Tractability of the Fredholm problem of the second kind

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Abstract

We study the tractability of computing ε-approximations of the Fredholm problem of the second kind: given $f \in F_d$ and $q \in Q_{2d}$, find $u \in L_2(I^d)$ satisfying

$$u(x) - \int_{I^d} q(x,y)u(y)\,dy = f(x) \qquad \forall\, x \in I^d = [0,1]^d.$$

Here, F_d and Q_{2d} are spaces of d-variate right hand functions and $2d$-variate kernels that are continuously embedded in $L_2(I^d)$ and $L_2(I^{2d})$, respectively. We consider the worst case setting, measuring the approximation error for the solution u in the $L_2(I^d)$-sense. We say that a problem is tractable if the minimal number of information operations of f and q needed to obtain an ε-approximation is sub-exponential in ε^{-1} and d. One information operation corresponds to the evaluation of one linear functional or one function value. The lack of sub-exponential behavior may be defined in various ways, and so we have various kinds of tractability. In particular, the problem is strongly polynomially tractable if the minimal number of information operations is bounded by a polynomial in ε^{-1} for all d.

We show that tractability (of any kind whatsoever) for the Fredholm problem is equivalent to tractability of the L_2-approximation problems over the spaces of right-hand sides and kernel functions. So (for example) if both these approximation problems are strongly polynomially tractable, so is the Fredholm problem. In general, the upper bound provided by this proof is essentially non-constructive, since it involves an interpolatory algorithm that exactly solves the Fredholm problem (albeit for finite-rank approximations of f and q). However, if linear functionals are permissible and that F_d and Q_{2d} are tensor

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product spaces, we are able to surmount this obstacle; that is, we provide a fully-constructive algorithm that provides an approximation with nearly-optimal cost, i.e., one whose cost is within a factor $\ln \varepsilon^{-1}$ of being optimal.

1 Introduction

The *Fredholm problem of the second kind* consists of finding a d-variate function u such that

$$u(x) - \int_{I^d} q(x, y)u(y)\,dy = f(x) \qquad \forall\, x \in I^d = [0, 1]^d. \tag{1}$$

Here, $f \in F_d$ and $q \in Q_{2d}$, where F_d and Q_{2d} are given classes of functions that are respectively defined over I^d and I^{2d}. We want to determine the *complexity* of computing the solution of (1) to within ε in the worst case setting. This means that we want to find an *algorithm* that solves this problem with *minimal* cost. Here, we measure *cost* by a weighted sum of the total number of function values or linear functionals of the specific right hand function and the kernel, and the total number of arithmetic operations.

The first paper on the complexity of the Fredholm problem of the second kind was published by Emelyanov and Ilin [3] already in 1967. The problem was to approximate the solution with right hand functions and kernels being r-times continuously differentiable. Their result was that the minimal worst case error of algorithms that use at most n function values is proportional to $n^{-r/(2d)}$. This means that the complexity of computing an ε-approximation is proportional to $\varepsilon^{-2d/r}$, with the proportionality factor depending on r and d. After a quarter-century hiatus, researchers in information-based complexity began looking once again at the complexity of this problem. A partial list of results includes Dick, Kritzer, Kuo, and Sloan [2], Frank, Heinrich, and Pereverzev [4], Heinrich [6], Heinrich and Mathé [7], Pereverzev [14], and [11, 17, 18, 19]). The results were also obtained for the solution at a point as well as for global solution and for various Sobolev spaces in the worst case and randomized settings.

The papers [4, 14, 18, 19] treated the worst case setting for Sobolev spaces, see also [17]. They found the complexity to be proportional to $(1/\varepsilon)^{d\,\alpha}$, with a positive α dependent on the smoothness parameters of the spaces but independent of d. Again, the proportionality factors depend on d and the smoothness parameters. Typically, it is *not* known if the dependence on d is exponential or maybe "only" polynomial.

These results are fine when d is so small that computing exponentially-many (in d) information or arithmetic operations doesn't faze us; so for many problems in science and engineering, in which we have $d \le 3$, these results are computationally relevant. But what happens when d is so large that we can no longer afford to calculate (say) 2^d function values or linear functionals or arithmetic operations? When this happens, we are stymied by the exponential (in d) behavior of the ε-complexity for the d-dimensional problem, which Bellman [1] called the "curse of dimensionality." In fact, there are many multivariate problems for which the curse of dimensionality is indeed present. Since we are dealing with complexity (minimal cost), there's no way that we can find a cleverer algorithm for the problem. If we really want to solve the problem, we have two choices:

1. We can weaken the assurance given by the worst case setting, typical choices being the average case, probabilistic, or randomized settings.

2. We can stay with the worst case setting, but reformulate the problem using different spaces for F_d and Q_{2d}.

The papers by Heinrich [6] and by Heinrich and Mathé [7] pursued the first choice, using the randomized setting. For the second choice, we usually[1] shrink the original spaces F_d and Q_{2d} by introducing "weights" that measured the importance of successive variables and groups of variables. Dick, Kritzer, Kuo, and Sloan [2] pursued this latter path, a choice we also follow in this paper.

Vanquishing the curse of dimensionality for multivariate problems forms the heart of research into *tractability* studies. A problem is tractable if the *information complexity* is sub-exponential in ε^{-1} and d. Information complexity is defined as the minimal number of information operations needed to compute an ε-approximation, with one information operation being understood as the evaluation of one function value or one linear functional. If we specify a particular non-exponential behavior, we get a specific kind of tractability. For example, *polynomial* tractability means that there exist non-negative C, p and q such that the information complexity is bounded by $C\varepsilon^{-p}d^q$ for all $\varepsilon \in (0,1)$ and all $d = 1, 2, \ldots$. If $q = 0$, then we have *strong polynomial* tractability. This is an especially challenging property since then the information complexity has a bound independent of d. It is good to know that strong polynomial tractability holds for many multivariate problems with properly decaying weights.

Obviously, the information complexity is a lower bound on the (total) complexity. Therefore, the complexity is sub-exponential in ε^{-1} and d only if the problem is tractable. If the complexity is more or less the same as the information complexity then the study of complexity and tractability coincide. The last assumption means that the total number of arithmetic operations needed to compute an ε-approximation is almost the same as the number of information operations. Interestingly enough, the last assumption holds for most *linear* problems and selected *nonlinear* problems. The current state of the art of tractability studies may be found in [9, 11, 12].

Since the Fredholm problem is *not* linear, it is not clear a priori whether its total complexity is essentially the same as its information complexity. Dick, Kritzer, Kuo, and Sloan [2] showed that these were essentially (i.e., to within a logarithmic factor) equal for the problem that they studied; we show that this is also the case for the problem studied in this paper, provided that linear functionals are permissible and that F_d and $Q_{@d}$ are tensor product spaces.

Dick, Kritzer, Kuo, and Sloan [2] were the first to address the tractability of the Fredholm problem of the second kind. They considered d-variate right hand functions and d-variate convolution kernels from the same space, a weighted Korobov space with *product weights*. They obtained a result that is within a logarithmic factor of being optimal, and proved strong polynomial and polynomial tractability under natural assumptions on the decay of product weights. The algorithm for which this holds is the lattice-Nyström method, which uses function values; the resulting $n \times n$ linear system has a special structure, allowing it to be solved in $\mathscr{O}(n \ln n)$ arithmetic operations. Tractability of the Fredholm problem of the second kind is also addressed in [11, Sect. 18.2].

In this paper, we study the Fredholm problem for kernel functions that may fully depend on all $2d$ variables. Moreover, we allow the spaces F_d and Q_{2d} to be independent of each other, up to the final section of this paper, in which we will need to impose some relations between these two spaces by assuming that they are certain tensor product spaces. That is, F_d is the d-fold and Q_{2d} is the $2d$-fold product space of some spaces of univariate functions.

The Fredholm problem is similar to the quasi-linear problems studied in [20, 21]. The main difference is that the function spaces defining the linear and nonlinear parts of the problems studied in [20, 21] are both defined over I^d, whereas for the Fredholm problem these spaces are respectively defined over I^d and I^{2d}, and in general are not related. Moreover, the papers [20, 21] only provided upper bounds on the complexity, and here we provide both upper and lower bounds.

[1] But not always, see [13].

We present two results in this paper. The first result exhibits relationships between the tractability of the Fredholm problem and the tractability of approximating the right-hand side and kernel function appearing in this Fredholm problem. Suppose that $F = \{F_d\}_{d=1,2...}$ and $Q = \{Q_d\}_{d=1,2,...}$ are families of right-hand sides and kernel functions for this problem. Under certain mild conditions on F and Q, we show that

$$\text{tract}_{\text{FRED}} \equiv \text{tract}_{\text{APP}_F} \wedge \text{tract}_{\text{APP}_Q} . \tag{2}$$

That is, tractability of the Fredholm problem is equivalent to tractability of the approximation problem for F and Q. We stress that this holds for all kinds of tractability. This result is useful since the tractability of approximation has been studied for many spaces and much is known about this problem, see again [9, 11, 12]. Due to the equivalence, all these results can be also applied for the Fredholm problem.

The lower tractability bounds for the Fredholm problem are obtained by taking first a special f or q and then showing that the Fredholm problem is equivalent to the approximation problem for functions q or f, respectively. We get the results in this paper by choosing the special functions $f = 1$ and $q = 0$.

The upper tractability bounds for the Fredholm problem are obtained by using an interpolatory algorithm that gives the exact solution of the Fredholm problem (1) with f and q replaced by their approximations. In general, this kind of algorithm will be impossible to implement. It does not matter for negative tractability results since, as we already mentioned, the total complexity is lower bounded by the information complexity. On the other hand, positive tractability results are in question since it may theoretically happen that although the information complexity is reasonable but the implementation cost may be too large.

So for our second result, we address the problem of how to actually implement a good algorithm for the Fredholm problem. Suppose that linear functionals can be used, and that F_d and Q_{2d} are tensor product function spaces. In this case, we develop a modified interpolatory algorithm whose total cost is roughly the same as the information complexity. More precisely, we exhibit a fixed-point iteration that produces an approximation having the same error as the interpolatory algorithm, with a penalty that is at worst a multiple of $\ln \varepsilon^{-1}$. This proves that the complexity and the information complexity are essentially the same for tensor-product spaces, as long as linear functionals can be used.

We briefly comment on the case when only function values can be used. Using the results that relate the power of function values and linear functionals, see [8, 16], it is possible to show that in many cases polynomial or strong polynomial tractability is preserved. However, the tractability and complexity exponents of ε^{-1} can be larger when function values are used. We omit the details of this study not to make our paper even longer.

We now give a brief overview of the paper. In Section 2, we define basic concepts, such as the problem to be solved and various kinds of tractability for the problem. In Section 3, we show relations between tractability of the Fredholm problem and tractability of the L_2-approximation problems over the spaces F_d and Q_d. In Section 4, we apply the results of Section 3. We first show that if either F_d or Q_d is a space of infinitely differentiable functions with the same role of all variables and groups of variables, then the Fredholm problem suffers from the curse of dimensionality. This means that even sufficiently high smoothness of functions does not imply tractability. Next, we look at the case where F_d and Q_{2d} are general unweighted tensor product spaces, finding both positive and negative tractability results. Then we examine the case of weighted Sobolev spaces, once again getting both positive and negative results. In Section 5, we define general weighted tensor product spaces. Finally, in Section 6 we suppose that continuous linear functionals are permissible and that the F_d and Q_{2d} are weighted tensor product spaces (as in Section 5). We then exhibit a modified interpolatory algorithm, studying its implementation cost, and showing that the total cost of this algorithm is nearly (i.e., to within a logarithmic factor) the same as the information complexity, so that this method is nearly optimal.

2 Basic concepts

Recall that $I = [0,1]$ is the unit interval[2], and that $d \in \mathbb{N} = \{1, 2, \dots\}$ is a positive integer. For $q \in L_2(I^{2d})$, let T_q be the compact Fredholm operator on $L_2(I^d)$ defined by

$$T_q v = \int_{I^d} q(\cdot, y) v(y)\, dy \qquad \forall\, v \in L_2(I^d).$$

We say that q is the *kernel* of T_q. Clearly,

$$\|T_q v\|_{L_2(I^d)} \le \|q\|_{L_2(I^{2d})} \|v\|_{L_2(I^d)} \qquad \forall\, q \in L_2(I^{2d}),\ v \in L_2(I^d).$$

Therefore

$$\|T_q\|_{\mathrm{Lin}[L_2(I^d)]} \le \|q\|_{L_2(I^{2d})} \qquad \forall\, q \in L_2(I^{2d}). \tag{3}$$

Moreover, if $\|q\|_{L_2(I^{2d})} < 1$ then the operator $I - T_q$ has a bounded inverse, with

$$\|(I - T_q)^{-1}\|_{\mathrm{Lin}[L_2(I^d)]} \le \frac{1}{1 - \|q\|_{L_2(I^{2d})}}. \tag{4}$$

Let F_d and Q_d be normed linear subspaces whose norms are denoted by $\|\cdot\|_{F_d}$ and $\|\cdot\|_{Q_d}$, respectively. We assume that F_d and Q_d are continuously embedded subspaces of $L_2(I^d)$ for all $d \in \mathbb{N}$. Without essential loss of generality, we also assume that

$$\|\cdot\|_{L_2(I^d)} \le \|\cdot\|_{F_d} \qquad \text{and} \qquad \|\cdot\|_{L_2(I^d)} \le \|\cdot\|_{Q_d}. \tag{5}$$

Given $M_1 \in (0,1)$, let

$$Q_d^{\mathrm{res}} = \left\{ q \in Q_d : \|q\|_{Q_d} \le M_1 \right\} \qquad \forall\, d \in \mathbb{N}.$$

We define a solution operator $S_d \colon F_d \times Q_{2d}^{\mathrm{res}} \to L_2(I^d)$ as

$$u = S_d(f, q) \qquad \text{iff} \qquad (I - T_q)u = f \qquad \forall\, (f, q) \in F_d \times Q_{2d}^{\mathrm{res}}.$$

Note that

$$S_d(\cdot, q) = (I - T_q)^{-1} \in \mathrm{Lin}[L_2(I^d)] \qquad \forall\, q \in Q_{2d}^{\mathrm{res}}.$$

In particular, for $q = 0$, we have $T_q = 0$, so that

$$S_d(f, 0) = f \qquad \forall\, f \in F_d.$$

The operator S_d is linear in its first variable, but nonlinear in its second variable. Using (4) and (5), we have the a priori bound

$$\|S_d(f, q)\|_{L_2(I^d)} \le \frac{\|f\|_{L_2(I^d)}}{1 - M_1} \qquad \forall\, (f, q) \in F_d \times Q_{2d}^{\mathrm{res}}. \tag{6}$$

Let BF_d denote the unit ball of F_d. We want to approximate $S_d(f, q)$ for $(f, q) \in BF_d \times Q_{2d}^{\mathrm{res}}$, using algorithms whose *information* $N(f, q)$ about a right-hand side f and a kernel q consists of finitely many information operations from a class Λ_d of permissible functionals of f and from a class Λ_{2d} of permissible functionals of q. These functionals can be either of the following:

[2] In fact, one can take I as a measurable subset of $\mathbb{R}$ with a positive Lebesgue measure and define $L_2(I)$ with a weight ρ such that $\int_I \rho(t)\, dt = 1$. We take $I = [0,1]$ for simplicity.

- *Linear Class*. In this case, we are allowing the class of all *continuous linear functionals*. We write $\Lambda_d = \Lambda_d^{\text{all}}$ or $\Lambda_{2d} = \Lambda_{2d}^{\text{all}}$.
- *Standard Class*. In this case, we are allowing only function values and choose the spaces F_d and Q_d such that function values are continuous linear functionals. We write $\Lambda_d = \Lambda_d^{\text{std}}$ or $\Lambda_{2d} = \Lambda_{2d}^{\text{std}}$.

That is, for some nonnegative integers n_1 and n_2 we have

$$N(f,q) = \big[L_1(f), L_2(f), \dots, L_{n_1}(f), L_{n_1+1}(q), L_{n_1+2}(q), \dots, L_{n_1+n_2}(q)\big],$$

where $L_i \in \Lambda_d$ for $i = 1, 2, \dots, n_1$ and $L_i \in \Lambda_{2d}$ for $i = n_1+1, n_1+2, \dots, n_1+n_2$. The choice of the functionals L_i and the numbers n_i may be determined adaptively.

An *algorithm* $A\colon BF_d \times Q_{2d}^{\text{res}} \to L_2(I^d)$ approximating the Fredholm problem S_d has the form

$$A(f,q) = \phi\big(N(f,q)\big),$$

where $N(f,q)$ is the information about f and q and $\phi\colon N(BF_d \times Q_{2d}^{\text{res}}) \to L_2(I^d)$ is a *combinatory function* that combines this information and produces an approximation to the exact solution. For further discussion, see (e.g.) [15, Sect. 3.2].

The (worst case) *error* of an algorithm is given by

$$e(A, S_d) = \sup_{(f,q)\in BF_d \times Q_{2d}^{\text{res}}} \|S_d(f,q) - A(f,q)\|_{L_2(I^d)}.$$

Let

$$e(n, S_d, \Lambda_{d,2d}) = \inf_{A_n} e(A_n, S_d)$$

denote the nth *minimal worst case error* for solving the Fredholm problem. Here, the infimum is over all algorithms A_n using at most n information operations of right-hand sides from Λ_d and of kernel functions from Λ_{2d}, which we indicate by the shortcut notation $\Lambda_{d,2d}$. That is, if we use n_1 and n_2 information operations for f and q then $n_1 + n_2 \le n$.

Finally, for $\varepsilon \in (0,1)$ we let

$$n(\varepsilon, S_d, \Lambda_{d,2d}) = \inf\{\, n \in \mathbb{N} : e(n, S_d, \Lambda_{d,2d}) \le \varepsilon \,\}$$

denote the *information complexity*, i.e., the minimal number of information operations needed to obtain an *ε-approximation*, i.e., an approximation with error at most ε.

Remark. The (total) complexity of a problem is defined to be the minimal cost of computing an approximation. We will discuss the total complexity of the Fredholm problem later. □

Remark. In this paper, we will only deal with the *absolute error criterion*. One could also use the *normalized error criterion*, in which

$$n^{\text{nor}}(\varepsilon, S_d, \Lambda_{d,2d}) = \inf\{\, n \in \mathbb{N} : e(n, S_d, \Lambda_{d,2d}) \le \varepsilon \cdot e(0, S_d, \Lambda_{d,2d}) \,\},$$

where $e(0, S_d, \Lambda_{d,2d})$ is the *initial error*, i.e., the minimal error we can achieve without doing any information operations whatsoever. Under the normalized error criterion, we would be trying to determine the minimal number of information operations needed to reduce the initial error by a factor of ε. For simplicity, we restrict ourselves to the absolute error criterion in this paper. See [9, Sect. 4.4] for further discussion of error criteria. □

How hard is it to solve our problem for large d? We have the following tractability hierarchy for the problem $S = \{S_d\}_{d\in\mathbb{N}}$, see (e.g.) [9, Sect. 4.4]:

1. The problem S is *strongly polynomially tractable* if there exist $C \geq 0$ and $p \geq 0$ such that
$$n(\varepsilon, S_d, \Lambda_{d,2d}) \leq C\,\varepsilon^{-p} \qquad \forall\, d \in \mathbb{N}, \varepsilon \in (0,1).$$
Should this be the case, the infimum of all p such that this holds is said to be the *exponent of strong (polynomial) tractability*.

2. The problem S is *polynomially tractable* if there exist $C \geq 0$ and $p, q \geq 0$ such that
$$n(\varepsilon, S_d, \Lambda_{d,2d}) \leq C\,\varepsilon^{-p}\, d^{q} \qquad \forall\, d \in \mathbb{N}, \varepsilon \in (0,1).$$
We can speak of ε^{-1}- and d-tractability exponents for a tractable problem. However, these need not be uniquely determined; for example, we can sometimes decrease one of the exponents by allowing the other exponent to increase.

3. The problem S is *quasi-polynomially tractable* if there exist $C \geq 0$ and $t \geq 0$ such that
$$n(\varepsilon, S_d, \Lambda_{d,2d}) \leq C \exp\left(t \left(1 + \ln \varepsilon^{-1}\right)(1 + \ln d)\right) \qquad \forall\, d \in \mathbb{N}, \varepsilon \in (0,1). \tag{7}$$
The infimum of all t such that (7) holds is said to be the *exponent of quasi-polynomial tractability*. Quasi-polynomially tractability was introduced in [5]. The function appearing on the right-hand side of (7) is in some sense the smallest non-exponential tractability function T for which the approximation problem for unweighted tensor product spaces is T-tractable (see below).

4. Let Ω be an unbounded subset of $[1,\infty) \times [1,\infty)$. Let $T\colon [1,\infty) \times [1,\infty) \to [1,\infty)$ be a function that is non-decreasing in both its variables and that exhibits sub-exponential behavior, i.e.,
$$\lim_{\substack{(\xi,\eta)\in\Omega\\ \xi+\eta\to\infty}} \frac{\ln T(\xi,\eta)}{\xi + \eta} = 0.$$
The set Ω is called a *tractability domain*, and T a *tractability function*.

   The problem S is (T,Ω)-*tractable* if there exist $C \geq 0$ and $t \geq 0$ such that
$$n(\varepsilon, S_d, \Lambda_{d,2d}) \leq C\, T(\varepsilon^{-1}, d)^t \qquad \forall\, (\varepsilon^{-1}, d) \in \Omega. \tag{8}$$
The infimum of all t for which this holds is said to be the *exponent of* (T,Ω)-*tractability*.

   If the right-hand side of (8) holds with $d = 1$, so that
$$n(\varepsilon, S_d, \Lambda_{d,2d}) \leq C\, T(\varepsilon^{-1}, 1)^t \qquad \forall\, (\varepsilon^{-1}, d) \in \Omega,$$
then S is *strongly* (T,Ω)-*tractable*. In such a case, the infimum of all t for which this holds is said to be the *exponent of strong* (T,Ω)-*tractability*.

5. The problem S is *weakly tractable* if
$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, S_d, \Lambda_{d,2d})}{\varepsilon^{-1} + d} = 0.$$
A weakly tractable problem is one whose information complexity grows sub-exponentially in both ε^{-1} and d.

If the problem S is not even weakly tractable, then its information complexity is exponential in either ε^{-1} or d. We say that S is *intractable*. If the information complexity is exponential in d, we follow [1] and say that it suffers from the *curse of dimensionality*.

3 Tractability of Fredholm vs. tractability of approximation

In this section, we show that tractability of the Fredholm problem is strongly related to tractability of the L_2-approximation problems over F_d and Q_d. Here, the *L_2-approximation problem over V_d*, where V_d is a normed linear space that is continuously embedded in $L_2(I^d)$, is defined as approximating the canonical injection $\text{APP}_{V_d}\colon V_d \to L_2(I^d)$ given by

$$\text{APP}_{V_d}\, v = v \qquad \forall\, v \in V_d.$$

We approximate v from the unit ball BV_d of V_d, with error being measured in the $L_2(I^d)$-norm. Algorithm errors, minimal errors, and information complexity for the L_2-approximation problem over V_d are all defined analogously to the way they were defined for the Fredholm problem; the same is true for the various kinds of tractability for $\text{APP}_V = \{\text{APP}_{V_d}\}_{d\in\mathbb{N}}$, as well as intractability. Our assumption (5) is equivalent to requiring that

$$\|\,\text{APP}_{F_d}\,\|_{\text{Lin}[F_d;L_2(I^d)]} \le 1 \qquad \text{and} \qquad \|\,\text{APP}_{Q_d}\,\|_{\text{Lin}[Q_d;L_2(I^d)]} \le 1, \tag{9}$$

so that the initial errors of the L_2-approximation problems over F_d and Q_d are at most one. Note that if the bounds in (5) are sharp, then we have equality in (9), and then the L_2-approximation problems over F_d and Q_d are properly scaled.

3.1 Lower bounds

We are ready to prove lower bounds for the Fredholm problem. First, we show that the Fredholm problem S_d is not easier than the L_2-approximation problem over F_d.

Proposition 3.1. *We have*

$$n(\varepsilon, S_d, \Lambda_{d,2d}) \ge n(\varepsilon, \text{APP}_{F_d}, \Lambda_d) \qquad \forall\, \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

Proof. Let A_n be an algorithm for approximating the Fredholm problem S_d such that $e(A_n, S_d) \le \varepsilon$, using n information operations from $\Lambda_{d,2d}$. Define an algorithm $\widetilde{A}_n$ for APP_{F_d} by

$$\widetilde{A}_n(f) = A_n(f, 0) \qquad \forall\, f \in BF_d.$$

Since $\text{APP}_{F_d} = S_d(\cdot, 0)$, we have

$$e(\widetilde{A}_n, \text{APP}_{F_d}) \le e(A_n, S_d) \le \varepsilon,$$

which suffices to establish the desired inequality. ☐

We now wish to show that the Fredholm problem S_d is not easier than the L_2-approximation problem over Q_d. Before doing so, we need a bit of preparation. For a function $q\colon I^d \to \mathbb{R}$, let us define functions $q_X, q_Y\colon I^{2d} \to \mathbb{R}$ by

$$q_X(x,y) = q(x) \quad \text{and} \quad q_Y(x,y) = q(y) \qquad \forall\, x, y \in I^d.$$

We say that the sequence of spaces $Q = \{Q_d\}_{d\in\mathbb{N}}$ satisfies the *extension property* if for all $d \in \mathbb{N}$, we have

$$q \in Q_d \implies q_X, q_Y \in Q_{2d} \qquad \forall\, q \in Q_d,$$

with

$$\|q_X\|_{Q_{2d}} \le \|q\|_{Q_d} \qquad \text{and} \qquad \|q_y\|_{Q_{2d}} \le \|q\|_{Q_d}. \tag{10}$$

Let

$$M_2 = \frac{2(1+M_1)(3-M_1^2)}{M_1(1-M_1)}\,. \tag{11}$$

Clearly, $M_2 > 1$ and goes to infinity as M_1 goes to zero. Using Mathematica, we checked that

$$M_2 \ge 32.7757\ldots,$$

taking its minimal value when

$$\begin{aligned} M_1 &= \frac{1}{2} - \frac{1}{2}\sqrt{-\frac{1}{3} + \frac{1}{3}\sqrt[3]{656-72\sqrt{83}} + \frac{2}{3}\sqrt[3]{82+9\sqrt{83}}} \\ &\quad + \frac{1}{2}\sqrt{\frac{1}{3}\left(-2 - \sqrt[3]{656-72\sqrt{83}} - 2\sqrt[3]{82+9\sqrt{83}} + \frac{42}{\sqrt{-\frac{1}{3}+\frac{1}{3}\sqrt[3]{656-72\sqrt{83}}+\frac{2}{3}\sqrt[3]{82+9\sqrt{83}}}}\right)} \\ &\doteq 0.455213, \end{aligned}$$

Proposition 3.2. *Suppose that Q satisfies the extension property, and that $1 \in BF_d$. Then*

$$n(\varepsilon, S_d, \Lambda_{d,2d}) \ge n\left(M_2\varepsilon, \mathrm{APP}_{Q_d}, \Lambda_d\right) \qquad \forall\, \varepsilon \in \left(0, \frac{1}{2(1+M_1)}\right],\ d \in \mathbb{N}.$$

Proof. For $q \in BQ_d$, the extension property tells us that $M_1 q_X, M_1 q_Y \in Q_{2d}^{\mathrm{res}}$. As in [11, Sect. 18.2.1], we have

$$S_d(1, M_1 q_Y) = \frac{1}{1 - M_1 \int_{I^d} q(y)\,dy}. \tag{12}$$

Moreover, it is easy to see that

$$S_d(1, M_1 q_X) = \frac{M_1 q}{1 - M_1 \int_{I^d} q(y)\,dy} + 1.$$

Combining these results and solving for q, we see that

$$q = \frac{S_d(1, M_1 q_X) - 1}{M_1 S_d(1, M_1 q_Y)} = \frac{S_d(1, M_1 q_X) - 1}{M_1 \int_{I^d} S_d(1, M_1 q_Y)\,dy}, \tag{13}$$

the latter holding because (12) tells us that $S_d(1, M_1 q_Y)$ is a number. Now let A_n be an algorithm for approximating S_d over $BF_d \times Q_{2d}^{\mathrm{res}}$ such that it uses n information operations from $\Lambda_{d,2d}$ and $e(A_n, S_d) \le \varepsilon$, where

$$\varepsilon \le \frac{1}{2(1+M_1)}\,.$$

Guided by (13), we define an algorithm $\widetilde{A}_n$ for approximating APP_{Q_d} by

$$\widetilde{A}_n q = \frac{A_n(1, M_1 q_X) - 1}{M_1 \int_{I^d} A_n(1, M_1 q_Y)(y)\,dy} \qquad \forall\, q \in BQ_d.$$

We now compute an upper bound on the error of $\widetilde{A}_n$. First, some algebra yields that

$$\begin{aligned} q - \widetilde{A}_n q = &\frac{1}{M_1 \int_{I^d} A_n(1, M_1 q_Y)(y)\,dy}\Bigg[S_d(1, M_1 q_X) - A_n(1, M_1 q_X) + \\ &\left(\frac{1 - S_d(1, M_1 q_X)}{S_d(1, M_1 q_Y)}\right) \int_{I^d} [S_d(1, M_1 q_Y) - A_n(1, M_1 q_Y)(y)]\,dy \Bigg]. \end{aligned} \tag{14}$$

Using the inequality

$$\left|\int_{I^d}[S_d(1, M_1q_Y) - A_n(1, M_1q_Y)(y)]\,dy\right| \le \|S_d(1, M_1q_Y) - A_n(1, M_1q_Y)\|_{L_2(I^d)},$$

along with the fact that $e(A_n, S_d) \le \varepsilon$, equation (14) yields the inequality

$$\begin{aligned}\|q - \widetilde{A}_nq\|_{L_2(I^d)} \le \frac{1}{M_1\left|\int_{I^d} A_n(1, M_1q_Y)(y)\,dy\right|}\Bigg[&\|S_d(1, M_1q_X) - A_n(1, M_1q_X)\|_{L_2(I^d)}+ \\ &\frac{1 + \|S_d(1, M_1q_X)\|_{L_2(I^d)}}{|S_d(1, M_1q_Y)|}\|S_d(1, M_1q_Y) - A_n(1, M_1q_Y)\|_{L_2(I^d)}\Bigg] \qquad (15)\\ \le \frac{1}{M_1\left|\int_{I^d} A_n(1, M_1q_Y)(y)\,dy\right|}\Bigg[&1 + \frac{1 + \|S_d(1, M_1q_X)\|_{L_2(I^d)}}{|S_d(1, M_1q_Y)|}\Bigg]\cdot\varepsilon.\end{aligned}$$

Since $M_1q_X \in Q_{2d}^{\mathrm{res}}$, we have

$$\|S_d(1, M_1q_x)\|_{L_2(I^d)} \le \frac{1}{1 - M_1}.$$

Since $M_1q_Y \in Q_{2d}^{\mathrm{res}}$ and $S_d(1, M_1q_Y) \in \mathbb{R}$, we have

$$\frac{1}{1 + M_1} \le S_d(1, M_1q_Y) \le \frac{1}{1 - M_1}.$$

Now our restriction on ε implies that

$$\begin{aligned}\left|\int_{I^d} A_n(1, M_1q_Y)(y)\,dy\right| &\ge S_d(1, M_1q_Y) - \left|\int_{I^d}[S_d(1, M_1q_Y) - A_n(1, M_1q_Y)(y)]\,dy\right| \\ &\ge \frac{1}{1 + M_1} - \varepsilon \ge \frac{1}{2(1 + M_1)}.\end{aligned}$$

Substituting these last three inequalities into (15), we find

$$\|q - \widetilde{A}_nq\|_{L_2(I^d)} \le \frac{2(1 + M_1)}{M_1}\left[1 + \left(1 + \frac{1}{1 - M_1}\right)(1 + M_1)\right]\varepsilon = M_2 \cdot \varepsilon\,.$$

Since q is an arbitrary element of BQ_d, we see that

$$e(\widetilde{A}_n, S_d) \le M_2 \cdot \varepsilon\,.$$

This suffices to establish the desired inequality. □

Using Propositions 3.1 and 3.2, we have the following corollary.

Corollary 3.1. *Suppose that Q satisfies the extension property. Then the approximation problems* APP_F *and* APP_Q *are at least as hard as the Fredholm problem S. That is:*

1. *If the Fredholm problem S is strongly polynomially tractable, then so are* APP_F *and* APP_Q*. Moreover, the exponents of strong polynomial tractability of the approximation problems are no larger than those for the Fredholm problem.*

2. *If the Fredholm problem S is polynomially tractable, then so are* APP_F *and* APP_Q*. Moreover, ε^{-1}- and d-exponents for the approximation problems are no larger than those for the Fredholm problem.*

3. *If the Fredholm problem S is quasi-polynomially tractable, then so are APP_F and APP_Q. The exponent of quasi-polynomial tractability for the approximation problem APP_F is no larger than this for the Fredholm problem. However, the exponent of quasi-polynomial tractability for the approximation problem APP_Q may be larger than this for the Fredholm problem by the factor $1 + \ln\ M_2$.*

4. *Suppose that for all $\alpha > 0$, the tractability function T satisfies*

$$T(\alpha\xi, \eta) = \mathcal{O}\big(T(\xi, \eta)\big) \qquad \text{as } \xi, \eta \to \infty. \tag{16}$$

If the Fredholm problem S is (strongly) (T, Ω)-tractable, then so are APP_F and APP_Q. Moreover, the exponents of (strong) (T, Ω)-tractability for the approximation problems are no larger than those for the Fredholm problem.

5. *If the Fredholm problem S is weakly tractable, then so are APP_F and APP_Q.*

6. *If either APP_F or APP_Q are intractable, then so is the Fredholm problem S.*

Proof. All these statements follow from Propositions 3.1 and 3.2. However the statements regarding quasi-polynomial tractability and (T, Ω)-tractability are a bit more subtle than the others, so we give some details for these cases.

Suppose first that the Fredholm problem S is quasi-polynomially tractable. This means that there exist $C > 0$ and $t \geq 0$ such that

$$n(\varepsilon, S_d, \Lambda_{d,2d}) \leq C \exp\left(t\left(1 + \ln \varepsilon^{-1}\right)(1 + \ln d)\right) \qquad \forall\, d \in \mathbb{N}, \varepsilon \in (0,1).$$

From Proposition 3.1, we immediately find that APP_F is quasi-polynomially tractable, with the same estimate

$$n(\varepsilon, \mathrm{APP}_{F_d}, \Lambda_d) \leq n(\varepsilon, S_d, \Lambda_{d,2d}) \leq C \exp\left(t\left(1 + \ln \varepsilon^{-1}\right)(1 + \ln d)\right) \qquad \forall\, d \in \mathbb{N}, \varepsilon \in (0,1).$$

What about APP_Q? Proposition 3.2 yields that

$$n(M_2\varepsilon, \mathrm{APP}_{Q_d}, \Lambda_d) \leq n(\varepsilon, S_d, \Lambda_{d,2d}) \qquad \forall\, d \in \mathbb{N}, \varepsilon \in (0,1).$$

Replacing $M_2\varepsilon$ by ε, and remembering that $M_2 > 1$, we get

$$\begin{aligned} n(\varepsilon, \mathrm{APP}_{Q_d}, \Lambda_d) &\leq n(M_2^{-1}\varepsilon, S_d, \Lambda_d) \\ &\leq C \exp\left[t\left(1 + \ln M_2 + \ln \varepsilon^{-1}\right)(1 + \ln d)\right] \\ &= C \exp\left[t\left(1 + \ln \varepsilon^{-1}\right)(1 + \ln d)\left(1 + \frac{\ln M_2}{1 + \ln \varepsilon^{-1}}\right)\right] \\ &\leq C \exp\left(t(1 + \ln\ M_2)\left(1 + \ln \varepsilon^{-1}\right)(1 + \ln d)\right). \end{aligned}$$

Hence APP_Q is quasi-polynomially tractable, with an exponent at most $t(1+\ln\ M_2)$. This exponent is clearly larger than that of the Fredholm problem.

Now suppose that the Fredholm problem S is (strongly) (T, Ω)-tractable, with a tractability function T satisfying (16). For APP_F, we find that

$$n(\varepsilon, \mathrm{APP}_{F_d}, \Lambda_d) \leq n(\varepsilon, S_d, \Lambda_{d,2d}) = T(\varepsilon^{-1}, d)^t \qquad \forall\, d \in \mathbb{N}, \varepsilon \in (0,1).$$

For APP_Q, we find that

$$n(\varepsilon, \mathrm{APP}_{Q_d}, \Lambda_d) \leq n(M_2^{-1}\varepsilon, S_d, \Lambda_d) = \mathcal{O}\big(T(M_2\varepsilon^{-1}, d)^t\big) = \mathcal{O}\big(T(\varepsilon^{-1}, d)^t\big) \qquad \forall\, d \in \mathbb{N}, \varepsilon \in (0,1).$$

Thus both approximation problems are (strongly) (T, Ω)-tractable, with exponents at most as large as the exponent for the Fredholm problem, as claimed. □

3.2 Upper bounds

Having found lower bounds, we now look for analogous upper bounds.

Lemma 3.1. *Let $u = S_d(f,q)$ and $\tilde{u} = S_d(\tilde{f},\tilde{q})$ for $(f,q), (\tilde{f},\tilde{q}) \in BF_d \times Q_{2d}^{\text{res}}$. Then*

$$\|u - \tilde{u}\|_{L_2(I^d)} \le \frac{1}{1-M_1}\left[\|f-\tilde{f}\|_{L_2(I^d)} + \|u\|_{L_2(I^d)}\|q-\tilde{q}\|_{L_2(I^{2d})}\right].$$

Proof. Since $(I - T_q)u = f$ and $(I - T_{\tilde{q}})\tilde{u} = \tilde{f}$, we find that

$$f - \tilde{f} = u - \tilde{u} - T_q u + T_{\tilde{q}}\tilde{u} = u - \tilde{u} - T_{q-\tilde{q}}u - T_{\tilde{q}}(u-\tilde{u}),$$

and so

$$(I - T_{\tilde{q}})(u-\tilde{u}) = f - \tilde{f} + T_{q-\tilde{q}}u.$$

Hence

$$u - \tilde{u} = (I - T_{\tilde{q}})^{-1}[f - \tilde{f} + T_{q-\tilde{q}}u].$$

Using (3) and (4), we get the desired inequality. □

We now use Lemma 3.1 to find upper bounds for the Fredholm problem, in terms of upper bounds for the L_2-approximation problems for F_d and Q_d.

Proposition 3.3. *For $\varepsilon > 0$ and $d \in \mathbb{N}$, we have*

$$n(\varepsilon, S_d, \Lambda_{d,2d}) \le n\left(\frac{(1-M_1)\,\varepsilon}{2}, \text{APP}_{F_d}, \Lambda_d\right) + n\left(\frac{(1-M_1)^2\,\varepsilon}{2M_1}, \text{APP}_{Q_{2d}}, \Lambda_{2d}\right). \tag{17}$$

Proof. Let $\widetilde{A}_{n(F),F_d}$ and $\widetilde{A}_{n(Q),Q_{2d}}$ (respectively) be algorithms using $n(F)$ and $n(Q)$ information operations for the L_2-approximation problems over F_d and Q_{2d} such that

$$e\left(\widetilde{A}_{n(F),F_d}, \text{APP}_{F_d}\right) \le \frac{(1-M_1)\,\varepsilon}{2} \qquad \text{and} \qquad e\left(\widetilde{A}_{n(Q),Q_{2d}}, \text{APP}_{Q_{2d}}\right) \le \frac{(1-M_1)^2\,\varepsilon}{2M_1}. \tag{18}$$

Let $n = n(F) + n(Q)$. Define an algorithm A_n for the Fredholm problem as

$$A_n(f,q) = S_d\left(\widetilde{A}_{n(F),F_d}(f), \widetilde{A}_{n(Q),Q_{2d}}(q)\right) \qquad \forall\,(f,q) \in BF_d \times Q_{2d}^{\text{res}}.$$

Clearly, A_n uses n information operations. To compute the error of A_n, let $(f,q) \in BF_d \times Q_{2d}^{\text{res}}$. By (18), we have

$$\|f - \widetilde{A}_{n(F),F_d}(f)\|_{L_2(I^d)} \le \frac{(1-M_1)\varepsilon}{2}\|f\|_{F_d} \le \frac{(1-M_1)\varepsilon}{2}$$

and

$$\|q - \widetilde{A}_{n(Q),Q_{2d}}(q)\|_{L_2(I^{2d})} \le \frac{(1-M_1)^2\varepsilon}{2M_1}\|q\|_{Q_{2d}} \le \frac{(1-M_1)^2\varepsilon}{2}.$$

Using Lemma 3.1 and inequality (6), we now have

$$\begin{aligned} e(A_n, S_d) &\le \frac{1}{1-M_1}\left[\|f - \widetilde{A}_{n(F),F_d}\|_{L_2(I^d)} + \|S_d(f,q)\|_{L_2(I^d)}\|q - \widetilde{A}_{n(Q),Q_{2d}}(q)\|_{L_2(I^{2d})}\right] \\ &\le \frac{1}{1-M_1}\left(\frac{(1-M_1)\varepsilon}{2} + \frac{1}{1-M_1}\frac{(1-M_1)^2\varepsilon}{2}\right) \\ &= \varepsilon. \end{aligned}$$

Since (f,q) is an arbitrary element of $BF_d \times Q_{2d}^{\mathrm{res}}$, we see that

$$e(A_n, S_d) \le \varepsilon.$$

The algorithms $\widetilde{A}_{n(F),F_d}$ and $\widetilde{A}_{n(Q),Q_{2d}}$ are arbitrary and satisfy (18). We can then take them to be algorithms using the minimal number of information operations needed to satisfy (18). Inequality (17) now follows. $\square$

We now discuss the arguments of $n(\cdot, \mathrm{APP}_{F_d}, \Lambda_d)$ and $n(\cdot, \mathrm{APP}_{Q_{2d}}, \Lambda_{2d})$ in (17). For all $\varepsilon \in (0,1)$, the argument $(1-M_1)\varepsilon/2$ is less than $1/2$; however, the argument $(1-M_1)^2\varepsilon/(2M_1)$ may be larger than one if M_1 is small enough and ε close enough to one. In this case, the second term

$$n\left(\frac{(1-M_1)^2\,\varepsilon}{2M_1}, \mathrm{APP}_{Q_{2d}}, \Lambda_{2d}\right) = 0 \qquad \text{for} \qquad \frac{(1-M_1)^2\,\varepsilon}{2M_1} \ge 1,$$

since we now can take $A_0 = 0$ with error at most 1.

Using Proposition 3.3, we have the following corollary.

Corollary 3.2. *The Fredholm problem S is no harder than than the approximation problems APP_F and APP_Q. That is:*

1. *If APP_F and APP_Q are strongly polynomially tractable, then so is the Fredholm problem S. Moreover, the exponent of strong polynomial tractability for S is no larger than the greater of those for APP_F and APP_Q.*

2. *If APP_F and APP_Q are polynomially tractable, then so is the Fredholm problem S. Moreover, the ε^{-1}-exponents and the d-exponents for S are no larger than the greater of the ε^{-1}-exponents and the d-exponents for APP_F and APP_Q.*

3. *If APP_F and APP_Q are quasi-polynomially tractable, then so is the Fredholm problem S. Moreover, the exponent t_S of quasi-polynomial tractability for S satisfies*

$$t_S \le t_S^* := \max\left\{ t_F\left(1+\ln\frac{2}{1-M_1}\right), t_Q\left(1+\max\left\{0, \ln\frac{2M_1}{(1-M_1)^2}\right\}\right)(1+\ln 2)\right\}. \tag{19}$$

4. *Suppose that the following are true:*

   (a) APP_F is (strongly) (T_F,Ω)-tractable, with (strong) exponent t_F.

   (b) APP_Q is (strongly) (T_Q,Ω)-tractable, with (strong) exponent t_Q.

   (c) For any $\alpha > 0$, the tractability functions T_F and T_Q satisfy

   $$T_F(\alpha\xi,\eta) = \mathcal{O}\big(T_F(\xi,\eta)\big) \quad \text{and} \quad T_Q(\alpha\xi,\eta) = \mathcal{O}\big(T_Q(\xi,\eta)\big) \qquad \text{as } \xi,\eta \to \infty.$$

   Then

   (a) The Fredholm problem S is (T_S,Ω)-tractable, with $T_S = \max\{T_F, T_Q\}$. Moreover, strong (T_S,Ω)-tractability holds for S iff it holds for both APP_F and APP_Q.

   (b) The (strong) exponent of (T_S,Ω)-tractability is at most $\max\{t_F, t_Q\}$.

5. *If APP_F and APP_Q are weakly tractable, then so is the Fredholm problem S.*

6. *If the Fredholm problem S is intractable, then either* APP_F *is intractable or* APP_Q *is intractable.*

Proof. All this follows from Proposition 3.3 (as mentioned above), along with the definitions of the various kinds of tractability. To illustrate, we prove the quasi-polynomial case (part 3), if for no other reason than to explain the somewhat odd-looking result for t_S^*.

Since APP_F and APP_Q are quasi-polynomially tractable, there exist positive C_F and C_Q, as well as nonnegative t_F and t_Q, such that

$$n(\varepsilon, \mathrm{APP}_{F_d}, \Lambda_d) \le C_F \exp\left(t_F(1 + \ln\, \varepsilon^{-1})(1 + \ln\, d)\right)$$

and

$$n(\varepsilon, \mathrm{APP}_{Q_{2d}}, \Lambda_{2d}) \le C_Q \exp\left(t_Q(1 + \ln\, \varepsilon^{-1})(1 + \ln\, 2d)\right).$$

By Proposition 3.3, we have

$$n(\varepsilon, S_d, \Lambda_{d,2d}) \le C_F \exp\left(t_F\left[1 + \ln\left(\frac{(1-M_1)\varepsilon}{2}\right)^{-1}\right](1 + \ln\, d)\right) + \delta_\varepsilon\, C_Q \exp\left(t_Q\left[1 + \ln\left(\frac{(1-M_1)^2\varepsilon}{2M_1}\right)^{-1}\right](1 + \ln\, 2d)\right), \quad (20)$$

where $\delta_\varepsilon = 0$ for $(1 - M_1)^2\varepsilon/(2M_1) \ge 1$, and $\delta_\varepsilon = 1$, otherwise.

Clearly, for $c \in (0, 1]$ we have

$$1 + \ln(c\,\varepsilon)^{-1} \le (1 + \ln\, \varepsilon^{-1})(1 + \ln\, c^{-1}) \qquad \forall\, \varepsilon \in (0, 1),$$

as well as

$$1 + \ln\, 2d \le (1 + \ln\, 2)\,(1 + \ln\, d) \qquad \forall\, d \in \mathbb{N}.$$

Applying these inequalities to (20) we conclude that

$$n(\varepsilon, S_d, \Lambda_{d,2d}) \le C_F \exp\left(t_F\left(1 + \ln\,\frac{2}{1-M_1}\right)(1 + \ln\, \varepsilon^{-1})(1 + \ln\, d)\right) + C_Q \exp\left(t_Q\left(1 + \max\left\{0, \ln\,\frac{2M_1}{(1-M_1)^2}\right\}\right)(1 + \ln\, 2)(1 + \ln\, \varepsilon^{-1})(1 + \ln\, d)\right).$$

Using this we get the formula for t_S^*.

The proof of the remaining parts of the corollary is easy. □

Remark. In Section 2, we said that there was no essential loss of generality in assuming that (5) (equivalently, (9)) holds. To see why this is true, note the following:

- If $\|\, \mathrm{APP}_{F_d} \|_{\mathrm{Lin}[F_d; L_2(I^d)]} > 1$, the bound (17) in Proposition 3.3 becomes

$$n(\varepsilon, S_d, \Lambda_{d,2d}) \le n\left(\frac{(1-M_1)\,\varepsilon}{2}, \mathrm{APP}_{F_d}, \Lambda_d\right) + n\left(\frac{(1-M_1)^2\,\varepsilon}{2M_1\,\|\, \mathrm{APP}_{F_d} \|_{\mathrm{Lin}[F_d; L_2(I^d)]}}, \mathrm{APP}_{Q_{2d}}, \Lambda_d\right).$$

Hence if $\sup_{d\in\mathbb{N}} \| \, \mathrm{APP}_{F_d} \|_{\mathrm{Lin}[F_d;L_2(I^d)]} < \infty$, then

$$n\left(\frac{(1-M_1)^2}{2\,\| \, \mathrm{APP}_{F_d} \|_{\mathrm{Lin}[F_d;L_2(I^d)]}}\varepsilon, \| \, \mathrm{APP}_{F_d} \|_{\mathrm{Lin}[F_d;L_2(I^d)]}, \Lambda_d\right) \le$$
$$n\left(\frac{(1-M_1)^2}{2M_1 \sup\limits_{d\in\mathbb{N}} \| \, \mathrm{APP}_{F_d} \|_{\mathrm{Lin}[F_d;L_2(I^d)]}}\varepsilon, \| \, \mathrm{APP}_{F_d} \|_{\mathrm{Lin}[F_d;L_2(I^d)]}, \Lambda_d\right).$$

Thus the tractability results of Corollary 3.2 hold as stated, but with a slight change in the denominator of the first argument of $n(\cdot, \mathrm{APP}_{Q_{2d}}, \Lambda_d)$. However, if

$$\sup_{d\in\mathbb{N}} \| \, \mathrm{APP}_{F_d} \|_{\mathrm{Lin}[F_d;L_2(I^d)]} = \infty,$$

then the approximation problem for F_d is badly scaled.

- If $\| \, \mathrm{APP}_{Q_d} \|_{\mathrm{Lin}[Q_d;L_2(I^d)]} > 1$, we can renormalize Q_d under the (equivalent) norm

$$\|q\|_{\widehat{Q}_d} = \sqrt{\|q\|^2_{L_2(I^d)} + \|q\|^2_{Q_d}} \qquad \forall\, q \in Q_d,$$

calling the resulting space $\widehat{Q}_d$. We now replace Q_d by $\widehat{Q}_d$ and Q_d^{res} by

$$\widehat{Q}_d^{\mathrm{res}} = \left\{ q \in \widehat{Q}_d : \|q\|_{\widehat{Q}_d} \le M_1 \right\}.$$

Since $q \in \widehat{Q}_d^{\mathrm{res}}$ implies that $\|q\|_{L_2(I^d)} \le M_1$ and $\|q\|_{Q_d} \le M_1$, we see that all our results go through as before under this relabelling. □

4 Some examples

We now study the tractability of the Fredholm for three examples, each being defined by choosing particular spaces of right-hand side functions and kernel functions. The first example shows us that we may be stricken by the curse of dimensionality even if the right-hand side or the kernel function is infinitely smooth. In the second example, we look at unweighted isotropic spaces, finding that the Fredholm problem is quasi-polynomially tractable, but not polynomially tractable. In the third example, we explore tractability for a family of weighted spaces, getting both positive and negative results for polynomial tractability.

4.1 Intractability for C^∞ functions

Let $C^\infty(I^d)$ be the space of infinitely many times differentiable functions with the norm

$$\|v\|_{C^\infty(I^d)} = \sup_{\boldsymbol{\alpha}\in\mathbb{N}_0^d} \|D^\alpha v\|_{L_2(I^d)}.$$

Here, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$ is a multi-index with $|\boldsymbol{\alpha}| = \sum_{j=1}^d \alpha_j$, and

$$D^{\boldsymbol{\alpha}} v = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \cdots \partial^{\alpha_d} x_d}.$$

Let $F_d = G_d = C^\infty(I^d)$. The L_2-approximation problems for F_d and G_d satisfy the assumption (5). Moreover, since $\|1\|_{F_d} = \|1\|_{Q_d} = \|1\|_{L_2(I_d)}$, we have

$$\| \text{APP}_{F_d} \|_{\text{Lin}[F_d;L_2(I^d)]} = \| \text{APP}_{Q_d} \|_{\text{Lin}[Q_d;L_2(I^d)]} = 1.$$

This also shows that $1 \in BF_d$, as needed in Proposition 3.2. Moreover, $Q = \{Q_d\}_{d \in]posints}$ satisfies the extension property, with equality holding in (10). This means that we can use all the results presented in the previous section.

The functions in F_d and Q_d are of unbounded smoothness. As in [10], it is easy to check that for $\Lambda_d \in \{\Lambda_d^{\text{all}}, \Lambda_d^{\text{std}}\}$, we have

$$e(n, \text{APP}_{F_d}, \Lambda_d) = \mathcal{O}(n^{-r}) \quad \text{and} \quad e(n, \text{APP}_{Q_d}, \Lambda_d) = \mathcal{O}(n^{-r}) \qquad \text{as } n \to \infty,$$

for *any* $r > 0$, no matter how large. This implies that we also have

$$e(n, S_d, \Lambda_{d,2d}) = \mathcal{O}(n^{-r}) \quad \text{as } n \to \infty,$$

and

$$n(\varepsilon, S_d, \Lambda_{d,2d}) = \mathcal{O}(\varepsilon^{-1/r}) \quad \text{as } \varepsilon \to 0$$

for the Fredholm problem. Since r can be arbitrarily large, this might lead one to hope that the Fredholm problem does not suffer from the curse of dimensionality in this case. We now crush this hope, showing that the Fredholm problem is intractable if either $F_d = C^\infty(I^d)$ or $Q_{2d} = C^\infty(I^{2d})$ and F_d satisfies (5) as well as $1 \in BF_d$. This holds for the class Λ^{all}, and therefore also for the class Λ^{std}.

First, suppose that $F_d = C^\infty(I^d)$. Using [10, Remark 3], we find that

$$e(n, \text{APP}_{F_d}, \Lambda_d^{\text{all}}) = 1 \qquad \text{for } n < 2^{\lceil d/4 \rceil}.$$

Hence, the L_2-approximation problem over F_d is intractable, with

$$n(\varepsilon, \text{APP}_{F_d}, \Lambda_d^{\text{all}}) \geq 2^{\lfloor d/4 \rfloor} \qquad \forall\, \varepsilon \in (0, 1).$$

From Proposition 3.1, we immediately see that

$$n(\varepsilon, S_d, \Lambda_{d,2d}) \geq n(\varepsilon, \text{APP}_{F_d}, \Lambda_d) \geq 2^{\lfloor d/4 \rfloor} \qquad \forall\, \varepsilon \in (0, 1).$$

Hence the Fredholm problem is also intractable.

Now suppose that $Q_d = C^\infty(I^d)$, and that F_d satisfies (5), with $1 \in BF_d$. Again, [10, Remark 3] tells us that

$$e(n, \text{APP}_{Q_{2d}}, \Lambda_{2d}^{\text{all}}) = 1 \qquad \text{for } n < 2^{\lceil d/2 \rceil},$$

and so the L_2-approximation problem over Q_d is intractable, with

$$n(\varepsilon, \text{APP}_{Q_d}, \Lambda_d^{\text{all}}) \geq 2^{\lfloor d/2 \rfloor} \qquad \forall\, \varepsilon \in (0, 1).$$

Noting that

$$\min\left\{\frac{1}{M_2}, \frac{1}{2(1+M_1)}\right\} = \frac{1}{M_2},$$

Proposition 3.2 yields that

$$n(\varepsilon, S_d, \Lambda_{d,2d}) \geq n(M_2\varepsilon, \text{APP}_{Q_{2d}}, \Lambda_d) \geq 2^{\lfloor d/2 \rfloor} \qquad \forall\, \varepsilon \in \left(0, \frac{1}{M_2}\right].$$

Thus the Fredholm problem is intractable also in this case.

In short, the Fredholm problem suffers from the curse of dimensionality if $F_d = C^\infty(I^d)$ or $Q_d = C^\infty(I^d)$ and F_d satisfies (5) as well as $1 \in BF_d$. Using these extremely smooth spaces avails us not.

4.2 Results for unweighted tensor product spaces

We now start to explore tractability for tensor product spaces. Our first step is to look at unweighted tensor product Hilbert spaces, as per [9, Sect. 5.2]. We will then look at weighted tensor product Hilbert spaces in Section 5.

Since the space for the univariate case is a building block for the tensor product space, we first start with the univariate case, and then go on to define the tensor product space for general d.

For the univariate case, let $H_1 \subseteq L_2(I)$ be an infinite-dimensional separable Hilbert space of univariate functions. Suppose that the embedding $\mathrm{APP}_1 \colon H_1 \to L_2(I)$ is compact. Then $W_1 = \mathrm{APP}_1^* \mathrm{APP}_1 : H_1 \to H_1$ is a compact, self-adjoint, positive definite operator. Let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis for H_1 consisting of eigenfunctions of $W_1 = \mathrm{APP}_1^* \mathrm{APP}_1$, ordered so that

$$W_1 e_j = \lambda_j e_j \qquad \forall\, j \in \mathbb{N}$$

with $\lambda_1 \geq \lambda_2 \geq \cdots > 0$. Clearly, $\|W_1\|_{\mathrm{Lin}(H_1)} = \lambda_1$. Since H_1 is infinite-dimensional, the eigenvalues λ_i are positive. Note that for $f \in H_1$ we have

$$\|f\|^2_{L_2(I)} = \langle f, f\rangle_{L_2(I)} = \langle \mathrm{APP}_1 f, \mathrm{APP}_1 f\rangle_{L_2(I)} = \langle f, W_1 f\rangle_{H_1} \leq \lambda_1 \|f\|^2_{H_1}.$$

Hence, the assumption (5) holds if we assume that $\lambda_1 \leq 1$. For simplicity, we also assume that $e_1 \equiv 1 \in H_1$, with $\|1\|_{H_1} = 1$, so that $\lambda_1 = 1$.

We now move on to the general case $d \geq 1$, defining the tensor product space $H_d = H_1^{\otimes d}$, which is a Hilbert space under the inner product

$$\left\langle \bigotimes_{j=1}^{d} v_j, \bigotimes_{j=1}^{d} w_j \right\rangle_{H_d} = \prod_{j=1}^{d} \langle v_j, w_j\rangle_{H_1} \qquad \forall\, v_1, \ldots, v_d, w_1, \ldots, w_d \in H_1,$$

where

$$\left(\bigotimes_{j=1}^{d} v_j\right)(x) = \prod_{j=1}^{d} v_j(x_j) \qquad \forall\, x = (x_1, x_2, \ldots, x_d) \in I^d.$$

Let APP_d denote the canonical embedding of H_d into $L_2(I^d)$ given by

$$\mathrm{APP}_d\, v = v \qquad \forall\, v \in H_d.$$

Clearly, $\|\mathrm{APP}_d\| = 1$. Let $W_d = \mathrm{APP}_d^* \mathrm{APP}_d$. For a multi-index $\boldsymbol{\alpha} \in \mathbb{N}^d$, let

$$e_{\boldsymbol{\alpha}} = \bigotimes_{j=1}^{d} e_{\alpha_j} \qquad \text{and} \qquad \lambda_{\boldsymbol{\alpha}} = \prod_{j=1}^{d} \lambda_{\alpha_j}.$$

Then

$$W_d e_{\boldsymbol{\alpha}} = \lambda_{\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}} \qquad \forall\, \boldsymbol{\alpha} \in \mathbb{N}$$

and

$$\langle e_{\boldsymbol{\alpha}}, e_{\boldsymbol{\beta}}\rangle_{H_d} = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}} \qquad \forall\, \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^d.$$

Thus $\{e_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathbb{N}^d}$ is an orthonormal system of eigenfunctions of W_d.

Knowing the eigensystem of W_d, we can determine the nth minimal error $e(n, \mathrm{APP}_{H_d}, \Lambda^{\mathrm{all}})$. Let

$$\{\lambda_{d,j}\}_{j\in\mathbb{N}} = \{\lambda_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathbb{N}^d},$$

with

$$\lambda_{d,1} \geq \lambda_{d,2} \geq \cdots > 0$$

and let $e_{d,j}$ be the eigenfunction corresponding to $\lambda_{d,j}$. It is well-known (see, e.g., [15, Sect. 4.5]) that

$$e(n, \mathrm{APP}_{H_d}, \Lambda^{\mathrm{all}}) = \sqrt{\lambda_{d,n+1}},$$

this error being attained by the algorithm

$$A_n(v) = \sum_{j=1}^{n} \langle v, e_{d,j}\rangle_{H_d} e_{d,j}.$$

We now let $F_d = H_d$ and $Q_d = H_{2d}$. Then the assumptions (5) and (10) hold and $1 \in BF_d$ with $\|1\|_{F_d} = \|1\|_{L_2(I^d)} = 1$. What can we say about the tractability of the Fredholm problem?

If $\lambda_2 = 1$, then [9, Theorem 5.5] tells us that the L_2-approximation problem for H_d is intractable for the class Λ^{all} (and thus also for Λ^{std}). Hence the Fredholm problem is also intractable for Λ^{all} (and Λ^{std}) by Corollary 3.1.

We now suppose that $\lambda_2 \in (0, 1)$. In addition, for the remainder of this subsection, we shall restrict our attention to the case where there exists some $p > 0$ such that

$$\lambda_j = \Theta(j^{-p}) \qquad \text{as } j \to \infty.$$

From [9, Theorem 5.5], we find that the L_2-approximation problem for H_d is not polynomially tractable for the class Λ^{all} (and so for Λ^{std}). Again using Corollary 3.1, we see that the Fredholm problem is also not polynomially tractable for Λ^{all} (and Λ^{std}). So let's see what we can say about quasi-polynomial tractability.

First, suppose that the class Λ^{all} is used. From [5, Sect. 3.1], we find that the L_2-approximation problem for H_d is quasi-polynomially tractable with

$$t = \max\left\{\frac{2}{p}, \frac{2}{\ln \lambda_2^{-1}}\right\}.$$

Hence Corollary 3.2 tells us that the Fredholm problem is also quasi-polynomially tractable and

$$n(\varepsilon, S_d, \Lambda^{\mathrm{all}}_{d,2d}) \leq C \exp\left(t_S^*(1 + \ln \varepsilon^{-1})(1 + \ln d)\right)$$

with

$$t_S^* = t \max\left\{1 + \ln \frac{1}{1 - M_1}, 1 + \max\left\{0, \ln \frac{2M_1}{(1 - M_1)^2}\right\}(1 + \ln 2)\right\}.$$

Now suppose that we use the class Λ^{std}. Unfortunately, there are currently no general results for the case of standard information; we only know of some examples. From [5, Sect. 3.2], we know that there is a piecewise-constant function space for which quasi-polynomial tractability is the same for Λ^{all} and Λ^{std}, and there is a Korobov space for which quasi-polynomial tractability does not hold. So in the former case, the Fredholm problem will be quasi-polynomially tractable; in the latter case, it will not be quasi-polynomially tractable.

4.3 Results for a weighted Sobolev space

The results reported in Section 4.2 tell us that if we want the Fredholm problem to be polynomially tractable, then the right-hand side and kernel must belong to *non-isotropic spaces*, in which different variables or groups of variables play different roles. In this section, we examine a particular weighted space $H_{d,m,\boldsymbol{\gamma}}$, where $m \in \mathbb{N}$ is a fixed positive integer that measures the smoothness of the space, and $\boldsymbol{\gamma}$ is a sequence of weights that measure the importance of groups of variables. This will motivate the general definition presented in Section 5.

Our analysis uses the results and ideas found in [22]. We build our space $H_{d,m,\boldsymbol{\gamma}}$ in stages, starting with a unweighted univariate space $H_{1,m}$, then going to an unweighted multivariate space $H_{d,m}$, and finally arriving at our weighted multivariate space $H_{d,m,\boldsymbol{\gamma}}$.

So we first look at the case $d = 1$. The space $H_{1,m}$ consists of real functions defined on I, whose $(m-1)$st derivatives are absolutely continuous and whose mth derivatives belong to $L_2(I)$, under the inner product

$$\langle v, w\rangle_{H_{1,m}} = \int_I v(x)w(x)\,dx + \int_I v^{(m)}(x)w^{(m)}(x)\,dx \qquad \forall\, v, w \in H_{1,m}.$$

For $d \in \mathbb{N}$, define $H_{d,m} = H_{1,m}^{\otimes d}$ as a d-fold tensor product of $H_{1,m}$, under the inner product

$$\langle v, w\rangle_{H_{d,m}} = \int_{I^d} v(x)w(x)\,dx + \sum_{\substack{\mathfrak{u}\subseteq[d]\\ \mathfrak{u}\neq\emptyset}} \int_{I^d} \frac{\partial^{m|\mathfrak{u}|}}{\partial^m x_{\mathfrak{u}}} v(x) \frac{\partial^{m|\mathfrak{u}|}}{\partial^m x_{\mathfrak{u}}} w(x)\,dx \qquad \forall\, v, w \in H_{d,m}.$$

Here, $|\mathfrak{u}|$ denotes the size of $\mathfrak{u} \subseteq [d] := \{1, 2, \ldots, d\}$, and $x_{\mathfrak{u}}$ denotes the vector whose components are those components x_j of x for which $j \in \mathfrak{u}$.

We are now ready to define our weighted Sobolev space. Let

$$\boldsymbol{\gamma} = \{\gamma_{d,\mathfrak{u}}\}_{\mathfrak{u}\subseteq[d]}$$

be a set of non-negative *weights*. For simplicity, we assume that $\gamma_{d,\emptyset} = 1$. Then we let

$$H_{d,m,\boldsymbol{\gamma}} = \left\{ v \in H_{d,m} : \gamma_{d,\mathfrak{u}} = 0 \implies \frac{\partial^{m|\mathfrak{u}|}}{\partial^m x_{\mathfrak{u}}} v \equiv 0 \right\},$$

under the inner product

$$\langle v, w\rangle_{H_{d,m,\gamma}} = \int_{I^d} v(x)w(x)\,dx + \sum_{\substack{\mathfrak{u}\subseteq[d]\\ \mathfrak{u}\neq\emptyset \\ \gamma_{d,\mathfrak{u}}>0}} \gamma_{d,\mathfrak{u}}^{-1} \int_{I^d} \frac{\partial^{m|\mathfrak{u}|}}{\partial^m x_{\mathfrak{u}}} v(x) \frac{\partial^{m|\mathfrak{u}|}}{\partial^m x_{\mathfrak{u}}} w(x)\,dx \qquad \forall\, v, w \in H_{d,m,\boldsymbol{\gamma}}.$$

Interpreting $0/0$ as 0, we may rewrite this inner product in the simpler form

$$\langle v, w\rangle_{H_{d,m,\gamma}} = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{-1} \int_{I^d} \frac{\partial^{m|\mathfrak{u}|}}{\partial^m x_{\mathfrak{u}}} v(x) \frac{\partial^{m|\mathfrak{u}|}}{\partial^m x_{\mathfrak{u}}} w(x)\,dx \qquad \forall\, v, w \in H_{d,m,\boldsymbol{\gamma}}. \tag{21}$$

Let $F_d = H_{d,m_F,\boldsymbol{\gamma}_F}$ and $Q_d = H_{d,m_Q,\boldsymbol{\gamma}_Q}$. Here, the weights $\boldsymbol{\gamma}_F = \{\gamma_{d,\mathfrak{u},F}\}$ and $\boldsymbol{\gamma}_Q = \{\gamma_{d,\mathfrak{u},Q}\}$ may be different but we have $\gamma_{d.\emptyset,F} = \gamma_{d,\emptyset,Q} = 1$. Again, the assumption (5) is satisfied; moreover, since $\|1\|_{F_d} = \|1\|_{L_2(I^d)} = 1$, we have $1 \in BF_d$.

Recall that if $Q = \{Q_d\}_{d\in\mathbb{N}}$ satisfies the extension property, then the Fredholm problem is no easier than the L_2-approximation problem for Q_d. So what does it take for Q to satisfy the extension property? The key inequality (10) clearly depends on the weights. For instance, (10) holds whenever

$$\gamma_{d,\mathfrak{u},Q} \le \gamma_{2d,\mathfrak{u},Q} \qquad \text{for all } d \in \mathbb{N}, \mathfrak{u} \subseteq [d].$$

As a particularly simple case, this inequality holds when the weights $\gamma_{d,\mathfrak{u},Q}$ are independent of d, a case that has been well-studied in many papers that have dealt with tractability. So although we cannot say that there is no lack of generality in assuming that the extension property holds, it is certainly not an unwarranted assumption.

So let us assume that Q satisfies the extension property. What can we say about the tractability of the Fredholm problem?

The first result is as follows:

If $m_F > 1$ or $m_Q > 1$, then the Fredholm problem is intractable for the class Λ^{all} (and obviously also for Λ^{std}), no matter how the weights are chosen.

The reason for this is that the $L_2(I^d)$-approximation problem is intractable for $H_{d,m,\gamma}$ whenever $m > 1$, see [22, Theorem 3.1]. This last result may seem somewhat counter-intuitive, since it tells us that increased smoothness (i.e., increasing m) is bad. The reason for this intractability is that $\|\cdot\|_{H_{d,m,\gamma}} = \|\cdot\|_{L_2(I^d)}$ on the m^d-dimensional space $\mathscr{P}_{d.m-1}$ of d-variate polynomials having degree at most $m-1$ in each variable, which implies that

$$e(n, \text{APP}_{H_{d,m,\gamma}}, \Lambda_d) = 1 \qquad \text{for all } n < m^d,$$

and therefore

$$n(\varepsilon, \text{APP}_{H_{d,m,\gamma}}, \Lambda^{\text{all}}) \ge m^d \qquad \text{for all } \varepsilon \in (0,1).$$

Thus in the remainder of this subsection, we shall assume that $m_F = m_Q = 1$, so that

$$F_d = H_{d,1,\gamma_F} \qquad \text{and} \qquad Q_d = H_{d,1,\gamma_Q}.$$

For simplicity, we only look at families γ of *bounded product weights*, which have the form

$$\gamma_{d,\mathfrak{u},X} = \prod_{j\in\mathfrak{u}} \gamma_{d,j,X} \qquad \forall\, \mathfrak{u} \subseteq [d]$$

for a non-negative sequence

$$\gamma_{d,1,X} \ge \gamma_{d,2,X} \ge \cdots \ge \gamma_{d,d,X},$$

for any $d \in \mathbb{N}$. Here $X \in \{F, Q\}$, which indicates that we may use different weights for the space sequences $F = \{F_d\}_{d\in\mathbb{N}}$ and $Q = \{Q_d\}_{d\in\mathbb{N}}$. The boundedness of these product weights means that

$$M := \sup_{d\in\mathbb{N}} \max\{\gamma_{d,1,F}, \gamma_{d,1,Q}\} < \infty.$$

It is easy to see that if

$$\gamma_{d,j,Q} \le \gamma_{2d,j,Q} \qquad \text{for all } \ d \in \mathbb{N}, \ \ j \in [d]$$

then Q satisfies the extension property. In particular, this inequality holds when the weights $\gamma_{d,j}$ do not depend on d.

We first consider Λ^{all}. Since tractability results for the Fredholm problem are tied to those of the approximation problem, we will use the results found in [22].

- Strong polynomial tractability: We know that the problem APP_F is strongly polynomially tractable iff there exists a positive number τ_F such that

$$\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j,F}^{\tau_F} < \infty. \tag{22}$$

Define τ_F^* to be the infimum of τ_F such that (22) holds. Then the strong exponent for APP_F is $\max\{1, 2\tau_F^*\}$. The situation for APP_Q is analogous. From Corollaries 3.1 and 3.2, we see that the Fredholm problem S is strongly polynomially tractable iff both (22) and its analog (with F replaced by Q) hold, in which case the strong exponent for the Fredholm problem is $\max\{1, 2\tau_F^*, 2\tau_Q^*\}$.

- Polynomial tractability: The problem APP_F is polynomially tractable iff there exists a positive number τ_F such that

$$\limsup_{d\to\infty} \frac{1}{\ln d} \sum_{j=1}^{d} \gamma_{d,j,F}^{\tau_F} < \infty. \tag{23}$$

The situation for APP_Q is analogous. From Corollaries 3.1 and 3.2, we see that the Fredholm problem S is polynomially tractable iff both (23) and its analog (with F replaced by Q) hold.

- Quasi-polynomial tractability: If we replace all $\gamma_{d,j,F}$ and $\gamma_{d,j,G}$ by their upper bound M then the approximation problem becomes harder. The latter approximation problem is unweighted with the univariate eigenvalues $\lambda_1 = 1 > \lambda_2$ and $\lambda_j = \mathcal{O}(j^{-2})$. Therefore it is quasi-polynomially tractable (see Section 4.2). This implies that the weighted case is quasi-polynomially tractable for any bounded product weights. Therefore the Fredholm is also quasi-polynomially tractable.

- Weak tractability: Since the Fredholm problem is quasi-polynomially tractable, it is also weakly tractable.

We now turn to the case of standard information Λ^{std}. We will use the results found in [22] for polynomial tractability for the approximation problem, upon which we will base the polynomial tractability results for the Fredholm problem.

- Strong polynomial tractability: The problem APP_F is strongly polynomially tractable iff

$$\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j,F} < \infty. \tag{24}$$

The situation for APP_Q is analogous. From Corollaries 3.1 and 3.2, we see that the Fredholm problem S is strongly polynomially tractable iff both (24) and its analog (with F replaced by Q) hold. When this holds, the strong exponents for all three problems lie in the interval $[1, 4]$.

- Polynomial tractability: The problem APP_F is polynomially tractable iff

$$\limsup_{d\to\infty} \frac{1}{\ln d} \sum_{j=1}^{d} \gamma_{d,j,F} < \infty. \tag{25}$$

The situation for APP_Q is analogous. From Corollaries 3.1 and 3.2, we see that the Fredholm problem S is polynomially tractable iff both (25) and its analog (with F replaced by Q) hold.

At this time, we do not have conditions that are necessary and sufficient for the approximation problem to be quasi-polynomially tractable or weakly tractable for standard information. This means that the same is true for the Fredholm problem.

5 Weighted tensor product spaces

In Section 4.2, we saw that that the Fredholm problem is not polynomially tractable if either F_d or Q_{2d} is from a family of unweighted tensor product spaces. However in Section 4.3, we saw that our problem can be polynomially tractable (or even strongly polynomially tractable) if both F_d and Q_{2d} are from families of weighted Sobolev spaces. This leads us to wonder whether replacing the unweighted tensor product spaces of Section 4.2 by weighted tensor product spaces can render the Fredholm problem polynomially tractable, or maybe even strongly polynomially tractable.

So with the spaces $H_{d,m,\boldsymbol{\gamma}}$ as a guide, we now give the general definition of a *weighted tensor product space*, which captures this idea that different variables or groups of variables can play different roles. In Section 6, we will study a modified interpolatory algorithm for the Fredholm problem, and our analysis of this algorithm will draw heavily on the properties of weighted tensor product spaces.

Our presentation is based on that found in [9, Sect. 5.3], which should be consulted for additional details.

Let $\{\gamma_{d,\mathfrak{u}}\}_{\mathfrak{u}\subseteq[d]}$ be a set of non-negative weights. We assume the following about these weights:

- $\gamma_{d,\emptyset} = 1$, and
- $\gamma_{d,\mathfrak{u}} \le 1$ for all $\mathfrak{u} \subseteq [d]$.
- There is at least one nonempty $\mathfrak{u} \subseteq [d]$ for which $\gamma_{d,\mathfrak{u}} > 0$.

Let H_1 be defined as in Section 4.2. That is, H_1 is an infinite dimensional space with $e_1 \equiv 1 \in H_1$ and $\|e_1\|_{H_1} = 1$. Let

$$\widetilde{H}_1 = \{ f \in H_1 : \langle f, e_1\rangle_{H_1} = 0 \}$$

be the subspace of H_1 of functions orthogonal to $e_1 \equiv 1$. We now define

$$H_{d,\boldsymbol{\gamma}} = \bigoplus_{\mathfrak{u}\subseteq[d]} \widetilde{H}_{1,\mathfrak{u}}, \tag{26}$$

where $\widetilde{H}_{1,\mathfrak{u}} = \widetilde{H}_1^{\otimes|\mathfrak{u}|}$ is the $|\mathfrak{u}|$-fold tensor product of H_1. That is, $v \in H_{d,\boldsymbol{\gamma}}$ has the unique decomposition

$$v(x) = \sum_{\mathfrak{u}\subseteq[d]} v_{\mathfrak{u}}(x_{\mathfrak{u}}) \qquad \forall\, x \in I^d, \tag{27}$$

where

$$v_{\mathfrak{u}} \in \widetilde{H}_{1,\mathfrak{u}} \qquad \forall\, \mathfrak{u} \subseteq [d].$$

Although $H_{d,\boldsymbol{\gamma}}$ can algebraically be identified with a subspace of the space H_d described in Section 4.2, the spaces H_d and $H_{d,\boldsymbol{\gamma}}$ generally have different topologies. The inner product for $H_{d,\boldsymbol{\gamma}}$ is given by

$$\langle v, w\rangle_{H_{d,\boldsymbol{\gamma}}} = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}}^{-1} \langle v_{\mathfrak{u}}, w_{\mathfrak{u}}\rangle_{H_d} \qquad \forall\, v, w \in H_{d,\boldsymbol{\gamma}}. \tag{28}$$

For this to be well-defined, we assume that $v_{\mathfrak{u}} = w_{\mathfrak{u}} = 0$ whenever $\gamma_{d,\mathfrak{u}} = 0$, interpreting $0/0$ as 0. (Compare with (21) in Section 4.3.) The decomposition (27) tells us that we write v as a sum of mutually orthogonal functions, each term $v_{\mathfrak{u}}$ depending only on the variables in $\mathfrak{u}$. The formula (28) tells us that the contribution made by $\|v_{\mathfrak{u}}\|_{H_d}$ to $\|v\|_{H_{d,\gamma}}$ is moderated by the weight $\gamma_{d,\mathfrak{u}}$.

Let

$$e_{\boldsymbol{\alpha}}(x) = \prod_{k=1}^{d} e_{\alpha_k}(x_k) \qquad \forall\, x = (x_1, x_2, \dots, x_d) \in I^d$$

for any multi-index $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_d] \in \mathbb{N}^d$. Note that if $\alpha_k = 1$, then $e_{\alpha_k} \equiv 1$, and so $e_{\boldsymbol{\alpha}}$ does not depend on x_k. Defining

$$\mathfrak{u}(\boldsymbol{\alpha}) = \{\, k \in [d] : \alpha_k \geq 2 \,\},$$

we may write

$$e_{\boldsymbol{\alpha}}(x) = \prod_{k \in \mathfrak{u}(\boldsymbol{\alpha})} e_{\alpha_k}(x_k) \qquad \forall\, x = (x_1, x_2, \dots, x_d) \in I^d.$$

For further details, once again see [9, Sect. 5.3].

Let $W_{d,\boldsymbol{\gamma}} = \mathrm{APP}^*_{H_{d,\gamma}}\, \mathrm{APP}_{H_{d,\gamma}}$. Defining

$$e_{\boldsymbol{\alpha},d,\boldsymbol{\gamma}} = \gamma_{d,\mathfrak{u}(\boldsymbol{\alpha})}^{1/2} e_{\boldsymbol{\alpha}} \qquad \forall\, \boldsymbol{\alpha} \in \mathbb{N}^d,$$

we see that $\{e_{\boldsymbol{\alpha},d,\boldsymbol{\gamma}}\}_{\boldsymbol{\alpha}\in\mathbb{N}^d}$ is an orthonormal basis of $H_{d,\boldsymbol{\gamma}}$, consisting of eigenfunctions of $W_{d,\boldsymbol{\gamma}}$, with

$$W_{d,\boldsymbol{\gamma}} e_{\boldsymbol{\alpha},d,\boldsymbol{\gamma}} = \lambda_{\boldsymbol{\alpha},d,\boldsymbol{\gamma}} e_{\boldsymbol{\alpha},d,\boldsymbol{\gamma}} \qquad \forall\, \boldsymbol{\alpha} \in \mathbb{N}^d,$$

where

$$\lambda_{\boldsymbol{\alpha},d,\boldsymbol{\gamma}} = \gamma_{d,\mathfrak{u}(\boldsymbol{\alpha})} \prod_{k=1}^{d} \lambda_{\alpha_k} \qquad \forall\, \boldsymbol{\alpha} \in \mathbb{N}^d.$$

Note that all eigenvalues $\lambda_{\boldsymbol{\alpha},d,\boldsymbol{\gamma}} \in [0, 1]$ since we assumed that all $\gamma_{d,\mathfrak{u}} \leq 1$ and all $\lambda_j \leq 1$. Furthermore, infinitely many $\lambda_{\boldsymbol{\alpha},d,\boldsymbol{\gamma}}$ are positive. Indeed, since there exists a nonempty $\mathfrak{u}$ for which $\gamma_{d,\mathfrak{u}} > 0$, it is enough to take indices $\boldsymbol{\alpha}$ such that $\mathfrak{u}(\boldsymbol{\alpha}) = \mathfrak{u}$; since $\lambda_{\boldsymbol{\alpha}_k} > 0$ for $k \in [d]$, all the $\lambda_{\boldsymbol{\alpha},d,\boldsymbol{\gamma}}$ are positive. The condition $\mathfrak{u}(\boldsymbol{\alpha}) = \mathfrak{u}$ holds if $\alpha_k \geq 2$ for $k \in \mathfrak{u}$, and $\alpha_k = 1$ for $k \notin \mathfrak{u}$. For a nonempty $\mathfrak{u}$, we have infinitely many such indices $\boldsymbol{\alpha}$, and therefore we have infinitely many positive eigenvalues, as claimed.

In what follows, it will be useful to order the eigenvalues of $W_{d,\boldsymbol{\gamma}}$ in non-increasing order. So we order the multi-indices in $\mathbb{N}^d$ as $\boldsymbol{\alpha}[1], \boldsymbol{\alpha}[2], \dots$, with

$$1 = \lambda_{\boldsymbol{\alpha}[1],d,\boldsymbol{\gamma}} \geq \lambda_{\boldsymbol{\alpha}[2],d,\boldsymbol{\gamma}} \geq \cdots > 0. \tag{29}$$

We stress the last inequality in (29), which holds since infinitely many eigenvalues are positive. This also implies that $\gamma_{d,\mathfrak{u}(\boldsymbol{\alpha}[j])} > 0$.

It will often be useful to write $\lambda_{j,d,\boldsymbol{\gamma}}$ and $e_{j,d,\boldsymbol{\gamma}}$, rather than $\lambda_{\boldsymbol{\alpha}[j],d,\boldsymbol{\gamma}}$ and $e_{\boldsymbol{\alpha}[j],d,\boldsymbol{\gamma}}$, so that

$$W_{d,\boldsymbol{\gamma}} e_{j,d,\boldsymbol{\gamma}} = \lambda_{j,d,\boldsymbol{\gamma}} e_{j,d,\boldsymbol{\gamma}}$$

with

$$1 = \lambda_{1,d,\boldsymbol{\gamma}} \geq \lambda_{2,d,\boldsymbol{\gamma}} \geq \cdots > 0.$$

We shall do so when this causes no confusion.

Remark. A sequence of weighted tensor product space $\{H_{d,\boldsymbol{\gamma}}\}_{d=1,2,\dots}$ defined in this section has the extension property if

$$\gamma_{d,\mathfrak{u}} \leq \gamma_{2d,\mathfrak{u}} \qquad \text{for all } d \in \mathbb{N},\ \mathfrak{u} \subseteq [d]. \quad \square$$

For tensor product spaces, the eigenfunctions $e_{j,2d,\boldsymbol{\gamma}}$ of $W_{2d,\boldsymbol{\gamma}_Q}$ are related to the eigenfunctions $e_{j,d,\boldsymbol{\gamma}}$ of $W_{d,\boldsymbol{\gamma}_Q}$. Indeed, the eigenfunctions of $W_{2d,\boldsymbol{\gamma}_Q}$ have the form

$$e_{j,2d,\boldsymbol{\gamma}_Q} = e_{\boldsymbol{\alpha}[j],2d,\boldsymbol{\gamma}_Q} = \gamma_{2d,\mathrm{u}(\boldsymbol{\alpha}[j]),Q}^{1/2} e_{\boldsymbol{\alpha}[j]},$$

where

$$\boldsymbol{\alpha}[j] = [(\alpha[j])_1, (\alpha[j])_2, \ldots, (\alpha[j])_{2d}] \in \mathbb{N}^{2d}$$

has $2d$ components. Let

$$\boldsymbol{\alpha}_1[j] = [(\alpha[j])_1, (\alpha[j])_2, \ldots, (\alpha[j])_d] \in \mathbb{N}^d$$

and

$$\boldsymbol{\alpha}_2[j] = [(\alpha[j])_{d+1}, (\alpha[j])_{d+2}, \ldots, (\alpha[j])_{2d}] \in \mathbb{N}^d.$$

Since $e_{\boldsymbol{\alpha}[j]} = e_{\boldsymbol{\alpha}_1[j]} \otimes e_{\boldsymbol{\alpha}_1[j]}$ we obtain

$$e_{\boldsymbol{\alpha}[j],2d,\boldsymbol{\gamma}} = \gamma_{2d,\mathrm{u}(\boldsymbol{\alpha}[j])}^{1/2}\, e_{\boldsymbol{\alpha}_1[j]} \otimes e_{\boldsymbol{\alpha}_2[j]},$$

$$e_{\boldsymbol{\alpha}[j],2d,\boldsymbol{\gamma}} = \frac{\gamma_{2d,\mathrm{u}(\boldsymbol{\alpha}[j])}^{1/2}}{\gamma_{d,\mathrm{u}(\boldsymbol{\alpha}_1[j])}^{1/2}\gamma_{d,\mathrm{u}(\boldsymbol{\alpha}_2[j])}^{1/2}}\, e_{\boldsymbol{\alpha}_1[j],d,\boldsymbol{\gamma}} \otimes e_{\boldsymbol{\alpha}_2[j],d,\boldsymbol{\gamma}}.$$

6 Interpolatory Algorithm for Tensor Product Spaces

We now define an *interpolatory algorithm* whose error for the Fredholm problem will be expressed in terms of the L_2-approximation errors for F_d and Q_d as in Lemma 3.1. Then we analyze the implementation cost of this algorithm. As we shall see, the implementation cost will be quite small as long as we use tensor product spaces for F_d and Q_d.

We first specify the spaces as $F_d = H_{d,\boldsymbol{\gamma}_F}$ and $Q_d = H_{d,\boldsymbol{\gamma}_Q}$, where $H_{d,\boldsymbol{\gamma}}$ is defined as in Section 5. This means that $\boldsymbol{\gamma}_F = \{\gamma_{d,\mathrm{u},F}\}$ and $\boldsymbol{\gamma}_Q = \{\gamma_{d,\mathrm{u},Q}\}$ are sequences of weights for the spaces $H_{d,\boldsymbol{\gamma}_F}$ and $H_{d,\boldsymbol{\gamma}_Q}$ satisfying the assumptions of Section 5. Note that the weight sequences $\boldsymbol{\gamma}_F$ and $\boldsymbol{\gamma}_Q$ may be different, or they may be the same. Thus $\{e_{j,d,\gamma_F}\}_{j\in\mathbb{N}}$ is a F_d-orthonormal system, consisting of the eigenfunctions for $W_{d,\boldsymbol{\gamma}_F}$, and $\{e_{j,2d,\gamma_Q}\}_{j\in\mathbb{N}}$ is a Q_{2d}-orthonormal system, consisting of the eigenfunctions for $W_{2d,\boldsymbol{\gamma}_Q}$. In both cases, the corresponding eigenvalues $\lambda_{j,d,\boldsymbol{\gamma}_F}$ and $\lambda_{j,2d,\boldsymbol{\gamma}_Q}$ are ordered.

Let $n(F)$ and $n(Q)$ be two positive integers. The information about f will be given as the first $n(F)$ inner product with respect to $\{e_{j,d,\gamma_F}\}_{j\in\mathbb{N}}$, and the information about q as the first $n(Q)$ inner products with respect to $\{e_{j,2d,\gamma_Q}\}_{j\in\mathbb{N}}$. That is, we use the class Λ^{all}, and for $(f,q) \in BF_d \times Q_{2d}^{\mathrm{res}}$ we compute

$$N_{n(F)}(f) = \left[\langle f, e_{1,d,\boldsymbol{\gamma}_F}\rangle_{H_{d,\boldsymbol{\gamma}_F}}, \langle f, e_{2,d,\boldsymbol{\gamma}_F}\rangle_{H_{d,\boldsymbol{\gamma}_F}}, \ldots, \langle f, e_{n(F),d,\boldsymbol{\gamma}_F}\rangle_{H_{d,\boldsymbol{\gamma}_F}}\right]^{\mathsf{T}}$$

$$N_{n(Q)}(q) = \left[\langle q, e_{1,2d,\boldsymbol{\gamma}_Q}\rangle_{H_{2d,\boldsymbol{\gamma}_Q}}, \langle q, e_{2,2d,\boldsymbol{\gamma}_Q}\rangle_{H_{2d,\boldsymbol{\gamma}_Q}}, \ldots, \langle q, e_{n(Q),2d,\boldsymbol{\gamma}_Q}\rangle_{H_{2d,\boldsymbol{\gamma}_Q}}\right]^{\mathsf{T}}.$$

Define the orthogonal projector operators

$$P_{n(F),d,\boldsymbol{\gamma}_F} = \sum_{j=1}^{n(F)} \langle \cdot, e_{j,d,\boldsymbol{\gamma}_F}\rangle_{H_{d,\gamma_F}} e_{j,d,\boldsymbol{\gamma}_F}$$

and

$$P_{n(Q),d,\boldsymbol{\gamma}_F} = \sum_{j=1}^{n(Q)} \langle \cdot, e_{j,2d,\boldsymbol{\gamma}_Q}\rangle_{H_{2d,\gamma_Q}} e_{j,2d,\boldsymbol{\gamma}_Q}.$$

Knowing $N_{n(F)}(f)$ and $N_{n(Q)}(q)$, we know

$$\widetilde{f} = P_{n(F),d,\gamma_F} f \qquad \text{and} \qquad \widetilde{q} = P_{n(Q),2d,\gamma_Q} q.$$

Observe that $(\widetilde{f}, \widetilde{q}) \in BF_d \times Q_{2d}^{\mathrm{res}}$. Furthermore, $(\widetilde{f}, \widetilde{q})$ *interpolate* the data, i.e,

$$N_{n(F)}(\widetilde{f}) = N_{n(F)}(f) \qquad \text{and} \qquad N_{n(Q)}(\widetilde{q}) = N_{n(Q)}(q).$$

We define the *interpolatory algorithm*

$$A^{\mathrm{INT}}_{n(F),n(Q)}(f,q) = S_d(\widetilde{f}, \widetilde{q}) \qquad \text{for all } (f,q) \in BF_d \times Q_{2d}^{\mathrm{res}}$$

as the exact solution of the Fredholm problem for $(\widetilde{f}, \widetilde{q})$. Lemma 3.1 gives an error bound for $A^{\mathrm{INT}}_{n(F),n(Q)}$ in terms of the errors of the L_2-approximation problems for F_d and Q_{2d}. As in the proof of Proposition 3.3, we can choose $n(F)$ and $n(Q)$ to make the approximation errors for F_d and Q_{2d} be at most $(1-M_1)\varepsilon/2$ and $(1-M_1)^2\varepsilon/(2M_1)$, respectively; this guarantees that the error of $A^{\mathrm{INT}}_{n(F),n(Q)}$ for the Fredholm problem is at most ε.

Our next step is to reduce the computation of $\widetilde{u} = A^{\mathrm{INT}}_{n(F),n(Q0}(f,q)$ to the solution of a linear system of equations. To do this, we will use the notation and results of Section 5, suitably modified to take account of the fact that we are dealing with two sequences of weights. Now $\alpha_F[j]$ is the d-component multi-index giving the jth-largest eigenvalue of $W_{d,\boldsymbol{\gamma}_F}$ and $\alpha_Q[j]$ is the $2d$-component multi-index giving the jth-largest eigenvalue of $W_{2d,\boldsymbol{\gamma}_Q}$. Thus

$$e_{j,d,\boldsymbol{\gamma}_F} = e_{\boldsymbol{\alpha}_F[j],d,\boldsymbol{\gamma}_F} = \gamma^{1/2}_{d,\mathrm{u}(\boldsymbol{\alpha}_F[j]),F}\, e_{\boldsymbol{\alpha}_F[j]}$$

and

$$e_{j,2d,\boldsymbol{\gamma}_Q} = e_{\boldsymbol{\alpha}_Q[j],2d,\boldsymbol{\gamma}_Q} = \gamma^{1/2}_{2d,\mathrm{u}(\boldsymbol{\alpha}_Q[j]),Q}\, e_{\boldsymbol{\alpha}_{1,Q}[j]} \otimes e_{\boldsymbol{\alpha}_{2,Q}[j]}.$$

Here, $\boldsymbol{\alpha}_{1,Q}[j]$ denotes the first d indices of $\boldsymbol{\alpha}_Q[j]$, and $\boldsymbol{\alpha}_{2,Q}[j]$ denotes the remaining indices of $\boldsymbol{\alpha}_Q[j]$, as at the end of Section 5.

We have

$$\langle e_{\boldsymbol{\alpha}}, e_{\boldsymbol{\beta}} \rangle_{H_d} = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}} \qquad \text{and} \qquad \langle e_{\boldsymbol{\alpha}}, e_{\boldsymbol{\beta}} \rangle_{L_2(I^d)} = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}\, \lambda_{\boldsymbol{\alpha}},$$

and so the functions $\{e_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathbb{N}^d}$ are orthogonal in the unweighted space H_d, as well as in the space $L_2(I^d)$. Since $A^{\mathrm{INT}}_{n(F),n(Q)}$ is an interpolatory algorithm, we see that $\widetilde{u}$ satisfies the equation

$$\widetilde{u} = \int_{I^d} \widetilde{q}(\cdot, y)\, \widetilde{u}(y)\, dy + \widetilde{f},$$

which can be rewritten as

$$\widetilde{u} = \sum_{j=1}^{n(Q)} \zeta_j \langle e_{\boldsymbol{\alpha}_{2,Q}[j]}, \widetilde{u} \rangle_{L_2(I^d)} e_{\boldsymbol{\alpha}_{1,Q}[j]} + \sum_{j=1}^{n(F)} \theta_j e_{\boldsymbol{\alpha}_F[j]}, \tag{30}$$

with

$$\zeta_j = \langle q, e_{j,2d,\boldsymbol{\gamma}_Q} \rangle_{H_{2d,\boldsymbol{\gamma}_Q}} \gamma^{1/2}_{2d,\mathrm{u}(\boldsymbol{\alpha}_Q[j]),Q} \qquad \text{and} \qquad \theta_j = \langle f, e_{j,d,\boldsymbol{\gamma}_F} \rangle_{H_{d,\boldsymbol{\gamma}_F}} \gamma^{1/2}_{d,\mathrm{u}(\boldsymbol{\alpha}_F[j]),F}.$$

This proves that

$$\widetilde{u} \in E_{n(F),n(Q)} = \mathrm{span}\left\{ e_{\boldsymbol{\alpha}_F[1]}, e_{\boldsymbol{\alpha}_F[2]}, \dots, e_{\boldsymbol{\alpha}_F[n(F)]}, e_{\boldsymbol{\alpha}_{1,Q}[1]}, e_{\boldsymbol{\alpha}_{1,Q}[2]}, \dots, e_{\boldsymbol{\alpha}_{1,Q}[n_q]} \right\}.$$

Note that the elements $e_{\alpha_F[j]}$ are orthogonal for $j = 1, 2, \dots, n(F)$. Moreover, the elements $e_{\alpha_{1,Q}[j]}$ are orthogonal for different $\alpha_{1,Q}[j]$. However, two kinds of "overlap" are possible:

- We might have $\boldsymbol{\alpha}_F[j] = \boldsymbol{\alpha}_{1,Q}[j']$ for some $j \in \{1, 2, \ldots, n(F)\}$ and $j' \in \{1, 2, \ldots, n(Q)\}$.
- We might have $\boldsymbol{\alpha}_{1,Q}[j] = \boldsymbol{\alpha}_{1,Q}[j']$ for some $j, j' \in \{1, 2, \ldots, n(F)\}$.

Therefore

$$m := \dim E_{n(F),n(Q)} \in \{n(F), n(F)+1, \ldots, n(F)+n(Q)\}.$$

We remove all redundant $e_{\boldsymbol{\alpha}_{1,Q}[j]}$, as well as all $e_{\boldsymbol{\alpha}_{1,Q}[j]}$ that belong to $\operatorname{span}\{e_{\boldsymbol{\alpha}_F[1]}, e_{\boldsymbol{\alpha}_F[2]}, \ldots, e_{\boldsymbol{\alpha}_F[n(F)]}\}$, calling the remaining elements $e_{\boldsymbol{\alpha}_{1,Q}[l_1]}, e_{\boldsymbol{\alpha}_{1,Q}[l_2]}, \ldots, e_{\boldsymbol{\alpha}_{1,Q}[l_{m-n(F)}]}$. Therefore

$$E_{n(F),n(Q)} = \operatorname{span}\{z_1, z_2, \ldots, z_m\},$$

where

$$z_j = \begin{cases} e_{\boldsymbol{\alpha}_F[j]} & \text{for } j \in \{1, 2, \ldots, n(F)\}, \\ e_{\boldsymbol{\alpha}_{1,Q}[l_{j-n(F)}]} & \text{for } j \in \{n(F)+1, n(F)+2, \ldots, m\}. \end{cases}$$

The elements $z_1 \ldots, z_m$ are $L_2(I^d)$-orthogonal, i.e., $\langle z_j, z_k \rangle_{L_2(I^d)} = 0$ for $j \neq k$, with

$$\|z_j\|_{L_2(I^d)} = \begin{cases} \lambda^{1/2}_{\boldsymbol{\alpha}_F[j],d,\boldsymbol{\gamma}_F} & \text{for } j \in \{1, 2, \ldots, n(F)\}, \\ \lambda^{1/2}_{\boldsymbol{\alpha}_{1,Q}[l_j-n(F)],d,\boldsymbol{\gamma}_Q} & \text{for } j \in \{n(F)+1, n(F)+2, \ldots, m\}. \end{cases}$$

We know that

$$\widetilde{u} = \sum_{k=1}^{m} \upsilon_k \, z_k$$

for some real coefficients $\upsilon_1, \upsilon_2, \ldots \upsilon_m$. From (30) we conclude that

$$\widetilde{u} = \sum_{k=1}^{m} \upsilon_k \left(\sum_{j=1}^{n(Q)} \zeta_j \langle e_{\boldsymbol{\alpha}_{2,Q}[j]}, z_k \rangle_{L_2(I^d)} \, e_{\boldsymbol{\alpha}_{1,Q}[j]} \right) + \sum_{j=1}^{n(F)} \theta_j e_{\boldsymbol{\alpha}_F[j]}.$$

This leads to the system

$$(\mathbf{I} - \mathbf{K})\mathbf{u} = \mathbf{b} \tag{31}$$

of linear equations, where $\mathbf{I}$ denotes the $m \times m$ identity matrix and the $m \times m$ matrix $\mathbf{K} = [\kappa_{i,k}]_{1 \le i,k \le m}$ is given by

$$\kappa_{i,k} = \sum_{j=1}^{n(Q)} \zeta_j \frac{\langle e_{\boldsymbol{\alpha}_{2,Q}[j]}, z_k \rangle_{L_2(I^d)} \, \langle e_{\boldsymbol{\alpha}_{1,Q}[j]}, z_i \rangle_{L_2(I^d)}}{\langle z_i, z_i \rangle_{L_2(I^d)}},$$

with

$$\mathbf{b} = \left[\frac{\theta_1}{\langle z_1, z_1 \rangle_{L_2(I^d)}}, \frac{\theta_2}{\langle z_2, z_2 \rangle_{L_2(I^d)}}, \ldots, \frac{\theta_{n(F)}}{\langle z_{n(F)}, z_{n(F)} \rangle_{L_2(I^d)}}, 0, 0, \ldots, 0 \right]^{\mathsf{T}} \in \mathbb{R}^m$$

and

$$\mathbf{u} = [\upsilon_1, \upsilon_2, \ldots, \upsilon_{n(F)}, \upsilon_{n(F)+1}, \ldots, \upsilon_m]^{\mathsf{T}} \in \mathbb{R}^m.$$

We can now look at some important properties of $\mathbf{K}$, including the structure of $\mathbf{K}$ and the invertibility of $\mathbf{I} - \mathbf{K}$.

Lemma 6.1. *Define*

$$\mathscr{I} = \{\, \boldsymbol{\alpha}_Q[j] = (\boldsymbol{\alpha}_{1,Q}[j], \boldsymbol{\alpha}_{2,Q}[j]) \in \mathbb{N}^{2d} : 1 \le j \le n(Q) \,\}.$$

1. *We have*

$$\kappa_{i,k} = \begin{cases} \zeta_j \lambda_{\boldsymbol{\alpha}_{2,Q}[j]} & \textit{if } (i,k) = (\boldsymbol{\alpha}_{1,Q}[j], \boldsymbol{\alpha}_{2,Q}[j]) \textit{ for some } j \in \{1, 2, \dots, n(Q)\} \\ 0 & \textit{if } (i,k) \notin \mathscr{I}, \end{cases}$$

   and so the matrix $\mathbf{K}$ *has at most* $n(Q)$ *non-zero elements.*

2. $\|\mathbf{K}\|_{\mathrm{Lin}[\ell_2(\mathbb{R}^m)]} \le M_1 < 1$.

3. *The matrix* $\mathbf{I} - \mathbf{K}$ *is invertible, with*

$$\|(\mathbf{I} - \mathbf{K})^{-1}\|_{\mathrm{Lin}[\ell_2(\mathbb{R}^m)]} \le \frac{1}{1 - M_1}.$$

Proof. For part 1, note that the coefficient $\kappa_{i,k}$ may be nonzero only if there exists an integer $j \in [1, n(Q)]$ such that

$$z_i = e_{\boldsymbol{\alpha}_{1,Q}[j]} \qquad \text{and} \qquad z_k = e_{\boldsymbol{\alpha}_{2,Q}[j]},$$

that is, when $(i,k) \in \mathscr{I}$. In this case, there is at most one nonzero term in the sum defining $\kappa_{i,k}$, since $\mathscr{I}$ consists of distinct elements. Then

$$\kappa_{i,k} = \zeta_j \|e_{\boldsymbol{\alpha}_{2,Q}[j]}\|^2_{L_2(I^d)} = \zeta_j \lambda_{\boldsymbol{\alpha}_{2,Q}[j]} = \langle q, e_{j,2d,\boldsymbol{\gamma}_Q} \rangle_{H_{2d,\boldsymbol{\gamma}_Q}} \gamma^{1/2}_{2d,\mathrm{u}(\boldsymbol{\alpha}_Q[j]),Q} \lambda_{\boldsymbol{\alpha}_{2,Q}[j]}.$$

Obviously, if $(i,k) \notin \mathscr{I}$ then $\kappa_{i,k} = 0$. Hence, the number of nonzero coefficients of the matrix $\mathbf{K}$ is at most $|\mathscr{I}| = n(Q)$, as claimed in part 1.

To see that part 2 holds, we estimate $\|\mathbf{K}\|^2_{\mathrm{Lin}[\ell_2(\mathbb{R}^{n(F)})]}$ by the square of the Frobenius norm $\sum_{i,k=1}^{n(Q)} \kappa^2_{i,k}$ and then apply part 1. Recall that L_2-approximation is properly scaled for Q, i.e., that $\lambda_{\boldsymbol{\alpha}_{2,Q}[j]} \le 1$ and $\gamma_{2d,\mathrm{u}(\boldsymbol{\alpha}),Q} \le 1$ for all eigenvalues and weights. Thus we have

$$\begin{aligned} \|\mathbf{K}\|^2_{\mathrm{Lin}[\ell_2(\mathbb{R}^{n(F)})]} &\le \sum_{i,k=1}^{n(Q)} \kappa^2_{i,k} \le \sum_{j=1}^{n(Q)} \zeta_j^2 \lambda^2_{\boldsymbol{\alpha}_{2,Q}[j]} = \sum_{j=1}^{n(Q)} \langle q, e_{j,2d,\boldsymbol{\gamma}_Q} \rangle^2_{H_{2d,\boldsymbol{\gamma}_Q}} \gamma_{2d,\mathrm{u}(\boldsymbol{\alpha}_Q[j]),Q} \lambda^2_{\boldsymbol{\alpha}_{2,Q}[j]} \\ &\le \sum_{j=1}^{n(Q)} \langle q, e_{j,2d,\boldsymbol{\gamma}_Q} \rangle^2_{H_{2d,\boldsymbol{\gamma}_Q}} = \|P_{n(Q),2d,\boldsymbol{\gamma}_Q} q\|^2_{H_{2d,\boldsymbol{\gamma}_q}} \le \|q\|^2_{Q_{2d}} \le M_1^2 < 1, \end{aligned}$$

which proves part 2. Part 3 follows immediately from part 2. ☐

We now discuss the implementation of the interpolatory algorithm $A^{\mathrm{INT}}_{n(F),n(Q)}$, which is equivalent to solving the linear equation $(\mathbf{I} - \mathbf{K})\mathbf{u} = \mathbf{b}$. Note that the $m \times m$ matrix $\mathbf{K}$ is sparse, in the sense that it has at most $n(Q)$ nonzero elements; moreover, its norm is at most $M_1 < 1$, independent of the size of m. Therefore, it seems natural to approximate the solution $\mathbf{u}$ via the simple fixed-point iteration

$$\begin{aligned} \mathbf{u}^{(\ell+1)} &= \mathbf{K}\mathbf{u}^{(\ell)} + \mathbf{b} \qquad (0 \le \ell < r), \\ \mathbf{u}^{(0)} &= \mathbf{0}. \end{aligned} \tag{32}$$

Letting

$$\mathbf{u}^{(r)} = [\upsilon_1^{(r)}, \upsilon_2^{(r)}, \dots, \upsilon_m^{(r)}]^{\mathsf{T}},$$

we shall write

$$u^{(r)}_{n(F),n(Q)} = \sum_{k=1}^{m} \upsilon_k^{(r)} z_k$$

for our r-step fixed-point approximation to the exact solution

$$\tilde{u} = A^{\text{INT}}_{n(F),n(Q)}(f,q) = \sum_{k=1}^{m} \upsilon_k z_k.$$

Let us write

$$u^{(r)}_{n(F),n(Q)} = A^{\text{INT}}_{n(F),n(Q),r}(f,q),$$

calling $A^{\text{INT}}_{n(F),n(Q),r}$ the *modified interpolatory algorithm*.

We now analyze the cost of computing $\widetilde{u} = A^{\text{INT}}_{n(F),n(Q)}(f,q)$. How much do we lose when going from the interpolatory algorithm to the modified interpolatory algorithm? The answer is, "not much," if the parameter r is properly defined. Let $\text{cost}(A)$ denote the overall cost of an algorithm A for approximating the Fredholm problem, including the cost of both information and combinatory operations. We shall make the usual assumption, commonly made in information-based complexity theory, that arithmetic operations have unit cost and that one information operation of f and q have a fixed cost $\mathbf{c}_d \geq 1$. Now let

$$\text{cost}(\varepsilon, A^{\text{INT}}_{\varepsilon,d}, \Lambda^{\text{all}}_{d,2d}) = \inf\left\{ \text{cost}(A^{\text{INT}}_{n_f,n(Q)}) \;:\; e\left(A^{\text{INT}}_{n_f,n(Q),}, S_d, \Lambda^{\text{all}}_{d,2d}\right) \leq \varepsilon \right\}$$

and

$$\text{cost}(\varepsilon, A^{\text{INT-MOD}}_{\varepsilon,d}, \Lambda^{\text{all}}_{d,2d}) = \inf\left\{ \text{cost}(A^{\text{INT}}_{n_f,n(Q),r}) \;:\; e\left(A^{\text{INT}}_{n_f,n(Q),r}, S_d, \Lambda^{\text{all}}_{d,2d}\right) \leq \varepsilon \right\}$$

respectively denote the minimal cost of using the interpolatory and modified interpolatory algorithms to find an ε-approximation of the Fredholm problem. That is, we minimize the cost by choosing proper parameters $n(F), n(Q)$ and r of the modified interpolatory algorithm, and the parameters $n(F)$ and $n(Q)$ of the interpolatory algorithm.

Proposition 6.1.

$$\text{cost}(\varepsilon, A^{\text{INT-MOD}}_{\varepsilon,d}, \Lambda^{\text{all}}_{d,2d}) = \mathbf{c}_d \cdot \Theta\left(n\left(\tfrac{1}{2}\varepsilon, A^{\text{INT}}_{\varepsilon,d}, \Lambda^{\text{all}}_{d,2d}\right) \ln\left(\frac{1}{\varepsilon}\right)\right),$$

where the Θ-factor is independent of d and ε. Hence if

$$n\left(\tfrac{1}{2}\varepsilon, A^{\text{INT}}_{\varepsilon,d}, \Lambda^{\text{all}}_{d,2d}\right) = \mathscr{O}\left(n(\varepsilon, A^{\text{INT}}_{\varepsilon,d}, \Lambda^{\text{all}}_{d,2d})\right) \tag{33}$$

with $\mathscr{O}$-factor independent of d and ε, then

$$\text{cost}(\varepsilon, A^{\text{INT-MOD}}_{\varepsilon,d}, \Lambda^{\text{all}}_{d,2d}) = \mathbf{c}_d \cdot \Theta\left(n\left(\varepsilon, A^{\text{INT}}_{\varepsilon,d}, \Lambda^{\text{all}}_{d,2d}\right) \ln\left(\frac{1}{\varepsilon}\right)\right).$$

Proof. Recall that $\mathbf{K}$ has $n(Q)$ non-zero elements, see Lemma 6.1. Hence each iteration of (32) can be done in $\Theta\big(n(F)+n(Q)\big)$ arithmetic additions and multiplications. Thus the total number of arithmetic operations needed to compute $u^{(r)}_{n(F),n(Q)}$ will be $\Theta\big((n(F)+n(Q))r\big)$.

For a given value of $\varepsilon \in (0,1)$, let us choose $n(F)$ and $n(Q)$ so that the solution $\tilde{u}$ of the interpolatory algorithm satisfies

$$\|u - \tilde{u}\|_{L_2(I^d)} \leq \tfrac{1}{2}\varepsilon.$$

Obviously, it is enough to choose r such that

$$\|\tilde{u} - u^{(r)}_{n(F),n(Q)}\|_{L_2(I^d)} \le \tfrac{1}{2}\varepsilon, \tag{34}$$

and then our approximation $u^{(r)}_{n(F),n(Q)} \in L_2(I^d)$ will satisfy

$$\|u - u^{(r)}_{n(F),n(Q)}\|_{L_2(I^d)} \le \varepsilon, \tag{35}$$

as required.

So let's analyze the convergence of the fixed-point iteration (32). From Lemma 6.1, we know that

$$\|\mathbf{K}\|_{\mathrm{Lin}[\ell_2(\mathbb{R}^m)]} \le M_1 < 1 \qquad \text{so that} \qquad \|(\mathbf{I} - \mathbf{K})^{-1}\|_{\mathrm{Lin}[\ell_2(\mathbb{R}^m)]} \le \frac{1}{1 - M_1}.$$

Each iteration of (32) reduces the error by a factor of M_1, i.e.,

$$\|\mathbf{u} - \mathbf{u}^{(\ell+1)}\|_{\ell_2(\mathbb{R}^m)} \le M_1 \|\mathbf{u} - \mathbf{u}^{(\ell)}\|_{\ell_2(\mathbb{R}^m)} \qquad (0 \le \ell < r),$$

and so

$$\|\mathbf{u} - \mathbf{u}^{(r)}\|_{\ell_2(\mathbb{R}^m)} \le M_1^r \|\mathbf{u}\|_{\ell_2(\mathbb{R}^m)} = M_1^r \|(\mathbf{I} - \mathbf{K})^{-1}\mathbf{b}\|_{\ell_2(\mathbb{R}^m)} \le \frac{M_1^r}{1 - M_1} \|\mathbf{b}\|_{\ell_2(\mathbb{R}^m)}.$$

Finally, since $f \in BF_d$, we have

$$\|\mathbf{b}\|^2_{\ell_2(\mathbb{R}^n)} = \sum_{j=1}^{n(F)} \langle f, e_{j,d,\gamma_F}\rangle^2_{F_d} \gamma_{d,\mathfrak{u}(\boldsymbol{\alpha}_F[j]),F} \le \sum_{j=1}^{n(F)} \langle f, e_{j,d,\gamma_F}\rangle^2_{F_d} = \|P_{n(F),d,\boldsymbol{\gamma}_F} q\|_{F_d} \le \|f\|^2_{F_d} \le 1,$$

and thus the previous inequality becomes

$$\|\mathbf{u} - \mathbf{u}^{(r)}\|_{\ell_2(\mathbb{R}^m} \le \frac{M_1^r}{1 - M_1}.$$

Taking

$$r = \left\lceil \frac{\ln\bigl(2/(1 - M_1)\bigr) + \ln 1/\varepsilon}{\ln 1/M_1} \right\rceil = \Theta\left(\ln \frac{1}{\varepsilon}\right), \tag{36}$$

we thus have

$$\|\mathbf{u} - \mathbf{u}^{(r)}\|_{\ell_2(\mathbb{R}^m} \le \tfrac{1}{2}\varepsilon. \tag{37}$$

We now claim that with r given by (36), we have (34). Indeed, note that since the $L_2(I^d)$ approximation problem is properly scaled over F_d and over Q_d, we have $\lambda_{\boldsymbol{\alpha}_F[j],d,\gamma_F}, \lambda_{\boldsymbol{\alpha}_{1,Q}[l_j - n(F)],d,\gamma_Q} \le 1$ for all $j \in \mathbb{N}$. Then

$$\begin{aligned}
\|\tilde{u} - u^{(r)}_{n(F),n(Q)}\|^2_{L_2(I^d)} &= \sum_{j=1}^{m} (\upsilon_j - \upsilon_j^{(r)})^2 \|z_j\|^2_{L_2(I^d)} \\
&= \sum_{j=1}^{n(F)} (\upsilon_j - \upsilon_j^{(r)})^2 \lambda_{\boldsymbol{\alpha}_F[j],d,\gamma_F} + \sum_{j=n(F)+1}^{m} (\upsilon_j - \upsilon_j^{(r)})^2 \lambda_{\boldsymbol{\alpha}_{1,Q}[l_j - n(F)],d,\gamma_Q} \\
&\le \sum_{j=1}^{m} (\upsilon_j - \upsilon_j^{(r)})^2 = \|\mathbf{u} - \mathbf{u}^{(r)}\|_{\ell_2(\mathbb{R}^n)},
\end{aligned}$$

and so

$$\|\tilde{u} - u^{(r)}_{n(F),n(Q)}\|_{L_2(I^d)} \le \|\mathbf{u} - \mathbf{u}^{(r)}\|_{\ell_2(\mathbb{R}^m} \le \tfrac{1}{2}\varepsilon,$$

establishing (34), as claimed.

Since (34) holds, we have our desired result (35). Hence we have computed an ε-approximation with information cost $\Theta(\mathbf{c}_d(n(F) + n(Q)))$ and combinatory cost $\Theta\big([n(F) + n(Q)] \ln(1/\varepsilon)\big)$, and so the result follows. □

Using Proposition 6.1, along with the results in Section 3, we see that when (33) holds, the modified interpolatory algorithm is within a logarithmic factor of being optimal. Such is the case when the Fredholm problem (or, alternatively, the L_2-approximation problems APP_F and APP_Q) is strongly polynomially tractable or polynomially tractable. Obviously, the extra factor $\ln(1/\varepsilon)$ does not change the exponents of strong polynomial or polynomial tractability.

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