STRATIFIED GRAPHS

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ABSTRACT. Two imbeddings of a graph G are considered to be adjacent if the second can be obtained from the first by moving one or both ends of a single edge within its or their respective rotations. Thus, the collection of imbeddings of G may be regarded as a "stratified graph", denoted SG. The induced subgraph of SG on the set of imbeddings into the surface of genus k is called the "kth stratum", and one may observe that the sequence of stratum sizes is precisely the genus distribution for the graph G. It is proved that the stratified graph is a complete isomorphism invariant for the category of graphs whose minimum valence is at least three and that the spanning subgraph of SG corresponding to moving only one edge-end is a cartesian product of graphs whose underlying isomorphism types depend only on the valence sequence for G.

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1. Introduction

The set of imbeddings of a graph G admits a natural concept of adjacency between imbeddings. We thereby obtain a graded "edge-colored" graph, denoted SG, that we call the "stratified graph" for G. A few preliminaries and the formal definition of SG appear in this section, shortly below.

The stratified graph SG is very much larger than G itself. Indeed, each point of SG typically has more neighbors than G has vertices. Some of the structure of such a neighborhood is described by Cayley graphs we call "circular arrangement graphs", which we examine in Section 2. In Section 3, we study the general structure of the neighborhood of any point in SG with particular attention paid to cliques. In Section 4, we show how to reconstruct a graph from a neighborhood of any point in the colored stratified graph, thereby established the colored stratified graph as a complete invariant of isomorphism type over the category of all graphs of minimum valence at least 3. The uncolored stratified graph is considered in Section 5 and related to the medial graph of an imbedding. The cubic case is analyzed completely and is shown to provide a large supply of constant link (Zykov regular) graphs.

Beyond the inherent topological interest in the formulation of this non-superficial complete invariant for isomorphism type, one might well wonder about the usefulness of something so large in isomorphism testing. In Section 6 we illustrate how two graphs might be "nearly isomorphic", yet distinguishable by accessible properties of their stratified graphs.

Throughout this paper, a <u>graph</u> is "simplicial", that is, it has no multiple adjacencies or self-adjacencies. It is taken to be connected, unless one can readily infer otherwise from the immediate context.

The closed orientable surface of genus j is denoted S_j. By an imbedding we mean a cellular imbedding into a closed orientable surface. In general, the methods described here are readily adaptable to the non-orientable surfaces and to the collection of all closed surfaces.

In the present exposition, it is assumed that the reader is familiar with the fundamentals of topological graph theory, as described by Gross and Tucker [1987], or -- with minor terminological differences -- by White [1984].

We regard two imbeddings as adjacent if one can be obtained from the other either by moving an edge-end in the rotation at its vertex to somewhere else in that rotation, or by moving both ends of the same edge within their respective rotations. The set of embeddings of a graph G may be regarded as the points of a stratified graph SG, in which we visualize a labeling of each point by the genus of the corresponding imbedding surface for G.

We think of the point-labels as altitudes. The two kinds of imbedding-adjacency (i.e., one edge-end or both) are called VM-lines and EM-lines, for "vertex modification" and "edge modification", respectively. (We like to draw them in the mnemonic colors "violet" and "ecru".) For clarity, we refer to "vertices" and "edges" in G, and to "points" and "lines" in SG.

The induced subgraph in SG on all the G-imbeddings into S_j is called the *jth stratum* for (the imbedding system of) the graph G and is denoted S_jG. Lines of SG that lie within a single stratum of G are called *level lines*. All other lines of SG run between consecutive strata and are called *transverse lines* (or *transversals*).

The size of the *jth* stratum is denoted $g_j(G)$, or simply g_j, if there is only one graph whose imbeddings are under consideration. Thus, the sequence

$$g_0, g_1, g_2, \cdots$$

of stratum sizes is just the genus distribution for the graph G. Conversely, we observe that the problems of describing the structure of the stratified graph SG is precisely a refinement of the problem of calculating the genus distribution of G.

Thus, stratified graphs are a proper member of the hierarchy of graph invariants that correspond to distributional information about the entire system of cellular embeddings of a graph, described by Gross and Furst [1988]. There are already several calculations of formulas for genus distribution, region-size distribution, and other invariants at the low end of that hierarchy.

The first such calculation for any infinite classes of graphs was the result of Furst, Gross, and Statman [1989] establishing the genus distributions of closed-end ladders and of "cobblestone paths". Gross, Robbins, and Tucker [1989] have derived the genus distributions of bouquets, by using a formula of Jackson [1987] concerning representations of the symmetric group.

Rieper's thesis [1987] includes a computation of the region-size distributions for bouquets and several other significant results, based on enumerative methods of Redfield [1927]. Mull, Rieper, and White [1987] enumerated the congruence classes of imbedding distributions of wheels and of complete graphs. Rieper [1987] has also calculated the stratified graphs for cobblestone paths.

2. On circular arrangement graphs and the VM structure of the stratified graph

Two cyclic permutations on d symbols are considered to be adjacent if the first can be transformed into the second by moving a single symbol. For instance, if we move the symbol i within the "standard d-cycle"

$$C = (1\ 2\ \dots\ d)$$

to a new location preceding the sumbol j, we obtain the adjacent d-cycle

$$(1 \dots i\text{-}1\ i\text{+}1 \dots j\text{-}1\ i\ j \dots d) \quad \text{if} \quad i < j$$

or

$$(1 \dots j\text{-}1\ i\ j \dots i\text{-}1\ i\text{+}1 \dots d) \quad \text{if} \quad i > j.$$

Under this notion of adjacency the collection of d-cycles form what we call the circular arrangement graph on d symbols, denoted CA_d. Circular arrangement graphs are highly symmetric: they are Cayley graphs. Recall that the Cayley graph for the group A given a generating set X has A as vertex set and edges between a and ax for all a in A and all x in X. Left multiplication by A on the vertices of any Cayley graph for A gives a subgroup of the automorphism group of the Cayley graph, and is transitive and fixed-point free on the vertex set. In fact, the existence of such an action of a group A on a graph G makes G a Cayley graph for A.

THEOREM 2.1 **The circular** arrangement graph CA_d is a Cayley graph for the **full symmetric** group $\sum_{d-1}$ on $d\text{-}1$ symbols with generating set the collection of all cycles of consecutive integers, that is cycles of the form $(i\ i\text{+}1\ \dots\ j\text{-}1\ j)$, $1 \le i < j < d$, together with all powers of the standard $d\text{-}1$ cycle $(1\ 2\ \dots\ d\text{-}1)$.

Proof. Write each d-cycle so that the symbol d appears last. Then each d-cycle written in this standardized cycle form can be described uniquely by a permutation σ of the symbols $1, 2, \ldots, d-1$, which tells the order in which these symbols occur in the d-cycle. We do not think of σ itself in cycle form, but rather as a rearrangement of the symbols $1, 2, \ldots, d-1$. Suppose that the n-cycles C_1 and C_2 are adjacent by moving symbol i, where $i < d$. Then the corresponding permutations σ_1 and σ_2 agree as arrangements of the symbols $1, \ldots, d-1$ when the symbol i is deleted. If π is any other permutation of $1, \ldots, d-1$, then $\pi \sigma_1$ and $\pi \sigma_2$ agree as arrangements when the symbol $\pi(i)$ is deleted. Thus $\pi \sigma_1$ and $\pi \sigma_2$ correspond to adjacent d-cycles. Suppose instead that C_1 and C_2 are adjacent by moving the symbol d. If C_1 is written as a cycle in standard form with d last and if C_2 is then obtained from C_1 by moving d to the position immediately before the ith symbol in the standard cycle form for C_1, then the arrangement σ_2 for C_2 is obtained from the arrangement σ_1 for C_1 by cyclicly shifting right $d-i$ times. If π is any permutation of $1, \ldots, d-1$, then $\pi \sigma_2$ as an arrangement is also obtained from $\pi \sigma_1$ by cyclicly shifting right $d-i$ times. Therefore $\pi \sigma_1$ and $\pi \sigma_2$ again correspond to adjacent d-cycles.

We conclude that the vertices of CA_d can be identified with elements of the full symmetric group $\sum_{d-1}$ and that left multiplication by an element of $\sum_{d-1}$ is a graph automorphism of CA_d. It follows that CA_d is a Cayley graph for $\sum_{d-1}$. The associated generating set is recovered by looking at vertices of CA_d adjacent to the identity element of $\sum_{d-1}$, which corresponds to the standard d-cycle C. Moving symbol i to the position immediately preceding j in C is achieved, when $i < j$, by applying the inverse of the cycle of consecutive integers $(i\ i+1\ \ldots\ j-1)$, and when $i > j$ by applying the cycle $(j\ \ldots\ i-1\ i)$. Cyclic right shifts of $1\ 2\ \ldots\ d-1$ are achieved by applying powers of the standard $(d-1)$ - cycle $(1\ 2\ \ldots\ d-1)$. □

The spanning subgraph of SG containing only the VM-lines is called the VM-subgraph. The proof (omitted) of the following structure theorem is an exercise in definitions.

THEOREM 2.2 Let G be a graph with valence sequence $d_1, \ldots, d_n$. Then the VM-subgraph of SG is isomorphic (as a graph, neglecting altitude labels) to the cartesian produce of n circular arrangement graphs on $d_1, d_2, \ldots, d_n$ symbols, respectively. □

Theorem 2.2 raises the recognition problem for stratified graphs: which labelings of cartesian products of circular arrangement graphs are realizable as VM-subgraphs of stratified graphs? Since CA_3 is just the complete graph K_2 on two vertices, the case of 3-regular graphs is of particular interest: which labelings of the n-cube Q_n are isomorphic to the VM-subgraph of the stratified graph for a 3-regular graph?

3. Links of points in the stratified graph

If v is a vertex of a graph G, then the link of v is the subgraph of G induced by the set of all vertices adjacent to v (this does not include v itself). Given a point p in the statified graph SG, let $L(p)$ and $VL(p)$ denote, respectively, the link of p in SG and the link of p in the VM-subgraph of SG. Call $L(p)$ the total link of p and $VL(p)$ the VM link of p. The purpose of the sections of this paper following this section is to show how to recover an underlying graph G from the total link $L(p)$ of any point in the stratified graph SG. In order to do this, we must understand the adjacency structure of $L(p)$.

If two points of $L(p)$ are obtained from p by moving one or both ends of the same edge e, then those two points are adjacent to each other, again by moving ends of the edge e. Call such an adjacency or such a line in $L(p)$ standard. The structure of $L(p)$ would be reasonably easy to describe if all lines in $L(p)$ were standard: each edge e in G gives rise to a clique of points in $L(p)$ corresponding to all the embeddings q which agree with p except for the placement of the ends of edge e. Call such a clique an edge clique. Every line in $L(p)$ is in some edge clique. Two edge cliques share a point q if and only if the two edge e_1 and e_2 corresponding to those cliques are consecutive at some vertex v in the imbedding p, (that is, e_2 immediately follows e_1 in the rotation at vertex v or vice versa) and q is obtained from p by switching e_1 and e_2 at vertex v. Call q a switch point.

Unfortunately, there are extra adjacencies, that is, adjacencies which are not standard. For example, suppose that p, q, and r are imbeddings which agree at every vertex except vertex v, where the rotations are

$$p: \quad \cdots e_0 e_1 e_2 e_3 e_4 \cdots$$
$$q: \quad \cdots e_0 e_2 e_3 e_1 e_4 \cdots$$
$$r: \quad \cdots e_0 e_3 e_1 e_2 e_4 \cdots \quad .$$

Then q and r are both in $L(p)$, the former by moving e_1 and the latter by moving e_3. However, q and r are also adjacent to each other by moving e_2. This is not a standard adjacency in $L(p)$. Call it an extra adjacency of type 1.

Suppose instead that G has edges uv, vw, and wu, and that in the embedding p the edges uv and vw are consecutive at vertex v, that vw and wu are consecutive at vertex w, and that wu and uv are consecutive at u; call such a triangle in the imbedding p a consecutive triangle. Let q_u be the imbedding obtained from p by switching edges uv and vw at vertex v and by switching edges vw and wu at vertex w. Thus q_u is obtained from p by moving edge vw. Define q_v and q_w similarly. Then q_u and q_v agree at vertex w but differ at vertices u and v. Thus q_u and q_v are adjacent by moving the edge uv. This is an extra adjacency in $L(p)$. There are also extra adjacencies between q_v and q_w and between q_w and q_u. Call these type 2 extra adjacencies.

The following theorem shows that the two types of extra adjacencies just described are the only extra adjacencies in $L(p)$. To help in the analysis, let us call a point in $L(p)$ a VM point if it is VM-adjacent to p and a EM point otherwise.

THEOREM 3.1 The only extra adjacencies in $L(p)$ are type 1 or type 2.

Proof. Let q and r be points in $L(p)$. If q and r are VM points differing from p at vertices u and v respectively, $u \neq v$, then the only way they can be adjacent is if they are obtained from p by moving opposite ends of edge uv; that is, they are standardly adjacent.

Suppose that q and r are VM points differing from p at a single vertex v and that q and r are obtained from p by moving edges e_1 and e_3, respectively. Suppose in addition that q and r are adjacent by moving an edge e_2 that is different from e_1 or e_3; that is, q and r are joined by an extra line in $L(p)$. If e_2 were deleted, then q and r would be identical and would agree with p except for edges e_1 and e_3. Thus without e_2, we would have edges e_1 and e_3 consecutive at vertex v in p, say in the order $e_1\,e_3$, and also consecutive in q and r, but in the opposite order $e_3\,e_1$ (recall that in q only edge e_1 is moved and in r only edge e_3 is moved). Now consider the placement of edge e_2 at vertex v in p. If the order is $e_1 e_2 e_3$ in p, it must be $e_2 e_3 e_1$ in q since only e_1 moves and e_1 goes after e_3 in q. Similarly the order must be $e_3 e_1 e_2$ in r, since only e_3 moves this time and again e_1 goes after e_3. Therefore if the order is $e_1 e_2 e_3$ in p, we have an extra adjacency of type 1. Suppose the order at p is $e_2 e_1 e_3$ instead. Then the order in q must be $e_2 e_3 e_1$ since only e_1 moves and e_1 goes after e_3. But then q is adjacent to p by moving e_3. Since r is already adjacent to p by moving e_3, it follows that q and r are standardly adjacent by moving e_3. Similarly, if the order in p is $e_1 e_3 e_2$, then q and r are standardly adjacent by moving edge e_1.

Suppose that q is a VM point and r is an EM point. If q agrees with p except at the vertex u and r agrees with p except at v and w, where $v \neq u$ and $w \neq u$, then there can be no way of changing

q at all three vertices u, v, and w simultaneously. Thus q and r are not adjacent. If q agrees with p except at vertex u and r agrees with p except at u and v, then the only adjacency between q and r is a standard one obtained by moving the ends of edge uv. We conclude there are no extra adjacencies between VM and EM points.

Finally suppose that both q and r are EM points and that q and r are obtained from p by moving both ends of edges uv and wx respectively, where at least u, v, and w are distinct. Suppose in addition that q and r are adjacent. Then $x = u$ or $x = v$, since the movement of a single edge cannot change an imbedding at four distinct vertices. Assume $x = v$. By hypothesis q and r differ at both vertices u and w. Thus in order for q and r to be adjacent, they must agree at vertex v and there must be an edge uw, which can be moved to change the embedding q at u and w into the embedding r. Since q moves edge uv and r moves edges vw, in order for q and r to be the same at vertex v the edges uv and vw must be consecutive in p, q and r at v. Since q agrees with p at w but is adjacent to r by moving edge uw, it must be that imbedding r at vertex w is obtained from p not only by moving vw, as hypothesized, but also by moving uw. Therefore vw and uw are consecutive at vertex w in both p and r. Similarly, uv and uw are consecutive at vertex u in both p and q. Therefore uv, vw, and wu form a consecutive triangle and the extra adjacency is type 2. □

With Theorem 3.1 in hand, we know where all the lines in $L(p)$ come from. To recover G from $L(p)$, we will also need to use the coloring of lines at VM or EM. The following lemmas describe the structure of the VM link of p, namely VL(p).

LEMMA 3.2 The link of a vertex in CA_3 is a single vertex. The link of a vertex in CA_4 is a 4-cycle. The link of a vertex in CA_d, for $d > 4$, consists of d copies $H_1, H_2, \ldots, H_d$ of the complete graph K_{d-2} arranged in a circle so that H_i shares exactly one vertex with H_{i-1} and exactly one vertex with H_{i+1}, and, in addition, for each i there is an extra edge joining a vertex of H_i with a vertex of H_{i+2} (the joined vertices are not shared vertices). In particular, the link of a vertex in CA_d, for all $d > 2$, is connected and nonempty.

Proof. Consider the general case of $d > 4$ first. Since CA_d is vertex symmetric, we can just look at the lines of the standard d-cycle C. There are $d - 2$ different positions the symbol i can occupy in an arrangement of the symbols $1, \ldots, d-1$ other than position i itself. Thus the set of vertices in CA_d obtained from C by moving symbol i induce in the link of C a complete graph H_i. The subgraphs H_i and H_j share a vertex if and only if i and j are consecutive in cycle C, that is $j = i + 1$ or $i = j + 1$. The extra edge joining vertices in H_i and H_{i+2} is that corresponding to an extra adjacency of type 1.

For $d = 3$, clearly CA_3 is a two-vertex graph so the link of a vertex is a single vertex (technically, the description for $d > 4$ still holds since 3 copies of K_1 each sharing a vertex with the other is simply a single vertex). For $d = 4$, one might expect the link to be a 4-cycle together with both diagonals as the extra edges, but again the description requires the extra edges to be between vertices in H_i and H_{i+2} that are not shared with another H_j. When $d = 4$, each of the two vertices in

H_i is a shared vertex. Alternatively, one can check that the two vertices (2314) and (3124) joined by an extra edge, although apparently obtained by moving 1 and 3 respectively, are also obtained by moving 4 and hence are already standardly adjacent. □

The clique structure of $L(p)$ is complicated by extra adjacencies, but it is still possible to give a complete description. The extra lines of type 1 join vertices in edge cliques which do not share a switch point. Hence each of these lines is a clique of size two. If t is a consecutive triangle in the imbedding p, then the three switch points in $L(p)$ corresponding to t form a standard triangle, and the three extra adjacencies of type 2 created by t form a second triangle in $L(p)$. Finally, a third type of triangle is created in $L(p)$ by t among any two EM points q and r joined by an extra line of type 2 together with the switch point shared by the edge cliques containing q and r. Call these three types of triangles in $L(p)$, respectively, the <u>VM triangle</u>, the <u>EM triangle</u> and the <u>VEM triangles</u> (there are three of them) created by the consecutive triangle t. As long as G is not K_4, it is impossible to have a configuration of four consecutive triangles in the imbedding p based on four vertices in G. It follows that each of the triangles created by a consecutive triangle is not contained in a larger clique. Thus each of these triangles is a clique by itself, and every clique of size larger than 3 is an edge clique. We summarize this discussion in the following theorem.

THEOREM 3.3 Let G be a graph of minimum valence 3 and p a point in SG. Then the cliques of $L(p)$, listed by size, are as follows:

1) there are no cliques of size 1;

2) every clique of size 2 is an extra line of type 1;

3) every clique of size 3 is a VM, EM, or VEM triangle created by a consecutive triangle in p, or the edge clique of an edge in G joining two vertices of valence 3;

4) all cliques of size 4 or greater are edge cliques. □

4. The complete invariance of colored stratified graphs

We will show that a graph G can be recovered in a canonical way from the link of any point in the stratified graph SG, if we are given the coloring of lines of SG as VM or EM. An edge uv in a graph G is contracted by deleting the edge, identifying u and v, and removing any resulting multiple adjacencies.

THEOREM 4.1 Let G be any graph of minimum valence 3 and let p be any point in the stratified graph SG. Then G is isomorphic to the graph obtained from the link $L(p)$ by deleting all EM points and then contracting all VM lines.

Proof. The link of a vertex in a cartesian product is the disjoint union of the links of the coordinates of that vertex in the factors of the cartesian product. Therefore the VM link $VL(p)$ consists of n disjoint graphs of the form described in Lemma 2.3, one for each of the n vertices of G. Since each of these graphs is connected, again by Lemma 2.3, each component of $VL(p)$ corresponds to a vertex of G. Moreover, there is a line joining points in different components of $VL(p)$ if and only if the corresponding vertices of G are joined by an edge (extra lines of type 1 only join points in the same component of $VL(p)$ and extra lines of type 2 only join EM points in $L(p)$). Thus if EM points are deleted and VM lines contracted, each component of $VL(p)$ will contract to a single point, corresponding to a single vertex of G, and the points will be joined by lines if and only if the corresponding vertices in G are joined by edges. □

COROLLARY 4.2 The VM/EM colored stratified graph is a complete isomorphism invariant for simplicial graphs of minimum valence 3. □

Although the interest of this paper is simplicial graphs, one can also recover multiple adjacencies and self adjacencies. The number of points in a component of VL(p) determines the degree of the corresponding vertex of G. The number of EM lines joining different components of VL(p) determines the number of edges joining the corresponding vertices of G. Once the degree of each vertex and number of multiple adjacencies have been determined, the number of self-adjacencies at each vertex is determined. The simplicial structure of G is already determined by Theorem 4.1. We therefore have the following corollary for non-simplicial graphs.

COROLLARY 4.3 The VM/EM colored stratified graph is a complete isomorphism invariant for all graphs of minimum valence 3. □

5. The uncolored stratified graph

We would like to be able to recover G from its stratified graph SG without using the VM coloring, but simply from the adjacency structure alone. In this section, we show how this can be done for cubic graphs, that is regular simplicial graphs of valence 3. We also consider the case when G has minimum valence 4.

Suppose that G is a cubic graph and that p is a point in SG. Each component of VL(p) consists of a single point, which means in Theorem 4.1 no edge contractions are necessary. Thus G is isomorphic to the subgraph of L(p) induced by the points of VL(p). The trouble is that without the VM coloring, we cannot directly detect which points in L(p) are in VL(p). Nevertheless, the entire adjacency structure of L(p) is not difficult to describe. Each EM point in L(p) is standardly adjacent to two VM points: each edge-clique in L(p) consists of a triangle containing one EM point (corresponding to moving both ends of the edge) and two VM points (corresponding to moving either end of the edge). There are no extra lines of type 1. However, since every triangle in a cubic graph is a consecutive triangle, for every triangle t in G there is a triangle of extra lines of type 2 in L(p) joining the three EM points corresponding to moving both ends of each of the three edges of t. We can summarize this discussion as follows.

THEOREM 5.1 Let G be a cubic graph and p any point in SG. Then the link of p can be obtained from G by doubling every edge, inserting an extra vertex of valence 2 in each added edge, and for each triangle

t in G, adding a triangle joining the three new vertices on the doubled edges of triangle t. In particular, if the graph G is not K_4, then G is the subgraph of $L(p)$ induced by all points of valence 6 which are themselves adjacent to at least three points of valence 6.

Proof. We must verify only the last statement about points of valence 6. Each VM point in $L(p)$ is adjacent to three other VM points and to three EM points. Thus each point in $VL(p)$ has valence 6 and is adjacent to at least three other points of valence 6. Each EM point has valence 2, 4, 6 depending on whether the edge of G moved by that EM point lies on no, one, or two triangles in G. However, an EM point is adjacent to any two VM points. Therefore, if an EM point of valence 6 is adjacent to three or more points of valence 6, at least one of these points must be EM as well. The resulting adjacent EM points, each of valence 6, must correspond to two edges of G, each of which lies in two triangles of G and both of which lie on the same triangle of G. It is easily verified that this can only happen if G is K_4. Therefore the points of valence 6, which are adjacent to at least three points of valence 6, are precisely the points of $VL(p)$. □

COROLLARY 5.2 If G is cubic, then the link of every point in p is the same. □

Thus for cubic graphs, the stratified graphs SG, all have constant link and are "Zykov regular" graphs (Brown and Connelly [1973].) In fact, every triangle-free cubic graph G is the link of some finite, constant-link graph, after each edge of G is doubled and a vertex of valence two is inserted in each added edge.

COROLLARY 5.3 The uncolored, unlabeled stratified graph is an isomorphism invariant for cubic graphs.

Proof. By Theorem 5.1, we need only distinguish the stratified graph for K_4 from the stratified graphs for other cubic graphs. This is simply a matter of counting vertices: if G has n vertices and $m = 3n/2$ edges, then SG has $m + m$ vertices. Therefore K_4 is the only cubic graph whose stratified graph has 10 vertices. □

For noncubic graphs, we are unable to recover G from the link of a point p in SG, but we can recover some interesting information about G and the imbedding p itself. Given any graph G, the <u>line graph</u> LG for G has a vertex for each edge in G and an adjacency between two vertices if and only if the corresponding edges in G are incident to the same vertex of G. Given an imbedding p of G, the <u>medial graph</u> MG_p has a vertex for each edge in G and an adjacency between two vertices if and only if the corresponding edges in G are consecutive at some vertex of G in the imbedding p. Clearly MG_p is a subgraph of LG, and is regular of valence p. If G is cubic, then $MG_p = LG$ for every imbedding p of G. In general, however, MG_p depends on the imbedding p. It also depends on G, but does not determine G; for example, the medial graphs of an imbedding and its dual imbedding are isomorphic. For an interesting application of medial graphs to self-dual graphs, see Archdeacon and Richter [1989]. Our main theorem shows how to recover the medial graph MG_p from the link of a point p in the uncolored stratified graph SG, for most graphs G.

THEOREM 5.4 Let G be a graph of minimum valence 3 such that no triangle in G contains more than one vertex of valence 3. Then for any point $\underline{p}$ in SG, the medial graph MG_p is isomorphic to the graph that has a vertex for each clique in $L(p)$ of size greater than 3 and each clique of size three including a point of valence 2, and has an edge between two vertices if and only if the corresponding cliques share a point.

Proof. By Theorem 3.5, every clique in $L(p)$ of size greater than 3 is an edge clique. Conversely, every edge in G incident to a vertex of valence greater than 3 gives rise to a clique in $L(p)$ of size greater than 3. By Theorem 3.5, every clique in $L(p)$ of size 3 containing a point of valence 2 is an edge clique corresponding to an edge in G between two vertices of valence 3. Since by hypothesis e does not lie on a triangle in G, the point in $L(p)$ corresponding to moving both ends of edge e is not involved in any extra adjacencies and hence has valence two in $L(p)$. It follows that the edge clique for e has size 3 and contains a point of valence 2. We conclude that the vertices in the graph constructed in the statement of this theorem correspond to the edges of G. Since two edge cliques in $L(p)$ share a vertex if and only if the corresponding edges are consecutive at some vertex in the imbedding p, the constructed graph is the medial graph MG_p. □

We believe that the restriction in Theorem 5.4 on triangles and vertices of valence 3 is not necessary, and that even for general G the clique structure described in Theorem 3.5, can be used to identify which cliques of size 3 in $L(p)$ are edge cliques. On the other hand, we do not see how to recover the original graph G, not just the medial graph, from L(p) alone. It is conceivable that nonisomorphic graphs may have some isomorphic links in their uncolored stratified graphs. If that is the case, we cannot count alone on the local structure of the uncolored stratified graph SG to determine the isomorphism type of G. We nevertheless conjecture that the uncolored stratified graph is a complete isomorphism invariant.

6. Strata for Two "Nearly Isomorphic" Graphs

To draw an entire stratified graph would be quite laborious. After all, the number of imbeddings of an n-vertex graph might be about as large as $(n!)^n$, the average VM-valence about n^3, and the average EM-valence about n^4. Even to draw the strata tends to be a formidable task, and to compute the strata sequence of a graph is evidently more difficult than to compute the genus distribution, which is simply the sequence of strata sizes. However, if our objective is to distinguish isomorphism types, we cannot content ourselves with genus distributions.

Although Gross, Klein, and Rieper [1989] used elementary methods to construct arbitrarily many non-isomorphic 2-connected graphs with the same genus distribution, the construction of non-isomorphic 3-connected simplicial graphs with the same genus distribution was resistant until Rieper [1987] successfully used Redfield enumeration. Even if such examples were not known, the similarity in the genus distribution

$$2,\ 38,\ 24$$

of the circular ladder CL_3 with three rungs (a.k.a. $K_3 \times K_2$, see Gross and Furst [1987]) and the genus distribution

$$0,\ 40,\ 24$$

of the Mobius ladder ML_3 on three rungs (a.k.a. $K_{3,3}$, see ibid.) is disquieting, since one could not expect to distinguish the two easily with a small sample of imbeddings. Moreover, McGeoch [1987] has proved that, in general, circular ladders and Mobius ladders with the same number of rungs have nearly identical genus distributions. In particular, they have the same number of imbeddings in all surfaces of genus two or larger, and differ elsewhere only in that the circular ladder has two sphere imbeddings and the Mobius ladder none, but two fewer toroidal imbeddings than the Mobius ladder.

Having explained our motivation for examining such large objects, we now consider the VM-strata of CL_3 and of ML_3. As illustrated by Figure 4.1 and Figure 4.2, the VM-strata are overtly different in various readily apparent respects. Details of the derivations of these illustrations are omitted because, although numerous, they are not difficult.

References

1. D. Archdeacon and R.B. Richter [1989], The construction and classification of self-dual polyhedra, preprint.

2. M. Brown and R. Connelly [1973], On graphs with constant link, in New Directions in Graph Theory (F. Harary, ed.), Academic Press, New York.

3. M.L. Furst, J.L. Gross, and L.A. McGeoch [1988], Finding a maximum-genus graph imbedding, J. Assoc. Comp. Mach. 35, 523-534.

4. M.L. Furst, J.L. Gross, and R. Statman [1989], Genus distributions for two classes of graphs, J. Combin. Theory Ser. B 46, 22-36.

5. J.L. Gross and M.L. Furst [1987], A hierarchy for imbedding-distribution invariants of a graph, J. Graph Theory 11, 205-220.

6. J.L. Gross, E.W. Klein, and R.G. Rieper [1989], On the average genus of a graph, Tech. Report., Columbia University.

7. J.L. Gross, D.P. Robbins, and T.W. Tucker [1989], Genus distributions for bouquets of circles, J. Combin. Theory Ser. B, to appear.

8. J.L. Gross and T.W. Tucker [1987], Topological Graph Theory, Wiley-Interscience, New York, 1987.

9. D.M. Jackson [1987], Counting cycles in permutations by group characters, with an application to a topological problem, Trans. Amer. Math. Soc. 299, 785-801.

10. L.A. McGeoch [1987], Ph.D. Thesis, Carnegie-Mellon Univ.

11. B.P. Mull, R.G. Rieper, and A.T. White [1987], Enumerating 2-cell imbeddings of complete graphs, Proc. Amer. Math. Soc. 103, 321-330.

12. H. Redfield [1927], The theory of group reduced distributions, Amer. J. Math 49, 433-455.

13. R.G. Rieper [1987], Ph.D. Thesis, Western Michigan Univ.

14. A.T. White [1984], Graphs, Groups and Surfaces, North-Holland, Amsterdam.