Converting from Spherical to Parabolic Coordinates

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We start with θ and φ, as defined in computer graphics so that θ is the angle from the pole, or latitude, and φ is the polar angle, or longitude. This is the opposite of mathematical convention.

Parabolic coordinates are defined for a hemisphere as normalized Cartesian vector, <x, y, z>, as:

$$u = \frac{x}{z+1} \text{ (1.1)}$$

$$v = \frac{y}{z+1} \text{ (1.2)}$$

Given a direction described as <θ, φ>, the normalized Cartesian vector is given as:

$$x = \cos\phi \sin\theta \text{ (2.1)}$$

$$y = \sin\phi \sin\theta \text{ (2.2)}$$

$$z = \cos\theta \text{ (2.3)}$$

Combining equations (1.x) with (2.x) we get:

$$u = \frac{\cos\phi \sin\theta}{\cos\theta + 1} \text{ (3.1)}$$

$$v = \frac{\sin\phi \sin\theta}{\cos\theta + 1} \text{ (3.2)}$$

For the reduction to a simpler form, we use the following trigonometric identities:

$$\frac{\sin t}{\cos t} = \tan t \text{ (4.1)}$$

$$\frac{1+\cos(2t)}{2} = \cos^2 t \text{ (4.2)}$$

$$\sin(2t) = 2\sin t \cos t \text{ (4.3)}$$

We define $\theta' = (\theta/2)$, and rewrite equations (3.x) as:

$$u = \frac{\cos\phi \sin(2\theta')}{\cos(2\theta')+1} \text{ (5.1)}$$

$$v = \frac{\sin\phi \sin(2\theta')}{\cos(2\theta')+1} \text{ (5.2)}$$

Using equation (4.2), in equations (5.x), we get:

$$u = \frac{\frac{1}{2}\cos\phi \sin(2\theta')}{\cos^2\theta'} \text{ (6.1)}$$

$$v = \frac{\frac{1}{2}\sin\phi \sin(2\theta')}{\cos^2\theta'} \text{ (6.2)}$$

Substituting (4.3) into (6.x), we further get:

$$u = \frac{\frac{1}{2}\cos\phi 2\sin\theta' \cos\theta'}{\cos^2\theta'} \text{ (7.1)}$$

$$v = \frac{\frac{1}{2}\sin\phi 2\sin\theta' \cos\theta'}{\cos^2\theta'} \text{ (7.2)}$$

Some algebraic simplification yields:

$$u = \frac{\cos\phi \sin\theta'}{\cos\theta'} \text{ (8.1)}$$

$$v = \frac{\sin\phi \sin\theta'}{\cos\theta'} \text{ (8.2)}$$

A final substitution of (4.1) into (8.x), and remembering the definition of θ', gives us:

$$u = \cos\phi \tan(\theta/2) \text{ (9.1)}$$

$$v = \sin\phi \tan(\theta/2) \text{ (9.2)}$$

As a final verification, we know that *u* and *v* range from [-1,1]. Looking at 9.x, we see that tan(θ/2) will be in the range [0,1] for θ in [0,π].

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Jason Lawrence for introducing me to parabolic coordinates as described in Heidrich and Seidel's *View-independent Environment Maps*, and http://www.sosmath.com/trig/Trig5/trig5/trig5.html for a table of trigonometric identities, and http://mathworld.wolfram.com/SphericalCoordinates.html for a definition of spherical coordinates.