An intractability result for multiple integration

I.H.Sloan † and H.Woźniakowski ††

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† School of Mathematics
University of New South Wales
Sydney 2052
Australia

and

†† Department of Computer Science
Columbia University
New York, NY 10027
USA
and
Institute of Applied Mathematics
University of Warsaw
ul. Banacha 2, 02-097 Warsaw
Poland

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1 Introduction

In this note we consider the approximation of integrals over the d-dimensional unit cube,

$$If = \int_{[0,1]^d} f(x_1, \dots, x_d)\, dx_1 \dots dx_d = \int_{[0,1]^d} f(x) dx, \tag{1.1}$$

under the assumption $f \in E_{\alpha,d}$, where, for arbitrary $\alpha > 1$, $E_{\alpha,d}$ is the set of complex-valued functions in $L_1([0,1]^d)$, whose Fourier coefficients satisfy

$$|\widehat{f}(h)| \le \frac{1}{(\overline{h}_1 \cdots \overline{h}_d)^\alpha}.$$

Here $h = (h_1, h_2, \dots, h_d)$ with integers h_j and

$$\widehat{f}(h) = \int_{[0,1]^d} f(x) e^{-2\pi i h \cdot x} dx,$$

$h \cdot x = \sum_{j=1}^d h_j x_j$, and $\overline{h}_j = \max(1, |h_j|)$.

Note that a function f belonging to $E_{\alpha,d}$ necessarily has a continuous 1-periodic extension, since the Fourier series for $f \in E_{\alpha,d}$ is absolutely convergent:

$$\sum_{h \in \mathbb{Z}^d} |\widehat{f}(h) e^{2\pi i h \cdot x}| \le \sum_{h \in \mathbb{Z}^d} (\overline{h}_1 \cdots \overline{h}_d)^{-\alpha} < \infty.$$

Our aim is to show that in the worst-case setting the integration problem is *intractable*: that is, to achieve a given error ε, $\varepsilon < 1$, for all $f \in E_{\alpha,d}$, the amount of information required is exponential in d. More precisely, we prove that the minimal error of any quadrature rule that uses $N < 2^d$ points is one. The bound on N is sharp, since the error of a quadrature rule that uses $N = 2^d$ may be arbitrarily small for large α.

The space $E_{\alpha,d}$ is a standard setting for the particular class of quadrature formulas known as lattice rules (for a survey see [4]). The implications of the intractability result for lattice methods are considered briefly in Section 3.

2 The intractability result

A quadrature rule approximating (1.1) is a linear functional of the form

$$Qf = Q(d, N, \omega, t) f := \sum_{j=1}^N \omega_j f(t_j),$$

where the 'weights' $\omega := (\omega_1, \dots, \omega_N)$ and 'points' $t := (t_1, \dots, t_N)$ satisfy

$$\omega_j \in \mathbb{C}, \qquad t_j \in [0,1)^d, \qquad \text{for } j = 1, \dots, N.$$

Without loss of generality we can assume that $t_1, \dots, t_N$ are distinct points. The worst-case error for the quadrature rule $Q = Q(d, N, \omega, t)$ for the class $E_{\alpha,d}$ is

$$P_\alpha(Q) = P_\alpha(d, N, \omega, t) := \sup \{|Qf - If| : f \in E_{\alpha,d}\}.$$

Since we are interested in a lower bound for $P_\alpha(Q)$, we define

$$e(\alpha, d, N) := \inf \{P_\alpha(d, N, \omega, t) : \omega \in \mathbb{C}^N, t \in ([0,1)^d)^N\}.$$

It is an elementary fact that

$$e(\alpha, d, N) \leq 1, \tag{2.1}$$

since by taking $\omega_1 = \omega_2 = \ldots = \omega_N = 0$ we obtain

$$P_\alpha(d, N, 0, t) = \sup \{|If| : f \in E_{\alpha,d}\} = \sup \{|\widehat{f}(0)| : f \in E_{\alpha,d}\} = 1.$$

The following theorem, which is our main result, states in effect that if $N < 2^d$ then, in the worst-case setting and for the class $E_{\alpha,d}$, the error is as bad as it can be, and the quadrature rule $Q = 0$ is a best possible rule.

Theorem 1 *If $N < 2^d$ then*

$$e(\alpha, N, d) = 1.$$

Proof.

Let $N < 2^d$, and suppose that points $t = (t_1, \ldots, t_N)$ and weights $\omega = (\omega_1, \ldots, \omega_N)$ are given. The theorem is proved by constructing a function $g \in E_{\alpha,d}$, depending on t and ω, such that $Ig = 1$ and $Qg = 0$. From this it will follow that $P_\alpha(Q) \geq |Qg - Ig| = |Ig| = 1$. Since this holds for any choice of points t and weights ω, it follows that $e(\alpha, N, d) \geq 1$, which together with (2.1) proves $e(\alpha, N, d) = 1$.

To accomplish the construction, let $B_d := \{0,1\}^d$, and define g to be a trigonometric polynomial of the form

$$g(x) = \theta(x) \sum_{h \in B_d} a_h e^{2\pi i h \cdot x}, \tag{2.2}$$

where $\{a_h \in \mathbb{C} : h \in B_d\}$ is a non-trivial solution of the linear system

$$\sum_{h \in B_d} a_h e^{2\pi i h \cdot t_j} = 0, \quad j = 1, \ldots, N, \tag{2.3}$$

and θ is a trigonometric polynomial which is yet to be specified. A crucial point in the proof is that, because the homogeneous linear system (2.3) has 2^d unknowns but fewer than 2^d equations, a non-trivial solution of (2.3) certainly exists. Let $h^* \in B_d$ be such that $|a_h| \leq |a_{h^*}|$ for $h \in B_d$. We scale our non-trivial solution of (2.3) so that

$$|a_h| \leq 1 \quad \text{for} \quad h \in B_d \quad \text{and} \quad a_{h^*} = 1.$$

It follows from (2.2) and (2.3) that

$$g(t_j) = \theta(t_j) \sum_{h \in B_d} a_h e^{2\pi i h \cdot t_j} = 0, \quad j = 1, \ldots, N,$$

from which it is clear that $Q(d, N, \omega, t) g = 0$. Now we choose

$$\theta(x) := e^{-2\pi i h^* \cdot x},$$

so that

$$g(x) = \sum_{h \in B_d} a_h e^{2\pi i (h-h^*)\cdot x}. \tag{2.4}$$

Clearly

$$Ig = \widehat{g}(0) = a_{h^*} = 1.$$

On the other hand g given by (2.4) is a trigonometric polynomial of degree ≤ 1 in each component of x, since for $h, h^* \in B_d$ we have

$$h_j - h_j^* = 0, 1 \text{ or } -1 \text{ for } j = 1, \ldots, d.$$

This implies

$$\overline{(h_1 - h_1^*)}\ \overline{(h_2 - h_2^*)} \cdots \overline{(h_d - h_d^*)} = 1 \text{ for } h, h^* \in B_d.$$

It therefore follows, since $|a_h| \leq 1$, that $g \in E_{\alpha,d}$, and so the theorem is proved. □

Remark 1

Theorem 1 remains valid if we permit more general quadrature rules. Namely, for fixed points t_j we may approximate the integral If by $\phi(f(t_1), \ldots, f(t_n))$, where ϕ is an arbitrary nonlinear mapping, $\phi : \mathbb{R}^n \to \mathbb{R}$. Since the class $E_{\alpha,d}$ is convex and symmetric (i.e., $f \in E_{\alpha,d}$ implies that $-f \in E_{\alpha,d}$) we may apply Smolyak's theorem, see e.g., [6] p. 76. This theorem states that the mapping ϕ which minimizes the worst-case error is linear. For linear ϕ we have just proven that the error is at least one.

We may also permit adaptive choice of points t_j. That is, assuming that the points $t_1, \ldots, t_{j-1}$ are already chosen and the function values $f(t_1), \ldots, f(t_{j-1})$ are already computed, the next point t_j may depend arbitrarily on $f(t_1), \ldots, f(t_{j-1})$. By Bakhvalov's theorem, see e.g., [6], p. 59, the worst-case error of arbitrary quadrature rule that uses adaptive points t_j cannot be smaller than the minimal worst-case error of linear quadrature rules that use nonadaptive (fixed) points. Again the latter error is at least one. Hence, Theorem 1 also holds for adaptive choice of points t_j.

3 Lattice rule results

Our purpose in this section is merely to mention some known lattice rule results that cast on interesting light on the result in Theorem 1. In particular, we will see that the condition $N < 2^d$ in the theorem cannot be improved, and indeed that the behaviour of $e(\alpha, N, d)$ changes dramatically when N reaches 2^d.

A lattice rule is an equal-weight rule of the form

$$Qf = \frac{1}{N} \sum_{j=1}^{N} f(t_j),$$

where

$$\{t_1, \ldots, t_N\} = L \cap [0,1)^d,$$

and L is an 'integration lattice'; that is to say, L is a geometrical lattice containing $\mathbb{Z}^d$ as a subset, where a geometrical lattice is a discrete subset of $\mathbb{R}^d$ that is closed under addition and subtraction. It is known [5] that if Q is a lattice rule that corresponds to an integration lattice L then

$$Qf - If = \sum_{\substack{h\in L^\perp,\\ h\neq 0}} \widehat{f}(h) \quad \text{for} \quad f \in E_{\alpha,d},$$

where $L^\perp$ is the 'reciprocal lattice' of L,

$$L^\perp := \{h \in \mathbb{R}^d : x\cdot h \in \mathbb{Z},\ \forall x \in L\} \subseteq \mathbb{Z}^d.$$

It follows in turn that

$$P_\alpha(Q) := \sup\ \{|Qf - If| : f \in E_{\alpha,d}\} = \sum_{\substack{h\in L^\perp,\\ h\neq 0}} \frac{1}{(\overline{h_1}\cdots\overline{h_d})^\alpha}. \tag{3.1}$$

One of the simplest of all lattice rules is the n^d-point product-rectangle rule

$$R_n f := \frac{1}{n^d}\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-1}\cdots\sum_{k_d=0}^{n-1} f\left(\frac{k_1}{n},\frac{k_2}{n},\ldots,\frac{k_d}{n}\right).$$

For this rule we can easily compute $P_\alpha(R_n)$, by using the fact that the corresponding integration lattice is $L = (n^{-1}\mathbb{Z})^d$, from which it follows that $L^\perp = (n\mathbb{Z})^d$. Specifically, from (3.1) we find

$$P_\alpha(R_n) = \prod_{j=1}^{d}\left(\sum_{h_j\in n\mathbb{Z}} \frac{1}{\overline{h_j}^\alpha}\right) - 1 = \left(1 + \frac{2\zeta(\alpha)}{n^\alpha}\right)^d - 1, \tag{3.2}$$

where $\zeta(x)$ is the Riemann zeta function,

$$\zeta(x) = \sum_{i=1}^{\infty} i^{-x}, \quad \text{for} \quad x > 1.$$

In particular, on setting $n = 2$ we find

$$P_\alpha(R_2) = (1 + \zeta(\alpha)/2^{\alpha-1})^d - 1. \tag{3.3}$$

In the product-rectangle rule with $n = 2$ we have $N = 2^d$, thus this example only just misses being covered by Theorem 1. On the other hand, we note from (3.3) that

$$P_\alpha(R_2) \longrightarrow 0 \quad \text{as} \quad \alpha \longrightarrow \infty,$$

from which it follows that

$$e(\alpha, 2^d, d) \longrightarrow 0 \quad \text{as} \quad \alpha \longrightarrow \infty.$$

And since $e(\alpha, N, d)$ is clearly non-increasing in N, it follows in turn that

$$e(\alpha, N, d) \longrightarrow 0 \quad \text{as} \quad \alpha \longrightarrow \infty \quad \text{for all} \quad N \geq 2^d. \tag{3.4}$$

The last result stands in striking contrast to the result from the theorem that

$$e(\alpha, N, d) = 1 \quad \forall\ \alpha > 1 \quad \text{if} \quad N < 2^d.$$

This result (3.4) shows that the condition $N < 2^d$ in the theorem cannot be weakened, at least for large values of α.

The product-rectangle rule is not usually thought of as an interesting lattice rule, because lattice rules are traditionally designed to have good asymptotic convergence properties as $N \longrightarrow \infty$ (for fixed d.) By this test the n^d-point product-rectangle rule performs poorly, since (3.2) gives the inferior result

$$P_\alpha(R_n) = O(n^{-\alpha}) = O(N^{-\alpha/d}).$$

Much greater interest attaches usually to the 'method of good lattice points', a class of lattice rules of the form

$$Q(z)f := \sum_{j=0}^{N-1} f\left(\left\{j\frac{z}{N}\right\}\right), \tag{3.5}$$

where $z \in \mathbb{Z}^d$ is a well chosen integer vector, with no nontrivial factor in common with N, and $\{x\}$ for $x \in \mathbb{R}^d$ means that each component of x is to be replaced by its fractional part in $[0, 1)$.

The classical theorems of the method of good lattice points (see, for example, [3] or [4]) assert the existence of $z = z(N)$ such that

$$P_\alpha(Q(z)) \leq c(\alpha, d)\,\frac{(\log N)^{\beta(\alpha,d)}}{N^\alpha},$$

for some positive functions c and β. Usually, $\beta(\alpha, d)$ is of order d. This result indicates on impressive rate of convergence for large N, but asymptotic bounds of this kind either do not give explicit values of $c(\alpha, d)$, or do not provide useful bounds for smaller values of N. Among the known explicit bounds, the authors of [2] assert that for prime values of N up to approximately 10^d the bound in the following theorem is as good as any known bound:

Theorem 2 *For N prime there exists a lattice rule $Q(z)$ of the form (3.5) such that*

$$P_\alpha(Q(z)) \leq \frac{(1 + 2\zeta(\alpha))^d}{N} + \frac{N-1}{N}\left(1 - \frac{2(1 - N^{1-\alpha})\zeta(\alpha)}{N-1}\right)^d - 1. \tag{3.6}$$

An analogous result for composite N is given in [1].

For $N \approx 2^d$ it can easily be seen that the right-hand side of (3.6) is bounded below by $(0.5 + \zeta(\alpha))^d - 1$ whenever

$$2^d \geq 2\zeta(\alpha) + 1. \tag{3.7}$$

Since $\zeta(\alpha) \geq 1 + 1/(\alpha - 1)$, it is easy to check that

$$1 + \frac{\zeta(\alpha)}{2^{\alpha-1}} \leq \frac{1}{2} + \zeta(\alpha).$$

Thus, except possibly for the small values of α and d that violate (3.7), the known theoretical result for the method of good lattice points is in fact worse than the result (3.3) for the humble 2^d-point product-rectangle rule. For still smaller values of N it can be seen that the right-hand side of (3.6) always exceeds 1, so there is no violation of Theorem 1.

References

[1] Disney, S.A.R. Error bounds for rank 1 lattice quadrature rules modulo composites. *Monatshefte für Mathematik,* **110,** 89-100 (1990).

[2] Disney, S.A.R. and Sloan, I.H. Error bounds for the method of good lattice points. *Mathematics of Computation*, **56**, 257-66 (1991).

[3] Niederreiter, H. *Random number generation and quasi-Monte Carlo methods.* Society for Industrial and Applied Mathematics, Philadelphia (1992).

[4] Sloan, I.H. and Joe, S. *Lattice methods for multiple integration.* Oxford University Press, Oxford (1994).

[5] Sloan, I.H. and Kachoyan, P.J. Lattice methods for multiple integration: theory, error analysis and examples. *SIAM Journal on Numerical Analysis*, **24**, 116-28 (1987).

[6] Traub, J.F., Wasilkowski, G. and Woźniakowski, H. *Information-based complexity.* Academic Press, Boston (1988).