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Abstract

In the mechanism design literature, collusion is often modelled as agents signing side contracts. This modelling approach is in turn implicitly justified by some unspecified repeated-interaction story. In this paper, we first second-guess what kind of repeated-interaction story these side-contract theorists (would admit that they) are having in mind. We then show that, within this repeated-interaction story, there is a big difference between communicative and tacit collusion. While communicative collusion hurts the mechanism designer, tacit collusion is exploitable.

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1 Introduction

In the mechanism design literature, collusion is often modelled as agents signing side contracts. A typical paper of this kind usually starts with a side-contract model, followed immediately by an apology, admitting that collusive side contracts are probably not enforceable in most economies, and hence the side-contract approach is at best a modelling shortcut of some repeated-interaction story.[1] However, exactly what kind of repeated-interaction story these side-contract theorists are having in mind is never specified. This is not a minor omission, because there are numerous perceivable ways to set up a repeated-interaction model, and many of them would not share the same implications of the side-contract approach. Should we assume that agents' private types being drawn anew every period? Or should we assume that these types are persistent over time? Should we assume that the mechanism designer can commit to long-term contracts? Or should we assume that the mechanism designer has to design a new mechanism every period?

In this paper, we shall try to second-guess what kind of repeated-interaction story these side-contract theorists (would admit that they) are having in mind. By elimination, and for lack of better imagination, we will settle at the following repeated-interaction story: the mechanism designer can only have one single chance to choose a grand mechanism, and after then agents will play the same grand mechanism again and again, with their private types drawn anew every period. The mechanism design question is then: foreseeing that agents can collude and coordinate on their favorite equilibrium in any induced repeated game, which grand mechanism should the mechanism designer choose in the first place in order to maximize her objective function?

We then show that, within this repeated-interaction story, there is a big difference between communicative and tacit collusion. While communicative collusion hurts the mechanism designer, tacit collusion is exploitable. We shall illustrate this difference with the example of auction design.

1.1 Related Literature

There are many studies on communicative collusion. If the communication channel is rich enough, pre-play communication can be modelled as a message game. Depending on whether the outcomes of the message game are to be enforced by side contracts or by repeated interaction, the literature can be divided into two corresponding categories. For example, Graham and Marshall (1987), McAfee and McMillan (1992), and Marshall and Marx (2002) belong to the side-contract category; whereas Aoyagi (2002) belongs to the repeated-interaction category. All these papers restrict their attention to fixed (subsets of) grand mechanisms.

A relatively smaller literature touches upon tacit collusion. How the mechanism designer manages to forbid agents from talking to each other before playing the grand mechanism

[1] See, for example, McAfee and McMillan (1992) and Laffont and Martimort (1997, 2000).

is usually not modelled. But the assumption that she somehow manages to do so is not unrealistic. Cramton (1995) documents probably the most entertaining example of tacit collusion in auction history, where bidders of the first FCC auction used the last few digits of their bids to communicate their preferences to each other. The fact that bidders needed to communicate their preferences *during* the auction can be viewed as an evidence that the auction designer (the FCC in this example) somehow managed to forbid bidders from communicating *before* the auction.

It seems that unless the grand mechanism in question has multiple equilibria (in which case we can label some equilibria as more "collusive"), it is difficult to talk about tacit collusion without setting up a model of repeated interaction. Examples of repeated-interaction model of tacit collusion include Athey, Bagwell, Sanchirico (2000), Blume and Heidhues (2001), and Skrzypacz and Hopenhayn (2001). All these papers restrict their attention to fixed (subsets of) grand mechanisms.

Athey and Bagwell (2001) look at both communicative and tacit collusion in a repeated-interaction model, but the differences between them are not dramatic. This is because they also restrict their attention to a fixed grand mechanism. As we will see in this paper, when the mechanism designer is allowed to optimize over grand mechanisms, the two kinds of collusion have very different implications.

Laffont and Martimort (1997, 2000) deal with mechanism design with communicative collusion. This paper is parallel to their papers, and deals with mechanism design with tacit collusion. Another difference is that they employ the side-contract approach as a modelling shortcut, but does not specify what that approach is a modelling shortcut of.

The paper that is closest in spirit to ours is Che and Yoo (2001), which we will say more about in Section 4.

1.2 What is the Side-Contract Approach a Modelling Shortcut of?

We shall first outline our model (but defer the formal exposition to Section 2), and then explain why some other arguably more natural (and possibly even more realistic) competing models are nevertheless incompatible with the side-contract approach.

More specifically, we study the special case of auction design. The repeated-interaction model we shall use is as follows. The auctioneer can only have one single chance to choose an auction game, and after then bidders will play the same auction game again and again, with their private valuations drawn anew every period. In the case of communicative collusion, bidders talk before playing the auction game in any particular period. In the case of tacit collusion, bidders do not talk, and go directly to play the auction game in any particular period.

Hence any auction game chosen by the auctioneer would induce a repeated game, which in turn would have infinitely many sequential equilibria. In this model, collusion is modelled as an equilibrium selection rule. Specifically, we model collusion by simply picking the bidders' favorite sequential equilibrium. The problem of "mechanism design with collusion" can be stated as the following: foreseeing that bidders can collude and coordinate on their favorite equilibrium in any induced repeated game, which auction game should the auctioneer choose in the first place in order to maximize her expected profit?

In this model, bidders' valuations are assumed to be drawn anew every period (i.e., IID-repetition). There is no a priori reason why IID-repetition is a more realistic assumption, nor that it is the only assumption that is compatible with the side-contract approach. We make this assumption partly because it is routinely assumed by almost all previous authors,[2] and partly because it drives the biggest wedge between communicative collusion and tacit collusion. To see how it works, notice that IID-repetition (by pooling individual rationality constraints) generates the opportunity for the auctioneer to extract more surplus than if the auction game were not repeated. However, the auctioneer would not be able to take advantage of this opportunity until she could exploit tacit collusion. The contrast between the one-buyer and multi-bidder cases will nicely illustrate this last point. We will show in Section 3 that, in the one-buyer case, the auctioneer cannot take advantage of the opportunity generated by IID-repetition. The one-buyer case is reminiscent to collusion with perfect communication which, in contrast to tacit collusion, is the kind of collusion the auctioneer cannot harness. We suspect that a similar wedge, albeit less dramatic, exists between communicative collusion and tacit collusion when bidders' private valuations are more persistent over time.

We perceive that there are two main competing models against the one we outlined above, but neither of them would be compatible with the side-contract approach. In both of these competing models, the auctioneer can choose different auctions in different periods. In the first competing model, the auctioneer can sign long term contracts with the bidders. Notice that if the auctioneer can commit to long term contracts, then her future choices of auction game can depend on bidders' past behavior (instead of being memoryless).

The reason why this competing model is incompatible with the side-contract approach is that full surplus extraction is immediate in such a model. The intuition is simple: suppose there is only one buyer and his private valuations are IID over time, then the auctioneer can simply sell the whole future surplus to him in the first period, and give the object to him for free in all future periods if and only if he pays in the first period. Since the buyer has no informational advantage over his future valuations, the auctioneer can extract all the future surplus. This intuition applies to the multi-bidder case as well, on which we provide a formal treatment in Appendix A. Although we only deal with tacit collusion and IID-repetition in Appendix A, it is easy to verify that this full-surplus-extraction result continues to hold with communicative collusion, and Kremer and Skrzypacz (2002) also argue that this result does not rely on the assumption of IID-repetition.

The fact that full surplus extraction is possible regardless of the extent of collusion pretty much turns the problem of "mechanism design with collusion" into a non-problem. We suspect that most side-contract theorists would vehemently deny that this is the repeated-

[2] See, for example, Athey, Bagwell, and Sanchirico (2000), Athey and Bagwell (2001), Blume and Heidhues (2001), Skrzypacz and Hopenhayn (2001), and Aoyagi (2002).

interaction story they are having in mind.

In the second competing model, the auctioneer has no commitment power over future choices of auction game at all. Hence she herself is one of the players in the following larger repeated game: in any particular period, the auctioneer first announces the auction-game-of-the-day, and then bidders play this announced auction game. Once again, there will be infinitely many sequential equilibria in this repeated game. If we continue to model collusion as an equilibrium selection rule, and proceed to pick the bidders' favorite sequential equilibrium, we will run into a second kind of problem. In the bidders' favorite sequential equilibrium, the auctioneer earns zero profit. We provide a formal treatment of this result in Appendix B. Once again, although we only deal with tacit collusion and IID-repetition in Appendix B, it is easy to verify that this zero-profit result continues to hold with communicative collusion and non-IID-repetition.

The fact that the auctioneer will be completely crushed by collusion, be it communicative or tacit, once again turns the problem of "mechanism design with collusion" into a non-problem, albeit for a different reason. Once again, we suspect that most side-contract theorists would vehemently deny that this is the repeated-interaction story they are having in mind.

By elimination, and for lack of better imagination, we hence postulate that our repeated-interaction story (as outlined above and formally described in the next section) is probably what many side-contract theorists are having in mind. In the rest of this paper, we shall proceed to analyze such a story, and verify our earlier claim that tacit collusion differs from communicative collusion in being exploitable.

The rest of this paper is organized as follows. Section 2 formally describes the model. Section 3 contains the one-buyer benchmark. The main result is formally stated and proved in Section 4. Section 5 discusses several extensions.

2 The Model

There are one auctioneer and a set N of $n \geq 2$ symmetric bidders. The auctioneer has one (perishable) object to sell every period. The object is indivisible, and is worth nothing to the auctioneer. Each bidder i has a private valuation $v_{it} \in [0, 1]$ over the object being sold in period t, where v_{it} is IID across i and t. For any i and t, v_{it} follows the probability distribution function F, which has a strictly positive density function f over $[0, 1]$.

There are infinitely many periods (i.e., $t = 1, 2, \ldots$). Before the beginning of period 1 (say in period 0), the auctioneer has one single chance to choose an auction game. Then, in each period t, bidders first learn their private valuations, and then play the auction game chosen in period 0.

It is important for the reader to distinguish two kinds of game here: (i) the auction game which is played period by period, and (i) the repeated game which has infinite horizon. Formally, an auction game is a selling mechanism (S, p, q, M, α), where:

- $S := \times_{i \in N} S_i$ is a vector of message spaces; each S_i is a set of possible messages bidder i can send to the auctioneer, with the restriction that $S_i = \{0_i\} \cup \hat{S}_i$, where 0_i is the "non-participation" message;[3]

- $p : S \to \{(x_1, \ldots, x_n) \in [0,1]^n \mid \sum_{i \in N} x_i \leq 1\}$ is an allocation function that specifies the probabilities with which each bidder will get the object, with the restriction that $\forall i$, $p_i(\cdots, 0_i, \cdots) \equiv 0$ (non-participating bidders never get the object);

- $q : S \to \mathbb{R}^n$ is a payment function that specifies the amount each bidder has to pay the auctioneer, with the restriction that $\forall i$, $q_i(\cdots, 0_i, \cdots) \equiv 0$ (non-participating bidders never pay the auctioneer);

- $M := \times_{i \in N} M_i$ is a vector of message spaces; each M_i is a set of possible messages the auctioneer can send to bidder i; and

- $\alpha : S \to \Delta(M)$ is an announcement function that specifies what information the auctioneer would disclose to each bidder after the auction.

The above definition of an auction game is standard except for the announcement function. The specification of the announcement function is not important in any one-shot auction game, as it will not affect bidders' incentives. It becomes important when we study repeated auctions, as it now affects bidders' ability to collude. In the definition above, α_i (the projection of α onto $\Delta(M_i)$) specifies the (potentially random) message the auctioneer tells bidder i after the auction, and these messages can be correlated across bidders.[4]

All players are risk neutral: the auctioneer's period-t payoff u_t is simply her revenue in period t; and each bidder i's period-t payoff u_{it} is equal to $p_i v_{it} - q_i$, where p_i is his probability of obtaining the object in period t, and q_i is the amount he pays the auctioneer in period t. Notice that if a bidder does not participate in the period-t auciton game, his period-t payoff will be zero.

All players have common discount factor δ.[5] We follow the convention in the repeated-game literature and normalize a player's discounted sum of payoffs by the factor $(1-\delta)$. For example, if bidder i's period-t payoffs are u_{it}, $t = 1, 2, \ldots$, then his normalized dicounted sum of payoffs will be $(1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} u_{it}$.

[3]It is not clear that we can appeal to the Revelation Principle and restrict our attention to direct revelation mechanisms (i.e., $S_i = [0,1]$). As is well known in the implementation literature, if agents can defy any wish of the mechanism designer and instead choose to play their own favorite equilibrium, it is then with loss of generality for the mechanism designer to restrict her attention to direct revelation mechanisms.

[4]More generally, we should allow the announcement function to depend on the realization of the winner when the winner is being picked randomly. This can easily be done with extra notations.

[5]That the auctioneer has the same discount factor bidders have is both important and unimportant. It is important because otherwise infinite surplus can be generated out of thin air by simply having players trade intertemporally. It is however *not* important for our main result that the auctioneer can turn bidders' ability to collude into her advantage and capture almost all the expected gain of trade. See also Footnote 10.

Once the auctioneer has chosen an auction game in period 0, the bidders are playing an induced repeated game. Typically, this repeated game will have multiple sequential equilibria. Our way of modelling tacit collusion is to focus on the sequential equilibrium that maximizes bidders' total payoff (hereafter the bidder-optimal equilibrium). In other words, we model collusive bidding by assuming that bidders can coordinate on their favorite sequential equilibrium. If there are more than one sequential equilibrium that maximizes bidders' total payoff, we focus our attention to the one that maximizes the auctioneer's payoff.

Finally, the following definitions will prove useful in subsequent sections. We shall use W^* to denote each period's expected social surplus; i.e., $W^* := \mathbf{E}\max\{v_{1t}, \ldots, v_{nt}\} = \int_0^1 v\mathrm{d}F(v)^n$ (after normalization W^* is also the discounted sum of social surplus). We shall use w^* to denote each period's per-bidder expected social surplus; i.e., $w^* := W^*/n$.

When we mention the "Myerson auction," we mean the direct revelation mechanism that implements Myerson's (1981) optimal auction, and the auctioneer discloses everything to every bidder after the auction. Although the Myerson auction is well understood in the literature, it may still be worthwhile to remind the reader two of its major properties. First, the Myerson auction is strategyproof, meaning that it is a dominant strategy for every bidder to honestly report his true valuation. Second, although truth-telling is not the only dominant strategy, all dominant strategy equilibria are outcome-equivalent.[6] When a bidder's true valuation is v, we shall use $D(v)$ to denote the set of all his dominant-strategy reports. Strategyproofness means that $v \in D(v)$. We say a bidder with true valuation v bids sincerely if his report b is in $D(v)$. Let π^* denote the auctioneer's expected revenue in a one-shot Myerson auction when bidders bid sincerely.

When we mention the "CGK auction," we mean the following auction: all participating bidders submit sealed bids, the good is transferred to the highest bidder, and each participating bidder i pays $b_i - \frac{1}{|B|-1}\sum_{i \neq j \in B} b_j$, where B is the set of participating bidders, and the auctioneer discloses everything to every bidder after the auction. Notice that, in the CGK auction, bidder's total payment to the auctioneer is always zero. On top of this property, Cramton, Gibbons, and Klemperer (1987) prove that the CGK auction also has an efficient equilibrium. In this efficient equilibrium, even a bidder with the lowest type (i.e., a bidder with valuation 0) will receive strictly positive expected payoff. Let $\underline{w} > 0$ denote this strictly positive expected payoff a bidder with the lowest type will receive in the CGK auction's efficient equilibrium. Define $\underline{W} := n\underline{w}$.

For any probability distribution function F, the three numbers (W^*, π^*, and $\underline{W}$) are always ranked as follows:

$$W^* > \pi^* > \underline{W}.$$

For example, if $n = 2$, and F is the uniform distribution on $[0,1]$, then $W^* = \frac{2}{3}$, $\pi^* = \frac{5}{12}$, and $\underline{W} = \frac{1}{3}$.

[6] For example, if the probability that a bidder gets the object is 1/10 whenever his reported valuation is between 1/3 and 2/3, then it is a dominant strategy for this bidder to report anything between 1/3 and 2/3 (and not necessarily his true valuation) as long as his true valuation is also between 1/3 and 2/3. However, this kind of mis-report is immaterial in the Myerson auction.

When we mention the null auction, we mean the seller simply refuses to transact; i.e., $\forall i$, $S_i = \{0_i\}$.

3 The One-Buyer Benchmark

Fix any auction game (S, p, q, M, α). In any period t, the buyer's problem is

$$\max_{s \in S}(1-\delta)(p(s)v_t - q(s)) + \delta(\text{CONTINUATION PAYOFF}).$$

Since neither the distribution of the buyer's future valuations nor the auction games he will face in the future depend on the past history (including the current-period message he is sending to the auctioneer), the second term is a constant. So the buyer reacts to any auction game as if it is not repeated. So the auctioneer's optimal auction design problem is the same as if the auction game is not repeated.

In the static problem, we know that the optimal selling mechanism is for the seller to post the monopoly price. Notice that, when the auctioneer posts the monopoly price, she earns the monopoly profit, which is bounded away from the expected gain of trade. In other words, although IID-repetition (by pooling individual rationality constraints) generates the opportunity for the auctioneer to extract full surplus, the auctioneer is not able to take advantage of this opportunity.

4 The Main Result

We shall now demonstrate how the auctioneer can exploit tacit collusion and capture almost all the expected gain of trade when $n \geq 2$. To understand the intuition, it would be helpful to recall a closely related paper by Che and Yoo (2001; hereafter CY), who design an optimal multi-agent incentive contract in a similar repeated-interaction setting. Although CY and we study different kinds of problem (CY's problem is a moral harzard one, whereas ours is an adverse selection one), both of our designs share the same property of creating *positive* externalities among agents. Notice that in both of our problems, when there is no repeated interaction, the optimal design is to create *negative* externalities among agents ("relative performance evaluation" in CY's problem, "Myerson auction" in our problem). But when agents collude in a repeated-interaction setting, playing agents against each other becomes ineffective. Instead, the optimal design should try to exploit tacit collusion by creating *positive* externalities among agents. CY's trick of creating positive externalities is to use "joint performance evaluation" in place of "relative performance evaluation," whereas our trick is to use a public good provision game in place of more traditional auction games. Although the tricks are different, the idea is the same. We believe that this same idea would have other applications as well.

Consider the following auction, which takes the form of a public good provision game:[7]

- Bidders simultaneously decide whether or not to contribute to a public good (say, building a temporary auction house).
- If every bidder contributes, the auctioneer runs the CGK auction.
- If at least one bidder refuses to contribute, the auctioneer runs the null auction.
- The amount of contribution is $q^* := (1-\delta)\underline{w} + \delta w^*$.
- The auctioneer discloses everything to every bidder after the auction.

Once the auctioneer has chosen this auction game in period 0, bidders will find themselves playing a repeated game. A typical approach to analyze such a repeated game would be to use the APS machinery (Abreu, Pearce, and Stacchetti (1990)) to characterize the set of equilibrium payoffs, and then identify the one that maximizes bidders' total payoff. However, since this repeated game is deliberately designed to have certain properties, we can actually derive the bidder's optimal sequential equilibrium by brute force without ever using of the APS machinery.

Let's ignore incentives for a moment and ask what the most efficient outcome (from bidders' point of view) is. We shall then show that this efficient outcome can be supported as a sequential equilibrium outcome. Finally, we shall show that, in that bidder-optimal equilibrium, the auctioneer's payoff is

$$(1-\delta)\underline{W} + \delta W^*.$$

Since W^* is the upper bound of the auctioneer's payoff, the auction described above is hence almost optimal when δ is close to 1.[8]

Ignore incentives for a moment. Suppose a social planner can mandate some behavorial rule for each bidder, and aims at maximizing the sum of bidders' payoffs. What should such a social planner mandate each bidder to do? Since the auction game will remain the same in the next period regardless of what the planner mandates the bidders to do in the current period, the planner's problem can be collapsed into a single-period problem. If the CGK auction is ever played, the social planner should tell each bidder to use the same strictly increasing bidding function. This would guarantee that the winner of the CGK auction is the highest-valuation bidder, and hence maximize bidders' total payoff. The social planner still has to decide when the bidders should contribute, which amounts to trading-off the benefit of running the CGK auction against the cost of paying the contribution. Since each

[7]Notice the slight abuse of terminology below: both the CGK and the null "auctions" are actually parts of the auction we are constructing, and hence strictly speaking do not by themselves qualify as real auctions (as defined in Section 2). However, we believe there should be no confusion.

[8]Actually the upper bound W^* is not achievable. As we saw in Appendix A, even when the auctioneer can sign long term contracts with the bidders, at most she can only achieve an expected payoff of $(1-\delta)\pi^* + \delta W^*$, which is strictly smaller than W^* (but still larger than $(1-\delta)\underline{W} + \delta W^*$).

bidder's contribution decision can only be measurable with respect to his own valuation but not the others', the optimal contribution decision should take the form of a threshold rule, such that a bidder contributes if and only if his valuation is above a certain threshold.

Let $\mathbf{a} := (a_1, \ldots, a_n)$ be a vector of thresholds, and

$$W(\mathbf{a}) := \sum_{i \in N}(1 - F(a_i))(-q^*) + \int_{v_1=a_1}^{1} \cdots \int_{v_n=a_n}^{1} \max\{v_1, \ldots, v_n\} \mathrm{d}F(v_n) \cdots \mathrm{d}F(v_1)$$

is the corresponding sum of bidders' payoffs. We shall first prove that the optimal thresholds must be symmetric (i.e., $a_1 = \cdots = a_n$). Suppose $\mathbf{a}$ is not symmetric. Without loss of generality assume $a_1 < a_2$. Let $b \in (a_1, a_2)$ be such that

$$2F(b) = F(a_1) + F(a_2),$$

and define $\mathbf{a}' = (b, b, a_3, \ldots, a_n)$. It suffices to prove that $W(\mathbf{a}') > W(\mathbf{a})$. Let

$$\phi(v_1, v_2) := \int_{v_3=a_3}^{1} \cdots \int_{v_n=a_n}^{1} \max\{v_1, v_2, v_3, \ldots, v_n\} \mathrm{d}F(v_n) \cdots \mathrm{d}F(v_3),$$

and notice that $\phi(\cdot,\cdot)$ is a symmetric function. Then, using the definition of b, we have

$$
\begin{aligned}
W(\mathbf{a}') &= \sum_{i\in N}(1-F(a_i))(-q^*) + \int_{v_1=b}^{1}\int_{v_2=b}^{1}\phi(v_1,v_2)\mathrm{d}F(v_2)\mathrm{d}F(v_1) \\
&= \sum_{i\in N}(1-F(a_i))(-q^*) \\
&\quad + \int_{v_1=b}^{1}\int_{v_2=a_2}^{1}\phi(v_1,v_2)\mathrm{d}F(v_2)\mathrm{d}F(v_1) + \int_{v_1=a_1}^{b}\int_{v_2=a_2}^{1}\phi(v_1,v_2)\mathrm{d}F(v_2)\mathrm{d}F(v_1) \\
&\quad + \int_{v_1=b}^{1}\int_{v_2=b}^{a_2}\phi(v_1,v_2)\mathrm{d}F(v_2)\mathrm{d}F(v_1) - \int_{v_1=a_1}^{b}\int_{v_2=a_2}^{1}\phi(v_1,v_2)\mathrm{d}F(v_2)\mathrm{d}F(v_1) \\
&= W(\mathbf{a}) + \int_{v_1=b}^{1}\int_{v_2=b}^{a_2}\phi(v_1,v_2)\mathrm{d}F(v_2)\mathrm{d}F(v_1) - \int_{v_1=a_1}^{b}\int_{v_2=a_2}^{1}\phi(v_1,v_2)\mathrm{d}F(v_2)\mathrm{d}F(v_1) \\
&= W(\mathbf{a}) + \int_{v_1=b}^{a_2}\int_{v_2=b}^{a_2}\phi(v_1,v_2)\mathrm{d}F(v_2)\mathrm{d}F(v_1) \\
&\quad + \int_{v_1=a_2}^{1}\int_{v_2=b}^{a_2}\phi(v_1,v_2)\mathrm{d}F(v_2)\mathrm{d}F(v_1) - \int_{v_1=a_1}^{b}\int_{v_2=a_2}^{1}\phi(v_1,v_2)\mathrm{d}F(v_2)\mathrm{d}F(v_1) \\
&> W(\mathbf{a}) + \int_{v_1=b}^{a_2}\int_{v_2=b}^{a_2}\phi(v_1,v_2)\mathrm{d}F(v_2)\mathrm{d}F(v_1) \\
&\quad + \int_{v_1=a_2}^{1}\int_{v_2=b}^{a_2}\phi(v_1,b)\mathrm{d}F(v_2)\mathrm{d}F(v_1) - \int_{v_1=a_1}^{b}\int_{v_2=a_2}^{1}\phi(b,v_2)\mathrm{d}F(v_2)\mathrm{d}F(v_1) \\
&= W(\mathbf{a}) + \int_{v_1=b}^{a_2}\int_{v_2=b}^{a_2}\phi(v_1,v_2)\mathrm{d}F(v_2)\mathrm{d}F(v_1) \\
&\quad + \int_{v_1=a_2}^{1}\phi(v_1,b)(F(a_2)-F(b))\mathrm{d}F(v_1) - \int_{v_2=a_2}^{1}\phi(b,v_2)(F(b)-F(a_1))\mathrm{d}F(v_2) \\
&= W(\mathbf{a}) + \int_{v_1=b}^{a_2}\int_{v_2=b}^{a_2}\phi(v_1,v_2)\mathrm{d}F(v_2)\mathrm{d}F(v_1) \\
&> W(\mathbf{a}),
\end{aligned}
$$

which verifies our claim.

From now on we can restrict our attention to symmetric thresholds, and we shall slightly abuse notation and write $W(a)$ instead of $W(a,\ldots,a)$. We want to prove that $W(a)$ is maximized at $a=0$.

For any $a\in[0,1]$, let

$$Z_n(a) := \int_{v_1=a}^{1}\cdots\int_{v_n=a}^{1}\max\{v_1,\ldots,v_n\}\mathrm{d}F(v_n)\cdots\mathrm{d}F(v_1) \geq 0,$$

and notice that $Z_n(0)=W^*$. Define $Z_{n-1}(a)$ similarly. Also notice that $\forall a\in(0,1)$,

$$Z_n'(a) = -f(a)nZ_{n-1}(a) < 0,$$

so $Z_n(a)$ is monotone decreasing in a. Similarly $Z_{n-1}(a)$ is monotone decreasing in a as well.[9]

We can now rewrite $W(a)$ as

$$W(a) = Z_n(a) - nq^*(1 - F(a)),$$

and notice that $\forall a \in (0,1)$,

$$W'(a) = f(a)[nq^* - nZ_{n-1}(a)].$$

Since $Z_{n-1}(a)$ is monotone decreasing in a, $[nq^* - nZ_{n-1}(a)]$ is monotone increasing in a, and hence $W(a)$ is quasi-convex. $W(a)$, being a quasi-convex function, is maximized either at $a = 0$ or $a = 1$. Since

$$W(0) = W^* - nq^* = (1-\delta)(W^* - \underline{W}) > 0 = W(1),$$

$W(a)$ is maximized at $a = 0$.

In summary, the social planner should tell the bidders to always contribute, and then play the efficient equilibrium in the CGK auction.

Can this planner's ideal outcome be supported as a sequential equilibrium outcome? The answer is affirmative, and the proof is constructive. Consider the following strategy:

1. In the first period, contribute regardless of the period-1 valuation, and if the CGK auction is run, play the efficient equilibrium.
2. In any subsequent period,
   (a) if there exists at least one bidder who once refused to contribute in the past, do not contribute regardless of the current-period valuation;
   (b) otherwise, contribute regardless of the current-period valuation, and if the CGK auction is run, play the efficient equilibrium.

We claim that it is a sequential equilibrium for every bidder to follow the above strategy. First notice that it is a sequential equilibrium for every bidder not to contribute in any period regardless of his valuation. So the strategy described above is essentially a grim-trigger strategy. Then, by the one-stage deviation principle, it suffices to check the following two kinds of one-stage deviations:

1. *A bidder cannot benefit from not contributing when he should.* This follows from the facts that (*i*) if he does not contribute, his current-period and continuation payoffs will both be zero, whereas (*ii*) if he contributes, his current-period payoff will be at least $(1-\delta)(\underline{w} - q^*)$, his continuation payoff will be $\delta(w^* - q^*)$, and these two sum up to at least $((1-\delta)\underline{w} + \delta w^*) - q^* = 0$.

[9] This is true even when $n = 2$.

2. *When a CGK auction is run, a bidder cannot benefit from not playing the efficient-equilibrium bidding strategy.* This follows from the facts that (i) his opponents' future behavior will not depend on how he bids in the current-period CGK auction, and (ii) the efficient-equilibrium bidding strategy is the best response to his opponent's current-period bidding strategies.

When bidders play the above mentioned strategies, they contribute every period. So the auctioneer's expected payoff is $(1-\delta)\underline{W}+\delta W^*$, which is arbitrarily close to the upper bound W^* when δ is arbitrarily close to 1 (i.e., when bidders are patient). This concludes that the auction described above is almost optimal when bidders are patient.[10]

Theorem 1 *When there are two or more bidders, and when bidders are patient, an almost optimal auction takes the form of a public good provision game (as described above), and the auctioneer captures almost all the expected gain of trade.*

It is illuminating to compare our auction with the Myerson auction. In the extreme case when $\delta = 0$, bidders are completely impatient and hence cannot tacitly collude at all. In that case, the Myerson auction is the optimal auction, whereas our auction reduces to the Vickrey auction. Apparently our auction is not even remotely optimal (not to mention "almost") in this case.

However, as δ increases from 0 to 1, our auction will catch up and finally surpass the Myerson auction. To see this, notice that for any $\epsilon > 0$, our auction will give the auctioneer a payoff within the ϵ-neighborhood of the upper bound W^* for δ big enough. Whereas the auctioneer's payoff in the Myerson auction is uniformly bounded away from W^* regardless of δ. This bound can be computed as follows. First notice that honest bidding is a stage-game equilibrium for the Myerson auction, and hence bidding honestly every period is a sequential equilibrium in the repeated game where bidders play the Myerson auction repeatedly. Let W be the sum of bidders' payoffs in this sequential equilibrium, and by construction W does not depend on δ. Now, the bidder-optimal sequential equilibrium (which depends on δ) may give bidders an even higher total payoff: $W(\delta) \geq W$. Therefore, for any δ, the auctioneer's payoff cannot be bigger than $W^* - W(\delta)$, which in turn is no bigger than $W^* - W$. So $W^* - W$ is a uniform bound for the auctioneer's payoff regardless of δ.

This argument is actually more general than it looked. Not only that it continues to apply even if we modify our definition of the Myerson auction so that the auctioneer discloses *nothing* to any bidder after the auction, it also applies to any other standard auctions that

[10] This is a good place to go back to our earlier claim that it is the bidders' (not the auctioneer's) patience that drives this result. (See Footnote 5.) Suppose the auctioneer has a different discount factor that is far away from 1. Suppose intertemporal trade contracts are not enforceable and hence we do not need to worry about the infinite surplus that can potentially be generated from players having different discount factors. If the auctioneer uses our auction, she will still collect the same contributions every period, and hence her normalized discounted sum of expected payoff–normalized and discounted by her own far-away-from-1 discount factor–will still be $(1-\delta)\underline{W}+\delta W^*$ and is arbitrarily close to the total gain of trade W^* as bidders' discount factor δ goes to 1.

have a Bayesian Nash equilibrium (regardless of the disclosure policies). For any such an auction and any such a Bayesian Nash equilibirum, it is a sequential equilibrium for bidders to play that equilibrium every period in the corresponding repeated game. Let W be the sum of bidders' payoffs in that sequential equilibirum. Then $W^* - W$ will be a uniform bound for the auctioneer's payoff regardless of δ. Hence any fixed auction will perform worse than our auction when δ is big enough.

This argument does not apply to our auction, because our auction is not a fixed auction. It explicitly incorporates δ, which was originally a primitive parameter of the environment, into the formula of the non-refundable contribution. So its rule actually changes when the environment changes. When the environment is such that bidders are patient, it takes a form that out-performs the Myerson auction as well as any other standard auctions.

5 Discussions

5.1 Renegotiation-Proofness

One apparent problem with our auction is that the bidder-optimal sequential equilibrium is not renegotiation-proof. Once a bidder deviates and refuses to contribute in a particular period, every bidder will receive zero payoff in the continuation equilibrium. This gives rise to the possibility of renegotiation among bidders after such a deviation. However, this problem can be fixed with some modification to our auction when $n \geq 3$. The idea is to allow any $(n-1)$-coalition to punish the remaining bidder by excluding him from the auction. If we can make the total expected payoff collected by this $(n-1)$-coalition the same as the total expected payoff collected by the grand coalition along the equilibrium path, then we can say that the planner's ideal outcome (i.e., to have all bidders contribute in every period regardless of their valuations) can be supported as a renegotiation-proof sequential equilibrium outcome.

Let z^* be the per-bidder expected payoff in an $(n-1)$-CGK auction (i.e., a CGK auction among a group of $(n-1)$ bidders), and $\underline{z}$ be the expected payoff a bidder with the lowest type will receive in an $(n-1)$-CGK auction. Recall that in our original auction, bidders' total expected payoff along the equilibrium path is $n(w^* - q^*)$. Let $\hat{q}$ be implicitly defined by

$$(n-1)(z^* - \hat{q}) = n(w^* - q^*).$$

Roughly speaking, $\hat{q}$ would be the amount we should ask each member of a punishing coalition to contribute in order to maintain renegotiation-proofness. But $\hat{q}$ may be too large, and a member of a punishing coalition may not be willing to contribute this much when his valuation is low. So we need to adjust it a little bit. Let $\tilde{q}$ be the amount of per-bidder contribution if we were to apply our original auction to a group of $(n-1)$ bidders; i.e., $\tilde{q} := (1-\delta)\underline{z} + \delta z^*$. Let $\hat{\Delta} := \min\{0, \hat{q} - \tilde{q}\}$. Now the adjusted amount of contribution, $\hat{q} - \hat{\Delta}$, would be small enough for any member of a punishing coalition to swallow. To maintain renegotiation-proofness, we need to adjust q^* as well. Let $\triangle^*$ be implicitly defined by

$$(n-1)[z^* - (\hat{q} - \hat{\triangle})] = n[w^* - (q^* - \triangle^*)].$$

Now we can describe our modified auction:

- Bidders simultaneously decide whether or not to contribute to a public good.
- If bidder i chooses to contribute, he has to report a pair $(b_i, x_i) \in \mathbb{R}_+ \times N$; i.e., he has to report a bid, and name a fellow bidder to punish.
- If at least two bidders refuse to contribute, then every contributing bidder pays a contribution of $(q^* - \triangle^*)$, and the auctioneer runs the null auction.
- If exactly one bidder (say j) refuses to contribute,
  - if $\forall i \neq j$, $x_i = j$, then every contributing bidder pays a contribution of $(\hat{q} - \hat{\triangle})$, and the auctioneer runs the $(n-1)$-CGK auction among the contributing bidders;
  - otherwise, every particpating bidders pays a contribution of $(q^* - \triangle^*)$, and the auctioneer runs the null auction.
- If every bidders contributes,
  - if there exists j such that $\forall i \neq j$, $x_i = j$, then every bidder (except j) pays a contribution of $(\hat{q} - \hat{\triangle})$, and the auctioneer runs the $(n-1)$-CGK auction among the $(n-1)$ bidders other than j (i.e., the auctioneer in effect treats j as not contributing);
  - if $\forall i$, $x_i = i$, then every bidder pays a contribution of $(q^* - \triangle^*)$, and the auctioneer runs the CGK auction;
  - otherwise, every bidder pays a contribution of $(q^* - \triangle^*)$, and the auctioneer runs the null auction.
- The auctioneer discloses everything to every bidder after the auction.

Using exactly the same argument as in Section 4, and using the definitions of $(\hat{q} - \hat{\triangle})$ and $(q^* - \triangle^*)$, we can continue to argue that one of the social planner's ideal outcomes is to have all bidders contribute in every period regardless of their valuations. A renegotiation-proof sequential equilibrium that supports this planner's ideal outcome is as follows:

- In any period that belongs to the normal state (to be specified below), every bidder i contributes regardless of his valuation, and reports $x_i = i$.
- In any period that belongs to the i-punishment state (to be specified below), i does not contribute, and every bidder $j \neq i$ contributes regardless of his valuation, and reports $x_j = i$.

- The first period belongs to the normal state.
- If in any period there exists i who deviates, then the next period belongs to the i-punishment state.

The proof that the above strategy profile is indeed a sequential equilibrium is largely the same as that in Section 4. On the equilibrium path, bidders contribute in every period regardless of their valuations. Ex post efficiency is achieved. Bidders' total expected payoff is at most $(n-1)(z^*-\tilde{q}) = (1-\delta)(n-1)(z^*-\underline{z}) \approx 0$. So the auctioneer continues to capture almost all the expected gain of trade with this modified auction.

5.2 Entry and Exit

It seems to us that any auction that explicitly exploits the cooperation (rather than competition) among bidders is bounded to be too inflexible to handle unanticipated entry and exit of bidders. Nevertheless, there exists certain situations where our original auction can be modified to accommodate entry and exit.

Imagine that there is a large number of potential bidders in any period, but only n of them are serious. The identities of these serious bidders may change over time, and individual bidders need not know the identities of these serious bidders in any particular period. In this situation, our original auction can be modified such that as long as the auctioneer manages to collect n contributions, she will run the CGK auction among those who contribute. The modified sequential equilibrium is that each bidder contributes if and only if (i) he is serious, and (ii) there was a CGK auction in the last period.

Alternatively, imagine that there is a pool of n bidders. But every once a while some bidders will die and being replaced by new comers. If bidders have constant probability of dying, then the above modification will also work.

5.3 Other Issues

As we mentioned at the end of Section 4, our auction is performing well only when bidders are patient enough. This prompts the questions of whether or not tacit collusion is still exploitable when δ is in the intermediate range, and if yes, how? This optimal auction design problem is highly non-trivial because the auctioneer may want to choose an auction that does not disclose information regarding bids, identity of winners, etc., to the bidders at the end of each auction game. Such a non-disclosure policy has natural appeal to the auctioneer because it can potentially inhibit collusion (as deviations will be more difficult to detect). So solving the optimal auction design problem inevitably requires analyzing how well bidders can do when they play auctions with this kind of non- or partial-disclosure policies repeatedly, which in turn requires analyzing repeated games with private monitoring–an area of repeated games theory that we still do not know much.[11] Notice how we circumvented the

[11] See, for example, Ely and Välimäki (2002) and Matsushima (2001) for the state-of-the-art in this area.

problem of private monitoring by always having the auctioneer discloses everything to every bidder after every auction. This did not hurt our case because we looked at cases where δ is large. But the same is unlikely to be true once we move beyond cases of large δ.

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Appendix A: Competing Model–Full Commitment

In this appendix, we shall look at the competing model where the auctioneer can sign long term contracts with the bidders. Implicitly, such a model requires that the auctioneer has full commitment power over her future choices of auctions contingent on the bidding history. In other words, the auctioneer can commit to a contingent plan specifying which auction game will be announced in which period after which bidding history. Once the auctioneer has committed to any such contingent plan, the bidders will be playing a dynamic game with infinite horizon. This dynamic game may have multiple equilibria. Once again, we focus our attention on the bidder-optimal sequential equilibrium (and if there is more than one we focus our attention on the one that maximizes the auctioneer's expected payoff). The auctioneer's problem is: foreseeing that bidders can coordinate on their favorite sequential equilibrium in any dynamic game, which contingent plan should the auctioneer choose?

Not surprisingly, full surplus extraction is immediate in such a model. The reason why the following otherwise standard proof may look longer than one would anticipate is that it needs to handle the possibility of tacit collusion among bidders.

First observe that, if there were only one period, then the Myerson auction would have been optimal. Let π^* denote the expected revenue from a one-shot Myerson auction. Second observe that, by the first time when bidders have a chance to move, they have already learned their period-1 valuations, but not yet their valuations in subsequent periods. Hence the auctioneer has to honor the interim individual rationality contraints in the first period, but not necessarily those in subsequent periods. This suggests that $(1-\delta)\pi^* + \delta W^*$ is an upper bound of the auctioneer's expected payoff. Since this upper bound can be achieved by the following contingent plan, the following plan must be optimal for the auctioneer:

- In the first period, announce the following modified version of the Myerson auction:
  1. Bidders simultaneously submit two numbers (b_i, e_i), where $b_i \in [0,1]$ is i's bid in the Myerson auction, and $e_i \in \{0,1\}$ is i's willingness to pay a contribution of $\frac{\delta}{1-\delta}w^*$.
  2. The object is allocated as in the Myerson auction.
  3. Bidders first pay the Myerson-auction payments. On top of that, if $[\forall i,\ e_i = 1]$, then every bidder also needs to pay a contribution of $\frac{\delta}{1-\delta}w^*$.
  4. The auctioneer discloses everything to every bidder after the auction.
- If $[\forall i,\ e_i = 1]$ in the first period, announce the CGK auction from the second period onward;
- If $[\exists i,\ e_i = 0]$ in the first period, announce the null auction from the second period onward.

In other words, in the first period, the auctioneer announces the Myerson auction plus the following public good provision game:[12] if every bidder agrees to contribute, the auctioneer will give away the object for free from the second period onward; and if at least one bidder refuses to contribute, the auctioneer will refuse to transact from the second period onward. Facing such a contingent plan, bidders in effect find themselves playing a dynamic game. There are multiple equilibria in this dynamic game. But we shall show that in any sequential equilibrium that does not involve dominated strategies, bidders will bid sincerely in the period-1 Myerson auction.

Let σ be any sequential equilibrium of this dynamic game. For each bidder i, the sequential equilibrium σ (together with the distribution function F) induces a joint probability distribution H_i over v_i, b_i, and e_i (i.e., i's period-1 valuation,[13] bid, and willingness to contribute) on the equilibrium path. Define H_{-i} and H_N similarly, and notice that these are product measures. Let $V_i \subseteq [0,1]$ be the set of i's period-1 valuation such that i contributes with positive probability; i.e.,

$$V_i := \{v_i \in [0,1] \mid H_i(e_i = 1 \mid v_i) > 0\}.$$

Pick any $v_i \in V_i$, and any b_i in the support of the conditional probability $H_i(b_i \mid v_i, e_i = \mathbf{1})$. When bidder i has valuation v_i, he must find that submitting the pair $(b_i, e_i = 1)$ is at least as good as submitting the pair $(b_i, e_i = 0)$. The difference between these two pairs is that, if bidder i submits the pair $(b_i, e_i = 1)$, on top of the same Myerson-allocation and Myerson-payment, with probability $H_{-i}(e_{-i} = \mathbf{1})$, he will also pay the contribution of $\frac{\delta}{1-\delta}w^*$ in exchange for the chance of playing the CGK auction from the second period onward. This is profitable only if

$$H_{-i}(e_{-i} = \mathbf{1})w^* \leq \int_{v_{-i}} \int_{b_{-i}} w_i(b_i, b_{-i}) H_{-i}(\mathrm{d}b_{-i}|v_{-i}, e_{-i} = \mathbf{1}) H_{-i}(e_{-i} = \mathbf{1}|v_{-i}) H_{-i}(\mathrm{d}v_{-i}), \tag{1}$$

where $w_i(b_i, b_{-i})$ is i's continuation payoff conditional on the bidding history (b_i, b_{-i}) and the event that every bidder contributes in the first period. Multiplying both sides with $H_i(e_i = 1|v_i)$, and integrating over b_i with respect to the conditional probability $H_i(b_i|v_i, e_i = 1)$, we can rewrite the above inequality as

$$\begin{aligned} & H_{-i}(e_{-i} = \mathbf{1}) \int_{b_i} w^* H_i(e_i = 1|v_i) H_i(\mathrm{d}b_i|v_i, e_i = 1) \\ \leq & \int_{v_{-i}} \int_{b_N} w_i(b_N) H_N(\mathrm{d}b_N|v_N, e_N = \mathbf{1}) H_N(e_N = \mathbf{1}|v_N) H_{-i}(\mathrm{d}v_{-i}). \end{aligned}$$

[12] More precisely, it is a game of public good provision *with* refund, and is different from the game of public good provision *without* refund that we will use in the main text of this paper.

[13] We suppress the time subscript for simplicity and write v_i instead of v_{i1}.

Integrating over v_i, and using the fact that $[v_i \notin V_i \Longrightarrow H_i(e_i = 1|v_i) = 0]$, we can rewrite the above inequality as

$$H_N(e_N = \mathbf{1})w^* \leq \int_{v_N} \int_{b_N} w_i(b_N) H_N(\mathrm{d}b_N|v_N, e_N = \mathbf{1}) H_N(e_N = \mathbf{1}|v_N) H_N(\mathrm{d}v_N).$$

Summing over i, and using the fact that

$$\forall b_N, \quad \sum_{i \in N} w_i(b_N) \leq W^* = nw^*,$$

we can rewrite the above inequality as

$$H_N(e_N = \mathbf{1})W^* \leq \int_{v_N} \int_{b_N} W^* H_N(\mathrm{d}b_N|v_N, e_N = \mathbf{1}) H_N(e_N = \mathbf{1}|v_N) H_N(\mathrm{d}v_N) = H_N(e_N = \mathbf{1})W^*. \tag{2}$$

Apparently the above weak inequalities hold as equalities. Since inequality (2) comes from integrating and summing a bunch of inequalities (1), we must have for all i, for all $v_i \in V_i$, and for $H_i(b_i|v_i, e_i = 1)$-almost all b_i, inequality (1) holds as an equality–bidder i is indifferent between submitting the pair $(b_i, e_i = 1)$ and submitting the pair $(b_i, e_i = 0)$ when his valuation is v_i. If $b_i \notin D(v_i)$, then submitted the pair $(b_i, e_i = 0)$ will be dominated, a contradiction. So we have $b_i \in D(v_i)$. The same domination argument holds for any b_i that is in the support of the conditional probability $H_i(b_i|v_i, e_i = 0)$ as well. Hence we have $b_i \in D(v_i)$ for all b_i in the support of $H_i(b_i|v_i)$.

Finally, consider any $v_i \notin V_i$. Using the same domination argument again, we have $b_i \in D(v_i)$ for all b_i in the support of $H_i(b_i|v_i)$. This completes our proof that in any sequential equilibrium that does not involve dominated strategies, bidders bid sincerely in the period-1 Myerson auction.

The above proof also shows that bidders are indifferent among all the sequential equilibria (because the contributions exactly cancel out their expected continuation payoffs on the equilibrium path of any sequential equilibrium) modulo those involve dominated strategies which we ignore for practical reason. So all these sequential equilibria are trivially bidder-optimal. According to our equilibrium selection criteria, when there are more than one bidder-optimal sequential equilibrium, we shall select the one that maximizes the auctioneer's payoff. It is easy to see which sequential equilibrium will maximize the auctioneer's payoff–the one that bidders agrees to pay the contributions regardless of their period-1 valuations:

- every bidder bids sincerely in the period-1 Myerson auction;
- every bidder submits $e_i = 1$ and pays the contribution of $\frac{\delta}{1-\delta}w^*$ regardless of his valuation in the first period; and
- in each of the CGK auctions from the second period onward, bidders play the CGK auction's efficient equilibrium.

It is straightforward to check that this is indeed a sequential equilibrium. In this sequential equilibrium, the auctioneer's expected payoff is

$$\begin{aligned} & (1-\delta)(\pi^* + n\frac{\delta}{1-\delta}w^*) \\ = \quad & (1-\delta)\pi^* + \delta W^*, \end{aligned}$$

which achieves the upper bound as we claimed earlier.

Appendix B: Competing Model–No Commitment

In the last appendix we saw that when the auctioneer has full commitment power and can sign long term contracts with the bidders, full surplus extraction is immediate. In this appendix we shall analyze another competing model of repeated interaction, namely that the auctioneer has no commitment power whatsoever on her future choices of auctions.[14] Such a model in effect turns the whole game into a repeated game, with the auctioneer one of the players. In every stage game of this repeated game, the auctioneer moves first, announces an auction, and then bidders play this auction game. This repeated game has multiple sequential equilibria. Once again, we focus our attention to the bidder-optimal sequential equilibrium. We shall show that the implication of such a model is exactly the opposite of that in the long-term-contract model: when bidders are patient enough, the sum of bidders' payoffs can achieve the upper bound of W^* in the bidder-optimal sequential equilibirum. This means the auctioneer's payoff is driven down to zero.

The proof that, if the auctioneer is herself a player of a bigger repeated game, her profit would be driven down to zero in the bidder-optimal sequential equilibrium is contructive. Consider the following strategy profile:

- In the first period, the auctioneer announces the CGK auction.
- In any period, *unless* it is in the punishment phase (to be specified below), the auctioneer announces the CGK auction.
- In any period, *if* it is in the punishment phase, the auctioneer announces the null auction.
- In any period, every bidder refuses to participate if the auctioneer announces any auction different from the CGK auction, and plays the CGK auction's efficient equilibrium if the auctioneer announces the CGK auction.[15]

[14] This is different from saying that the auctioneer cannot commit to the auction game she announces. We maintain the assumption that, within any given period, the auctioneer's commitment power is complete. In other words, even if the announced auction is not ex post efficient, she can stick to her gun and insist not to transact anymore within that period.

[15] Notice that bidders' strategies do not depend on whether the current period belongs to the normal state or the punishment phase, and hence bidders do not need to know what other bidders have done in the past in order to follow these strategies. In particular, bidders can follow the prescribed strategies regardless of the disclosure policy employed by the auctioneer.

- The transitional dynamics between the normal state and the punishment phase is as follows:
  1. The first period belongs to the normal state.
  2. In any normal-state period,
     (a) if the auctionner announces the CGK auction, the next period will belongs to the normal state;
     (b) if the auctioneer announces an auction different from the CGK auction,
        i. if no bidders participate, then the next period belongs to the normal state;
        ii. if two or more bidders participate, then the next period belongs to the normal state;
        iii. if exactly one bidder participates, then the next period becomes the first period of an m-period-long punishment phase, where m is any large but finite integer that satisfies

$$\delta w^* \geq (1-\delta) + \delta^{m+1} w^*. \tag{3}$$

  3. In any period during a punishment phase,
     (a) if the auctioneer announces the CGK auction, then the next period belongs to the normal state;
     (b) if the auctioneer announces an auction different from either the CGK auction or the null auction,
        i. if no bidders participate, then the next period belongs to the normal state;
        ii. if two or more bidders participate, then the next period belongs to the normal state;
        iii. if exactly one bidder participates, then the next period becomes the first period of an m-period-long punishment phase.
     (c) if the auctioneer announces the null auction, and if it is the m'-th consecutive period in the current punishment phase, where $m' < m$, then the next period remains in the punishment phase;
     (d) if the auctioneer announces the null auction, and if it is the m'-th consecutive period in the current punishment phase, where $m' \geq m$, then the next period belongs to the normal state.

Notice that inequality (3) will be satisfied by some m if bidders are patient enough; i.e., if δ is large enough such that $\delta w^* > 1 - \delta$.

If players follow this strategy profile, then the auctioneer will announce the CGK auction every period and get zero payoff, whereas each bidder will get an expected payoff of w^*. To see that this strategy profile is indeed a sequential equilibrium, all we need is to invoke the one-stage deviation principle and check all possible one-stage deviations.

1. If the auctioneer announces the CGK auction, bidder i knows that (i) his fellow bidders will play the CGK auction's efficient equilibrium strategies, and (ii) the next period

will belong to the normal state. So it does not pay to deviate from also playing the CGK auction's efficient equilibrium strategy.

2. If the auctioneer announces the null auction, bidder i has no ways to deviate anyway.

3. If the auctioneer announces an auction different from either the CGK auction or the null auction, bidder i knows that his fellow bidders will not participate. If he does not participate, the next period will belong to the normal state, and hence his continuation payoff will be δw^*. If he participates, his current-period payoff is at most 1, but he has to endure an m-period-long punishment phase. So his (normalized) payoff is at most $(1-\delta)1+\delta^{m+1}w^*$, which by inequality (3) is no bigger than δw^*. So it does not pay to deviate from not participating.

4. In any period, the auctioneer is indifferent between announcing the CGK auction and announcing the null auction. Deviating from announcing either of these will not make any difference as bidders will not participate anyway.

This exhausts all possible one-stage deviations and verifies that the above strategy profile is indeed a sequential equilibrium.