Generalized Tractability for Multivariate Problems

Part II: Linear Tensor Product Problems, Linear Information, and Unrestricted Tractability

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Abstract

We continue the study of generalized tractability initiated in our previous paper "Generalized tractability for multivariate problems, Part I: Linear tensor product problems and linear information", J. Complexity, 23, 262-295 (2007). We study linear tensor product problems for which we can compute linear information which is given by arbitrary continuous linear functionals. We want to approximate an operator S_d given as the d-fold tensor product of a compact linear operator S_1 for $d = 1, 2, \ldots,$ with $\|S_1\| = 1$ and S_1 has at least two positive singular values.

Let $n(\varepsilon, S_d)$ be the minimal number of information evaluations needed to approximate S_d to within $\varepsilon \in [0, 1]$. We study *generalized tractability* by verifying when $n(\varepsilon, S_d)$ can be bounded by a multiple of a power of $T(\varepsilon^{-1}, d)$ for all $(\varepsilon^{-1}, d) \in \Omega \subseteq [1, \infty) \times \mathbb{N}$. Here, T is a *tractability* function which is non-decreasing in both variables and grows slower than exponentially to infinity. We study the *exponent of tractability* which is the smallest power of $T(\varepsilon^{-1}, d)$ whose multiple bounds $n(\varepsilon, S_d)$. We also study *weak tractability*, i.e., when $\lim_{\varepsilon^{-1}+d\to\infty, (\varepsilon^{-1},d)\in\Omega} \ln\, n(\varepsilon, S_d)/(\varepsilon^{-1} + d) = 0$.

In our previous paper, we studied generalized tractability for proper subsets Ω of $[1,\infty)\times\mathbb{N}$, whereas in this paper we take the unrestricted domain $\Omega^{\rm unr}=[1,\infty)\times\mathbb{N}$.

We consider the three cases for which we have only finitely many positive singular values of S_1, or they decay exponentially or polynomially fast. Weak tractability holds for these three cases, and for all linear tensor product problems for which the singular values of S_1 decay slightly faster that logarithmically. We provide necessary and sufficient conditions on the function T such that generalized tractability holds. These conditions are obtained in terms of the singular values of S_1 and mostly limiting properties of T. The tractability conditions tell us how fast T must go to infinity. It is known that T must go to infinity faster than polynomially. We show that generalized tractability is obtained for $T(x,y)=x^{1+\ln\,y}$. We also study tractability functions T of product form, $T(x,y)=f_1(x)f_2(x)$. Assume that $a_i=\liminf_{x\to\infty}(\ln\,\ln f_i(x))/(\ln\,\ln\,x)$ is finite for $i=1,2$. Then generalized tractability takes place iff

$$a_i>1\ \text{ and }\ (a_1-1)(a_2-1)\ge 1,$$

and if $(a_1-1)(a_2-1)=1$ then we need to assume one more condition given in the paper. If $(a_1-1)(a_2-1)>1$ then the exponent of tractability is zero, and if $(a_1-1)(a_2-1)=1$ then the exponent of tractability is finite. It is interesting to add that for T being of the product form, the tractability conditions as well as the exponent of tractability depend only on the second singular eigenvalue of S_1 and they do *not* depend on the rate of their decay.

Finally, we compare the results obtained in this paper for the unrestricted domain $\Omega^{\rm unr}$ with the results from our previous paper obtained for the restricted domain $\Omega^{\rm res}=[1,\infty)\times\{1,2,\dots,d^*\}\cup[1,\varepsilon_0^{-1})\times\mathbb{N}$ with $d^*\ge 1$ and $\varepsilon_0\in(0,1)$. In general, the tractability results are quite different. We may have generalized tractability for the restricted domain and no generalized tractability for the unrestricted domain which is the case, for instance, for polynomial tractability $T(x,y)=xy$. We may also have generalized tractability for both domains with different or with the same exponents of tractability.

1 Introduction

Tractability of multivariate problems has been extensively studied in information-based complexity and the recent account of the tractability research can be found in the forthcoming book [3]. Tractability is the study of approximating operators S_d defined on spaces of functions with k_d variables with k_d proportional to d. Problems with huge d occur in many applications, see [5]. We approximate S_d by computing linear information which is given by finitely many, say n, continuous linear functionals, and the error of an algorithm is defined in the worst case setting. Before tractability study, the errors of algorithms were studied as functions of n and the main point was to find the best possible rate of convergence as n tends to infinity. For large d, the errors of algorithms crucially depend also on d, and for some problems this dependence is *exponential* in d.

Let $n(\varepsilon,S_d)$ denote the information complexity of S_d which is the minimal number of continuous linear functionals needed to approximate S_d to within ε. The main point of tractability is to check whether $n(\varepsilon, S_d)$ does *not* depend exponentially on ε^{-1} and d. Since there are different ways to measure the lack of exponential behavior, we have different types of tractability. The first type of tractability is *polynomial tractability* which has been extensively studied in many papers. In this case we want to verify whether $n(\varepsilon, S_d)$ can be bounded by a multiple of powers of ε^{-1} and d for all $(\varepsilon^{-1}, d) \in [1,\infty) \times \mathbb{N}$. There are many positive and negative results for polynomial tractability. Usually, positive results are for problems for which the successive variables or groups of variables of large cardinality play a diminishing role, and negative results are for problems for which all variables and groups of variables play the same role. The primary example leading to negative results is approximation of linear tensor product problems. In this case, S_d is a d-fold tensor product of a compact linear operator S_1, where S_1 is defined between Hilbert spaces, $\|S_1\| = 1$ and S_1 has at least two positive singular values. Let $\{\sqrt{\lambda_j}\}$ denote the sequence of the ordered singular values of S_1, $0 < \lambda_2 \le \lambda_1 = 1$. It is well known, see [4], that the information complexity of S_d is

$$n(\varepsilon, S_d) \;=\; |\{\, (i_1, i_2, \dots, i_d) \in \mathbb{N}^d \mid \lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_d} \;>\; \varepsilon^2 \,\}|.$$

Clearly, if $\lambda_2 = 1$ then $n(\varepsilon, d) \ge 2^d$ for all $\varepsilon < 1$, and we have exponential dependence on d causing intractability of the problem. That is why we need to assume that $\lambda_2 < 1$. Still, as long as λ_2 is positive, $n(\varepsilon, d)$ goes faster to infinity that any power of d, see [6], and that is why polynomial tractability does not hold for linear tensor product problems.

In [1], we propose to study *generalized tractability* by verifying whether $n(\varepsilon, S_d)$ can be bounded by a multiple of a power of $T(\varepsilon, d)$ for all $(\varepsilon^{-1}, d) \in \Omega \subseteq [1,\infty) \times \mathbb{N}$. Here T is a *tractability* function which means that $T : [1,\infty)^2 \to [1,\infty)$ is non-decreasing in both variables and grows slower than exponentially to infinity, i.e.,

$$\lim_{x+y\to\infty} \frac{\ln\, T(x,y)}{x+y} \;=\; 0.$$

The set Ω is called *tractability domain*, and can be a proper subset of $[1,\infty) \times \mathbb{N}$ but at least one of the parameters ε^{-1} or d is allowed to go to infinity. The *exponent of tractability* is defined as the smallest (or more precisely as the infimum) power of $T(\varepsilon^{-1}, d)$ whose multiple bounds $n(\varepsilon, S_d)$. There is also the notion of *weak tractability* when

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln\, n(\varepsilon, S_d)}{\varepsilon^{-1}+d} \;=\; 0,$$

see [2, 3], and it is a necessary condition on the lack of exponential behavior of $n(\varepsilon, S_d)$.

Of course, the hope is that by taking reasonable restricted domains Ω or by allowing tractability functions T that tend to infinity faster than polynomially, we may enlarge the class of tractable problems including linear tensor product problems. Indeed, this is the case. In [1] we showed that polynomial tractability of linear tensor product problems holds if we assume that the singular values tend to zero polynomially fast, and we take the restricted tractability domain

$$\Omega \;=\; \Omega^{\mathrm{res}} \;:=\; [1,\infty) \times \{1, 2, \dots, d^*\} \;\cup\; [1, \varepsilon_0^{-1}) \times \mathbb{N}$$

with $d^* \geq 1$ and $\varepsilon_0 \in (0, 1)$.

In this paper, we study the second option and we take the unrestricted domain

$$\Omega \,=\, \Omega^{\rm unr} \,=\, [1, \infty) \,\times\, \mathbb{N},$$

but we allow tractability functions T which go to infinity faster than polynomially.

We study linear tensor product problems for three cases depending on the behavior of the singular values of S_1. In the first case we assume that only finitely many of the singular values are positive, in the second case we assume that they decay exponentially fast, and in the third case that they decay polynomially fast.

For each of these three cases, we have weak tractability. In fact, weak tractability holds if the singular values behave as $o((\ln(j)\,\ln(\ln(j)))^{-1})$ and it is also "almost" a necessary condition.

We provide necessary and sufficient conditions on T such that generalized tractability holds. These conditions are satisfied if T goes sufficiently fast to infinity. We also provide the formulas for the corresponding exponents of tractability. We illustrate these conditions and formulas for specific tractability functions. For example, take $T(x, y) = x^{1+\ln y}$. Then we have tractability for the three cases of singular values. For finitely many positive singular values and for exponentially decaying singular values, the exponent of tractability is $2/\ln(\lambda_2^{-1})$. Hence it only depends on the second singular value and is independent of how many of them are positive. For polynomially decaying singular values, $\lambda_j = \Theta(j^{-\beta})$ for $\beta > 0$, the exponent of tractability is $\max\{2/\beta, 2/\ln(\lambda_2^{-1})\}$.

We also illustrate our results for tractability functions of product form, that is when $T(x, y) = f_1(x)f_2(y)$ with finite $a_i = \liminf_{x\to\infty}(\ln\ln f_i(x))/(\ln\ln x)$, $i = 1, 2$. Then generalized tractability holds iff $a_i > 1$ and $(a_1-1)(a_2-1) \geq 1$, and if $(a_1-1)(a_2-1) = 1$ then we need to assume additionally condition (12) for $k = 2$ which depends only on the second singular value. For $(a_1 - 1)(a_2 - 1) > 1$, the exponent of tractability is zero, whereas for $(a_1 - 1)(a_2 - 1) = 1$, the exponent of tractability is positive. In fact, in the last case, depending on specific functions f_i for which a_i are fixed, the exponent of tractability can be arbitrary. Note that a_i only depends on the limiting behavior of f_i, and is independent on the behavior of the singular values. Hence, for $(a_1-1)(a_2-1) > 1$, we have the zero exponent of tractability independently of the behavior of the singular values, whereas for $(a_1-1)(a_2-1) = 1$, the exponent of tractability depends only on the second singular value and is independent of the rest of them.

In the final section, we compare the results obtained in this paper for the unrestricted domain $\Omega^{\rm unr}$ with the results from our previous paper obtained for the restricted domain $\Omega^{\rm res}$. The tractability results for the unrestricted and restricted domains may be quite different. We may have generalized tractability for the restricted domain and no generalized tractability for the unrestricted domain which is the case, as we already mentioned, for polynomial tractability $T(x, y) = xy$. We may also have generalized tractability for both domains, however, the exponents of tractability may depend on the domain and can be much larger for the unrestricted domain than for the restricted domain.

2 Preliminaries

2.1 Multivariate Problems

For $m, d \in \mathbb{N}$, let F_d be a normed linear space of functions

$$f \,:\, D_d \subseteq \mathbb{R}^{d\,m} \to \mathbb{R}$$

and let G_d be a normed linear space. We consider in this paper sequences $S = \{S_d\}$ of linear operators $S_d : F_d \to G_d$. We call S a *multivariate problem*.

By *linear information* $\Lambda_d^{\mathrm{all}} = F_d^*$ we mean the class of all continuous linear functionals defined on F_d. Let $\Lambda_d \subseteq F_d^*$ be a class of admissible continuous linear functionals.

Without loss of generality, see e.g., [4], we consider linear algorithms that use finitely many admissible information evaluations. An algorithm $A_{n,d}$ has the form

$$A_{n,d}(f) = \sum_{i=1}^{n} g_i L_i(f) \tag{1}$$

for some $L_i \in \Lambda_d$ and some $g_i \in G_d$.

In this paper we restrict ourselves to the worst case setting. The *worst case error* of the algorithm $A_{n,d}$ is defined as

$$e^{\mathrm{wor}}(A_{n,d}) = \sup_{f \in F_d, \|f\|_{F_d} \le 1} \|S_d(f) - A_{n,d}(f)\|_{G_d}\,. \tag{2}$$

The initial error is

$$e^{\mathrm{init}}(S_d) \;=\; \|S_d\| = e^{\mathrm{wor}}(A_{0,d}^*)\,,$$

where $A_{0,d}^* = 0$ is the zero algorithm. Let

$$n(\varepsilon, S_d, \Lambda_d) = \min\{\, n \;\mid \exists\, A_{n,d} \,:\; e^{\mathrm{wor}}(A_{n,d}) \le \varepsilon\, e^{\mathrm{init}}(S_d)\,\} \tag{3}$$

denote the minimal number of admissible information evaluations from Λ_d needed to reduce the initial error by a factor $\varepsilon \in [0, 1]$. The number $n(\varepsilon, S_d, \Lambda_d)$ is called the *information complexity* of the problem S_d.

2.2 Generalized Tractability

A *tractability domain* Ω is a subset of $[1, \infty) \times \mathbb{N}$ satisfying

$$[1, \infty) \times \{1, \ldots, d^*\} \cup [1, \varepsilon_0^{-1}) \times \mathbb{N} \;\subseteq\; \Omega \tag{4}$$

for some $d^* \in \mathbb{N} \cup \{0\}$ and some $\varepsilon_0 \in (0, 1]$ such that $d^* + (1 - \varepsilon_0) > 0$. In this paper we focus on the unrestricted tractability domain $\Omega^{\mathrm{unr}} := [1, \infty) \times \mathbb{N}$.

A function $T :\; [1, \infty)\times[1, \infty) \;\to\; [1, \infty)$ is a *tractability function* if T is non-decreasing in x and y and

$$\lim_{(x,y) \,\in\, \Omega,\; x+y \to \infty} \frac{\ln T(x, y)}{x + y} = 0\,. \tag{5}$$

Let now Ω be a tractability domain and T a tractability function. The multivariate problem $S = \{S_d\}$ is (T,Ω)*-tractable in the class* $\Lambda = \{\Lambda_d\}$ if there exist non-negative numbers C and t such that

$$n(\varepsilon, S_d, \Lambda_d) \le C\, T(\varepsilon^{-1}, d)^t \quad \text{for all } (\varepsilon^{-1}, d) \in \Omega. \tag{6}$$

The *exponent* t^{tra} *of* (T,Ω)*-tractability* in the class Λ is defined as the infimum of all non-negative t for which there exists a $C = C(t)$ such that (6) holds.

The multivariate problem S is *strongly* (T,Ω)*-tractable* in the class $\Lambda = \{\Lambda_d\}$ if there exist non-negative numbers C and t such that

$$n(\varepsilon, S_d, \Lambda_d) \le C\, T(\varepsilon^{-1}, 1)^t \quad \text{for all } (\varepsilon^{-1}, d) \in \Omega. \tag{7}$$

The *exponent* t^{str} *of strong* (T,Ω)*-tractability* in the class Λ is the infimum of all non-negative t for which there exists a $C = C(t)$ such that (7) holds.

An extensive motivation of the notion of generalized tractability and many examples of tractability domains and functions can be found in [1].

We say that a multivariate problem S is *weakly tractable* if

$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, S_d, \Lambda_d)}{d + \varepsilon^{-1}} = 0\,.$$

Obviously, if S is $(T, \Omega^{\mathrm{unr}})$-tractable then S is also weakly tractable. If S is weakly tractable and $n(\varepsilon, S_d, \Lambda_d)$ is at least one and non-decreasing in ε^{-1} and d, then S is also $(T, \Omega^{\mathrm{unr}})$-tractable for any non-decreasing extension $T : [1,\infty) \times [1,\infty) \to [1,\infty)$ of $n(\varepsilon, S_d, \Lambda_d)$.

2.3 Linear Tensor Product Problems

We describe the setting we want to study in this paper in more details. Let F_1 be a separable Hilbert space of real valued functions defined on $D_1 \subseteq \mathbb{R}^m$, and let G_1 be an arbitrary separable Hilbert space. Let $S_1 : F_1 \to G_1$ be a compact linear operator. Then the non-negative self-adjoint operator

$$W_1 := S_1^* S_1 :\, F_1 \,\to\, F_1$$

is also compact. Let $\{\lambda_i\}$ denote the sequence of non-increasing eigenvalues of W_1, or equivalently let $\{\sqrt{\lambda_i}\}$ be the sequence of the singular values of S_1. If $k = \dim(F_1)$ is finite, then W_1 has just finitely many eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then we formally put $\lambda_j = 0$ for $j > k$. In any case, the eigenvalues λ_j converge to zero. Without loss of generality, we assume that S_1 is not the zero operator, and normalize the problem by assuming that $\lambda_1 = 1$. Hence,

$$1 = \lambda_1 \ge \lambda_2 \ge \cdots \ge 0\,.$$

This implies that $\|S_1\| = 1$ and the initial error is also one.

For $d \geq 2$, let

$$F_d \;=\; F_1 \otimes \cdots \otimes F_1$$

be the complete d-fold tensor product Hilbert space of F_1 of real valued functions defined on $D_d = D_1 \times \cdots \times D_1 \subseteq \mathbb{R}^{dm}$. Similarly, let $G_d = G_1 \otimes \cdots \otimes G_1$, d times.

The linear operator S_d is defined as the tensor product operator

$$S_d \;=\; S_1 \otimes \cdots \otimes S_1 : F_d \to G_d.$$

We have $\|S_d\| = \|S_1\|^d = 1$, so that the initial error is one for all d. We call the multivariate problem $S = \{S_d\}$ a *linear tensor product problem.*

In this paper we analyze the problem S only for the class of linear information $\Lambda^{\text{all}} = \{\Lambda_d^{\text{all}}\}$. For convenience we write $n(\varepsilon, S_d)$ instead of $n(\varepsilon, S_d, \Lambda_d^{\text{all}})$. It is known, see e.g., [4], that

$$n(\varepsilon, S_d) = |\{(i_1, \ldots, i_d) \in \mathbb{N}^d \;\mid\; \lambda_{i_1} \ldots \lambda_{i_d} > \varepsilon^2\}|, \tag{8}$$

with the convention that the cardinality of the empty set is zero. Thus the linear tensor product problem S is trivial if $\lambda_2 = 0$, since $n(\varepsilon, S_d) = 1$ for all $\varepsilon \in [0, 1)$. On the other hand, $n(\varepsilon, S_d)$ grows exponentially in d if $\lambda_2 = 1$, since $n(\varepsilon, S_d) \geq 2^d$ for all $\varepsilon \in [0, 1)$. Therefore we assume $\lambda_2 \in (0, 1)$.

We consider here the unrestricted case, i.e.,

$$\Omega = \Omega^{\text{unr}} = [1, \infty) \times \mathbb{N}\,.$$

We know from [1, Lemma 3.1] that for this tractability domain the linear tensor product problem S is *not* strongly (T, Ω^{unr})-tractable, regardless of the tractability function T.

For $\varepsilon \in (0, 1]$ we define

$$\alpha(\varepsilon) = \lceil 2 \ln(1/\varepsilon) / \ln(1/\lambda_2) \rceil - 1\,. \tag{9}$$

Notice that $\alpha(\varepsilon)$ is the largest integer n satisfying $\lambda_2^n > \varepsilon^2$. We stress that $\alpha(\varepsilon)$ depends on λ_2. It tends to infinity as λ_2 approaches 1, and is zero iff $\sqrt{\lambda_2} \leq \varepsilon$. From (8) it follows that $n(\varepsilon, S_d) = 1$ if $\alpha(\varepsilon) = 0$.

For $\varepsilon \in (0, 1)$ and $\lambda_2 \in (0, 1)$, let $a := \min\{\alpha(\varepsilon), d\}$. Then it is easy to show, see also [1, Lemma 3.2], that

$$\binom{d}{a} \leq n(\varepsilon, S_d) \leq \binom{d}{a} n(\varepsilon, S_1)^a\,. \tag{10}$$

3 Finitely Many Eigenvalues

In this section we consider the case when $W_1 = S_1^* S_1$ has only finitely many positive eigenvalues λ_i. First we consider the case where W_1 has $k \geq 2$ eigenvalues different from zero and $k - 1$ of them are equal. We now prove an auxiliary lemma which will be helpful in the course of the proof of our first theorem.

Lemma 3.1. *Let $d, k \in \mathbb{N}$ and let α be an integer satisfying $0 \le \alpha \le \frac{k-1}{k}(d+1)$. Then*

$$\max_{0\le\nu\le\alpha} \binom{d}{\nu}(k-1)^\nu = \binom{d}{\alpha}(k-1)^\alpha. \tag{11}$$

Proof. For $0 \le \nu \le \alpha$ the inequality

$$\binom{d}{\nu-1}(k-1)^{\nu-1} \le \binom{d}{\nu}(k-1)^\nu$$

holds iff $\nu \le (d-\nu+1)(k-1)$, and the last inequality holds iff $\nu \le \frac{k-1}{k}(d+1)$. This shows that the function

$$\nu \mapsto \binom{d}{\nu}(k-1)^\nu$$

is non-decreasing on $[0,\alpha] \cap \mathbb{N}$. □

Theorem 3.2. *Let T be a tractability function. Let*

$$\lambda_1 = 1,\ 0 < \lambda_2 = \ldots = \lambda_k < 1, \quad and \quad \lambda_l = 0 \quad for \quad l > k \ge 2.$$

Then the linear tensor product problem $S = \{S_d\}$ is $(T, \Omega^{\rm unr})$-tractable in the class of linear information iff

$$B_k := \liminf_{d\to\infty} \inf_{1\le\alpha(\varepsilon)\le\frac{k-1}{k}d} \frac{\ln T(\varepsilon^{-1}, d)}{m_k(\varepsilon, d)} \in (0, \infty]\,, \tag{12}$$

where $m_k(\varepsilon, d) := \alpha(\varepsilon)\ln(\frac{d}{\alpha(\varepsilon)}(k-1)) + (d-\alpha(\varepsilon))\ln(\frac{d}{d-\alpha(\varepsilon)})$.

If $B_k > 0$, then the exponent $t^{\rm tra}$ of tractability is given by

$$t^{\rm tra} = B_k^{-1}\,. \tag{13}$$

Proof. For the eigenvalues specified in Theorem 3.2, it is easy to check that (8) yields

$$n(\varepsilon, S_d) = \sum_{\nu=0}^{\min\{\alpha(\varepsilon),d\}} \binom{d}{\nu}(k-1)^\nu\,. \tag{14}$$

Let us first assume that S is $(T, \Omega^{\rm unr})$-tractable, i.e., that there exist $C, t > 0$ such that $n(\varepsilon, S_d) \le CT(\varepsilon^{-1}, d)^t$. Let $1 \le \alpha(\varepsilon) \le \frac{k-1}{k}d$. From (11) and (14) we get the estimate

$$\binom{d}{\alpha(\varepsilon)}(k-1)^{\alpha(\varepsilon)} \le n(\varepsilon, S_d) \le (\alpha(\varepsilon)+1)\binom{d}{\alpha(\varepsilon)}(k-1)^{\alpha(\varepsilon)}\,. \tag{15}$$

Using Stirling's formula for factorials $m! = m^{m+1/2}\,\mathrm{e}^{-m}\sqrt{2\pi}\,(1+o(1))$, we obtain

$$\begin{aligned}
&\ln\left(\binom{d}{\alpha(\varepsilon)}(k-1)^{\alpha(\varepsilon)}\right)\\
&=\ln(d!)-\ln(\alpha(\varepsilon)!)-\ln((d-\alpha(\varepsilon))!)+\alpha(\varepsilon)\ln(k-1)\\
&=\left(d+\frac{1}{2}\right)\ln(d)-\left(\alpha(\varepsilon)+\frac{1}{2}\right)\ln(\alpha(\varepsilon))-\left(d-\alpha(\varepsilon)+\frac{1}{2}\right)\ln(d-\alpha(\varepsilon))\\
&\quad-\ln(\sqrt{2\pi})+\ln(\mathcal{O}(1))+\alpha(\varepsilon)\ln(k-1)\\
&=m_k(\varepsilon,d)+\frac{1}{2}\ln\left(\frac{d}{\alpha(\varepsilon)(d-\alpha(\varepsilon))}\right)+\mathcal{O}(1).
\end{aligned}$$

Thus

$$\frac{\ln T(\varepsilon^{-1},d)}{m_k(\varepsilon,d)} \geq \frac{1}{t}+\frac{\ln\left(\frac{d}{\alpha(\varepsilon)(d-\alpha(\varepsilon))}\right)}{2t\,m_k(\varepsilon,d)}-\frac{\ln(C)}{t\,m_k(\varepsilon,d)}+\frac{\mathcal{O}(1)}{t\,m_k(\varepsilon,d)}. \qquad (16)$$

Let $\{(\varepsilon_\nu^{-1},d_\nu)\}$ be a sequence in Ω^{unr} such that $1\leq\alpha(\varepsilon_\nu)\leq(k-1)d/k$, and $\lim_{\nu\to\infty}(\alpha(\varepsilon_\nu)/d_\nu)$ exists (and obviously is at most $(k-1)/k$) with $\lim_{\nu\to\infty}d_\nu=\infty$.

If $\lim_{\nu\to\infty}(\alpha(\varepsilon_\nu)/d_\nu)>0$ then $m_k(\varepsilon_\nu,d_\nu)=\Theta(d_\nu)$ and the right hand side of (16) tends to $1/t$ for $\nu\to\infty$.

If $\lim_{\nu\to\infty}(\alpha(\varepsilon_\nu)/d_\nu)=0$ then

$$m_k(\varepsilon_\nu,d_\nu)=\Theta\left(\alpha(\varepsilon_\nu)\ln\left(\frac{d_\nu}{\alpha(\varepsilon_\nu)}(k-1)\right)\right),$$

since

$$(d_\nu-\alpha(\varepsilon_\nu))\ln\left(\frac{d_\nu}{d_\nu-\alpha(\varepsilon_\nu)}\right)=\Theta(\alpha(\varepsilon_\nu)).$$

Furthermore,

$$\left|\ln\left(\frac{d_\nu}{\alpha(\varepsilon_\nu)(d_\nu-\alpha(\varepsilon_\nu))}\right)\right|=\Theta(\ln(\alpha(\varepsilon_\nu))).$$

Hence, again, the right hand side of (16) tends to $1/t$. Since an arbitrary sequence $\{(\varepsilon_\nu^{-1},d_\nu)\}$ with $\lim_{\nu\to\infty}d_\nu=\infty$ and $1\leq\alpha(\varepsilon_\nu)\leq\frac{k-1}{k}d$ has a sub-sequence $\{(\varepsilon_\mu^{-1},d_\mu)\}$ for which $\{\alpha(\varepsilon_\mu)/d_\mu\}$ converges, we conclude that

$$B_k\geq\frac{1}{t}>0\,,\qquad\text{and}\qquad t^{\mathrm{tra}}\geq B_k^{-1}\,. \qquad (17)$$

Assume now $B_k>0$. We want to show that for all $t>B_k^{-1}$ there exists a $C=C(t)>0$ such that $n(\varepsilon,S_d)\leq CT(\varepsilon^{-1},d)^t$ for all $d\in\mathbb{N}$, $\varepsilon\in(0,1)$. From (14) we see that this inequality is trivial if $\alpha(\varepsilon)=0$, and, since $T(\varepsilon^{-1},d)$ is non-decreasing in ε^{-1}, that the case $\alpha(\varepsilon)>d$ is settled if we have the inequality for $\alpha(\varepsilon)=d$. Thus it remains to consider the following two cases:

Case 1: $1 \le \alpha(\varepsilon) \le \frac{k-1}{k}\, d$. We now show that for all $t > B_k^{-1}$ there exists a $C = C(t) > 0$ such that for all $d \in \mathbb{N}$,

$$\frac{\ln T(\varepsilon^{-1},d)}{m_k(\varepsilon,d)} \ge \frac{1}{t} + \frac{\ln(1+\alpha(\varepsilon))}{t\, m_k(\varepsilon,d)} + \frac{\ln\left(\frac{d}{\alpha(\varepsilon)(d-\alpha(\varepsilon))}\right)}{2t\, m_k(\varepsilon,d)} - \frac{\ln(C)}{t\, m_k(\varepsilon,d)} + \frac{\mathcal{O}(1)}{t\, m_k(\varepsilon,d)}. \qquad (18)$$

Due to (15) and the formula for $\ln\left(\binom{d}{\alpha(\varepsilon)}(k-1)^{\alpha(\varepsilon)}\right)$, we conclude that $n(\varepsilon, S_d) \le C T(\varepsilon^{-1}, d)^t$ for all $d \in \mathbb{N}$, $1 \le \alpha(\varepsilon) \le \frac{k-1}{k} d$.

To prove (18), observe that $\ln\left(\frac{d}{\alpha(\varepsilon)(d-\alpha(\varepsilon))}\right) \le \ln(k)$. Obviously,

$$m_k(\varepsilon,d) \ge \alpha(\varepsilon)\ln(\frac{d}{\alpha(\varepsilon)}(k-1)).$$

For a given $l \in \mathbb{N}$ let $x_l \in \mathbb{R}$ be so large that for all $x \ge x_l$ we have

$$\frac{\ln(1+x)}{x\,\ln(k)} \le \frac{1}{l}\,.$$

Let now $d_l \ge (1+x_l)^{l+1}$. Then we get for $d \ge d_l$ and $1 \le \alpha(\varepsilon) \le \frac{k-1}{k}\, d$,

$$\frac{\ln(1+\alpha(\varepsilon))}{\alpha(\varepsilon)\ln(\frac{d}{\alpha(\varepsilon)}\,(k-1))} \le \frac{1}{l}.$$

Furthermore, let $\{\tilde{B}_l\}$ be a sequence in $(0, B_k)$ that converges to B_k. For each l we find a d_l' such that for all $d \ge d_l'$ and all $1 \le \alpha(\varepsilon) \le \frac{k-1}{k}\, d$

$$\frac{\ln T(\varepsilon^{-1},d)}{m_k(\varepsilon,d)} \ge \tilde{B}_l.$$

Choose $t_l := (1+\frac{1}{l})\tilde{B}_l^{-1}$. For all $d \ge \max\{d_l, d_l'\}$ and all $1 \le \alpha(\varepsilon) \le \frac{k-1}{k}\, d$ we have

$$\frac{\ln T(\varepsilon^{-1},d)}{m_k(\varepsilon,d)} \ge \frac{1}{t_l}\left(1 + \frac{\ln(1+\alpha(\varepsilon))}{m_k(\varepsilon,d)}\right).$$

It is now easy to see that (18) holds for $t = t_l$ and all $d \in \mathbb{N}$ and all $1 \le \alpha(\varepsilon) \le \frac{k-1}{k}\, d$ if we just choose $C = C(t_l)$ suitably large. Observe that t_l converges to B_k^{-1} as l tends to infinity.

Case 2: $\frac{k-1}{k}\, d < \alpha(\varepsilon) \le d$. Let $\delta \in (0, B_k^{-1})$ and $t \ge (B_k - \delta)^{-1}$. There exists a d_δ such that for all $d \ge d_\delta$ and all ε_* with $1 \le \alpha(\varepsilon_*) \le \frac{k-1}{k}\, d$, we have

$$\frac{\ln T(\varepsilon_*^{-1},d)}{m_k(\varepsilon_*,d)} \ge B_k - \delta.$$

For $d \ge d_\delta$ and $\alpha(\varepsilon) \ge \frac{k-1}{k}\, d$, choose $\varepsilon_* \in [\varepsilon, 1)$ such that $\alpha(\varepsilon_*) = \lfloor \frac{k-1}{k}\, d \rfloor = d - \lceil \frac{d}{k} \rceil$. Then

$$m_k(\varepsilon_*,d) \ge d\ln(k) - \left\lceil \frac{d}{k} \right\rceil \ln\left(1+\frac{k}{d}\right)\,,$$

and

$$t\,\ln T(\varepsilon^{-1},d) \ge (B_k-\delta)^{-1}\ln T(\varepsilon_*^{-1},d) \ge m_k(\varepsilon_*,d)\,.$$

We find a number C not depending on d such that $\ln(C) \ge \lceil \frac{d}{k}\rceil \ln(1+\frac{k}{d})$. From (14) we know that $n(\varepsilon,S_d) \le k^d$, and this yields

$$t\,\ln T(\varepsilon^{-1},d) \ge m_k(\varepsilon_*,d) \ge \ln n(\varepsilon,S_d) - \ln(C)\,,$$

implying $CT(\varepsilon^{-1},d)^t \ge n(\varepsilon,S_d)$. Choosing $C=C(t)$ sufficiently large the last inequality extends to all d and all ε with $\frac{k-1}{k}\,d \le \alpha(\varepsilon) \le d$.

The statement of the theorem follows from Cases 1 and 2. □

We illustrate Theorem 3.2 by two tractability functions.

- Let $T(x,y)=xy$ which corresponds to polynomial tractability. Then it is easy to check that $B_k=0$ for all $k\ge 2$. This means that we do not have polynomial tractability for any linear tensor product problem with at least two positive eigenvalues for $d=1$. This result has been known before.
- Let $T(x,y)=x^{1+\ln y}$. Then it can be checked that

  $$B_k \;=\; \tfrac{1}{2}\ln(\lambda_2^{-1}) \;\text{ for all }\; k\;\ge\;2, \;\text{ and }\; t^{\rm tra} \;=\; \frac{2}{\ln(\lambda_2^{-1})}.$$

  Hence, the exponent of tractability only depends on the second largest eigenvalue and is independent of its multiplicity. Note that the exponent of tractability goes to infinity as λ_2 approaches one.

We now consider the general case of finitely many positive eigenvalues.

Corollary 3.3. *Let T be a tractability function. Let $k\ge 2$ and $\lambda_1=1$, $\lambda_2\in(0,1)$, and $\lambda_l=0$ for $l>k$. Then the linear tensor product problem $S=\{S_d\}$ is $(T,\Omega^{\rm unr})$-tractable in the class of linear information iff for some (and thus for all) $j\in\{2,3,\dots,k\}$*

$$B_j = \liminf_{d\to\infty}\ \inf_{1\le\alpha(\varepsilon)\le\frac{j-1}{j}\,d}\ \frac{\ln T(\varepsilon^{-1},d)}{m_j(\varepsilon,d)} \in (0,\infty]\,, \tag{19}$$

where $m_j(\varepsilon,d)=\alpha(\varepsilon)\ln(\frac{d}{\alpha(\varepsilon)}(j-1))+(d-\alpha(\varepsilon))\ln(\frac{d}{d-\alpha(\varepsilon)})$. In this case the exponent $t^{\rm tra}$ of tractability satisfies

$$B_2^{-1} \le t^{\rm tra} \le B_k^{-1}\,. \tag{20}$$

Proof. Obviously we have $B_2\ge B_3\ge\cdots\ge B_k$. We need to show that $B_2>0$ implies that $B_k>0$. We first show that $B_2>0$ implies

$$\liminf_{d\to\infty}\ \inf_{1\le\alpha(\varepsilon)\le\frac{k-1}{k}\,d}\ \frac{\ln T(\varepsilon^{-1},d)}{m_2(\varepsilon,d)} > 0\,. \tag{21}$$

Let ε_* satisfy $\alpha(\varepsilon_*) = \lfloor \frac{d}{2} \rfloor$. We have $m_2(\varepsilon_*, d) \geq d \ln(2) - \lceil \frac{d}{2} \rceil \ln(1 + \frac{2}{d})$. Thus for $\frac{d}{2} < \alpha(\varepsilon) \leq \frac{k-1}{k}\, d$ we get

$$m_2(\varepsilon, d) \leq d \ln(2) + \frac{d}{2} \ln(k) \leq C m_2(\varepsilon_*, d)$$

for d and C sufficiently large. Since T is non-decreasing with respect to the first variable, it is easy to see that $B_2 > 0$ implies (21).

Now we prove that

$$\frac{m_k(\varepsilon, d)}{m_2(\varepsilon, d)} = 1 + \frac{\alpha(\varepsilon) \ln(k-1)}{\alpha(\varepsilon) \ln(\frac{d}{\alpha(\varepsilon)}) + (d - \alpha(\varepsilon)) \ln(\frac{d}{d - \alpha(\varepsilon)})} \tag{22}$$

is bounded uniformly for all $d \in \mathbb{N}$ and all ε with $1 \leq \alpha(\varepsilon) \leq \frac{k-1}{k}\, d$. This follows easily from

$$m_2(\varepsilon, d) \geq \alpha(\varepsilon) \ln\left(\frac{d}{\alpha(\varepsilon)}\right) \geq \alpha(\varepsilon) \ln\left(\frac{k}{k-1}\right).$$

Thus $B_2 > 0$ implies

$$B_k \geq \left(\liminf_{d \to \infty} \inf_{1 \leq \alpha(\varepsilon) \leq \frac{k-1}{k} d} \frac{\ln T(\varepsilon^{-1}, d)}{m_2(\varepsilon, d)} \right) \left(\inf_{d \in \mathbb{N}\,;\, 1 \leq \alpha(\varepsilon) \leq \frac{k-1}{k} d} \frac{m_2(\varepsilon, d)}{m_k(\varepsilon, d)} \right) > 0.$$

Since the linear tensor product problem S' having only the two non-zero eigenvalues $\lambda_1' = \lambda_1$ and $\lambda_2' = \lambda_2$ is at most as difficult as S and the problem S'' having eigenvalues $\lambda_1'' = \lambda_1$, $\lambda_2'' = \ldots = \lambda_k'' = \lambda_2$ and $\lambda_l'' = 0$ for $l > k$ is at least as as difficult as S, the corollary follows from Theorem 3.2. □

Remark 3.4. Theorem 3.2 shows that in the case $\lambda_3 = \ldots = \lambda_k = 0$ we have $t^{\mathrm{tra}} = B_2^{-1}$, while in the case $\lambda_2 = \lambda_3 = \ldots = \lambda_k$ we have $t^{\mathrm{tra}} = B_k^{-1}$.

If we consider a fixed tractability function T, a sequence $\{S^{(n)}\}$ of tensor product problems whose eigenvalues $\{\lambda_i^{(n)}\}$ satisfy $\lambda_1^{(n)} = \lambda_1 = 1$, $\lambda_2^{(n)} = \lambda_2 \in (0,1)$, $\lambda_3^{(n)}, \ldots, \lambda_k^{(n)} > 0$, and $\lim_{n \to \infty} \lambda_3^{(n)} = 0$, then we do not necessarily have that the corresponding exponents of tractability t_n^{tra} converge to B_2^{-1} as the following counterexample shows. Let

$$T(\varepsilon^{-1}, d) = \sum_{\nu=0}^{\min\{\alpha(\varepsilon), d\}} \binom{d}{\nu}.$$

Then it is not hard to see that T is indeed a tractability function and that $B_2 = 1$ (we showed that implicitly in the proof of Theorem 3.2). According to Corollary 3.3 each problem $S^{(n)}$ is $(T, \Omega^{\mathrm{unr}})$-tractable. For $d \in \mathbb{N}$ we obviously have $\sup_{\varepsilon \in (0,1)} T(\varepsilon^{-1}, d) = 2^d$. If we choose $\varepsilon = \varepsilon_d^{(n)} = \frac{1}{2}(\lambda_k^{(n)})^{d/2}$, we get $n(\varepsilon, S_d^{(n)}) = k^d$. This implies $t_n^{\mathrm{tra}} \geq \ln(k)/\ln(2)$ for all n. This shows that the sequence $\{t_n^{\mathrm{tra}}\}$ does not converge to $B_2^{-1} = 1$.

Example 3.5. Let the conditions of Corollary 3.3 hold. We consider the special tractability function $T(x,y) = \exp(f_1(x)f_2(y))$, where $f_i : [1,\infty) \to (0,\infty)$, $i = 1,2$, are non-decreasing functions. Let

$$a_i := \liminf_{x\to\infty} \frac{f_i(x)}{\ln x} \quad \text{for } i = 1,2.$$

Let us assume that S is $(T,\Omega^{\mathrm{unr}})$-tractable. According to Corollary 3.3 we have $B_2 > 0$, and from $m_2(\varepsilon,d) \geq \alpha(\varepsilon)(\ln(d) - \ln(\alpha(\varepsilon)))$ for all ε satisfying $1 \leq \alpha(\varepsilon) \leq d/2$ we get

$$\begin{aligned} 0 < B_2 &\leq \liminf_{d\to\infty} \frac{f_1(\varepsilon^{-1})f_2(d)}{\alpha(\varepsilon)(\ln(d) - \ln(\alpha(\varepsilon)))} \\ &\leq \frac{f_1(\varepsilon^{-1})}{\alpha(\varepsilon)} \left(\liminf_{d\to\infty} \frac{f_2(d)}{\ln(d)}\right) \left(\limsup_{d\to\infty} \frac{\ln(d)}{\ln(d) - \ln(\alpha(\varepsilon))}\right) = \frac{f_1(\varepsilon^{-1})}{\alpha(\varepsilon)} a_2. \end{aligned}$$

Thus $a_2 > 0$, and

$$0 < \frac{B_2}{a_2} \leq \liminf_{\varepsilon\to 0} \frac{f_1(\varepsilon^{-1})}{\alpha(\varepsilon)} = \liminf_{\varepsilon\to 0} \left(\frac{\ln(\varepsilon^{-1})}{\alpha(\varepsilon)} \frac{f_1(\varepsilon^{-1})}{\ln(\varepsilon^{-1})}\right) = \frac{\ln(\lambda_2^{-1})}{2}\, a_1.$$

Hence, $a_1 > 0$ and $a_2 > 0$ are necessary conditions for the problem S to be $(T,\Omega^{\mathrm{unr}})$-tractable, and the exponent of tractability is bounded from below by

$$t^{\mathrm{tra}} \geq B_2^{-1} \geq \frac{2}{a_1\, a_2\, \ln(\lambda_2^{-1})}.$$

In Corollary 5.2 we will show in particular that the conditions $a_1 > 0$, $a_2 > 0$ are also sufficient for $(T,\Omega^{\mathrm{unr}})$-tractability.

Remark 3.6. Under the conditions of Corollary 3.3 we can state a slightly simpler criterion to characterize $(T,\Omega^{\mathrm{unr}})$-tractability. The linear tensor product problem $S = \{S_d\}$ is $(T,\Omega^{\mathrm{unr}})$-tractable in the class of linear information iff

$$B := \liminf_{d\to\infty} \inf_{1\leq\alpha(\varepsilon)\leq d/2} \frac{\ln T(\varepsilon^{-1},d)}{\alpha(\varepsilon)\ln(d/\alpha(\varepsilon))} \in (0,\infty]\,. \tag{23}$$

The necessity and sufficiency of $B > 0$ follows from (19) and the (easy to check) inequalities

$$\frac{1}{2} m_2(\varepsilon,d) \leq \alpha(\varepsilon)\ln\left(\frac{d}{\alpha(\varepsilon)}\right) \leq m_2(\varepsilon,d)$$

for all ε satisfying $1 \leq \alpha(\varepsilon) \leq \frac{d}{2}$ and large d. A drawback of (23) is that the quantity B is not related to the exact exponent of tractability as B_k in Theorem 3.2.

Example 3.7. The tractability criteria (19) and (23) depend on the second largest eigenvalue λ_2 via $\alpha(\varepsilon)$. In fact, for a given tractability function T, a linear tensor product problem $S = \{S_d\}$ with only two positive eigenvalues for $S_1^* S_1$ may be $(T,\Omega^{\mathrm{unr}})$-tractable, but if we increase the value of λ_2 this may not necessarily be the case any more. Choose, e.g.,

$$T(x,y) := \begin{cases} 1 & \text{if } x \in [1, \lambda_2^{-1/2}], \\ \mathrm{e}^{\ln(x)(1+\ln(y))} & \text{otherwise.} \end{cases}$$

From criterion (23) it follows easily that S is $(T, \Omega^{\mathrm{unr}})$-tractable. But if we consider the problem $\tilde{S}$ where we only increase the second eigenvalue to $\tilde{\lambda}_2 > \lambda_2$, we see that for $\tilde{\lambda}_2^{-1/2} < \varepsilon^{-1} \le \lambda_2^{-1/2}$ we have

$$n(\varepsilon, \tilde{S}_d) \ge \sum_{\nu=0}^{\min\{\tilde{\alpha}(\varepsilon),d\}} \binom{d}{\nu} \ge \binom{d}{1} = d, \quad \text{where } \tilde{\alpha}(\varepsilon) := \left\lceil \frac{2\ln(\varepsilon^{-1})}{\ln(\tilde{\lambda}_2^{-1})} \right\rceil - 1 \ge 1.$$

Thus the problem $\tilde{S}_d$ is obviously not $(T, \Omega^{\mathrm{unr}})$-tractable since $CT(\varepsilon^{-1}, d)^t = C$ cannot be larger than d for $d > C$.

The counterexample above motivates us to state a sufficient condition on T ensuring $(T, \Omega^{\mathrm{unr}})$-tractability of all linear tensor product problems S with finitely many eigenvalues regardless of the specific value of λ_2.

Corollary 3.8. *Let T be a tractability function. If*

$$\widetilde{B} := \liminf_{d\to\infty} \inf_{1<\varepsilon^{-1}\le e^d} \frac{\ln T(\varepsilon^{-1}, d)}{\ln(\varepsilon^{-1})\,(1 + \ln(d/\ln(\varepsilon^{-1})))} \in (0, \infty] \tag{24}$$

then arbitrary linear tensor product problem S with finitely many eigenvalues is $(T, \Omega^{\mathrm{unr}})$-tractable. However, the exponent of tractability goes to infinity as λ_2 approaches one.

Proof. The proof of the corollary is easy. For values of $\varepsilon \in [e^{-d}, 1)$ satisfying $\alpha(\varepsilon) \in [1, d/2]$ one can simply show that $\alpha(\varepsilon)\ln(d/\alpha(\varepsilon)) \le C\ln(\varepsilon^{-1})(1 + \ln(d/\ln(\varepsilon^{-1})))$, where the constant C depends only on λ_2. If we substitute the upper bound on $\alpha(\varepsilon)$ in the definition of B in (23) by the minimum of $d/2$ and $\lceil 2d/\ln(\lambda_2^{-1})\rceil - 1$ we therefore see that this modified quantity is strictly positive. From that we can deduce similarly as in Case 2 in the proof of Theorem 3.2 that $B > 0$, and due to Remark 3.6, the problem S is $(T, \Omega^{\mathrm{unr}})$-tractable. Obviously, $n(\varepsilon, S_d) \ge 2^d$ for $\varepsilon^2 < \lambda_2^d$. Hence, the exponent of tractability must go to infinity as λ_2 goes to one. □

Remark 3.9. Condition (24) in the corollary above is sufficient for $(T, \Omega^{\mathrm{unr}})$-tractability for all linear tensor product problems S with finitely many eigenvalues, but not necessary as the example $T(\varepsilon^{-1}, d) = \exp(\ln(\varepsilon^{-1})(1 + \ln(d)))$ shows, see Corollary 5.2.

4 Exponential Decay of Eigenvalues

We begin to study linear tensor problems with infinitely many positive eigenvalues. As we shall see, tractability results depend on the behavior of the eigenvalues for $d = 1$. In this section we assume that they are exponentially decaying whereas in the next section that they are polynomially decaying.

Theorem 4.1. *Let T be a tractability function. Let S be a linear tensor product problem with exponentially decaying eigenvalues λ_j,*

$$\exp(-\beta_1(j-1)) \le \lambda_j \le \exp(-\beta_2(j-1)) \quad \text{for all} \quad j \in \mathbb{N},$$

for some positive numbers β_1, β_2. For $i = 1, 2$, define

$$B_{\mathrm{e}}^{(i)} := \liminf_{\substack{\varepsilon^{-1}+d\to\infty \\ \varepsilon<\sigma_i}} \frac{\ln T(\varepsilon^{-1}, d)}{m_{\mathrm{e}}^{(i)}(\varepsilon, d)},$$

where $\sigma_1 = e^{-\beta_1/2}$, $\sigma_2 = \sqrt{\lambda_2}$, and

$$m_{\mathrm{e}}^{(i)}(\varepsilon, d) := \lceil z_i \rceil \ln\left(1 + \frac{d}{\lceil z_i \rceil}\right) + d \ln\left(1 + \frac{\lceil z_i \rceil}{d}\right),$$

with

$$z_i = z_i(\varepsilon) := \frac{2}{\beta_i} \ln(\varepsilon^{-1}) - 1.$$

Then

$$S \text{ is } (T, \Omega^{\mathrm{unr}})\text{-tractable} \quad \text{iff} \quad B_{\mathrm{e}}^{(2)} \in (0, \infty].$$

Furthermore, $B_{\mathrm{e}}^{(2)} > 0$ is equivalent to $B_{\mathrm{e}}^{(1)} \in (0, \infty]$ and $B_2 \in (0, \infty]$ with B_2 given by (12) for $k = 2$.

If S is $(T, \Omega^{\mathrm{unr}})$-tractable then the exponent t^{tra} of tractability satisfies

$$\left(\min\{B_2, B_{\mathrm{e}}^{(1)}\}\right)^{-1} \le t^{\mathrm{tra}} \le \left(B_{\mathrm{e}}^{(2)}\right)^{-1}.$$

If $\beta_1 = \beta_2$ then

$$t^{\mathrm{tra}} = (B_{\mathrm{e}}^{(2)})^{-1}.$$

Before we prove Theorem 4.1, we state an auxiliary lemma.

Lemma 4.2. *For $d \in \mathbb{N}$ and $x > -1$ let*

$$\mu_{\mathrm{e}}(x, d) := \left| \left\{ (i_1, \ldots, i_d) \in \mathbb{N}^d \;\middle|\; \sum_{j=1}^{d} i_j < x + d + 1 \right\} \right|.$$

Then

$$\mu_{\mathrm{e}}(x, d) = \binom{\lceil x \rceil + d}{d}.$$

Proof. For $d = 1$ we have

$$\mu_{\mathrm{e}}(x, 1) = |\{i \in \mathbb{N} \,|\, i < x + 2\}| = \lceil x \rceil + 1\,.$$

Assume by induction that

$$\mu_{\mathrm{e}}(y, d) = \binom{\lceil y \rceil + d}{d}$$

for some $d \in \mathbb{N}$ and all $y > -1$. If $x > -1$ then

$$\mu_e(x, d+1) = \sum_{k=1}^{\lceil x \rceil + 1} \mu_e(x+1-k, d) = \sum_{k=1}^{\lceil x \rceil + 1} \binom{\lceil x \rceil + 1 - k + d}{d}$$
$$= \sum_{\nu=0}^{\lceil x \rceil} \binom{\nu + d}{d} = \binom{\lceil x \rceil + d + 1}{d+1}.$$

□

Proof of Theorem 4.1. Let $\mu_{\mathrm{e}}(x, d)$ be defined as in Lemma 4.2. Then

$$\mu_{\mathrm{e}}(z_1, d) = \left| \left\{ (i_1, \ldots, i_d) \in \mathbb{N}^d \;\middle|\; \prod_{j=1}^{d} \exp(-\beta_1(i_j - 1)) > \varepsilon^2 \right\} \right| \le n(\varepsilon, S_d).$$

Similarly, we get $n(\varepsilon, S_d) \le \mu_{\mathrm{e}}(z_2, d)$.

Let us first assume that S is $(T, \Omega^{\mathrm{unr}})$-tractable, i.e., that there exist positive t, C such that

$$n(\varepsilon, S_d) \le C T(\varepsilon^{-1}, d)^t \quad \text{for all } (\varepsilon^{-1}, d) \in \Omega^{\mathrm{unr}}.$$

Let us assume that $\varepsilon < \mathrm{e}^{-\beta_1/2}$, which implies that $\lceil z_1 \rceil \ge 1$. From this inequality we get due to Lemma 4.2

$$\frac{\ln T(\varepsilon^{-1}, d)}{m_{\mathrm{e}}^{(1)}(\varepsilon, d)} \ge \frac{\ln(C^{-1}) + \ln \binom{\lceil z_1 \rceil + d}{d}}{t\, m_{\mathrm{e}}^{(1)}(\varepsilon, d)}.$$

Similarly as in the proof of Theorem 3.2 we use Stirling's formula for factorials, and conclude

$$\ln \binom{\lceil z_1 \rceil + d}{d} = m_{\mathrm{e}}^{(1)}(\varepsilon, d) + \frac{1}{2} \ln \left(\frac{\lceil z_1 \rceil + d}{\lceil z_1 \rceil\, d} \right) + \mathcal{O}(1). \tag{25}$$

We have

$$-\min\{\ln(d), \ln\lceil z_1 \rceil\} \le \ln \left(\frac{\lceil z_1 \rceil + d}{\lceil z_1 \rceil\, d} \right) = \ln \left(\frac{1}{\lceil z_1 \rceil} + \frac{1}{d} \right) \le \ln(2).$$

So it is easy to check that we get $B_{\mathrm{e}}^{(1)} \ge 1/t$, implying $B_{\mathrm{e}}^{(1)} > 0$ and $t^{\mathrm{tra}} \ge (B_{\mathrm{e}}^{(1)})^{-1}$. Furthermore, we get from Corollary 3.3 that $B_2 > 0$ and $t^{\mathrm{tra}} \ge B_2^{-1}$.

Let us now show that $B_2 > 0$ and $B_{\mathrm{e}}^{(1)} > 0$ imply $B_{\mathrm{e}}^{(2)} > 0$. As a careful analysis reveals, we get

$$K := \liminf_{\varepsilon^{-1} + d \to \infty} \inf_{\varepsilon < e^{-\beta_1/2}} \frac{m_{\mathrm{e}}^{(1)}(\varepsilon, d)}{m_{\mathrm{e}}^{(2)}(\varepsilon, d)} > 0,$$

which gives us

$$\liminf_{\substack{\varepsilon^{-1} + d \to \infty \\ \varepsilon < e^{-\beta_1/2}}} \frac{\ln T(\varepsilon^{-1}, d)}{m_{\mathrm{e}}^{(2)}(\varepsilon, d)} \ge B_{\mathrm{e}}^{(1)} K > 0.$$

In the case $e^{-\beta_1/2} \le \varepsilon < \sqrt{\lambda_2}$ both functions $\alpha(\varepsilon)$ and $z_2(\varepsilon)$ are bounded. Thus we have $m_2(\varepsilon, d) = \Theta(\ln(d)) = m_{\rm e}^{(2)}(\varepsilon, d)$, where m_2 is given in Theorem 3.2. Hence

$$L := \liminf_{d\to\infty} \inf_{e^{-\beta_1/2}\le\varepsilon<\sqrt{\lambda_2}} \frac{m_2(\varepsilon, d)}{m_{\rm e}^{(2)}(\varepsilon, d)} > 0,$$

which yields

$$\liminf_{\substack{\varepsilon^{-1}+d\to\infty \\ e^{-\beta_1/2}\le\varepsilon\le\sqrt{\lambda_2}}} \frac{\ln T(\varepsilon^{-1}, d)}{m_{\rm e}^{(2)}(\varepsilon, d)} \ge B_2 L > 0.$$

This means that $B_{\rm e}^{(2)}$ is positive, as claimed.

Now let us assume that $B_{\rm e}^{(2)} > 0$ and let $t_\delta := ((1-\delta)B_{\rm e}^{(2)})^{-1}$ for a given $\delta \in (0, 1)$. Then there exists an $R(\delta)$ such that for any pair (ε, d) with $\varepsilon^{-1} + d > R(\delta)$ (and $\varepsilon < \sqrt{\lambda_2}$, but for convenience we will not mention this restriction in the rest of the proof) we get

$$\frac{\ln T(\varepsilon^{-1}, d)}{m_{\rm e}^{(2)}(\varepsilon, d)} > \left(1 - \frac{\delta}{2}\right) B_{\rm e}^{(2)}.$$

We want to show that there exists a number C_δ such that

$$n(\varepsilon, S_d) \le C_\delta\, T(\varepsilon^{-1}, d)^{t_\delta} \quad \text{for all } (\varepsilon^{-1}, d) \in \Omega^{\rm unr}.$$

Since $n(\varepsilon, S_d) \le \mu_{\rm e}(z_2, d)$, it is sufficient to verify the inequality

$$\frac{\ln T(\varepsilon^{-1}, d)}{m_{\rm e}^{(2)}(\varepsilon, d)} \ge \frac{\ln(C_\delta^{-1}) + \ln \binom{\lceil z_2 \rceil + d}{d}}{t_\delta\, m_{\rm e}^{(2)}(\varepsilon, d)}. \tag{26}$$

The left hand side is at least $(1-\delta/2)B_{\rm e}^{(2)}$. Using Stirling's formula (25) for z_2 instead of z_1, we see that the right hand side can be written as

$$(1-\delta)B_{\rm e}^{(2)} + \frac{\ln(\frac{\lceil z_2\rceil + d}{\lceil z_2 \rceil\, d})}{2t_\delta\, m_{\rm e}^{(2)}(\varepsilon, d)} + \frac{\ln(C_\delta^{-1})}{t_\delta\, m_{\rm e}^{(2)}(\varepsilon, d)} + \frac{\mathcal{O}(1)}{t_\delta\, m_{\rm e}^{(2)}(\varepsilon, d)}.$$

The limes superior of all the summands, except of $(1-\delta)B_{\rm e}^{(2)}$, goes to zero as $\varepsilon^{-1}+d$ tends to infinity. Hence, there exists an $\tilde{R}(\delta)$ such that for all pairs (ε, d) with $\varepsilon^{-1} + d > \tilde{R}(\delta)$ inequality (26) holds. Choosing C_δ sufficiently large, we see therefore that (26) holds for all $(\varepsilon^{-1}, d) \in \Omega^{\rm unr}$. This shows that we have $(T, \Omega^{\rm unr})$-tractability and, since $\delta \in (0, 1)$ was arbitrary, the exponent of tractability $t^{\rm tra}$ satisfies $t^{\rm tra} \le (B_{\rm e}^{(2)})^{-1}$. As we already have seen, tractability implies $B_{\rm e}^{(1)} > 0$ and $B_2 > 0$.

Finally, if $\beta_1 = \beta_2$ then $B_{\rm e}^{(1)} = B_{\rm e}^{(2)}$, and therefore $(\min\{B_2, B_{\rm e}^{(1)}\})^{-1} \le t^{\rm tra} \le (B_{\rm e}^{(2)})^{-1}$ implies that $B_2 \ge B_{\rm e}^{(1)}$ and $t^{\rm tra} = (B_{\rm e}^{(2)})^{-1}$. □

We illustrate Theorem 4.1 by taking again the tractability function $T(x, y) = x^{1+\ln y}$. For $\beta_1 = \beta_2 = \beta > 0$, we have $\lambda_2 = \exp(-\beta)$. It can be checked that

$$B_{\rm e}^{(2)} = \frac{\beta}{2} = \frac{\ln(\lambda_2^{-1})}{2}.$$

Thus the exponent of tractability is

$$t^{\rm tra} \;=\; (B_{\rm e}^{(2)})^{-1} \;=\; \frac{2}{\beta} \;=\; \frac{2}{\ln(\lambda_2^{-1})}.$$

We can simplify the necessary and sufficient conditions in Theorem 4.1 for $(T,\Omega^{\rm unr})$-tractability at the expense of getting good estimates on the exponent of tractability.

Corollary 4.3. *Let T be a tractability function. Let S be a linear tensor product problem with $0<\lambda_2<\lambda_1=1$, and with exponentially decaying eigenvalues λ_j,*

$$K_1\exp(-\beta_1 j) \le \lambda_j \le K_2\exp(-\beta_2 j) \quad \text{for all} \quad j\in\mathbb{N},$$

for some positive numbers β_1,β_2,K_1 and K_2. Then S is $(T,\Omega^{\rm unr})$-tractable iff

$$\liminf_{\substack{\varepsilon^{-1}+d\to\infty\\ \varepsilon<\sqrt{\lambda_2}}} \frac{\ln T(\varepsilon^{-1},d)}{\min\{d,\alpha(\varepsilon)\}\,(1+|\ln(d/\alpha(\varepsilon))|)} \in (0,\infty]\,. \tag{27}$$

Proof. Since $\lambda_j \le \min\{\lambda_2, K_2\exp(-\beta_2 j)\}$ for $j\ge 2$, we can choose positive $\beta_1'\ge\beta_1$, $\beta_2'\le\beta_2$ such that

$$\exp(-\beta_1'(j-1)) \le \lambda_j \le \exp(-\beta_2'(j-1)) \quad \text{for all} \quad j\in\mathbb{N}.$$

Thus we can apply Theorem 4.1. There we showed that $B_{\rm e}^{(2)}>0$ is necessary and sufficient for $(T,\Omega^{\rm unr})$-tractability. For $1\le\alpha(\varepsilon)\le d/2$ and large d, we have $m_2(\varepsilon,d)/2\le \alpha(\varepsilon)\ln(d/\alpha(\varepsilon))\le m_2(\varepsilon,d)$. Furthermore, one can also verify that

$$\liminf_{d\to\infty}\ \inf_{d/2\le\alpha(\varepsilon)} \left(\frac{m_{\rm e}^{(2)}(\varepsilon,d)}{\min\{d,\alpha(\varepsilon)\}(1+|\ln(d/\alpha(\varepsilon))|)}\right)^q > 0,$$

where $q\in\{-1,+1\}$. Thus (27) holds iff $B_{\rm e}^{(2)}\in(0,\infty]$, which proves the corollary □

5 Polynomial Decay of Eigenvalues

In this section we study tractability for linear tensor product problems with polynomially decaying eigenvalues for $d=1$. We believe that such behavior of eigenvalues is typical and therefore the results of this section are probably more important than the results of the previous sections.

Theorem 5.1. *Let T be a tractability function. Let S be a linear tensor product problem with $1=\lambda_1>\lambda_2>0$ and $\lambda_j=\mathcal{O}(j^{-\beta})$ for all $j\in\mathbb{N}$ and some positive β. A sufficient condition for $(T,\Omega^{\rm unr})$-tractability of S is*

$$F := \liminf_{\substack{\varepsilon^{-1}+d\to\infty\\ \varepsilon<\sqrt{\lambda_2}}} \frac{\ln T(\varepsilon^{-1},d)}{\ln(\varepsilon^{-1})(1+\ln(d))} \in (0,\infty].$$

If $F \in (0,\infty]$, then the exponent of tractability satisfies

$$B_2^{-1} \,\le\, t^{\rm tra} \,\le\, \max\left\{\frac{2}{\beta}, \frac{2}{\ln(\lambda_2^{-1})}\right\} F^{-1},$$

with B_2 given in (12) for $k=2$.

Proof. Let C_1 be a positive constant satisfying $\lambda_j \le C_1 j^{-\beta}$ for all j. With $C_2 := C_1^{1/\beta}$ we have

$$n(\varepsilon, S_1) = \max\{j \mid \lambda_j > \varepsilon^2\} \le \max\{j \mid C_1 j^{-\beta} > \varepsilon^2\} \le C_2\, \varepsilon^{-2/\beta} \le C_2\, \varepsilon^{-p}$$

for all $p > 2/\beta$. From the identity

$$n(\varepsilon, S_d) = \sum_{i=1}^{\infty} n\left(\varepsilon/\sqrt{\lambda_i},\, S_{d-1}\right)$$

it now follows by simple induction that

$$n(\varepsilon, S_d) \le C_2 \left(\sum_{j=1}^{\infty} \lambda_j^{p/2}\right)^{d-1} \varepsilon^{-p} \quad \text{for all } p > 2/\beta. \tag{28}$$

Thus for each $d_0 \in \mathbb{N}$ and all $p > 2/\beta$ there exists a number $C(d_0,p)$ such that

$$n(\varepsilon, S_d) \le C(d_0,p)\, \varepsilon^{-p} \quad \text{for all } d \le d_0 \text{ and } \varepsilon \in (0, \sqrt{\lambda_2}).$$

Let now $\delta \in (0,1)$ and $\varepsilon_\delta < \sqrt{\lambda_2}$ such that for all $\varepsilon \in (0, \varepsilon_\delta)$ and all $d \le d_0$

$$\frac{\ln T(\varepsilon^{-1}, d)}{\ln(\varepsilon^{-1})(1+\ln(d))} \ge (1-\delta)\, F,$$

where F is assumed to be positive. Then for $t = t(\delta, p, d_0) := p(1-\delta)^{-1}F^{-1}$ and $C = C(d_0,p)$ we have

$$\ln(CT(\varepsilon^{-1},d)^t) \ge \ln C + p(1+\ln(d))\ln(\varepsilon^{-1}) \ge \ln n(\varepsilon, S_d)$$

for all $d \le d_0$ and $\varepsilon \in (0, \varepsilon_\delta)$. This implies that for each $t > (2/\beta)F^{-1}$ there exists a sufficiently large number $C = C_t$ such that

$$n(\varepsilon, S_d) \le C\, T(\varepsilon^{-1}, d)^t \quad \text{for all } d \le d_0 \text{ and } \varepsilon \in (0, \sqrt{\lambda_2}). \tag{29}$$

We now consider arbitrarily large d. Let us estimate the sum on the right hand side of inequality (28). For this purpose we choose $k \in \mathbb{N}$ such that $\lambda_2 > C_1 k^{-\beta}$. Since $\lambda_2 \le C_1 2^{-\beta}$, we have obviously $k > 2$. We have

$$\sum_{j=1}^{\infty} \lambda_j^{p/2} \le 1 + \lambda_2^{p/2} + \cdots + \lambda_k^{p/2} + C_1^{p/2} \sum_{j=k+1}^{\infty} j^{-\frac{p\beta}{2}},$$

and

$$\sum_{j=k+1}^{\infty} j^{-\frac{p\beta}{2}} \le \int_k^\infty x^{-\frac{p\beta}{2}}\,dx = \frac{k^{-\frac{p\beta}{2}+1}}{(p\beta/2)-1}\,.$$

Now we choose $p = p(d)$ such that

$$k\left(\lambda_2^{p/2} + \frac{(C_1 k^{-\beta})^{p/2}}{(p\beta/2)-1}\right) = \frac{1}{d}\,.$$

From $k\lambda_2^{p/2} \le 1/d$ we conclude

$$p \ge 2\left(\frac{\ln d + \ln k}{\ln(\lambda_2^{-1})}\right)\,.$$

From $\lambda_2 > C_1 k^{-\beta}$ we get

$$k\left(1 + \frac{1}{(p\beta/2)-1}\right)\lambda_2^{p/2} \ge \frac{1}{d}\,,$$

implying

$$p \le 2\left(\frac{\ln d + \ln k + \ln\left(1+\frac{1}{(p\beta/2)-1}\right)}{\ln(\lambda_2^{-1})}\right)\,.$$

Thus we have

$$p = \frac{2\ln(d)}{\ln(\lambda_2^{-1})}(1 + o_d(1)) \quad \text{as } d \to \infty.$$

Let now $\sigma \in (0,1)$ and $d_\sigma \in \mathbb{N}$ such that $o_d(1) \le \sigma$ and

$$\frac{\ln T(\varepsilon^{-1}, d)}{\ln(\varepsilon^{-1})(1+\ln(d))} \ge (1+\sigma)^{-1}F$$

for all $d \ge d_\sigma$ and all $\varepsilon \in (0, \sqrt{\lambda_2})$. For these d and ε we have

$$\begin{aligned} n(\varepsilon, S_d) &\le C_2\left(1+\frac{1}{d}\right)^{d-1}\varepsilon^{-p} \le e\,C_2 \exp\left(\frac{2\ln(d)}{\ln(\lambda_2^{-1})}(1+\sigma)\ln(\varepsilon^{-1})\right) \\ &\le C_3 \exp\left(\frac{2}{\ln(\lambda_2^{-1})}F^{-1}(1+\sigma)^2 \ln T(\varepsilon^{-1}, d)\right), \end{aligned}$$

where $C_3 := e\,C_2$. Hence for $\tau = \tau(\sigma, p, d_\delta) := 2(\ln(\lambda_2^{-1}))^{-1}(1+\sigma)^2F^{-1}$ we get

$$n(\varepsilon, S_d) \le C_3 T(\varepsilon^{-1}, d)^\tau \quad \text{for all } d \ge d_\sigma \text{ and } \varepsilon \in (0, \sqrt{\lambda_2}). \tag{30}$$

The estimates (29) and (30) show that we have (T, Ω^{unr})-tractability. Choosing $d_0 = d_\sigma$ in (29) and letting σ tend to zero yields the claimed upper bound for t^{tra}.

Since our problem is at least as hard as the problem with only two positive eigenvalues $0 < \lambda_2 < \lambda_1 = 1$ for $d = 1$, the lower bound $t^{\text{tra}} \ge B_2^{-1}$ follows from Theorem 3.2 for $k = 2$. □

The upper bound on the exponent $t^{\rm tra}$ in Theorem 5.1 is, in general, sharp. Indeed, assume that $\lambda_j = \Theta(j^{-\beta})$ and take $T(x,y) = x^{1+\ln y}$. Then $n(\varepsilon, S_1) = \Theta(\varepsilon^{-2/\beta})$ which easily implies that $t^{\rm tra} \ge 2/\beta$. In this case, we have $F = 1$ and $B_2 = \frac{1}{2}\ln(\lambda_2^{-1})$. This shows that the upper bound on $t^{\rm tra}$ in Theorem 5.1 is sharp and

$$t^{\rm tra} \;=\; \max\left\{\frac{2}{\beta}, \frac{2}{\ln(\lambda_2^{-1})}\right\}.$$

Hence, for $\beta \ge \ln(\lambda_2^{-1})$ the exponent of tractability is the same as for the problem with only two positive eigenvalues $0 < \lambda_2 < \lambda_1 = 1$. For this tractability function, the problem S with polynomially decaying eigenvalues is as hard as the problem with only two positive eigenvalues. However, for $\beta < \ln\,\lambda_2^{-1}$, the exponent of tractability depends on β and the problem S is harder than the problem with only two positive eigenvalues.

Corollary 5.2. *Let $1 = \lambda_1 > \lambda_2 > 0$ and $\lambda_j = \mathcal{O}(j^{-\beta})$ for all $j \in \mathbb{N}$ and some fixed $\beta > 0$. Let $f_i : [1,\infty) \to (0,\infty)$, $i = 1,2$, be non-decreasing functions such that*

$$\lim_{x+y\to\infty} \frac{f_1(x) f_2(y)}{x+y} = 0\,.$$

For $T(x,y) = \exp(f_1(x)f_2(y))$, we have $(T,\Omega^{\rm unr})$-tractability iff

$$a_i := \liminf_{x\to\infty} \frac{f_i(x)}{\ln x} \in (0,\infty] \quad \textit{for} \;\; i = 1,2.$$

If a_1, $a_2 \in (0,\infty]$, then the exponent of tractability satisfies

$$\frac{2}{a_1\,a_2\,\ln(\lambda_2^{-1})} \le t^{\rm tra} \le \max\left\{\frac{2}{\beta}, \frac{2}{\ln(\lambda_2^{-1})}\right\} \frac{1}{\min\{a_1 b_2, b_1 a_2\}},$$

where

$$b_1 = \inf_{\varepsilon<\sqrt{\lambda_2}} \frac{f_1(\varepsilon^{-1})}{\ln(\varepsilon^{-1})} \quad \textit{and} \quad b_2 = \inf_{d\in\mathbb{N}} \frac{f_2(d)}{1+\ln(d)}.$$

Proof. We have already seen in Example 3.5 that even for two non-zero eigenvalues λ_1, λ_2 and $0 = \lambda_3 = \lambda_4 = \dots$ the condition $a_1, a_2 > 0$ is necessary for S to be $(T,\Omega^{\rm unr})$-tractable, and that $t^{\rm tra} \ge 2/(a_1 a_2 \ln(\lambda_2^{-1}))$.

Let us now assume that a_1, $a_2 \in (0,\infty]$. It is easy to see that

$$F = \liminf_{\substack{\varepsilon^{-1}+d\to\infty\\ \varepsilon<\sqrt{\lambda_2}}} \frac{\ln T(\varepsilon^{-1},d)}{\ln(\varepsilon^{-1})(1+\ln(d))} = \liminf_{\substack{\varepsilon^{-1}+d\to\infty\\ \varepsilon<\sqrt{\lambda_2}}} \frac{f_1(\varepsilon^{-1})f_2(d)}{\ln(\varepsilon^{-1})(1+\ln(d))} = \min\{a_1b_2, b_1a_2\},$$

and that a_1, $a_2 > 0$ implies b_1, $b_2 > 0$. Thus $F > 0$ and due to Theorem 5.1 we have $(T,\Omega^{\rm unr})$-tractability and the stated upper bound for $t^{\rm tra}$. $\square$

We illustrate Corollary 5.2 again for $T(x,y) = x^{1+\ln y} = \exp((\ln x)(1+\ln y))$. We now have $a_1 = a_2 = b_1 = b_2 = 1$. If we assume that $\lambda_j = \Theta(j^{-\beta})$ then, as we have already checked, $t^{\rm tra} = \max\{2/\beta, 2/\ln \lambda_2^{-1}\}$. Hence, the upper bound on $t^{\rm tra}$ in Corollary 5.2 is, in general, sharp. This proves that for tractability functions T of the form $T(x,y) = \exp(f_1(x)f_2(x))$, the exponent of tractability may depend on β, i.e., on how fast the eigenvalues decay to zero for $d=1$.

We now consider different tractability functions of the form $T(x,y) = f_1(x)f_2(x) = \exp(\ln\, f_1(x) + \ln\, f_2(x))$ and show that for such functions the exponent of tractability does *not* depend on β. The following theorem generalizes a result from [7] which corresponds to $f_i(x) = \exp(\ln^{1+\alpha_i}(1+x))$.

Theorem 5.3. *Let S be a linear tensor product problem with $1 = \lambda_1 > \lambda_2 > 0$ and $\lambda_j = \mathcal{O}(j^{-\beta})$ for all $j \in \mathbb{N}$. For $i = 1,2$ let $f_i : [1,\infty) \to [1,\infty)$ be a non-decreasing function with*

$$a_i := \liminf_{x\to\infty} \frac{\ln\ln f_i(x)}{\ln\ln x} < \infty.$$

Then the function T defined by $T(x,y) = f_1(x)f_2(y)$ is a tractability function.

S is $(T,\Omega^{\rm unr})$-tractable iff

$$a_1 > 1,\ a_2 > 1,\ (a_1-1)(a_2-1) \ge 1, \quad \text{and} \quad B_2 \in (0,\infty],$$

where B_2 is given by (12) for $k=2$.

If $a_1 > 1$, $a_2 > 1$ and $(a_1-1)(a_2-1) > 1$ then $B_2 = \infty$ and the exponent of tractability $t^{\rm tra}$ is zero.

If $a_1 > 1$, $a_2 > 1$, $(a_1-1)(a_2-1) = 1$ and $B_2 > 0$ then the exponent of tractability is

$$t^{\rm tra} \;=\; B_2^{-1} \;=\; \left(\liminf_{\substack{\varepsilon^{-1}+d\to\infty \\ \varepsilon<\sqrt{\lambda_2}}} \frac{\ln f_1(\varepsilon^{-1}) \;+\; \ln f_2(d)}{\alpha(\varepsilon)\;\ln(d)} \right)^{-1}.$$

Proof. Since $a_1,\ a_2 < \infty$, it is obvious that T is a tractability function. Let first S be (T,Ω)-tractable, i.e., there exist positive constants C,t such that

$$n(\varepsilon,S_d) \;\le\; C\, f_1(\varepsilon^{-1})^t f_2(d)^t \quad \text{for all} \quad (\varepsilon^{-1},d) \in \Omega^{\rm unr}.$$

Due to (10) we have

$$n(\varepsilon,S_d) \ge \binom{d}{\alpha(\varepsilon)} \ge \left(\frac{d}{\alpha(\varepsilon)}\right)^{\alpha(\varepsilon)},$$

which implies

$$\alpha(\varepsilon)\ln\left(\frac{d}{\alpha(\varepsilon)}\right) \le \ln(C) + t\ln f_1(\varepsilon^{-1}) + t\ln f_2(d). \tag{31}$$

Keeping ε fixed and letting d grow, we see that for any $\delta > 0$ there exists a $d' = d'(\delta,\varepsilon)$ such that for all $d \ge d'$ we have $\alpha(\varepsilon)\ln(d) \le (t+\delta)\ln f_2(d)$, and therefore

$$1 + \frac{\ln\alpha(\varepsilon)}{\ln\ln(d)} \le \frac{\ln\ln f_2(d)}{\ln\ln(d)} + \frac{\ln(t+\delta)}{\ln\ln(d)}.$$

Thus $a_2 \geq 1$. Let now ε vary and take $d = 2\alpha(\varepsilon)$. Since $\ln f_2(d) = o(d) = o(\alpha(\varepsilon))$, we get from (31) for arbitrary $\delta > 0$, for $\varepsilon' = \varepsilon'(\delta)$ sufficiently small, and for all $\varepsilon \leq \varepsilon'$ that $\alpha(\varepsilon)\ln(2) \leq (t+\delta)\ln f_1(\varepsilon^{-1})$. Since

$$\ln \alpha(\varepsilon) = \ln(2) + \ln\ln(\varepsilon^{-1}) - \ln\ln(\lambda_2^{-1}) + \mathcal{O}(1) \quad \text{as } \varepsilon \text{ tends to zero,}$$

the estimate $a_1 \geq 1$ easily follows. Let now $\eta > a_1 - 1$. Define

$$d = d(\varepsilon) = \alpha(\varepsilon)^{\alpha(\varepsilon)^\eta}.$$

Then (31) yields

$$(\alpha(\varepsilon)^{\eta+1} - \alpha(\varepsilon))\ln(\alpha(\varepsilon)) \leq \ln(C) + t\ln f_1(\varepsilon^{-1}) + t\ln f_2(d).$$

Due to the choice of η and the fact that $\alpha(\varepsilon) = 2\ln(\varepsilon^{-1})/\ln(\lambda_2^{-1}) + \mathcal{O}(1)$, the function $\ln\, f_1(\varepsilon^{-1})$ is of order $o(\alpha(\varepsilon)^{\eta+1})$. We thus have for arbitrary δ, for $\varepsilon(\delta)$ sufficiently small, and for all $\varepsilon \leq \varepsilon(\delta)$,

$$\alpha(\varepsilon)^{\eta+1}\ln(\alpha(\varepsilon)) \leq (t+\delta)\ln f_2(d),$$

leading to

$$\eta + 1 + \frac{\ln\ln(\alpha(\varepsilon))}{\ln(\alpha(\varepsilon))} \leq \frac{\ln(t+\delta)}{\ln(\alpha(\varepsilon))} + \frac{\ln\ln f_2(d)}{\ln\ln(d)}\frac{\ln\ln(d)}{\ln(\alpha(\varepsilon))}.$$

This implies

$$\eta + 1 \leq \left(\liminf_{d\to\infty}\frac{\ln\ln f_2(d)}{\ln\ln(d)}\right)\left(\lim_{\varepsilon^{-1}\to\infty}\frac{\eta\ln(\alpha(\varepsilon)) + \ln\ln(\alpha(\varepsilon))}{\ln(\alpha(\varepsilon))}\right) = a_2\eta.$$

Thus $\eta(a_2 - 1) \geq 1$. Letting η tend to $a_1 - 1$ we get $(a_1-1)(a_2-1) \geq 1$. This proves that $a_1 > 1$ and $a_2 > 1$. Furthermore, due to Theorem 3.2, B_2 has to be positive or infinite for any tractable problems with two positive eigenvalues $0 < \lambda_2 < \lambda_1 = 1$.

Assume now that $a_1 > 1$, $a_2 > 1$, $(a_1-1)(a_2-1) \geq 1$, and $B_2 > 0$. Due to Theorem 5.1, to prove (T, Ω^{unr})-tractability it is enough to verify that

$$F = \liminf_{\substack{\varepsilon^{-1}+d\to\infty \\ \varepsilon<\sqrt{\lambda_2}}} \frac{\ln f_1(\varepsilon^{-1}) + \ln f_2(d)}{\ln(\varepsilon^{-1})(1+\ln(d))} \in (0,\infty].$$

Assume we have an arbitrary sequence $\{(\varepsilon_m^{-1}, d_m)\}$ such that $\{\varepsilon_m^{-1} + d_m\}$ tends to infinity, $\varepsilon_m < \sqrt{\lambda_2}$, and the sequence $\{F_m\}$, where

$$F_m := \frac{\ln f_1(\varepsilon_m^{-1}) + \ln f_2(d_m)}{\ln(\varepsilon_m^{-1})(1+\ln(d_m))},$$

converges to F. Then we find a sub-sequence $\{(\varepsilon_n^{-1}, d_n)\}$ for which $\{\ln\ln(d_n)/\ln\ln(\varepsilon_n^{-1})\}$ converges to an element $x \in [0,\infty]$. For this sub-sequence we show that $F > 2B/\ln(\lambda_2^{-1})$. If the sequence $\{\varepsilon_n^{-1}\}$ or $\{d_n\}$ is bounded, then $\{F_n\}$ tends to infinity, since a_1 and a_2 are both strictly larger than 1. So we can assume that $\{\varepsilon_n^{-1}\}$ as well as $\{d_n\}$ tend to infinity.

First, let us assume that $x \in [0,(a_1-1))$. Then $\ln(d_n) \le \ln(\varepsilon_n^{-1})^{a_1-1-\delta}$ for δ sufficiently small and sufficiently large $n \ge n(\delta)$. Thus

$$F \ge \liminf_{n\to\infty} \frac{\ln f_1(\varepsilon_n^{-1})}{\ln(\varepsilon_n^{-1})^{a_1-\delta}} = \infty.$$

If $x \in ((a_2-1)^{-1},\infty]$, we just change the roles of the parameters ε^{-1} and d to get

$$F \ge \liminf_{n\to\infty} \frac{\ln f_2(d_n)}{\ln(d_n)^{a_2-\delta}} = \infty.$$

If $(a_1-1)(a_2-1) > 1$, then we have considered all possible values of x in $[0,\infty]$ since then $[0,(a_1-1)) \cup ((a_2-1)^{-1},\infty] = [0,\infty]$, and we have shown that $F=\infty$. Theorem 5.1 implies then that the exponent of tractability is $t^{\rm tra}=0$ and $B_2=\infty$.

If $(a_1-1)(a_2-1)=1$, we still have to consider the case $x=a_1-1$. Then

$$\ln(\alpha(\varepsilon_n)) = \ln\ln(\varepsilon_n^{-1}) + \ln(2) - \ln\ln(\lambda_2^{-1}) + \mathcal{O}(1) \in [(a_2-1)-\delta,(a_2-1)+\delta]\ln\ln(d_n)$$

for arbitrary δ and sufficiently large $n \ge n(\delta)$. Then $\alpha(\varepsilon_n) \le (\ln d_n)^{a_2-1+\delta} = o(d_n)$. Hence we have

$$F = \liminf_{n\to\infty} \frac{\ln f_1(\varepsilon_n^{-1}) + \ln f_2(d_n)}{\alpha(\varepsilon_n)(1+\ln(d_n/\alpha(\varepsilon_n)))} \frac{\alpha(\varepsilon_n)(1+\ln(d_n/\alpha(\varepsilon_n)))}{\ln(\varepsilon_n^{-1})(1+\ln(d_n))} = B_2 \frac{2}{\ln(\lambda_2^{-1})} > 0.$$

To obtain the formula for the exponent $t^{\rm tra}$ we can use the bound on $t^{\rm tra}$ from Theorem 5.1. For $\beta \ge \ln \lambda_2^{-1}$ we get $t^{\rm tra} = B_2^{-1}$. To obtain the same result for $\beta < \ln \lambda_2^{-1}$ we proceed as follows. In the proof of Theorem 5.1 we showed that for small positive δ there is a positive number $C_{\beta,\delta}$ depending only on β and δ such that

$$n(\varepsilon,S_d) \;\le\; C_{\beta,\delta}\,\exp\left(-\max\left\{\frac{2+\delta}{\beta},\frac{2(1+\delta)\ln(d)}{\ln(\lambda_2^{-1})}\right\}\ln(\varepsilon^{-1})\right) \quad \text{for all} \quad (\varepsilon^{-1},d) \in \Omega^{\rm unr}.$$

To show that the last right side function is at most $C\left(f_1(\varepsilon^{-1})f_2(d)\right)^t$ it is enough to check that

$$\frac{2(1+\delta)}{\ln(\lambda_2^{-1})}\,\ln(\varepsilon^{-1})\,\ln(d) \;\le\; t\left(\ln(f_1(\varepsilon^{-1})) \,+\, \ln(f_2(d))\right)$$

for large ε^{-1} and d. Or equivalently that

$$t \;\ge\; (1+\delta)\left(\liminf_{\substack{\varepsilon^{-1}+d\to\infty\\ \varepsilon<\sqrt{\lambda_2}}} \frac{\ln(f_1(\varepsilon^{-1})) \,+\, \ln(f_2(d))}{\alpha(\varepsilon)\,\ln(d)}\right)^{-1}.$$

The last limit inferior is achieved if $\alpha(\varepsilon)$ is a power of $\ln(d)$, and therefore it is the same as B_2. Since δ can be arbitrarily small we conclude that $t^{\rm tra} \le B_2^{-1}$. The lower bound on $t^{\rm tra}$ from Theorem 5.1 then implies $t^{\rm tra} = B_2^{-1}$, as claimed. This completes the proof of Theorem 5.3. □

Remark 5.4. Let the conditions of Theorem 5.3 hold and assume that $a_1 > 1$, $a_2 > 1$ and $(a_1-1)(a_2-1) = 1$. Then condition $B_2 \in (0, \infty]$ does not necessarily hold as the following example shows. Let $\delta : [1, \infty) \to [0, \infty)$ be a decreasing function with $\lim_{x\to\infty} \delta(x) = 0$. Define

$$f_i(x) = \exp\left(\ln(x)^{2-\delta(x)}\right) \quad \text{for } i = 1, 2.$$

Then we have obviously $a_1 = 2 = a_2$ and $(a_1 - 1)(a_2 - 1) = 1$. But

$$\begin{aligned}\left(\ln\, \lambda_2^{-1}\right)^{-1}\, B_2 \le \liminf_{\substack{\varepsilon^{-1}+d\to\infty \\ \varepsilon^{-1}=d}} \frac{\ln(\varepsilon^{-1})^{2-\delta(\varepsilon^{-1})} + \ln(d)^{2-\delta(d)}}{\ln(\varepsilon^{-1})\ln(d)} &= 2 \liminf_{d\to\infty} \ln(d)^{-\delta(d)} \\ = 2 \liminf_{d\to\infty} \exp\left(-\delta(d) \ln\ln(d)\right).\end{aligned}$$

If we choose, e.g., $\delta(x) = (\ln\ln\ln(x))^{-1}$, then we see that $B_2 = 0$.

We stress again that the exponent of tractability in Theorem 5.3 does not depend on β and it is B_2^{-1} for all polynomial decaying eigenvalues with the same two largest eigenvalues $0 < \lambda_2 < \lambda_1 = 1$. However, B_2 depends on particular functions f_i satisfying the conditions of Theorem 5.3. We now show that B_2 can take any positive value or even be infinite. Indeed, take $f_i(x) = \exp\left(c_i\,[\ln\, x]^{(1+\alpha_i)}\right)$ for positive c_i and α_i. Then $a_i = 1 + \alpha_i$. For $\alpha_1\alpha_2 = 1$ it can be checked that

$$B_2 \;=\; c_2(1+\alpha_2)\left(\frac{c_1\,\alpha_1}{c_2}\right)^{1/(1+\alpha_1)}\, \frac{\ln(\lambda_2^{-1})}{2}. \tag{32}$$

Taking, $c_2 = c_1 = c$ and varying c for fixed α_i, we see that B_2 can be any positive number with the same limits a_i.

On the other hand, for $f_i(x) = \exp\left(\ln(e + \ln\, x)\,[\ln\, x]^{1+\alpha_i}\right)$, and $\alpha_1\alpha_2 = 1$ we get $a_i = 1 + \alpha_i$ as before, but $B_2 = \infty$.

We also stress that in Theorem 5.3 we assume that the eigenvalues decay *at least* polynomially. This assumption holds, in particular, for finitely many positive or exponentially decaying eigenvalues. We summarize this discussion in the following remark.

Remark 5.5. As long as a tractability function T is of product form, $T(x, y) = f_1(x)f_2(x)$, then (T, Ω^{unr})-tractability of S as well as the exponent of tractability depend *only* on the functions f_1, f_2 and the second eigenvalue λ_2 as long as the eigenvalues λ_j decay at least polynomially. Hence, if we have two problems, one with only two positive eigenvalues $0 < \lambda_2 < \lambda_1 = 1$, and the second with the same two eigenvalues and the rest of them are non-negative and decaying polynomially, then these two problems lead to the same tractability conditions and to the same exponents of tractability.

We stress that this property does *not* hold for more general tractability functions. For instance, if we consider $T(x, y) = \exp(g_1(x)g_2(y))$, i.e, when $\ln T$ is of product form, then the exponent of tractability may depend on the rate of decay of eigenvalues. This holds, for instance, for $T(x, y) = \exp(\ln(x)\,(1 + \ln(d)))$ as shown after Corollary 5.2.

6 Weak Tractability

So far we discussed (T, Ω^{unr})-tractability of linear tensor product problems with exponentially and polynomially decaying eigenvalues. We now verify what we have to assume about the decay of eigenvalues to obtain weak tractability. As we shall see, in particular, exponential or polynomial decay of eigenvalues implies weak tractability.

Let us consider a logarithmic decay of the eigenvalues, i.e., $\lambda_j = \Theta((1+\ln j)^{-\beta})$ for all j and some fixed $\beta > 0$. In [1] we proved that $\ln n(\varepsilon, S_1) = \Theta(\varepsilon^{-2/\beta}(1+o(1)))$. Thus for $\beta \le 2$ not even the one-dimensional problem S_1 is tractable. For $\beta > 2$, we characterized (T, Ω^{res})-tractability for $\Omega^{\text{res}} = [1,\infty) \times [d^*] \cup [1, \varepsilon_0^{-1}) \times \mathbb{N}$, with $d^* + (1-\varepsilon_0) > 0$, see [1]. Here we consider the unrestricted tractability domain and prove, in particular, weak tractability for $\beta > 2$.

Theorem 6.1.

- *Let* $\lambda_1 = 1$, $\lambda_2 \in (0,1)$ *and*

$$\lambda_j = o\Big((\ln j)^{-2} (\ln(\ln j))^{-2}\Big) \quad \text{as} \quad j \to \infty. \tag{33}$$

  Then the linear tensor product problem S is weakly tractable.

- *If S is weakly tractable then* $\lambda_2 < 1$ *and*

$$\lambda_j = o\Big((\ln j)^{-2}\Big) \quad \text{as} \quad j \to \infty.$$

Proof. To prove the first point, we may assume without loss of generality that $\lambda_j > 0$ for all $j \in \mathbb{N}$. Then there exists a function $f : \mathbb{N} \to (0,\infty)$ with $\lim_{j\to\infty} f(j) = 0$ and

$$\lambda_j = f(j)(1+\ln j)^{-2}(1+\ln(1+\ln j))^{-2} \quad \text{for all } j \in \mathbb{N}.$$

We now show that $\ln n(\varepsilon, 1) = o(\varepsilon^{-1}(\ln(\varepsilon^{-1}))^{-1})$. According to (8) we have

$$n(\varepsilon, 1) = \max\{j \mid g(j)(1+\ln j)(1+\ln(1+\ln j)) < \varepsilon^{-1}\}\,, \tag{34}$$

where $g(j) := f(j)^{-1/2}$. Now let $j = j(\varepsilon) = \exp(\lceil c\varepsilon^{-1}(\ln(\varepsilon^{-1}))^{-1}\rceil - 1)$ for some $c > 0$. Then

$$\begin{aligned}&\varepsilon g(j)(1+\ln j)(1+\ln(1+\ln j)) \ge \\ &g(j(\varepsilon))(c + (1+\ln c)c(\ln(\varepsilon^{-1}))^{-1} - c\ln\ln(\varepsilon^{-1})(\ln(\varepsilon^{-1}))^{-1})\,,\end{aligned}$$

which tends to infinity as ε approaches zero. From this calculation and from (34) we conclude that $\ln n(\varepsilon, 1) = o(\varepsilon^{-1}(\ln(\varepsilon^{-1}))^{-1})$. With $a := \min\{\alpha(\varepsilon), d\}$ we get from (10)

$$\frac{\ln n(\varepsilon,d)}{d+\varepsilon^{-1}} \le \frac{\ln\binom{d}{a} + a\ln n(\varepsilon,1)}{d+\varepsilon^{-1}} \le \frac{a(\ln(d/a) + 1 + \ln n(\varepsilon,1))}{d+\varepsilon^{-1}}\,.$$

Case 1: $\alpha(\varepsilon) \le d$. Then

$$\frac{\ln n(\varepsilon, d)}{d+\varepsilon^{-1}} \le \frac{\alpha(\varepsilon)(\ln(d/\alpha(\varepsilon))+1)}{d+\varepsilon^{-1}} + \frac{o(\alpha(\varepsilon)\varepsilon^{-1}(\ln(\varepsilon^{-1}))^{-1})}{d+\varepsilon^{-1}}\,. \tag{35}$$

Since $\alpha(\varepsilon) \sim \ln(\varepsilon^{-1})$, the second term on the right hand side of (35) goes to zero as $d+\varepsilon^{-1}$ tends to infinity. If $\alpha(\varepsilon) = \Theta(d)$, the first term goes obviously also to zero if $d+\varepsilon^{-1}$ tends to infinity. So let us consider the case $\alpha(\varepsilon) = o(d)$. If $\alpha(\varepsilon) = \Omega(d/(\ln d))$, then $\varepsilon^{-1} = \exp(\Omega(d/(\ln d)))$, and $\alpha(\varepsilon)(\ln d)\varepsilon \to 0$ as $d+\varepsilon^{-1} \to \infty$. If $\alpha(\varepsilon) = o(d/(\ln d))$, then $\alpha(\varepsilon)(\ln d)/d \to 0$ as $d+\varepsilon^{-1} \to \infty$.

Case 2: $\alpha(\varepsilon) > d$. Then

$$\frac{\ln n(\varepsilon, d)}{d+\varepsilon^{-1}} \le \frac{d}{d+\varepsilon^{-1}} + \frac{d(o(\varepsilon^{-1}(\ln(\varepsilon^{-1})^{-1})))}{d+\varepsilon^{-1}} = \frac{o(\varepsilon^{-1})}{\varepsilon^{-1}} \to 0$$

as $d+\varepsilon^{-1} \to \infty$. Altogether we proved $\lim_{d+\varepsilon^{-1}\to\infty} \ln n(\varepsilon,d)/(d+\varepsilon^{-1}) = 0$.

We switch to the second point and assume that S is weakly tractable. Then $\lambda_2 < 1$ since otherwise $n(\varepsilon, S_1) \ge 2^d$ for all $\varepsilon \in (0,1)$. For $d = 1$ we have

$$n(\varepsilon, S_1) \;=\; \min\{\, j \mid \lambda_{j+1} \le \varepsilon^2 \,\} \;=\; \exp\left(o\left(\varepsilon^{-1}\right)\right).$$

This can happen only if $\lambda_j = o((\ln j)^{-2})$, as claimed. This completes the proof. □

7 Comparison

We briefly compare tractability results of this paper for the unrestricted domain

$$\Omega^{\text{unr}} \;=\; [1,\infty) \times \mathbb{N}$$

with tractability results of [1] for the restricted domain

$$\Omega^{\text{res}} \;=\; [1,\infty) \times \{1, 2, \ldots, d^*\} \,\cup\, [1, \varepsilon_0^{-1}) \times \mathbb{N}$$

for $d^* \ge 1$ and $\varepsilon_0 \in (0,1)$.

We consider linear tensor product problems S with $\varepsilon_0^2 < \lambda_2 < \lambda_1 = 1$.

- Strong (T, Ω^{unr})-tractability of S as well as strong (T, Ω^{res})-tractability of S does not hold regardless of the tractability function T, see [1, Lemma 3.1].

- Consider finitely many, say k, positive eigenvalues as in Section 2. This case has not been formally studied in [1] for Ω^{res}. However, it is easy to see from (14) that for (ε, d) with $d \le d^*$, the information complexity $n(\varepsilon, S_d)$ is uniformly bounded in ε^{-1}. Therefore the more interesting case is when $(\varepsilon^{-1}, d) \in [1, \varepsilon_0^{-1}) \times \mathbb{N}$. Then $n(\varepsilon, S_d) = \Theta(d^{\alpha(\varepsilon)})$ with the factors in the Theta-notation only dependent on ε_0,

λ_2 and k. So we have a polynomial dependence on d which obviously implies weak tractability. It follows from [1, Theorem 4.1] that $(T,\Omega^{\rm res})$-tractability of S holds iff

$$B_{\rm res} := \liminf_{d\to\infty} \inf_{1\le\alpha(\varepsilon)\le\alpha(\varepsilon_0)} \frac{\ln T(\varepsilon^{-1},d)}{\alpha(\varepsilon)\,\ln(d)} \in (0,\infty], \tag{36}$$

and the exponent of tractability is $1/B_{\rm res}$.

In particular, we have polynomial tractability, i.e., when $T(x,y)=xy$, with the exponent

$$\alpha(\varepsilon_0) = \left\lceil \frac{2\,\ln(\varepsilon_0^{-1})}{\ln(\lambda_2^{-1})} \right\rceil - 1.$$

This exponent can be arbitrarily large if ε_0 is small or λ_2 close to one. On the other hand, it is interesting that the exponent does not depend on the total number k of positive eigenvalues.

As we already said, for the unrestricted domain $\Omega^{\rm unr}$ we do *not* have polynomial tractability of S. This agrees with the fact that the exponent of polynomial tractability for the restricted domain goes to infinity as ε_0 approaches zero, and for the unrestricted domain formally $\varepsilon_0 = 0$.

- Consider exponentially decaying eigenvalues $\lambda_j = \exp(-\beta(j-1))$ for a positive β. Then [1, Theorem 4.8] states that $(T,\Omega^{\rm res})$-tractability of S holds iff

$$A_{\rm e,res} := \liminf_{x\to\infty} \frac{\ln T(x,1)}{\ln\,\ln(x)} \in (0,\infty] \quad\text{and}\quad B_{\rm res} \in (0,\infty],$$

where $B_{\rm res}$ is given by (36). Furthermore, if $A_{\rm e,res}=\infty$ then the exponent of tractability is $B_{\rm res}^{-1}$.

Hence, we again have polynomial tractability, and indeed since $A_{\rm e,res}=\infty$ and $\lambda_2 = \exp(-\beta)$, the exponent of polynomial tractability is

$$\alpha(\varepsilon_0) = \left\lceil \frac{2\ln\,\varepsilon_0^{-1}}{\beta} \right\rceil - 1.$$

As we know, for the unrestricted domain $\Omega^{\rm unr}$ we do not have polynomial tractability.

Take now $T(x,y) = x^{1+\ln y}$. Then $A_{\rm e,res}=\infty$ and $B_{\rm res}=\beta/2$. Furthermore, as we already know, $B_{\rm e}^{(2)} = \beta/2$. So we have $(T,\Omega^{\rm res})$-tractability as well as $(T,\Omega^{\rm unr})$-tractability with the same exponents $2/\beta$. Hence, there is no much difference between the restricted and unrestricted domains for this particular tractability function.

Note also the difference in the exponents for the last two tractability functions and for the restricted domain. For polynomial tractability, the exponent depends on ε_0 and goes to infinity as ε_0 approaches zero. For the second tractability function, the exponent does not depend on ε_0.

- Consider polynomially decaying eigenvalues $\lambda_j = \Theta(j^{-\beta})$ for a positive β. Then [1, Theorem 4.8] states that (T, Ω^{res})-tractability of S holds iff

  $$A_{\text{p,res}} := \liminf_{x\to\infty} \frac{\ln T(x,1)}{\ln(x)} \in (0,\infty] \quad \text{and} \quad B_{\text{res}} \in (0,\infty].$$

  If this holds then the exponent of tractability is $t^{\text{tra}} = \max\{2/(\beta\, A_{\text{p,res}}), 1/B_{\text{res}}\}$.

  Let us consider polynomial tractability, i.e., $T(x,y) = xy$. Then $A_{\text{p,res}} = 1$ and, as stated above, $B_{\text{res}} = \alpha(\varepsilon_0)^{-1}$. Due to [1, Theorem 4.8] we have (T, Ω^{res})-tractability with $t^{\text{tra}} = \max\{2/\beta, \alpha(\varepsilon_0)\}$ but, as already mentioned, no (T, Ω^{unr})-tractability.

  Take now $T(x,y) = \exp(\ln^2 x)\exp(\ln^2 y)$. Then $A_{\text{p,res}} = B_{\text{res}} = \infty$, and S is (T, Ω^{res})-tractable with $t^{\text{tra}} = 0$. For the unrestricted case, we conclude from (32) that S is (T, Ω^{unr})-tractable with $t^{\text{tra}} = (\ln(\lambda_2^{-1}))^{-1}$. Hence, we have tractability in both cases but the exponents are quite different.

  Let now $T(x,y) = x^{1+\ln y}$. Then $A_{\text{p,res}} = 1$ and $B_{\text{res}} = \beta/2$. Thus S is (T, Ω^{res})-tractable with $t^{\text{tra}} = 2/\beta$, see also [1, Theorem 4.8]. As already stated, we have also (T, Ω^{unr})-tractability with the exponent of tractability $t^{\text{tra}} = \max\{2/\beta, 2/\ln(\lambda_2^{-1})\}$.

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