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The Complexity of the Poisson Problem for Spaces of Bounded Mixed Derivatives

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Abstract. We are interested in the complexity of the Poisson problem with homogeneous Dirichlet boundary conditions on the d-dimensional unit cube Ω. Error is measured in the energy norm, and only standard information (consisting of function evaluations) is available. In previous work on this problem, the standard assumption has been that the class F of problem elements has been the unit ball of a Sobolev space of fixed smoothness r, in which case the ε-complexity is proportional to $\varepsilon^{-d/r}$. Given this exponential dependence on d, the problem is intractable for such classes F. In this paper, we seek to overcome this intractability by allowing F to be the unit ball of a space $\mathring{H}^{\rho}(\Omega)$ of bounded mixed derivatives, with ρ a fixed multi-index with positive entries. We find that the complexity is proportional to $c(d)(1/\varepsilon)^{1/\rho_{\min}}[\ln(1/\varepsilon)]^{b}$, and we give bounds on $b = b_{\rho,d}$. Hence, the problem is tractable in $1/\varepsilon$, with exponent at most $1/\rho_{\min}$. The upper bound on the complexity (which is close to the lower bound) is attained by a modified finite element method (MFEM) using discrete blending spline spaces; we obtain an explicit bound (with no hidden constants) on the cost of using this MFEM to compute ε-approximations. Finally, we show that for any positive multi-index ρ, the Poisson problem is strongly tractable, and that the MFEM using discrete blended piecewise polynomial splines of degree ρ is a strongly polynomial time algorithm. In particular, for the case $\rho = \mathbf{1}$, the MFEM using discrete blended piecewise linear splines produces an ε-approximation with cost at most

$$0.839262\,(c(d)+2)\left(\frac{1}{\varepsilon}\right)^{5.07911}.$$

1. Introduction

Second-order elliptic problems over d-dimensional domains arise often in scientific computation. The most well-known examples of such problems occur in engineering calculations, such as elasticity problems and steady-state heat flow.

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Note that for these examples, we have $d = 2$ or $d = 3$. However, high-dimensional problems with large d occur in other areas of scientific computation. Common examples of such problems include high-dimensional random walks and simultaneous Brownian motion of many non-interacting particles. Another example of a high-dimensional problem is given by Schrödinger's equation from quantum mechanics, the solution of a p-particle problem in three dimensions being a function of $d = 3p$ variables. For instance, if we wanted to find the wave function of the electron cloud in an uranium atom, we would have $p = 92$ electrons, so the dimension of the problem is $d = 276$.

We wish to find the computational complexity of such problems, as well as optimal algorithms for their solution. To make this more precise, let F be a given class of functions and let L be a given second-order elliptic operator. Then we wish to calculate an ε-approximation to the solution, i.e., for any $f \in F$, we wish to calculate an approximation U to the solution u of the problem $Lu = f$ such that $\|u - U\| \le \varepsilon$, the norm being the standard energy norm. Of course, algorithms that produce approximations U are constrained to using only finitely many information operations about the f. In this paper, we suppose that only standard information is permissible, i.e., values of f at points in the domain. Then the ε-complexity is the minimal cost of calculating an ε-approximation, and an algorithm that computes an ε-approximation with minimal cost is an optimal algorithm.

Let us review what is known about the complexity of such problems. Suppose we first we make the standard assumption that F is the unit ball of a standard Sobolev space consisting of all functions having fixed smoothness r. If we let $c(d)$ denote the cost of any information operation, then we find that the ε-complexity is proportional to $c(d)\varepsilon^{-d/r}$. Moreover, finite element methods of fixed degree using refined meshes are nearly optimal. (See [**14**, Chapter 5] for details.)

Note that the complexity increases exponentially in d. This is well-known as the hallmark of intractability (see [**4**]). Moreover, this effect can be seen even for the engineering examples with $d = 3$. To be specific, let us only consider the case of standard information with $r = 1$. Then the number of information operations required to compute an ε-approximation is proportional to ε^{-3}. If we need four-place accuracy ($\varepsilon = 10^{-4}$), this means that we must use roughly 10^{12} operations. On a megaflop machine, this takes about a week and a half; on a gigaflop machine, this takes around 15 minutes.

Note that cleverer algorithms won't help defeat this "curse of dimensionality," since this is a result about the inherent problem complexity. If we are going to make such problems tractable, we need to somehow change the problem formulation.

Since one of this paper's main themes is to determine whether elliptic problems can be made tractable, we should recall that there are several different kinds of tractability.

1. If the complexity is at most $(c(d) + 2)K(d)(1/\varepsilon)^p$, then the problem is tractable in $1/\varepsilon$, and the smallest p is called the $1/\varepsilon$-exponent of the problem. An algorithm that computes an ε-approximation at cost at most $(c(d) + 2)K(d)(1/\varepsilon)^p$ is said to be an $1/\varepsilon$-polynomial time algorithm.
2. If the complexity is at most $(c(d) + 2)K(\varepsilon)d^q$, then the problem is *tractable in* d, and the smallest q is called the d-exponent of the problem. An algorithm that computes an ε-approximation at cost at most $(c(d) + 2)K(\varepsilon)d^q$ is said to be a d-polynomial time algorithm.

3. If the complexity is at most $(c(d)+2)K(1/\varepsilon)^p$, then the problem is *strongly tractable*, and the smallest p is called the strong exponent of the problem. An algorithm that computes an ε-approximation at cost at most $(c(d)+2)K(1/\varepsilon)^p$ is said to be a strongly polynomial time algorithm.

This is discussed in [**19**]. Note that rather than having the cost $c(d)$ of an information operation appear in these definitions, it is more convenient to use $c(d)+2$.

Recall that to make the problem tractable, we must reformulate it somehow. There are two ways to do this. The first is to no longer require that the error be at most ε for all $f \in F$. In other words, we change the "setting." One way of doing this is to replace the worst case setting by an average case or probablistic setting; another is to allow nondeterminism. There is some discussion of this approach in [**14**, Chapters 7 and 8]; more recent results may be found in [**17**] and [**18**].

The only other way we can overcome intractability is to change the class F of problem elements. One idea is to note that the admissible problem elements f are analytic or (even more often) piecewise analytic functions. We allowed F to be a class of analytic functions in [**16**], where we find that the complexity is proportional to $(c(d)+2)(\log 1/\varepsilon)^d$. (Moreover, finite element methods with fixed mesh using increasing degree of approximation are nearly optimal.) In the terminology of [**12**], the problem is tractable in $1/\varepsilon$, with exponent 0. However, no d-tractability or strong tractability results are known. Unfortunately, the results for piecewise analytic functions are negative, unless we know the locations of the breaks; see [**15**] for the details.

In this paper, we return to the idea of using problem elements having limited smoothness, but in a different sense. Rather than assume that all Sobolev derivatives of a given order are bounded, we assume that f has a given number ρ_i of derivatives in the ith coordinate direction for each $i \in \{1, \dots, d\}$. In other words, we follow the lead of [**12**] in using spaces of bounded mixed derivatives, such as those studied by Temlyakov (see [**8**], [**9**], [**10**]).

Note that much of our previous work on elliptic problems dealt with the case of arbitrary elliptic operators L. Of course, we did this so that the results would apply to as wide a class of elliptic problems as possible. However, there is a downside to this approach; most of the results involved constants whose explicit expressions are hard to obtain. Since we are interested in what happens as the dimension d varies, we need to know how these constants change with d. Hence, we only investigate a specific model problem in this paper, namely, the Poisson problem $-\Delta u = f$ on the d-dimensional unit cube $\Omega = I^d$, with homogeneous Dirichlet boundary conditions. Of course, there are other problems that we could have chosen instead (e.g., the Helmholtz problem, or a Neumann problem); for many of them, the techniques for their investigation should be similar to those contained in this paper.

We now outline the contents of this paper, section by section, describing the main results.

In Section 2, we formally describe our problem. As in [**12**], the class of problem elements is the unit ball in a tensor product $\mathring{H}^{\rho}(\Omega) = \bigotimes_{i=1}^d \mathring{H}^{\rho_i}(I)$ of one-dimensional spaces. However, the problem itself is not a tensor product problem, i.e., the solution operator is not a tensor product of one-dimensional operators. We note that as formulated, the solution to our problem always satisfies periodic homogeneous boundary conditions; this means that singularities will not occur at the corners.

In Section 3, we determine complexity results for standard information. The problem complexity is proportional to $c(d)(1/\varepsilon)^{1/\rho_{\min}}[\ln(1/\varepsilon)]^{b_{\rho,d}}$. Here, we only know that $b_{\rho,d} \in [(k^* - 1)/(2\rho_{\min}), d-1]$, where k^* is the number of times the entry $\rho_{\min}$ appears in the multi-index ρ. This means that we know the problem complexity to within a polylogarithmic factor. We also investigate a modified finite element method (MFEM) using discrete blending spline spaces of degree ρ_i in the ith coordinate direction ([**1**], [**2**], [**3**]). We determine that this MFEM can compute an ε-approximation with cost at most

$$\beta_1(c(d)+2)\left(\beta_2 + \beta_3 \frac{\ln(1/\varepsilon)}{d-1}\right)^{\frac{(\rho_{\min}+1)(d+1)}{\rho_{\min}}} \left(\frac{1}{\varepsilon}\right)^{1/\rho_{\min}}.$$

Moreover, we have explicit expressions for the constants β_1, β_2, and β_3. For example, in the case $\rho = \mathbf{1}$, we consider the MFEM using discrete blended piecewise linear splines; the cost of using this algorithm to find an ε-approximation is at most

$$0.192705\,(c(d)+2)\left(-0.961691 + 2.05964\frac{\ln(1/\varepsilon)}{d-1}\right)^{2(d+1)} \left(\frac{1}{\varepsilon}\right).$$

It now follows that the problem is tractable in $1/\varepsilon$ with exponent $1/\rho_{\min}$, and the MFEM is a $1/\varepsilon$-polynomial time algorithm. Finally in Section 4, we show that the problem is strongly tractable when standard information is permissible. The precise form of our strong exponent for arbitrary ρ is somewhat complicated, requiring the solution of a nonlinear equation. However, since the complexity decreases as ρ increases, we can use explicit values for the case $\rho = \mathbf{1}$ to find that the problem complexity is at most

$$0.839262\,(c(d)+2)\left(\frac{1}{\varepsilon}\right)^{5.07911}$$

for any ρ. Hence the Poisson problem is strongly tractable when standard information is permissible, with a strong exponent of at most 5.07911. We are certain that this estimate of the strong exponent is pessimistic, since the $1/\varepsilon$ exponent is $1/\rho_{\min}$.

Finally, we note in closing that the results in this paper were obtained under the assumption that only standard information is permissible. We could well ask what happens if we allow arbitrary continuous linear information, i.e., allowing any continuous linear functional of the problem element. Since standard information is a proper subset of continuous linear information, one might expect the problem complexity to be significantly less and the strong exponent of the problem to be smaller, when continuous linear information is permissible. This is the subject of ongoing research, and will be reported in a future paper.

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2. Problem description

In what follows, we assume that the reader is familiar with the usual terminology and notations arising in the variational study of elliptic boundary value problems. See Chapter 5 and the Appendix of [**14**] for further details, as well as the references cited therein. We use $\mathbb{N}$ and $\mathbb{P}$ to respectively denote the nonnegative and positive integers. Unless stated otherwise, Greek letters will be used to denote multi-indices, i.e., vectors in $\mathbb{N}^d$.

Our first step in the problem description will be to introduce some special Sobolev spaces, which will be closed subspaces of the usual Sobolev spaces. The main novelty in these spaces is that they satisfy certain boundary conditions. Our reason for choosing them is that they have a convenient orthonormal basis. (See also [**13**] for another example of using this technique, although in a different situation.)

First, we consider the spaces of functions defined over a one-dimensional interval. Let $I = (0,1)$, and set $z_j(t) = \sqrt{2}\sin j\pi t$ for $j \in \mathbb{P}$. Of course, $\{z_j\}_{j\in\mathbb{P}}$ is an orthonormal basis in $L_2(I)$. For $r \in \mathbb{R}$, we define

$$\mathring{H}^r(I) = \{\, v \in \operatorname{span}\{z_j\}_{j=1}^\infty : \|v\|_{\mathring{H}^r(I)} < \infty \,\},$$

under the norm

$$\|v\|^2_{\mathring{H}^r(I)} = \sum_{j=1}^\infty (\pi j)^{2r} \langle f, z_j\rangle^2_{L_2(I)}.$$

It is easy to verify the following facts:

Lemma 2.1.
1. *$\mathring{H}^r(I)$ is a Hilbert space.*
2. *For any $f, g \in \mathring{H}^r(I)$, we have $\langle f, g\rangle_{\mathring{H}^r(I)} = \langle D^r f, D^r g\rangle_{L_2(I)}$.*
3. *Let $w_j(t) = (\pi j)^{-r} z_j(t)$ for $j \in \mathbb{P}$. Then $\{w_j\}_{j\in\mathbb{P}}$ is an orthonormal basis for $\mathring{H}^r(I)$.* □

Next, we define the spaces over the d-dimensional hypercube $\Omega = I^d$. For any multi-index $\rho \in \mathbb{N}^d$, we let

$$\mathring{H}^\rho(\Omega) = \bigotimes_{i=1}^d \mathring{H}^{\rho_i}(I),$$

under the tensor product norm

$$\|v_1 \dots v_d\|_{\mathring{H}^\rho(\Omega)} = \prod_{i=1}^d \|v_i\|_{\mathring{H}^{\rho_i}(I)} \qquad \forall\, v_i \in \mathring{H}^{\rho_i}(I), 1 \le i \le d.$$

The properties of the space $\mathring{H}^\rho(\Omega)$ are given by the following

Lemma 2.2.
1. *$\mathring{H}^\rho(\Omega)$ is a Hilbert space.*
2. *For any $f, g \in \mathring{H}^\rho(\Omega)$, we have $\langle f, g\rangle_{\mathring{H}^\rho(\Omega)} = \langle D^\rho f, D^\rho g\rangle_{L_2(\Omega)}$.*
3. *For any multi-index α, let $w_\alpha(x) = \pi^{-|\rho|}\alpha^{-\rho} z_\alpha(x)$, where*

$$z_\alpha(x_1, \dots, x_d) = z_{\alpha_1}(x_1)\dots z_{\alpha_d}(x_d).$$

Then $\{w_\alpha\}_{\alpha\in\mathbb{P}^d}$ is an orthonormal basis for $\mathring{H}^\rho(\Omega)$. □

In short, $\mathring{H}^\rho(\Omega)$ is a space of mixed derivatives, analogous to those studied in [**9**].

We are now ready to define our solution operator. Let $\rho \in \mathbb{P}^d$ be a fixed multi-index. Define

$$F = \{\, f \in \mathring{H}^\rho(\Omega) : \|f\|_{\mathring{H}^\rho(\Omega)} \le 1 \,\},$$

i.e., F is the unit ball of $\mathring{H}^{\rho}(\Omega)$. Then we let $S\colon F \to H^1_0(\Omega)$ be given by

$$\langle Sf, v\rangle_{H^1_0(\Omega)} = \langle f, v\rangle_{L_2(\Omega)} \qquad \forall\, v \in H^1_0(\Omega). \tag{2.1}$$

Thus $u = Sf$ is the variational solution to

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega,\\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Note that we have the representation formula

$$Sf = \frac{1}{\pi^2} \sum_{\alpha\in\mathbb{P}^d} \frac{\langle f, w_\alpha\rangle_{\mathring{H}^{\rho}(\Omega)}}{\alpha_1^2+\dots+\alpha_d^2} w_\alpha = \frac{1}{\pi^2} \sum_{\alpha\in\mathbb{P}^d} \frac{\langle f, z_\alpha\rangle_{L_2(\Omega)}}{\alpha_1^2+\dots+\alpha_d^2} z_\alpha.$$

Next, we recall the usual concepts of information-based complexity, see, e.g., [**11**] for a fuller development. We assume that only *standard information* is permissible, i.e., for any problem element f, we only know information of the form

$$Nf = [f(x_1), \dots, f(x_n)] \qquad \forall f \in F,$$

where $x_1, \dots, x_n \in \overline{\Omega}$. Note that if the sample points are distinct, then N is information of *cardinality* n, i.e., $\operatorname{card} N = n$. Note that the information used is nonadaptive, i.e., the number and choice of evaluation points defining N is determined in advance, independently of any particular $f \in F$. Since the solution operator is linear and F is a convex balanced set, there is no loss of generality in doing this; see [**11**, Chapter 4.5] for further discussion.

Our model of computation is the standard one given in [**11**]. The evaluation of $f(x)$ for any $x \in \overline{\Omega}$ and $f \in F$ has cost $c(d)$, and the cost of basic combinatory operations is 1. Typically, $c(d) \gg 1$.

In this paper, we consider the worst case setting. Hence, the error of any algorithm ϕ using information N is given by

$$e(\phi, N) = \sup_{f\in F} \|Sf - \phi(Nf)\|_{H^1_0(\Omega)}.$$

The radius of information N is

$$r(N) = \inf_{\phi} e(\phi, N).$$

and the nth minimal radius is

$$r(n) = \inf\{\, r(N) : \operatorname{card} N \le n \,\}.$$

The cost of an algorithm ϕ using N is given by

$$\operatorname{cost}(\phi, N) = \sup_{f\in F} \operatorname{cost}(\phi, N, f),$$

with $\operatorname{cost}(\phi, N, f)$ denoting the cost of computing ϕ for a particular problem element f. As always, the ε-complexity

$$\operatorname{comp}(\varepsilon) = \inf\{\, \operatorname{cost}(\phi, N) : e(\phi, N) \le \varepsilon \,\}$$

of our problem is the minimal cost of computing an ε-approximation, for $\varepsilon \ge 0$. In this paper, we make frequent use of the inequality

$$c(d)\, m(\varepsilon) \le \operatorname{comp}(\varepsilon) \le (c(d)+2) m(\varepsilon),$$

where

$$m(\varepsilon) = \min\{\, n \in \mathbb{N} : r(n) \le \varepsilon \,\}$$

is the ε-cardinality number.

In the sequel, we may sometimes wish to stress the dependence of various quantities on parameters such as the dimension d of the region $\Omega = I^d$ or the multi-index ρ defining the smoothness of the problem elements. We shall do this by writing, e.g., $\mathrm{comp}(\varepsilon, d, \rho)$ for the ε-complexity of a problem on a d-dimensional domain for which the problem elements have smoothness given by ρ.

3. Complexity results

In this section, we establish bounds on the nth minimal radius and the ε-complexity of our problem. These bounds are fairly tight in ε. We shall use these results in the next section, where we will establish that our problem is strongly tractable.

We first establish a lower bound on the nth minimal radius. In what follows, we let

$$\rho_{\min} = \min_{1 \le i \le d} \rho_i$$

be the minimal component of ρ, and let

$$k^* = \mathrm{card}\,\{\, i \in \{1, \ldots, d\} : \rho_i = \rho_{\min} \,\}$$

denote the number of times $\rho_{\min}$ appears in ρ.

THEOREM 3.1.

1. *There exists a constant C, depending on d, such that*

$$r(n) \ge C \frac{(\lg n)^{(k^*-1)/2}}{n^{\rho_{\min}}}.$$

2. *There exists a constant C, depending on d, such that*

$$\mathrm{comp}(\varepsilon, d) \ge C \cdot c(d) \left(\frac{1}{\varepsilon}\right)^{1/\rho_{\min}} \left(\lg \frac{1}{\varepsilon}\right)^{(k^*-1)/(2\rho_{\min})},$$

where $\rho_{\min}$ is the minimal component of ρ, appearing k^* times.

PROOF. To prove the first part of this Theorem, we will show that our problem is no easier than the integration problem studied in [**10**]. Let

$$Nf = [f(x_1), \ldots, f(x_n)]$$

for points $x_1, \ldots, x_n \in \Omega$. Choose a fixed δ for which $0 < \delta < 1$; for instance, $\delta = \frac{1}{2}$ is a good choice. Let $\Omega_\delta = [-1+\delta, 1-\delta]^d$. In what follows, there is no essential loss of generality if we assume that $x_1, \ldots, x_n \in \Omega_\delta$. From [**10**], there exists a function $h \in C_0^\infty(\Omega_\delta)$ such that

$$\begin{aligned} Nh &= 0, \\ \|h\|_{\mathring{H}(\Omega_\delta)} &= 1, \\ \int_{\Omega_\delta} h(x)\, dx &\ge C \frac{(\lg n)^{(k^*-1)/2}}{n^{\rho_{\min}}}. \end{aligned}$$

Here, C is a constant that is independent of n. Now extend h to the whole region Ω by taking $h = 0$ outside Ω_δ. Then $\|h\|_{\mathring{H}^\rho(\Omega)} = 1$. Choose a nonnegative function $g \in H_0^1(\Omega)$ such that $g \equiv 1$ in Ω_δ. Using [**11**, Theorem 5.53] and (2.1), we find that

$$\begin{aligned} r(N) &\geq \|Sh\|_{H_0^1(\Omega)} \geq \frac{\langle h, g\rangle_{L_2(\Omega)}}{\|g\|_{H_0^1(\Omega)}} \geq \frac{1}{\|g\|_{H_0^1(\Omega)}} \int_{\Omega_\delta} h(x)\,dx \\ &\geq \frac{C}{\|g\|_{H_0^1(\Omega)}} \frac{(\lg n)^{(k^*-1)/2}}{n^{\rho_{\min}}}. \end{aligned}$$

Since N is arbitrary standard information of cardinality n, the result follows immediately.

To prove the second part of this Theorem, we need only observe that by the first part, there exists C, independent of ε, such that

$$m(\varepsilon, d) \geq C \cdot \left(\frac{1}{\varepsilon}\right)^{1/\rho_{\min}} \left(\lg \frac{1}{\varepsilon}\right)^{(k^*-1)/(2\rho_{\min})}. \qquad \square$$

We now seek an algorithm whose performance will be close to the lower bounds established in Theorem 3.1. The main idea here is to use a discrete blending spline approximation (see [**1**], [**2**], [**3**]). As [**5**] and [**6**] point out, this is the same as Smolyak's algorithm (see [**7**], [**8**], [**9**], [**10**], [**12**]).

For future reference, we recall the approximation properties of one-dimensional polynomial spline spaces. Choose positive integers m and r; we assume (without essential loss of generality) that 2^r is a multiple of m. Let $\mathcal{S}_{m,r}$ denote the space of continuous functions on I, whose restriction to the interval $[(i-1)m2^{-r}, im2^{-r}]$ is a polynomial of degree at most m, for $1 \leq i \leq 2^r/m$. For any function $v\colon I \to \mathbb{R}$, we let $U_{m,r}v$ be the unique element of $\mathcal{S}_{m,r}$ of degree m that interpolates v at the points

$$X_r = \{i2^{-r}\}_{i=0}^{2^r}.$$

It will be convenient to let $U_{m,0} = 0$. The following standard result may be found in (e.g.) [**13**, p. 309]:

Lemma 3.1. *Let $\ell_1 \leq \ell_2$, with $\ell_1 \in \{0,1\}$. There exists $C_{\min\{m,\ell_2-1\}} > 0$ such that*

$$\|(v - U_{m,r}v)^{(\ell_1)}\|_{L_2(I)} \leq C_{\min\{m,\ell_2-1\}} 2^{-(\min\{\ell_2,m+1\}-\ell_1)r} \|v^{(\ell_2)}\|_{L_2(I)}$$

for any $v \in H^{\ell_2}(I)$.

We now describe the construction of our multidimensional blending spline spaces, as well as the sparse grid on which they are defined and interpolation operators for these spaces. For $\kappa \in \mathbb{P}^d$, let

$$\Delta_{\kappa_i,r} = U_{\kappa_i,r} - U_{\kappa_i,r-1} \qquad \text{for } 1 \leq i \leq d.$$

For $q \geq d$, let

$$S_{q,d,\kappa} = \sum_{|\alpha| \leq q} \bigotimes_{i=1}^{d} \Delta_{\kappa_i,\alpha_k}.$$

Note that this is is a mild variation of the approximation operator found in [**12**]; the novelty is that we allow different one-dimensional operators in the different coordinate directions, whereas the same operator was used for each direction in [**12**].

We can find a more explicit representation of this approximation operator by using the ideas found in [**6**] and in [**12**, Section 3]. First, we find that

$$S_{q,d,\kappa} = \sum_{q-d+1\le|\alpha|\le q} (-1)^{q-|\alpha|} \binom{d-1}{q-|\alpha|} \bigotimes_{i=1}^{d} U_{\kappa_i,\alpha_i}.$$

Moreover, for any function v defined on Ω, the approximation $S_{q,d,\kappa}v$ depends on v only through the values of v at the union

$$H(q,d) = \bigcup_{q-d+1\le|\alpha|\le q} X_{\alpha_1} \times \cdots \times X_{\alpha_d} = \{x_1, \ldots, x_{n(q,d)}\}.$$

Note that since the sets $\{X_i\}_{i=1}^{\infty}$ are nested, the sets $\{H(q,d)\}_{q=1}^{\infty}$ are also nested. The points belonging to $H(q,d)$ are called *hyperbolic cross points*, and the set $H(q,d)$ is called a *sparse grid*.

We let

$$n = n(q,d) = \operatorname{card} H(q,d)$$

denote the number of points in the sparse grid $H(q,d)$. We also let

$$\mathcal{S}_{q,d,\kappa} = \bigcup_{q-d+1\le|\alpha|\le q} \bigotimes_{i=1}^{d} \operatorname{Range}(U_{\kappa_i,\alpha_i})$$

denote the resulting *discrete blending spline space*.

Now we are ready to describe our approximation algorithm, which will use the particular blending spline space $\mathcal{S}_{q,d,\rho}$. Note that this is the blending spline space $\mathcal{S}_{q,d,\kappa}$ with $\kappa = \rho$. In other words, we choose the polynomial degree of the blending spline space in a given direction as the number of derivatives that a problem element has in that direction.

Let q and d be given. For $f \in F$, let

$$N_{q,d}f = [f(x_1), \ldots, f(x_{n(q,d)}).]$$

denote *hyperbolic cross point information* about f. Note that since $f \in \mathring{H}^{\rho}(\Omega)$ and $\rho \in \mathbb{P}^d$, the Sobolev embedding theorem guarantees that the information $N_{q,d}f$ is well-defined for any $f \in F$. Then Lemma 7 of [**12**] states that

$$\operatorname{card} N_{q,d} = n(q,d) \le 2^{q-d+1}\binom{q-1}{d-1}. \tag{3.1}$$

For $f \in F$, we calculate $u_{q,d} \in \mathcal{S}_{q,d,\rho}$ such that

$$\langle \nabla u_{q,d}, \nabla s\rangle_{L_2(\Omega)} = \langle S_{q,d,\rho}f, s\rangle_{L_2(\Omega)} \qquad \forall\, s \in \mathcal{S}_{q,d,\rho}. \tag{3.2}$$

Note that this is a modified version of Galerkin's method, with the test and trial spaces both being $\mathcal{S}_{q,d,\rho}$; the change is that we are not using inner products with f, but with $S_{q,d,\rho}f$. It is easy to see that for any $f \in F$, there exists a unique $u_{q,d} \in \mathcal{S}_{q,d,\rho}$ satisfying (3.2) and that $u_{q,d}$ depends on f only through the information $N_{q,d}f$; see [**11**, p. 161] for further discussion. Hence we may write

$$u_{q,d} = \phi_{q,d}(N_{q,d}f),$$

where $\phi_{q,d}$ is the *modified finite element method* (MFEM) based on $\mathcal{S}_{q,d,\rho}$.

We now give an abstract error estimate for our MFEM; a proof is contained in the beginning of the proof of [**14**, Theorem 5.7.4].

LEMMA 3.2. *For any q and d, we have*

$$e(\phi_{q,d}, N_{q,d}) \leq \sup_{f \in F} \left[\|Sf - S_{q,d,\rho}Sf\|_{H_0^1(\Omega)} + \|f - S_{q,d,\rho}f\|_{H_0^{-1}(\Omega)} \right]. \qquad \square$$

We now develop Sobolev error bounds for using Smolyak's algorithm to approximate the identity operator. Once we have such bounds, we can directly apply Lemma 3.2. These bounds are given in

LEMMA 3.3. *For $\kappa \in \mathbb{P}^d$, let*

$$C_\kappa = \max_{1 \leq i \leq d} C_{\kappa_i}.$$

Let $\beta, \theta \in \mathbb{N}^d$, with $\beta \leq \theta$. Define

$$\delta_i = \min\{\theta_i, \kappa_i + 1\} \qquad \text{for } 1 \leq i \leq d.$$

Let

$$s = \min_{1 \leq i \leq d} \theta_i - \beta_i,$$

$$t = \min_{1 \leq i \leq d} \min\{\theta_i, \kappa_i + 1\} - \beta_i,$$

$$H_\kappa = \max\left\{ \left(\frac{2}{\pi}\right)^s, 1 + 2^{(\theta-\beta)_{\max}} C_\kappa \right\}.$$

Then

$$\|D^\beta(v - S_{q,d,\kappa}v)\|_{L_2(\Omega)} \leq C_\kappa H_\kappa^{d-1} \binom{q}{d-1} 2^{-tq} \|D^\theta v\|_{L_2(\Omega)}.$$

PROOF. We first establish some notation that we will need to prove this lemma. In such cases where it will cause no confusion, we shall write $S_{q,d}$ instead of $S_{q,d,\kappa}$ in what follows. For any q and d, we let

$$Q(q,d) = \{\, \alpha \in \mathbb{P}^d : |\alpha| \leq q \,\}.$$

We also let id_d denote the identity operator for functions that are defined on I^d. If $L\colon X \to Y$ is a linear transformation of normed linear spaces, we let $\|L\|_{\mathcal{L}(X \to Y)}$ denote the usual operator norm. Finally, we let

$$e_{q,d} = \|D^\beta(\mathrm{id}_d - S_{q,d})\|_{\mathcal{L}(\mathring{H}^\theta(\Omega) \to L_2(\Omega))}.$$

We seek bounds on $e_{q,d}$.

As in [**12**, p. 13], we have

$$S_{q,d} = \sum_{\alpha \in Q(q-1,d-1)} \left(\bigotimes_{i=1}^{d-1} \Delta_{\kappa_i,\alpha_i} \right) \otimes U_{\kappa_d, q-|\alpha|}.$$

Hence if $\kappa \in \mathbb{P}^{d+1}$, we find that

$$\mathrm{id}_{d+1} - S_{q+1,d+1} = \sum_{\alpha \in Q(q,d)} \left(\bigotimes_{i=1}^{d} \Delta_{\kappa_i,\alpha_i} \right) \otimes (\mathrm{id}_1 - U_{d+1,q+1-|\alpha|}) + (\mathrm{id}_d - S_{q,d}) \otimes \mathrm{id}_1.$$

Thus

$$\partial_1^{\beta_1} \dots \partial_{d+1}^{\beta_{d+1}} (\mathrm{id}_{d+1} - S_{q+1,d+1}) = \sum_{\alpha \in Q(q,d)} \left(\bigotimes_{i=1}^{d} \partial_i^{\beta_i} \Delta_{\kappa_i,\alpha_i} \right) \otimes \partial_{d+1}^{\beta_{d+1}} (\mathrm{id}_1 - U_{d+1,q+1-|\alpha|}) + \partial_1^{\beta_1} \dots \partial_d^{\beta_d} (\mathrm{id}_d - S_{q,d}) \otimes \mathrm{id}_1.$$

From this we find that

$$e_{q+1,d+1} \le \sum_{\alpha \in Q(q,d)} \left(\prod_{i=1}^{d} \|\partial_i^{\beta_i} \Delta_{\kappa_i,\alpha_i}\|_{\mathcal{L}(\mathring{H}^{\theta_i}(I) \to L_2(I))} \times \|\partial_{d+1}^{\beta_{d+1}} (\mathrm{id}_1 - U_{d+1,q+1-|\alpha|})\|_{\mathcal{L}(\mathring{H}^{\theta_{d+1}}(I) \to L_2(I))} \right) + e_{q,d} \|\partial_{d+1}^{\beta_{d+1}}\|_{\mathcal{L}(\mathring{H}^{\theta_{d+1}}(I) \to L_2(I))}. \tag{3.3}$$

Using a Fourier series expansion, it is straightforward to check that

$$\|\partial_{d+1}^{\beta_{d+1}}\|_{\mathcal{L}(\mathring{H}^{\theta_{d+1}}(I) \to L_2(I))} = \left(\frac{1}{\pi}\right)^{\theta_{d+1} - \beta_{d+1}}. \tag{3.4}$$

Let $E_\kappa = 2^{(\theta-\beta)_{\max}} C_\kappa$. Using (3.4) and Lemma 3.1 in (3.3), we find that

$$e_{q+1,d+1} \le C_\kappa E_\kappa^d \sum_{\alpha \in Q(q,d)} 2^{-[(\delta_1-\beta_1)\alpha_1 + \cdots + (\delta_d - \beta_d)\alpha_d + (\delta_{d+1} - \beta_{d+1})(q+1-|\alpha|)]} + \left(\frac{1}{\pi}\right)^{\theta_{d+1} - \beta_{d+1}}.$$

Since

$$\operatorname{card} Q(q,d) = \binom{q}{d},$$

the previous inequality implies that

$$e_{q,d} \le C_\kappa E_\kappa^d \binom{q}{d} 2^{-t(q+1)} + \left(\frac{1}{\pi}\right)^{\theta_{d+1} - \beta_{d+1}}.$$

As in [**12**, Lemma 2], we now find that

$$e_{q,d} \le C_\kappa H_\kappa^{d-1} \binom{q}{d-1} 2^{-tq},$$

as required. □

We can now give an error bound for the MFEM based on the space $\mathcal{S}_{q,d,\rho}$ of discrete blending splines:

THEOREM 3.2. *Let*

$$C_\rho = \max_{1 \le i \le d} C_{\rho_i}$$

and

$$H_\rho = \max \left\{ \left(\frac{2}{\pi}\right)^{\rho_{\min}}, (1 + 2^{\rho_{\max}}) C_\rho \right\}.$$

For any $d \in \mathbb{P}$ and $q \geq d$, we have

$$e(\phi_{q,d}, N_{q,d}) \leq (1 + \pi^{-1}) C_\rho H_\rho^{d-1} \binom{q}{d-1} 2^{-q\rho_{\min}}.$$

PROOF. Let $f \in F$, and then let $u = Sf$. We need to find upper bounds for $\|u - S_{q,d,\rho} u\|_{H_0^1(\Omega)}$ and for $\|f - S_{q,d,\rho} f\|_{H_0^{-1}(\Omega)}$. In what follows, we will make use of the notation given in Lemma 3.3, letting $\kappa = \rho$.

We first estimate $\|f - S_{q,d,\rho} f\|_{H_0^{-1}(\Omega)}$. Choose $\beta = \mathbf{0}$ and $\theta = \rho$. Then $\delta = \rho$, so that $s = t = \rho_{\min}$. From Lemma 3.3, it follows that

$$\begin{aligned} \|f - S_{q,d,\rho} f\|_{H_0^{-1}(\Omega)} &\leq \|f - S_{q,d,\rho} f\|_{L_2(\Omega)} \\ &\leq C_\rho H_\rho^{d-1} \binom{q}{d-1} 2^{-q\rho_{\min}} \|D^\rho f\|_{L_2(\Omega)}. \end{aligned} \tag{3.5}$$

We are now left with estimating $\|u - S_{q,d,\rho} u\|_{H_0^1(\Omega)}$. Choose $i \in \{1, \ldots, d\}$, and let $\beta = e_i$, where e_i is the ith standard unit vector in $\mathbb{R}^d$. Now set $\theta = \rho + e_i$. Since $\delta - \beta = \rho$, we find that $t = \rho_{\min}$. Moreover, $\theta - \beta = \rho$, so that $s = \rho_{\min}$. This gives the estimate

$$\|\partial_i (u - S_{q,d,\rho} u)\|_{L_2(\Omega)} \leq C_\rho H_\rho^{d-1} \binom{q}{d-1} 2^{-q\rho_{\min}} \|D^\rho \partial_i u\|_{L_2(\Omega)},$$

so that

$$\|u - S_{q,d,\rho} u\|_{H_0^1(\Omega)} \leq C_\rho H_\rho^{d-1} \binom{q}{d-1} 2^{-q\rho_{\min}} \left[\sum_{i=1}^d \|D^\rho \partial_i u\|_{L_2(\Omega)}^2 \right]^{1/2}. \tag{3.6}$$

We must estimate the sum on the right-hand side of (3.6). Expanding

$$f = \sum_{\alpha \in \mathbb{P}^d} c_\alpha z_\alpha, \tag{3.7}$$

we have

$$u = Sf = \pi^{-2} \sum_{\alpha \in \mathbb{P}^d} \frac{c_\alpha}{\alpha_1^2 + \cdots + \alpha_d^2} z_\alpha. \tag{3.8}$$

Hence for any $i \in \{1, \ldots, d\}$, we have

$$\|D^\rho \partial_i u\|_{L_2(\Omega)}^2 = \pi^{2(|\rho|-1)} \sum_{\alpha \in \mathbb{P}^d} \frac{c_\alpha^2 \alpha^{2\rho} \alpha_i^2}{(\alpha_1^2 + \cdots + \alpha_d^2)^2},$$

so that

$$\sum_{i=1}^d \|D^\rho \partial_i u\|_{L_2(\Omega)}^2 = \pi^{2(|\rho|-1)} \sum_{\alpha \in \mathbb{P}^d} \frac{c_\alpha^2 \alpha^{2\rho}}{\alpha_1^2 + \cdots + \alpha_d^2} \leq \pi^{2(|\rho|-1)} \sum_{\alpha \in \mathbb{P}^d} c_\alpha^2 \alpha^{2\rho}. \tag{3.9}$$

On the other hand, we have

$$\|D^\rho f\|_{L_2(\Omega)}^2 = \sum_{\alpha \in \mathbb{P}^d} \|D^\rho z_\alpha\|_{L_2(\Omega)}^2 = \pi^{2|\rho|} \sum_{\alpha \in \mathbb{P}^d} c_\alpha^2 \alpha^{2\rho}. \tag{3.10}$$

Comparing (3.9) with (3.10), we see that

$$\left[\sum_{i=1}^{d} \|D^\rho \partial_i u\|_{L_2(\Omega)}\right]^{1/2} \le \pi^{-1}\|D^\rho f\|_{L_2(\Omega)}. \tag{3.11}$$

Using this inequality in (3.6), we get

$$\|u - S_{q,d,\rho}u\|_{H_0^1(\Omega)} \le \pi^{-1} C_\rho H_\rho^{d-1} \binom{q}{d-1} 2^{-q\rho_{\min}} \|D^\rho f\|_{L_2(\Omega)}. \tag{3.12}$$

The desired result now follows from (3.5), (3.12), and Lemma 3.2. □

Using this result, we now find an upper bound on the error of using the MFEM with n evaluations.

THEOREM 3.3. *For any d, let*

$$A_d = \frac{(1+\pi^{-1})C_\rho H_\rho^{d-1}}{2^{\rho_{\min}(d-2)}((d-1)!)^{\rho_{\min}+1}}$$

and

$$B_d = 2^{d-1}(d-1)!.$$

Let n be given. Choose

$$q = q(n,d) = \lfloor \lg(B_d n) - (d-1)\lg\lg(B_d n)\rfloor.$$

Then

$$\operatorname{card} N_{n,q} \le n$$

and

$$e(\phi_{q,d}, N_{q,d}) \le A_d \frac{[\lg(B_d n)]^{(d-1)(\rho_{\min}+1)}}{n^{\rho_{\min}}}.$$

Hence, there exists $a_{\rho,d} \in [\frac{1}{2}(k^*-1), (d-1)(\rho_{\min}+1)]$ such that

$$r(n) = \Theta\left(\frac{(\lg n)^{a_{\rho,d}}}{n^{\rho_{\min}}}\right).$$

PROOF. Use (3.1) to find that

$$\operatorname{card} N_{q,d} \le 2^{q-d+1}\frac{(q-1)^{d-1}}{(d-1)!} \le \frac{2^q q^{d-1}}{B_d}.$$

From the definition of $q(n,d)$, we find that

$$2^q \le \frac{B_d n}{[\lg(B_d n)]^{d-1}}$$

and that

$$q \le \lg(B_d n). \tag{3.13}$$

We now find that

$$\operatorname{card} N_{q,d} \le n.$$

On the other hand, we also see that there exists $y \in [0,1)$ such that

$$q = \lg\left(\frac{B_d n}{[\lg(B_d n)]^{d-1}}\right) - y,$$

and hence

$$2^{-q} \le 2\frac{\lg(B_d n)^{d-1}}{B_d n}.$$

Using this inequality and (3.13) in Theorem 3.4, we immediately find the desired bound on the error. □

REMARK. Note that there is a mild gap between the lower and upper bounds on the nth minimal error as given (respectively) by Theorem 3.1 and Theorem 3.4. That is, the exponents of n are the same, but the exponents of $\lg n$ differ.

There are two reasons for this. The first is that we are using the integration problem to establish our lower bound, whereas we are using a variant of the L_2-approximation problem to find our upper bound. The difficulty here is that the minimal error and complexity are different for these two problems, and so a gap is unavoidable, unless we can find a different technique of establishing our lower and upper bounds.

The second reason for the gap may be found in Lemma 3.3. If we consider the bound on the L_2-error contained in (3.5) as a function of the cardinality n of the information, we find an upper bound $O(n^{-\rho_{\min}}(\lg n)^{(d-1)(\rho_{\min}+1)})$ on the error of using discrete blending splines to approximate $\mathring{H}^\rho(\Omega)$-functions in the L_2-norm. But the nth minimal error for this approximation problem is proportional to $(\lg n)^{(k^*-1)(\rho_{\min}+\beta(n))}n^{-\rho_{\min}}$, where k^* is the number of minimal components in ρ and $\beta(n) \in [0,1]$. This implies one of two possibilities: either discrete blending spline approximation is not an optimal algorithm (even for the L_2-approximation problem) or the estimate given in Lemma 3.3 is pessimistic. □

Hence we have found reasonably tight bounds on the nth minimal error. The next task is to find an upper bound on the cost of using the MFEM to compute an ε-approximation, said upper bound being reasonably close (i.e., to within a polylogarithmic factor of $1/\varepsilon$) to the lower bound given in Theorem 3.1. Our approach follows that of [**12**, Section 6].

THEOREM 3.4. *Let $\varepsilon > 0$ and $d \in \mathbb{P}$ be given.*

1. *Suppose that $\varepsilon \ge \pi^{-(|\rho|+1)}d^{-1}$. The error of the zero algorithm $\phi_{\text{zero}} \equiv 0$ (which uses the zero information $N_{\text{zero}} \equiv 0$) is at most ε. Hence*

$$\text{comp}(\varepsilon, d) = \text{cost}(\phi_{\text{zero}}, N_{\text{zero}}) = 0 \qquad \textit{for } \varepsilon \ge \pi^{-(|\rho|+1)}d^{-1}.$$

2. *Suppose that $d = 1$. Then*

$$e(\phi_{q,1}, N_{q,1}) \le \varepsilon$$

for

$$q = \left\lceil \frac{1}{\rho_1} \ln \frac{(1+\pi^{-1})C_{\rho_1}}{\varepsilon} \right\rceil.$$

Hence

$$\text{comp}(\varepsilon, 1) \le \text{cost}(\phi_{q,1}, N_{q,1}) \le 2(c(1)+2)\left(\frac{(1+\pi^{-1})C_{\rho_1}}{\varepsilon}\right)^{1/\rho_1}.$$

3. *Suppose that $d \ge 2$ and $\varepsilon \le \pi^{-(|\rho|+1)}d^{-1}$. Recall that*

$$C_\rho = \max_{1 \le i \le d} C_{\rho_i}$$

and

$$H_\rho = \max\left\{ \left(\frac{2}{\pi}\right)^{\rho_{\min}}, (1+2^{\rho_{\max}})C_\rho \right\}.$$

Let

$$h = h(\varepsilon, d) = \frac{eH_\rho}{\rho_{\min}\ln 2}\left(\frac{C_\rho(1+\pi^{-1})}{\varepsilon\sqrt{2\pi(d-1)}}\right)^{1/(d-1)}.$$

Recursively define the sequence $\{t_k\}_{k=0}^{\infty}$ *as*

$$t_k = \begin{cases} \ln(ht_{k-1}) & \text{if } k \geq 1, \\ \dfrac{e\ln h}{e-1} & \text{if } k = 0. \end{cases}$$

Pick any $k \in \mathbb{N}$*, and let*

$$q = q_{\varepsilon,d} = \left\lceil \frac{t_{k+1}(d-1)}{\rho_{\min}\ln 2} \right\rceil.$$

Define

$$\alpha_0(d) = \left(\frac{1}{2\pi(d-1)}\right)^{(\rho_{\min}+1)/(2\rho_{\min})} \frac{2}{\rho_{\min}\ln 2},$$

$$\alpha_2 = \frac{e^2 H_\rho^{1/(\rho_{\min}+1)}}{(e-1)\rho_{\min}\ln 2}\left(\frac{1}{2}\right)^{\rho_{\min}/(\rho_{\min}+1)},$$

$$\alpha_1 = \alpha_2 \cdot \ln\frac{eH_\rho}{\rho_{\min}\ln 2}.$$

Then

$$e(\phi_{q,d}, N_{q,d}) \leq \varepsilon$$

and

$$\operatorname{comp}(\varepsilon, d) \leq \operatorname{cost}(\phi_{q,d}, N_{q,d}) \leq (c(d)+2)\alpha_0(d) \times \left(\alpha_1 + \alpha_2\frac{\ln\sqrt{\dfrac{1}{2\pi(d-1)}} + \ln\dfrac{C_\rho(1+\pi^{-1})}{\varepsilon}}{d-1}\right)^{\frac{(\rho_{\min}+1)(d-1)}{\rho_{\min}}} \times \left(\frac{C_\rho(1+\pi^{-1})}{\varepsilon}\right)^{1/\rho_{\min}}.$$

Hence, there exists $b_{\rho,d} \in [(k^*-1)/(2\rho_{\min}), d-1]$ *such that*

$$\operatorname{comp}(\varepsilon, d) = \Theta\left(\left(\frac{1}{\varepsilon}\right)^{1/\rho_{\min}}\left(\ln\frac{1}{\varepsilon}\right)^{b_{\rho,d}}\right).$$

PROOF. To prove part 1, we recall that the error of the zero algorithm equals the operator norm $\|S\|_{\mathcal{L}(\mathring{H}^\rho(\Omega),H_0^1(\Omega))}$. To calculate this, we expand an arbitrary $f \in \mathring{H}^\rho(\Omega)$ as in (3.7), with $u = Sf$ given by (3.8). We then have

$$\begin{aligned}\|Sf\|^2_{H_0^1(\Omega)} &= \pi^{-2} \sum_{\alpha\in\mathbb{P}^d} \frac{c_\alpha^2}{(\alpha_1^2+\cdots+\alpha_d^2)^2} \le \pi^{-2} d^{-2} \sum_{\alpha\in\mathbb{P}^d} c_\alpha^2 \alpha^{2\rho} \\ &= \pi^{-2(|\rho|+1)} d^{-2} \sum_{\alpha\in\mathbb{P}^d} c_\alpha^2 \alpha^{2\rho} = \pi^{-2(|\rho|+1)} d^{-2} \|f\|_{\mathring{H}^\rho(\Omega)},\end{aligned}$$

see (3.10). Since this bound is sharp (choose $f = z_{\mathbf{1}}$), we find that

$$\|S\|_{\mathcal{L}(\mathring{H}^\rho(\Omega),H_0^1(\Omega))} = \pi^{-(|\rho|+1)} d^{-1}.$$

Hence the zero algorithm yields an ε-approximation for any $\varepsilon \le \pi^{-(|\rho|+1)} d^{-1}$, as claimed.

To prove part 2, we need only use Theorem 3.2 to see that

$$e(\phi_{q,1}, N_{q,1}) \le (1+\pi^{-1}) C_{\rho_1} 2^{-q\rho_1}.$$

It now follows that $e(\phi_{q,1}, N_{q,1}) \le \varepsilon$ with q as given. Using (3.1), we see that

$$\operatorname{cost}(\phi_{q,d}, N_{q,d}) \le (c(1)+2)2^q.$$

We now turn to the proof of part 3, following the approach used in the proof of [**12**, Theorem 1]. Let $d \ge 2$ and $\varepsilon \le \pi^{-(|\rho|+1)} d^{-1}$. Let $x = q/(d-1)$. Since $m! \ge (m/e)^m \sqrt{2\pi m}$, we have

$$\binom{q}{d-1} \le \frac{q^{d-1}}{(d-1)!} \le \frac{(xe)^{d-1}}{\sqrt{2\pi(d-1)}}. \tag{3.14}$$

From Theorem 3.2, we find that $e(\phi_{q,d}, N_{q,d}) \le \varepsilon$ if

$$\frac{(xe)^{d-1}}{\sqrt{2\pi(d-1)}} \le \frac{\varepsilon}{(1+\pi^{-1}) C_\rho H_\rho^{d-1} 2^{-q\rho_{\min}}}. \tag{3.15}$$

We change variables, letting $t = x\rho_{\min} \ln 2$. Then (3.15) may be rewritten as

$$t \ge \ln t + \ln h, \tag{3.16}$$

which is a sufficient condition for $e(\phi_{q,d}, N_{q,d}) \le \varepsilon$. From the definition of H_ρ, we know that $C_\rho \ge \pi^{-\rho_{\min}}$. Hence we have

$$\varepsilon \le \pi^{-(\rho_{\min} d+1)} \le C_\rho \pi^{-1} \left(\frac{H_\rho}{2^{\rho_{\min}}}\right)^{d-1} \le C_\rho \pi^{-1} \left(\frac{eH_\rho}{2^{\rho_{\min}}}\right)^{d-1} \sqrt{\frac{1}{2\pi(d-1)}},$$

see [**12**, p. 29]. From this, it follows that

$$h \ge \frac{2^{\rho_{\min}}}{\rho_{\min} \ln 2} > e.$$

Using this inequality, it is easy to prove (via induction) that $t_k \ge \ln t_k + \ln h$ for all $k \in \mathbb{N}$, with t_k monotonically decreasing (as $k \to \infty$) to the unique solution t^* of the equation $t^* = \ln t^* + \ln h$. In what follows, we fix a choice for k. Let $q = q_{\varepsilon,d}$, with $q_{\varepsilon,d}$ as in the statement of the Theorem. Since (3.16) holds for $t = t_k$, it now follows that $e(\phi_{q,d}, N_{q,d}) \le \varepsilon$.

To complete the proof of (3), we let

$$q = q_{e,d} \le 1 + \frac{t_{k+1}(d-1)}{\rho_{\min} \ln 2}$$

and then we calculate an upper bound on $n(q,d) = \operatorname{card} N_{q,d}$. Using (3.1), (3.14), and (3.15), we see that

$$n(q,d) \le 2^{q-d+1}\binom{q-1}{d-1} = 2^{q-d+1}\frac{q-d+1}{q}\binom{q}{d-1} \le \frac{2^{q(\rho_{\min}+1)}\varepsilon}{(1+\pi^{-1})C_\rho H_\rho^{d-1}2^{d-1}}.$$

Substituting the upper bound for q into this inequality, we find (after some calculations) that

$$n(q,d) \le \frac{2^{d-1}}{H_\rho^{d-1}}\left(\frac{eH_\rho t_k}{\rho_{\min}\ln 2}\right)^{\frac{(\rho_{\min}+1)(d-1)}{\rho_{\min}}} \times \left(\frac{1}{\sqrt{2\pi(d-1)}}\right)^{\frac{\rho_{\min}+1}{\rho_{\min}}}\left(\frac{C_\rho(1+\pi^{-1})}{\varepsilon}\right)^{1/\rho_{\min}}.$$

Since $t_k \le t_0$, this inequality also holds when t_k is replaced by t_0. After a few more calculations, we find that

$$n(q,d) \le \alpha_0(d)\left(\frac{C_\rho(1+\pi^{-1})}{\varepsilon}\right)^{1/\rho_{\min}} \times \left(\alpha_1 + \alpha_2 \frac{\ln\sqrt{\frac{1}{2\pi(d-1)}} + \ln\frac{C_\rho(1+\pi^{-1})}{\varepsilon}}{d-1}\right)^{\frac{(\rho_{\min}+1)(d-1)}{\rho_{\min}}}.$$

Since $\phi_{q,d}$ is a linear algorithm using information $N_{q,d}$, we immediately get the desired bound on $\operatorname{cost}(\phi_{q,d}, N_{q,d})$. □

As in [**12**, p. 32], we can simplify the form of the upper bound in (3) of Theorem 3.4.

THEOREM 3.5. *Let*

$$\beta_1 = \alpha_0(2)(C_\rho(1+\pi^{-1}))^{1/\rho_{\min}},$$
$$\beta_2 = \alpha_1 + \alpha_2 \ln\left(\frac{C_\rho(1+\pi^{-1})}{\sqrt{2\pi}}\right),$$
$$\beta_3 = \alpha_2.$$

Then

$$\operatorname{comp}(\varepsilon,d) \le \operatorname{cost}(\phi_{q,d},N_{q,d}) \le \beta_1(c(d)+2)\left(\beta_2 + \beta_3\frac{\ln(1/\varepsilon)}{d-1}\right)^{\frac{(\rho_{\min}+1)(d+1)}{\rho_{\min}}}\left(\frac{1}{\varepsilon}\right)^{1/\rho_{\min}}.$$

Here $\alpha_0(2)$, α_1, α_2, and q are as in Theorem 3.4. □

This is the bound given in the Introduction to this paper.

REMARK: *The case $\rho = \mathbf{1}$.* Let us exhibit the bound given by Theorem 3.5 when $\rho = \mathbf{1}$. Note that for this special case, our MFEM uses discrete blended spaces of piecewise linear splines. We find that

$$\begin{aligned} C_{\mathbf{1}} &\doteq 0.318310, \\ H_{\mathbf{1}} &\doteq 0.954930, \\ \alpha_0(2) &\doteq 0.459224, \\ \alpha_1 &\doteq 2.71954, \\ \alpha_2 &\doteq 2.05964. \end{aligned}$$

Hence the MFEM computes an ε-approximation with cost at most

$$0.192705\,(c(d)+2)\left(-0.961691 + 2.05964\frac{\ln(1/\varepsilon)}{d-1}\right)^{2(d+1)}\left(\frac{1}{\varepsilon}\right). \qquad \square$$

4. Strong tractability results

In this section, we show that the Poisson problem is strongly tractable when standard information is permissible. We use the notation (and techniques) in [**12**].

Recall the definition of $\alpha_0(d)$, α_1, and α_2 in Theorem 3.4. Using the result of Theorem 3.4, we see that it is no loss of generality to restrict our attention to the case of $\varepsilon \le \pi^{-(|\rho|+1)}$ and $d \ge 2$.

Let $q^{**} \in (0,1)$ be the unique solution of the equation

$$q\alpha_1 = 1 + \ln q.$$

Let

$$\gamma_1 = \alpha_1 + \alpha_2 \ln \pi$$

and

$$\gamma_2 = \pi^{\alpha_2 \rho_{\min}}.$$

Define

$$q^* = \begin{cases} \dfrac{\ln \gamma_1}{\ln \gamma_2} & \text{if } \gamma_1 \ln \gamma_1 \ge \ln \gamma_2, \\ q^{**} & \text{if } \gamma_1 \ln \gamma_1 < \ln \gamma_2. \end{cases}$$

Let

$$p = 1/\rho_{\min} + \alpha_2(\rho_{\min}+1)q^*.$$

Finally, let

$$K = \max\{2[C(1+\pi^{-1})]^{\rho_{\min}}, K_1\},$$

where

$$K_1 = \alpha_0(2)[C_\rho(1+\pi^{-1})\pi^{\rho_{\min}}]^{\rho_{\min}}\left(\frac{1}{\pi}\right)^{p\rho_{\min}} \times \max\left\{1, \left(\frac{C_\rho(1+\pi^{-1})}{\sqrt{2}}\pi^{\rho_{\min}-1/2}\right)^{\alpha_2(\rho_{\min}+1)/\gamma_1}\right\}.$$

THEOREM 4.1. *For $d \geq 2$ and $0 < \varepsilon \leq \pi^{-(|\rho|+1)}$, define q as in part (3) of Theorem 3.4. Then*

$$e(\phi_{q,d}, N_{q,d}) \leq \varepsilon$$

and

$$\mathrm{comp}(\varepsilon,) \leq \mathrm{cost}(\phi_{q,d}, N_{q,d}) \leq (c(d)+2)K\left(\frac{1}{\varepsilon}\right)^p.$$

PROOF SKETCH. We merely follow the steps used in deriving [**12**, Theorem 2] from [**12**, Theorem 1], except that we start with our own Theorem 3.4. Note that this is possible since our Theorem 3.4 can be rewritten as a special form of [**12**, Theorem 1]. That is, the error formula in (3) of our Theorem 3.4 has the form of [**12**, Theorem 1], with appropriate changes in parameter values. □

Hence the algorithm $\phi_{q,d}$, with q as given, is a strongly polynomial time algorithm, and the strong exponent of the problem is at most p.

REMARK: *The case $\rho = \mathbf{1}$ (continued).* We now use the case $\rho = \mathbf{1}$ to illustrate Theorem 4.1. We find that

$$\begin{aligned}
\gamma_1 &\doteq 10.3268,\\
\gamma_2 &\doteq 120.958,\\
q^* &\doteq 0.486867,\\
p &\doteq 5.07911,\\
K_1 &\doteq 2.50506 \times 10^{-3},\\
K &\doteq 0.839262.
\end{aligned}$$

Hence

$$\mathrm{comp}(\varepsilon, d, \mathbf{1}) \leq 0.839262\,(c(d)+2)\left(\frac{1}{\varepsilon}\right)^{5.07911}.$$

Hence the algorithm $\phi_{q,d}$ is a strongly polynomial time algorithm, and our problem is strongly tractable, with a strong exponent of at most 5.07911. □

We close this paper by noting that for any multi-index $\rho \in \mathbb{P}^d$, we have $\rho \geq \mathbf{1}$. Hence the class F of problem elements is always a subset of the unit ball of $\mathring{H}^{\mathbf{1}}(\Omega)$. Thus for any $\rho \in \mathbb{P}^d$, we have

$$\mathrm{comp}(\varepsilon, d, \rho) \leq \mathrm{comp}(\varepsilon, d, \mathbf{1}) \leq 0.839262\,(c(d)+2)\left(\frac{1}{\varepsilon}\right)^{5.07911}.$$

Hence the problem is strongly tractable for any ρ.

References

1. Baszenski, G. and Delvos, F. J., *Remainders for boolean interpolation*, Progress in Approximation Theory (P. Nevai and A. Pinkus, eds.), Academic Press, Boston, 1991, pp. 31–41.
2. ———, *Multivariate boolean trapezoidal rules*, Approximation, Probability, and Related Fields (G. Anastassiou and S. T. Rachev, eds.), Plenum Press, New York, 1994, pp. 109–117.
3. Delvos, F. J. and Schempp, W., *Boolean Methods in Interpolation and Approximation*, Pitman, Harlow, 1989.
4. Garey, M. R. and Johnson, D. S., *Computers and intractability*, Freeman, San Francisco, 1979.
5. Novak, E., *Optimal error bounds for adaptive and nonadaptive numerical methods*, in preparation.

6. Novak, E. and Ritter, K., *High dimensional integration based on Smolyak's construction applied to the Clenshaw-Curtis rule*, in preparation.
7. Smolyak, S. A., *Quadrature and interpolation formulas for tensor products of certain classes of functions*, Dokl. Akad. Nauk SSSR **4** (1963), 240–243.
8. Temlyakov, V. N., *Approximate recovery of periodic functions of several variables*, Math. USSR Sbornik **56** (1987), 249–261.
9. ______, *Approximation of functions with bounded mixed derivative*, Proc. Steklov Inst. Math. **178** (1989), 1–121.
10. ______, *On a way of obtaining lower estimates for the errors of quadrature formulas*, Math. USSR Sbornik **71** (1990), no. 1, 247–257.
11. Traub, J. F., Wasilkowski, G. W., and Woźniakowski, H., *Information-Based Complexity*, Academic Press, New York, 1988.
12. Wasilkowski, G. W. and Woźniakowski, H., *Explicit cost bounds of algorithms for multivariate tensor product problems*, J. Complexity **11** (1995), no. 1, 1–56.
13. Werschulz, A. G., *What is the complexity of related elliptic, parabolic, and hyperbolic problems?*, Math. Comp. **47** (1986), 461–472.
14. ______, *The Computational Complexity of Differential and Integral Equations: An Information-Based Approach*, Oxford University Press, Oxford, 1991.
15. ______, *The complexity of two-point boundary-value problems with piecewise analytic data*, J. Complexity **10** (1994), 367–383.
16. ______, *The complexity of multivariate elliptic problems with analytic data*, J. Complexity **11** (1995), 154–173.
17. Woźniakowski, H., *Average case complexity of linear multivariate problems: I. Theory*, J. Complexity **8** (1992), 337–372.
18. ______, *Average case complexity of linear multivariate problems: II. Applications*, J. Complexity **8** (1992), 373–392.
19. ______, *Tractability and strong tractability of linear multivariate problems*, J. Complexity **10** (1994), 96–128.

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