d_{i}^{\prime} \times d d_{k-1-i}^{\prime \prime}+\left(2 d d_{i}^{\prime \prime}\right.\right. \\
& \left.\left.+2 s d_{i}^{\prime}\right) \times d s_{k-1-i}^{\prime}\right]
\end{aligned}
$$

Theorem 38. Let $(X, e, f)=(G, e, d) *(H, g, f)$, where $e, d, g, f$ all have two 2-valent endpoints. Then

$$
s s_{k}^{2}(X)=\sum_{i=0}^{k} 2 s s_{i}^{2} \times s s_{k-i}^{2}+\sum_{i=0}^{k-1} 2 d d_{i}^{\prime \prime} \times d d_{k-1-i}^{\prime \prime}
$$

