Fostering Confidence and Competence in Early Childhood Mathematics Teachers

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Abstract

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This study aimed to increase efficacy for teaching mathematics in pre-service early childhood teachers through the presentation of five video lessons on topics in early childhood mathematics. Each lesson entailed reading a short essay on a topic related to children's mathematical thinking and then watching a short video of a child engaged in a relevant task and a clinical interview with an adult. This study also examined pre-service teachers' knowledge of mathematical development (KMD) and intellectual modesty, or the awareness of the limits of one's knowledge, as possible mediators of change in efficacy. Results showed that the video lessons did significantly increase efficacy for teaching mathematics, but that KMD and intellectual modesty were not significant mediators of the change in efficacy. In effect, confidence appeared disconnected from competence. Follow-up analyses revealed the importance of rich mathematical content within the videos in producing increased confidence and competence.

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For Monte, who can count to 1

Chapter 1: Introduction

"I'm not a math person," is something heard too frequently in U.S. elementary classrooms. Perhaps most troubling is that this sentiment is held not only by many students, but also by their teachers. In fact, there is consistent evidence that most early childhood teachers have limited knowledge both of mathematics and of the thinking strategies that children use to learn early childhood mathematics (Clements, Copple, & Hyson, 2002). As these teachers are responsible for children's first formal exposure to mathematics, this ignorance is problematic and serves as a barrier to high quality, and high quantity, early childhood mathematics instruction.

The National Council of Teachers of Mathematics (NCTM) has called attention to the importance of providing high quality early childhood mathematics instruction, saying that,

> Developing a solid mathematical foundation from pre-kindergarten…is essential for every child. In these grades, students are building beliefs about what math is, about what it means to know and do mathematics, and about themselves as mathematics learners. These beliefs influence their thinking about, performance in, and attitudes toward mathematics and decisions related to studying mathematics in later years. (NCTM, 2000, p. 98)

Thus, a critical question is: What can be done to promote high quality early childhood mathematics instruction?

A key route to improved early childhood mathematics instruction is through enhancing teachers' confidence and competence in teaching math (Kilpatrick et al., 2001). To do this, early childhood teacher training must focus not only on mathematics content and its concomitant pedagogical content knowledge, but also on teachers' dispositions towards math (Clements et al., 2004; Beilock et al., 2010), which is referred to as "efficacy" for teaching mathematics. This is an especially important focus in the pre-service training of early childhood teachers because the efficacy of this population has been found to be even lower than the already low efficacy levels found with in-service teachers, and because the pre-service training period tends to be marked by major fluctuations in teacher efficacy (Ross, 1994; Tschannen-Moran & Woolfolk-Hoy, 2007), making it a malleable period in efficacy development.

This study examines the effects of video-based methods on pre-service teachers' efficacy for teaching early childhood math. The methods involve a series of five video lessons in which a young child is interviewed about his or her knowledge and understanding of an early math concept. These videos demonstrate mathematical content, but also go beyond this to demonstrate knowledge of mathematical development (i.e., what students know mathematically and how they learn mathematics) and the nature of effective pedagogical tools for teaching early childhood mathematics. As such, these videos are intended to provide pre-service teachers with tools to feel efficacious in teaching math to young students.

The major hypothesis is that completing these video lessons increases efficacy for teaching math. Secondarily, this study examines variables that might partially explain this change in efficacy, including knowledge of mathematical development (KMD) and one's "intellectual modesty" in regard to the limits of one's knowledge of students and mathematical development. KMD and modesty are each hypothesized to partially mediate the change in efficacy, albeit in opposite directions. That is, as subjects gain additional knowledge of how students learn mathematics (i.e., KMD), their efficacy for teaching math is also expected to increase. However, as subjects come to appreciate the complexities of interpreting children's mathematical behaviors and are thereby confronted with the limits of their knowledge in this area (i.e., intellectual modesty), their efficacy is expected to decrease. Consequently, intellectual modesty is expected to create a more conservative, or more realistic, perception of one's capacity to teach early childhood mathematics. Hopefully this more realistic perception drives subjects to learn more about students' mathematical development in order to become better teachers, which would then increase their efficacy and further promote positive teaching behaviors associated with high efficacy.

Chapter 2: Background

Teacher Efficacy

Bandura (1977) categorizes teacher efficacy as a type of self-efficacy. It is a cognitive process in which teachers construct ideas about their capacity to perform a given teaching task at a given level. Others have expanded upon this definition by relating these beliefs about teaching ability to the outcome of student learning. For example, Guskey and Passaro state that teacher efficacy is "teachers' belief or conviction that they can influence how well students learn, even those who may be difficult or unmotivated" (1994, p. 4), and Pajares (1996) defines teacher efficacy as the personal belief about one's capabilities to help students learn. Moreover, Tschannen-Moran and her colleagues (1998) argue that teacher efficacy is context-specific in that it is always judged in relation to the available skills and resources required for a specific teaching task. Researchers (e.g., Hoy & Woolfolk, 1993) have also differentiated the independent factors of general (GTE) and personal teaching efficacy (PTE). GTE is viewed as teachers' beliefs about the external factors that overwhelm their own power to influence student achievement, while PTE is viewed as teachers' confidence in their abilities as teachers to overcome those external factors that have the potential to interfere with student learning (Tschannen-Moran et al., 1998).

Bandura (1997) proposed that teacher self-efficacy should influence the same types of activities that student self-efficacy affects, namely choice of activities, effort, persistence, and achievement. For example, teachers with low self-efficacy may avoid planning activities they believe exceed their capabilities, they may not persist with students who are having difficulties, they may expend little effort to find materials, and they may not re-teach in ways that allow students to better understand the subject matter. Teachers with high self-efficacy, on the other hand, are more apt to develop challenging activities, help students succeed, and persist with students who have problems. As such, self-efficacy can reinforce itself through its influence on performance, which provides new input into evaluations of efficacy (Tschannen-Moran & McMaster, 2009).

Ashton & Webb (1986) found evidence to support Bandura's hypotheses. They found that teacher self-efficacy is associated with teachers' choice of instructional activities, the amount of effort expended in teaching, student encouragement, and the degree of teachers' persistence maintained when confronted with difficulties in the classroom. Similarly, teacher efficacy has been linked to teachers' enthusiasm while teaching, commitment to teaching, willingness to embrace innovation, resilience in the face of failure, as well as to student outcomes like achievement, student motivation, students' self-efficacy beliefs, and problem solving abilities (Akinsola, 2009). Furthermore, Gibson and Dembo (1984) found that teachers with high self-efficacy are more effective in leading students to correct responses in classroom discussions, and Akinsola (2009) found that high efficacy teachers use a greater variety of instructional strategies and are more likely to use inquiry and student-centered teaching strategies than are low efficacy teachers.

In brief, there are many ways in which teachers' efficacy predicts their teaching behaviors. Just as teachers' expectations of students and what they can learn impacts the tasks posed, the questions asked, the time given for responding, and the encouragement given to students (Kilpatrick et al., 2001), so teachers' efficacy for teaching impacts these same behaviors. Teachers with high efficacy are less overtly controlling of student behavior in the classroom (Hoy & Woolfolk, 1990), expend more effort to help students learn, and set challenging goals for their students (Akinsola, 2009), while teachers with low efficacy treat high and low achieving students differently, calling less on low-achieving students, assigning them more busy work, and giving more appropriate praise and feedback to high-achieving students (Ashton & Webb, 1986). In fact, teachers' efficacy is so powerful that students who moved from elementary teachers with high efficacy for teaching math to middle school teachers with low efficacy for teaching math had lower expectancy beliefs about their own math ability than did other students (Midgley et al., 1989).

The Relationship Between Perceived and Real Abilities: Efficacy and Competence

Research has clearly demonstrated the importance of teacher efficacy for instruction and student learning, but has often ignored the relationship between this self-perception and the reality of a teacher's skills. That is, does confidence in one's abilities as a teacher correspond to competence in teaching?

While intuition might point to a positive relationship between teachers' confidence in their abilities and their actual competence at teaching, little is actually known about this relationship. However, the field of learning research has examined students' confidence judgments of learning and their relation to academic performance, and can potentially shed light on the relationship between these variables for teachers. For example, Nietfeld and Schraw (2002) found that math learners tend to accurately assess their learning, as evidenced by the positive correlation between such evaluations and students' performance. Similarly, Williams (1994) found that students who rated themselves as having high efficacy for learning math actually performed at higher levels than those who rated themselves as having low efficacy, thus finding that efficacy was in line with competence. Moreover, Pajares and Miller (1994) found that math self-efficacy was a better predictor of math problem solving ability than was prior experience with problem solving.

Nonetheless, a weak or inverse link between self-perceptions of learning and actual performance has been more typical in the literature. In fact, a very recent study found that students who studied through self-testing methods (e.g., recalling text, reviewing it and then recalling again) believed they would do worse on a subsequent test than students who made concept maps or who simply reviewed material multiple times, while in reality the reverse was true (Karpicke & Blunt, 2011). The study's authors believe that the self-testing method of studying points out a lack of knowledge to students, which shakes their confidence and leads to lower predictions of performance. However, when a student found that he or she did not know or remember a piece of information during the self-testing, the result was often a mental review of that information, which subsequently produced better performance (Richland et al., 2009). The other methods of studying lacked feedback, and therefore were less effective at detecting and correcting a mismatch between perceived and actual knowledge. Kornell and Bjork (2009) similarly found a lack of correlation between confidence and performance on a cued-recall test of word pairs. Participants expected no improvement over time when in fact they did improve. Furthermore, Collins and Bissell (2004) found a lack of correlation between confidence and performance on a grammar task at the beginning of a college semester, noting that students are fairly poor judges of their own knowledge due to lack of practice in such an evaluation. In addition, low-achieving students tend to overestimate their abilities more than high-achieving students, and giving students individual feedback on their performance appears to temper their confidence judgments in the correct direction (Zechmeister, 1983).

Feedback has consistently been found to increase monitoring of learning, as has greater domain knowledge (Nietfield & Schraw, 2002). Surprisingly though, while domain knowledge has been found to increase monitoring as well as actual performance, it remains unrelated to judgments of confidence in learning (Nietfeld & Schraw, 2002). In other words, expertise provides a basis on which to evaluate learning, but also likely points out knowledge that is lacking, in a form of feedback similar to the self-testing method described above, which impacts confidence. Thus, domain knowledge may be a prerequisite to accurate self-monitoring, which may be a precursor to achievement, but such competence may either hinder or be unrelated to confidence.

Looking at this relationship in the opposite direction, lack of feedback has been found to distort the relationship between confidence and competence. For example, Collins and Bissell (2004) found no correlation between grammar self-efficacy and performance at the beginning of a semester, but found a positive correlation at the end of the semester. Importantly, grammar performance had not actually changed over the course of the semester, and students were not provided with any feedback on which to evaluate their learning trajectory. Thus, the perception that students had of learning grammar increased their efficacy, despite the reality of their lack of learning. That is, confidence had increased without a corresponding increase in competence. So while increased competence can hinder confidence, a lack of increase in competence can still promote confidence. In effect, the ignorant remain unaware of their lack of knowledge, and feel more confident in their abilities!

Assuming that this inaccurate link between confidence and competence is also true for teachers, there is a possibly dangerous illusion of knowledge that might hinder teachers' motivation to learn and improve. Consequently, it is beneficial for teachers to have a form of modesty that allows for recognition of the possibility that there are limits to what they know at a particular moment. Such intellectual modesty would involve a "grounded skepticism" that leads "teachers to want to gather more information through further probing and assessment" of their students, and "increase the likelihood that they will revise their thinking when new evidence becomes available" (Preston, 2010). Furthermore, research suggests that such modesty might develop through feedback and self-assessment of knowledge. Still, it remains unclear how such modesty would relate to judgments of confidence or efficacy, though presumably both modesty and efficacy would promote positive teacher behaviors like increased effort and persistence, helping to align confidence and competence.

Teacher Efficacy and Knowledge of Mathematical Development

Looking more closely at the relationship between confidence and competence in mathematics specifically, Hill (2010) points out that little is known about whether teachers' efficacy for mathematics is related to their mathematical knowledge. If little is known about the relationship between teachers' mathematical efficacy and content knowledge, then virtually nothing is known about the relationship between teachers' efficacy for mathematics and knowledge of mathematical development (KMD). However, this is a key relationship since KMD is essential to a math teachers' competence, and confidence without competence could lead to the perpetuation of bad teaching. Thus, like teacher efficacy, KMD is a key construct in the evaluation of good teaching.

KMD has been linked to student achievement (Hill et al., 2005; Baumert et al., 2010) and mathematical quality of instruction (Hill et al., 2008). This relationship makes sense since instruction ought to take into account students' current understandings, and KMD encompasses the knowledge needed to accomplish this task. KMD not only includes a deep understanding of the mathematical ideas students are learning, but it also includes an appreciation for the ways in which children learn mathematics naturally and with instruction. It necessitates knowing about typical sequences and ways that children learn ideas, as well as knowing about the misconceptions children espouse along the way. It therefore involves knowing *what*, knowing *how*, and knowing *why*. For example, KMD includes knowing mathematical explanations for why common rules or procedures work, the ability to link and sequence representations of mathematical subject matter, and the ability to select student responses to highlight and move mathematical discussions forward (Hill, 2010; Ball et al., 2008; Shulman, 1986). Thus, KMD lies at the heart of pedagogical content knowledge, which requires knowing how to act in the teaching moment to address the “preconceptions and background knowledge that students typically bring to each subject” (NBPTS, 2006, p. vi, cited in Hill et al., 2008). It requires knowing about mathematics, students, pedagogy, and curricula.

To date, the only study uncovered in a review of the literature that addresses the relationship between efficacy and KMD found a correlation of 0.25 ($p < .001$) in a sample of 625 elementary teachers (Hill, 2010). This is a fairly weak relationship, and the author suggests that the placement of the efficacy questions at the end of a 1-hour session of working math problems may have produced the weak association. However, it is also possible that there is only a weak relationship between subjects’ perceived and actual abilities, as already discussed. Hill reports that the majority of teachers perceived their teaching abilities in mathematics as “adequate,” which does not leave much room for differentiating between teachers with different actual levels of KMD. Along these lines, Hill (2010) also suggests that this weak relationship may be the result of infrequent opportunities for teachers to receive feedback about their mathematical strengths and weaknesses.

Enhancing Teacher Efficacy

Bandura (1997) reports four primary influences on the development of efficacy: prior (hopefully mastery) experiences in relevant situations, vicarious experiences (e.g., witnessing another teacher succeeding or failing on a teaching task), verbal persuasion of influential peers or superiors, and physiological arousal anticipating a relevant task. The strongest source of efficacy is prior performance. If teachers have a positive prior experience with a student around the mathematical topic to be taught, then they should feel that their future experiences should also be positive, and vice versa for negative prior experiences. In other words, teacher efficacy should rise when students display learning progress (Tschannen-Moran et al., 1998). Furthermore, teachers' efficacy can increase if they believe they are in control of improving future student outcomes, for example, by understanding the issues of the teaching and learning situation more fully and having alternative teaching strategies that may address these issues to produce better results. In this way, the experience of watching others succeed can also be a powerful influence on efficacy since such models transmit KMD in the form of ideas about student mathematics learning and effective teaching skills and strategies, and demonstrate the necessary thinking for accomplishing a desired teaching task (Tschannen-Moran & McMaster, 2009). Similarly, verbal persuasion can provide feedback on one's skills that will influence perceptions of efficacy, while simultaneously increasing modesty, grounding that self-perception of efficacy in reality.

Thus, providing teachers with positive teaching experiences, examples of peers teaching effectively, and/or positive feedback on their teaching capabilities should enhance efficacy. However, we cannot ensure positive teaching experiences, especially for pre-service teachers, nor would it be wise to provide unfounded praise. Consequently, giving pre-service teachers positive vicarious experiences with teaching appears the most viable way of influencing their efficacy. For example, pre-service teachers might observe successful teachers in classrooms or working individually with children, watch videos of effective teaching techniques, or read summaries or transcripts of positive teaching experiences. Several studies have shown that vicarious experiences have increased teachers' efficacy (e.g., Sparks, 1988; Gorrell & Capron, 1990; Hagen, 1998). Furthermore, this strategy seems particularly useful with pre-service teachers because they currently lack an objective measure of their abilities, and instead tend to evaluate themselves in relation to the performance of their peers (Bandura, 1997). By watching a peer succeed on a relevant task, it therefore seems likely that pre-service teachers will relate that experience to themselves, and subsequently use it in their assessment of their own capabilities.

The Use of Video in Teacher Learning

As videos have the potential to create positive vicarious experiences for teachers and thereby enhance their efficacy, it is important to consider the research on the use of videos in teacher learning. Videos have a long history of use with teachers, and have primarily focused on the observation and assessment of teacher behaviors in the classroom (Fuller & Manning, 1973). However, improved technology for both capturing and sharing videos have begun to change the landscape of teacher learning from videos (Rich & Hannafin, 2009). There has been a recent shift to the study of teacher "noticing," which focuses on students' behaviors rather than the teacher's behavior during the viewing of the video (van Es, 2008). This shift converges with a movement in the mathematics education community to attend to students' strategies, interpret their understanding, and decide how to respond effectively based on these understandings (Jacobs et al., 2009).

Video cases have increasingly become a part of the pre-service training of teachers

(Carter & Anders, 1996) because they can provide specific and realistic teaching situations and demonstrate student misconceptions and strategies very clearly, allowing teachers the time to reflect on their responses to such situations (Ball & Cohen, 1999). As Harrington (1995) notes, video cases can present teachers, especially pre-service teachers, with alternative perspectives and practices and prompt them to look at a teaching situation from both the teacher's and the student's perspective. These cases can encourage viewers to become increasingly aware of their own knowledge, beliefs, values, and feelings as they consider how they would have acted in the portrayed situation. However, these video experiences must include focused activities and interaction with the videos to avoid passive viewing and the failure to relate the experience to one's own beliefs and practice (Nemirovsky & Galvis, 2004).

Videos have also started to become an integral part of some of the most successful professional development efforts. For example, the charter network of schools, Success Academies, uses video to share best practices and hone teachers' skills within the classroom. Similarly, in some versions of lesson study, teachers analyze a video of the lesson being taught with regard to the teacher's interactions with the students and their engagement with the content of the lesson (Mast, 2008).

Very few studies have looked at the use of video to enhance efficacy, but those doing so have found an increase in efficacy as a result of the vicarious experience. For example, Hagan and colleagues (1998) found that viewing videos of effective classroom management techniques improved teachers' efficacy for managing a classroom. Similarly, in a study involving "eSupervision," in which prospective teachers were videotaped and the supervisors as well as peers in an online forum gave feedback on the performance, student teachers were found to have higher efficacy for teaching at the end of the term than did prospective teachers in a control group receiving typical supervision (Kopcha & Alger, 2011).

Professional Development & Changes in Teacher Efficacy

Studies linking specific experiences to enhanced teacher efficacy are lacking, particularly within the domain of mathematics. In fact, in an extensive review of the literature, fewer than 10 studies were uncovered that investigate the relationship between teacher efficacy and mathematics-specific professional development. Narrowing this search to early childhood teachers produces only 1 paper, which references 4 unpublished dissertations that also address this topic. Still, teachers' self-efficacy has consistently been found to be among the most powerful influences on receptivity to change, and it is therefore essential to consider it when designing professional development. For example, McKinney et al. (1999) found that low efficacy teachers focused on the impact that a new teaching method for reading would have on *them,* rather than the impact it might have on their teaching or their students, which was the focus of high efficacy teachers. By contrast, the high efficacy teachers tended to view the new instructional program as important and feasible, and were enthusiastic about implementing it in their classrooms, which was not the case for the low efficacy teachers. Similarly, Scribner (1999) found that high efficacy teachers sought out new knowledge and skills while low efficacy teachers did not view such professional learning as appropriate to their current needs.

In addition, when researchers do focus on the link between professional development and teacher efficacy, they tend to concentrate on mathematical content knowledge. It has been thought that increasing domain knowledge will lead to better teaching and improved student outcomes since teachers who lack their own conceptual understanding of the material cannot impart that knowledge to students. However, some researchers (e.g., Ross, 1994) have found that new knowledge alone is often insufficient to produce significant changes in teacher efficacy because the knowledge remains disconnected from practice. Furthermore, a number of researchers have found an "implementation dip" in efficacy as content knowledge is initially applied to teaching tasks. This is likely the result of a lack of insight into the actual level of skills that a teacher possesses. Cunningham et al. (2004) found that teachers are not particularly accurate in their assessment of their own content knowledge, and that they consistently tend to overestimate their abilities. Therefore, when they are confronted with the reality of their knowledge and skills in their own classrooms, they often have a decline in efficacy.

Still, such a dip in efficacy may increase modesty, which may provide the necessary motivation to learn and improve (Wheatley, 2002), assuming the dip is not so low as to produce avoidance (Tschannen-Moran & McMaster, 2009). In a recent study, Tschannen-Moran & McMaster (2009) found that awareness of a new instructional strategy that had been shown to impact struggling students caused some teachers to reassess their definition of good teaching and to consequently reassess their own teaching abilities, setting higher standards for acceptable practice. These teachers saw the new strategy as a goal to achieve rather than as a threat to avoid. Similarly, Schorr & Ginsburg (2000) found that as prospective teachers became aware of the prevalence and variety of invented strategies among their students, they realized that teaching only one mathematical strategy might shut out a method that is more understandable to students. Consequently, these teachers decided they would be more open to the possibility of multiple solution strategies in their classrooms and would attempt to lead discussions around why a particular method might be more appropriate or effective in certain situations than in others. In effect, by learning more about how students know and learn mathematics, these pre-service teachers then devised better ways of teaching such students.

In sum, research has shown that vicarious experience is a powerful source of teacher efficacy because it demonstrates knowledge and strategies that can be adopted in order to succeed at a teaching task. In addition, a mismatch between real and perceived teaching skills can produce a temporary decline in teacher efficacy or can create an illusion of knowledge that hinders efforts to improve. What remains to be investigated is what specific knowledge and understanding is acquired through this vicarious experience, particularly within the domain of mathematics and particularly with pre-service early childhood teachers, and how this specific knowledge produces a change in efficacy. Furthermore, how do real and perceived skills interact to change efficacy?

This study investigated the relationship between efficacy, knowledge, and intellectual modesty as participants viewed a series of five video lessons in which a young child was interviewed about his or her knowledge and understanding of an early math concept. Each video supplied the subject with a vicarious experience of helping a child learn and understand a mathematics topic as the subject watched a similar other successfully working with children on topics in mathematics. These videos demonstrated mathematical content, but also went beyond this to demonstrate knowledge of mathematical development (i.e., what students know mathematically and how they learn mathematics) and what effective questioning and exploration of children's mathematical understanding actually looks like. As such, these videos provided participants with tools for them to feel they can be efficacious in teaching math to young students and were therefore hypothesized both to increase participants' efficacy and to increase their knowledge of mathematical development.

Chapter 3: Methods

Design

Table 1 shows an overview of the study. The study took place in the middle of the fall semester and lasted seven weeks. Each week involved one 60-minute session. The first and last weeks were reserved for pre- and post-testing (described in detail below), which included a measure of Teacher Efficacy and a video analysis of a child being interviewed about her understanding of patterns, which was not a topic of any of the video lessons though it is still an important topic of early childhood math. There were two different pattern videos, each approximately 6 minutes in length, and they were counterbalanced so that half of the participants viewed one of the videos at pretest while the other half viewed that same video at posttest. After viewing the video, subjects were asked two summative questions: (1) What does the child really know about patterns? (2) What would you do next in teaching? Responses to the first question were used to derive a measure of Intellectual Modesty, and responses to the second question were used to derive a measure of KMD, as described below. In the five weeks in between, non-control subjects completed five lessons (one lesson per week) on various mathematical topics, including addition, counting, enumeration, equivalence, and subtraction. Each lesson included a short text on the topic (typically 2 pages, derived from Children's Arithmetic (Ginsburg, 1989)) and a short video of a 4-6 year old child with an interviewer that illustrated the topic in some way.

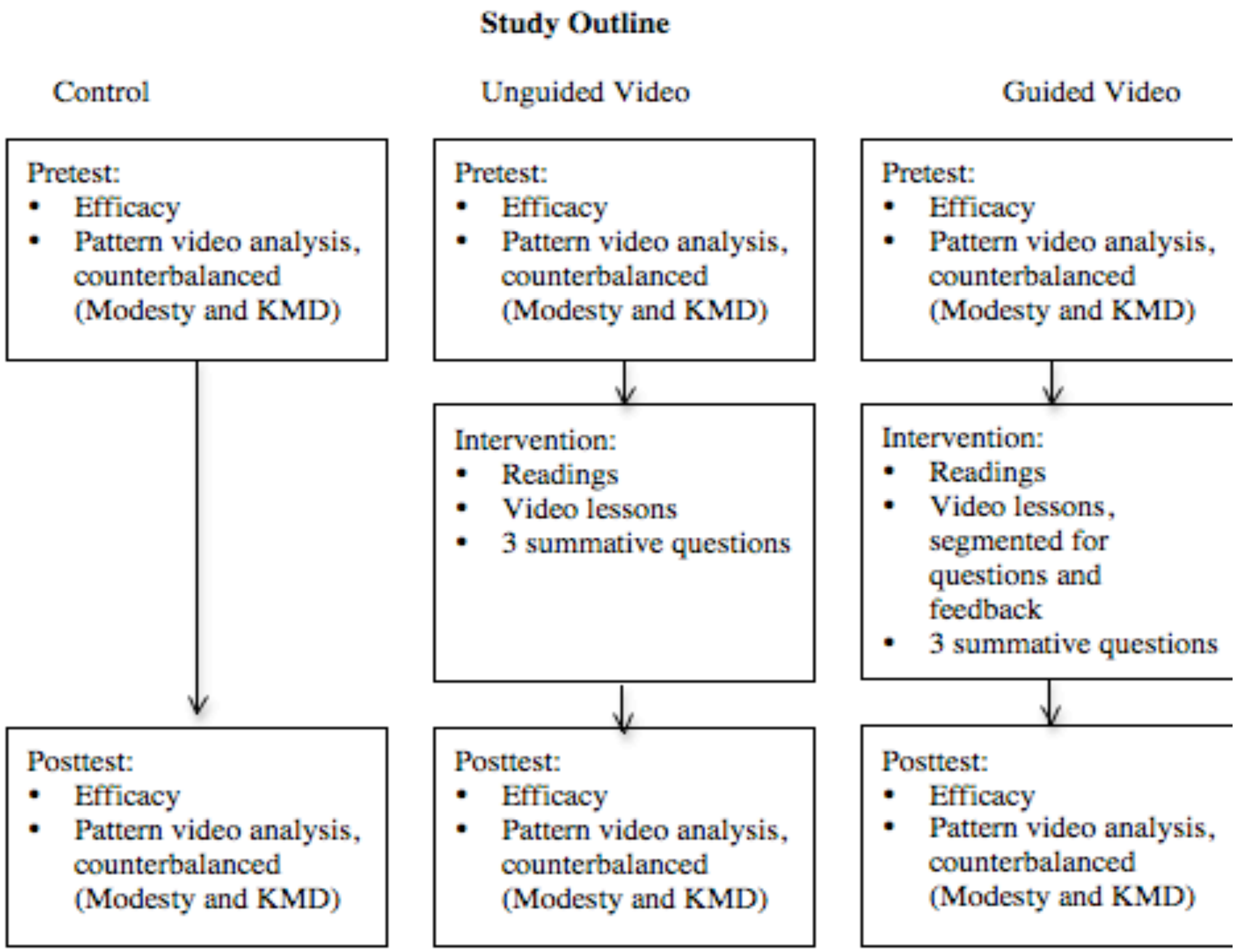


Table 1: Overview of study design

Subjects were randomly assigned to one of three conditions: “guided video,” “unguided video,” and control. In the control condition, participants only completed the pre- and posttests. In the unguided video condition, participants also completed the pre- and posttests, as well as completing a 2-page reading on each of the topics that appeared in the videos, then viewing the associated video lesson and completing three summative questions: (1) What does the child really know about [topic of the video lesson]? (2) What would you do next in teaching? (3) What did you learn from this exercise? Finally, unguided participants then completed the posttests. In the guided video condition, participants again completed the pretest, the readings, and the video lessons with the three summative questions, but in addition, their video lessons were segmented into smaller clips of approximately 10 seconds in length, each with their own questions. The interspersed questions were of several varieties, such as describing what was observed and interpreting the child's behavior, offering a hypothesis for what the child knows, predicting what will happen next, providing further questions to ask or tasks to try with the child, and suggesting strategies for teaching. Each lesson contained an average of 15 interspersed questions. Upon answering these interspersed questions, standardized responses would appear that served as feedback to the participant. These sample responses were intended to provide a range of possible interpretations of the content of the video, and to give subjects an opportunity to self-assess their knowledge by benchmarking their responses against those of their peers. Importantly, participants could not navigate back to their original responses once they viewed the feedback. Finally, these participants also completed the posttests. (See Table 1 above for an overview of the study design.) Both the guided and unguided video groups read the text, watched the video, and answered the questions within the Survey Monkey web interface. Furthermore, the order in which subjects viewed the five topics within the video lessons was counterbalanced for both the unguided and guided video groups. (Appendix B contains the readings and questions from the guided video condition.)

Participants

Sixty-three undergraduate pre-service early childhood teachers (57 female and 6 male) from ten universities (Barnard College, Fordham University, Georgia State University, Michigan State University, Oklahoma State University, Rider University, Rutgers University, Slippery

Rock University, University of Dayton, University of Georgia) completed this study (see Table 2). The recruitment materials described the study as a test of new video-based lessons in early childhood mathematics education and specifically said that all materials would be delivered online and there were no pre-requisites. Therefore, subjects did not have prior coursework in mathematics methods or prior teaching experience, which ensured consistent levels of familiarity with the subject matter. Subjects from each site were randomly assigned to one of the three conditions described below, and participants were not told of the existence of other groups nor of the condition to which they were assigned. All subjects were compensated for their participation to ensure completion of all sessions. Ten subjects had to be dropped due to failure to complete the sessions in a timely manner.

School	# of Subjects	Percent
Barnard College	3	4.8
Fordham University	1	1.6
Georgia State University	1	1.6
Michigan State University	15	23.8
Oklahoma State University	15	23.8
Rider University	2	3.2
Rutgers University	4	6.3
Slippery Rock University	7	11.1
University of Dayton	8	12.7
University of Georgia	7	11.1

Table 2: Participants' and their colleges and universities

Measures

The Teacher Efficacy Scale

Several measures of teachers' self-efficacy are available. Most were used rarely and were not adopted by the research community. For example, the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI; Enochs et al., 2000), has had very limited use, mainly due to its subject specificity. However, two measures have gained more widespread acceptance: the

Teachers' Sense of Efficacy Scale (TSES), developed by Tschannen-Moran and Hoy (2001), and the Teacher Efficacy Scale (TES), developed by Gibson and Dembo (1984). Short versions of both of these scales have also been developed, and both have been used with pre-service and in-service teachers. Reliability (α reported below) and validity information, including convergent and discriminant validity, is available for both scales, with values meeting recommended social science standards (Tschannen-Moran & Hoy, 2001; Gibson & Dembo, 1984). For example, Gibson & Dembo (1984) tested validity of the TES with a heterotrait-heteromethod correlation matrix and found significant positive correlations along the diagonal for the three traits tested, including verbal ability ($r = 0.30, p < .05$), flexibility ($r = 0.39, p < .05$), and teacher efficacy ($r = 0.42, p < .05$), which is evidence for convergent validity. Additionally, the correlations of teacher efficacy when measured by two methods exceeds both the correlations between teacher efficacy and verbal ability not having method in common ($r = .08, r = 0.09$) and between teacher efficacy and flexibility not having method in common ($r = 0.21, r = -0.06$), which is evidence for discriminant validity.

The TSES contains items that differentiate between three subsections of efficacy: efficacy in student engagement ($\alpha = 0.87$), efficacy in instructional practices ($\alpha = 0.91$), and efficacy in classroom management ($\alpha = 0.90$). Since this study is concerned with efficacy in instructional practices of teachers, it might make sense to only use this subsection for the study. On the other hand, the authors recommend using all three sections with pre-service teachers because this population tends not to distinguish among these different types of efficacy. Consequently, the TSES was not used in this study and instead the TES was used, which has consistently found two independent factors: General Teaching Efficacy ($\alpha = 0.72$) and Personal Teaching Efficacy ($\alpha = 0.84$). As I am primarily interested in PTE and the authors allow for the use of just this subscale, the TES offers greater face validity than the TSES.

Items were slightly adapted to be math teaching specific. For example, one item was changed from: "When I really work at teaching, I can get through to most difficult students." to "When I really work at teaching math, I can get through to most difficult students." Subjects rated their agreement with each statement on a six-point Likert scale, ranging from "Strongly Disagree" to "Strongly Agree." (See Appendix A for all items.) Informal clinical interviews were conducted to ensure that the minor adaptations did not alter participants' responses to the items. These cognitive interviews entailed asking participants why they chose a particular answer and whether the question was clear. Feedback from the cognitive interviews was used to make final adjustments to the items prior to use in the study. Thus, the modified set of items seemed to measure the constructs in question, though the previously reported psychometrics of reliability and validity varied slightly (for the PTE subscale, $\alpha = 0.70$ on the pretest and 0.81 on the posttest). The TES was administered as part of the pretest and posttest. (See Appendix A)

Intellectual Modesty Coding Scheme (Preston, 2010)

The number of modest statements was tallied for each subject in response to the question, "What does this child really know about patterns?" following the viewing of the counterbalanced videos on patterns that were presented during the pre- and posttests. Preston, the designer of the coding scheme for intellectual modesty, and a colleague obtained an inter-rater reliability of .87, and disagreements were discussed and a single code decided upon.

Modest statements evaluate the adequacy of a claim, propose alternatives, or acknowledge the limits of the evidence. Modesty demonstrates recognition of the limits of what is knowable given the evidence. For example, a modest statement might read, "At this point it seems that the child understands the idea of pattern." Modest language includes explicit statements in which the author assesses the relative certainty of a specific interpretation, as demonstrated by the use of conditional words like "might" and "could," perception words like "appears" and "seems," temporal words like "now" and "before" that acknowledge interpretations can change with new evidence, and metacognitive words like "we realize" and "leads one to believe" that insert the author's thinking into the response. Modesty can also identify missing evidence, as in, "Because the interviewer changed tasks, we did not see whether Gabriella could continue the pattern on her own," as well as suggestions for obtaining more evidence, as in, "I would have asked the child to make her own pattern to see whether she understood the repeating concept."

Knowledge of Mathematical Development Coding Scheme

Following the pre- and post-test pattern video, subjects were asked, "What would you do next in teaching?" From their responses to this question, the number of statements about declarative, procedural, and conceptual knowledge in teaching mathematics was tallied and a weighted total was computed. As KMD includes knowing *what*, knowing *how*, and knowing *why*, these three types of knowledge encompass the full representation of participants' KMD. That is, declarative knowledge describes *what* factual-type mathematical knowledge is known, procedural knowledge involves knowing *how* to do something mathematically, and conceptual knowledge involves an understanding of *why* mathematical facts and procedures work the way they do. More specifically, declarative knowledge statements refer to responses that deal with mathematical facts (e.g., 2 + 2 = 4), typically acquired through memorization. Declarative knowledge does not include an understanding of *how* to use the fact or *why* the fact is true, but is simply a statement of *what* a child knows or should be taught. Procedural statements refer to responses that deal with mathematical skills (e.g., double-digit multiplication), and conceptual knowledge statements refer to responses that deal with deeper understanding of mathematical principles (e.g., place value and its relation to standard algorithms or the inverse relationship of addition and subtraction). Coders were particularly strict in applying the code of conceptual knowledge, and responses that included the word "understanding" without further explanation were not given this code. Responses needed to indicate the participant's own conceptual understanding of the mathematical idea or the desire to pass on to a student an understanding of *why* something works mathematically in order to achieve this code. Consequently, this code was rarely used, with most participants achieving less than two such statements within their pretest or posttest responses. Table 3 contains sample responses to the question, "What would you do next in teaching?" that followed the viewing of the pattern video and the codes they received. Inter-rater reliability was .90 for two coders, and disagreements were discussed and a single code decided upon.

Code	Examples
Declarative Knowledge (DK)	• "I would tell her that green, orange, green, orange is a pattern." • "I would tell her what a family is within a pattern."
Procedural Knowledge (PK)	• "I might try and have her separate the bears into color groups." • "I would ask her to make her own pattern or to continue mine."
Conceptual Understanding (CU)	• "I would help her understand that patterns can be of other things than colors and are in different contexts. Maybe she could even teach me a new pattern." • "I would explain to her that a pattern is like a trail and has

	rules.”

Table 3: KMD sample responses

Based on research that declarative mathematical knowledge is simpler and easier for students than procedural knowledge, which itself is simpler than conceptual knowledge, the total KMD score weighted procedural statements as two times the value of a declarative statement, and conceptual statements as three times the value of a declarative statement. For example, the seminal work Adding It Up (Kilpatrick et al., 2001) discusses five strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. Conceptual understanding involves understanding why algorithms work and connecting concepts in an intelligent way. Procedural fluency involves mathematical computations. Strategic competence involves the ability to formulate, represent, and solve math problems. Adaptive reasoning includes the capacity for logical thought, reflection, explanation, and justification. A productive disposition pertains to the habitual inclination to see math as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy. In the KMD coding scheme, conceptual understanding encompasses the strands of conceptual understanding, strategic competence, and adaptive reasoning, while procedural knowledge encompasses only procedural fluency. Declarative knowledge is so basic as to not even be mentioned within the strands of mathematical proficiency.

Similarly, Ball and colleagues’ (2008) definition of pedagogical content knowledge encompasses conceptual understanding in the current coding scheme. Declarative knowledge and procedural fluency are assumed of most teachers, and distinctions are made between teachers ability to describe *why* mathematical concepts and procedures are true, represent these understandings in multiple ways and connect these representations. Thus, the code of conceptual understanding received the greatest weight in the overall KMD total.

Unfortunately, upon analyzing the data, it became apparent that the two counterbalanced pattern videos were not actually equivalent in terms of the KMD demonstrated by the children within the videos. While the videos did match in terms of the setting (a table in the back of a preschool classroom) and topic of the video, as well as the interviewer and gender of the child, it turned out that one of the videos contained no examples of conceptual knowledge of patterns while the other video contained seven examples of such knowledge. In the less mathematically rich video, the child appears distracted by the goings-on within the classroom and often requires the interviewer to repeat herself. Still, this child seemed to understand significantly less about patterns, as she often was unable to continue an ABAB pattern generated by the interviewer or to describe an example of a pattern. In the more mathematically rich video, the child not only continues patterns, but she also spontaneously generates examples of patterns and explains that patterns "go on and on" and "repeat." Consequently, the planned pretest-posttest comparison of KMD scores was not appropriate and data was separated by which video was viewed first, and then group differences in KMD were examined for the control, unguided, and guided groups.

The next chapter describes the statistical analyses conducted on the data, including comparisons within and between groups on the pre- and posttests for efficacy and modesty. While similar analyses were also conducted for KMD, post-hoc analyses of KMD based on which video participants viewed at pretest were also conducted to account for the lack of comparability between the two pattern videos used in the coding of KMD. Relationships among all three variables were also addressed, and factors affecting the results are presented.

Chapter 4: Results

This chapter presents the results of the study. For each research questions, the initial hypothesis is re-stated, followed by all relevant results and a conclusion relating back to the original hypothesis. The end of this section presents an alternative look at the data that helps to put many of the findings into perspective.

Research Question 1: Does completion of video lessons on early childhood math topics increase pre-service teachers' efficacy for teaching math?

Hypothesis 1: Completing the video lessons will increase efficacy. The two video conditions will have higher post-efficacy scores than the control condition.

Results: The mean post-efficacy score for the combined video groups was significantly higher than for the control group ($F(1,62) = 4.495$, $p < .05$). When the two video groups were separated, the 2-way ANOVA of all 3 groups remained significant. Tukey's HSD post-hoc test revealed that the unguided and control group posttest means were significantly different ($p < .05$) (see Table 4).

Post-efficacy	Minimum	Maximum	Mean	SD
Control	2.20	6.00	4.45*	.83
Unguided	4.00	6.00	5.01*	.58
Guided	2.80	5.80	4.73	.76

Table 4: Post-efficacy comparison for control, unguided, and guided groups

Because the efficacy change scores were not significantly different for the video versus control subjects while the post-efficacy scores were, an ANCOVA was also run, with pre-test scores as a covariate. In other words, pre-test scores were controlled, which theoretically was done by the random assignment, in estimating group differences in post-efficacy scores. Predictably, this caused the effect of group on post-efficacy scores to diminish ($F(62,2) = 1.381$, $p = .259$). Still, as Table 5 indicates, while not significant, change in efficacy was in the right direction to coincide with the hypothesis that the video groups would increase in efficacy more than the control group.

	Pre-efficacy mean	Post-efficacy mean	Efficacy Change mean
Control	4.24	4.45	.22
Unguided	4.63	5.01	.38
Guided	4.47	4.73	.27

Table 5: Efficacy comparison for control, unguided, and guided groups

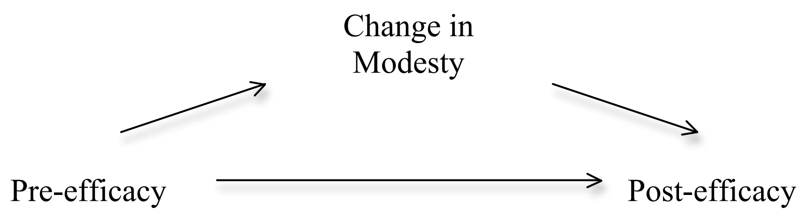
Pre-efficacy and post-efficacy scores were significantly correlated within each group (see Table 6), indicating that most subjects stayed in the same approximate rank order from pre- to posttest. That is, subjects who had lower than average pre-efficacy scores also had lower than average post-efficacy scores. Based on the Fisher r-to-z transformation, it was found that the correlations for each group were not significantly different from each other: Control vs. Unguided: $z = .39$, $p = .70$; Control vs. Guided: $z = .12$, $p = .90$; Unguided vs. Guided: $z = -.26$, $p = .79$. Scatterplots were made to ensure that there was a linear relationship and the Pearson correlation coefficient was an appropriate measure.

	Control	Unguided	Guided
Pearson Correlation of Pre-efficacy with Post-efficacy	.570**	.476*	.542*
Sig. (2-tailed)	.006	.034	.011
N	22	20	21

Table 6: Correlation of pre- and post-efficacy for control, unguided, and guided groups

Conclusion: Efficacy did increase as a result of the video lessons. The unguided group had a significantly higher mean post-efficacy score than the control group. The guided group had a higher, though non-significant, post-efficacy score than the control group.

Research Question 2: Does intellectual modesty mediate the change in efficacy?



Hypothesis 2: Intellectual modesty will partially mediate the increase in efficacy. As participants increase in modesty, they will decrease in efficacy. Modesty is expected to increase for the guided subjects who receive feedback on their responses within the video lessons, and therefore guided subjects are expected to have a smaller increase in efficacy than the unguided subjects. That is, the feedback could help the guided subjects to realize their limitations with respect to analyzing children's mathematical thinking. The increased modesty could temper the increase in efficacy expected by the video lesson experience. Still, both video groups are expected to have higher post-efficacy than the control group.

Results: Change in intellectual modesty was significantly different between groups ($F(58,2) = 6.985$, $p < .01$). Tukey's HSD test revealed that the guided group was significantly different from both the unguided group ($p<.01$) and the control group ($p=.01$). Table 7 shows that the guided group mostly increased in efficacy, while the other groups mostly had no change ($\chi^2(4, N=59)=16.939$, $p<.01$).

Modesty Change by Group Cramer's V = .38		Group			Total
		Control	Unguided	Guided	
Modesty Change	Decrease	4	1	0	5
	No change	11 (55%)	17 (90%)	9	37
	Increase	5	1	11 (55%)	17
Total		20	19	20	59

Table 7: Modesty change for control, unguided, and guided groups

There was no evidence of a correlation between efficacy change and modesty change within each group (see Table 8). Seventy-one percent of participants who increased in modesty also increased in efficacy, while 18% decreased in efficacy (see Table 9).

	Control	Unguided	Guided
Pearson Correlation of Δ Efficacy with Δ Modesty	.25	.11	.13
Sig. (2-tailed)	.29	.67	.60
N	20	19	20

Table 8: Correlation of efficacy and modesty change for control, unguided, and guided groups

Efficacy & Modesty χ^2 (4, N = 59) = 3.703, p=.45 Cramer's V = .18		Modesty Change			Total
		Decrease	No change	Increase	
Efficacy Change	Decrease	3	10	3 (18%)	16
	No change	0	4	2	6
	Increase	2	23	12 (71%)	37
Total		5	37	17	59

Table 9: Frequency of participants who increased or decreased in efficacy and modesty

As the hypothesis predicted that the participants whose modesty increased would have smaller increases in efficacy than participants whose modesty did not increase, a comparison of participants whose modesty increased versus those whose modesty did not increase was conducted. However, there was not a significant difference in efficacy change between participants who increased in modesty and other subjects ($F(62,1)=1.316$, $p>.05$), nor was there a significant difference between subjects who increased in modesty, who did not change in modesty, and who decreased in modesty ($F(58, 2) = 1.073$, $p>.05$). Though not significant, modesty increasers had a larger mean increase in efficacy (.45) than other participants (.23), which was counter to the hypothesis that increased modesty would diminish the increase in efficacy.

The Baron & Kenny model of mediation was used to test the hypothesis that modesty would mediate the change in efficacy. While pre-efficacy did significantly predict post-efficacy, pre-efficacy did not predict change in modesty (see Table 10), and change in modesty did not significantly predict post-efficacy, controlling for pre-efficacy (see Table 11). The tables below first show this relationship for just the guided group, since only this group had a significant change in modesty, and then show this relationship for the entire data.

Model (Guided Group only)	Unstandardized Coefficients		Standardized Coefficients		
	B	Std. Error	Beta	T	Sig.
(Constant)	1.991	1.882		1.058	.304
Pre-efficacy	-.199	.415	-.112	-.480	.637

a. Dependent Variable: Δ Modesty

Table 10: Regression predicting change in modesty from pre-efficacy for the guided group

Model (Guided Group only)	Unstandardized Coefficients		Standardized Coefficients		
	B	Std. Error	Beta	T	Sig.
(Constant)	2.129	.974		2.186	.043
Δ Modesty	.041	.118	.070	.345	.735
Pre-efficacy	.570	.210	.553	2.713	.015

a. Dependent Variable: Post-efficacy

Table 11: Regression predicting post-efficacy from pre-efficacy and change in modesty for the guided group

Model (All Data)	Unstandardized Coefficients		Standardized Coefficients		
	B	Std. Error	Beta	T	Sig.
(Constant)	.657	.949		.692	.492
Pre-efficacy	-.052	.210	-.033	-.249	.804

a. Dependent Variable: Δ Modesty

Table 12: Regression predicting change in modesty from pre-efficacy for all groups combined

Model (All Data)	Unstandardized Coefficients		Standardized Coefficients		
	B	Std. Error	Beta	T	Sig.
(Constant)	2.304	.520		4.431	.000
Pre-efficacy	.544	.114	.535	4.758	.000
Δ Modesty	.063	.072	.098	.874	.386

a. Dependent Variable: Post-efficacy

Table 13: Regression predicting post-efficacy from pre-efficacy and change in modesty for all groups combined

Conclusion: The results did not confirm the hypothesis. While modesty did significantly increase for the guided group, it was found that those subjects whose modesty increased experienced a larger, though non-significant, increase in efficacy than subjects whose modesty did not increase. In other words, as subjects became more aware of the limits of their own knowledge, they also tended to become increasingly confident in their own capacities, which was the opposite of what was hypothesized. Furthermore, there was no evidence for a correlation between change in efficacy and change in modesty, and change in modesty did not significantly predict change in efficacy. Thus, there was no evidence that modesty and efficacy are dependent constructs, and change in modesty does not appear to mediate the change in efficacy.

Research Question 3: Does change in knowledge of mathematical development mediate the change in efficacy?

Hypothesis 3: KMD change will partially mediate the change (i.e., increase) in efficacy. As participants increase in KMD, they will also increase in efficacy. KMD is expected to increase the most for the guided group and the least for the control group, so efficacy is also expected to increase the most for the guided group and least for the control group. KMD is expected to increase the most for the guided group because they have the greatest amount of time and interaction with the material of the video lessons.

Results: There was a significant between group difference in change in KMD ($F(62,2)=3.563$, $p<.05$). Tukey's HSD post-hoc test revealed that the difference was between the control group and the unguided video group ($p=.05$) and the guided video group ($p=.08$) (see Table 14). However, the unguided and guided groups both decreased in KMD, while the control group increased ($\chi^2(4, N=63)=8.235$, $p=.08$) (see Table 15).

	Mean Pre-KMD	Mean Post-KMD	Mean KMD change
Control (N=22)	3.41	4.27	.86*
Unguided (N=20)	3.90	3.05	-.85*
Guided (N=21)	3.90	3.19	-.71

Table 14: KMD comparison for control, unguided, and guided groups

	KMD Decrease	KMD No Change	KMD Increase
Control (N=22)	6	5	11 (50%)
Unguided (N=20)	13 (65%)	1	6
Guided (N=21)	13 (62%)	2	6
Total	32 (51%)	8	23 (37%)

Table 15: Frequency of participants who decreased or increased in KMD for control, unguided, and guided groups

However, it turned out that there was fatigue at posttest, such that response lengths that were used in the coding of KMD were significantly shorter at posttest than at pretest (see Tables 16 and 17). In other words, the decrease in KMD in the two video groups might have been caused by fatigue since participants in these two groups had been writing many responses within the video lessons in between the pre- and posttest, while the control group had not been doing this writing.

	Paired Differences					T	df	Sig. (2-tailed)
	Mean	Std. Deviation	Std. Error Mean	95% Confidence Interval of the Difference: Lower	95% Confidence Interval of the Difference: Upper			
Response Length_Pre – Response Length_Post	12.42857	48.14384	6.06555	.30370	24.55344	2.049	62	.045*

Table 16: Dependent samples t-test for mean response length to KMD item at pre- and posttest

	N	Minimum	Maximum	Mean	Std. Deviation
Response Length_Pre	63	13.00	193.00	62.6667	33.72278
Response Length_Post	63	15.00	252.00	50.2381	37.48751
Valid N (listwise)	63				

Table 17: Descriptive statistics for response length to KMD item at pre- and posttest

Surprisingly however, despite the overall decrease in response length from pre- to posttest, there did not turn out to be group differences in response length at pre- and posttest (see Table 18). Thus, all three groups, including the control, decreased in response length at posttest, and therefore response length in and of itself cannot fully explain the decrease in KMD for the two video groups since the control group increased in KMD despite a decrease in response length at posttest.

		Sum of Squares	Df	Mean Square	F	Sig.
Response Length_Pre	Between Groups	1319.290	1	1319.290	1.176	.283
	Within Groups	67338.903	60	1122.315		
	Total	68658.194	61			
Response Length_Post	Between Groups	240.065	1	240.065	.168	.683
	Within Groups	85766.774	60	1429.446		
	Total	86006.839	61			

Table 18: Comparison of response length to KMD item at pre- and posttest for the control, unguided, and guided groups

In looking further into the cause of the declining KMD in the two video groups, it turned out that the order of videos on patterns viewed at pre- and posttest, which were counterbalanced for the entire sample, made a difference in participants' KMD scores (see Table 19). (Note that the responses to these videos were used in the coding of KMD.) One of the videos turned out to be much richer in mathematical and cognitive content than the other, and therefore subjects who viewed this video at posttest tended to have increases in KMD (see Table 20), while subjects who viewed this video at pretest tended to have decreases in KMD (see Table 21). In fact, one of the videos had no examples of a student's conceptual knowledge, while the other video had seven examples of a student's conceptual knowledge. Thus, the content of responses to the video filled with conceptual knowledge statements contain more mathematically and cognitively rich material, regardless of response length.

		Sum of Squares	Df	Mean Square	F	Sig.
KMD_Change	Between Groups	339.171	1	339.171	26.966	.000*
	Within Groups	767.242	61	12.578		
	Total	1106.413	62			
KMD_Post	Between Groups	235.699	1	235.699	22.868	.000*
	Within Groups	628.714	61	10.307		

	Total	864.413	62			
KMD_Pre	Between Groups	10.645	1	10.645	1.167	.284
	Within Groups	547.242	60	9.121		
	Total	557.887	61			

Table 19: KMD comparison for participants who viewed the richer pattern video at pretest to those who viewed it at posttest

	N	Minimum	Maximum	Mean	Std. Deviation
KMD_Pre	30	4.00	14.00	8.2333	3.00211
KMD_Post	31	4.00	20.00	9.8065	3.70962
KMD_Change	31	-5.00	9.00	1.5161	3.76715
Valid N (listwise)	30				

Table 20: Descriptive statistics for participants who viewed the richer pattern video at posttest

	N	Minimum	Maximum	Mean	Std. Deviation
KMD_Pre	32	5.00	16.00	9.0625	3.03674
KMD_Post	32	2.00	13.00	5.9375	2.63888
KMD_Change	32	-10.00	3.00	-3.1250	3.31906
Valid N (listwise)	32				

Table 21: Descriptive statistics for participants who viewed the richer pattern video at pretest

Moreover, the KMD scores at posttest were not significantly different among the control, unguided, and guided groups when only the subjects who watched the richer video at posttest were examined (see Table 22). In effect, the intervention did not appear to impact KMD for these subjects, and therefore is unlikely to have impacted KMD in the opposite direction as hypothesized. (And interestingly, the intervention also did not appear to impact these subjects' efficacy or modesty [see Table 23], which were also not significantly different at posttest between groups.) Still, these results should be interpreted with caution given the small sample size (roughly half of the original sample).

		Sum of Squares	df	Mean Square	F	Sig.
KMD_Pre	Between Groups	3.347	2	1.674	.571	.571
	Within Groups	82.072	28	2.931		
	Total	85.419	30			
KMD_Post	Between Groups	5.076	2	2.538	.553	.581
	Within Groups	128.472	28	4.588		
	Total	133.548	30			
KMD_Change	Between Groups	14.497	2	7.248	1.554	.229
	Within Groups	130.600	28	4.664		
	Total	145.097	30			

Table 22: KMD comparison of control, unguided, and guided groups for participants who viewed the richer pattern video at posttest

Efficacy_Post	Between Groups	.978	2	.489	1.052	.363
	Within Groups	13.010	28	.465		
	Total	13.987	30			
Efficacy_Change	Between Groups	.058	2	.029	.050	.951
	Within Groups	16.156	28	.577		
	Total	16.214	30			
Modesty_Change	Between Groups	6.410	2	3.205	2.884	.074
	Within Groups	28.900	26	1.112		
	Total	35.310	28			

Table 23: Efficacy and modesty comparison of control, unguided, and guided groups for participants who viewed the richer pattern video at posttest

Going back to the original finding that KMD increased for the control group, but decreased for the two video groups, the component scores of KMD were also examined between groups to help make sense of this surprising finding. Looking within the total KMD scores, there was a significant between group difference in the change in number of statements containing declarative math knowledge ($F(2,62)=3.102$, $p=.05$), but not procedural or conceptual math knowledge. The guided video group was decreasing their declarative statements while the control subjects were not (p=.05). Thus, it appears that the increase in KMD for the control group was driven by the increase in declarative and conceptual statements, while the decrease for the unguided group was driven by the decrease in procedural and conceptual statements and the decrease for the guided group was driven by the decrease in declarative and (slightly) procedural statements (see Table 24).

	Mean Δ Declarative	Mean Δ Procedural	Mean Δ Conceptual
Control	.14*	-.05	.27
Unguided	.00	-.20	-.15
Guided	-.43*	-.14	.00

Table 24: Comparison of change in each type of KMD statement for the control, unguided, and guided groups

A correlation of the three types of knowledge statements within each group (and for the whole data set) was also conducted (see Tables 25 through 28). Procedural and conceptual statements were consistently significantly correlated with the weighted total KMD score, which is not surprising given the weighting these types of statements received. Interestingly, concepts and procedures were inversely correlated in the unguided group at pre-test, such that the more of one type of statement, the less of the other. At post-test in the guided group, procedural and declarative statements were inversely related, despite the fact that both types of statements tended to be decreasing from pre- to post-test. Overall, though, it appears that the three types of knowledge statements are fairly independent. Ideally this indicates that they each add their own value to an understanding of a subject's KMD.

(Note that above the diagonal are pre-test scores and below the diagonal are post-test scores. Only significant correlations are reported.)

All Data (N=63)	Declarative	Procedural	Conceptual	KMD
Declarative	1			
Procedural	-.276*	1	-.315*	.621**
Conceptual			1	.495**

KMD		.581**	.716**	1

Table 25: Correlation of each type of KMD statement for all data

Control Group (N=22)	Declarative	Procedural	Conceptual	KMD
Declarative	1			
Procedural		1		.695**
Conceptual			1	.679**
KMD		.465*	.717**	1

Table 26: Correlation of each type of KMD statement for the control group

Unguided Group (N=20)	Declarative	Procedural	Conceptual	KMD
Declarative	1			
Procedural		1	-.496*	
Conceptual			1	.612**
KMD		.738**	.606**	1

Table 27: Correlation of each type of KMD statement for the unguided group

Guided Group (N=21)	Declarative	Procedural	Conceptual	KMD
Declarative	1			
Procedural	-.438*	1		.75**
Conceptual			1	.495**
KMD		.611**	.776**	1

Table 28: Correlation of each type of KMD statement for the guided group

The correlation of mean pre-KMD and post-KMD scores was only significant for the control group (see Table 29). This may indicate that the control group was appropriately unaffected by the intervention (since they did not participate in the intervention), while the unguided and guided groups were affected.

	Control	Unguided	Guided
Pearson Correlation of Pre-KMD with Post-KMD	.463*	.037	.112
Sig. (2-tailed)	.03	.88	.63
N	22	20	21

Table 29: Correlation of pre- and posttest KMD for the control, unguided, and guided groups

There was no evidence of a correlation between change in KMD and change in efficacy within each group (see Table 30), leading to the disappointing possibility that competence and confidence may be unrelated.

	Control	Unguided	Guided
Pearson Correlation of Δ Efficacy with Δ KMD	.20	.09	.31
Sig. (2-tailed)	.37	.72	.17
N	22	20	21

Table 30: Correlation of change in efficacy with change in KMD for the control, unguided, and guided groups

Despite the lack of evidence for a correlation between KMD and efficacy, a regression model was used to examine whether the change in KMD had any significant impact on the change in efficacy. As expected, pre-efficacy did not predict change in KMD, and change in KMD was not a significant predictor of post-efficacy, controlling for pre-efficacy (see Table 31).

Model		Unstandardized Coefficients		Standardized Coefficients		
		B	Std. Error	Beta	T	Sig.
1	(Constant)	1.951	.515		3.787	.000
	Pre-efficacy	.627	.115	.576	5.459	.000
	Δ KMD	.039	.033	.126	1.191	.238

a. Dependent Variable: Post-efficacy

Table 31: Regression predicting post-efficacy from pre-efficacy and change in KMD

Conclusion: The results did not confirm the hypothesis. While change in KMD was significantly different between groups, the control group tended to increase in KMD while the two video groups tended to decrease in KMD. Furthermore, change in efficacy was not correlated with change in KMD, nor did change in KMD predict change in efficacy, meaning that confidence and competence appeared disconnected.

Alternative Explanation: Profiles

To help interpret the relationship between the variables that were uncovered, an analysis of participants' tendencies to focus on student learning versus teaching was conducted. In this analysis, each subject was categorized as having either a student/learning or a teaching/instruction profile. (A small minority of subjects was categorized as having a focus on both students and teaching, and they were dropped from the following analyses.) This analysis was suggested by the fact that the measure of efficacy was for teaching, and the measure of KMD was also set in the context of teaching, being in response to the question, "What would you do next in teaching?" Therefore, it is more likely that subjects who were focused on teaching would experience changes in these variables.

The profile was determined by coding the response to the final question of the video lessons, "What did you learn from this exercise?" If the response focused on students, the subject received this profile code, while responses that focused on teaching received the teaching code. For example, a response of, "I learned how children interpret counting numbers early on in their life. I realized what misconceptions children have when learning how to count." received the code of a student profile, while a response of, "I learned how to use the child's experience and add to it, use what they know and build on it." received the code of a teaching profile. This coding was done for the 1st, 3rd, and 5th lesson that a subject completed. As only the unguided and guided video groups completed the lessons, only these subjects received a profile code. An overall profile code was also assigned to each subject, which was determined by the majority code for all 3 lessons. In the case where a subject received one student code, one teaching code, and one combination code, the subject's overall code was combination. Again, combination subjects were excluded from the analyses due to their very small numbers.

The results of this qualitative analysis demonstrated that participants trended more and more towards a student/learning profile as they progressed through the video lessons (see Tables 32 and 33). Thus, perhaps the mismatch between what the video lessons were highlighting for the subjects and what was being picked up by the efficacy and KMD measures, which were focused on teaching, produced the disconfirming results described above.

Changes from Lesson 1 to 3 χ^2 (1, N = 32) = 3.72, p=.05 Cramer's V = .34		Lesson 3 Profile		
		Student	Teaching	Total
Lesson 1 Profile	Student	11	1	12
	Teaching	12	8	20
Total		23	9	32

Table 32: Frequency of profiles at lessons 1 and 3

Changes from Lesson 1 to 5 χ^2 (1, N = 32) = 5.54, p<.05 Cramer's V = .42		Lesson 5 Profile		
		Student	Teaching	Total
Lesson 1 Profile	Student	13	2	15
	Teaching	8	9	17
Total		21	11	32

Table 33: Frequency of profiles at lessons 1 and 5

Looking at these results by group, the unguided group started with only 5 teaching-focused subjects while the guided group started with 16, and the unguided group ended up with 3 teaching-focused students while the guided group ended up with 9. Notice that by Lesson 5, 11 subjects were focused on teaching, while 21 were focused on students. A comparison of KMD scores for participants who had a teaching profile, which aligned more with the KMD measure, was also conducted. It turned out that, at posttest, subjects with a teaching profile made significantly more conceptual statements than subjects with a student profile (see Table 34). That is, participants who focused on teaching while viewing the video lessons tended to think at a deeper mathematical and cognitive level about the content and its implications for teaching. While the absolute difference in means (.9 versus .4; see Tables 35 and 36) is not that big for subjects with each profile, remember that the criteria for achieving a conceptual statement label was quite high and the numbers of these types of statements were very low, generally less than 2 statements for any subject.

		Sum of Squares	Df	Mean Square	F	Sig.
Declarative_Post	Between Groups	1.348	1	1.348	1.767	.193
	Within Groups	24.417	32	.763		
	Total	25.765	33			
Procedural_Post	Between Groups	.456	1	.456	.180	.674
	Within Groups	80.985	32	2.531		
	Total	81.441	33			
Conceptual_Post	Between Groups	2.375	1	2.375	5.425	.026*
	Within Groups	14.008	32	.438		
	Total	16.382	33			
Declarative_Change	Between Groups	.015	1	.015	.011	.917
	Within Groups	42.227	31	1.362		
	Total	42.242	32			
Procedural_Change	Between Groups	.015	1	.015	.006	.939
	Within Groups	79.500	31	2.565		
	Total	79.515	32			
Conceptual_Change	Between Groups	.545	1	.545	.762	.389
	Within Groups	22.182	31	.716		
	Total	22.727	32			
KMD_Post	Between Groups	4.456	1	4.456	.318	.577
	Within Groups	448.985	32	14.031		
	Total	453.441	33			

KMD_Change	Between Groups	5.470	1	5.470	.315	.579
	Within Groups	538.773	31	17.380		
	Total	544.242	32			

Table 34: KMD comparison for student versus teaching profiles at lesson 5

	N	Minimum	Maximum	Mean	Std. Deviation
Unweighted Post-conceptual	22	.00	2.00	.3636	.58109
Valid N (listwise)	22				

Table 35: Descriptive statistics for lesson 5 student profile participants' KMD conceptual statements

Descriptive Statistics (Teaching Profile)

	N	Minimum	Maximum	Mean	Std. Deviation
Unweighted Post-conceptual	12	.00	2.00	.9167	.79296
Valid N (listwise)	12				

Table 36: Descriptive statistics for lesson 5 teaching profile participants on KMD conceptual statements

Finally, as a primary goal of this study was for subjects to link what they were learning about students to their teaching, it may in fact be that the very few subjects who were assigned a combination of student and teacher profiles were the "ideal" subjects and had higher KMD and efficacy scores at posttest. However, by Lesson 5 there were only 6 such subjects, and they did not look significantly different from other subjects in any of their posttest scores.

Summary

Efficacy tended to increase, regardless of condition, but did so the most for the unguided group. KMD tended to decrease for the video conditions, while it increased for the control condition. Modesty tended to stay the same, except in the guided condition, where it tended to increase. Thus, the hypothesis about the positive impact of the video lessons on efficacy was confirmed; the hypothesis that modesty would temper the increase in efficacy was not confirmed; and the hypothesis that increasing KMD would drive an increase in efficacy was also not confirmed. In other words, condition did matter for all three variables, but not always in the ways that were hypothesized. One possible explanation for these findings is that efficacy and KMD were both set within the context of, and application to, teaching in this study, and subjects in the video conditions tended to increasingly focus on students rather than teaching as they progressed through the video lessons, diminishing the potential for change in efficacy and KMD in these conditions.

Chapter 5: Discussion

Summary of Findings

This study aimed to increase pre-service early childhood teachers' efficacy for teaching mathematics through video lessons of children engaging in mathematical activities and discussing their ideas. The vicarious experiences offered through the videos of children engaged in classroom-like activities imparted information about how children understand and develop early childhood mathematical concepts, offered ideas for teaching children, and offered ideas about correcting children's misconceptions. As such, they were successful in increasing participants' efficacy for teaching math.

In addition, this study investigated how and why efficacy increased, looking at whether the increase in efficacy corresponded to an increase in KMD and also whether this increase in efficacy related to increased intellectual modesty in the form of greater awareness of the limits of one's knowledge. Unfortunately, participants' KMD did not increase as a result of the videos and there was no evidence of a relationship between change in KMD and change in efficacy. That is, competence did not appear to be related to confidence! Furthermore, participants' intellectual modesty only increased when they received feedback on their responses within the video lessons, and the change in modesty also appeared unrelated to the change in efficacy.

However, one important factor can account for the unexpected findings in regard to KMD. Participants appeared to focus their attention more and more on the students as they progressed through the video lessons, rather than on the teacher and teaching. KMD was measured through participants' responses to the question, "What would you do next in teaching?" following the viewing of the video on children's understandings of patterns. There was therefore a mismatch between the focus of participants' attention during the intervention and the way that KMD was measured in this study. Consequently, participants may have been increasing their awareness and understanding of how children develop mathematical ideas, but were not yet able to use this new knowledge in their considerations of teaching and instruction.

Furthermore, the decrease in KMD appeared to be the result of decreasing declarative statements. Declarative statements relate to imparting mathematical facts to students and are the most basic form of understanding students' mathematical thinking. Mathematical procedures take these facts and apply them, and conceptual understanding provides insight into why procedures work in the way they do and how they generalize across contexts. Thus, while participants in the video lesson conditions were not increasing in their conceptual statements, they were decreasing in declarative statements, indicating a move in the right direction towards less focus on memorization and fact-based instruction.

There was also a significant decrease in response length to the question used for coding KMD from pre- to posttest. Perhaps as a result, KMD scores also decreased from pre- to posttest. Still, this does not offer a full explanation for the decreasing KMD scores because control subjects actually increased in KMD, despite also having a decrease in response length from pre- to posttest. Upon further exploration, it became clear that one of the videos on children's understandings of pattern that was used for the pre- and posttest was richer in mathematical content than the other video with which it was counterbalanced. As a result, participants who viewed this richer video at pretest declined in KMD as they responded to the less rich video at posttest, while the opposite was true for participants who viewed the richer video at posttest. While this is a welcome finding in helping to explain why KMD appeared to decrease as a result of the video lessons, the results were complex. When only the participants who had the richer video at posttest were examined, there were no longer any group differences in KMD at posttest.

In other words, these participants increased in KMD, regardless of whether they were in the control, unguided video, or guided video conditions. Simply having the opportunity to respond to a richer video made the difference in what these subjects were able to demonstrate about their KMD. Similarly, for these participants, group differences in efficacy and modesty also disappeared at posttest. That is, all participants tended to increase in efficacy and increase in modesty, regardless of whether they were in the control, unguided video, or guided video group. This was not the case for the subjects who viewed the richer video at pretest, who maintained their group differences in efficacy and modesty at posttest. Still, these results point to the value of the videos in imparting knowledge about mathematical content and students and increasing efficacy of teachers as a result. Nonetheless, the small sample sizes for these analyses also require caution in regard to the interpretation of the results.

Implications

Teachers are crucial change agents in education reform, and teachers' beliefs about themselves and about the content they are teaching are precursors to change. In fact, research has consistently found teachers' efficacy to be among the most powerful influences on receptivity to change (Tschannen-Moran & McMaster, 2009). We know that early childhood teachers are often scared of math and often resist or struggle in its teaching. Therefore, finding a way to increase their efficacy is critical to improving the quantity and quality of early childhood mathematics instruction. However, in doing this, it is also critical to find ways to increase teachers' efficacy in a way that is consistent with their actual skill and knowledge level. That is, as teachers gain confidence, they should also gain competence. Increased efficacy should go hand-in-hand with more sophisticated hypotheses about children's thinking and pedagogical tools for using those hypotheses to direct questioning and discussions, pose tasks, and guide learning in the classroom, because efficacy in the absence of accurate knowledge is overconfidence, which can be detrimental to effective teaching.

Teacher efficacy is critical to high quality instruction and is therefore an important consideration in the training and support of teachers. Teacher efficacy has been linked to teachers' engagement, enthusiasm, effort, persistence, resilience, and commitment to teaching. It corresponds with the encouragement teachers offer students, their openness to innovation in the classroom, and overall achievement. It also influences students' efficacy, motivation, and achievement. However, there is a crisis of low efficacy for teaching math in early childhood teachers, which is particularly important given that they are responsible for children's initial exposure to math as a formal activity. As a result, early childhood teachers must be equipped with knowledge of the math they will teach, their students' current mathematical understandings, and strategies for fostering mathematical ideas in a way that matches students' learning needs so that they can feel confident and teach effectively (Kilpatrick et al., 2001).

Of course, helping teachers become proficient in understanding their students' reasoning and giving them the appropriate pedagogical tools for effective questioning and scaffolding in the classroom is quite difficult. Of key importance in helping teachers is devising ways to link their knowledge of content, students, and curriculum with the act of teaching. "After interpreting students' work, teachers need to be able to use their interpretations productively in making specific instructional decisions: what to ask, tasks to pose, homework to assign." (Kilpatrick et al., 2001, p. 350) Teachers' must possess a psychological understanding of children's mathematical learning and thinking, employ sensitive assessment techniques for determining what children already know and what they need to learn, and use developmentally appropriate teaching methods based on that understanding (NAEYC-NCTM, 2002). When teachers are able to do this, their instruction is clearer, more focused, and more effective (Kilpatrick et al., 2001).

Providing teachers with such understandings demands fundamental changes to teacher preparation and professional development (Kilpatrick et al., 2001). Teachers' opportunities to learn more about mathematics, students' learning and thinking, and their teaching practice have been very limited (Kilpatrick et al., 2001). Video offers a promising way of bridging the divide between knowledge and teaching, as it can act as a vicarious experience from which to gain knowledge and see it applied in context. As Ginsburg et al. (2009) note, video is like a manipulative that can be controlled, segmented, organized, reviewed, discussed, and debated to make the connection between knowledge and practice. Not only can it bridge knowledge and practice, but it also can slow down the on-the-spot instructional decision-making that teachers must constantly do. Videos are also a cost effective way to reach large numbers of teachers. But what are the key features to be included in such videos to be most effective at promoting efficacy and enhancing instruction?

Such videos must promote KMD, which effective teachers consistently possess. KMD is not just mathematical content knowledge, but is also knowledge of what students know, how they know it, and how they learn it. It's knowledge of how to teach mathematical ideas to children to address current understandings and misunderstandings and for conceptual understanding. It's knowledge of curriculum, of what models, diagrams, manipulatives, activities, and materials most help students learn new ideas or formalize their knowledge. In effect, it is the knowledge that will bridge research on teaching with the practice of teaching.

This study also points to the need for these videos to concretely connect knowledge of students to the pedagogy of teaching these students, an area that remains challenging.

Unfortunately, research has struggled to find ways of linking knowledge with practice. However, it is critical to improve teachers' limited specialized content knowledge for teaching, and therefore research is needed to uncover the critical aspects of teacher preparation and professional development that will impact teacher knowledge and thereby teaching and student achievement. This study attempted to do this, but like the prior research, struggled to find significant changes (Robelen, 2011). Like other prior research, this study pointed to teachers' ability to identify gaps in students' knowledge or identify students' needs, but struggled to find evidence of teacher understanding of instructional materials or ways to act on information about students. But "effective programs of teacher preparation and professional development cannot stop at simply engaging teachers in acquiring knowledge; they must challenge teachers to develop, apply, and analyze that knowledge in the context of their own classrooms so that knowledge and practice are integrated" (Kilpatrick et al., 2001, p. 380).

Finally, especially when working with in-service teachers, intellectual modesty is important to consider because without it teachers are much less open to changing viewpoints and practices, to incorporating new ideas into their existing practice. Thus, videos must be interactive and include feedback so that teachers have the opportunity to reflect on their current knowledge, beliefs, and practices in light of best practices.

Limitations & Future Directions

A major limitation of this study was the mismatch between the focus of the video lessons and the measures of efficacy and KMD. The video lessons focused the majority of participants' attention on the student and her thinking, while the measures of efficacy and KMD both focused on teaching. Thus, this study likely did not capture everything related to efficacy and KMD that was changing as participants progressed through the intervention. This was surprising, given that the videos attempted to focus participants on both students and teaching. Each video lesson asked, “What would you do next in teaching?” after the lesson was completed, and the guided video condition included similar questions within the video lessons.

Another major limitation was the fact that the two videos used for the pre- and posttests were not as equivalent as predicted. One of the videos appeared to be significantly richer in mathematical and cognitive content and therefore produced greater evidence of KMD than did the other video. As a result, analyses of the entire sample may not be as “clean” as desired. Furthermore, the sample size of this study was rather small and further studies should aim to recruit more participants and from a greater range of universities. Similarly, this sample included only pre-service early childhood teachers, and including in-service teachers within such a study or in a study of their own would prove useful and interesting.

Future studies should focus on the features of the videos that most capture participants' attention and that would most help them to connect their knowledge of students to their pedagogy. “Alternative forms of teacher education and professional development that attempt to teach mathematical content, psychology of learning, and methods of teaching need to be developed and evaluated to see whether prospective and practicing teachers from such programs can draw appropriate connections and apply the knowledge they have acquired to teach mathematics effectively.” (Kilpatrick et al., 2001, p. 382) Professional development must help teachers analyze teaching problems in the moment. Videos have proven to be an effective and practical medium for such teacher preparation and learning.

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Appendix A: Teacher Efficacy Scale (Short Form)

A number of statements about organizations, people, and teaching are presented below. The purpose is to gather information regarding the actual attitudes of educators concerning these statements. There are no correct or incorrect answers. We are interested only in your frank opinions. Your responses will remain confidential.

INSTRUCTIONS: Please indicate your personal opinion about each statement by checking the appropriate box to the left of each statement.

Strongly Agree	Moderately Agree	Agree slightly more than Disagree	Disagree slightly more than Agree	Moderately Disagree	Strongly Disagree	
☐	☐	☐	☐	☐	☐	1. The amount of math a student can learn is primarily related to family background.
☐	☐	☐	☐	☐	☐	2. If students aren't disciplined at home, they aren't likely to accept any discipline.
☐	☐	☐	☐	☐	☐	3. When I really work at teaching math, I can get through to most difficult students.
☐	☐	☐	☐	☐	☐	4. A teacher is very limited in what he/she can achieve in teaching math because a student's home environment is a large influence on his/her achievement.
☐	☐	☐	☐	☐	☐	5. If parents would do more for their children in math, I could do more.
☐	☐	☐	☐	☐	☐	6. If a student did not remember math information I gave in a previous lesson, I would know how to increase his/her retention in the next lesson.
☐	☐	☐	☐	☐	☐	7. If a student in my class becomes disruptive and noisy, I feel assured that I know some techniques to redirect him/her quickly.

Strongly Agree	Moderately Agree	Agree slightly more than Disagree	Disagree slightly more than Agree	Moderately Disagree	Strongly Disagree	
☐	☐	☐	☐	☐	☐	8. If one of my students couldn't do a class math assignment, I would be able to accurately assess whether the assignment was at the correct level of difficulty.
☐	☐	☐	☐	☐	☐	9. If I really try hard to teach math, I can get through to even the most unmotivated students.
☐	☐	☐	☐	☐	☐	10. When it comes right down to it, a teacher really can't do much for students' math learning because most of a student's motivation and performance depends on his or her home environment.

Note: Items 3, 6, 7, 8, and 9 measure PTE, while items 1, 2, 4, 5, and 10 measure GTE.

Appendix B: Readings and Guided Video Lessons

Learning to Count

Saying the number words "One, two, three..." or even "One thousand fifty six, one thousand fifty seven, one thousand fifty eight…" seems to us so easy and trivial that we fail to appreciate how complex is the task of learning them in the first place. For children, learning number words that they frequently hear poses several difficulties. It involves both memory and creativity; the child must first memorize the small basic numbers, and this having been accomplished, can then learn rules for generating larger and even novel numbers.

One of the first aspects of number children try to learn is the ordered counting words, "One, two, three..." Virtually all children engage in this learning process and typically encounter two distinct problems. First, there are a lot of numbers to learn. Second, they must be said in a certain way and not others. Children solve the first problem—the apparently endless number of number words—by limiting what they try to learn at any one time. Thus, they usually struggle with one through five before they try six through ten. They carefully regulate their own learning and attempt only a little more than they can master at any given time. The second problem is more difficult. Why is it wrong to say "one, three, two" but right to say "one, two, three?" Gradually, children resolve this difficulty too and with practice learn the sequence of number words slowly. They begin to see a pattern in the apparent chaos of number words. They perceive that numbers are like a song: the number words involve sequence of sounds in the same order all the time. Parents try to make the task simpler by imposing on it some rhyme: "One, two, buckle my shoe; three, four, close the door..." This is intended as a device to reduce the load on brute memory and thereby make the numbers easier to learn. Young children make a great discovery when they learn that "two" always comes after "one" and "three" always after "two." Unfortunately the beginning of the sequence—in English, the first 12 or so numbers—is completely arbitrary. There is no rational basis for predicting what comes after a certain number. Children have a lot to memorize, and it's not until they are about three years old that they learn to say the beginning numbers in the proper sequence (Baldwin & Stecher, 1925).

After a period of time, they discover that the numbers after about 13 contain an underlying pattern. Using it, children develop a few simple rules by which to generate the numbers up to about 100. We can ourselves get some appreciation for the young child's difficulty by performing a little experiment on our knowledge of Jack and Jill. "Let us ask a friend to test us in this fashion: What word comes before down? What word comes after his? Go through the verse saying only every second word. Say only every third word. Say the whole verse backward." The task is quite hard, and it is the same kind of task children have to do with "one, two, three."

Children also acquire some beliefs about the number words. At the age of three and a half, Josh demonstrated that he could count to 12 without error. The interviewer then asked Josh to listen to him (the adult) count, and indicate if any mistakes were made. The opportunity to correct an adult is no doubt novel for a young child, and so Josh listened with intense interest.

Interviewer (I): Listen to me now and tell me if I make a mistake. One, two, three, red.

Josh (J), interrupting with a laugh: No, red was color.

I: OK, I'll start again. One, two, three, four, five, seven.

J, laughing again: No. You're wrong!

I: Sorry. One, two, three, four, five, six, seven, eight, nine, ten, nine, eight.

J: No, you made a mistake. After ten comes something else.

I: OK. I'll do it right this time. Nine, ten.

J: No. That's wrong. You have to start with "one."

So Josh not only learned to say the number words in the conventional order, but also believed that number words are different from color words; that you cannot skip numbers; that numbers cannot be said in backwards order; and that you must always start counting with the number "one." Children do not learn simply to execute behavior; they also develop theories about their behavior.

A few general principles about learning to count:

1. Children search for meaning. Children try to make sense of the world by looking for an underlying pattern, for a deeper meaning. Sometimes it does not exist, as in the case of the first 12 or so numbers. But often it does: the larger numbers display clear underlying regularities. Having searched these out, children use them to develop rules for producing the larger numbers themselves. The exploitation of underlying patterns makes intellectual work easier and more efficient, and allows children to avoid the drudgery of rote memorization.

2. Errors are meaningful and informative. Children's errors usually make sense. Often they provide insight into what children are really trying to do. Thus, they say "twenty- ten" because they are trying to capture the underlying structure of the counting numbers, not because they weren't thinking.

3. Children can use different learning strategies depending on environmental circumstances. When there is no underlying pattern, children memorize the numbers in a rote fashion. When the pattern is there to be found, they often exploit it. Some learning is done by rote; some is meaningful. Children do not learn in only one way.

4. Children can learn in a wide range of circumstances. Children learn a great deal about numbers outside of school, without instruction or special help; indeed their parents are often unaware that they are trying to learn numbers. Also, they manage to learn even though their experience is often confusing and unplanned. For example, children may hear an adult counting by twos or by fives before they have mastered counting by ones. Nevertheless, they manage to learn.

COUNTING LESSON

This lesson will focus on the topic of counting. You will watch a series of short video clips and answer questions about them. After you submit your responses, you will be shown examples of other students' successful responses so you can compare them to your own. These responses are intended to provide you with some ideas about the material, but they are not necessarily comprehensive.

This lesson features a boy named Lateek, age 4. As you proceed through the video clips and questions, keep in mind: "What does the child know?"

Clip 1 (00:32-00:53)

1. Watch the video closely two times. What stood out to you in this clip?

2. What does Lateek know about counting?

Here are some comments from other students:

- He counts using his fingers.
- He appears to know that each number goes with a different finger.
- He skips numbers 7 through 10.
- He picks up at 11 and counts 11, 12, 13, 14, and 16, omitting 15.
- He counts most numbers without hesitating, although he hesitates before 14.
- He appears confident in his counting abilities.

Clip 2 (00:48-00:59)

3. Lateek skipped the number 15 when he counted. Why didn't the interviewer correct him?

Here are some comments from other students:

- The interviewer was most likely demonstrating a particular teaching approach.
- Perhaps she is planning to continue to test his knowledge in other ways, thus she avoids correcting and teaching him at this time.
- She may want to facilitate a positive learning environment, free of corrections at this point.

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4. Do you think the interviewer should have said "good job" or not? What would you have done?

Here are some comments from other students:

- Saying "good job" is a way for the interviewer to build rapport with the student and provide encouragement.
- It is important to facilitate a neutral learning environment where the child is not feeling evaluated on a continuum of good or bad.

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5. What do you think of her question, "Can I start counting?"

Here are some comments from other students:

- By asking the child permission to count, perhaps she intends to create a rapport in which the child perceives he is on equal footing with the adult.
- She is giving him the chance to say "no," which could be a potential problem if she wants to continue the interaction.
- Another way to phrase this question might be as a statement: "Now it's my turn to count."

Clip 3 (00:54-01:16)

6. What does the interviewer do here? Why? What would you have done?

Here are some comments from other students:

- She counts without using her fingers, but skips the numbers between 6 and 11 as he did, perhaps to see if he would accept this version as correct.
- She includes 15, which he omitted.
- One thing teachers can learn from this interaction is that it's OK to make mistakes in front of children.
- Mistakes can provide room for interesting discussion about the mistakes and possibly reveal a lot about a child's reasoning.

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7. How does Lateek react? What does it mean?

Here are some comments from other students:

- He seems to count along with the interviewer, sometimes mouthing the words that go with the numbers.
- When she jumps from 6 to 11 his face changes expression. Perhaps he recognized that something was missing, or perhaps he is getting ready to express the bigger numbers?

Clip 4a (01:11-01:16)

8. Now we are going to focus on what the interviewer did. What do you think about the question she asks? What would you have asked?

Here are some comments from other students:

- She asks, "Is that good?" to invite him to react to what she did, without giving away her own opinion.
- By asking for his opinion, she is trying to get him to respond to her answer and encourage discussion about his ideas.
- She is putting him on equal footing with her.
- I may have asked, "Was that right or did you notice anything wrong" to initiate a discussion about the missing numbers.

Clip 4b (01:15-01:21)

9. What did Lateek mean that you count "one 16"?

Here are some comments from other students:

- He said that you should count 16 only once.
- He is instructing her to count each number only once and may have thought she said 16 twice.
- Perhaps he does not know 15 is a number in the counting sequence and thinks 15 and 16 are the same number when he listens to her count.

Clip 4c (01:17-01:31)

10. Why did the interviewer accept his answer? What do you think would have happened if she had challenged him? Would you have approached this situation differently, and if so, how?

Here are some comments from other students:

- She is verifying his comment and perhaps giving him a chance to correct her.
- Had she disagreed with him, she may have learned more about what he does or doesn't know about 15.
- Note that she interpreted his statement to mean that she said "16" twice.

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11. What is the interviewer trying to find out? What does Lateek know about counting?

Here are some comments from other students:

- The interviewer is trying to figure out if Lateek recognizes that he skipped number 15.
- By his correction, it appears he thought it was OK to skip those numbers or still does not realize the mistake in his counting.
- Lateek knows that each number occurs only once in the counting sequence.

Clip 5 (01:27-01:49)

12. What do you think of the interviewer's questions? Why does she ask if it's OK to use her fingers?

Here are some comments from other students:

- The questions involve him in what she is about to do and make him feel like he's in charge.
- The interviewer is exploring his competence and his reactions to her counting.
- She asks to use her fingers perhaps to see if he might express any ideas about using fingers to count.

Clip 6 (01:43-02:01)

13. What is the most important thing that happened here? How do you explain it?

Here are some comments from other students:

- At the number 6, he took over the counting from the interviewer, and counted correctly from 7 to 10.
- It turns out he may actually know his numbers through 10, but just needed a little help to get past 6, or needed someone else's fingers to guide him.

- He may have just remembered these numbers after several trials.

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14. In what ways, if any, did you see the interviewer help him with 7?

Here are some comments from other students:

- She did not actually help him with 7.
- Starting with 6, she gradually stopped counting and let him take over.
- At first she is simply modeling counting, but as he starts to participate, she drops out.
- Watching her fingers (instead of his own) may have helped him.

Clip 8 (02:17-02:41)

15. What did Lateek do this time? Why do you think he was able to do this?

Here are some comments from other students:

- He was able to count to 16 successfully, missing only 15.
- Practice seems to have helped him gain confidence in counting to 10.
- He stopped trying to use his fingers after 10, which may have helped him count to 16—although he continues to skip 15, perhaps as a duplicate of 16 and still not aware that 15 and 16 are different numbers, or that 15 even exists as a number.

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16. What did the interviewer do? Why do you think she did this?

Here are some comments from other students:

- She counted along with him, but said the number words only after he did, probably to avoid guiding him too much.
- Her technique is a good idea for teachers because the student feels supported while the teacher is not giving away the answer.

Clip 9 (02:50-03:01)

17. How would you interpret what he did here? What would you do next?

Here are some comments from other students:

- He probably interpreted her expression about number to mean "height" rather than higher numbers.
- This is a humorous example of how a child's egocentrism leads them to interpret a question from their own point of view.
- After his response, I would clarify the question by asking, "How high can you count without using your fingers?"

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18. What does Lateek know about counting?

Here are some comments from other students:

- At first, it appears Lateek does not to know how to count correctly to 10 or 16, but with repeated attempts, he demonstrates more competence.
- Children's performance can be variable at this age and with repetition and instructors guidance, they may reveal knowledge or skills not previously seen.

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19. What does this video tell you about assessment?

Here are some comments from other students:

- It's important to be patient and give the child multiple opportunities to express him or herself.
- Sometimes a poor performance doesn't mean poor knowledge.

Clip 10 (00:00-03:01)

20. Now watch the video clip again in its entirety. What do you think this child really knows about counting? Explain how you know.

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21. What would you do next in teaching? You can watch the video again.

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22. What did you learn from this exercise? How did your thinking change during the lesson?

Congratulations, you have completed the lesson on counting. Good job!

Enumeration: Learning to Count Things

For several years, young children engage in the pleasant struggle of learning to attach the numbers they can already say to the objects they can see. They begin by learning to count small numbers of objects. Then they must repeat the learning process with larger numbers. The typical preschool child has no difficulty in counting very small collections, but when a collection's number surpasses a certain value (which varies with age), the child's enumeration falls apart. For example, Vicky, a 4 year old, counted two objects accurately, and then four objects accurately, but when the two and the four were combined she enumerated in a very unsystematic fashion and got ten for an answer. Indeed, her behavior appeared so incompetent when she was counting the larger collection that she seemed like a different child from the one who counted the small collections with ease and accuracy. When larger numbers were involved, she seemed not to understand enumeration at all.

Vicky is not atypical. Young preschoolers achieve sufficient mastery of these concepts and skills to enumerate small sets with considerable success. But they make mistakes in enumerating larger sets, mainly because they are not skilled at considering things once and only once. This is hard for them because they lack a systematic plan for keeping track of things and therefore rely on rote memory, which soon becomes overburdened. Young children's enumeration of "small" sets is often quite accurate, whereas their counting of "large" sets involves sloppiness and inconsistency. The first time children count a large set, they may get one answer; the next time, another. They do not know which answer is right, or may even think that both are. At 4-6, Samantha was presented with a collection of candies randomly arranged on a table. Her job was to count the candies. A very precocious girl, she had no difficulty in saying the number words; indeed, she could easily reach 100. Yet in counting the candies, she made many errors. On one try, she would get 23; on another, 24; on yet another, 22. Which was right? She had no idea. Her procedure was to point to each candy in its original location; she did not bother to push any candies aside after counting them. Because of this, she forgot which were counted and which were not. She counted several candies twice and several not at all, and as a result got different results each time. This inconsistency did not disturb her in the slightest. Children sometimes believe the same collection can be characterized by two or more numbers: yes, it has 14, and it also has 15!

Why are there so many errors on a task that to us seems trivially easy? One reason is that young children seem to forget which items have been touched and which have not. They lack a systematic plan for making sure that each picture has been considered once and only once. Using a haphazard procedure, they touch the circle first, then the triangle, then the trapezoid, and so on. Doing this puts a great strain on children's memories; to be correct, they must remember exactly which items have been counted and which have not. For example, on the fourth choice, they must remember that they have already touched the circle, triangle, and trapezoid. Since there is so much to keep in mind at the same time, they frequently forget and therefore count some things twice and some not at all. Young children are not very methodical or organized. They do not make it easy for themselves to consider things once and only once.

To count things accurately, children must learn how to do at least the following:

1. To realize that you can count anything, objects or even ideas. You can count peas or elephants; you can even count the number of candies you ate yesterday or images of unicorns.

2. To say the number words in their proper order. You cannot count things accurately if you do not know the number words. You will be wrong if you say, "One, three, four, seven…"

3. To count each member of the relevant collection once and only once. That is, you must not count one object two times or forget to count an object.

4. To match up each number word with each thing. The word "one" has to go with this object and "two" with that object. "One" cannot go with both. Each object must be assigned one and only one number word.

We will see that children master many of these principles at a very young age, and can apply them successfully to relatively small collections. Problems occur mainly when larger numbers are involved.

In summary, young children spend several years learning how to attach numbers to things. Counting is a complex activity requiring several concepts and skills. The child must realize that anything can be counted, objects or even ideas; must have the ability to say the number words in their proper order; must be able to count each member of the relevant collection once and only once; and must be able to match up each number word with each thing. Young preschoolers achieve sufficient mastery of these concepts and skills to enumerate small sets with considerable success. But they make mistakes in enumerating larger sets, mainly because they are not skilled at considering things once and only once. This is hard for them since they lack a systematic plan for keeping track of things and hence rely on rote memory, which soon becomes overburdened. With development, children overcome these difficulties, first with smaller numbers and then with larger ones. They develop several useful strategies for counting. They learn to push objects aside as they count, thereby reducing the strain on memory and making it easier to count things once and only once. Perhaps partly as a result of repeated counting, they learn to perceive small numbers directly and no longer need to count them. Then they learn shortcuts for counting: instead of considering individual objects, children group them, counting for example by twos or using arithmetic operations to get the totals. Gradually children learn to enumerate absent or imaginary objects, often by enumerating various substitutes for real objects: images, fingers, or written symbols.

ENUMERATION LESSON

This lesson will focus on the topic of enumeration. You will watch a series of short video clips and answer questions about them. After you submit your responses, you will be shown examples of other students' successful responses so you can compare them to your own. These responses are intended to provide you with some ideas about the material, but they are not necessarily comprehensive.

This lesson features a boy named Harry, age 4. As you proceed through the video clips and questions, keep in mind: "What does the child know?"

Clip 1: 00:01-00:05

1. The interviewer asks, "Can you count how many red bears there are?" These three highlighted words/phrases might play an important role in the child's interpretation of this question. What is one of the possible meanings of each?

Here are some possible interpretations of the highlighted text:

- The word "can" may connote either a request or a question about whether the child is able, and may not elicit the response intended by the interviewer (if the child answers "yes" or "no").
- The word "count" most likely refers to the process of enumeration.
- The phrase "how many" refers to cardinality and figuring out the total value of the set.

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2. How do you think the child will count the bears? What method(s) might he use to get the answer?

Clip 2: 00:06-00:15

3. He was clearly trying to count. What else did you see?

Here is what a student saw in this clip:

- He started out counting, "1, 2 ..." and then paused for a moment. He decided to start again, and after "2" he proceeded directly to "3 ... 4, 5, 6." He clearly did not count two objects in the group.

Clip 2: 00:06-00:15

4. Now watch the clip again. What do you make of the above interpretation? Were there any important things you saw this time that you didn't see before?

Here is what an "expert" saw in this clip:

- Here are a few things that I saw that might be important. Harry started out by counting "1, 2," and then paused for a moment. He decided to start again, and after "2" he proceeded to count "3 ... 4, 5, 6." He touched each object in order, and he said the numbers correctly from 1 to 6. At the end, he said two numbers that were out of place, one smaller than the total and one larger. He is very careful about touching each object and counting it.

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5. At the end, he said "3" and then "10." Why do you think he did this?

Here is how an "expert" interpreted this clip:

- Harry may have been confused by the arrangement of the bears, as they were not laid out in any order. As he moved on after the second bear, he may have drawn a conclusion as to an order in which he would count the bears, and therefore he started over. When he reached the last bear, he must have been guessing at a number, as he did point to the bear, but maybe didn't know the number after "6."

Clip 2: 00:06-00:15

6. What does this clip say about Harry's enumeration skills? What about his understanding of number? Watch the clip again if you like.

7. At this point, what would you do next? What task would you use and/or what question would you ask?

In summary, one could say that Harry is intent on listing each bear once and only once, and he seems to have a good grasp of the sequence of numbers 1 through 6. He may or may not understand that the last number counted tells how many there are altogether.

As for a follow-up task, one could ask him to count again and be more careful, give him a smaller number of bears to count, or have him line them up for more systematic counting.

Clip 3: 00:18-00:22

8. What did the interviewer do before asking another question? Why?

Here is how an "expert" interpreted this clip:

- The interviewer cleared away the bears that Harry had been counting and removed two, presumably to test something about the hypothesis that smaller numbers are easier to count, and would therefore give a better measure of Harry's competence with enumeration than the larger set would.

The factors to consider here are; the quantity of objects used (why 5?), and the phrasing of the question to get the total number versus just counting:

- Can you count these for me?
- How many are there?
- Can you count to find out how many there are?

Clip 4: 00:19-00:27

9. What did you see?

Harry points to each bear as he counts, 1 to 5. He has one-to-one correspondence and is consistent and accurate with the number sequence up to 5. He says five with some authority.

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10. What does this clip say about Harry's enumeration skills? What does he understand about number?

He seemed to enumerate OK when the number was smaller, even though the objects were in a haphazard arrangement.

Clip 5: 00:28-00:47

11. Why do you think Harry proposes putting the bears in a line?

To organize them, which makes a simpler task of one-to-one correspondence while counting.

Clip 6: 00:48-01:02

12. Why did the interviewer ask the question, "How many bears did you have before ... when you counted them?"

To confirm whether Harry remembered the last number counted. If he did, then it is possible to determine whether he knows that it indicates cardinality; if he doesn't, then any response is ambiguous.

Clip 7: 01:03-01:12

13. Why did the interviewer rearrange the bears?

To see if Harry understood cardinality -- that is, that the last number indicates the value of the set as a whole -- and therefore could answer without counting. This is also sometimes referred to as conservation of number.

Clip 8: 01:13-01:22

14. Why did the interviewer cover the bears?

To see whether Harry did not need to count each time he is asked how many. If he understands cardinality, he does not need to count again and could rely on the last number counted if he can remember it.

Clip 9: 01:22-1:38

15. Why did the interviewer ask this question? What is another way to find out what Harry knows?

The interviewer may have been trying to determine whether Harry did in fact remember the last number counted and would use it as the cardinal number.

Clip 10: 00:00-01:38

16. Now watch the video clip again in its entirety. What do you think this child really knows about counting and number? Explain how you know.

17. What would you do next in teaching? You can watch the video again.

18. What did you learn from this exercise? How did your thinking change during the lesson?

Congratulations, you have completed the lesson on enumeration. Good job!

Simple Arithmetic: Learning to Calculate

Children's initial approach to addition and subtraction is concrete. At the age of 3 or so, children cannot deal with addition and subtraction problems unless they have concrete objects immediately before them. Such problems typically take the following form. The child is presented with a toy rabbit that has two toy carrots, right in front of it. Another toy rabbit is introduced that has three carrots, again clearly visible. The child is told how many carrots each rabbit has and asked how many carrots the rabbits have "all together."

There are of course many possible variations on these problems. The story might involve, for example, a rabbit that has three carrots and then gets two more. The child can then be asked, "How many does he have now?" But in all cases, the key feature is that the problem involves objects the child can see and simple instructions using everyday words like "all together" and "take away," not "addition" and "subtraction."

Given problems like these, very young children—2 and 3 year olds—may not even interpret the problem correctly. They apparently do not realize that the answer can be obtained by combining the sets (literally, or by counting), and therefore do not attempt to count them. By about 4 years of age, some children are able to interpret the problem correctly and achieve a solution by counting the objects one by one, although they are not always successful. If there are three objects plus two more to be counted, the child begins by counting the first set—"one, two, three"—and then continues with the second—"four, five." This technique is called "counting all."

For the young child, subtraction also involves a process of counting objects. Suppose the problem is: "Johnny had five candies and lost two of them. How many does he have left?" In this case, the child actually removes two of the five candies and simply counts the remainder to get the result.

The development of early calculation has several interesting features. One is that there is a thin line between the child's counting and addition (or subtraction). For the young child, addition is simply an extension of counting objects (enumeration). From the child's point of view, adding is the counting of things in two (or more) sets combined. That is, if you want to know how many objects are in some sets, count all of the objects in all of the sets. This may seem very simple and very concrete, and it is. But this approach to addition is quite legitimate: it is an informal expression of the interpretation of addition as the union of sets. Subtraction too is tied to counting: Subtracting is the counting of what is left after things have been removed from a set.

A second interesting point is that over time the young child spontaneously invents strategies like "counting on," which are more efficient and easier to use than the "counting all" strategies employed at the outset. Note that this development often occurs without the benefit of instruction or adult intervention. Indeed, most adults are not even aware that the child is constructing any strategies for addition. Intellectual development in the natural environment (but not necessarily in school) is characterized by a trend towards mental efficiency and economy. On their own, children tend to develop more and more effective mental procedures.

A third point is that the child's early calculation suffers from severe limits. For one thing, it is often not very accurate. The child may know generally what to do but frequently makes mistakes in implementation, so that answers are wrong. For another thing, the child does not yet fully understand the relations between addition and subtraction. At 4 years of age, Andy was given a series of concrete addition and subtraction problems. The first problem involved adding four candies to a set of three. Using a counting all method, he easily determined that there were seven candies all together. Now what happens when you take away three candies? He took away three and counted the results, getting four again. He did not spontaneously make the inference that if you put back what you took away you will get the amount that you started with. Then the interviewer repeated the process. What happens if you take away three? Again Andy counted. And what happens if you put back three? Again Andy counted to get the result.

Why does this happen? The preschool child seems to be struggling so hard to execute each of the operations separately that he does not examine the relations between them. If his attention is exclusively centered on doing correct addition, he cannot easily see how it relates to subtraction. And if he does not see that addition undoes the subtraction, then he has to compute each result separately each time. The young child must do unnecessary work because, mired in the operational details, he has missed the larger picture. Only as the child becomes more and more skillful at the operations of addition and subtraction, does he gradually learn to perceive the relations between them. In this case, efficient performance seems to be a prerequisite to basic understanding.

Suppose that children are asked to add imaginary objects: how many are three Martians and two Martians? Now children cannot simply count objects that they can see in front of them. Instead they have to represent the imaginary objects. They have to use something—fingers, or blocks, or mental images, or tallies—to stand for the imaginary objects. The children may then perform the arithmetic operations on the representations, such as by counting all. That is, since it is not possible to work with imaginary objects directly, the child must create substitutes or surrogates for them, which may then be manipulated in various ways.

Summary

In the natural environment, children are frequently faced with problems that adults can solve by means of written mathematics. Lacking this, children gradually develop from their intuition and their counting skills a practical arithmetic. This involves at least two steps: the interpretation of problems and the implementation of a solution. Given concrete objects to add, 2 and 3 year olds may not interpret the problem correctly: they may not even realize that it is necessary to combine the objects in some way. By 4 years of age, the spirit is willing but the flesh is weak: many children realize that things must be combined but are not always good at doing the necessary counting. By 5 years of age, concrete problems present little difficulty and children even invent shortcuts (like counting on from the larger number) to make the calculations easier.

Children also develop proficiency in dealing with imaginary objects. At first, the attempted solution is to represent the imaginary objects by real things and then count them up. Another solution is finger counting. Cross-cultural research shows that counting methods are extremely widespread, and history shows that they have taken elaborate and effective forms; indeed, finger counting was once considered the mark of an educated man. Eventually, children may bypass the use of concrete substitutes and carry out counting operations on the mental level. They also learn to use elementary methods of written symbolism, like tallying, although the use of written numbers presents special difficulties.

Children are quite comfortable with their self-developed practical arithmetic. Such procedures are extremely widespread, and can be put to good use.

Principles

1. Before entrance to school, almost all children engage in arithmetic problem solving of an elementary nature. In the natural environment, children develop informal ways of dealing with problems of addition, subtraction, and perhaps multiplication. It is not true that schooling introduces children to arithmetic; their practical, non-written arithmetic originates earlier.

2. Counting forms the core of children's practical arithmetic. Children add and subtract by counting, sometimes on their fingers. Their practical arithmetic depends on counting as its basic computational technique.

3. Written work presents special difficulties. Children find it hard to represent on paper collections of objects and especially events taking place over time, like adding and subtracting. Children have difficulty in using even so elementary a device as the tally. It is easier for them to count objects or even mental images than to work on paper.

SIMPLE ARITHMETIC LESSON

This lesson will focus on the topic of simple arithmetic. You will watch a series of short video clips and answer questions about them. After you submit your responses, you will be shown examples of other students' successful responses so you can compare them to your own. These responses are intended to provide you with some ideas about the material, but they are not necessarily comprehensive.

This lesson features a girl named Rachel, age 6. As you proceed through the video clips and questions, keep in mind: "What does the child know?"

Clip 1

1. What did you observe about Rachel's response to the question?

2. What was the method she used to get the answer?

3. Can you think of any other ways she might have solved the problem?

Here are some hypotheses offered by other students. You may agree or disagree with what they wrote.

- "She knows how to do mental subtraction and mental addition."
- "She makes a quick calculation in her head to solve the subtraction word problem."
- "She can imagine these pictures in her head and that is how she was able to get the answer."
- "She may already be familiar with basic subtraction facts, to the point where giving answers is rote."
- "She was just repeating the number that she last heard the interviewer say."
- "She was using her fingers in order to reach a solution. I noticed that one of her fingers moved slightly."

Clip 1 again

4. Three students proposed the following hypotheses to explain what Rachel did.

Student A: "Rachel most likely has her number facts memorized. She probably learned the subtraction fact '6 - 3 = 3' in school. This question is easy to her because she has easily retrieved it from her memory. There was no counting involved in her mind."

Student B: "[deleted "Obviously,"] Rachel had the carrots pictured in her mind. She mentally moved three of the carrots away from the group of six that she was picturing, and then counted out the carrots that were left. Of course, she did this rather quickly because her response was fast, but I could tell that she 'saw' them in her head because her eyes flickered a bit. She was also doing some of the visual calculating as Prof. Ginsburg was finishing the question; in this way, she was predicting the question he was about to ask."

Student C: "Rachel probably has '3 + 3 = 6' memorized, but not the related subtraction fact. However, she was able to reason about this problem enough to do the reversal in her mind, and came to the conclusion that '6 - 3 = 3'. Therefore, she had a related problem memorized that allowed her to come to the answer very quickly, with the help of a little logical thinking."

Do you think that these hypotheses are plausible, based on the evidence they used? Why or why not?

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5. If you were the interviewer, what would you do next?

Clip 2

6. Now what did you observe about Rachel and how she got the answer to this problem?

7. Does this support your previous hypothesis about Rachel's thinking and understanding? Please explain.

Here is an example of a successful response to this question: "After watching the video clip I learned that Rachel was doing something more complicated than I initially thought. She didn't simply memorize or count; instead, she reasoned to get the answer by using the inverse. In summary, she knew 3 + 3 = 6, and she also knew the inverse, 6 - 3 = 3." So it turns out she didn't get her answer by memorizing or counting!

Clip 3

8. How do you think Rachel got her answer?

Clip 4

9. What do you think Rachel meant by "I did the same thing"? What would you ask next?

Clip 5

10. Describe and evaluate Rachel's solution.

Rachel did not correctly remember the number fact 4 + 3. Nevertheless, her reasoning was very sophisticated; she used the same inverse principle as before.

Clip 6

11. What would you do next if you wanted to help her learn the way to get the answer?

12. What do you think Rachel will do next?

Clip 7

13. What was the interviewer trying to accomplish by giving Rachel the blocks?

14. How did she learn from this?

Rather than correct Rachel, the interviewer suggests another method for solving the problem that enables Rachel to discover her error. One effective teaching strategy is to place Rachel in a situation in which she will discover the contradiction between what she said at the beginning and the results of her block counting. This approach seemed to work; Rachel saw the contradiction between the count and her previous answer. She hesitated but finally did accept the right answer.

Clip 8

15. Now watch the video clip again in its entirety. What do you think this child really knows about simple arithmetic? Explain how you know.

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16. What would you do next in teaching? You can watch the video again.

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17. What did you learn from this exercise? How did your thinking change during the lesson?

Congratulations, you have completed the lesson on simple arithmetic. Good job!

Subtraction: Strategies and Common Errors

Many of the same strategic and developmental trends described for children's addition also apply to children's subtraction. Early on in development, children count to solve simple subtraction problems and rely on the use of manipulatives and fingers to help them represent the problem and to help them keep track of the counting. With experience and maturation, children are better able to mentally keep track of the counting process and thus abandon the use of manipulatives and fingers for verbal counting. Children also rely on their knowledge of addition facts to solve subtraction problems. Here, a child might solve 8 – 3 by retrieving 5 + 3 = 8. The most sophisticated problem-solving strategy that can be used to solve subtraction problems involves decomposing the problems into a series of smaller problems.

Simple Subtraction

Many 4- and 5-year-old children can solve formally presented subtraction problems; for instance, "If you had three cookies and gave one to your brother, how many would you have left?" There are three common subtraction procedures that involve the use of manipulatives. The first of these procedures is called *separating from*. For this problem, the child first gets three blocks to represent the three cookies, removes one block, and then counts or simply states the number of remaining blocks. The second procedure, *adding on*, involves stating with a number of blocks stated by the *subtrahend* (the smaller number) and then adding the number of block until the value of the *minuend* (the larger number) is reached. The number of blocks added to the subtrahend represents the answer. So, for this example, the child would place one block in front of him- or herself and add two more blocks while counting "2, 3." Because two blocks were added to the first block, the answer would be 2. The final procedure involves matching in a one-to-one fashion the number of blocks represented by the minuend and the subtrahend. The number of unmatched blocks represents the answer. For this example, the child would have one row consisting of a single block aligned with a second row consisting of three blocks. The two unmatched blocks would represent the answer. The use of these different procedures varies with how the problem is presented to the child.

Most 5- and 6-year-olds typically use counting to solve simple subtraction problems. As with addition, counting is sometimes done with the aid of fingers and sometimes done without fingers. Again, finger counting allows the child to represent the numbers to be manipulated and to keep track of the subtraction process and is more likely to be used to solve problems with larger numbers, such as 7 – 3, than for problems with smaller numbers, such as 3 – 1. For solving subtraction problems, counting—whether on fingers or verbally—can involve one of two procedures, *counting up* and *counting down*. Counting down involves counting backward from the minuend a number of times represented by the value of the subtrahend. To solve the problem 7 – 3, the child would count, "6, 5, 4; the answer is 4." If the child cannot keep track of how many values have been counted while counting backward, then she or he will first lift seven fingers and then fold down three fingers in succession. He or she might count backward, "6, 5, 4," while folding down the fingers or first fold them down and then count the remaining fingers, "1, 2, 3, 4."

Complex Subtraction

When first learning to solve multicolumn subtraction problems, such as 17 – 3 or 48 – 27, children rely on the knowledge and strategies developed for solving simple subtraction problems.

In particular, children count and refer to related addition problems. For example, to solve 17 – 3, the child might count down, "16, 15, 14; the answer is 14." Children also use the addition-reference strategy, if they have the complementary addition fact memorized. Moreover, with the introduction of these more complex problems, children also begin to use a problem-solving rule: *the delete-10s rule*. Deleting 10s involves a type of decomposition in which children treated the 10s value separately from the 1s value. For example, on 15 – 3, they might explain their answer by saying, "5 – 3 = 2, and you put back the 1, so 12."

Children also use a columnar-processing strategy, in which the units-column information is processed first, followed by the tens-column information. To solve 48 – 27, the child might first retrieve, or count, to get the answer to 8 – 7 and then process in a similar manner the 4 – 2 in the tens column.

In keeping with the research on children's addition, the decomposition strategy is also used to solve subtraction problems. There are two common decomposition strategies, and both are based on the base-10 structure of the number system. The first is called the *down-over-the-ten method* and is used to solve problems with minuends greater than 10, such as 14 – 6. Here, 10 is first subtracted from the minuend, 14 – 10; the difference, 4, is then subtracted from the subtrahend, 6 – 4; and this difference, 4, is then subtracted from 10 to yield the answer, 10 – 2 = 8.

The second decomposition procedure is called the *take-from-the-ten method*. Here, the first operation involves subtracting the subtrahend from 10. So, for the problem 14 – 6, the first operation would involve 10 – 6. The child then notes the difference, 4 in this example, and then subtracts 10 from the minuend, 14 – 10. The child then notes the difference, 4 in this example, and then subtracts 10 from the minuend, 14 – 10. Finally, the two provisional differences are added together to give the answer, 4 + 4 = 8.

Summary

The same types of errors that were described for children's addition are also evident in children's subtraction. Counting errors are often due to losing track of the counting process, which often results in under- or over-counting. Another common counting error involves starting the counting procedure at the wrong value. For instance, to solve 9 – 7, the child counts, "7, 8, 9; the answer is 3" rather than "8, 9; the answer is 2." The child includes the number representing the value of the subtrahend as part of the counting-up process. As with addition, retrieval errors in subtraction are often the result of operation confusions, for instance, retrieving 16 to solve 8 – 8. Errors in complex subtraction typically result from the inappropriate use of a subtraction procedure. The procedure is often appropriate for some subtraction problems, but is indiscriminately applied to problems where it is not appropriate. These types of errors reflect a general lack of understanding of place value and the base-10 structure of the number system.

SUBTRACTION LESSON

This lesson will focus on the topic of subtraction. You will watch a series of short video clips and answer questions about them. After you submit your responses, in some cases you will be shown examples of other students' successful responses so you can compare them to your own. These responses are intended to provide you with some ideas about the material, but they are not necessarily comprehensive.

This lesson features a boy named Ricardo, age 8. He is looking at this 100s chart. As you proceed through the video clips and questions, keep in mind: "What does the child know?"

Clip 1: 12:09 - 12:57

1. What does Ricardo know about the 100s chart?

Here are some comments from other students:

- He knows that adding 10 to a number on the chart makes it the number directly below it.
- He knows that this rule is valid anywhere on the chart.
- He seems to know that counting can help when adding a number that is slightly more or less than 10, such as counting up one from 10 when adding 11.

Clip 2: 15:38 - 16:07

2. A little later, the interviewer introduces a subtraction problem. Watch this video closely. What are some of the things you notice about his approach?

Here are some observations and hypotheses from other students:

- He seems to think about the number chart as he is listening to the question.
- I noticed he first touched his fingers to a higher number than either of the numbers in the problem.
- He appears to be counting by two's.
- He mistakenly adds instead of subtracts.
- He might have thought he needed to subtract 5.

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3. What did you notice about the interviewer's method? How would you evaluate it?

Here are some comments from other students:

- She repeated the question many times.
- She touched each number in the question.
- She didn't ask him why he touched the numbers he did.
- She didn't modify her question to respond to his behavior, or clarify its meaning.

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4. As the interviewer, what might you have done differently? What would you have done next? You can watch the video again.

Clip 3: 16:00 - 16:21

5. Now watch this video. How does Ricardo solve the problem?

Here are some observations and hypotheses from other students:

- He appears to count back by 1's.
- He touches each number on the chart as he counts back.
- He counts up to 15 as he touches the numbers going back, and then assesses how many he has left.
- He verifies that there are 10 squares remaining after he counts down 15 squares.
- He touches the 1 perhaps because he didn't know which number to touch next.

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6. What are some other ways he could have solved it? You can watch the video again.

Here are some comments from other students:

- If he had used the rule he already knew from addition, he would have realized there are 10 squares between 25 and 15.
- He could have used the number chart to find 25 and 15 and count the difference.
- He might have counted back by 10's.
- He could have subtracted in his head.
- He could have used manipulatives.

Clip 4: 16:20 - 16:52

7. What does he mean by "count down from back"?

Here are some comments from other students:

- He might have confused his words.
- He appears to know to move down or back, not forward on the number line.
- It's his way of saying "count backwards."

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8. What is the difference in his mind about counting by one's vs. counting by two's? You can watch the video again.

Here are some comments from other students:

- He thinks counting by two's is a more powerful method.
- He's aware that there's a risk of skipping a number, but he associates this with counting by one's rather than two's.
- He's confused about the difference between counting by one's and by two's.
- His statement makes no sense.

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9. What would you have asked Ricardo? You can watch the video again.

Here are some comments from other students:

- What did you mean by "count down from back"?
- Why is counting by two's better than counting by one's?
- Why might 11 be the right answer?
- Is there a way you can check whether 10 or 11 is the right answer?

Clip 5: 16:48 - 17:15

10. How does Ricardo get this answer? You can watch the video again.

Here are some comments from other students:

- He counts back by two's, and when he gets to 11, he stops and says his answer: 11.
- The number chart may be confusing him since he would have to move across the page to go from 11 to 10.
- It appears he answers 11 because it is the last number he touches, and he does not realize he must refer to the quantity that's left at the end to get his answer.
- Using an even counting strategy when subtracting an odd amount is a faulty strategy that is confusing him.
- He is silent when he works, which makes it hard to know what he is thinking.

Clip 5a: 16:48 - 17:15

11. Now watch the video again, at half-speed (sound removed). Do you change your interpretation?

Here are some comments from other students:

- It appears that he did know to "fix" his even counting strategy by counting 25 by itself (apparently having "thrown away" 26) before beginning to count by two's.
- It is more clear that he did not know to refer to the remaining quantity at the end, rather than the number he landed on with his fingers.

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12. What would you have asked Ricardo at this point? You can watch the video again.

Here are some comments from other students:

- I would have asked him to show me how he solved the problem or encouraged him to repeat what he had done and talk me through it as he solved it.
- I would have asked him why he put fingers on 26 as well as 25 before he began counting by two's.
- I would have asked him why he changed his mind about the answer.
- I might have told him that his answer before was 10 and instead of saying "you changed your mind," I would asked him if there was a way he could check to be sure it was 10 or 11.

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13. One student wrote,

After watching the video clip I learned that Ricardo was not able to solve a problem this complicated. He appeared to understand that subtraction was taking away, but as he was holding all the information for the problem in his head, he lost sight of what the question was. Perhaps there were too many pieces for him to hold in his memory. Perhaps the strategy of counting by two's is confusing him as well. The number chart can be useful here if he were to count back by one's, not two's.

Do you agree with this interpretation? Why or why not? You can watch the video again.

Clip 6: 12:09 - 12:57; 15:38 - 17:15

14. Now watch the video in its entirety. What does Ricardo know about subtraction?

Here are some comments from other students:

- He knows how to relate a spoken number sentence to a number chart.
- He appears to know subtraction means reducing the number, and he knows to take the second number from the first number.
- He knows the strategy of counting by two's or one's can be used when subtracting.
- He makes the common mistake of considering as the answer the last number counted back, rather than the number remaining.

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15. What would you do next in teaching? You can watch the video again.

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16. What did you learn from this exercise? How did your thinking change during the lesson?

Congratulations, you have completed the lesson on subtraction. Good job!

Equivalence: The Meaning of Written Symbols

In the first few grades of school, children are taught the conventional symbols +, –, = and written numbers like 13 in the context of simple problems like 8 + 5 = 13. Usually the instruction seems to go pretty well: children are proficient at solving problems like 3 + 4 = ☐ in workbooks. Consequently, we usually conclude that they have learned what they were taught and now understand the elementary symbols much as we do. But this is not necessarily the case. Children construct their own meanings for the symbols—meanings often strikingly different from the adult's. Consider first + and =.

Young preschoolers have difficulty using writing to represent the events of addition and subtraction. For example, preschool children who are shown a situation in which a toy is joined by two or more may understand the situation well, but they are likely to be unable to use written means—for example, tallies—to represent the act of addition.

How would older children handle this situation? It seems reasonable to think that they should have no trouble since in school they have learned the conventional symbols for adding and subtracting. Thus, shown two frogs joined by one more, they should simply write 2 + 1 = 3. Yet, this is not necessarily so. For example, in one study, children in the first few grades of school were shown toy bricks added to others and asked to use paper and pencil to show what had happened. Under these circumstances, children used some ingenious methods to represent adding and subtracting. For example, one five-year-old drew a hand adding bricks to the original pile. But not one of the children use the conventional + and – symbols to describe the situation even though these were taught in school. As the author of the study writes, "It seems that the whole notion of representing these transformations [addition and subtraction] on paper is something which children find very hard to grasp."

Suppose that now we reverse the situation in this way: The child is first shown symbols like 4 + 3 and then asked to show what these mean in terms of the toy bricks. A simple solution might involve putting four bricks on the table, then adding three more. The point of this task is to see whether the child can translate from arithmetic symbolism to concrete situations. We already know that the child has trouble picturing events in terms of arithmetic symbols. Can he or she describe the symbols in terms of events?

This problem seems so obvious that it is difficult to believe the results. Children from 6 to 10 years of age in a British school experienced considerable difficulty with this task, even though they were exposed almost every day to the correct use of the symbols involved. What can we conclude from this? Apparently, what the children learn in school is of limited generality. Children assume that the symbols + and – are to be used in connection with written arithmetic without seeing an application beyond that domain. The meaning of the symbols is restricted to the context in which they were learned.

Other observations show that children interpret the symbols in distinctive ways. Suppose the child is shown some simple mathematical sentences and asked what the symbols mean. Kenneth, a first grader (about 6 or 7 years), was shown the mathematical sentence 2 + 4 = ☐.

I: You read that [+], will you? What does that sign say?

K: Plus.

I: What does it tell you?

K: It tells you to add this [2] and this [4].

I: OK. What can you tell me about that [=] symbol?

K: Equals. That means, like this 2 plus this 4 equals 6. There has to be an equal there.

Kenneth interpreted + and = in terms of actions to be performed. So do other children. Presented with the same problem, Evelyn, also a first grader, maintained that 2 + 4 = □ means "to put number 6 in the box." She said that 2 and 4 are numbers, but 2 + 4 is not a number. A second grader, Donna, said that in 3 + 4 = □, "the equals sign means what it adds up to." Another first grader, Tammy, said that the = sign means that "you're coming to the end." The children's understanding of symbols refers to actions—calculational operations. The form a + b = □ means that you do something with a + b to get the answer, namely, the sum.

This, of course, is one legitimate interpretation. In fact, most often when children are presented with sentences of that kind, the two numbers are supposed to be added up. Yet, the interpretation is limited and can lead to trouble:

I: How do you think you would read this [□ = 3 + 4]?

K: ... Blank equals 3 plus 4.

I: OK. What can you say about that, anything?

K: It's backwards! [He changed it to a 4 + 3 = □.] You can't go, 7 equals 3 plus 4.

Given the same problem, Tammy also changed it because, "it's backward," and asked the interviewer, "Do you read backwards?"

So one consequence of the child's interpretation in terms of action is that he finds it difficult to read legitimate sentences that do not directly reflect the order of his calculations. The child first does 4 and then 3 to get 7, but that is not what the sentence says, so the sentence must be wrong.

The child's interpretation also leads him to distort some sentences. Kenneth was asked to read □ = 2 + 5. He said, in effect, that he had to switch around the + and = signs, because they were in the wrong places. This results in "trying to add up to five," namely, □ + 2 = 5. Later, Kenneth was shown 3 = 5.

I: What can you say about that?

K: Cross that line out. [Kenneth wrote over the = sign to change it to 3 + 5.]

I: Can I write this [3 = 3]? Does it make sense?

K: Nope. Now you could fix that by going like this. [He changed it to 0 + 3 = 3.]

When Charles was shown 3 = 5, he counted on his fingers and wrote 3 = 85. "Five ... there's no plus." That makes it wrong. I'll put a plus in the middle." He wrote 3 + $_{8}$5.

If the child always interprets symbols in terms of actions on numbers, he cannot make sense of sentences that express relationships, like 3 = 3 or 4 = 4.

What is the reason for children's tendency to interpret symbols in terms of actions? Part of the reason seems to be that their arithmetic is based on actions, particularly actions of counting. For young children, arithmetic is the activities of adding and taking away, of counting backwards and forwards. Given this orientation, children interpret symbols in terms of action. Furthermore, their action interpretation may simply reflect the ordinary demands of the classroom. Most often, sentences do ask children to perform a calculation; if so, why should they interpret them otherwise? In classrooms that do not stress addition as action but instead promote a relational view (that 4 + 3 is the same as or another way of saying 7), children eventually learn to take a different approach, seeing that other interpretations are also possible. But this learning comes hard, because children's natural tendency is to see arithmetic in terms of action.

EQUIVALENCE LESSON

This lesson will focus on the topic of equivalence, and symbols in general. You will watch a series of short video clips and answer questions about them. After you submit your responses, in some cases you will be shown examples of other students' successful responses so you can compare them to your own. These responses are intended to provide you with some ideas about the material, but they are not necessarily comprehensive.

This lesson features a boy named Jordan, age 8. As you proceed through the video clips and questions, keep in mind: "What does the child know?"

The lesson begins with an extended conversation about fractions. The interviewer has Jordan use blocks to demonstrate various groupings. Toward the end of this task, Jordan spontaneously introduces a written symbol.

Clip 1: 38:21-38:27

1. You have blocks and paper and pencil. Give an overview about how you would test Jordan's knowledge of the equals sign. Provide four questions you'd like to ask, in sequence.

Clip 2: 39:26-39:36

2. What would you ask Jordan now?

- I would ask him what "equals" means.
- I would ask him to show me what equals means using the blocks around the symbol.
- I would ask him what a "sign" or "symbol" means and if he knows of other symbols.

Clip 3: 39:35-39:56

3. What is he describing here?

- He's describing equivalence.
- He's saying that two numbers are equal if they are the same.
- He's using 12 and 12 as an example of two numbers that are equal.
- He's saying the relationship between the sets is equal because they are the same amount of elements.

4. Do you think the interviewer should have helped him with his verbal explanation of the concept of equals? You can watch the video again.

- Yes, his language was not quite adequate for the idea, and she could have helped him clarify.
- I would have asked him to elaborate on the sentence, "Those two together are the same number."

Clip 4: 39:52-39:59

5. What do you think of the interviewer's question and gesture at the end of this video clip?

- It's good that she confirms what he's saying.
- I thought she could have asked him to explain further.
- She's affirming his answer and leaving her question open-ended to see what he knows; it's child-centered.
- Her gesture mirrored what he did, to help build rapport by showing that she was following him and could repeat what he was doing and saying.

Clip 5: 39:57-40:01

6. What does Jordan mean by that? What does it imply about his thinking?

- Could mean that he's flexible in his thinking.
- For him, symbols are arbitrary and then don't mean anything to him.
- Trivial generalization -- could do it with four blocks instead of six.
- Could be an interesting new set of ideas.

Clip 6: 39:59-40:41

7. What is the distinction Jordan is making?

- He calls his equation a "number sentence."
- He knows the symbol for addition and that it has a function in the number sentence.
- He says that "equals" is interchangeable with "makes."
- He believes that having both the addition and equals signs in a number sentence indicates

"makes."

- He may consider the addition and equal signs to be different from numbers. The symbols tell you what to do and the numbers are the objects used for the process.

Clip 7: 40:41-40:53

8. What would you ask Jordan here?

- Why is it the same symbol as before?
- Why doesn't it just mean the two sides are the same?
- Does equal have more than one meaning?
- Didn't you say before that the addition sign was important?

Clip 8: 40:47 - 40:59

9. What do you think he will say?

- No, I think equals means "makes."
- Yes, I was wrong, equals means "the same as."
- Yes, but in this case it means "makes," and in the other one, "equals."
- It means "the same as" only after you make the two sides the same.

Clip 9: 40:54-41:06

10. When he maintains that the equals sign does not mean "the same as" in this instance," what would you ask or do next?

- I would ask, "Why not?"
- I would have him show me using the blocks or on the paper.
- I would talk to him about how the two types of equations were different.

Clip 10: 41:08-41:20

11. What is the interviewer trying to accomplish here, and how does she go about doing it?

- She wants him to see that the equals sign can mean either "the same as" or "makes."

- She first establishes equivalence, and then she separates the blocks a little.
- She's showing him the link between addition and equivalence by demonstrating the process.

Clip 11: 41:08-41:30

12. What do you think happened?

- It appears that he saw the interviewer's simpler equation and realized that the two sides were equivalent.
- I think the interviewer led him to see what it was he was trying to explain.

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13. Do you think that Jordan really understands that the equals sign means "the same as" even in a number sentence? You can watch the video again.

- It's hard to say, because he agrees with her before she finishes explaining, and he doesn't really look at the blocks.
- He probably does understand because he agrees so readily. He also looked back at the blocks indicating 3 + 3.

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14. Would you say that the interviewer was "teaching" him? If so, how was she teaching him? You can watch the video again.

Clip 12: 38:21-41:30 (complete)

15. Now watch the interview clip in its entirety. What do you conclude about his understanding of the equals sign?

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16. What would you do next in teaching? You can watch the video again.

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17. What did you learn from this exercise? How did your thinking change?

Congratulations, you have completed the lesson on equivalence. Good job!