Tractability of multivariate approximation over a weighted unanchored Sobolev space: Smoothness sometimes hurts

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Abstract

We study d-variate L_2-approximation for a weighted unanchored Sobolev space having smoothness $m \geq 1$. Folk wisdom would lead us to believe that this problem should become easier as its smoothness increases. This is true if we are only concerned with asymptotic analysis: the nth minimal error is of order $n^{-(m-\delta)}$ for any $\delta > 0$. However, it is unclear how long we need to wait before this asymptotic behavior kicks in. How does this waiting period depend on d and m? We prove that no matter how the weights are chosen, the waiting period is at least m^d, even if the error demand ε is arbitrarily close to 1. Hence, for $m \geq 2$, this waiting period is exponential in d, so that the problem suffers from the curse of dimensionality and is intractable. In other words, the fact that the asymptotic behavior improves with m is irrelevant when d is large. So, we will be unable to vanquish the curse of dimensionality unless $m = 1$, i.e., unless the smoothness is minimal. We then show that our problem *can* be tractable if $m = 1$. That is, we can find an ε-approximation using polynomially-many (in d and ε^{-1}) information operations, even if only function values are permitted. When $m = 1$, it is even possible for the problem to be *strongly*

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tractable, i.e., we can find an ε-approximation using polynomially-many (in ε^{-1}) information operations, independent of d. These positive results hold when the weights of the Sobolev space decay sufficiently quickly or are bounded finite-order weights, i.e., the d-variate functions we wish to approximate can be decomposed as sums of functions depending on at most ω variables, where ω is independent of d.

1 Introduction

It is widely believed that as a problem becomes smoother, the easier it is to solve. In this paper, we show that this belief is not always well-founded, by providing a natural counterexample.

We first consider the problem of approximating a real function defined over the unit interval, with error to be measured in the L_2-sense. Suppose that the functions f to be approximated have m derivatives for some $m \geq 1$, satisfying

$$\int_0^1 f^2(x)\,dx + \int_0^1 \left(f^{(m)}(x)\right)^2\,dx \leq 1.$$

This class of functions is the unit ball of a reproducing kernel Hilbert space (RKHS) $H_{1,m}$ studied in [7]. For $m = 1$, the reproducing kernel has an intriguing explicit representation, see (39) in Section 4.2.

Let us consider the nth minimal error $e(n, H_{1,m})$ for this approximation problem, which is defined to be the minimal worst case error of any algorithm using at most n linear functionals (or function values) of the function to be approximated. It is well-known that the nth minimal error for this approximation problem is

$$e(n, H_{1,m}) = \begin{cases} 1 & \text{if } n < m, \\ \Theta(n^{-m}) & \text{as } n \to \infty, \end{cases}$$

where the Θ-factor depends on m and is independent of n. Note that this asymptotic error behavior improves as the smoothness m increases. So smoothness helps for univariate approximation.

What happens in the multivariate L_2-approximation problem? We now assume that the functions f to be approximated belong to the unit ball of a Sobolev space $H_{d,m}$ (defined as the d-fold tensor product of $H_{1,m}$, see §2), so that

$$\int_{[0,1]^d} f^2(\mathbf{x})\,d\mathbf{x} + \sum_{\substack{\mathfrak{u}\subseteq[d]\\ \mathfrak{u}\neq\emptyset}} \int_{[0,1]^d} \left(\frac{\partial^{m|\mathfrak{u}|}}{\partial^m \mathbf{x}_{\mathfrak{u}}} f(\mathbf{x})\right)^2 d\mathbf{x} \leq 1.$$

Here, $[d] := \{1, 2, \ldots, d\}$ and $\mathbf{x}_{\mathfrak{u}}$ denotes the vector whose components are those components x_j of $\mathbf{x}$ for which $j \in \mathfrak{u}$. That is,

$$\frac{\partial^{m|\mathfrak{u}|}}{\partial^m \mathbf{x}_{\mathfrak{u}}} = \prod_{j\in\mathfrak{u}} \frac{\partial^m}{\partial^m x_j}.$$

It is easy to see that the nth minimal error $e(n, H_{d,m})$ of this approximation problem satisfies[1]

$$e(n, H_{d,m}) = \mathscr{O}(n^{-(m-\delta)}) \qquad \forall\,\delta > 0, \tag{1}$$

where the $\mathscr{O}$-factor depends on d, m, and δ and is independent of n. Hence the exponent of n improves as m increases, and so smoothness helps asymptotically for the multivariate approximation problem as well.

[1]For $d = 1$, the space $H_{1,m}$ is equivalent to the standard Sobolev space $H^m([0, 1])$, for which it is known that the nth minimal error is $\Theta(n^{-m})$, regardless of whether continuous linear functionals or function values are used. For $d \geq 2$, we can use Smolyak's algorithm as in [9] to see that the nth minimal error is $\mathscr{O}(n^{-m}(\log n)^{c(m-1)})$, where c is independent of both d and n.

How long do we need to wait until we see the asymptotic behavior of (1) in action? We already know that for $d = 1$, we have to wait for n to be at least as big as m. For general d, we have

$$e(n, H_{d,m}) = 1 \qquad \text{if } n < m^d.$$

So our wait period is exponentially large in d for $m \geq 2$. In this case, one might blame the long wait period on the fact that the space $H_{d,m}$ is too big. To shrink $H_{d,m}$, we introduce a family $\Gamma = \{\gamma_{d,\mathfrak{u}}\}$ of non-negative weights, and we consider functions f belonging to the unit ball of a space $H_{d,m,\Gamma}$ (also defined in §2), so that

$$\int_{[0,1]^d} f^2(\mathbf{x})\, d\mathbf{x} + \sum_{\substack{\mathfrak{u}\subseteq[d]\\ \mathfrak{u}\neq\emptyset}} \frac{1}{\gamma_{d,\mathfrak{u}}} \int_{[0,1]^d} \left(\frac{\partial^{m|\mathfrak{u}|}}{\partial^m \mathbf{x}_{\mathfrak{u}}} f(\mathbf{x})\right)^2 d\mathbf{x} \leq 1.$$

(If $\gamma_{d,\mathfrak{u}} = 0$ for some $\mathfrak{u}$, then we will require that $(\partial^{m|\mathfrak{u}|}/\partial^m \mathbf{x}_{\mathfrak{u}}) f(\mathbf{x}) \equiv 0$, and interpret $0/0$ as 0.) Clearly if all $\gamma_{d,\mathfrak{u}} \leq 1$, we shrink the class of functions that we are trying to approximate. The most extreme case is when all the weights $\gamma_{d,\mathfrak{u}}$ are zero, in which case the space $H_{d,m,\Gamma}$ is now the space of polynomials of degree at most $m - 1$ in each variable; this space has dimension m^d.

The weights Γ cannot worsen the asymptotic behavior of the nth minimal error, and so we again have

$$e(n, H_{d,m,\Gamma}) = \mathcal{O}(n^{-(m-\delta)}) \qquad \forall\, \delta > 0. \tag{2}$$

Once again, it is natural to ask how long we must wait for this asymptotic behavior to take hold. It is relatively easy to prove that

$$e(n, H_{d,m,\Gamma}) = 1 \qquad \text{if } n < m^d, \tag{3}$$

see Theorem 3.1. We stress that this result holds for *any* family Γ of weights, even if the weights are all zero. The reason for this is that the norms in the source and target spaces are the same over the m^d-dimensional subspace of polynomials mentioned previously; that is, the $H_{d,m,\Gamma}$ and $L_2([0, 1]^d)$ norms coincide on this polynomial subspace.

Let $\mathrm{card}(\varepsilon, d, H_{d,m,\Gamma})$ be the smallest number of information operations needed to guarantee that the worst case error of an approximation is at most ε. By an information operation, we mean either the evaluation of a linear functional or the evaluation of a function at some point in its domain. Our approximation problem is properly normalized, in the sense that we have the sharp a priori bound $\|f\|_{L_2([0,1])^d} \leq 1$ for all f in the unit ball of $H_{d,m,\Gamma}$. This implies that

$$\mathrm{card}(\varepsilon, d, H_{d,m,\Gamma}) = 0 \qquad \text{for } \varepsilon = 1.$$

From (3), we see that

$$\mathrm{card}(\varepsilon, d, H_{d,m,\Gamma}) \geq m^d \qquad \text{for } \varepsilon < 1.$$

(Note that this cardinality is discontinuous at $\varepsilon = 1$.) We emphasize that this result holds for *any* set of weights, and for ε arbitrarily close to 1. Hence for $m \geq 2$, our problem suffers from the curse of dimensionality, and is intractable.

So, the only possibility of a positive tractability result occurs when $m = 1$. By tractability, we mean one of the following:

- *strong polynomial tractability*, i.e., $\mathrm{card}(\varepsilon, d, H_{d,m,\Gamma})$ is bounded by a polynomial in $1/\varepsilon$, independent of d,
- *polynomial tractability*, i.e., $\mathrm{card}(\varepsilon, d, H_{d,m,\Gamma})$ is bounded by a polynomial in $1/\varepsilon$ and d, or

- *weak tractability*, i.e., $\mathrm{card}(\varepsilon, d)$ does not depend exponentially on $1/\varepsilon + d$.

We show such positive results in §4; these results depend on our assumptions about the weights, as well as on whether we allow the class Λ^{all} of arbitrary continuous linear functionals or the class Λ^{std} of function values as information operations:

- For *equal weights* $\gamma_{d,\mathfrak{u}} \equiv 1$, we have

Class Λ^{all}	Class Λ^{std}
polynomially intractable	polynomially intractable
weakly tractable	not weakly tractable

- For bounded *product weights* $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}$, we have

Class Λ^{all}	Class Λ^{std}
always weakly tractable	weakly tractable iff $\lim_{d\to\infty} \frac{1}{d}\sum_{j=1}^{d} \gamma_{d,j} = 0$
polynomially tractable iff $\exists\, \tau > 0$ such that $\limsup_{d\to\infty} \frac{1}{\ln d}\sum_{j=1}^{d} \gamma_{d,j}^{\tau} < \infty$	polynomially tractable iff $\limsup_{d\to\infty} \frac{1}{\ln d}\sum_{j=1}^{d} \gamma_{d,j} < \infty$
strongly polynomially tractable iff $\exists\, \tau > 0$ such that $\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j}^{\tau} < \infty$	strongly polynomially tractable iff $\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j} < \infty$

- For bounded *finite order weights* $\gamma_{d,\mathfrak{u}} = 0$ for $|u| > \omega$, we have

Class Λ^{all}	Class Λ^{std}
always polynomially tractable	always polynomially tractable

In summary, this multivariate approximation problem suffers from the curse of dimensionality when the smoothness m is at least 2; however, this curse may be lifted when $m = 1$. This leads us to the counter-intuitive realization that increasing smoothness $m \geq 2$ always makes our problem intractable, whereas the problem is tractable for the smallest smoothness $m = 1$, under properly decaying weights.

For $m = 1$, the space $H_{d,1,\Gamma}$ is an RKHS, whose reproducing kernel has the form

$$K_{d,1,\Gamma}(\mathbf{x}, \mathbf{y}) = \sum_{k\in\mathbb{N}^d} \frac{2^{|\{ j\in[d]: k_j > 1 \}|}}{1 + \sum_{\substack{\mathfrak{u}\subseteq[d]\\ \mathfrak{u}\neq\emptyset}} \gamma_{d,\mathfrak{u}}^{-1} \prod_{j\in\mathfrak{u}}[\pi(k_j-1)]^2} \prod_{j=1}^{d} \cos[\pi(k_j-1)x_j]\cos[\pi(k_j-1)y_j]. \qquad (4)$$

For product weights $\gamma_{d,\mathrm{u}} = \prod_{j\in\mathrm{u}} \gamma_{d,j}$, this formula simplifies to

$$K_{d,1,\Gamma}(\mathbf{x},\mathbf{y}) = \prod_{j=1}^{d} \left(1 + 2\sum_{k=1}^{\infty} \frac{\gamma_{d,j}}{\gamma_{d,j}+\pi^2k^2} \cos(\pi k x_j)\cos(\pi k y_j) \right)$$
$$= \prod_{j=1}^{d} \frac{\sqrt{\gamma_{d,j}}}{\sinh\sqrt{\gamma_{d,j}}} \cosh[\sqrt{\gamma_{d,j}}(1-\max\{x_j,y_j\})]\cosh[\sqrt{\gamma_{d,j}}\min\{x_j,y_j\}]\,,$$

where the last equality follows from [7]. Note that this reproducing kernel has a form that is different than that previously studied in [12, 13]. This means that we cannot use the results of [12, 13] for our problem.

We stress that this counterintuitive result holds for the specific space $H_{d,m,\Gamma}$. There are other function spaces for which increasing smoothness may help. For example, suppose that our functions f belong to the unit ball of the usual Sobolev space $H^m([0,1])$, so that

$$\sum_{j=0}^{m} \int_0^1 \left(f^{(j)}(x)\right)^2 dx \le 1.$$

Letting $e(n,H^m)$ denote the nth minimal error for this space, we find that $e(n,H^{m+1}) \le e(n,H^m)$ for all m and n, and that the same asymptotic behavior $e(n,H^m) = \Theta(n^{-m})$ holds as before. So if we take the d-variate version of this approximation to be over the d-fold tensor product $(H^m)^{\otimes d}$ of H^m, then the nth minimal errors $e(n,(H^m)^{\otimes d})$ satisfy

$$e(n,(H^{m+1})^{\otimes d}) \le e(n,(H^m)^{\otimes d}) \qquad \text{for all } m \text{ and } n,$$

as well as

$$e(n,(H^m)^{\otimes d}) = \mathscr{O}(n^{-(m-\delta)}) \qquad \forall\,\delta > 0.$$

Hence we find that smoothness definitely helps in this case. More precisely, the general results of [5] tell us that the L_2-multivariate approximation problem over $(H^m)^{\otimes d}$ for the class Λ^{all} is weakly tractable, but not polynomially tractable, for any m.

The tractability of L_2-multivariate approximation for the weighted version of $(H^m)^{\otimes d}$ remains to be studied. However, suppose that we choose a weighted space $(H^m_\Gamma)^{\otimes d}$ that is a subset of $H_{d,m,\Gamma}$. For example, suppose that Γ consists of product weights and that we take $(H^m_\Gamma)^{\otimes d} = \otimes_{j=1}^d H^m_{\gamma_{d,j}}$, where the space H^m_γ is equipped with the norm

$$\|f\|^2_{H^m_\gamma} = \int_0^1 f^2(x)\,dx + \frac{1}{\gamma}\sum_{j=1}^{m}\int_0^1 \left(f^{(j)}(x)\right)^2 dx.$$

Then sufficient conditions for L_2-approximation to be tractable over $H_{d,m,\Gamma}$ will also be sufficient for tractability over $(H^m)^{\otimes d}$.

2 The approximation problem

In this section, we define the approximation problem to be studied and recall some basic concepts of information-based complexity.

First, we establish some notational conventions. We let $\mathbb{N}$ denote the strictly positive integers. Moreover, we will let I denote the closed unit interval $[0, 1]$. Finally, we will use a slightly more elaborate notation than we used in the Introduction, in which we stress how the results depend on the problem to be solved.

As we mentioned in the Introduction, we consider the L_2-approximation of functions belonging to a Γ-weighted reproducing kernel Hilbert space $H_{d,m,\Gamma}$ of functions over I^d, having smoothness m. We shall define this space in several stages.

Let us first start with the unweighted case:

- First, suppose that $d = 1$. The space $H_{1,m}$ consists of real functions defined on I, whose $(m-1)$st derivatives are absolutely continuous and whose mth derivatives belong to $L_2(I)$, under the inner product
$$\langle f, g\rangle_{H_{1,m}} = \int_0^1 f(x)g(x)\,dx + \int_0^1 f^{(m)}(x)g^{(m)}(x)\,dx \qquad \forall\, f, g \in H_{1,m}.$$
Note that $H_{1,m}$ is simply the standard Sobolev space $H^m(I)$, under a different norm $\|\cdot\|_{H_{1,m}}$ that is equivalent to the standard norm $\|\cdot\|_{H^m(I)}$.

  Since $m \geq 1$, we see that function evaluation $\delta_x\colon f \in H_{1,m} \mapsto f(x)$ is a continuous linear functional for $x \in I$. Hence, $H_{1,m}$ is a reproducing kernel Hilbert space.

- We now consider the case of general $d \in \mathbb{N}$. We define $H_{d,m} = H_{1,m}^{\otimes d}$ as a d-fold tensor product of $H_{1,m}$, under the inner product
$$\langle f, g\rangle_{H_{d,m}} = \int_{I^d} f(\mathbf{x})g(\mathbf{x})\,d\mathbf{x} + \sum_{\substack{\mathfrak{u}\subseteq[d]\\ \mathfrak{u}\neq\emptyset}} \int_{I^d} \frac{\partial^{m|\mathfrak{u}|}}{\partial^m \mathbf{x}_\mathfrak{u}} f(\mathbf{x}) \frac{\partial^{m|\mathfrak{u}|}}{\partial^m \mathbf{x}_\mathfrak{u}} g(\mathbf{x})\,d\mathbf{x} \qquad \forall\, f, g \in H_{d,m}.$$
Here, $|\mathfrak{u}|$ denotes the size of $\mathfrak{u} \subseteq [d] := \{1, 2, \dots, d\}$, and $\mathbf{x}_\mathfrak{u}$ denotes the vector whose components are those components x_j of $\mathbf{x}$ for which $j \in \mathfrak{u}$.

We now move on to the weighted case. Let
$$\Gamma = \{\, \gamma_{d,\mathfrak{u}} \geq 0 : \text{nonempty } \mathfrak{u} \subseteq [d],\, d \in \mathbb{N}\,\}$$
be a given set of non-negative *weights* $\gamma_{d,\mathfrak{u}}$.

Remark 2.1. Although some of our results hold for any family Γ of weights, we will pay special attention to several particularly important specific families:

1. If $\gamma_{d,\mathfrak{u}} > 0$ for all d and all nonempty $\mathfrak{u} \subseteq [d]$, then Γ is a set of *positive weights*, which will be denoted as $\Gamma > \mathbf{0}$.

2. If $\gamma_{d,\mathfrak{u}} = 1$ for all d and all nonempty $\mathfrak{u} \subseteq [d]$, then Γ is a set of *equal weights*, which will be denoted as $\Gamma = \mathbf{1}$.

3. If $\gamma_{d,\mathfrak{u}} = 0$ for all d and all nonempty $\mathfrak{u} \subseteq [d]$, then Γ is a set of *zero weights*, which will be denoted as $\Gamma = \mathbf{0}$.

4. If $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}$ for all d and all nonempty $\mathfrak{u} \subseteq [d]$, where $\gamma_{d,j} \geq 0$ for $j \in [d]$, then Γ is a set of *product weights*.

5. If there exists ω such that for all d, we have $\gamma_{d,\mathfrak{u}} = 0$ whenever $\mathfrak{u} \subseteq [d]$ and $|\mathfrak{u}| > \omega$, then Γ is a set of *finite-order weights* introduced in [1]. The smallest value of ω such that this relation holds is said to be the *order* of Γ. Note that if Γ is a set of finite-order weights of order ω, then Γ contains at most

$$\sum_{j=0}^{\omega} \binom{d}{j} = \frac{d^{\omega}}{\omega!}\bigl(1 + o(1)\bigr) \le 2\,d^{\omega}. \tag{5}$$

nonzero weights, see [13].

6. If there exists ω such that for all d, we have $\gamma_{d,\mathfrak{u}} = 0$ whenever $\mathfrak{u} \subseteq [d]$ and $\operatorname{diam}\mathfrak{u} \ge \omega$, then Γ is a set of *finite-diameter* weights. Here, the diameter of a set $\mathfrak{u}$ is defined to be $\operatorname{diam}\mathfrak{u} = \max_{i,j\in\mathfrak{u}} |i - j|$, as usual. The smallest value of ω such that this relation holds is said to be the *order* of Γ. Note that if Γ is a set of finite-diameter weights of order ω, then Γ contains at most

$$2^{\min\{\omega,d\}-1} - (\min\{\omega, d\} - 2)2^{\min\{\omega,d\}-1} = \Theta(d)$$

nonzero weights. (This concept was introduced by J. Creutzig, see [5, §5.3].) Clearly, finite-diameter weights are always finite-order weights; however, the converse is not true. □

We are now finally ready to define our weighted reproducing kernel Hilbert space $H_{d,m,\Gamma}$ to be

$$H_{d,m,\Gamma} = \left\{ f \in H_{d,m} : \frac{\partial^{m|\mathfrak{u}|}}{\partial^m \mathbf{x}_{\mathfrak{u}}} f \equiv 0 \text{ whenever } \gamma_{d,\mathfrak{u}} = 0 \right\},$$

under the inner product

$$\langle f, g\rangle_{H_{d,m,\Gamma}} = \int_{I^d} f(\mathbf{x})g(\mathbf{x})\,d\mathbf{x} + \sum_{\substack{\mathfrak{u}\subseteq[d]\\ \mathfrak{u}\neq\emptyset\\ \gamma_{d,\mathfrak{u}}>0}} \frac{1}{\gamma_{d,\mathfrak{u}}} \int_{I^d} \frac{\partial^{m|\mathfrak{u}|}}{\partial^m \mathbf{x}_{\mathfrak{u}}} f(\mathbf{x}) \frac{\partial^{m|\mathfrak{u}|}}{\partial^m \mathbf{x}_{\mathfrak{u}}} g(\mathbf{x})\,d\mathbf{x} \qquad \forall\, f, g \in H_{d,m,\Gamma}.$$

Note that the norms $\|\cdot\|_{H_{d,m}}$ and $\|\cdot\|_{H_{d,m,\Gamma}}$ are equivalent when the weights are positive, with

$$\min\left\{1, \min_{\substack{\mathfrak{u}\subseteq[d]\\ \mathfrak{u}\neq\emptyset}} \gamma_{d,\mathfrak{u}}^{-1/2}\right\} \|\cdot\|_{H_{d,m}} \le \|\cdot\|_{H_{d,m,\Gamma}} \le \max\left\{1, \max_{\substack{\mathfrak{u}\subseteq[d]\\ \mathfrak{u}\neq\emptyset}} \gamma_{d,\mathfrak{u}}^{-1/2}\right\} \|\cdot\|_{H_{d,m}}.$$

However when the weights Γ are not necessarily positive, the space $H_{d,m,\Gamma}$ might be a proper subspace of $H_{d,m}$. In the most extreme case $\Gamma = \mathbf{0}$, we see that $H_{d,m,\Gamma}$ is the space $[P_{m-1}(I)]^{\otimes d}$ of polynomials of degree at most $m - 1$ in the variables $x_1, x_2, \ldots, x_d$, with $\|\cdot\|_{H_{d,m,\Gamma}} = \|\cdot\|_{L_2(I^d)}$.

We wish to approximate functions belonging to the unit ball $\mathscr{B}H_{d,m,\Gamma}$ of $H_{d,m,\Gamma}$, measuring the quality of an approximation in the $L_2(I^d)$-norm. This approximation problem is described by the embedding operator $\mathrm{App}_d \colon H_{d,m,\Gamma} \to L_2(I^d)$, which is defined as

$$\mathrm{App}_d\, f = f \qquad \forall f \in H_{d,m,\Gamma}.$$

Such an approximation is given by an algorithm $A_{d,n}$ using at most n information operations from a class Λ of linear functionals on $H_{d,m,\Gamma}$. Here, Λ will be either $\Lambda^{\text{all}} = [H_{d,m,\Gamma}]^*$ of all continuous linear functionals on $H_{d,m}$ (*continuous linear information*) or the class Λ^{std} consisting of function evaluations on I^d (*standard information*). The worst case *error* of $A_{d,n}$ is given by

$$e(A_{d,n}, \Lambda) = \sup_{f \in \mathscr{B}H_{d,m,\Gamma}} \|f - A_{d,n}f\|_{L_2(I^d)}.$$

We define the nth *minimal error* as

$$e(n, \text{App}_d, \Lambda) = \inf_{A_{d,n}} e(A_{d,n}, \Lambda), \tag{6}$$

the infimum being over all algorithms using at most n information operations from Λ. An algorithm $A_{d,n}$ using at most n operations from Λ and for which $e(A_{d,n}, \Lambda) = e(n, \text{App}_d, \Lambda)$ is said to be an *nth minimal error algorithm*.

There is a well-known explicit formula for $e(n, \text{App}_d, \Lambda^{\text{all}})$. Let $W_d = (\text{App}_d)^*(\text{App}_d)$, which is a compact self-adjoint positive definite linear transformation on $H_{d,m,\Gamma}$. Let $\alpha_{d,1} \geq \alpha_{d,2} \geq \cdots > 0$ be the ordered eigenvalues of W_d. Then

$$e(n, \text{App}_d, \Lambda^{\text{all}}) = \sqrt{\alpha_{d,n+1}}\,. \tag{7}$$

Moreover, the algorithm

$$A^*_{d,n}(f) = \sum_{j=1}^{n} \langle f, e_{d,j} \rangle_{H_{d,m,\Gamma}} e_{d,j} \qquad \forall f \in \mathscr{B}H_{d,m,\Gamma}$$

is an nth minimal error algorithm. For further discussion, see (e.g.) [8, §4.5].

If $\varepsilon \in (0, 1]$, we say that the algorithm $A_{d,n}$ provides an *ε-approximation* to if

$$e(A_{d,n}, \Lambda) \leq \varepsilon. \tag{8}$$

Let

$$\text{card}(\varepsilon, \text{App}_d, \Lambda) = \min\{\, n \geq 0 : e(n, \text{App}_d, \Lambda) \leq \varepsilon \,\} \tag{9}$$

denote the *ε-cardinality number*, i.e., the minimal number of information operations from Λ needed to compute an ε-approximation. From (7), we see that

$$\text{card}(\varepsilon, \text{App}_d, \Lambda^{\text{all}}) = \min\{\, n \geq 0 : \alpha_{d,n+1} \leq \varepsilon^2 \,\}. \tag{10}$$

We are now ready to describe various notions of tractability, see (e.g.) [5] for discussion. The approximation problem is said to be *weakly tractable* in the class Λ if

$$\lim_{\varepsilon^{-1}+d \to \infty} \frac{\ln \text{card}(\varepsilon, \text{App}_d, \Lambda)}{\varepsilon^{-1} + d} = 0. \tag{11}$$

Note that the approximation problem is weakly tractable iff the cardinality number grows subexponentially in ε^{-1} and d. We say that this problem is *intractable* if it is not weakly tractable.

The problem is said to be (polynomially) *tractable* in the class Λ if there exist non-negative numbers C, p, and q such that

$$\text{card}(\varepsilon, \text{App}_d, \Lambda) \leq C \left(\frac{1}{\varepsilon}\right)^p d^q \qquad \forall\, \varepsilon \in (0, 1), d \in \mathbb{N}. \tag{12}$$

Numbers $p = p(\mathrm{App}, \Lambda)$ and $q = q(\mathrm{App}, \Lambda)$ such that (12) holds are called ε- and *d-exponents of tractability*; these need not be uniquely defined. Finally, the problem is said to be *strongly* (polynomially) *tractable* in the class Λ if $q = 0$ in (12); in this case, we define

$$p(\Lambda) = \inf \left\{ p \geq 0 : \ \exists\, C \geq 0 \text{ such that } \operatorname{card}(\varepsilon, \mathrm{App}_d, \Lambda) \leq C \left(\frac{1}{\varepsilon}\right)^p \ \forall\, \varepsilon \in (0,1), d \in \mathbb{N} \right\}$$

to be the *exponent of strong tractability*.

Remark 2.2. Note that we are formally using an *absolute* error criterion in (8). There has also been a stream of work using a *normalized* error criterion

$$e(A_{d,n}, \Lambda) \leq \varepsilon \cdot e(0, \mathrm{App}_d, \Lambda)$$

to define an ε-approximation (once again, see [5] and the references cited therein). Here, $e(0, \mathrm{App}_d, \Lambda)$ is the *initial error* that can be obtained by algorithms using no information operations whatsoever. For the spaces used in this paper, it is easy to see that the absolute and normalized error criteria coincide. Indeed, we have $e(0, \mathrm{App}_d, \Lambda) = 1$; this follows from the fact that $\| f \|_{L_2(I^d)} \leq \| f \|_{H_{d,m,\Gamma}}$ for any $f \in H_{d,m,\Gamma}$, with equality holding for $f \equiv 1$. Hence $e(0, \mathrm{App}_d, \Lambda) = \| \mathrm{App}_d \|_{\mathrm{Lin}[H_{d,m,\Gamma}, L_2(I^d)]} = 1$, which implies that the absolute and normalized error criteria coincide, as claimed.

3 Intractability for $m \geq 2$

In this section, we prove that our approximation problem is intractable whenever $m \geq 2$.

Theorem 3.1. *For* any *weights* Γ *and* any $\varepsilon \in [0, 1)$*, we have*

$$\operatorname{card}(\varepsilon, \mathrm{App}_d, \Lambda^{\mathrm{std}}) \geq \operatorname{card}(\varepsilon, \mathrm{App}_d, \Lambda^{\mathrm{all}}) \geq m^d.$$

Hence multivariate approximation App_d *suffers from the curse of dimensionality and is* intractable *whenever* $m \geq 2$.

This result could be easily obtained from general results of information-based complexity. However, we prefer the short direct proof presented below.

Proof of Theorem 3.1. Since $\Lambda^{\mathrm{std}} \subseteq \Lambda^{\mathrm{all}}$, it suffices to prove the result for Λ^{all}. So let Γ be any set of weights. Note that $[P_{m-1}(I)]^{\otimes d}$ is an m^d-dimensional subspace of $H_{d,m,\Gamma}$, with equality when $\Gamma = \mathbf{0}$. Furthermore, the $L_2(I^d)$- and $H_{d,m,\Gamma}$-norms coincide on $[P_{m-1}(I)]^{\otimes d}$.

Suppose that $n < m^d$, and consider any algorithm using at most n functionals from Λ^{all}. For $f \in \mathscr{B}H_{d,m,\Gamma}$, such an algorithm produces an approximation

$$A_{d,n}(f) = \phi(\lambda_1(f), \lambda_2(f), \ldots, \lambda_n(f)),$$

where $\lambda_j = \lambda_j(\cdot; \lambda_1(f), \ldots, \lambda_{j-1}(f))$ are adaptively-chosen continuous linear functionals (i.e., $\lambda_j \in H^*_{d,m,\Gamma}$) for $j \in \{1, 2, \ldots, n\}$. We want to find a nonzero $g \in [P_{m-1}(I)]^{\otimes d}$ such that

$$\begin{aligned} &\lambda_1(g) = 0 \\ &\lambda_2(g; 0) = 0 \\ &\quad\vdots \\ &\lambda_n(g; 0, \ldots, 0) = 0. \end{aligned} \tag{13}$$

Since (13) is a system of n homogeneous linear equations in at least $n+1$ unknowns, such a function g exists. We can normalize g by requiring that $\|g\|_{H_{d,m,\Gamma}} = \|g\|_{L_2(I^d)} = 1$. Observe that $-g$ also satisfies (13), so that $A_{d,n}(\pm g) = a = \phi(0,\ldots,0) \in L_2(I^d)$. Hence

$$\begin{aligned} e(A_{d,n}, \Lambda^{\text{all}}) &\geq \max_{\theta\in\{-1,1\}} \|\theta g - a\|_{L_2(I^d)} \\ &\geq \tfrac{1}{2}\left(\|g-a\|_{L_2(I^d)} + \|g+a\|_{L_2(I^d)}\right) \geq \|g\|_{L_2(I^d)} \\ &= 1. \end{aligned}$$

Since $A_{n,d}$ can be any arbitrary algorithm using n operations from Λ^{all}, we conclude that $e(n, \text{App}_d, \Lambda^{\text{all}}) \geq 1$ if $n < m^d$. Hence to obtain an algorithm with error $\varepsilon < 1$, we must use at least m^d information operations. This means that $\text{card}(\varepsilon, \text{App}_d, \Lambda^{\text{all}}) \geq m^d$, as claimed. □

4 Tractability results for $m = 1$

Since our approximation problem is intractable for $m \geq 2$, we shall restrict our attention to the case $m = 1$ in the rest of this paper. We will study two classes of information: Λ^{all} and Λ^{std}. As we shall see, the approximation problem can be intractable, weakly tractable, polynomially tractable, or even strongly polynomially tractable; the level of tractability will depend on properties of the weight sequence Γ.

4.1 Results for Λ^{all}

Recall that for the class Λ^{all}, the minimal errors and cardinality numbers are determined by the ordered eigenvalues of W_d, and that we can use the eigenvectors of W_d to construct minimal error algorithms, see (7)–(10). So, it behooves us to determine the eigensystem of W_d. We do this in several steps.

First, suppose that $d = 1$. For $\gamma > 0$, the inner product in $H_{1,1,\gamma}$ is given by

$$\langle f, g\rangle_{H_{1,1,\gamma}} = \int_0^1 f(x)g(x)\,dx + \gamma^{-1}\int_0^1 f'(x)g'(x)\,dx \qquad \forall\, f, g \in H_{1,1,\gamma}.$$

We now have

Lemma 4.1. *Suppose that $\gamma > 0$. For $k \in \mathbb{N}$, let*

$$e_k(x) = \cos[\pi(k-1)x] \qquad \forall x \in [0,1]$$

and

$$\alpha_k = \frac{\gamma}{\gamma + \pi^2(k-1)^2}. \tag{14}$$

Then

$$W_1 e_k = \alpha_k e_k \qquad \forall\, k \in \mathbb{N}. \tag{15}$$

Moreover $\{e_k\}_{k\in\mathbb{N}}$ is an orthogonal basis for $H_{1,1,\gamma}$, with

$$\|e_k\|_{H_{1,1,\gamma}} = \left(\delta_{1,k} + \tfrac{1}{2}\sqrt{2}(1-\delta_{1,k})\right)\left(\sqrt{1+\gamma^{-1}[\pi(k-1)^2]}\right) \qquad \forall\, k \in \mathbb{N}.$$

Proof. We first show that the eigenpairs of W_1 are $\{(e_k, \alpha_k)\}_{k\in\mathbb{N}}$. Since $W_1 = (\mathrm{App}_1)^*(\mathrm{App}_1)\colon H_{1,1,\gamma} \to H_{1,1,\gamma}$, we see that (e, α) is an eigenpair of W_1 iff e is a nonzero element of $H_{1,1,\gamma}$ for which $W_1 e = \alpha e$, the latter holding iff

$$\langle e, w\rangle_{L_2(I)} = \langle W_1 e, w\rangle_{H_{1,1,\gamma}} = \alpha\langle e, w\rangle_{H_{1,1,\gamma}} = \alpha\left[\langle e, w\rangle_{L_2(I)} + \gamma^{-1}\langle e', w'\rangle_{L_2(I)}\right] \qquad \forall\, w \in H_{1,1,\gamma}.$$

Setting $w = e$ in this equation and using the fact that $e \neq 0$, we see that $\alpha > 0$. Collecting terms and multiplying by γ, we see that the previous equation can be rewritten as

$$\langle e', w'\rangle_{L_2(I)} = \beta\langle e, w\rangle_{L_2(I)} \qquad \forall\, w \in H_{1,1,\gamma}, \tag{16}$$

where

$$\beta = \frac{\gamma(1-\alpha)}{\alpha} \qquad \text{so that} \qquad \alpha = \frac{\gamma}{\gamma+\beta}. \tag{17}$$

From (16), we see that (e, β) is the variational solution of the classical eigenproblem

$$e''(x) + \beta e(x) = 0 \qquad \forall\, x \in (0, 1),$$

subject to the boundary conditions

$$e'(0) = e'(1) = 0.$$

Note that

$$\langle e_k', w'\rangle_{L_2(I)} = [\pi(k-1)]^2\langle e_k, w\rangle_{L_2(I)}.$$

Hence the kth eigenpair of this eigenproblem is given by (e_k, β_k), where $\beta_k = \pi^2(k-1)^2$. From (17), we see that (15) holds with α_k given by (14). Hence the eigenpairs of W_1 are as claimed.

Clearly, the set $\{e_k\}_{k\in\mathbb{N}}$ is orthogonal in $H_{1,1,\gamma}$, the formula for each $\|e_k\|_{H_{1,1,\gamma}}$ being given by a straightforward calculation. Since $\{e_k\}_{k\in\mathbb{N}}$ are the eigenvectors of a compact self-adjoint positive definite linear transformation on $H_{1,1,\gamma}$, they form an orthogonal basis for $H_{1,1,\gamma}$. □

Remark 4.1. One of our hypotheses in Lemma 4.1 is that $\gamma > 0$. What can we say about the eigensystem of W_1 when $\gamma = 0$? In Section 2, we saw that if $\gamma = 0$, then $H_{1,1,0} = P_0(I)$, the one-dimensional space of constant functions over the unit interval. In this case, we only have the eigenvalue $\alpha = 1$. □

Next, we consider the case of general d. We need to find the eigenpairs of the operator $W_d = (\mathrm{App}_d)^*(\mathrm{App}_d)$ on $H_{d,1,\Gamma}$.

Lemma 4.2. *Suppose that* $\Gamma > \mathbf{0}$. *Let* $d \in \mathbb{N}$. *For* $k \in \mathbb{N}^d$, *let*

$$e_{d,k}(\mathbf{x}) = \prod_{j=1}^{d}\cos[\pi(k_j-1)x_j] \qquad \forall\, \mathbf{x} \in [0, 1]^d$$

and

$$\alpha_{d,k} = \left(1 + \sum_{\substack{\mathfrak{u}\subseteq[d]\\ \mathfrak{u}\neq\emptyset}} \gamma_{d,\mathfrak{u}}^{-1}\prod_{j\in\mathfrak{u}}[\pi(k_j-1)]^2\right)^{-1}. \tag{18}$$

Then

$$W_d e_{d,k} = \alpha_{d,k} e_{d,k} \qquad \forall\, k \in \mathbb{N}^d.$$

Moreover $\{e_{d,k}\}_{k\in\mathbb{N}^d}$ *is an orthogonal basis for* $H_{d,1,\Gamma}$*, with*

$$\|e_{d,k}\|_{H_{d,1,\Gamma}} = 2^{-\left|\{j\in[d]\,:\,k_j>1\}\right|/2}\left(1 + \sum_{\substack{\mathfrak{u}\subseteq[d]\\ \mathfrak{u}\neq\emptyset}} \gamma_{d,\mathfrak{u}}^{-1} \prod_{j\in\mathfrak{u}} [\pi(k_j-1)]^2\right)^{1/2}. \tag{19}$$

Proof. Note that since $\Gamma > \mathbf{0}$, the space $H_{d,1,\Gamma}$ is algebraically and topologically equivalent to the space $H_{d,1,\mathbf{1}} = H_{1,1,1}^{\otimes d}$, for which $\Gamma = \mathbf{1}$. Hence we may use Lemma 4.1, along with the tensor-product structure of $H_{d,1,\Gamma}$, to see that $\{e_{d,k}\}_{k\in\mathbb{N}^d}$ is an orthogonal basis for $H_{d,1,\Gamma}$. Furthermore, a straightforward calculation shows that the given formula for $\|e_{d,k}\|_{H_{d,1,\Gamma}}$ holds. So, we only need to check that the eigensystem of W_d is as claimed.

Observe that

$$\langle v, w\rangle_{L_2(I^d)} = \langle v, W_d w\rangle_{H_{d,1,\Gamma}} \qquad \forall\, v, w \in H_{d,1,\Gamma}.$$

Taking $v = e_{d,j}$ and $w = e_{d,k}$ for arbitrary $j, k \in \mathbb{N}^d$ and using the fact that $\{e_{d,k}\}_{k\in\mathbb{N}^d}$ is an orthogonal sequence in $L_2(I^d)$, we conclude that $W_d e_{d,k}$ is orthogonal to $e_{d,j}$ whenever $j \neq k$. Hence, $e_{d,k}$ is an eigenvector of W_d whose eigenvalue $\alpha_{d,k}$ is given by $\alpha_{d,k} = \|e_{d,k}\|^2_{L_2(I^d)}/\|e_{d,k}\|^2_{H_{d,1,\Gamma}}$. Since $\|e_{d,k}\|^2_{L_2(I^d)} = 2^{-\left|\{j\subseteq[d]\,:\,k_j>1\}\right|}$, we see that $\alpha_{d,k}$ has the given formula. □

It is known that the reproducing kernel of any separable RKHS H has the form

$$K(x, y) = \sum_{j=1}^{\infty} b_j(x) b_j(y),$$

where $\{b_j\}_{j\in\mathbb{N}}$ is an arbitrary orthonormal basis of H. In our case $H = H_{d,1,\Gamma}$, we therefore have

$$K_{d,1,\Gamma}(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{\infty} \frac{e_{d,k}(\mathbf{x}) e_{d,k}(\mathbf{y})}{\|e_{d,k}\|^2_{H_{d,1,\Gamma}}},$$

which yields the formula (4) in the Introduction.

Remark 4.2. One of our hypotheses in Lemma 4.2 is that $\Gamma > \mathbf{0}$. What can we say about the eigensystem of W_d when some weight is zero? In Section 2, we saw that $H_{d,1,\Gamma}$ is now only a subspace of the tensor product space $H_{d,1,\mathbf{1}}$, rather than the full space itself. The results of Lemma 4.2 remain true, provided that the multi-index k is restricted by requiring that $k_j = 1$ for some $j \in \mathfrak{u}$ whenever if $\gamma_{d,\mathfrak{u}} = 0$. .

The most extreme example occurs when $\Gamma = \mathbf{0}$. In this case, we see that $H_{d,1,\Gamma} = P_0(I^d)$, the one-dimensional space of constant functions over the unit cube. The only surviving multi-index is $k = \mathbf{1}$. The resulting eigenvalue is $\alpha = 1$.

Now suppose we have finite-order weights of order $\omega \geq 1$, so that $\gamma_{d,\mathfrak{u}} = 0$ whenever $|\mathfrak{u}| > \omega$. We claim that there are at most $2^\omega - 1$ positive terms in the sums (18) and (19). Indeed, let

$$\mathfrak{u}_k = \{j \subseteq [d] : k_j > 1\} = \{k_{j_1}, k_{j_2}, \ldots, k_{j_s}\}, \tag{20}$$

where $s_k = |\mathfrak{u}_k|$. Note that if $|\mathfrak{u}_k| > \omega$ then $\gamma_{d,\mathfrak{u}_k} = 0$, so that the norm $\|e_{d,k}\|_{H_{d,1,\Gamma}}$ would be infinite. Since $s_k \leq \omega$, this means that at most ω components of k can be greater than 1, i.e., at least $d - \omega$ components of k are 1. So we cannot have a positive term in the sum (19) unless $\mathfrak{u} \subseteq \mathfrak{u}_k$. Since $|\mathfrak{u}_k| \leq \omega$, we know that $\mathfrak{u}_k$ has at most $2^\omega - 1$ nonempty subsets. Hence we can have at most $2^\omega - 1$ terms in the sum above, as claimed. □

From equation (10), we see that

$$\operatorname{card}(\varepsilon, \mathrm{App}_d, \Lambda^{\mathrm{all}}) = |\{k \in \mathbb{N}^d : \alpha_{d,k} > \varepsilon^2\} = |\{k \in \mathbb{N}^d : \alpha_{d,k}^{-1} < \varepsilon^{-2}\}|.$$

Hence using Lemma 4.2, we now have

$$\operatorname{card}(\varepsilon, \mathrm{App}_d, \Lambda^{\mathrm{all}}) = \left| \left\{ k \in \mathbb{N}^d : 1 + \sum_{\substack{\mathfrak{u} \subseteq [d] \\ \mathfrak{u} \neq \emptyset}} \gamma_{d,\mathfrak{u}}^{-1} \prod_{j \in \mathfrak{u}} [\pi(k_j - 1)]^2 < \varepsilon^{-2} \right\} \right|. \tag{21}$$

We are now ready to talk about tractability for various kinds of weights.

4.1.1 Equal weights

First, we consider equal weights $\Gamma = \mathbf{1}$. In this case, $H_{d,1,\Gamma}$ is a tensor product space whose eigenvalues $\alpha_{d,k}$ are of the form

$$\alpha_{d,k} = \prod_{j=1}^{d} \frac{1}{1 + \left[\pi(k_j - 1)\right]^2} \qquad \text{for all } k \in \mathbb{N}^d.$$

Hence the eigenvalues $\alpha_{d,k}$ are products of the eigenvalues $\alpha_k = \alpha_{1,k}$ for the univariate case. Clearly we have $\alpha_1 = 1$ and $\alpha_2 = (1 + \pi^2)^{-1}$, i.e., the largest eigenvalue is simple. Moreover, $\alpha_n = \Theta(n^{-2})$. From [5, Theorem 5.5], we find that

- the approximation problem *is not* polynomially tractable, but
- the approximation problem *is* weakly tractable

for equal weights.

4.1.2 Bounded product weights

We now consider bounded product weights, i.e., $\gamma_{d,\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_{d,j}$ for all nonempty $\mathfrak{u} \subseteq [d]$, with

$$M := \sup_{d \in \mathbb{N}} \max_{1 \le j \le d} \gamma_{d,j} < \infty. \tag{22}$$

Hence $H_{d,1,\Gamma} = \otimes_{j=1}^{d} H_{1,1,\gamma_{d,j}}$. Once again, we see that $H_{d,1,\Gamma}$ is a tensor product space whose eigenvalues now have the form

$$\alpha_{d,k} = \prod_{j=1}^{d} \frac{\gamma_{d,j}}{\gamma_{d,j} + \left[\pi(k_j - 1)\right]^2} \qquad \text{for all } k \in \mathbb{N}^d. \tag{23}$$

Weak tractability: Note that

$$\alpha_{d,k} \le \prod_{j=1}^{d} \frac{M}{M + [\pi(k_j - 1)]^2}$$

and so [5, Theorem 5.5] applies. Hence approximation is weakly tractable whenever (22) holds. This boundedness condition (22) is also necessary in some sense, since there exist sequences Γ for which (22) does not hold and for which the approximation problem is intractable. One such sequence is given by $\gamma_{d,j} = d$. Since the second-largest eigenvalue for the d-dimensional case is now $1/(1 + \pi^2/d)$, it follows that $\mathrm{card}(\varepsilon, \mathrm{App}_d, \Lambda^{\mathrm{all}}) \geq 2^d$ for

$$\varepsilon \leq \varepsilon_d := \frac{1}{2}\frac{1}{(1+\pi^2/d)^{d/2}} = \tfrac{1}{2}\exp(-\tfrac{1}{2}\pi^2)(1+o(1)).$$

Hence the approximation problem, subject to this specific set of unbounded weights, suffers from the curse of dimensionality.

Strong tractability: Using [5, Theorem 5.2] and standard proof techniques, we will show that approximation is strongly tractable iff there exists $\tau > 0$ for which

$$\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j}^{\tau} < \infty. \tag{24}$$

Note that we may assume that $\tau > \frac{1}{2}$ without loss of generality, since if (24) holds for some $\tau = \tau_0$, it also holds for any $\tau > \tau_0$. Moreover, [5, Theorem 5.2] also tells us that the exponent of strong tractability is given by

$$p(\Lambda^{\mathrm{all}}) = 2\tau^*, \tag{25}$$

where τ^* is the infimum of all $\tau > \frac{1}{2}$ for which (24) holds.

Indeed, [5, Theorem 5.2] tells us that that our problem is strongly tractable and that (25) holds iff there exists $\tau > 0$ such that

$$\Sigma_{d,\tau} := \left(\sum_{k\in\mathbb{N}^d} \alpha_{d,k}^{\tau}\right)^{1/\tau} = \prod_{i=1}^{d}\left(1 + \sum_{j=1}^{\infty}\left(\frac{\gamma_{d,i}}{\gamma_{d,i}+\pi^2 j^2}\right)^{\tau}\right)^{1/\tau} \tag{26}$$

is uniformly bounded over all $d \in \mathbb{N}$. If Γ is not a set of zero weights, it is clear that the condition $\tau > \frac{1}{2}$ is necessary and sufficient for the last sum to converge. For

$$c = \frac{\pi^2}{\pi^2 + M}, \tag{27}$$

we see that

$$c\,\frac{\gamma_{d,i}}{\pi^2 j^2} \leq \frac{\gamma_{d,i}}{\gamma_{d,i}+\pi^2 j^2} \leq \frac{\gamma_{d,i}}{\pi^2 j^2}. \tag{28}$$

Letting ζ denote the Riemann zeta function, we use the upper bound of (28) to see that

$$\Sigma_{d,\tau} \leq \prod_{i=1}^{d}\left(1 + \sum_{j=1}^{\infty}\frac{\gamma_{d,i}^{\tau}}{\pi^{2\tau} j^{2\tau}}\right)^{1/\tau} = \prod_{i=1}^{d}\left(1 + \gamma_{d,i}^{\tau}\frac{\zeta(2\tau)}{\pi^{2\tau}}\right)^{1/\tau},$$

and so

$$\ln \Sigma_{d,\tau} \leq \frac{1}{\tau}\sum_{i=1}^{d} \ln\left(1 + \gamma_{d,i}\frac{\zeta(2\tau)}{\pi^{2\tau}}\right) \leq \frac{\zeta(2\tau)}{\tau\pi^{2\tau}}\sum_{i=1}^{d}\gamma_{d,i}^{\tau}. \tag{29}$$

Hence (24) implies strong tractability. To prove the reverse implication, simply start with the lower bound of (28) and proceed as above, noting that $\ln(1+x) \geq \tilde{c}x$ for all $x \in [0, c_\tau]$ with $\tilde{c} = \ln(1 + c_\tau)/c_\tau$ and $c_\tau = (Mc/\pi^2)^\tau \zeta(2\tau)$, where c is given by (27).

Polynomial tractability: Again using [5, Theorem 5.2] and standard proof techniques, we find that the approximation problem is tractable iff there exists $\tau > 0$ for which

$$\limsup_{d\to\infty} \frac{1}{\ln d} \sum_{j=1}^{d} \gamma_{d,j}^{\tau} < \infty. \tag{30}$$

As before, we can assume that $\tau > \frac{1}{2}$. To obtain the exponents of tractability, define

$$A_\tau = \limsup_{d\to\infty} \frac{1}{\ln d} \sum_{j=1}^{d} \min\left\{1, \gamma_{d,j}^{\tau} \frac{\zeta(2\tau)}{\pi^{2\tau}}\right\} < \infty. \tag{31}$$

Note that (30) and (31) are equivalent. When A_τ is finite, then for any $q_\tau > A_\tau$, we have

$$\mathrm{card}(\varepsilon, \mathrm{App}_d, \Lambda^{\mathrm{all}}) = \mathcal{O}(d^{q_\tau} \varepsilon^{-2\tau}),$$

where the $\mathcal{O}$-factor is independent of d and ε. Hence we we may take

$$p(\mathrm{App}, \Lambda^{\mathrm{all}}) = 2\tau \qquad \text{and} \qquad q(\mathrm{App}, \Lambda^{\mathrm{all}}) = q_\tau$$

for any $q_\tau > A_\tau$ and any τ satisfying (31).

Indeed, [5, Theorem 5.2] tells us that our problem is tractable iff there exist positive numbers q and τ such that $d^{-q}\Sigma_{d,\tau}$ is uniformly bounded over all $d \in \mathbb{N}$, where $\Sigma_{d,\tau}$ is given by (26). Using the fact that $\xi = d^{\ln \xi / d}$ and the inequality

$$c\,\frac{\gamma_{d,i}}{\pi^2 j^2} \le \frac{\gamma_{d,i}}{\gamma_{d,i} + \pi^2 j^2} \le \min\left\{1, \frac{\gamma_{d,i}}{\pi^2 j^2}\right\},$$

the proof is analogous to that of the strongly tractable case.

Of course, strong tractability always implies tractability. What about the reverse implication?

1. Suppose that our weights are independent of d, i.e., $\gamma_{d,j} = \gamma_j$ for $j \in \{1, \ldots, d\}$ and $d \in \mathbb{N}$. From [10], we find that (24) and (30) are equivalent. Hence in this case, tractability implies strong tractability.

2. Suppose that for some $q > 0$, we have weights

$$\gamma_{d,j} = \begin{cases} 1 & \text{for } j \in \{1, \ldots, \lceil \ln d \rceil^q\}, \\ 0 & \text{for } j \in \{\lceil \ln d \rceil^q + 1, \ldots, d\}. \end{cases} \qquad j \in \{1, \ldots, d\}, d \in \mathbb{N}.$$

   Then the approximation problem is *never* strongly tractable. However, this problem is tractable iff $q \le 1$. To wit:

   - If $q < 1$, then $A_\tau = 0$ for all $\tau > \frac{1}{2}$. This implies that the exponent of d can be arbitrarily small and the exponent of ε^{-1} can be arbitrarily close to 1.

- If $q = 1$, then

$$A_\tau = \min\left\{1, \frac{\zeta(2\tau)}{\pi^{2\tau}}\right\}.$$

This means that as $\lim_{\tau\to 1/2+} A_\tau = \infty$. Hence, when the exponent of ε^{-1} goes to 1, the exponent of d goes to infinity. On the other hand, if we choose $\tau \doteq 0.635564$ to be the solution of

$$\zeta(2\tau) = \pi^{2\tau},$$

then the exponent of ε^{-1} is approximately 1.27113 and the exponent of d takes its minimal value of 1. This illustrates the tradeoff between the exponents of ε^{-1} and d.

4.1.3 Bounded finite-order and finite-diameter weights

Since finite-diameter weights are a special case of finite-order weights, we will consider them together. Our main result is that the approximation problem is tractable if a boundedness condition (similar to (22) above) holds. Moreover, this boundedness condition is necessary, since there exist unbounded families of finite-diameter weights for which approximation is intractable. Our first positive result is

Theorem 4.1. *Suppose that*

$$M = \sup_{d\in\mathbb{N}} \max_{\substack{\mathfrak{u}\subseteq[d]\\ \mathfrak{u}\neq 0\\ |\mathfrak{u}|\le\omega}} \gamma_{d,\mathfrak{u}} < \infty\,. \tag{32}$$

1. *Let Γ be a family of finite-order weights of order ω. Then for any $\tau > 1$, there exists $C_{\tau,\omega} > 0$ such that*

$$\mathrm{card}(\varepsilon, \mathrm{App}_d, \Lambda^{\mathrm{all}}) \le C_{\tau,\omega} M^{\tau/2} d^{\omega} \varepsilon^{-\tau} \qquad \forall\, \varepsilon \in (0,1], d \in \mathbb{N}.$$

   The multiplicative factor $C_{\tau,\omega}$ is independent of M, d, and ε.

2. *Let Γ be a family of finite-diameter weights of order ω. Then for any $\tau > 1$, there exists $C_{\tau,\omega} > 0$ such that*

$$\mathrm{card}(\varepsilon, \mathrm{App}_d, \Lambda^{\mathrm{all}}) \le C_{\tau,\omega} M^{\tau/2} d \varepsilon^{-\tau} \qquad \forall\, \varepsilon \in (0,1], d \in \mathbb{N}.$$

   The multiplicative factor $C_{\tau,\omega}$ is independent of M, d, and ε.

Proof. We first consider finite-order weights. It suffices to consider $d > \omega$. For $k \in \mathbb{N}^d$, we have

$$\frac{1}{\alpha_{d,k}} = 1 + \sum_{\substack{\mathfrak{u}\subseteq[d]\\ \mathfrak{u}\neq\emptyset}} \gamma_{d,\mathfrak{u}}^{-1} \prod_{j\in\mathfrak{u}} [\pi(k_j - 1)]^2,$$

using the fact that for $\gamma_{d,\mathfrak{u}} = 0$, there exists $j \in \mathfrak{u}$ such that $k_j = 1$ and interpreting $0/0$ as 0. From Remark 4.2, we see that this sum may be rewritten as

$$\frac{1}{\alpha_{d,k}} = 1 + \sum_{\substack{\mathfrak{u}\subseteq\mathfrak{u}_k\\ \mathfrak{u}\neq\emptyset}} \gamma_{d,\mathfrak{u}}^{-1} \prod_{j\in\mathfrak{u}} [\pi(k_j - 1)]^2,$$

where $\mathfrak{u}_k$ is given by (20). Letting $s_k = |\mathfrak{u}_k|$ and using the fact that $\gamma_{d,\mathfrak{u}} \le M$, we have

$$\frac{1}{\alpha_{d,k}} \ge 1 + \frac{1}{M} \sum_{\substack{\mathfrak{u} \subseteq \mathfrak{u}_k \\ \mathfrak{u} \neq \emptyset}} \prod_{j \in \mathfrak{u}} [\pi(k_j - 1)]^2$$
$$= 1 + \frac{1}{M} \left[\prod_{i=1}^{s_k} [1 + \pi^2 (k_{j_i} - 1)^2] - 1 \right]$$
$$\ge 1 + \frac{1}{M} \left[\pi^{2s_k} \prod_{i=1}^{s_k} (k_{j_i} - 1)^2 - 1 \right].$$

Since $\alpha_{d,k}^{-1} < \varepsilon^{-2}$, this inequality implies that

$$\prod_{i=1}^{s_k} (k_{j_i} - 1)^2 < \frac{M(\varepsilon^{-2} - 1) + 1}{\pi^{2s_k}},$$

which may be rewritten as

$$\sum_{i=1}^{s_k} \ln(k_{j_i} - 1) < \sigma_k := \ln \sqrt{\frac{M(\varepsilon^{-2} - 1) + 1}{\pi^{2s_k}}}\,. \tag{33}$$

Let $\ell = [\ell_1, \ell_2, \dots, \ell_s]$ with $\ell_i = k_{j_i} - 1 \ge 1$. From Remark 4.2, we see that the number of sets $\mathfrak{u}_k$ for which $\gamma_{d,\mathfrak{u}_k} > 0$ is at most equal to the cardinality of the nonzero finite-order weights, which is at most $2\,d^\omega$ by (5). Hence we may use (21), along with (33), to see that

$$\begin{aligned} \mathrm{card}(\varepsilon, \mathrm{App}_d, \Lambda^{\mathrm{all}}) &\le \sum_{\substack{k \in \mathbb{N}^d \\ \gamma_{d,\mathfrak{u}_k} > 0}} \left| \left\{ \ell \in \mathbb{N}^{s_k} : \sum_{i=1}^{s_k} \ln \ell_i < \sigma_k \right\} \right| \\ &\le \sum_{\substack{k \in \mathbb{N}^d \\ \gamma_{d,\mathfrak{u}_k} > 0}} \left| \left\{ \ell \in \mathbb{N}^{\omega} : \sum_{i=1}^{\omega} \ln \ell_i < \sigma \right\} \right|, \end{aligned} \tag{34}$$

where

$$\sigma := \ln \sqrt{M(\varepsilon^{-2} - 1) + 1}\,.$$

Using this last inequality and (5), we have

$$\mathrm{card}(\varepsilon, \mathrm{App}_d, \Lambda^{\mathrm{all}}) \le 2\,d^\omega \left| \left\{ \ell \in \mathbb{N}^{\omega} : \sum_{i=1}^{\omega} \ln \ell_i < \sigma \right\} \right|.$$

From [2], we know that for any $\tau > 1$, there exists $C(\tau, \omega) > 0$ such that

$$\left| \left\{ \ell \in \mathbb{N}^{\omega} : \sum_{i=1}^{\omega} \ln \ell_i < \sigma \right\} \right| \le C(\tau, \omega) e^{\sigma\tau}.$$

Hence

$$\mathrm{card}(\varepsilon, \mathrm{App}_d, \Lambda^{\mathrm{all}}) \le 2\,d^\omega C(\tau, \omega) e^{\sigma\tau} = \mathscr{O}(d^\omega M^{\tau/2} \varepsilon^{-\tau}).$$

The $\mathscr{O}$-factor is independent of M, d, and ε, depending only on τ and ω.

We now turn to finite-diameter weights. In this case, we find that the previous analysis applies, noting that the last sum in (34) is over a set of terms having cardinality $\mathscr{O}(d)$. □

We finally show that the boundedness condition (32) is necessary for approximation to be tractable for finite-diameter (and thus finite-order) weights. Consider the following simple example. Choose

$$\gamma_{d,\mathfrak{u}} = \begin{cases} 1 & \text{if } \mathfrak{u} = \emptyset, \\ 2^d & \text{if } \mathfrak{u} = \{1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\omega = 1$, but $M = \infty$. Then App_d reduces to the univariate approximation problem, so that

$$\mathrm{card}(\varepsilon, \mathrm{App}_d, \Lambda^{\mathrm{all}}) = |\{\,\ell \in \mathbb{N} : \ell^2 < 2^d(\pi\varepsilon^{-1})^2\,\}| = \pi 2^{d/2}\varepsilon^{-1}(1 + o(1)).$$

Hence this approximation problem suffers from the curse of dimensionality, and is intractable.

4.2 Results for Λ^{std}

We now examine tractability when only standard information Λ^{std} is available. In particular, we will consider the same three families of weights studied in §4.1.

4.2.1 Equal weights

We first consider equal weights $\Gamma = \mathbf{1}$. For this case, we claim that multivariate approximation is *intractable* for Λ^{std}. To see this, consider the integration problem $\mathrm{Int}_d \colon H_{d,1,\Gamma} \to \mathbb{R}$ defined by

$$\mathrm{Int}_d\, f = \int_{I^d} f(\mathbf{x})\, d\mathbf{x} \qquad \forall\, f \in H_{d,1,\Gamma}.$$

Since $|\,\mathrm{Int}_d\, f| \le \|f\|_{L_2(I^d)}$ for any $f \in H_{d,1,\Gamma}$, with equality for $f \equiv 1$, we know that $\|\,\mathrm{Int}_d\,\| = 1$. Since we already know that $\|\,\mathrm{App}_d\,\| = 1$, the two problems of multivariate approximation and integration have the same initial error. It is easy to see that multivariate integration is at least as hard as multivariate approximation, i.e., that

$$\mathrm{card}(\varepsilon, \mathrm{App}_d, \Lambda^{\mathrm{std}}) \ge \mathrm{card}(\varepsilon, \mathrm{Int}_d, \Lambda^{\mathrm{std}}) \qquad \forall\, \varepsilon \in (0,1), d \in \mathbb{N}. \tag{35}$$

So, it suffices to show that multivariate integration is intractable.

From [6, pp. 492–493], we know that we may make a rank-one modification to the reproducing kernel $K_{1,1,\gamma}$ for $H_{1,m,\gamma}$, with the resulting kernel $\widetilde{K}_{1,1,\gamma}$ being decomposable at the point $\frac{1}{2}$, i.e., that

$$\widetilde{K}_{1,1,\gamma}(x,y) = 0 \qquad \text{if} \qquad 0 \le x \le \tfrac{1}{2} \le y \le 1.$$

We may now use [4, Theorem 4]. This theorem requires several conditions to hold, which are established in [6]. Part (4) of [4, Theorem 4] tells us that there exists $b > 1$ such that

$$\lim_{d\to\infty} e(\lfloor b^d \rfloor, \mathrm{Int}_d, \Lambda^{\mathrm{all}}) = 1.$$

This means that for any $\varepsilon < 1$, we must have

$$\mathrm{card}(\varepsilon, \mathrm{Int}_d, \Lambda^{\mathrm{std}}) \ge \lfloor b^d \rfloor$$

for large d. Hence the integration problem is intractable, which implies that the approximation problem is also intractable.

4.2.2 Bounded product weights

We now consider bounded product weights, i.e., $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}$ for all nonempty $\mathfrak{u} \subseteq [d]$, with the $\gamma_{d,j}$ satisfying (22).

Weak tractability: We claim that the approximation problem is weakly tractable for Λ^{std} iff

$$\lim_{d\to\infty} \frac{1}{d} \sum_{j=1}^{d} \gamma_{d,j} = 0. \tag{36}$$

To see that this condition is necessary, we can use the proof techniques of [2, Theorem 1] and [6, pg. 492 ff.] to conclude that (36) is necessary for multivariate integration to be weakly tractable. Since multivariate approximation is at least as hard as multivariate integration, see (35), we see that that (36) is also necessary for weak tractability of multivariate approximation.

We now show that (36) is sufficient for multivariate approximation to be weakly tractable. Using (23), we see that for any $n \in \mathbb{N}$, we have

$$n\alpha_{d,n} \le \sum_{k\in\mathbb{N}^d} \alpha_{d,k} = \prod_{j=1}^{d} \left(1 + \gamma_{d,j} \sum_{i=1}^{\infty} \frac{1}{\gamma_{d,j} + \pi^2 i^2}\right) \le \prod_{j=1}^{d} \left(1 + \min\{1, \tfrac{1}{6}\gamma_{d,j}\}\right),$$

and so

$$e^2(n, \text{App}_d, \Lambda^{\text{all}}) \le \alpha_{d,n} \le \frac{1}{n} \prod_{j=1}^{d} \left(1 + \min\{1, \tfrac{1}{6}\gamma_{d,j}\}\right). \tag{37}$$

Now [11, Theorem 1] tells us that

$$e(n, \text{App}_d, \Lambda^{\text{std}}) \le \min_{l\ge 0} \left(e^2(l, \text{App}_d, \Lambda^{\text{all}}) + \frac{M_d\, l}{n}\right)^{1/2}, \tag{38}$$

where

$$M_d = \int_{I^d} K_{d,1,\Gamma}(\mathbf{x}, \mathbf{x})\, d\mathbf{x}$$

with $K_{d,1,\Gamma}$ being the reproducing kernel of $H_{d,1,\Gamma}$. Since $H_{d,1,\Gamma} = \otimes_{j=1}^{d} H_{1,1,\gamma_{d,j}}$, we have

$$K_{d,1,\Gamma}(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^{d} K_{d,1,\gamma_j}(x_j, y_j) \qquad \forall\, \mathbf{x}, \mathbf{y} \in I^d,$$

where the reproducing kernel $K_{d,1,\Gamma}$ for the univariate case is given by

$$K_{d,1,\Gamma}(x, y) = \frac{\sqrt{\gamma}}{\sinh\sqrt{\gamma}} \cosh[\sqrt{\gamma}(1 - \max\{x, y\})] \cosh[\sqrt{\gamma} \min\{x, y\}] \qquad \forall\, x, y \in [0, 1], \tag{39}$$

see [7]. It can be checked that

$$M_d = \prod_{j=1}^{d} \left[\tfrac{1}{2}\left(1 + \sqrt{\gamma_{d,j}}\, \frac{\cosh\sqrt{\gamma_{d,j}}}{\sinh\sqrt{\gamma_{d,j}}}\right)\right] \le \prod_{j=1}^{d} \left(1 + \min\{1, \tfrac{1}{6}\gamma_{d,j}\}\right). \tag{40}$$

Setting $l = \lceil \sqrt{n} \rceil$ in (38) and using (37), we see that

$$e^2(n, \mathrm{App}_d, \Lambda^{\mathrm{std}}) \le \frac{1}{\sqrt{n}} \prod_{j=1}^{d} \big(1 + \tfrac{1}{6}\gamma_{d,j}\big) + \frac{2}{\sqrt{n}} \prod_{j=1}^{d} (1 + \tfrac{1}{6}\gamma_{d,j}) = \frac{3}{\sqrt{n}} \prod_{j=1}^{d} (1 + \tfrac{1}{6}\gamma_{d,j}).$$

Hence

$$\mathrm{card}(\varepsilon, \mathrm{App}_d, \Lambda^{\mathrm{std}}) \le \frac{9}{\varepsilon^4} \prod_{j=1}^{d} (1 + \tfrac{1}{6}\gamma_{d,j})^2 + 1.$$

Since $\ln(1+\xi) \le \xi$ for $\xi \ge 0$, we may now use (36) and the previous inequality to see that that (11) holds, and so the approximation problem is weakly tractable, as claimed.

Strong tractability: We will show that the approximation problem is strongly tractable for Λ^{std} iff

$$\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j} < \infty. \tag{41}$$

To see that this condition is necessary, we note that [6, Theorem 1] tells us that (41) is necessary for the integration problem to be strongly tractable.[2] Since multivariate integration is at least as hard as multivariate approximation, we see that (41) is also necessary for multivariate approximation to be strongly tractable.

To see that (41) implies that approximation is strongly tractable for Λ^{std}, first note that we may use (29) with $\tau = 1$, along with (41), to see that $\Sigma_{d,1}$ is uniformly bounded in d. This implies that approximation is strongly tractable for Λ^{all}, with $p(\Lambda^{\mathrm{all}}) \le 2$, so that

$$e(n, \mathrm{App}_d, \Lambda^{\mathrm{all}}) = \mathscr{O}\left(\frac{1}{n^{1/2}}\right),$$

with the $\mathscr{O}$-factor being independent of both d and n, see § 4.1.2. From (40), we see that M_d is also uniformly bounded in d. If we once again let $l = \lceil \sqrt{n} \rceil$ in (38), we now find that

$$e(n, \mathrm{App}_d, \Lambda^{\mathrm{std}}) = \mathscr{O}\left(\frac{1}{n^{1/4}}\right), \tag{42}$$

with the $\mathscr{O}$-factor being independent of both d and n. Hence the problem is strongly tractable, as claimed.

Now that we have shown that (41) is necessary and sufficient for strong tractability, let us say more about the strong exponent $p(\Lambda^{\mathrm{std}})$. We claim that

$$p(\Lambda^{\mathrm{all}}) \le p(\Lambda^{\mathrm{std}}) \le p(\Lambda^{\mathrm{all}})\big(1 + \tfrac{1}{2}p(\Lambda^{\mathrm{all}})\big). \tag{43}$$

Indeed, since the left-hand inequality is a consequence of the trivial inclusion $\Lambda^{\mathrm{std}} \subseteq \Lambda^{\mathrm{all}}$, it only remains to prove the right-hand inequality. First, suppose that $p(\Lambda^{\mathrm{all}}) = 2$; then (42) immediately tells us that $p(\Lambda^{\mathrm{std}}) \le 4$, which is the value of $p(\Lambda^{\mathrm{all}})\big(1 + \frac{1}{2}p(\Lambda^{\mathrm{all}})\big)$ when $p(\Lambda^{\mathrm{all}}) = 2$. Hence, we

[2] The paper [6] only considers product weights $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_j$. However, all the results of [6] hold for product weights of the form $\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}$ that we are using in this paper.

need only consider the case $p(\Lambda^{\text{all}}) < 2$. Choose $\delta > 0$ such that $r_\delta = 1/(p(\Lambda^{\text{all}}) + \delta) > \frac{1}{2}$. We see that

$$e(n, \text{App}_d, \Lambda^{\text{all}}) = \mathcal{O}\left(\frac{1}{n^{r_\delta}}\right),$$

with the $\mathcal{O}$-factor being independent of both d and n. Since $r_\delta > \frac{1}{2}$, we may now use [3, Theorem 8] to see that

$$e(n, \text{App}_d, \Lambda^{\text{std}}) = \mathcal{O}\left(\frac{(\ln\,\ln\,n)^{2r_\delta/(2r_\delta+1)+1/2}}{n^{r_\delta \cdot 2r_\delta/(2r_\delta+1)}}\right),$$

with the $\mathcal{O}$-factor being independent of both d and n. Letting $\delta \to 0$, we immediately obtain the right-hand inequality in (43), as required.

Polynomial tractability: We now show that the approximation problem is polynomially tractable for Λ^{std} iff

$$\limsup_{d\to\infty} \frac{1}{\ln\,d} \sum_{j=1}^{d} \gamma_{d,j} < \infty. \tag{44}$$

To see that this condition is necessary, we note that [6, Theorem 1] tells us that (44) is necessary for the integration problem to be tractable. Since multivariate integration is at least as hard as multivariate approximation, we see that (41) is also necessary for multivariate approximation to be strongly tractable.

To prove the reverse implication, suppose that (44) holds. Recalling the definition of A_τ from (31), we see that A_1 is finite. From our tractability results for Λ^{all}, we see that

$$\text{card}(\varepsilon, \text{App}_d, \Lambda^{\text{all}}) = \mathcal{O}(d^q \varepsilon^{-2})$$

for any $q > A_1$, where the $\mathcal{O}$-factor is independent of both d and ε. Equivalently, we have

$$e(n, \text{App}_d, \Lambda^{\text{all}}) = \mathcal{O}(d^{q/2} n^{-1/2}),$$

where the $\mathcal{O}$-factor is independent of both d and n. Using (38) with $l = \lceil \sqrt{n} \rceil$, along with (40), we see that

$$e(n, \text{App}_d, \Lambda^{\text{std}}) = \mathcal{O}(n^{-1/4} d^{A_1/2+\delta})$$

for any $\delta > 0$, where the $\mathcal{O}$-factor is independent of both d and n, but may depend on δ. It now follows that for any $\delta > 0$, we have

$$\text{card}(\varepsilon, \text{App}_d, \Lambda^{\text{std}}) = \mathcal{O}(d^{2A_1+\delta} \varepsilon^{-4}),$$

where the $\mathcal{O}$-factor is independent of both d and ε, but may depend on δ. Hence our approximation problem is tractable, as claimed.

4.2.3 Bounded finite-order and finite-diameter weights

Recall from §4.1.3 that the boundedness of finite-order (or finite-diameter) weights (32) is needed for tractability in the class Λ^{all}. Since $\Lambda^{\text{std}} \subseteq \Lambda^{\text{all}}$, we see that this condition is also needed for tractability in the class Λ^{std}. So without loss of generality, we shall assume that (32) holds.

As we did for the class Λ^{all}, we now prove that if our finite-order (or finite-diameter) weights are uniformly bounded, then the approximation problem is tractable. The proof technique is analogous as that of §4.1.3, and relies on the relation between the approximation problem for both classes.

Let

$$\overline{\omega} = \begin{cases} \omega & \text{for finite-order weights of order } \omega, \\ 1 & \text{for finite-diameter weights.} \end{cases}$$

From part (1) of Theorem 4.1, we conclude that the ordered eigenvalues $\{\alpha_{d,k}\}_{k\in\mathbb{N}}$ satisfy

$$\alpha_{d,k} = \mathscr{O}\left(\frac{C_{\tau,\overline{\omega}}^{2/\tau} M d^{2\overline{\omega}/\tau}}{k^{2/\tau}}\right)$$

for all $\tau > 1$. If we take $\tau < 2$, then [3, Theorem 3] applies, and we have

$$e(n, \text{App}_d, \Lambda^{\text{std}}) = \mathscr{O}\left(d^{\overline{\omega}/\tau} \frac{(\ln \ln n)^{2/(\tau+2)+1/2}}{n^{2/(\tau^2+2\tau)}}\right),$$

with the $\mathscr{O}$-factor being independent of both d and n, and depending only on τ. This implies that

$$\text{card}(\varepsilon, \text{App}_d, \Lambda^{\text{std}}) = \mathscr{O}\left(\frac{d^{\overline{\omega}(1+\tau/2)}}{\varepsilon^{\tau(1+\tau/2)}}\right), \tag{45}$$

for any $\tau > 1$, with the $\mathscr{O}$-factor being independent of both d and ε^{-1}, and depending only on τ. Hence the approximation problem is polynomially tractable, as claimed.

It may be possible to improve the d- and ε^{-1}-exponents in (45) by using a modified version of the results found in [6] and [13], but we do not pursue this issue here.

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