NOTES AND CORRESPONDENCE

A Note on Low-Frequency Equatorial Basin Modes

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ABSTRACT

A new low-frequency standing equatorial wave mode is described. It is composed solely of long Kelvin and Rossby waves, whereas previously described low-frequency modes involved short, eastward propagating Rossby waves. It is argued that these short waves travel too slowly to allow such modes to set up. A simple closed form expression is given for the new basin mode; this is also a new form for the sum of a Kelvin wave and its eastern boundary reflection.

Standing modes in an equatorial ocean basin have been considered previously by Moore (1968) and Gent (1979). Their investigation, and this one, begin with the linear, inviscid equations describing the horizontal structure of the vertical mode with equivalent depth H_n on an infinite equatorial beta plane:

$$u_t - yv + h_x = 0, \tag{1}$$

$$v_t + yu + h_y = 0, \tag{2}$$

$$h_t + u_x + v_y = 0. \tag{3}$$

These equations have been nondimensionalized by an equatorial length scale $L = (c/\beta)^{1/2}$ and time scale $T = (c\beta)^{-1/2}$, where $c = (gH_n)^{1/2}$ is a scale for wave speeds. As is well known (e.g., Moore and Philander, 1977) the wave solutions of (1)–(3) are standing waves in the meridional direction: they have the form $e^{i(kx-\omega t)}$ times a function of y. For a given frequency ω, these solutions are: Kelvin waves (with dispersion relation $k = \omega$); a Yanai wave; and for meridional structure index $n = 1, 2, 3, \ldots$, waves with the dispersion relation

$$k_\pm(n) = \frac{-1}{2\omega} \pm \left[\omega^2 + \frac{1}{4\omega^2} - (2n+1)\right]^{1/2}. \tag{4}$$

If the ocean is bounded by meridional walls at $x = 0$ and $x = X_E$ the boundary conditions $u(0) = u(X_E) = 0$ also must be satisfied. As shown by Moore (1968), these conditions can be satisfied for any frequency by an infinite sum of waves, but this generally requires an energy source at the western side at $|y| = \infty$ and a matching sink at the eastern side. Moore went on to show how a true basin mode could be constructed, but he explicitly considered only frequencies in the range $1 - ½\sqrt{2} < \omega < 1 + ½\sqrt{2}$, where all waves except the Kelvin and Yanai waves are trapped to the meridional boundaries [i.e., in this range the k's given by (4) are complex].

Gent (1979) showed that for certain special frequencies and basin lengths it is possible to construct basin modes out of a finite number of waves, obtaining solutions for which all the energy is equatorially confined. Low-frequency basin modes are made up of 1) westward traveling, long Rossby waves; 2) an eastward traveling long Kelvin wave; and 3) eastward traveling short Rossby waves. [For low frequencies the Yanai wave behaves like a short Rossby wave]. Gent points out that his long-period basin modes cannot also have long space scales. The short, eastward traveling waves are always significant components.

It is unlikely that these short Rossby waves are able to propagate across the ocean basin from the western to the eastern side. Their group velocities are so slow [$O(\omega^2)$] that even small frictional or inertial influences will keep them trapped to the western boundary, forming a western boundary current (Pedlosky, 1965; Lighthill, 1969; Cane and Sarachik, 1977). This trapping means that the low-frequency basin mode solutions of Gent will not be realized.

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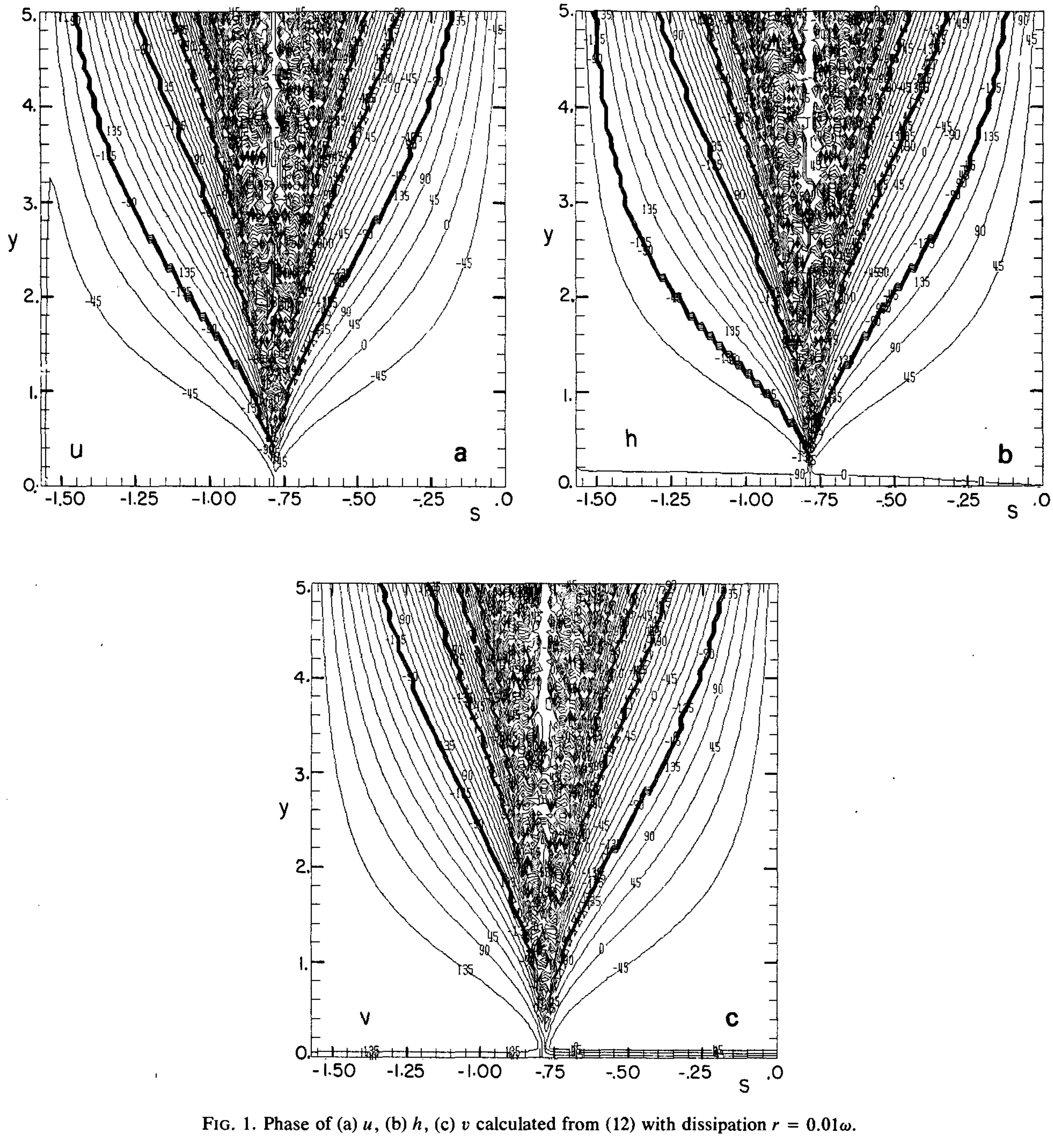

FIG. 1. Phase of (a) u, (b) h, (c) v calculated from (12) with dissipation $r = 0.01\omega$.

However, there are slightly leaky basin modes that do not make use of these short-wave components. These may be obtained as follows, using the results and notation of Cane and Sarachik (1977, hereafter CSII). We begin with a Kelvin wave of the form

$$u_K, h_K = \frac{\pi^{1/4}}{\sqrt{2}} \psi_0(y) e^{i\omega t} e^{-i\omega(x - X_E)}, \tag{5}$$

where ψ_n is the nth Hermite function. Letting

$$s = \omega(x - X_E) \tag{6}$$

and making use of the approximate long Rossby wave dispersion formula

$$k = -(2n + 1)\omega \tag{7}$$

[cf. (4) for $\omega \ll 1$] and Eqs. (22) and (8) of CSII, the

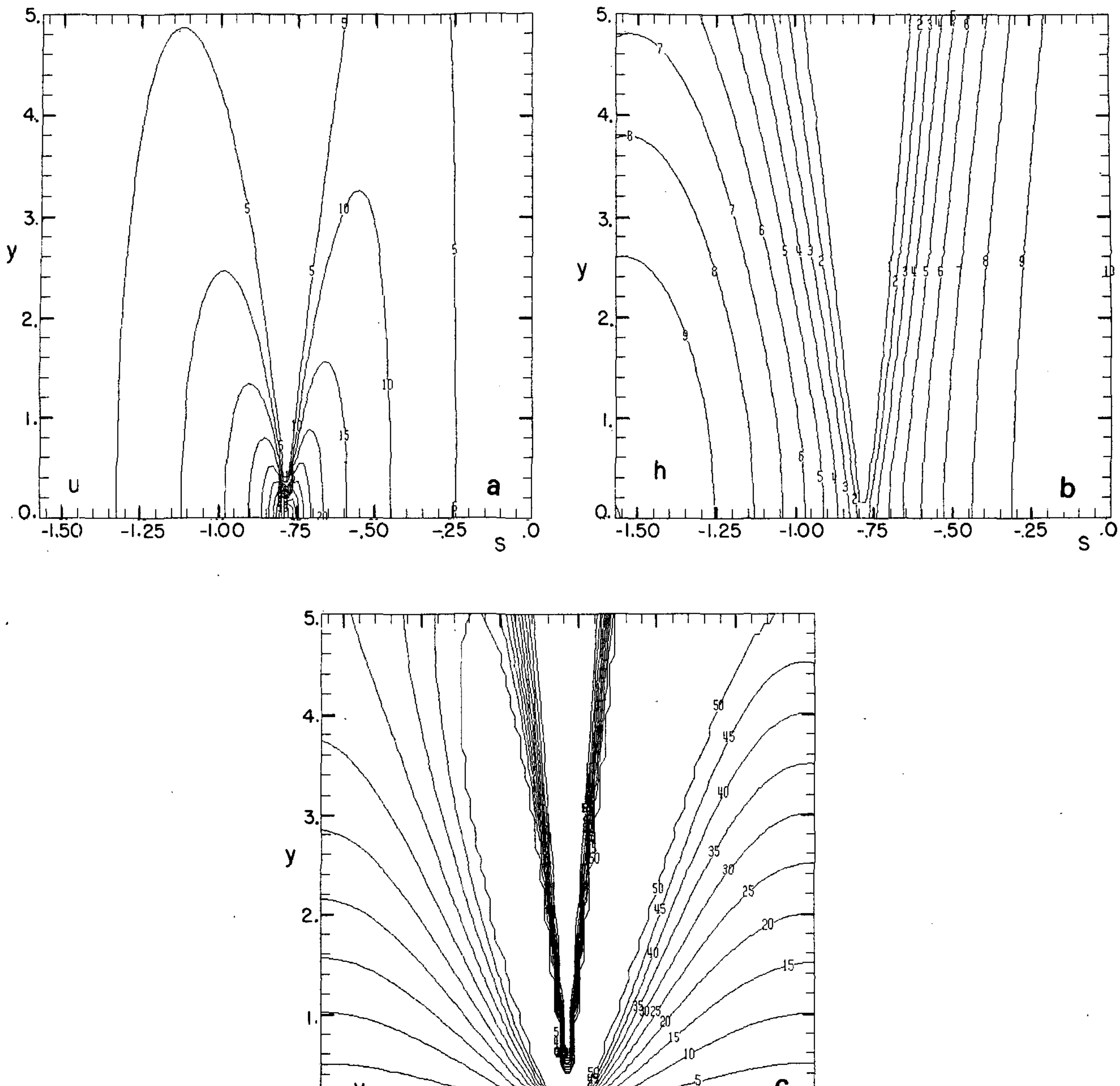

FIG. 2. Amplitude of (a) u (contour interval = 0.5), (b) h (contour interval = 0.1), (c) v (contour interval = 0.05), calculated from (12) with dissipation $r = 0.01\omega$.

sum of this Kelvin wave and its eastern boundary reflection is

$$u,\, h = \frac{\pi^{1/4}}{\sqrt{2}} e^{i\omega t}\Big\{\psi_0(y)e^{-is} + \sum_{\substack{n=1\\ n\ \text{odd}}}^{\infty} \alpha_1^n[(n+1)^{-1/2}\psi_{n+1}(y) \mp n^{-1/2}\psi_{n-1}(y)]e^{i(2n+1)s}\Big\}, \quad (8)$$

where the minus sign is for u, the plus sign for h, and α_1^n is a number defined by CSII Eq. (16b). From (8)

$$u,\, h = \frac{\pi^{1/4}}{\sqrt{2}} e^{i\omega t}\Big\{\psi_0 e^{-is} + \sum_{\substack{n=2\\ n\ \text{even}}}^{\infty} n^{-1/2}\alpha_1^{n-1}\psi_n e^{is(2n-1)} \mp \sum_{\substack{n=0\\ n\ \text{even}}}^{\infty} (n+1)^{-1/2}\alpha_1^{n+1}\psi_n e^{is(2n+3)}\Big\}.$$

With the aid of the relation $(n+1)^{-1/2}\alpha_1^{n+1} = n^{-1/2}\alpha_1^{n-1}$, the first sum may be rewritten to obtain

$$u, h = \frac{\pi^{1/4}}{\sqrt{2}} e^{i\omega t}\{ \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} (n+1)^{-1/2}\alpha_1^{n+1}\psi_n e^{is(2n+1)} \times [e^{-2is} \mp e^{+2is}]\};$$

or

$$h = \sqrt{2}\pi^{1/4} \cos 2s e^{i\omega t} \times \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} (n+1)^{-1/2}\alpha_1^{n+1}\psi_n e^{is(2n+1)}, \quad (9)$$

$$u = -ih \tan 2s. \quad (10)$$

A closed form expression for h may be found by making use of the integral representation for the ψ_n's. The same results may be obtained in a more elementary manner by noticing that each of the terms in (8) and, hence, the whole sum, satisfies the geostrophic relation

$$yu + h_y = 0.$$

Combining this with (10) yields

$$-[iy \tan 2s]h + h_y = 0$$

so

$$h = \eta(s) \exp i[y^2/2 \tan 2s].$$

The still unknown function $\eta(s)$ may be determined by considering (1) evaluated at $y = 0$. Making use of (10)

$$i\omega[-i \tan 2s \eta(s)] + \omega \frac{d}{ds}\eta(s) = 0,$$

$$\eta(s) = [\cos 2s]^{1/2}.$$

Hence [evaluating v from (1) or (3)],

$$(u, v, h) = e^{i\omega t}(-i \tan 2s, i\omega y \sec^2 2s, 1) \times [\cos 2s]^{1/2} \exp i[y^2/2 \tan 2s] \quad (12)$$

expresses the sum of a Kelvin wave and its eastern boundary reflection. This expression is new, and extends the results of Moore (see Moore and Philander, 1977), Anderson and Rowlands (1976), and CSII.

Eq. (12) is physically unreasonable near $s = \pi/4$ (and $s = 3\pi/4$, $5\pi/4$, etc.) where $\tan 2s$ becomes infinite [at $s = \pi/4$ the series in (9) becomes $e^{i\pi/4}\delta(y)$, where δ is the Dirac δ function]. This unrealistic behavior is a consequence of the assumptions and approximations that have been made; the most straightforward and physically plausible modification is to include a small amount of friction in (1)–(3). Adding a Rayleigh friction term ru, rv, rh to these equations alters (12) by replacing ω by $\omega - ir$.

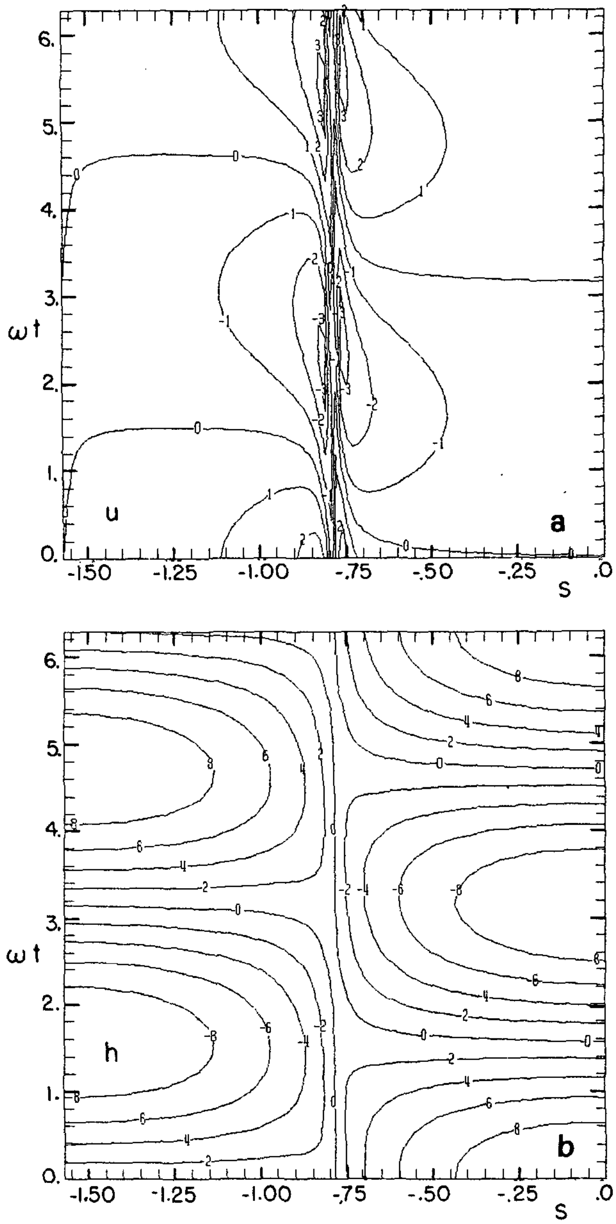

Fig. 3. Values of (a) u and (b) h along the equator ($y = 0$) for one full period.

[The term rv is negligible compared to the terms retained in (11).] The term in the exponent becomes

$$E \equiv \tan 2[s - ir(x - X_E)] = \frac{(1-\mu^2)\tan 2s + i\mu \sec^2 2s}{1 + \mu^2 \tan^2 2s},$$

where $\mu = -\tanh 2r(x - X_E)$. Consider r small enough so that $\mu \ll 1$. Then away from $s = \pi/4$, etc., $E \approx \tan 2s$ while near these points (i.e., where $|\tan 2s| \geq \mu^{-1}$) $E \approx i\mu^{-1}$ so the solution is heavily

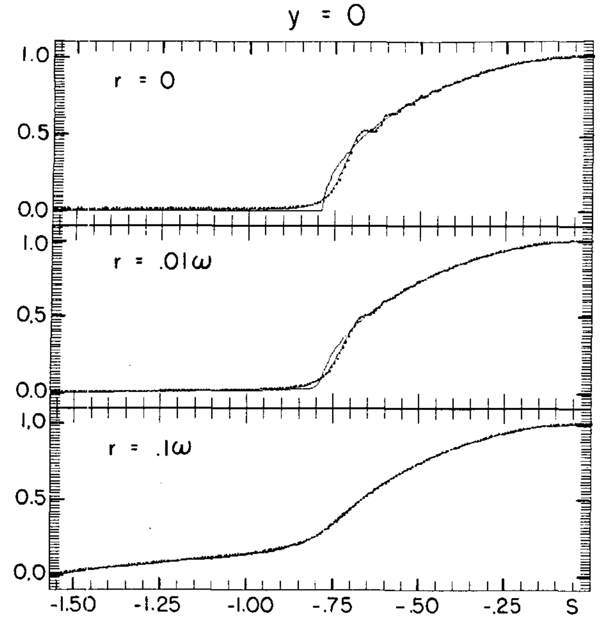

FIG. 4. Comparison of Re(h) given by the approximate solution (12) with the exact solution for a Kelvin wave and its eastern boundary reflection. For the latter $\omega = 1/30$ and all modes with $n < 400$ are included. Shown at $y = 0$. Exact: •–•–•; approximate: ——.

damped. [The equator, $y = 0$, is an exceptional point: v and h are small but u is $O(\mu^{-1/2})$.]

In order that (12) be a basin mode it also must satisfy a boundary condition at $x = 0$ ($s = -\omega X_E$). The obvious condition, $u = 0$ for all y, applies only for a model with a complete set of low-frequency modes, but the long-wave approximation (7) excludes the eastward traveling short Rossby waves [*viz.*, k_- in (4)]. When only long waves are included it has been shown [CSII (19) ff] that the proper condition is $\int_{-\infty}^{+\infty} u\ (x = 0)\ dy = 0$. For $u(y)$ of the particular form (12) this does imply $u = 0$ for all y or

$$-s\,(x = 0) = \omega X_E = m\,\frac{\pi}{2}\,; \quad m = 0, 1, 2, \ldots . \quad (13)$$

For $\omega = 0$ we have the degenerate solution $u = v = 0$ and $h = 1$: no motion and a uniform sea level [cf. CSII Eq. (23)]. Otherwise, the lowest frequency motion satisfying (13) has period $P = 4X_E$, four times the time for a Kelvin wave to cross the basin. (Alternately, the time for a Kelvin wave to cross from west to east plus the time for the $n = 1$ Rossby wave to return.) This period played a crucial role in the analysis of spin-up in CSII.

Figs. 1 and 2 are phase and amplitude plots of u, v, h for $-\pi/2 \leq s \leq 0$. Putting the western boundary at $-\pi/2$ corresponds to $m = 1$ in (13); the period is $P = 4X_E$. It follows from (9) that $h(s - \pi/2) = ih(s)$ and similarly for u and v. Hence solutions for $m > 1$ may be visualized by repeating the pattern of Figs. 1 and 2 with the appropriate phase shift. At the (possible) boundaries $s = m\pi/2$, $u = 0$, $v = \omega y$, and h is independent of y, with $h = i^m$. For the mode with period P there is thus a 90° phase difference in dynamic topography between the two ends of the basin. Aside from the phase shifts at $m\pi/2$, lines of constant phase are given by $y^2 \tan 2s$ = constant. The latter term causes no phase variation at the equator; zonal phase variations become increasingly rapid as one moves poleward. This reflects the fact that as y increases high n modes with short zonal wavelengths [cf. (7)] make the largest contribution to the sum (8). The solution displayed in Figs. 1 and 2 exhibits the same features pointed out by Schopf *et al.* (1981) in their ray theoretic analysis of a related problem. There is a focus at $s = -\pi/4$, $y = 0$. Emanating from this point are lines of high-energy density, or caustics. The caustics bound a shadow zone of rapid phase variation and small amplitude. Although the structure of our solution is quite complex its time-dependent behavior near the equator is rather straightforward (Fig. 3).

A key step in obtaining (12) was the long-wave approximation (7) to the exact dispersion relation (4). The error increases with increasing n, with the approximation becoming invalid for $n = O(\omega^{-2})$. It is easy to show that the long-wave low-frequency assumption (k, $\omega \ll 1$) implies that $v = O(\omega u)$ [a scaling that modifies the system of Eqs. (1)–(3) by replacing (2) with the geostrophic relation (11)]. From (12) the required relation between the magnitudes of v and u breaks down at a latitude $y = O(\omega^{-1})$. Since the mode making the major contribution to the sum (8) at this latitude is the one

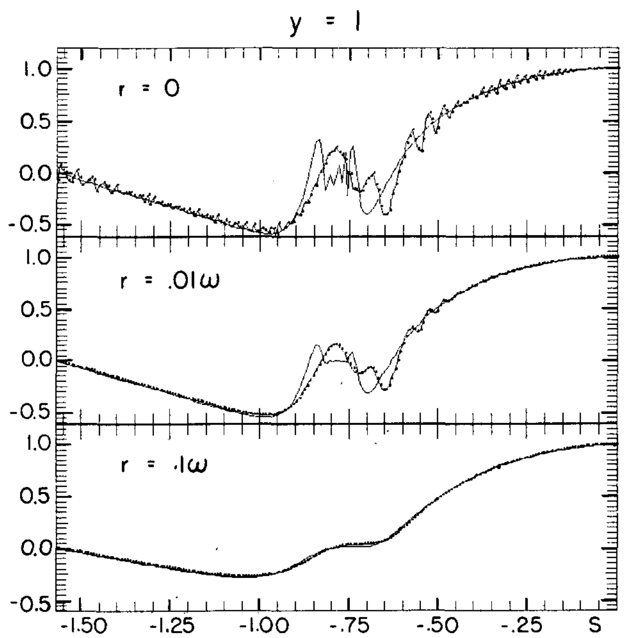

FIG. 5. As in Fig. 4 at $y = 1$.

for which $n \sim \frac{1}{2}y^2$ this again implies the assumption is invalid for $n = O(\omega^{-2})$.

This analysis implies that the approximate solution (12) will deviate least from the exact solution where the high n modes are unimportant: at low latitudes, or if dissipation is large enough to suppress these rapidly varying modes. Figs. 4–7 compare Re$\{h\}$[1] as given by the approximation (12) with the value obtained from the exact solution for a Kelvin wave and its eastern boundary reflection. A procedure for calculating the latter appears in Moore and Philander (1977). At low latitudes (Figs. 4 and 5) the exact solution shows small-scale low-amplitude wiggles attributable to short wavelength high n modes. The addition of dissipation eliminates this feature and makes the approximate and exact solutions nearly coincident. As latitudes increase (Figs. 6 and 7) so does the relative importance of the high n modes. The result is that the differences between the exact and approximate solutions also increase, although the two remain quite similar as far as $y = 5$ (about 15° of latitude for the first baroclinic mode). Agreement again improves with increasing friction, but since the solutions are now highly oscillatory they are noticeably damped away from the eastern boundary even for small values of r. [High values of r may be more appropriate at high latitudes because with the Rayleigh friction form we have used dissipation depends only linearly on wavenumber.]

For $n \geq N(\omega) = (8\omega^2)^{-1}$ Eq. (4) shows that k is complex and the Rossby wave amplitudes decay away from the eastern side. The trapping of high n waves makes the basin mode "leaky": when the Kelvin wave is reflected at the east some of its energy goes into trapped waves; this energy then does not reach the western boundary where the energy of the propagating Rossby waves is returned in the Kelvin wave. Using CSII Eq. (A1) we calculate that in a time P the amplitude of this Kelvin wave is reduced from 1 to $1 - (N + 1)(\alpha_1^N)^2 \approx 1 - 2(2\pi N)^{-1/2}$. We convert this loss to a decay time by estimating the Rayleigh friction coefficient r that gives the same amplitude loss. The resulting decay times, expressed in multiples of the standing mode period P, are given in Table 1. If, as seems likely, the frictional decay time for the ocean is order of several years or less, standing modes will lose energy to viscous dissipation faster than it will leak away to high latitudes.

Nonlinear effects are a more serious impediment to the existence of such low frequency basin modes in the world's oceans. The mean currents near the equator are fast compared to Rossby wave propagation speeds and profoundly alter the characteristics of these waves (Cane, 1979; Philander, 1979; McPhaden and Knox, 1979). It remains possible that the real oceans admit basin modes that resemble those described here in being composed solely of an eastward Kelvin wave and westward long Rossby waves, but their structure is likely to be rather different.

Whatever the case may be for the real ocean, the free mode (12) is an essential component of the solu-

[1] Similar results were obtained for Im$\{h\}$; note that in (12) Im$\{h(\pi/4 \pm s)\}$ = Re$\{h(\pi/4 \pm s)\}$.

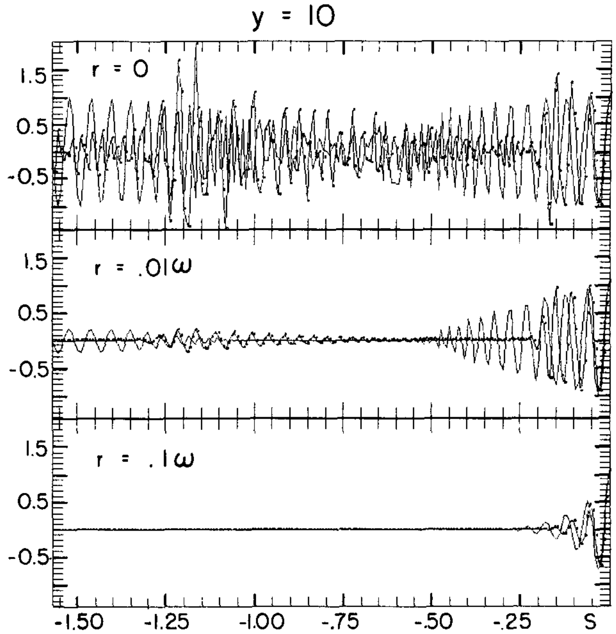

FIG. 7. As in Fig. 4 at $y = 10$.

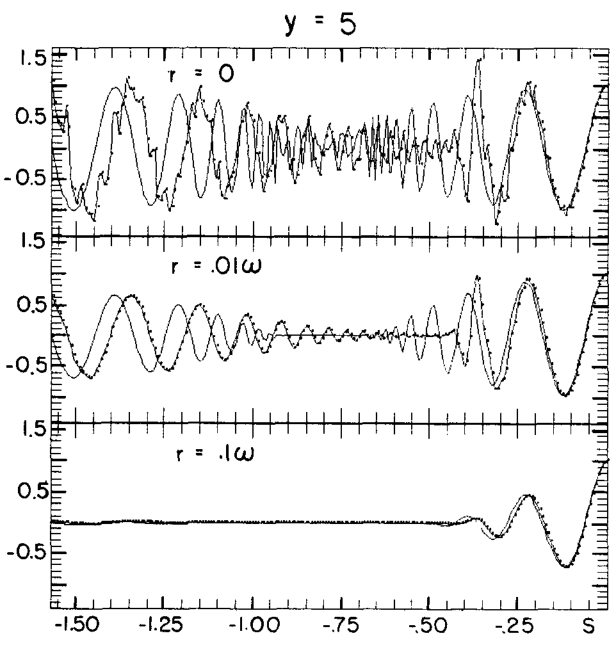

FIG. 6. As in Fig. 4 at $y = 5$.

TABLE 1. For the four gravest vertical modes in each of the world's oceans: H is the equivalent depth of the mode; L is the corresponding equatorial length scale; X_E is the zonal extent of the basin in nondimensional units; P is the period of the gravest basin mode [$m = 1$ in Eq. (13)]; $N(\omega)$ is the number of propagating Rossby waves of period P; r^{-1} is a decay time, as a multiple of P, due to the loss of energy to non-propagating modes (see text). Values of H from Moore and Philander (1977) for the Atlantic and from Eriksen (private communication) for the Indian and Pacific Oceans.

	H (cm)	L (km)	X_E	P (days)	N	r^{-1} (periods)
Atlantic						
1	60	326	20	126	21	6
2	20	248	27	220	38	8
3	8	197	34	346	59	10
4	4	166	40	489	82	11
Pacific						
1	87	357	47	263	108	13
2	27	267	62	472	200	18
3	11	214	78	740	313	22
4	7	187	89	964	386	25
Indian						
1	78	349	18	102	15	5
2	32	277	22	162	25	6
3	12	219	28	260	40	8
4	6	181	34	378	59	10

tion to any linear periodic problem that involves equatorial Kelvin waves. For example, Cane and Sarachik (1981) first find the forced response satisfying the boundary condition at $x = X_E$ and then add a free mode (12) with the amplitude needed to satisfy the boundary condition at $x = 0$. [If the frequency and basin length satisfy (13) then a resonance exists.] Since the free mode (12) is part of the response in practically all forced periodic linear equatorial problems the characteristics discussed above (foci, caustics, shadow zones) will be nearly ubiquitous.

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