The Design of Monte Carlo Experiments for VAR Models

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Abstract

This paper deals with the design of Monte Carlo experiments in the context of cointegrated VAR models. Such experiments often seek to establish the applicability of asymptotic distributional results for samples of size 100 to 200, which are typical of macro economic time series. Hithertofore, the design of such experiments has relied on certain simple models given in Bannerjee *et al.* (1986), Engle and Granger (1987), and Phillips (1991). Here we provide the framework for designing experiments based on much more general models, of which the designs above are special cases.

Key Words: Monte Carlo; nonstationary processes; stationary processes; mixing processes; companion matrix; stable roots; unit roots; cointegration.

1 Introduction

The literature on the detection of cointegration in nonstationary, $I(1)$, processes is rich with general results on the asymptotic properties of various test statistics or estimators in a great variety of models. In particular, see Dickey and Fuller (1978), (1981), Phillips and Durlauf (1986), Stock and Watson (1988), Phillips and Ouliaris (1990), Phillips and Loretan (1991), Phillips and Perron (1991), to mention but a few. There are also a number of Monte Carlo studies, which are either illustrative of a particular theoretically investigated procedure, as for example Stock and Watson (1988), Engle and Granger (1987), or aim at comparing the performance of various estimators or test procedures, see for example Haug (1996), Toda (1995), Gonzalo (1994), Hooker (1993), Gardeazabal and Regulez (GR) (1992), and several others. In the latter genre one typically deals, see e.g. GR, with models of the form

$$y_t - x_t b = z_t, \quad z_t = z_{t-1}\rho + \epsilon_{t1}$$

$$y_t a - x_t c = w_t, \quad w_t = \gamma w_{t-1} + \epsilon_{t2}, \tag{1}$$

preserving for the moment the original notation of the authors. Similarly, Phillips (1991), in a theoretical context, and Toda (1995), in a Monte Carlo context, deal with the model

$$y_{1t} = By_{2t} + u_{1t}, \qquad \Delta y_{2t} = u_{2t}, \quad \text{where} \quad u_t = (u_{1t}^{'}, u_{2t}^{'})^{'} \tag{2}$$

is a **stationary, or a suitably specified mixing process**. As portrayals of a VAR model, the virtue of the first representation is that it immediately discloses the nature of the process, through the magnitude of the parameters ρ, γ. Its disadvantage is that this is possible **only** in the case of a single lag. The virtue of the second representation is its extreme simplicity; its disadvantage is that it contains **only** unit roots, and all stationary roots are **null**. Both of these features will disappear if we introduce more lags. In fact, this system may be thought of, somewhat informally, as a singular system, an interpretation we shall give below.

2 A More General Formulation

Consider the general $VAR(n)$ model

$$X_{t\cdot}\Pi(L) = \epsilon_{t\cdot}, \quad t \geq 1, \text{ and } \epsilon_{s\cdot} = 0, \text{ for } s \geq 0, \tag{3}$$

where $X_{t\cdot}$ is q-element (row) vector denoting a cointegrated $I(1)$ process, and $\{\epsilon_{t\cdot} : t \geq 1\}$ is, similarly, a q-element row vector denoting either an i.i.d. sequence with mean zero and covariance matrix $\Sigma > 0$, or a (strictly) stationary process, or a suitably specified mixing process. In addition,

$$\Pi(L) = \sum_{j=0}^{n} \Pi_j L^j, \quad \Pi_0 = I_q, \quad L^0 = I, \tag{4}$$

where I, without a dimension denoting subscript is always the identity **operator**, and L is the usual lag operator.

The object of the Monte Carlo design is to generate a $VAR(n)$ with prespecified roots; for example, we may wish to investigate a system that has three **unit roots** and relatively small stationary roots, say not exceeding .5 in absolute value. Or, we may wish to examine a system that has five unit roots, and stationary roots that range between .75 and .98. If $n > 1$, it is not simple to construct such systems by the two devices above, nor is it feasible to do so by trial and error.

We begin by noting that the model in Eq. (3) may be rewritten as a first order difference equation with forcing function $g_{t\cdot} = e_{1\cdot} \otimes \epsilon_{t\cdot}$, where $e_{1\cdot}$ is a q-element **row** vector all of whose elements are zero except the first, which is unity. The system in question is

$$\zeta_{t\cdot} = \zeta_{t-1\cdot}C + g_{t\cdot}, \quad C = \begin{bmatrix} -\Pi_1 & I_q & 0 & 0 & \cdots & 0 \\ -\Pi_2 & 0 & I_q & & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \\ -\Pi_{n-1} & 0 & 0 & 0 & \cdots & I_q \\ -\Pi_n & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{5}$$

where $\zeta_{t\cdot} = (y_{t\cdot}, y_{t-1\cdot}, y_{t-2\cdot}, \cdots, y_{t-n+1\cdot})$. The matrix C is generally referred to as the **companion matrix**. It is shown in Dhrymes (1984), pp. 133-139, that the (ordered) characteristic roots of the companion matrix, say λ_i, are related to the characteristic roots of the system $|\Pi(z)| = 0$, say z_i, by $\lambda_i = z_i^{-1}$. In the literature of this topic we consider **exclusively** models that have **only real positive unit roots**, and stationary roots that are less than unity in absolute value, i.e. we consider a system with the properties

$$\lambda_i = 1, \quad i = 1, 2, 3, \ldots, r_0, \quad |\lambda_i| < 1, \quad \text{for } i > r_0; \tag{6}$$

thus, if we denote the roots of $|\Pi(z)| = 0$ by z_i, we have

$$z_i = 1, \quad i = 1, 2, 3, \ldots, r_0, \quad |z_i| > 1, \quad \text{for } i > r_0. \tag{7}$$

Note that the number of roots in question is nq, so that it grows as the product of the number of lags and the dimension of the system. Thus a **five** variable system with **four** lags necessitates the specification of **twenty** roots, which is not a trivial matter.

In view of the preceding we may write

$$\prod_{i=1}^{nq}(I - z_i L) = |I_q + \sum_{j=1}^{n} \Pi_j L^j|. \tag{8}$$

Given the specification of the roots, z_i, the left member above yields

$$\prod_{i=1}^{nq}(I - z_i L) = \sum_{i=0}^{nq} a_i L^i, \tag{9}$$

where

$$a_0 = 1, \quad a_i = \sum (z_{i_1} z_{i_2} \cdots z_{i_{i-1}} z_{i_i}), \quad i > 1, \tag{10}$$

i.e. the coefficient a_i is the sum over all products of the roots of the system in Eq. (8) taken i at a time.

If the matrices Π_j obey

$$\Pi_j = A T_j A^{-1} \tag{11}$$

where A is an arbitrary nonsingular matrix of order q, and T_j, $j = 1, 2, 3, \ldots, n$ are **lower triangular** matrices of order q, the right member of Eq. (8) yields

$$\pi(L) = |I_q + \sum_{j=1}^{n} \Pi_j L^j| = |I_q + \sum_{j=1}^{n} T_j L^j| = \sum_{k=0}^{nq} b_k L^k. \tag{12}$$

The determinant in the third member above depends only on the diagonal elements, and it is their product. The diagonal elements are given by

$$d_{ii} = 1 + \sum_{j=1}^{n} t_{ii}^{(j)} L^j, \quad i = 1, 2, 3, \ldots, q. \tag{13}$$

Consequently,

$$\begin{aligned}\pi(L) \;=\; & 1 + \sum_{j=1}^{n} (\mathrm{tr} T_j)\, L^j + \sum_{j=2}^{2n} \left(\sum_{\binom{q}{2}} a^{(j)}_{(2;r_1,r_2)} \right) L^j + \sum_{j=3}^{3n} \left(\sum_{\binom{q}{3}} a^{(j)}_{(3;r_1,r_2,r_3)} \right) L^j \\ & + \sum_{j=4}^{4n} \left(\sum_{\binom{q}{4}} a^{(j)}_{(3;r_1,r_2,r_3,r_4)} \right) L^j + \cdots + \sum_{j=q}^{nq} \left(\sum_{\binom{q}{q}} a^{(j)}_{(q;1,2,3,\ldots,q)} \right) L^j. \end{aligned} \tag{14}$$

In the preceding, the notation $\sum_{\binom{q}{s}}$ indicates that the summation is over the $\binom{q}{s}$ terms resulting by taking s out of q (diagonal) elements[1] from the matrices T_k, $k = 1, 2, \ldots, n$, such that the sum of the superscripts is j. To be more precise, let $t_{ii}^{(k)}$ be the i^{th} diagonal element of T_k, and consider the product $t_{i_1 i_1}^{(k_1)} t_{i_2 i_2}^{(k_2)} \cdots t_{i_s i_s}^{(k_s)}$ such that the sum of the superscripts is zero; fixing the superscripts, the elements in this product may be chosen in $\binom{q}{s}$, hence the notation $\sum_{\binom{q}{q}}$. The latter, however, is somewhat misleading

[1] Note that this notation means that we may take more than one diagonal element from a **given** matrix T_k, but cannot take a diagonal element in the **same position** from more than one matrix T_k.

since there may be, and typically are, several ways in which the diagonal elements of the matrices T_k are chosen so that the sum of the superscripts is[2] j. Similarly, the notation $a^{(j)}_{(m;r_1,r_2,\ldots,r_m)}$ denotes the sum of all possible terms of the form

$$t^{(j-s_1-s_2-\ldots-s_{m-1})}_{r_1r_1}\prod_{i=2}^{m} t^{(s_i-1)}_{r_ir_i}, \quad \text{i.e.} \quad \sum_{s_1}\sum_{s_2}\cdots\sum_{s_{m-1}} t^{(j-s_1-s_2-\ldots-s_{m-1})}_{r_1r_1}\prod_{i=2}^{m} t^{(s_i-1)}_{r_ir_i},$$

such that the superscripts add to j.

Comparing Eqs. (12) and (14) we determine

$$b_0 = 1, \quad b_1 = \mathrm{tr}T_1,$$

$$b_k = \mathrm{tr}T_k + \sum_{s=2}^{q}\left(\sum_{\binom{q}{s}} a^{(k)}_{(s;r_1,r_2,\cdots,r_s)}\right), \quad \text{for } 2 < k \le n,$$

$$b_k = \sum_{s=2}^{q}\left(\sum_{\binom{q}{s}} a^{(k)}_{(s;r_1,r_2,\cdots,r_s)}\right), \quad \text{for } n < k \le 2n;$$

$$b_k = \sum_{s=3}^{q}\left(\sum_{\binom{q}{s}} a^{(k)}_{(s;r_1,r_2,\cdots,r_s)}\right), \quad \text{for } 2n < k \le 3n, \quad \text{and generally}$$

$$b_k = \sum_{s=m}^{q}\left(\sum_{\binom{q}{s}} a^{(k)}_{(s;r_1,r_2,\cdots,r_s)}\right), \text{ for } (m-1)n < k \le mn, \ m = 4,5,\ldots,q-1;$$

$$b_k = \sum_{\binom{q}{q}} a^{(k)}_{(q;1,2,\ldots,q)}, \quad \text{for } (q-1)n < k \le nq. \tag{15}$$

Eqs. (10) and (15) imply

$$b_k = a_k, \quad k = 1,2,3,\cdots,nq, \tag{16}$$

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[2] To give an example, suppose $n = 12$ and we wish to compute $\sum_{\binom{q}{3}} a^{(12)}_{(3;r_1,r_2,r_3)}$. To do so, we first choose three matrices T_{j_s}, $s = 1,2,3$, such that $j_1 + j_2 + j_3 = j = 12$. This may be done in severl ways, say T_1, T_1, T_{10}; or, T_1, T_2, T_9; or, T_2, T_2, T_8, etc. For **each such choice** there are $\binom{q}{3}$ ways of choosing the diagonal positions to be employed, e.g. in the first case $t^{(1)}_{11}, t^{(1)}_{22}, t^{(10)}_{33}$ etc.

which is thus a system[3] of nq **nonlinear equations in the unknowns** $t_{ii}^{(j)}$, $i = 1,2,3,\ldots,q$, $j = 1,2,3,\ldots,n$.

The prodecure above obtains the diagonal elements of the matrices T_j; in view of Eq. (9), and the fact that the lower diagonal elements of the T_j, as well as the matrix A, are **arbitrary**, the Π_j are thus completely determined.

Because numerical accuracy is extremely important in $I(1)$ systems, and particularly so for large samples, it is desirable to verify the numerical accuracy of the nonlinear solution by comparing the inverse of the characteristic roots of the companion matrix, which is solely determined by the Π_j, with the prespecified characteristic roots of the system, z_i, $i = 1,2,3,\cdots,nq$.

Finally, one may use the matrices Π_j, determined by the procedure above, in conjunction with the specification of the error process to compute, recursively,

$$X_{t\cdot} = -\sum_{i=1}^{n} X_{t-i\cdot}\Pi_i + \epsilon_{t\cdot}, \quad t = 1,2,3,\cdots,T, \tag{17}$$

using the initial conditions $X_{-i\cdot}$, $i \geq 0$. Noting that the first n observations do not fully embody the dynamics of the VAR, one then uses the **abbreviated sample**,

$$X_{t\cdot} = -\sum_{i=1}^{n} X_{t-i\cdot}\Pi_i + \epsilon_{t\cdot}, \quad t = n+1, n+2, n+3, \cdots, T,$$

on the basis of which the Monte Carlo experiments are to be carried out.

3 Special Cases

In this section we shall show that the representations in Eqs. (1) and (2) are special cases of the representation in Eqs. (3) and (9). To this end, define $X_{t\cdot} = (y_t, x_t)$, $D_1 = \mathrm{diag}(-\rho, -\gamma)$, $\epsilon_{t\cdot}^* = (\epsilon_{t1}, \epsilon_{t2})$, and note that the model in Eq. (1) is simply

$$X_{t\cdot}A = \epsilon_{t\cdot}^*(I + D_1 L)^{-1}, \quad A = \begin{bmatrix} 1 & a \\ -b & -c \end{bmatrix}. \tag{18}$$

Reducing the operator on the right, and multiplying on the right by A^{-1}, yields

$$X_{t\cdot} + X_{t-1\cdot}AD_1A^{-1} = \epsilon_{t\cdot}^*A^{-1}. \tag{19}$$

[3] A code in GAUSS, for solving such a system is available, on request, from the author.

Comparing with Eq. (3), we see that $\Pi_1 = AD_1A^{-1}$, $\Pi_i = 0, i \geq 2$, and $\epsilon_{t\cdot} = \epsilon^*_{t\cdot}A^{-1}$, which is indeed a special case. Notice, further, that introducing more lags, i.e. another diagonal matrix, say D_2, destroys the simplicity of the model and leads only to a slightly less complex case than that considered in Eqs. (3) and (9). Finally, the characteristic equation of this model is

$$\pi_0(z) = |I_2 + AD_1A^{-1}z| = (1-\rho z)(1-\gamma z) = 0. \tag{20}$$

Thus, specifying $\rho = 1$ and $|\gamma| < 1$ creates a bivariate $I(1)$ process, which is cointegrated of rank one.

Next, consider the model in Eq. (2), define $X_{t\cdot} = (y'_{1t}, y'_{2t})$, $u_{t\cdot} = (u'_{1t}, u'_{2t})$, and multiply on the right by the matrix

$$F = \begin{bmatrix} I_r & 0 \\ B' & I_{r_0} \end{bmatrix},$$

where it is assumed that y_{2t} is an r_0 element vector and, thus, y_{1t} has $r = q - r_0$ elements. This operation yields

$$X_{t\cdot} + X_{t-1\cdot}P = u_{t\cdot}F, \quad P = \begin{bmatrix} 0 & 0 \\ -B' & -I_{r_0} \end{bmatrix}, \tag{21}$$

which is, evidently, a special case of Eqs. (3) and (9), with

$$\Pi_1 = P, \quad A = I_q, \text{ or } \Pi_1 = T_1, \quad \Pi_i = 0, \quad i \geq 2, \quad \epsilon_{t\cdot} = u_{t\cdot}F. \tag{22}$$

To show that this is a sort of a singular system, introduce a lag in the first equation by means of a diagonal matrix D; if the lag in question is $y_{1,t-1}$ the matrix P becomes

$$P = \begin{bmatrix} -D & 0 \\ -B' & -I_{r_0} \end{bmatrix},$$

which preserves the general character of the system. The characteristic equation is

$$\pi_1(z) = |I_q + Pz| = (1-z)^{r_0} \prod_{i=1}^{r}(1 - d_{ii}z), \tag{23}$$

which has r_0 unit roots, and r roots $z_i = d_{ii}^{-1}$, for $i > r_0$. Choosing $|d_{ii}| < 1$, we have that the system has r_0 **unit** roots and $r = q - r_0$ **stationary** roots. Since $\pi_1(z)$ is continuous in the parameters d_{ii}, the roots of the system become **unbounded** as $|d_{ii}| \to 0$. The companion matrix is $-P$; thus, looking at the same issue from the point of view of the **companion matrix,** we note that its characteristic roots are given as the solution to

$$0 = \left| \lambda I_q - \begin{bmatrix} D & 0 \\ B' & I_{r_0} \end{bmatrix} \right| = (\lambda - 1)^{r_0} \prod_{i=1}^{r} (\lambda - d_{ii}), \tag{24}$$

which is also continuous in the parameters d_{ii}. Thus, if $|d_{ii}| < 1$ we have an $I(1)$ system, which is cointegrated of rank $r = q - r_0$. By contrast with the previous discussion, in the companion matrix context there is no peculiarity as $|d_{ii}| \to 0$. The system remains well defined, and remains transparently cointegrated of rank $r = q - r_0$. The stationary roots are **all null,** and the dynamics of the transients of the system are eliminated. Thus, such a system is more suitable for studying the asymptotic properties of the system, and is not particularly appropriate for studying the suitability of asymptotic distributional results for the typically small samples encountered in applications.

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