SMOOTHNESS ASSUMPTIONS IN HUMAN AND MACHINE VISION, AND THEIR IMPLICATIONS FOR OPTIMAL SURFACE INTERPOLATION

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CUCS-244-86

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by

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Submitted in fulfillment of the area qualification requirement of the Doctoral degree in Computer Science at Columbia University.

In this paper we shall examine what smoothness assumptions are made about object surfaces, object motion, and image intensities. We begin by looking into the physiological limits of vision and how these might influence our perception of smoothness. We then look at a sampling of the computer vision and psychology literature, inferring smoothness constraints from the mathematical assumptions tacitly presumed by researchers. This look at computer vision and psychology of vision is not meant to be an inclusive study, but rather representative of the assumptions made, and in part representative of the mathematical models used therein. We shall conclude that prevalent assumptions are that surfaces, motion, and intensity images are functions in C^2, C^1 and C^2 respectively.

In the latter portion of this paper we examine one use of explicit assumptions on smoothness in the definition of existing method for obtaining "optimal" surface interpolation. We briefly introduce the nomenclature of information-based complexity originated by Traub, Woz'niakowski, and their colleagues, which is the mathematical machinery used in obtaining these "optimal" surfaces. This theory requires that we know the class of functions from which our desired surface comes, and part of the definition of a class is the degree of smoothness. We then survey many possible classes for the visual interpolation problem of two dimensional surfaces, and state formulas from which one can obtain the optimal surface interpolating given depth data.

§1 Some Definitions and the Relation between Derivatives and Smoothness.

Throughout most of this paper we shall be looking at smoothness as conditions on the number of derivatives of surfaces, and occasionally as upper bounds on the magnitude of derivatives of the surface. Let D be a closed, simply connected two set (region) in $\Re^2$, and define the following classes of functions:

$$C^0(D) = \{ f : D \to \Re \text{ such that } f \text{ is continuous} \}$$
$$C^1(D) = \{ f : D \to \Re \text{ such that } f, \partial f/\partial x \text{ and } \partial f/\partial y \text{ are continuous.} \}$$

and for k = 1,2,....

$$C^k(D) = \left\{ \begin{array}{c} f : D \to \Re \text{ such that } f \in C^{k-1}(D) \text{ and } \forall\ \alpha_1, \alpha_2 \text{ s.t. } k = \alpha_1 + \alpha_2 \\ \dfrac{\partial^k f}{(\partial^{\alpha_1} x)\ (\partial^{\alpha_2} y)} \text{ is continuous.} \end{array} \right\}$$

and finally,

$$C^\infty(D) = \{ f : D \to \Re \text{ such that } \forall\ k, f \in C^k(D). \}$$

Observe that by definition we trivially have $C^\infty \subset \ldots \subset C^k \subset C^{k-1} \subset \ldots \subset C^1 \subset C^0$, i.e. functions which are of the class C^k are also of the class C^{k-1}. Similar definitions can be made for functions of a single variable on one dimensional sets.

In vector calculus one says that a function (surface) f is "smooth" if and only if $f \in C^1$ and at each point on the surface the normal to the surface is defined and $\neq 0$, see [Marsden 76]. Intuitively this definition says a surface is smooth if it has no edges or corners. Unfortunately, the requirements on the surface normal do not generalize to "higher degrees of smoothness", therefore we extend this definition of smoothness and say that function f has "smoothness of degree k" iff $f \in C^k$.

We now consider that derivatives are a mathematical abstraction, and that in the physical world, we do not have formulae, but physical objects (or their motion in space). Therefore we address the problems of what it means for an object to have a surface with smoothness of some degree; what it means for an object to move with smoothness of some degree; and how can an image (which is registered at a finite number of points) have any degree of smoothness. The solution to all three problems is that we *assume* that the derivatives are approximated by difference quotients and that the denominator of the difference does not go to zero, but is bounded from below by some "small" value. However this assumption creates a problem of its own; in particular, what does "small" mean. The answer is that "small" is relative to the accuracy of the measurements of both the numerator and denominator, and is *scale dependent*. It is important that "small" be less than the limit of accuracy of measurement (or perception) of the numerator of the difference quotient. This places some limit on the degree of smoothness that can be measured. Recalling that the k^{th} derivative is defined by the limit of a difference quotient with the numerator involving the $(k-1)^{th}$ derivative we see that approximations of higher order derivatives are limited by the quality of the approximation of the lower derivatives. Marr discusses exactly this problem, and present a theorem originally due to Bernstein (see [Marr 82, p289]) which states that for any function f with band limit ß (and any finite physical measurements result in band limited data) then

$$\sup_{x \in D} |f'(x)| * \beta \geq \sup_{x \in D} |f(x)|.$$

Throughout this paper we shall usually infer from an author's use of k^{th} order derivatives that the image, object, or motion is in the class C^k. This, however, is often not the case; often a model, or an algorithm will use the fact that under certain circumstances there is a violation of the class assumption (e.g. a point is only k-1 times differentiable or not even continuous). This generally results in an approximation to the k^{th} derivative that is very large. If this is the case we shall say that they imply piecewise smoothness of degree k, (or piecewise C^k), i.e. smoothness of degree k except at a small finite number of points or curves. Some authors use a type piecewise C^k that is globally C^{k-1}, while others employ a piecewise C^k that may not even be globally C^0. We shall not distinguish greatly between these two different uses, since in both cases the authors generally assume they can segment the surfaces, images, or motions, into regions that are truly C^k.

We end this section with a final remark on the class C^∞. The reader should be aware that there is a theorem due to Weirstrass [Rudin 64] that states that on any compact domain D, any continuous function can be approximated *arbitrarily closely*, including its derivatives, with a C^∞ function (in fact with a polynomial). This theorem touches the heart of Fourier series approximations inasmuch as the components of Fourier series (sin ◊x, cos ◊x) span (are dense in) C^∞. One result of this theorem is that if one uses the Fourier series approximation, and includes all components from the lowest perceivable frequency to and including the highest frequency, then the recovered curve (surface in 2D) should not be visually distinguishable from the original, and is in C^∞.

Part 1: Smoothness Assumptions in Vision.

§2 Spatial and Temporal Limits of Human Visual Perception.

Given the above discussion on derivatives, it is important to examine the limits of human visual perception. We break the discussion of these limits into two categories, spatial and temporal. Note that this is not meant to be an implication that they are separable components of perception (although they are generally believed to be), but it is a classification long used in psychology.

Acuity is the term used in perception literature to refer to the lower limit of spatial resolution. "For very many years (from Lord Rayleigh's time) it was common practice to define visual performance in terms of one of several forms of resolution criteria." [Hartridge, p52] It has been experimentally determined that approximately .15 mrad (milliradian) is the lower spatial limit of perception for both detection (in good light) of a black disk on a white background, and for the minimal separation of two point sources of light to be perceived as two (not one) points. Unfortunately there is no single "limit" on spatial perception. The measurements are a function of the stimuli used (e.g. points of light, dark points, lines, bars...), the contrast between objects, the viewing time, the observer's experience, and age. See [Overington 76] for a discussion of how these and other properties effect perception.

For motion, there are both upper and lower bounds on the speed that an object can move and still be perceived as a moving object (though the shape may be distorted). Again, the actual values of these limits are influenced by target size, distance and contrast, the orientation of targets, and the direction of movement, structure (or lack thereof) of the background, and the mode of perception (static, or tracking). A lower bound of about 0.2 cm/sec for a 7.5cm x 2.5cm source at 2.0m from the observer was obtained by [Bonnet, 77], and the bound was shown to vary greatly with different display formats. Upper bounds range from about 30° to about 100° of visual angle per second [Kolers 72].

Due to their wide context sensitivity, the limits presented here are not universally accepted in the psychology field, and are meant to give the reader an intuitive feel of the limits. The results are a factor of many aspects of vision, and these figures model only a small number of possible experiments. More information on the exact apparatus used can be found in the cited sources.

§3 What We are Not Going to Consider.

There is a large volume of literature in both the psychology of perception and in machine vision, and only some of this literature makes assumptions on the smoothness of objects or their motion.

In psychology many of the early models dealt with restricted aspects of vision, for example with the size of an object vs contrast threshold for detection. Therefore we shall not consider such restricted models, including (but not limited to): Ricco's law, Piper's law, Bloch's law, the Blondel-Rey theory, quantum theory, circuit theory, and element contribution theory. We also shall not examine models (neural or otherwise) of the higher level vision process, since they do not in general require smoothness assumptions.

In the plethora of computer vision algorithms there are many that offer no assumptions on smoothness, including (but not limited to): template matching (for either edge detection, or object recognition), edge relaxation, edge following, Hough transforms, region growing algorithms, splitting and merging methods, quad-tree algorithms, algorithms for the blocks' world, spatial gray level dependence algorithms, model matching algorithms, medial axis transform recognition, and in general inhibition / excitation networks. Most of these methods are are not continuous models of the world but rather discrete algorithms, treating the image purely as pixels, dealing with 'objects' as members of a finite set. (Often they have so restricted their domains as to offer no useful assumption about the real world.) These algorithms, though important to machine vision as a whole, shall not be pursued further in this paper.

What we shall examine is the assumption on "smoothness" in both computer vision and psychology of vision. From psychology we shall mostly examine work on modeling the intensity image, and on visual motion. With respect to intensity images we shall examine the work of [Békésy 60], [Fry 48], [Huggins and Licklider 51], [Marr 79], and [Overington 76]. In the area of motion we shall examine the work of [Bahill 83], [Clocksin 81], [Fennema and Thompson 79], [Hoffman 80], [Prazdny 81, 83].

From computer vision we shall examine assumptions implicit and explicit in: internal object representations, shape from X algorithms, surface reconstruction algorithms, subjective contours, imaging operators, object motion, optical flow, and photometric stereo.

§4 Assumptions on the Smoothness of Object.

In this section we examine the assumptions in both psychology and computer vision about the smoothness of objects. We shall conclude that in general the surfaces of objects are assumed to be at least piecewise C^2, possibly with some upper bound on the magnitude of the second derivative. In section 4.1 we examine the smoothness assumptions in the work on stereoscopic contours, which will tell us something about smoothness of object boundaries. In 4.2 we continue with contours by examining the work on subjective contours. In section 4.3 we examine the assumptions which can be inferred about surfaces from their representations and from the work of Marr et al.. In section 4.4 we examine the smoothness assumptions about objects that can be inferred from work on shape from stereo. Section 4.5 deals with the assumptions implicit and explicit in work on shape from texture. Finally in section 4.6 we examine the assumptions on object smoothness implied by the use of second order differential operators on the intensity image.

§4.1 Smoothness of stereoscopic contours.

We begin with a discussion on the shape of stereoscopic contours. These contours are not edges of actual surfaces in $\Re^3$, but rather the edges of a perceived shape of a surface generated by stereograms and random dot stereograms. Random dot stereograms were proposed by Julesz [Julesz 60] as a means of studying stereo vision, because they give rise to perceived depth when viewed stereoscopically even though they contain no image intensity "edges" of the perceived surface. Random dot stereograms have attained a widespread acceptance in the study of stereo vision. (The interested reader is referred to [Marr 80] or [Grimson 81] for more information on the computational aspects of random-dot stereograms, and for a more complete psychophysical treatment see [Julesz 71] or [Gulick and Lawson 76].) In there study of human stereopsis, Gulick and Lawson noted:

> "... matrix targets (stereograms) occasionally led to the perception of well-defined contours that served to delimit the surfaces of sub-matrices seen in stereoscopic depth. These contours occurred in the absence of a continuous abrupt luminance gradient, and they seem to depend on the perception of surface depth brought about by the binocular abstraction of a disparate form." [Gulick and Lawson 76, p 121.]

Through a number of experiments they found that stereoscopic contours could be straight, or piecewise circular segments (which they refer to as curved). However they report "...*straight stereoscopic contours took precedence over curved* [their italics], and the presence of some curved contours based on luminance gradients did not appear to be influential." [Gluck 76, p.150] They found that the straight contours could cross homogeneous space (where there are no borders of dots) but could find no evidence that curved stereoscopic contours could cross these homogeneous spaces. In fact, their results suggested that in such circumstances observers reported curved portions of boundaries connected with straight segments. It was not made explicit in their work if the curved segments must be circular arcs or rather twice differentiable curves. It was also not clear however if the observers could detect where the straight regions ended and the curved ones began. If the observers could detect the "connection" points, it is probably safe to infer that they felt that stereoscopic contours are perceived as piecewise C^{∞} (if we can see only circular segments) or C^2 (if we can see arbitrary twice differentiable curves). If observers cannot detect where straight contours meet curved segments, then the contours should be assumed to be C^1, i.e. they have continuous slope, but not continuous curvature.

§4.2 Smoothness of Subjective Contours

A problem related to stereoscopic contours is that of subjective contours. These contours are generated by the subjective "filling in" of missing portions of diagrams, for example see Figure 4.2.1. On the computational side of this problem [Ullman 76] proposed a local computation network for the calculation of subjective contours utilizing assumptions on isotropy, smoothness, minimum curvature and a localization hypothesis. In particular the smoothness assumption he made was "except for some special cases of filled in corners, the generated curves are smooth, that is differentiable at least once." [Ullman 76, p.1] He also express the assumption, inspired by the similarity of subjective contours to a thin doubly cantilevered beam, that these contours should possess minimum curvature. That this implies the curves are piecewise twice differentiable (i.e. piecewise C^2) follows directly from his definition of curvature, $\int (d\beta/ds)^2$, where β is the curve slope and s is arc length, (this is generally called total curvature).

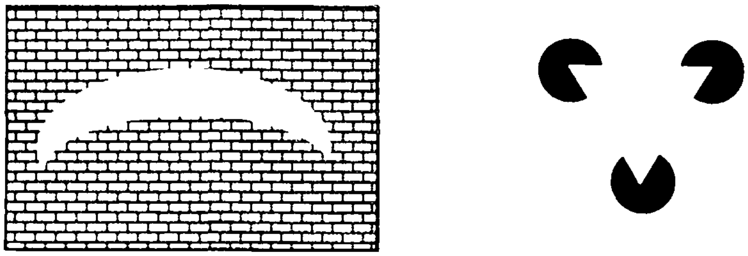

Figure 4.2.1: Examples of subjective contours.

In his treatment on subjective contours (of which stereoscopic contours are one type), Marr [Marr 82] proposes that the shape of curved subjective contours are piecewise circular arcs with minimal curvature. Mathematically this implies that they are C^∞ (if they are really circular arcs). If the circular arcs are just an approximation to some other cures, the fact that the contours are to minimize curvature implies that they are at least piecewise C^2.

§4.3 Surface Smoothness.

Some of the strongest assumptions on the smoothness of an object can often be derived from the representation of that object (e.g. from the representations internal to a computer vision algorithm). Unfortunately, most of the work in psychology and psychophysics does not attempt to model the internal representation of an object surface, and therefore it is hard to infer smoothness assumptions (on surfaces) from their work. In particular almost any neural model, especially those driven by inhibitory / excitatory connection networks, will not offer any implications on the degree of smoothness of object surfaces.

However, there are a number of computer vision algorithms that utilize specific internal representations having implications on the smoothness of surfaces. Three examples are: the class of generalized cylinders used in ACRONYM, [Brooks 79,80,81] and similar classes of generalized cylinders all of which imply that surfaces are piecewise C^∞; representations with polynomial patches or polynomial splines, e.g. bi-cubic or Coons patches [Allen 83] [Coons 76] which imply that surfaces are globally C^1 (and piecewise C^∞); and representation as finite element patches as in [Terzopoulos 83] which implies the surface is piecewise C^2 (however this representation is not even globally C^0). The smoothness assumptions in internal representations are not difficult to infer, and unfortunately do have far reaching implications. For these reasons we shall not consider the assumptions from representation further, but instead turn our attention to the assumptions implicit in the mathematical models on which much of computer vision work is based.

We begin with a computational theory of human visual perception as presented in [Marr 82], noting that this book draws heavily upon other sources including [Marr and Poggio 77], [Marr and Ullman 80], [Marr and Hildreth 81] [Grimson 81]. In this book, Marr explicitly expresses some assumptions involving smoothness. Unfortunately only a few assumptions are *mathematically* precise: most are defined only intuitively. The latter include the following statements from [Marr 82]:

> "Stated precisely, it is *that the visible world can be regarded as being composed of smooth surfaces having reflectance functions whose spatial structure may be elaborate.*" (p. 44, their italics)
>
> "The basic feature is that markings often form smooth contours on a surface, ..." (p. 49)
>
> "... *the loci of discontinuities in depth or in surface orientation are smooth almost everywhere.* This is probably the physical constraint that makes the mechanism of smooth subjective contours a useful one." (p. 50, their italics)

Clearly Marr was well aware that there are advantages to making assumptions that constrain the smoothness of objects, edges, and motion. But what should we infer for their meaning of "smooth" in the above statements (and others in the work)? One clue is given in Marr's discussion of the physical constraints for the matching in stereopsis, where he says "continuity means that the disparity varies smoothly almost everywhere" [Marr 82, p 114]. As stated, this implies that smoothness is equivalent to continuity; in other words, smooth objects are elements of C^0. Again when discussing the problem of "smooth motion" (see Marr 82, p.184), he implies that being continuous is equivalent to being smooth. However, when discussing subjective contours, Marr states, "In both cases, continuity and smoothness (minimum curvature) seem to be important criteria" [Marr 82, P 288]This implies that a smooth object is at least C^1 and piecewise C^2. Lastly, when discussing discontinuities and interpolation methods [Marr 82, pp 289-290], Marr states that in the discrete case "underlying discontinuity can no longer be discriminated from a very high curvature change".

Thus one could infer two separate definitions of the word smooth. We feel, however, that the correct inference is that Marr's definition of smooth is equivalent to piecewise C^2 objects that do not have too high a curvature. This last condition can be viewed as placing an upper bound on the magnitude of the second derivative.

As an aside, we note that in a recent paper Mayhew [Mayhew 82] interprets the Marr and Poggio principle of continuity (quoted in the above discussion), as implying that surfaces are piecewise planar. While this might hold if the planar patches are extremely small (so small as to not be detectable), this does not seem to fit well with the overall impression of Marr's use of the word "smooth".

§4.4 Assumptions about Object Smoothness in Shape from Stereo.

There have been many computer vision algotithms for surface reconstruction from stereo depth data, each with its own set of implicit smoothness assumptions. Much of the earlier work treated the surface as a discrete set of depth points in space, and made almost no mathematical assumptions about smoothness.

The assumptions that can be drawn from many stereo algorithms are generally derived from their internal representations, e.g. the work of Allen, [Allen 83] (which incorporates tactile depth data and stereo) uses an internal representation (bi-cubic patches) which implies that surfaces are truly C^1 and piecewise C^∞.

The assumptions on smoothness implied by the work of [Grimson 81], [Terzopoulos 83], and [Kender, Lee and Bout 85], are somewhat more fundamental. Their assumptions are not implicit in their representations, but rather in the formulation of the problem of reconstructing a "smooth" surface. These authors assume that a "smooth" surface is one that minimizes the functional given by

$$\Theta(f) \equiv \iint_{\Re^2} \left(\left(\frac{\partial^2 f}{\partial x^2}\right)^2 + 2 \cdot \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 f}{\partial y^2}\right)^2 \right)^{1/2}$$

Obviously the fact that this functional is well defined implies that the surface is at least C^2. Furthermore, their methods use $\Theta(\cdot)$ as a measure of the "unreasonableness" of a surface and attempt to find the surface that minimizes this functional. The dependence of $\Theta(\cdot)$ on the second derivative of the surface and the idea of minimizing the functional also supports the argument that smooth surfaces have small second derivatives.

§4.5 Smoothness Assumptions in Shape from Texture.

Of all the "shape from X" algorithms, "shape from texture" probably makes the most restrictive assumption on surfaces. The idea that shape can be derived from textural cues comes from a long known [Gibson 50] phenomenon that humans can perceive surface orientation solely from the distortion of certain textures on the surface. The principle involved in most shape from texture algorithms is the computation (at least locally) of the surface gradient associated with the (local) texel properties. To see some of the assumptions we examine shape from texture as discussed in [Kender 80] and [Witkin 81].

In Kender's development and subsequent use of the Normalized Texture Property Map, for the recovery of shape from texture he explicitly assumes the continuity of local surface orientation. The NTPM "is a way of 'deprojecting' the effects that surface orientation has one primitive textural properties such as slope in the image, length of major axis of elongation, etc." [Kender 80, p9]. He uses the continuity assumption to allow the local approximation of surface orientation by planes; this approximation following his meta-heuristic that assumptions should make the mathematics of a problem as simple as possible. We defer the discussion of this assumption until we examine the work of Witkin (who makes similar assumptions).

With a statistically based algorithm, using projective geometry and a few assumptions about texture geometry, Witkin was also able to recover the shape and orientation of both planar and curved objects, (see [Witkin 81]). In the same manner as Kender, his methods are developed for planer objects, with curves objects represented locally as tangent planes. His technique of shape recovery is to use a maximal likelihood estimator (MLE) for the slant and the angle of the slant with respect to the observer. For planes this is all the orientation information necessary to orient the plane. The MLE is developed from a geometric model expressing surface tangent as a function of: the angle between projected tangent and tilt direction, the amount of tilt, and the angle of tilt with respect to the observer.

Witkin states that it is not too difficult to obtain shape from texture with strong assumptions about the surfaces and the texture geometry, but that the real trick is to recover shape and orientations with minimal assumptions that are not only formally adequate, but are approximately true in nature. While his work does try to make minimal assumptions on the texture properties, he does impose limitations on smoothness of objects.

In two separate methods for shape from texture, we have seen that the assumptions of object smoothness were that curved surfaces could be reasonably well approximated by a tangent plane. Though this can be construed to imply that surfaces must be at least differentiable (which is equivalent to the continuity of surface orientation that Kender assumed), it is better stated as a limitation on curvature, or a bound on the second derivative. To see this, consider the surface generated by $\sin(1/(\varepsilon+x^2+y^2))$, for some small $\varepsilon>0$ and note that the surface orientation is a continuous function, but that for very small ε the surface will oscillate wildly near the origin, and therefore not be well approximated by a tangent plane. The function, however has very high second derivatives (and hence curvature) near the origin. This suggests that these methods require more than C^1, they also require that the surface orientation not change too quickly, or more simply that the surface be C^2 and have bounded second derivative.

§4.6 Object Smoothness Assumptions In Image Operators.

We end the discussion of the assumption of object smoothness with a short exposition on an assumption implicit in the choice of a second order differential operators as the input to the other components of theories (e.g. stereopsis, motion). [Grimson 81] employed a second order differential operator known as the Laplacian of a Gaussian ($\nabla^2 G$) . (This is application of a Laplace operator to the result of the convolution of the input image with a two dimensional Gaussian filter.) In [Grimson 81] he states a condition on surfaces that has been paraphrased as *no news is good news*. In this he meant that the lack of zero crossing of $\nabla^2 G$ at some location constrained the surface in a neighborhood of that area. We shall see that in general this does not place constraints on the 3rd or higher derivatives of the surface. Note that this discussion applies to any theory in which the most powerful differential operator applied to the input is of second order (e.g. the model of Huggins and Licklider [Huggins 51], or the theory of Marr and Poggio discussed above).

Given that an algorithm is using an approximation to at most a second order differential (i.e. it approximates at most the second derivative of the surface), one can infer that the theory or model being implemented by the algorithm does not make any assumptions about the continuity of 3rd or higher derivatives. To wit, because imaging is a continuous transformation (assuming no occlusion), then if the surface reflectance is isotropic, lighting from a single point source at infinity, and viewing angle is such that the object is not self occluding, then any second order differential (in particular an approximation to $\nabla^2 G$) can not detect discontinuities in the 3^{rd} or higher derivatives of that surface. If they were trying to do so, they would be 3^{rd} order differentials). This implies, that given any theory with at most a second order differential operator on the input image, that unless some combination of viewing angle, surface reflectance properties, lighting, etc., interact to highlight a 3^{rd} or higher derivatives, they should be invisible to the human observer.

§4.7 Conclusions on Object Smoothness.

In conclusion, the bulk of the literature in psychology of vision and computer vision alike, assume that the surface of objects are at least piecewise twice differentiable (i.e. in piecewise C^2 and in general globally in C^1), with a possible bound on the magnitude of the curvature or second derivative. As a point of interest we present two observations from fields other than psychology and machine vision that also support this definition of smoothness.

The first point of interest is linguistic in nature. Although we have a number of words (in English) that refer to the second derivative of an object surface or object motion (these are "curvature", and "acceleration" respectively), we do not have such general terms for the third derivative of either object surfaces or object motion. (Actually there is "jerk" for motion and torision for surfaces neither of these terms is exactly a third derivative, nor are they commonly used).

The second is an observation from a recent paper by Jan Koenderink and Andrea van Doorn [Keonderink and van Doorn 82]. In their paper, they cite numerous papers from the academic-art literature to support their claim (which they also support mathematically) that the end of a visible contour must be concave, and that the corresponding portion of the rim must be from a hyperbolic patch, see Figure 4.7.1. Koenderink and van Doorn note that throughout much of the art literature authors seem to neglect the possibility of a shape being hyperbolic and instead discussed shapes only as a combination of concave and / or convex elliptic patches. We note that ellipticity implies that objects are piecewise at least C^2 (they have curvature defined), and may imply piecewise C^∞. We also note that in their discussion, Koenderink and van Doorn never mention a difference between objects that are composed of patches that meet with continuous slope, and those that meet with continuous curvature (i.e. with continuous first or second order derivatives respectively).

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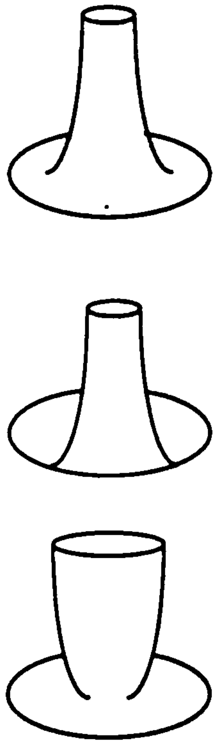

A proper drawing of an object: end of the contours are concave and the surface near the ends of the contours form hyperbolic patches (at both the top and bottom of column).

An imporper drawing: at the end of the contours (bottom of column) the surface is not hyperbolic . (Note that this could a valid object in occlusion).

An improper drawing: the end of the contours is convex. If the contours were correct, there should be a visible rim connecting them.

Figure 4.7.1 Examples of valid and invalid drawings.

§5 Assumptions on Smoothness of Motion.

There are two types of visual motion to be considered. The first, considered in section 5.1, is visual tracking, where the eyes move in response to a moving stimuli. The second, considered in section 5.2, is trefered to as optic flow, where there is motion of intensity patterns across the retina. This is the result of the projection of a moving object, or of a fixed object and a moving observer, or of a moving image (such as a television program). We examine in turn the assumptions encountered in the study of each of these types of motion.

§5.1 Smoothness of Visual Tracking.

In the study of visual tracking the subject is generally presented with a point which moves with some pattern unknown to the observer. It has been reported ([Rashbass 61] referenced in [Bahill 83]) that the human smooth pursuit eye movements have a latency of about 150 msec. However, Bahill and McDonald [Bahill 83] report that given a suitably predictable target, humans can learn to perform zero latency tracking. "The target motion predominantly used [by others] to test the smooth pursuit system is predictable sinusoidal motion." [Bahill 83, p. 1574] In their paper they present the observers with a number of targets moving in patterns with: triangular waveforms (i.e. continuous piecewise linear motion), piecewise parabolic waveforms, picewise cubic waveforms, sinusoidal waves or with pseudo-random discontinuities in acceleration. These are motions of smoothness C^0, C^1, C^2, C^∞, and C^1 (with the distance between discontinuities of acceleration small with respect to motion period) respectively. (Note that if considered piecewise they are all C^∞ if we do not require continuity of lower derivatives across peice boundaries. If we take this into consideration they are piecewise C^1, C^2, C^3, C^∞, and C^2 respectively.)

They report that there are problems in tracking at the discontinuities of velocity in triangular waveforms. In the case of the parabolic waveform, they found that the subjects could track the target with high accuracy, with the velocity of eye movements correctly following the velocity of the target (thus being insensitive to the discontinuities in the acceleration). Note that motion with parabolic waveforms is a very natural motion (objects subject to constant acceleration (e.g. gravity) follow such patterns. In the case of cubic motion pattern, which they comment does not seem to have any natural physical realization, the subjects initially tracked the target with mild success and after a short learning period the target was tracked with great fidelity. For the pseudo-random acceleration target waveform, the subjects initially tracked with "saccades every 200 msec, which is typical of poor tracking ability. The subject's velocities were widely dispersed from the target velocity. This waveform was the most difficult to learn." [Bahill 83, p. 1579].
The results of the pseudo-random acceleration target waveform suggests that subjects have problems with motion that is not at least piecewise C^2. The result that subjects initially tracked parabolic motion better than cubic motion may however suggest that (global) C^1 motion is more "natural" to track than (global) C^2 motion. These conclusions are weak and the subject requires further research.

§5.2 Smoothness Assumptions in Research on Optical Flow.

Now we examine what assumptions psychologists and computer scientists make about the smoothness of flow fields generated by object motion. In optic flow, we shall consider two separate types of smoothness assumptions: assumptions on the differentiability of the image with respect to time (i.e. whether patterns in the image move with continuous velocities, whether they accelerate continuously, etc..), and assumptions on the spatial smoothness of the fields of motion (i.e. whether there is spatial differentiability of the velocity or acceleration fields). Recently there has been a variety of work on the subject of recovery of shape from optic flow, e.g. [Clocksin 81], [Hoffman 80], [Horn and Schunk 81], and [Prazdny 81,83]. Though many of these works also make assumptions about the smoothness of the intensity image, we shall consider only their assumptions on the smoothness of object motion.

The idea that depth information, and hence shape, could be recovered from patterns of angular velocities of light on the retina is due to Helmholtz [Helmholtz 25]. Additionally, in his studies of different stimulus to the human visual system, Gibson [Gibson 50] concluded that optic flow was a principle source of information for surface layout and ego motion (motion of the observer with respect to the world). There are two main camps on how one calculates optic flow. Some feel that velocities are measured directly from a continuously changing retinal image, and others feel that optic flow can be calculated from a sequence of static images. We shall not go into this debate, for two reasons. The first reason is that the continuous approach generally uses neural motion or velocity receptors, which yield few assumptions about motion smoothness. The second is that although there is something intrinsically discrete about the latter approach, when the separate images are presented at a sufficiently high rate, they approach continuous motion. From our point of view the later approach encompasses the former, varying only in temporal scaling.

The problem of shape from motion has two components: the calculation of the optic flow fields, and the recovery of shape and depth from the flow fields, each of which we discuss in turn. For the calculation of optic flow, either there are detectors for motion built into the eye which can be thought of as returning optic flow, or there is some process that takes a sequence of images, finds corresponding points in each, and calculates the velocities of each point.

[Marr and Ullman 79] proposed a neural model for the calculation of directionally sensitive motion detection cells, and explicitly state that they assume sequences of intensity images which are continuously differentiable with respect to time, and make no assumptions on the spatial smoothness of the velocity field.

The statistically based cluster analysis algorithm developed in [Fennema and Thompson 79] assumes that an image sequence is temporally differentiable, and deals only with velocity flow fields that spatially are piecewise constant. The latter assumption rules out many objects, for example, a stationary but rotating object, where the optic flow yields a field with a continuum of velocities.

Horn and Schunck, see [Horn and Schunck 81], explicitly assume that the images are at least once differentiable with respect to time. They go on to consider the square of the magnitude of the spatial gradient of the velocity as a measure of the spatial smoothness of the velocity flow fields. However the implementation of their algorithm uses the spatial Laplacian of the velocity fields, which implies they assume the velocity flows are spatially at least piecewise C^2.

Finally, we examine the assumptions in [Hildreth 83]. In this work, Hildreth reconstructs the "smoothest" flow field. Again, the assumption is that the intensity images are continuously differentiable with respect to time (actually, she assumes that $\nabla^2 G$ (I) is differentiable with respect to time). For the spatial smoothness of the velocity fields (which exist only along the edge contours returned from the $\nabla^2 G$ operation), she considers the use of a functional to measure the smoothness, which then would be minimized. She considers functionals of the form (using her notation):

$$\int \left| \frac{\partial^k V(s)}{\partial s^k} \right|^j ds \, , \quad \int \frac{\partial^k |V(s)|^j}{\partial s^k} ds \quad \text{and} \quad \int \left| \frac{\partial^k \phi(s)}{\partial s^k} \right|^j ds \, ,$$

for k, j = 0, 1, 2, ... Where V(s) is the full velocity vector (parameretrized with respect to arc length of the contour), and *f* is the angle of the velocity with respect to arc length.

Among this class of functional she settles upon $\int |\partial V(s)/\partial s|^2 ds$ as the appropriate choice. In an experimental test, she found that the algorithm (minimizing this functional) results in flow fields that are similar to those produced by humans. In a number of cases where the algorithm resulted in a physically incorrect flow field, its results were very similar to human responses, which were also physically incorrect. The form of the functional to be minimized implies she is assuming that the velocity field is spatially at least piecewise C^1.

In conclusion, the psychologists and computer scientists generally assume that motion is temporally at least C^1, and spatially at least C^2.

§6 Assumptions on the Smoothness of the Intensity Image .

First we examine the differentiability assumptions on the "image" (light input to the system) in a number of psychological models of human visual perception. Then we examine the assumptions on the image made by the theory behind a few machine vision algorithms. We note that because imaging is a continuous transfer one might feel that smoothness of objects should imply corresponding smoothness of the image (and visa versa). However, the problems of object occlusion, and reflection and/or refraction of light invalidate this assumption for the general case. But as we shall see, the assumptions people make (without referring to the smoothness of objects) are about the same as for object smoothness.

Before we begin our look into the smoothness assumptions on the image we note that many targets used in psychophysical / psychological experiments are projected patterns, comprised of constant or linearly varying luminescence. The "projected patterns" are generally not meant to be the two dimensional projection of a three dimensional object viewed under particular conditions, but rather are purely two dimensional stimuli. These patterns are then generated to be piecewise C^{∞} in intensity (usually some type of gratings).

From the psychology literature we examine a number of historically significant models surveyed in [Overington 76], and extract the assumptions present in each. All of these psychological models are "inhibition" theories developed to explain the Mach effect (The Mach effect, named for Earnst Mach, a physiologist in the late 19th century, is the appearance of bands at or near a step edge that form an accentuation of the light and dark portions of a scene near the high contrast step edge.)

> "Mach himself, having noted the strange distortions mentioned above, questioned whether the relationship between local stimulus intensity, I, and neural response Q_s might be expressed in terms of the second derivative of the two dimensional luminance distribution in the original stimulus. He thus proposed the possible relationship:
>
> $$Q_s = I - m \cdot \left(\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} \right)$$
>
> where m is a constant." [Overington 76, p. 121].

This approach also implies that there is some neural interconnections that have the ability to approximate the second derivative, and that the image is at least twice differentiable.

Another inhibitory model was proposed by Fry. [Fry 48]. Fry's work is based on an assumed inhibitory electric field component, first studied by Granit, in the electric potential difference between the front and back of the retina. Fry assumed that the potential at one point would have contributions (graded with the distance) from neighboring points (physically nearby points in the retina). Then, since each stimulated point in the retina would set up its own potential field, the total response would be the simple local component field sum. Fry assumed that the form of component field distribution function is Gaussian i.e.

$$P \propto a \cdot E_R \cdot e^{\left[-d_s^2 / \sigma^2 \right]}$$

where σ expresses the extent of the inhibitory field, a is the area of a stimulated receptor and P is the potential difference at any point a distance d_s from the point of stimulation. Fry experimentally determined that $\sigma \approx 6$ mrad. This approach differs from Mach's model in that there is no assumed neural interactions, the inhibition potentials being directly controlled by the retinal illuminance distribution. We can infer that Fry assumed an 'intensity' image that is the sum of a number of gaussians and therefore is C^∞ (not piecewise C^∞, but globally C^∞).

A neural model of inhibition which assumes that the intensity image is C^2 was put forward in [Huggins and Licklider 51]. In their model, the input is processed by neurons outputting a discrete set of values for input and the first and second derivatives (actually differences) thereof. Huggins and Licklider investigated what weighting functions necessary to "smooth" the outputs and to approximate the true input function. They then define the neural response as a convolution of the *approximated* input function with a weighing function, where the weighting has a positive central lobe, surrounded by negative lobes.

Békésy [Békésy 60] proposed a neural model to explain the Mach effect, and contrast effects similar to Mach effects in auditorial and tactile perception. He claims his model is equivalent to a continuous functional, in that it predicts Mach bands whose strength depends on the magnitude of the intensity gradient, and rate of change of gradient (this is the second derivative of the intensity). Thus Békésy's model assumes that the intensity image (and the input to the auditorial and tactile perception systems) are at least C^2.

We would like to point out that one of the more recent computational theories of human visual, that of Marr and Poggio [Marr 79], does not actually make assumptions on the input intensity image per se. This model convolves the input image with Gaussian filters of varying sizes then applies a Laplacian (a second order differential operator). (Actually they use the linearity of convolution to first apply the Laplicaian to the Gaussian and then to convolve the Laplacian of the Gaussian ($\nabla^2 G$) with the image.) Convolution with a Gaussian results in a smoothed intensity image that is C^∞, hence the fact that they apply the Laplacian does not make assumptions on the smoothness of the original image. Thus they do not even assume that the input is C^0!

In machine vision, of the most common early processing steps is the extraction of edges from the image. There are a wide variety of operators for this, many of them are discrete template matching algorithms. One of the earliest however was the use of the maximum of the gradient of the intensity image (using the fact that edges are discontinuities of the intensity and hence have infinite gradient). This clearly assumes that the image is piecewise C^1.

In the optic flow work of [Horn and Schunck, 81] described in section 5, we note that the work makes assumptions not only about the temporal differentiability but also about the differentiability of the intensity image. Namely, they explicitly assume that the brightness E(x,y,t) at point (x,y) at time t is differentiable with respect to x,y and t. (The optic flow work of [Hildreth 83] does not make these assumptions since she applies $\nabla^2 G$ to the image before attempting to calculate the temporal derivatives.)

Yet another assumption that the image is C^1 can be found in [Woodham 81]. Here, in showing the relationship between the reflectance map, R(p,q), the hessian matrix H, and the image intensity I(x,y), Woodham (by differentiating the image irradiance equation [Horn 77]) obtains the equation $[I_x, I_y]^T = H\,[R_p, R_q]^T$ which clearly implies that he assumes that I is at least C^1.

The assumptions of the smoothness of images can be broken in to two categories: those derived from models of image in the visual system, and those inferred from the application of mathematical operators to the intensity image. In the former case we found that in general the assumptions were that the intensity image was at least C^2. In the latter case we found the assumptions were generally C^1 (or none at all if the image was first smoothed). There may be some rational for this difference, the application of a differential operator tends to increase the noise present in an image.

Furthermore, the higher the order of the operator the greater the magnification of the noise. Thus while it may be convenient to mathematically model the intensity image as twice differentiable, the application of a second order differential to an actual digitized may be impractical because of the inherent noise. (This also fits well into the fact that those theories that apply a second order differential first smooth the image to reduce the noise (e.g. the $\nabla^2 G$ operator.)

Part 2: Optimal Surface Interpolation

§7 Introduction to Surface Interpolation.

Up to this point, we have been considering what assumptions on the degree of smoothness have been made explicitly or implicitly by both computer vision and psychology researchers. Although these assumptions are interesting because of the implications they pose for the human visual system, they also have some practical import for a number of problems in computer vision. The remainder of this paper studies one such problem: surface interpolation given a sparse depth map. In recent years this topic has become the subject of fervent research, e.g. see [Grimson 81], [Grimson 80], [Terzopoulos 83,84], [Kender, Lee and Boult 85].

Through out this paper we have and shall continue to refer to the problem of *visual surface interpolation*. Although we are actually interested a method of approximating the surface, we refer to it as surface interpolation because we *restrict* the surface to interpolate the data. More preceisly the problem of surface interpolation is to find the "best" approximation to a surface passing through (i.e. interpolating) a (generally sparse) set of depth values. The trick here, of course, is in the proper definition of the term "best" . There are infinitely many surfaces passing through a given set of points, yet, in general, human observers perceive only one (there are certain cases in which there is more then one interpretation, see [Julesz 71].) In all of the afore mentioned work, the overwhelming consensus is that finding the "best" surface is equivalent to finding which surface humans would perceive, and how it can be specified.

One way to specify which surface is perceived, is to find some functional, say Θ, such that for any set of input depth data the interpolating function f^* perceived by human observers, minimizes $\Theta(f)$ for all f from some class of functions that "represent" physical or perceived objects that have the same values as f^* at the data points. An interpolation algorithm could then attempt to find f^* by minimizing this functional over the given class of functions. Note that there are a number of ***major*** assumptions here:

- The "best" surface is the surface seen by humans;
- There is some class of functions accurately representing physical or perceived objects;

* A functional measuring reasonableness of a function exists;

* Given that such a functional exists, that it can be minimized.

The first of these assumptions is a statement about the goals of computer vision. If taken as a true assumption, it implies that we are interested not is perception of true physical world, but in the visual world ala Gibson. The second of these assumptions has, up to this point, been the reason for most of this paper. A conclusion of this paper is that psychologists and computer scientists alike assume the world (perceived or physical) is at least piecewise C^2, possibly with a bound on the second derivative. Therefore any class modeling the world should be at least C^2, and be such that the second derivatives is bounded. The third of these assumptions, i.e. that there exists a functional measuring the reasonableness of a surface is a matter that needs further study. Though a number of functionals have been proposed, their relationship to the surfaces perceived by humans should be examined. The final assumption mentioned above, (i.e. that such a functional can be minimized), is a purely mathematical assumption, though it is of course dependent on the choice of the functional. The reader should keep in mind that the last two assumptions present a trade off between the physiological / psychological plausibility of some functional form, and the mathematical properties of that functional. This survey shall not, however, consider this points further.

Though all of the afore mentioned work on surface interpolation is in part grounded in this concept of minimization of a "measure of smoothness", [Kender, Lee and Boult 85] arrive at this approach from a different point of view. They show that given certain assumptions this problem fits into the framework of a Information-Based Complexity developed by Traub and Woz'niakowski and their colleagues. This general theory then yields a spline algorithm that is *optimal* in the sense that it has minimal error among all algorithms.

We begin with a brief introduction to the nomenclature of the information-based complexity. This in hand, the "needed" assumptions for application of the general theory shall be examined. Then we shall present a number classes of functions which, based on the assumptions reported in the earlier sections of this paper, are candidate classes for the interplaotry surface. For each class we shall also present an associated measure of smoothness, and some other information necessary to construct the optimal algorithm.

§8 Terminology of Information-Based Complexity.

To use the results of the information-based complexity we must first cast the problem of surface interpolation in their terminology. To do this, we must exactly define the problem, in terms of: the allowed class of information operators, and the allowed class of algorithms, the class of functions the information operator is applied to, the class of functions that the solution belongs to, and a restriction operator to restrict the class of functions to consider as solutions. We briefly discuss the definition of each of these in turn; the interested reader should consult [Kender, Lee and Boult 85], [Woz'niakowski 85], [Traub and Woz'niakowski 80] or [Traub, Wasilkowski and Woz'niakowski 83]. (Information-based complexity is a unified framework for discussing many previous, and numerous new theories dealing with optimal algorithms.) Then we discuss the definition of an algorithm, and the measure of error of an approximation. Please note that the definitions in this section are given in generality, in the hope that other vision problems may more easily be cast in this framework. This section may be skiped without significant loss of continuity.

First we recall the definition of a linear problem. Let F_1 and F_3 be linear spaces, and F_2 and F_4 be normed linear spaces (with norms $\|\cdot\|_2$ and $\|\cdot\|_4$ respectively). Then we define S: $F_1 \rightarrow F_2$, (S is called the solution operator), N: $F_1 \rightarrow F_3$, (N is called the information operator), T: $F_1 \rightarrow F_4$, (T is called the restriction operator) be *linear* operators. We also define the class of functions $F \equiv \{f \in F_1 \text{ such that } \| Tf \|_4 \leq 1\}$.

We make this more concrete by describing the linear spaces and operators for the problem of visual surface interpolation. F_1 is the class of all surfaces that we wish to approximate. Different definitions of F_1 give formally different problems: see later sections of this paper for examples. $F_2 \equiv \{f|D: f \in F_1\}$ is the class of surfaces restricted to some domain D. D maybe a circle or a square, or $\Re^2$ if we do not wish to restricted the domain at all. We also assume that there exists a norm on F_2 denoted by $\|\cdot\|_2$. This norm might be the L^2, L^∞ or any other norm, and will be generally depend on the definition of F_1. F_3 is simply $\Re^k$, where k is the the number of depth samples. In our case F_4 will be = F_2, with the definition of smoothness being reflected in the norm $\|\cdot\|_4$.

Now let us consider the linear operators S, T, and N for our problem. The solution operator S is = I (I being the Identity operator), i.e. we are trying to recover the physical surface which generated the information. (If for example we were instead trying to approximate the orientation of the interpolating surface, S would be the first partial derivatives.) The solution operator will be used later in the definition of error. In our treatment, we shall use $T = c \cdot I$, where the constant c is simply a scaling factor so that $\| Tf \|_4 \leq 1$ for all $f \in F_1$ that are "smooth" for the current scale of resolution. Without loss of generality we assume that the resolution scale is such that the constant $c = 1$, hence $T \equiv I$ and the class of functions F_4 is given by $\{ f: \| f \|_4 \leq 1\}$. The information operator N_k consists of k adaptively chosen function evaluations (depth values), that is $N_k(f) = [f(x_1,y_1), ..., f(x_k,y_k)]$ where in general x_i and y_i depend in an arbitrary manner on $[f(x_1,y_1), ..., f(x_{i-1},y_{i-1})]$. If there is no dependence of x_i and y_i on $[f(x_1,y_1), ..., f(x_{i-1},y_{i-1})]$ we term the information nonadaptive. Nonadaptive information has many nice properties, the foremost of which is that the information can be collected in parallel. It turns out that for our problem (in fact for linear problems in general, see [Traub and Woz'niakowski 80, p57-63]) that adaptive information is no more powerful that nonadaptive information. That is to say that for any adaptive information there exists nonadptive information (with the same number of function evaluations) such that the intrinsic error in the nonadaptive information is not greater than that in the adaptive information. Therefore without loss of generality we may restrict ourselves to nonadaptive information, $N_k(f)$ $[f(x_1,y_1), ..., f(x_k,y_k)]$ where x_i and y_i, $i = 1..k$ are given a priori. The implication of this is that nothing is lost (as far as amount of information needed) by a system that gathers all of its depth data simultaneously. The fact that the human eye collects its stereo information in parallel suggest that evolution has already learned this unintutative lesson.

Now we examine the definition of an algorithm. In the general theory, an algorithm $\phi:N(f)\rightarrow F_2$ is an arbitrary mapping from N(f) (the information) to F_2, the class of restricted surfaces. This definition is then augmented by restricting the algorithm ϕ to be an element of a class of realizable algorithms Φ, which may, for example, be algorithms implementable in a digital computer. Statements about an algorithm ϕ (e.g. about ϕ's optimality) made without such a restriction on the class of algorithms are much farther reaching. They have implications for *any* algorithm to solve the problem, even those implemented with vastly different properties than ϕ: even algorithms implemented within biological organisms.

Finally, we can address the definition of error. We consider only the worst case absolute error. All the results presented here turn out to be independent of the choice of relative or absolute error , see [Traub, Wasilkowski and Woz'niakowski 83, appendix E]. Then we define the error of a algorithm ϕ using information N_k to be:

$$e(N_k, \phi, f) = \sup_{g \in V(N_k, f)} \| g - \phi(N_k(f)) \|_2$$

where $V(N_k,f) \equiv \{g \in F: N_k(g) = N_k(f)\}$ is the set of functions from F, with the same information as f. Since we know only N_k we can not distinguish f from any other element of $V(N_k,f)$, thus we take the supremum to handle the worst case.

Now we define an algorithm ϕ^* to be strongly optimal *if and only if*

$$e(N_k, \phi^*, f) = \inf_{\phi \in \Phi} e(N_k, \phi, f) \quad \forall f \in F$$

where we place no restrictions on Φ. In other words, we define an algorithm ϕ^* to be strongly optimal if and only if for each surface f, ϕ^* has error less than or equal to any other algorithm. Observe that the error then depends on both $\|\bullet\|_2$ and $\|\bullet\|_4$. The above definitions of error are however independent of the choice of these norms and in fact the optimal algorithm ϕ^* is also independent of this choice. This is a non trivial result of the general theory, see [Traub, Wasilkowski, and Woz'niakowski 83, appendix E]. However, the *error* of the optimal algorithm, ϕ^*, is *dependent* on the choice of $\|\bullet\|_2$ and $\|\bullet\|_4$, and if they are not restrictive enough, it may be happen that the error of the optimal algorithm is infinite. In such a case, *no* algorithm could solve the problem for every f in F.

§9 Optimal Surface Interpolation: A General Setting.

In this section we present a form of the optimal algorithm, given a semi-norm and reproducing kernel of the space F_2, and the semi-norm of the space F_4. To be precise, given the reproducing kernel and semi-norm $\|\bullet\|_2$ (with null space $\tilde{N}$) of a space χ, and letting $\|\bullet\|_4 = \|\bullet\|_2$ we define F_2 and F_4 to be the quotient space $\chi/\tilde{N}$. Then F_2 and F_4 are normed linear space with norm $\|\bullet\|_2$. For the sake of brevity we shall often refer to $\|\bullet\|_2$ and $\|\bullet\|_4$ as norms even though they can be semi-norms. If it is important that they are not semi-norms we shall explicitly state this.

[Traub, Wasilkowski and Woz'niakowski 83] (see [Kender, Lee and Boult 85] for a less mathematical treatment) proves that the optimal algorithm for the problem defined as above is given by the spline function $\sigma_{N(f)}$ interpolating the data. We recall that the spline $\sigma_{N(f)}$ is such that :

$$\| \sigma_{N(f)} \|_4 = \inf_{f \in V(N,f)} \| f \|_4 .$$

Now that we know, at least in a mathematical sense, what the optimal algorithm is, we need to express it in a form that can be realized. This problem has, for a number of classes, already been answered in the approximation literature. Atteia [Atteia 66a, 66b, 70] and Thomann [Thomann 70a, 70b] first dealt with the case when F_4 was a Hilbert space, and their results were generalized by Duchon [Duchon 75, 76] to the case of semi-hilbert cases. We note that the following realization of the spline algorithm $\sigma_{N(f)}$ is not the only possible realization. In fact the work of [Grimson 81] is actually an attempt to obtain the spline by direct minimization of the semi-norm which is used to define $\sigma_{N(f)}$.

We follow the summary of Duchon's work given in [Schumaker 80]. If the space χ is a semi-Hilbert space, and $\tilde{N} = \{ f \in \chi : \| f \|_4 = 0 \}$ then the optimal algorithm can be written down in terms of K(x,y), the reproducing kernel of χ. (Note that these reproducing kernels always exist, though in practice they may be very difficult to derive.) Namely letting $N_k(f) = z = [z_1, \ldots z_k] = [f(x_1,y_1), \ldots , f(x_k,y_k)]$ we can write the interpolating spline as

$$\sigma_z(x,y) = \sum_{i=1}^{k} \alpha_i \cdot K((x,y);(x_i,y_i)) + \sum_{i=1}^{d} \beta_i \cdot p_i(x,y)$$

where $\{p_i\}_{i=1 \ldots d}$ is a basis for the set $\tilde{N}$, and the coefficients $\{\alpha_i\}$ and $\{\beta_i\}$ can be determined from the linear system of equations

$$\sum_{i=1}^{k} \alpha_i \cdot K((x_j,y_j);(x_i,y_i)) + \sum_{i=1}^{d} \beta_i \cdot p_i(x_j,y_j) = z_j , \qquad j = 1, \ldots k$$

$$\sum_{i=1}^{k} \alpha_i \cdot p_j(x_i,y_i) = 0, \quad j = 1, \ldots, d$$

There are two basic approaches for finding the appropriate classes, the associated norms and the reproducing kernels needed to find optimal spline for visual surface interpolation. The first approach is to derive (some how) from psychological models of human perception the classes of functions, and the associated norms for measuring smoothness. Given these two components, one must then derive the reproducing kernel(s) for the class(es) derived. This approach has the drawback that deriving mathematical classes and norms may be difficult, based on current models of the human visual system. Even given these classes and norms, obtaining the reproducing kernel remains technically difficult. The second approach for arriving at optimal visual surface interpolation algorithms is to take classes that reasonably fit the known psychological process for which we already know the norm and reproducing kernel, and to find the optimal algorithms for these classes. That is we compile a collection of reasonable (class, norm, reproducing kernel) triples that currently exist, and apply (9.1) and (9.2) to obtain the optimal algorithm. Given an implementation of these algorithms, we then experimentally determine which if any of these classes best approximates human perception of smooth surfaces. It is of course imperative to use optimal algorithms, so that any differences in the reconstructions are due to the (class, norm) choice and not a poor algorithm.

There are many properties that we might wish the interpolation algorithm to possess including: smoothness that coincides with the smoothness known to be present in human visual interpolation, time and space efficient implementations, biologically feasible implementations, high degree of parallelism, exact solution for low order polynomials, etc..

Keeping these properties in mind, the rest of this paper shall present a number of (class, norm, reproducing kernel) triples that have been used elsewhere in the approximation literature and meet at least the first of the above properties (i.e. they have acceptable smoothness.) We shall present each of these first in the general form, and then give a particular example of the formulas.

§10 The "Bending Energy Norm" and a little Notation.

In a number of the following (classes, norms, kernel) triples we shall encounter the following semi-norm (here after we shall refer to is as $\Theta(f)$) :

$$\Theta(f) \equiv \|f\|_4 = \left[\iint_{\Re^2} \left(\frac{\partial^2 f}{\partial x^2}\right)^2 + 2 \cdot \left(\frac{\partial^2 f}{\partial x \partial y}\right) + \left(\frac{\partial^2 f}{\partial y^2}\right)^2 \right]$$

This semi-norm has a nice physical interpretation as the amount of bending energy in a thin elastic plate. Hence minimizing $\Theta(f)$ is equivalent to finding the function from a given class, which passes through the data points, and has minimum bending energy. The reader familiar with the work of Grimson [Grimson 79,80,81] will recognize this as the same functional that his relaxation algorithm attempted to minimize. Grimson argues that this functional is minimized by the human visual system, though he does not present any psychophysical basis for his conclusions. It is also important to note that although Grimson's algorithm attempted to minimize a discrete version of this functional, it was not doing so with respect to any particular class of functions, and hence does not possess the optimality properties of the algorithms presented here. This functional has a null space spanned by $\{1,x,y\}$. This latter information will be used in the construction of the optimal algorithm for interpolation with respect to any class, with $\Theta(f)$ as a semi-norm or norm.

We present a little notation that will aid in the discussion of the classes, norm, and the associated reproducing kernel. First, we shall at times use the variable Γ locally in to denote the class of functions we are currently examining. We shall also use the notation $D_x{}^i$ to represent the differential operator $\partial^i(\cdot)/\partial x^i$, and similarly we shall use $D_y{}^i$. For the case $i = 1$ we shall often drop the superscript. In what follows the notation $D_x f(\alpha,\cdot)$ should be interpreted as a function of y defined as $\partial f/\partial x$ evaluated at the point $(x,y) = (\alpha,y)$. (We assume similar definitions for derivatives with respect to y.) Also $D_x D_y f(\alpha,\beta)$ should be interpreted as D_x applied to $D_y f(x,\beta)$ and evaluated at α. We shall use $K(x,y;s,t)$ as the notation for a reproducing kernel in cartesian coordinates, and $K(\rho_1, \alpha_1; \rho_2, \alpha_2)$ for reproducing kernels in polar coordinates. Finally we Use the standard notation Π_p for the space of bivariate polynomials with degree $\leq p$ in each variable.

§11 Atteia's C^2 functions on a disk or on a rectangle.

Here we investigate some of the mathematical details defining the two reproducing kernels derived in [Atteia 70]. Both classes of functions have the standard Sobelev semi-norm given by (10.1) , but have their domains restricted to circular or rectangular regions. These classes are considered because they contain the appropriate smoothness, use a intuitively pleasing semi-norm, and are defined over convenient regions. As we shall see however, they do not lead to very efficient implementations.

Let Ω be an open bounded region of $\Re^2$. Then let H be the set of twice differentiable continuous functions $f:\Omega \rightarrow \Re^2$, such that the first and second order partial derivatives are elements of $L^2(\Omega)$. Atteia shows that $\Theta(f)|_\Omega$ (that is the function defined in 10.1, where the integral is restricted to Ω) is a norm on the (quotient) space $H / \prod_1$. He examined two separate classes.

First, let Ω be a disk with radius r centered at the origin, and let our class of functions, Γ be defined as

$$(11.1) \qquad \Gamma = \left\{ f \in H : \oint_{\partial\Omega} f(x,y) = \oint_{\partial\Omega} x \cdot f(x,y) = \oint_{\partial\Omega} y \cdot f(x,y) = 0 \right\}$$

(The mathematical restrictions in definition of this class is similar to restricting the reconstructed surfaces to be such that if a thin wire was placed along the space curve given the value of the surface along the boundary of Ω, then that wire would have its center of gravity at the origin.)

Then the reproducing kernel given by [Atteia 70] is

$$(11.2) \qquad K(\rho_1, \alpha_1; \rho_2, \alpha_2) = \frac{1}{2\pi} \cdot \left(\frac{\rho_1^2}{2r} - \frac{r}{2} \right) \cdot \left(\frac{\rho_2^2}{2r} - \frac{r}{2} \right)$$

$$+ \frac{1}{12\pi} \cdot \left(\frac{\rho_1^3}{r} - \rho_1 \right) \cdot \left(\frac{\rho_2^3}{r} - \rho_2 \right) \cdot \cos(\alpha_1 - \alpha_2)$$

$$+ \frac{1}{2\pi} \sum_{n=2}^{\infty} \left(\left(\frac{2\sqrt{n-1}}{n^2+2} \cdot \frac{\rho_1^{n+2}}{r^{n+1}} - \frac{1}{n \cdot \sqrt{n-1}} \cdot \frac{\rho_1^n}{r^{n-1}} \right) \cdot \left(\frac{2\sqrt{n-1}}{n^2+2} \cdot \frac{\rho_2^{n+2}}{r^{n+1}} - \frac{1}{n\sqrt{n-1}} \cdot \frac{\rho_2^n}{r^{n-1}} \right) \cdot \cos(n \cdot (\alpha_1 - \alpha_2)) \right)$$

$$+ \frac{1}{8\pi} \left| z_1 - z_2 \right|^2 \cdot \operatorname{Log}_e \left| \frac{z_1 \cdot z_2 - 1}{z_1 - z_2} \right| + \frac{1}{2} \left(|z_1|^2 - 1 \right) \left(|z_2|^2 - 1 \right)$$

where $z_1 = \rho_1 \cdot e^{i \cdot \alpha_1}$ and $z_2 = \rho_2 \cdot e^{i \cdot \alpha_2}$.

Now let us take Ω to be [a,b] x [c,d], then Atteia shows that if we take the class of functions to be

$$(11.3) \qquad \Gamma = \left\{ f \in H: \oint_{\partial\Omega} f(x,y) = \oint_{\partial\Omega} (x \cdot f(x,y) + D_x f(x,y)) = \oint_{\partial\Omega} (y \cdot f(x,y) + D_y f(x,y)) = 0 \right\}$$

then $\Theta(f)|_\Omega$ is a norm on Γ / Π_1, and the reproducing kernel is given by:

$$(11.4) \qquad K(x,y; s,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Big(\gamma_{m,n} \cdot \sin \left(m \cdot \pi \cdot \frac{x-a}{b-a} \right) \cdot \sin \left(n \cdot \pi \cdot \frac{y-c}{d-c} \right)$$

$$\cdot \sin \left(m \cdot \pi \cdot \frac{s-a}{b-a} \right) \cdot \sin \left(n \cdot \pi \cdot \frac{t-c}{d-c} \right) \Big)$$

where $\gamma_{m,n} = \dfrac{2 \cdot \sqrt{((b-a) \cdot (d-c))^3}}{\pi^2 (m^2 \cdot (d-c)^2 + n^2 \cdot (b-a)^2)}$.

It is unfortunate that both of the above reproducing kernels are infinite series. In practice, they would of course be truncated after some number of terms (note that both series go to zero strictly faster than $n^{-1.5}$). It is very unlikely that the human visual system would be reconstructing a surface using truncated infinite series, but it may be reconstructing a surface that can be very closely approximated by one of the above approaches.

§12 Duchon's Classes of C^m functions on $\Re^2$.

Here we actually examine three infinite families of classes from [Duchon 70]. In all three families the domain of the functions is the entire space $\Re^2$, though when actually approximating we shall restrict this space to some finite domain Ω. All three classes have the standard m^{th} Sobelev semi-norm (given by (10.1) for the case m = 2) , and are considered because they have the appropriate smoothness (for any $m \geq 2$). Although these classes seem to differ in mostly mathematical terms, these differences have an effect on the efficiency of implementation as well as the visual appearance of the optimal interpolatory surface.

The first families of spaces Duchon called $D^{-m} L^2$, and for a given $m \geq 2$ each class is defined as the space of functions which have all partial derivatives of order m in $L^2(\Re^2)$. Then given that $m \geq 2$ we have that $D^{-m} L^2 / \Pi_{m-1}$ is a Hilbert space with the m^{th} Sobelev semi-norm as a norm. The reproducing kernel for $D^{-m} L^2(\Re^2)$ is given by

(12.1)

$$K(x,y;s,t) = \left((x-s)^2 + (y-t)^2\right)^{m-1} \cdot \mathrm{Log}_e \sqrt{(x-s)^2 + (y-t)^2}$$

If we define $D^{-m}H^{\mu}$, for $\mu \leq 1$, to be the space of functions which have all partial derivatives of order m in H^{μ}, where H^{μ} is the Hilbert space of functions such that their tempered distributions v have Fourier transform $\underline{v}$ that satisfy

$$\int_{\Re^2} |\tau|^{2\mu} \cdot |\underline{v}(\tau)|^2 \, d\tau < \infty$$

Then for $\mu > 1 - m$ the reproducing kernel of the space $D^{-m}H^{\mu}$ is given by :

(12.2)

$$K(x,y;s,t) = \begin{cases} \left((x-s)^2 + (y-t)^2\right)^{m+\mu-1} \cdot \mathrm{Log}_e \sqrt{(x-s)^2 + (y-t)^2} & \text{if } \mu \text{ is an even integer,} \\ \left((x-s)^2 + (y-t)^2\right)^{m+\mu-1} & \text{otherwise.} \end{cases}$$

As an example of these classes, let us choose m = 2, hence our example is for functions from C^2 such that their second derivatives are in H^μ. If we take $\mu = 0$, we get a reproducing kernel = $((x-s)^2 + (y-t)^2) \cdot .5 \cdot \mathrm{Log}((x-s)^2 + (y-t)^2))$, and if $\mu = 1$ we have a reproducing kernel that is simply the cube of the euclidean distance. It is this last case that was reported upon in [Kender, Lee and Boult 85].

All of the classes considered so far have had a Sobelev semi-norm (given by 10.1 for m = 2), which as we mentioned has a nice physical interpretation. We now consider a few classes that are more mathematically abstract. However, they are all of the appropriate smoothness, and some have other desirable properties.

§13 Arthur's Classes of functions in $R^{m,n}$ on [a,b] x [c,d].

The classes and their associated norms are considered in this section because have the appropriate smoothness, should have reasonable efficient implementations, and allows modeling of non-isotropic smoothness. Although many researchers in computer vision have assumed that the visual space is isotropic (in terms of smoothness or number of continuous derivatives), this is not an established psychophysical fact. In fact, given the horizontal placement of the eyes, which has many effects including yielding less stereo depth data along horizontal contours, one might infer that he visual space is in fact non-isotropic.

Let $R^{m,n}$ be the Hilbert space containing all functions defined on $\Omega \equiv [a,b] \times [c,d]$ such that for every $f \in R^{m,n}$ we have

$D_x^i f$ is continuous for $i = 1, \dots, m-1$;
$D_y^j f$ is continuous for $j = 1, \dots, n-1$;
$D_x^{m-1} f$ and $D_y^{n-1} f(x,y)$ are absolutely continuous;
and finally $D_x^m f$, $D_y^n f$, and $D_x^{m-1} D_y^{n-1} f$ are in $L^2(\Omega)$.

Let $\{x_1, \dots, x_m\}$ and $\{y_1, \dots, y_n\}$ be distinct points in [a,b] and [c,d] respectively. Then Arthur [Arthur 74], shows that $(f,f)^{1/2}$ is a semi-norm on $R^{m,n}$, where the inner product (f,g) for $R^{m,n}$ is given by:

$$(f,g) \equiv \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i,y_j)\cdot g(x_i,y_j) + \sum_{j=1}^{n}\int_a^b D_x^m f(x,y_j)\cdot D_x^m g(x,y_j)\,dx$$

$$+ \sum_{i=1}^{m}\int_c^d D_y^n f(x_i,y)\cdot D_y^n g(x_i,y)\,dy$$

(13.1)

$$+ \int_a^b\int_c^d D_x^m D_y^n f(x,y)\cdot D_x^m D_y^n g(x,y)\,dy\,dx$$

The null space associated with this semi-norm is

$$\left\{\ f\in R^{m,n}:\ f(x,y) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \beta_{i,j}\cdot x^i\cdot y^j;\quad \text{for some constants } \beta_{i,j}\ \right\}$$

which is of finite dimension $m\cdot n$.

Before we can define the reproducing kernel for $R^{m,n}$ with the above norm, we need a few auxiliary functions. Let

$$c_i(x) = \prod_{\substack{j=1\\ j\neq i}}^{m} \frac{x-x_j}{x_i-x_j} \quad i=1,\ldots,m \qquad \text{and} \qquad d_i(y) = \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{y-y_j}{y_i-y_j} \quad i=1,\ldots,n$$

Then define

$$K_1(x,s) = \sum_{i=1}^{m} c_i(x)\cdot c_i(s) + \frac{(-1)^m}{(2m-1)!}\cdot\left((x-s)_+^{2m-1} + \sum_{i=1}^{m}\sum_{j=1}^{m} c_i(x)\cdot c_j(s)\cdot (x_i-x_j)_+^{2m-1}\right.$$

(13.2)

$$- \sum_{i=1}^{m} \left(c_i(s) \cdot (x - x_i)_+^{2m-1} + c_i(x) \cdot (x_i - s)_+^{2m-1} \right) \Bigg)$$

and similarly define

$$K_2(y,t) = \sum_{j=1}^{n} d_j(y) \cdot d_j(t) + \frac{(-1)^n}{(2n-1)!} \cdot \Bigg((y-t)_+^{2n-1} + \sum_{i=1}^{n}\sum_{j=1}^{n} d_i(y) \cdot d_j(t) \cdot (y_i - y_j)_+^{2n-1}$$

(13.3)

$$- \sum_{j=1}^{n} \left(d_j(t) \cdot (y - y_j)_+^{2n-1} + d_j(y) \cdot (y_j - t)_+^{2n-1} \right) \Bigg),$$

where

$$(Z)_+ \equiv \begin{cases} Z & \text{if } Z \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Finally the reproducing function derived by Arthur for the class $R^{m,n}$ with norm derived from the inner product (13.1) is $K(x,y;s,t) = K_1(x,s) \cdot K_2(y,t)$. This separability significantly decreases the cost (both in time and space) necessary to obtain the optimal interpolatory surface for this class.

Using the above formulas, one can chose different m and n (they need not be the same) to obtain optimal interpolations for many classes, for example $R^{2,2}$, $R^{2,3}$, $R^{3,3}$, etc.. Note that changing the location of the points $\{x_1,\dots x_m\}$, $\{y_1, \dots, y_n\}$ results in a different norm as well as the associated kernel. At the present time we do not fully understand the implications of these changes. We now examine the particular case with $\Omega=[0,1] \times [0,1]$, with m=2, n=2, $x_1=0$, $x_1=1$, $y_1=0$, $y_2=1$. Then the class of functions for our example contains those functions defined on Ω with first partial derivatives absolutely continuous, and with second partial derivatives in $L^2(\Omega)$. Then the appropriate semi-norm (from 13.1) is

$$\| f \| \equiv \Big(f(0,0)^2 + f(0,1)^2 + f(1,0)^2 + f(1,1)^2 + \int_0^1 \left(D_x^2 f(x,0)\right)^2 dx + \int_0^1 \left(D_x^2 f(x,1)\right)^2 dx$$

(13.5)

$$\int_0^1 \left(D_y^2 f(0,y)\right)^2 dy \; + \int_0^1 \left(D_y^2 f(1,y)\right)^2 dy \; + \int_0^1\!\!\int_0^1 \left(D_x^2 D_y^2 \; f(x,y)\right)^2 dy \; dx \quad)$$

Then, if we restrict $0 \le x, y, s, t \le 1$, we obtain (from (13.2 and (13.3))

$$K_1(x,s) = x \cdot s + (x-1)\cdot(s-1) + \frac{1}{6}\cdot \left((x-s)_+^3 - (x-1)\cdot s + s \cdot x^3 + (1-s)_+^3 \cdot (1-x) \right)$$

$$K_2(y,t) = y \cdot t + (y-1)\cdot(t-1) + \frac{1}{6}\cdot \left((y-t)_+^3 - (y-1)\cdot t + t \cdot y^3 + (1-t)_+^3 \cdot (1-y) \right)$$

Then for this example the reproducing kernel $K(x,y;t,s) = K_1(x,s) \cdot K_2(y,t)$, and the null space for the semi-norm 13.4 is spanned by $\{1, x, y, xy\}$.

§14 Mansfield's classes $R_c^{p,q}$ on [a,b] x [c,d].

In this section we present a family of class and their associated semi-norm and reproducing kernel developed by Lois Mansfield [Mansfield 71]. We consider these classes because they have the appropriate amount of smoothness for the visual surface interpolation problem. Again these classes allow experiments to see if the perception of smoothness is non-isotropic, since we can vary the degree of smoothness in the x and y directions separatly. These families of classes also allow us to incorporate prior knowledge (on a set of points or lines) into the norm we will minimize. Finally, the second of the two families of classes examined, has the property that the optimal algorithm is exact for information generated by low order polynomials.

We point out that given the right choice of parameters, the norms and reproducing kernels developed by Mansfield totally subsumes the (chronologically later) work of [Arthur 74]. However, the Mansfield families of class are more general, and the equations arising from it, though equivalent for the right choice of parameters, are more complex. Before we can define the class $R_c^{p,q}$ we need a few auxiliary definitions.

Let our domain of interest be the two dimensional rectangle $\Omega \equiv [a,b] \times [c,d]$. Let $p \ge 1$ and define

$$F^p[a,b] \equiv \{ f : \Re \to \Re \text{ such that } D_x{}^{p-1} \text{ is absolutely continuous, and } D_x{}^p \in L^2[a,b] \}.$$

Let $G_1, \ldots, G_p$ be a set of linear functionals which are linearly independent over Π_{p-1}, such that each G_i is of the form

$$G_i(f) \equiv \sum_{j=0}^{p-1} \int_a^b D_x^j f(x)\ d\omega_j(x),$$

where the ω_j's are generalized functions of bounded variation (in the standard mathematical sense not in the sense of Grimson), for example, delta functions (in which case the integral reduces to a function evaluation at the spike of the delta function), constant functions, etc..

Similarly let q be ≥ 1, define

$$F^q[c,d] \equiv \{ \underline{f} : \Re \to \Re \text{ such that } D_y{}^{q-1} \text{ is absolutely continuous, and } D_y{}^q \in L^2[c,d] \},$$

and $\underline{G}_1, \ldots, \underline{G}_q$ be linear functionals of the form:

$$\underline{G}_i(\underline{f}) \equiv \sum_{j=0}^{q-1} \int_c^d D_y^j\ \underline{f}(y)\ d\underline{\omega}_i(y), \qquad \underline{\omega}_i(y) \text{ functions of bounded variation,}$$

which are linearly independent over Π_{q-1}.

Now we can define our class of functions $R_c{}^{p,q}$, $R_c{}^{p,q}: \Re^2 \to \Re$ as

$$R_c^{p,q} = \begin{cases} D_x^i D_y^j f \text{ is continuously differentiable on } \Omega,\ \forall\ i < p \text{ and } \forall\ j < q; \\ D_x^{p-1} \underline{G}_j f \text{ is AC}[\Omega], \text{ and } D_x^p \underline{G}_j f \in L^2[\Omega],\ \forall\ j = 1..q; \\ D_y^{q-1} G_j f \text{ is AC}[\Omega], \text{ and } D_y^q G_j f \in L^2[\Omega],\ \forall\ j = 1..p; \\ D_x^p D_y^q f \in L^2[\Omega]. \end{cases}$$

where AC[Ω] is the set of absolutely continuous functions on Ω. Here we note that if G_i and $\underline{G}_j$

are taken as function evaluations at points within [a,b] and [c,d] respectively (that is by choosing $\omega(x)$ and $\underline{\omega}(y)$ as delta functions), then by setting m=p, n=q , we have $R_c{}^{p,q} = R^{m,n}$ the class defined in (13.2).

Again, on the way to defining the semi-norm for this space, we develop a few other functions, to wit, let $r_j \in \Pi_{p-1}$, j = 1..p be functions such that

$$G_i(r_j) = \delta_{ij},\ i = 1..p,\ j = 1..p,$$

where ∂_{ij} is the Kroneker delta. Similarly let $\underline{r}_j \in \Pi_{q-1}$, j = 1..q be functions such that

$$\underline{G}_i(\underline{r}_j) = \delta_{ij},\ i = 1..q,\ j = 1..q.$$

Now define P and Q to the linear projections on $F^p[a,b]$ and $F^q[c,d]$ respectively with:

$$P(f(x)) = \sum_{i=1}^{p} G_i\,(f(x)) \cdot r_i \qquad \text{and} \qquad Q(\underline{f}(y)) = \sum_{j=1}^{q} \underline{G}_j\,(\underline{f}(y)) \cdot \underline{r}_j\,.$$

Now with the notation I as the indenty operator, $(L)_x f(x,y)$ meaning to apply the linear functional L to f as a function of x, and $(z)_+$ defined as in (13.3) define the two functions:

$$(14.1) \qquad G(x,t) = (I\text{-}P)_x \cdot \frac{(x-t)_+{}^{p-1}}{(p-1)!} \quad \text{and} \quad \underline{G}(y,s) = (I\text{-}Q)_y \cdot \frac{(y-s)_+{}^{q-1}}{(q-1)!}$$

and note that:

(14.2)

$$(I - P)f(x) = \int_a^b G(x,t) \cdot D_t^p\, f(t)\ dt$$

and

(14.3)

$$(I - Q)\,\underline{f}(y) = \int_c^d \underline{G}(y,s) \cdot D_s^q\, \underline{f}(s)\; ds.$$

Using these last two equations, we have for all $f \in R_c{}^{p,q}$

$$f(x,y) = (P \otimes Q)f(x,y) + ((I-P) \otimes Q)f(x,y) + (P \otimes (I-Q))f(x,y) + ((I-P) \otimes (I-Q))f(x,y),$$

where $\otimes$ is the normal tensor product. This directly leads to the following semi-norm on $R_c{}^{p,q}$:

$$\| f \|^2 = \int_a^b \int_c^d \left(D_x^p D_y^q f(x,y) \right)^2 dy\; dx + \sum_{j=1}^{q-1} \int_a^b \left(D_x^p \underline{G}_j\, f(x,y) \right)^2 dx$$

(14.4)

$$+ \sum_{i=1}^{p-1} \int_c^d \left(D_y^q\, G_i\, f(x,y) \right)^2 dy + \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} \left(G_i\, \underline{G}_j\, f(x,y) \right)^2$$

And we can define [f,f] as

$$[f,f] \equiv \| f \|^2 - \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} \left(G_i\, \underline{G}_j\, f(x,y) \right)^2$$

where $[f,f]^{1/2}$ is also a semi-norm on $R_c{}^{p,q}$. Both the semi-norm $[f,f]^{1/2}$ and the semi-norm (14.4) have a null space of dimension $p \cdot q$ given by

$$\left\{ f :\; f(x,y) = \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} \beta_{i,j} \cdot x^i \cdot y^j; \quad \text{for some constants } \beta_{i,j}. \right\}$$

Finally the (separable) reproducing kernel, $K^*(x,y;s,t) = K_1(x,s) \cdot K_2(y,t)$, for $R_c{}^{p,q}$ with semi-norm (14.4) is given by

(14.5)

$$K_1(x,s) = \sum_{i=1}^{p} r_i(x) \cdot r_i(s) + \int_a^b G(x,z) \cdot G(s,z)\, dz$$

(14.6)

$$K_2(y,t) = \sum_{j=1}^{q} \underline{r}_j(y) \cdot \underline{r}_j(t) + \int_c^d \underline{G}(y,z) \cdot \underline{G}(t,z)\, dz$$

and Mansfield notes that

(14.7)

$$\int_a^b G(x,z) \cdot G(s,z)\, dz = (I - P)_x (I - P)_s \left((-1)^p \cdot \frac{(x-s)_+^{2p-1}}{(2p-1)!} \right)$$

With a similar statement holding for $\int G(y,z) \cdot G(t,z)\, dz$. It can be shown that if we choose the G_i's and G_j's to pointwise functional evaluations then the above formulas would reduce to those recounted in section 13.

Now, we present an example of a particular class, say $R_C^{3,2}$, and develop the associated semi-norm and kernel. Let our domain be $\Omega = [0,1]\mathrm{x}[0,1]$ (that is a=c=0 and b=d=0). Take $G_1(f) = f(0)$, $G_2(f) = f(1)$, $G_3(f) = D_x f\,|_{x=0}$, $\underline{G}_1(\underline{f}) = \underline{f}(0)$, and $\underline{G}_2(\underline{f}) = \underline{f}(1)$. (Note that if we let p=q=2 and drop $G_3(f)$ then we recover exactly the example from section 13.)

Given these definitions out semi-norm is given by

$$\|f\|^2 \equiv \int_a^b\int_c^d \left(D_x^3 D_y^2 f(x,y)\right)^2 dy\, dx + \int_a^b \left(D_x^3 f(x,0)\right)^2 dx + \int_a^b \left(D_x^3 f(x,1)\right)^2 dx$$

$$+ \int_c^d \left(D_y^2 f(0,y)\right)^2 dy + \int_c^d \left(D_y^2 f(1,y)\right)^2 dy + \int_c^d \left(D_y^2 (D_x f(x,y)|_{x=0})\right)^2 dy$$

$$f(0,0)^2 + f(0,1)^2 + f(1,0)^2 + f(1,1)^2 + \left(\frac{\partial f(x,y)}{\partial x}\Big|_{x=0,y=0}\right)^2 + \left(\frac{\partial f(x,y)}{\partial x}\Big|_{x=0,y=1}\right)^2$$

For this class we have $r_1(x) = 1 - x^2$, $r_2(x) = x^2$, $r_3(x) = x - x^2$, $r_1(y) = 1 - y$, and $r_2(y) = y$. Then by use of equations (14.5) - (14.7) we obtain the kernel $K(x,y;\, s,t) = K_1(x,s) \cdot K_2(y,t)$ where $K_1(x,s)$ and $K_2(y,t)$ are given by:

$$K_1(x,s) = (1 - x^2)\cdot(1-s^2) + x^2 \cdot s^2 + (x - x^2)\cdot(s - s^2) + \frac{1}{5!} \cdot \Big((x - s)_+^5 - (x)_+^5 \cdot (1 - s^2)$$

$$(x - 1)_+^5 \cdot s^2 + 5\cdot(x)_+^4 \cdot (s - s^2) - (1 - s)_+^5 \cdot x^2 - (1-s^2)\cdot(1-x^2) + 5\cdot(s - s^2) - 5\cdot(s - s^2)\cdot x^2 \Big)$$

and

$$K_2(y\,,t) = (1 - y)\cdot(1 - t) + y\cdot t + \frac{1}{3!} \cdot \Big((y - t)_+^3 - (y)_+^3 \cdot (1 - t) - (1 - t)_+^3 \cdot y + (1 - t) \cdot y \Big)$$

Mansfield notes that the optimal interpolation with respect to the semi-norm given by 14.4 "will not usually be exact for low degree polynomial" [Mansfield 71, p119]. Since there is some belief that this may be a good property of an interpolation algorithm, she develops a new semi-norm, and associated reproducing kernel that will be exact for low degree polynomials. We present a simplification of her treatment here, for more details see [Mansfield 71]. Some vision researchers may not consider the property of exactness for low degree polynomials to be useful, yet in the work of others (e.g. [Marr 82], [Ullman 76]) there are assumptions that surfaces (or their boundaries) can be represented as conics (or circular arcs). The role of this property in a vision system for interpolation has not been shown; we present the following in the interest of completeness.

Let $F_1, \ldots, F_n$ be bounded linear functionals (in the usual mathematical sense), which are linearly independent over $\tilde{N} \equiv \Pi_{p-1}(x) \otimes \Pi_{q-1}(y)$ and of the form

$$Lf = \sum_{i=1}^{p-1} \sum_{j=1}^{q} \int_a^b \int_c^d D_x^i D_y^j \; f(x,y) \; d\omega_{i,j}(x,y),$$

where $\omega_{i,j}(x,y)$ is of bounded variation (in the mathematical sense). Also let $F_1, \ldots, F_n$ be such that there exists constants A_i i = 1..n such that $f = \sum_{i<n} A_i \cdot F_i(f)$ for all $f \in \tilde{N}$. (Note [Mansfield 71] considers the case when the F_i's are not linearly independent.) Then $\| f \| = [f,f] + \sum_{i<n} (F_i f)^2$ is a semi-norm on $R_c{}^{p,q}$ and the reproducing function is given by

$$K(x,y;s,t) = (I-P)_x (I-P)_q K^*(x,y;s,t) \quad + \sum_{i=1}^{n} \eta_i(x,s) \cdot \eta_i(y,t)$$

where $K^*(x,y;s,t)$ is defined as above, $P(f)$ is the projection from $R_c{}^{p,q}$ onto N defined by

$$Pf(x,y) = \sum_{i=1}^{n} F_i(f(x,y)) \cdot \eta(x,y)$$

and where $\eta_i(\zeta,\xi)$, $i = 1..n$ are elements of $\Pi_{p-1}(\zeta) \otimes \Pi_{q-1}(\xi)$ such that

$$F_i(\eta_j) = \delta_{ij},\ 1 \le i,j \le n.$$

These definitions are simply mathematically in nature. It has yet to be shown if these classes are useful for interpolation (especially for computer vision). The nature of these classes however, is complex and the correct choice of the parameters (degree of continuity p and q, as well as the functionals G_i or F_i) is not obvious.

§15 Mansfield's class $T^{p,q}[\alpha,\beta]$ on $[a,b] \times [c,d]$.

In [Mansfield 72] Mansfield derives the reproducing kernel for a class that is slightly smoother than $R_c{}^{p,q}$. This class, $T^{p,q}[\alpha,\beta]$, has properties similar to the space of functions $B_{p,q}(\alpha,\beta)$ discussed by Sard [Sard 63, Chapter 4]. It allows the incorporation of prior knowledge (along the lines $x = \alpha$ and $y = \beta$) into the definition of the norm to minimize. Again we consider this space because it is sufficiently smooth (for $p,q \ge 2$) and may have an efficient implementation.
Let $\Omega = [a,b] \times [c,d]$ be our domain of interest. Then letting $m = p+q$, we define $T^{p,q}[\alpha,\beta]$, as the set of functions $f : \Re^2 \to \Re$ such that:

$D_x{}^i D_y{}^j f(x,y)$ is contuious on Ω, for $i = 0..p-1$, $j = 0..q-1$,
$D_x{}^{m-j-1} D_y{}^j f(\cdot,\beta)$ is absoutly continuous on Ω, for $j = 1..q-1$,
$D_x{}^{m-j} D_y{}^j f(\cdot,\beta) \in L^2[a,b]$ for $j = 1..q-1$,
$D_y{}^{m-i-1} D_x{}^i f(\alpha,\cdot)$ is absoutly continuous on Ω, for $i = 1..p-1$,
and $D_y{}^{m-i} D_x{}^i f(\alpha,\cdot) \in L^2[c,d]$ for $i = 1..p-1$.

Note that by this definition, the partial derivatives $D_x{}^{p+i}D_y{}^j f(\cdot,\beta)$, $i = 0..q-j$, $j = 1..q-1$, need only exist with respect to y along the line $y = \beta$, hence all partials with respect to y must be taken before any partial of order $\geq p$ with respect to x. Similarly for $D_y{}^{q+j}D_x{}^i f(\alpha,\cdot)$, $j = 1..p-i$, $i = 1..p-1$ we take all partials with respect to x before any partials with respect to y of order $\geq q$. Mansfield builds this class as the direct sum of a number of complete classes, and hence it is complete. By equipping it with the semi-norm given by:

$$\|f\| = \int_a^b \int_c^d \left(D_x^p D_y^q f(x,y)\right)^2 dy\, dx + \sum_{j=1}^{q-1} \int_a^b \left(D_x^{m-1} D_y^j f(x,\beta)\right)^2 dx$$

(15.1)

$$+ \sum_{i=1}^{p-1} \int_c^d \left(D_y^{m-i} D_x^i f(\alpha,y)\right)^2 dy + \sum_{i+j<m} \left(D_x^i D_y^j f(\alpha,\beta)\right)^2$$

$T^{p,q}[\alpha,\beta]$ becomes a Hilbert space. The null space of this semi-norm has finite dimension $m\cdot(m+1)/2$, and is given by

$$\left\{ f:\ f(x,y) = \sum_{i+j<m} \gamma_{i,j} \cdot x^i \cdot y^j;\ \text{for some constants } \gamma_{i,j} \right\}$$

Given the semi-norm (15.1), the reproducing kernel for the space $T^{p,q}[\alpha,\beta]$ is given by:

$$K_m(x,y;s,t) = \psi^{2p-1}(x,s) \cdot \underline{\psi}^{2q-1}(y,t) + \sum_{i=0}^{m\cdot\frac{m+1}{2}} \eta_i(x,s) \cdot \eta_i(y,t)$$

(15.2)

$$+ \sum_{j=0}^{q-1} \frac{(t-\beta)^j}{j!} \cdot \frac{(y-\beta)^j}{j!} \psi^{2(m-j)-1}(x,s) + \sum_{i=0}^{p-1} \frac{(s-\alpha)^i}{i!} \cdot \frac{(x-\alpha)^i}{i!} \underline{\psi}^{2(m-i)-1}(y,t),$$

Though these last few classes may seem formidable, when restricted to particular examples, they result in polynomial (and therefore relatively efficient) algorithms. Of course it is important to verify the usefulness of the resulting interpolation algorithms. Because of their rather complex definitions it is difficult to infer from these definitions any implications on human vision. Given that the methods have yet to prove there applicability we defer in any attempts to make inferences.

§16 Conclusions

In the first part of this paper we attempted to better understand what is generally implied (in computer and human vision research) by the term smooth. We found that in general smooth objects were assumed to have piecewise C^2 boundaries with an upper bound on the magnitude of the second derivative, smooth visual motion was assumed to be was assumed at least piecewise C^1, and intensity images are assumed to be C^2. Another conclusion about the term smooth which can be infered from the works surveyed in this paper is that increasing the number of continuous derivatives does not corrisponding to increasing the smoothness. In fact, no researcher even slightly implied this property of the term smooth. This should be contrasted with work in approximation theory where the number of continuous derivatives of a class of functions is often refered to as the degree of smoothness (we also used this definition throughout this survey) and increasing the number of continuous derivatives corresponds to increasing the "smoothness".

All of these conclusions are, however, not based on direct experimental research, but rather on the assumptions implicit or explicit in the mathematics of the researchers. Therefore the conclusions may be more a reflection of the researchers mathematical bias than of the true meaning of smooth or of the smoothness possessed by objects, motion or images.

We then examined some of the terminology, and the results of application of information-based complexity. Application of this theory (or any theory which would yield a "optimal" interpolation algorithm) force us to make explicit the global assumptions on the smoothness of object, requiring us to choose a class of functions in which the surfaces exist, and to choose a norm (or semi-norm) on that class. However biased the conclusions of the first part of this paper, they serve as a useful starting place for choosing a class in in which to do optimal visual surface interpolation, suggesting that we should consider classes which are at least C^2. Given this starting point we examined a number of possible classes of functions giving the information necessary to implement the optimal error algorithm for that class.

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