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The Relation Between Implementability and the Core

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Abstract

This paper proves a simple and general theorem on resource-allocation mechanisms that achieve Pareto efficiency. We say that a mechanism (game form) is normal if at any action profile, an agent who obtains his endowments neither pays nor receives a positive amount. In the context of auctions, this simply means that losers receive no bill. We prove that for any normal mechanism, if its Nash equilibrium allocations are Pareto efficient for all preference profiles, then the equilibrium allocations are necessarily in the core. The result holds for a large class of allocation problems in which monetary transfers are feasible and the consumption space is discrete except for the space of transfers. Examples include auctions with any number of objects, economies with indivisible public goods, marriage problems, and coalition formation.

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1 Introduction

This paper studies mechanisms or institutions for allocating resources. A mechanism is modeled as a game form, which specifies a set of admissible actions for each agent and a feasible allocation for each profile of actions. This paper proves a simple and general theorem on mechanisms whose Nash equilibrium allocations are Pareto efficient.

We focus on allocation problems in which monetary transfers are feasible. We say that a mechanism is *normal* if at any profile of actions, the mechanism neither charges nor pays an agent who obtains his endowments. In the context of auctions, this simply means that a loser neither pays nor receives a positive amount. We show that for any normal mechanism, if its Nash equilibrium allocations are Pareto efficient for all profiles of preferences, then the equilibrium allocations are necessarily in the core. In other words, if a sub-correspondence of the Pareto correspondence can be implemented by a normal mechanism in Nash equilibrium, then the implemented correspondence is a sub-correspondence of the core correspondence. This implies that achieving Pareto efficient allocations is *equivalent* to achieving core allocations provided that only normal mechanisms are admissible and the equilibrium concept is Nash equilibrium.

The result holds for a large class of allocation problems in which monetary transfers are admissible and the consumption space is discrete except for the space of transfers. Examples include auctions of an arbitrary number of objects, economies with indivisible public goods, matching problems, and coalition formation. Examples are given in Section 6.

For allocation problems in which the core is empty for some preference profiles, our result implies that there exists no normal mechanism whose Nash equilibrium allocations are Pareto efficient for all preference profiles. Thus, to ensure efficiency, the planner has to use a mechanism that sometimes charges or pays agents who keep their endowments.

The next section illustrates the basic argument of the result by using examples in the context of auctions. The main section of the paper is Section 5, in which we prove two theorems that produce the result stated above as a corollary. The theorems divide normality into *no-tax-for-endowments* and *no-subsidy-for-endowments*, and derive their implications separately. We will see that the driving force of our main result is no-subsidy-for-endowments. The theorems are stated in terms of Maskin monotonicity (Maskin, 1999), which implies that the theorems also apply to other notions of implementation for which Maskin monotonicity is a necessary condition. Our results are also relevant for allocation problems in which monetary transfers are not feasible, which is discussed in Section 7. Appendix A proves parallel results for social choice functions that are strategy-proof and non-bossy.

2 Examples

This section illustrates the basic argument of the main result by using examples in the context of auctions. Suppose that there are three agents. One of them (the seller) initially owns one indivisible object, and the other agents (buyers 1 and 2) are interested in buying the object. Suppose that preferences are quasi-linear and the buyers' valuations for the object are $(v_1, v_2) = (5, 3)$. The seller's valuation for the object is assumed to be zero, i.e., he is a revenue-maximizer.

A mechanism (game form) specifies an action set of each buyer and the outcome for each profile of actions. A mechanism is called *normal* if at any action profile, the mechanism neither charges nor pays a buyer who does not obtain the object.[1]

The main result states that for any normal mechanism whose Nash equilibrium allocations are Pareto efficient for all profiles of valuations, the equilibrium allocations are actually in the core.

This is not to say that the Pareto set and the core coincide; of course, the core is a proper subset of the Pareto set for generic valuations. In our example above, any allocation in which buyer 1 wins the object is Pareto efficient, while the core requires that buyer 1 pay between 3 and 5. What our result says is that a normal mechanism cannot implement a correspondence that always chooses Pareto efficient allocations but sometimes chooses allocations outside of the core.

We use Nash equilibrium with complete information, which may strike some readers as unreasonable given that the auction literature usually uses Bayesian equilibrium. However, it should be noted that our result extends beyond the simple auction problem and holds for a rather large class of allocation problems.

To see our result in the above example, take any normal mechanism whose equilibrium allocations are Pareto efficient for all profiles of valuations. We first derive individual rationality. Suppose that for the above valuation profile, buyer 1 wins the object and pays $6 > 5$ at some Nash equilibrium (m_1, m_2). Buyer 1's utility is negative in this equilibrium, while normality guarantees a zero utility

[1] An important auction mechanism that violates normality is the all-pay mechanism.

for a loser. This implies that 1 cannot lose the auction by changing his action as long as 2 chooses m_2. For if 1 could lose, m_1 would not be a best response to m_2. Thus, given m_2, all 1 can do is to win the object and pay at least 6, and m_1 achieves the minimum payments. This trivially implies that (m_1, m_2) is also a Nash equilibrium when 1's valuation is $w_1 = 2$. However, this equilibrium is not Pareto efficient, a contradiction.

We now show that equilibrium outcomes are in the core. Suppose then that for valuation profile $(v_1, v_2) = (5, 3)$, buyer 1 wins the object and pays $2.5 < 3$ at some Nash equilibrium (m_1, m_2). This implies that given m_2, buyer 1 can win the object only by paying at least 2.5, and he pays 0 if he loses. It trivially follows that (m_1, m_2) is also a Nash equilibrium when 1's valuation is $w_1 = 2.7$, since he is still willing to pay 2.5 for the object and he cannot lower the price given that 2 chooses m_2. However, this equilibrium is not Pareto efficient, a contradiction.[2]

The result also holds when there exist more than one object. To see how the argument extends to the multi-object case, suppose that there are two objects (a and b) and buyers' valuation functions are given by

	$\{a,b\}$	$\{a\}$	$\{b\}$	$\emptyset$
v_1	9	2	6	0
v_2	5	4	2	0

We continue to assume that the seller is a revenue-maximizer, i.e., his valuation is zero for all bundles of objects. In this example, Pareto efficiency means that buyer 1 obtains b and buyer 2 obtains a. Let p_i denote buyer i's payments. Then it is easily checked that the core allocations are the Pareto efficient allocations in which $1 \leq p_1 \leq 6$ and $3 \leq p_2 \leq 4$. For example, if $p_2 < 3$, then the allocation is blocked by buyer 1 and the seller; the seller would take a from buyer 2 and give it to buyer 1 in exchange for 3 units of money.

Now, take any normal mechanism whose equilibrium outcomes are Pareto efficient. To see why equilibrium outcomes must belong to the core, suppose that $p_2 = 3 - \varepsilon < 3$ at some Nash equilibrium (m_1, m_2) for the above valuation profile. Let $P(\{a,b\})$ denote the infimum amount that 2 must pay to obtain both objects given that 1 chooses m_1; if 2 cannot obtain both objects given that 1 chooses

[2] The argument breaks down if the mechanism is not normal since buyer 1 may then be able to receive a positive amount if he loses. For example, suppose that he could receive 1 dollar if he loses. This does not destroy the initial equilibrium since by winning the object, 1 obtains a surplus of $5 - 2.5 > 1$. However, (m_1, m_2) is no longer an equilibrium when 1's valuation is $w_1 = 2.7$ since by winning the object, 1 only obtains a surplus of $2.7 - 2.5 < 1$.

m_1, then $P(\{a, b\}) = \infty$. Define $P(b)$ and $P(\emptyset)$ analogously. Note that since the mechanism is normal, if $P(\emptyset)$ is finite (i.e., if 2 can lose), then $P(\emptyset) = 0$. The fact that m_2 is a best response to m_1 means that

$$v_2(a) - (3 - \varepsilon) \geq \max\{v_2(\{a, b\}) - P(\{a, b\}), v_2(b) - P(b), -P(\emptyset)\}.$$

Now, an obvious but important observation is that since $P(\emptyset)$ is either infinite or zero, the above inequality continues to hold if we decrease buyer 2's valuation for each *non-empty* bundle by the same number provided that it does not make the left-hand side negative. In particular, we can decrease 2's valuation for each non-empty bundle by $v_2(a) - (3 - \varepsilon) = 1 + \varepsilon$ without violating the inequality. Thus, (m_1, m_2) remains a Nash equilibrium when the valuation functions are given by

	$\{a, b\}$	$\{a\}$	$\{b\}$	$\emptyset$
v_1	9	2	6	0
w_2	$4 - \varepsilon$	$3 - \varepsilon$	$1 - \varepsilon$	0

However, the equilibrium is not Pareto efficient because for (v_1, w_2) both objects should go to buyer 1 at any Pareto efficient allocation.

The result holds for more general allocation problems. The next section presents a class of allocation problems for which the result holds.

3 Preliminaries

3.1 Allocation Problems

Let $N = \{1, 2, \ldots, n\}$ denote the set of agents ($n \geq 2$). There exists a divisible commodity, which is referred to as "money." The consumption space of agent $i \in N$ is given by $X_i \times \mathbb{R}$ where X_i is an *arbitrary* non-empty set. A generic element of $X_i \times \mathbb{R}$ is denoted by $a_i = (x_i, t_i)$ where $t_i \in \mathbb{R}$ denotes the amount of money that i receives and $x_i \in X_i$ denotes the remaining part of the consumption bundle.

3.1.1 Preferences

Each agent $i \in N$ has a complete and transitive binary relation R_i defined over $X_i \times \mathbb{R}$. As usual, $a_i \, R_i \, a_i'$ means that a_i is at least as good as a_i' for agent i. The strict preference and indifference relations are denoted by P_i and I_i, respectively.

The universal set of complete and transitive binary relations over $X_i \times \mathbb{R}$ is denoted by $\mathcal{R}_i^U$. Let $\mathcal{R}_i \subseteq \mathcal{R}_i^U$ be a non-empty subset of admissible preferences for agent i. We impose the following assumptions on $\mathcal{R}_i$.

P1* (Strict Monotonicity in Money). For all $i \in N$, all $R_i \in \mathcal{R}_i$, all $x \in X_i$, and all $t, t' \in \mathbb{R}$, if $t' > t$, then $(x, t') \; P_i \; (x, t)$.

P2* (Compensability). For all $i \in N$, all $R_i \in \mathcal{R}_i$, all $x, x' \in X_i$, and all $t \in \mathbb{R}$, there exists $t' \in \mathbb{R}$ such that $(x', t') \; I_i \; (x, t)$.

The second assumption says that a sufficient amount of money can make up for any bundle of non-monetary goods. That is, no bundle of non-monetary goods is "infinitely desirable" nor "infinitely undesirable."

P1* and P2* are imposed throughout the paper (which is why we put asterisks). The following is not always imposed but important.

R1 (Richness for Agent *i*). $\mathcal{R}_i$ contains all quasi-linear preferences.[3]

This says that any quasi-linear preferences are admissible. This is a condition of the richness of the set of admissible preferences. This condition is not necessarily imposed for all agents. The set of agents for whom the richness condition is satisfied is denoted by $N_r \subseteq N$. R1 can be weakened; see Remark 4 below.

When X_i is a "continuous" set (e.g., $\mathbb{R}^\ell$), R1 is not very reasonable because it says that even discontinuous quasi-linear preferences are admissible. Similarly, R1 may not be very interesting when X_i is ordered in a way that is economically meaningful since then conditions such as monotonicity and single-peakedness would be relevant. The interesting case is therefore when X_i is discrete and there exists no clear economic ordering on X_i, in which case R1 is not unreasonable. Section 6 gives a few such examples of allocation problems.

To see why one may not want to assume R1 for all agents, recall that in the examples in the previous section, we assumed that the seller is a revenue-maximizer. The assumption, which is often imposed in the auction literature, implies that $|\mathcal{R}_i| = 1$ for the seller. Thus R1 is violated for him.

We let $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n$ denote the set of preference profiles, whose elements are denoted by $R = (R_1, R_2, \ldots, R_n)$. As usual, (R_i', R_{-i}) denotes the preference profile that is equivalent to R except that agent i's preferences are R_i'. We define $(R_S', R_{N \setminus S})$ similarly.

[3] A preference relation R_i is *quasi-linear* if it can be represented by a utility function of the form $u(x_i, t_i) = v(x_i) + t_i$ for some function $v \colon X_i \to \mathbb{R}$.

3.1.2 Feasible Allocations

We denote agent i's initial consumption bundle by $(\omega_i, 0)$, where ω_i is a predetermined element of X_i. In what follows, ω_i is called i's ***endowments***, while $(\omega_i, 0)$ is called i's ***initial consumption bundle***.

There is an exogenously given non-empty set $A \subseteq \prod_{i\in N}(X_i \times \mathbb{R})$ of ***feasible allocations***. A generic feasible allocation is denoted by a, and as usual, a_i denotes agent i's consumption bundle in allocation a.

Similarly, for each coalition $S \subseteq N$, there is an exogenously given set $A_S \subseteq \prod_{i\in S}(X_i \times \mathbb{R})$ of allocations that S can achieve on its own. Elements $a_S \in A_S$ are referred to as ***feasible S-allocations***. Feasible allocations are equivalent to feasible N-allocations, i.e., $A = A_N$. While we assume $A \neq \emptyset$, we allow $A_S = \emptyset$ for $S \subsetneq N$.

We consider the following assumptions on A and A_S.

F1. For all $i \in N$, there exists $a \in A$ such that $a_i = (\omega_i, 0)$.

F2 (Transferability). For all $(x_i, t_i)_{i\in N} \in A$ and all vectors $m \in \mathbb{R}^N$, if $\sum_{i\in N} m_i = 0$, then $(x_i, t_i + m_i)_{i\in N} \in A$.

F3 (No Negative Externality). For all coalitions $\emptyset \neq S \subsetneq N$ and all $a_S \in A_S$,

$$(a_S, (\omega_i, 0)_{i\in N\setminus S}) \in A.$$

F1 says that for each agent, there exists at least one feasible allocation in which he consumes his initial consumption bundle. This is trivially satisfied if the "initial allocation" $(\omega_i, 0)_{i\in N}$ is feasible, which is the case for most of the applications.

F2 simply says that it is feasible to transfer money among agents. While this assumption is reasonable and standard, it is not completely innocuous because it implies that there are no bounds to monetary transfers, which is a rather strong assumption.

F3 says that if a_S is a feasible allocation for S, then it is feasible that S consumes a_S and the other agents consume their initial consumption bundles. This is satisfied for typical private-good economies (e.g., auctions, marriage problems, house-allocation problems, etc). For example, in marriage problems, if a man and a woman can match with each other, then the matching in which this pair forms and the other agents remain single is feasible.

F3 does not hold if a coalition can produce public "bads" that are necessarily consumed by other agents. An example is a situation in which a coalition can emit pollution that inevitably harms other agents; the other agents cannot consume $(w_i, 0)$ because "clean air" is no longer available. On the other hand, F3 is not necessarily violated when a coalition can produce public goods and other agents can *refuse* to consume them. For example, libraries are usually non-rival and refusable. A formal example is given in Section 6.3.

F1–F3 are not always imposed. Our results specify which of them are required.

3.2 Social Choice Correspondences and Implementation

This section introduces a few standard definitions on social choice correspondences and implementation.[4]

A ***social choice correspondence*** is a correspondence $\varphi\colon \mathcal{R} \twoheadrightarrow A$ that associates with each admissible preference profile $R \in \mathcal{R}$ a subset of feasible allocations $\varphi(R) \subseteq A$.[5]

A feasible allocation $a \in A$ is ***Pareto efficient*** for $R \in \mathcal{R}$ if there exists no feasible allocation $a' \in A$ such that $a'_i \; R_i \; a_i$ for all $i \in N$ with strict preference holding for some $i \in N$. The set of Pareto efficient allocations for R is denoted by $P(R)$, and the correspondence P is called the ***Pareto correspondence***.

A feasible allocation $a \in A$ is ***improved upon*** by a coalition $S \subseteq N$ under $R \in \mathcal{R}$ if there exists a feasible S-allocation $a'_S \in A_S$ such that $a'_i \; R_i \; a_i$ for all $i \in S$ with strict preference holding for some $i \in S$. When this is the case, we also say that a'_S ***dominates*** a for S under R. The ***core*** for R is the set of feasible allocations that cannot be improved upon by any coalition $S \subseteq N$ under R. Let $C(R)$ denote the core for R. The correspondence C is called the ***core correspondence***, which is not necessarily nonempty-valued.

A feasible allocation $a \in A$ is ***individually rational*** for $R \in \mathcal{R}$ if $a_i \, R_i \, (\omega_i, 0)$ for all $i \in N$. The set of individually rational allocations for R is denoted by $I(R)$.

A ***sub-correspondence*** of a social choice correspondence φ is a social choice correspondence φ' such that $\varphi'(R) \subseteq \varphi(R)$ for all $R \in \mathcal{R}$. We denote this by $\varphi' \subseteq \varphi$.

A ***mechanism*** (or game form) is a list $G = ((M_i)_{i \in N}, g)$ where M_i is a non-empty set of strategies for agent i and $g\colon \prod_{i \in N} M_i \to A$ assigns a feasible

[4] Excellent surveys of the implementation literature include Moore (1991), Corchon (1996), Palfrey (2000), and Jackson (2001).

[5] The arrow $\twoheadrightarrow$ is used for a correspondence. That is, $f\colon X \twoheadrightarrow Y$ means that f associates with each element of X a subset of Y.

allocation to each strategy profile.

We say that a social choice correspondence φ is (fully) ***implementable in Nash equilibrium*** if there exists a game form G such that, for all $R \in \mathcal{R}$, the set of Nash equilibrium allocations of game (G, R) coincides with $\varphi(R)$.

4 Axioms

This section introduces our main axioms. We first define them for allocations and then extend the definitions to mechanisms.

Say that a feasible allocation satisfies *no-tax-for-endowments* if no agent who consumes his endowments pays a positive amount. Similarly, a feasible allocation satisfies *no-subsidy-for-endowments* if no one who consumes his endowments *receives* a positive amount. Finally, a feasible allocation satisfies *no-transfer-for-endowments* or *normality* if no transfer is made neither to nor from an agent who keeps his endowments. Although we used the term normality in the first two sections, we use no-transfer-for-endowments in what follows.

Definition. A feasible allocation $(x_i, t_i)_{i \in N}$ satisfies

(i) ***no-tax-for-endowments*** if for all $i \in N$, if $x_i = \omega_i$, then $t_i \geq 0$,

(ii) ***no-subsidy-for-endowments*** if for all $i \in N$, if $x_i = \omega_i$, then $t_i \leq 0$,

(iii) ***no-transfer-for-endowments***, or ***normality***, if (i) and (ii) are both satisfied.

Let A_{nt}, A_{ns}, and A_n denote the sets of feasible allocations that satisfy (i), (ii), and (iii), respectively.

The above definitions refer to allocations. We also define corresponding axioms for mechanisms.

Definition. A mechanism $G = ((M_i)_{i \in N}, g)$ satisfies ***no-tax-for-endowments***, ***no-subsidy-for-endowments***, and ***no-transfer-for-endowments*** if for all $m \in \prod_{i \in N} M_i$, $g(m) \in A_{nt}$, $g(m) \in A_{ns}$, and $g(m) \in A_n$, respectively.

Notice that these definitions do not refer to equilibrium. For example, a mechanism satisfies no-tax-for-endowments if at *any* strategy profile, the mechanism does not charge an agent who keeps his endowments.

It may appear that no-tax-for-endowments rules out *entry fees.* Under mechanisms that charge entry fees (e.g., matching services), it is possible that an agent pays an entry fee and ends up consuming ω_i (e.g., finding no job). However, no-tax-for-endowments does not necessarily rule out entry fees if N is the set of *participants*, who have already paid entry fees, and $G = ((M_i)_{i \in N}, g)$ describes the *post-entry* stage of the mechanism. With this interpretation, what is ruled out by no-tax-for-endowments is a situation in which an agent pays an entry fee, keeps his endowments ($x_i = \omega_i$), and then pays an amount in addition to the entry fee, where the additional fee depends on other agents' actions.

It might be helpful to compare our axioms with individual rationality. First, our axioms refer to purely physical features of allocation or mechanism, while individual rationality involves information about preferences. Second, our axioms apply only to agents who obtain their endowments and does not say anything about the others, while individual rationality applies to all agents. Third, individual rationality does imply no-tax-for-endowments, but the converse does not hold. Finally, individual rationality has no direct logical relation with no-subsidy-for-endowments (or no-transfer-for-endowments).

5 Results

Before we state and prove our results, we first introduce important definitions.

Given a consumption bundle $a_i \in X_i \times \mathbb{R}$ and a preference relation $R_i \in \mathcal{R}_i$, we define the lower-contour set of R_i at a_i by

$$L(R_i, a_i) = \{a'_i \in X_i \times \mathbb{R} : a_i \; R_i \; a'_i\}.$$

Given a subset $B \subseteq A$ of feasible allocations, we denote its projection on agent i's consumption space by $Proj_i(B)$. That is,

$$Proj_i(B) = \{a_i \in X_i \times \mathbb{R} : (a_i, a_{-i}) \in B \text{ for some } a_{-i}\}.$$

Definition. Given a subset $B \subseteq A$, a social choice correspondence φ is ***Maskin monotonic with respect to B*** if for all $R, R' \in \mathcal{R}$ and all $a = (a_i)_{i \in N} \in \varphi(R) \cap B$, if for all $i \in N$,

$$L(R_i, a_i) \cap Proj_i(B) \subseteq L(R'_i, a_i) \cap Proj_i(B), \tag{1}$$

then $a \in \varphi(R')$.

That is, Maskin monotonicity with respect to B means that if an allocation

$a \in B$ is φ-optimal for some R and another preference profile R' is obtained by expanding each agent's lower-contour set within $Proj_i(B)$ at a_i, then a remains φ-optimal for R'.

When (1) holds, we say that R'_i is a ***Maskin monotonic transformation*** of R_i at a_i with respect to $Proj_i(B)$. When (1) holds for *all* agents, then we say that R' is a ***Maskin monotonic transformation*** of R at a with respect to B.

Remark 1. If φ is Maskin monotonic with respect to B, then φ is also Maskin monotonic with respect to any superset $B' \supseteq B$. In particular, if φ is Maskin monotonic with respect to A_n, then φ is also Maskin monotonic with respect to A_{nt} and A_{ns}.

Maskin monotonicity is introduced by Maskin (1999) in the case of $B = A$. The following is a simple reinterpretation of Maskin's basic result on implementation.

Fact 1 (Maskin, 1999). *Let $B \subseteq A$ be given. If a social choice correspondence φ is implementable in Nash equilibrium by a game form $((M_i)_{i \in N}, g)$ such that the range of g is contained in B, then φ is Maskin monotonic with respect to B.*

Our first result states that Maskin monotonicity with respect to A_{nt} together with Pareto efficiency implies individual rationality for agents who satisfy the richness condition.

Theorem 1. *Assume F1 and F2 (transferability). Then, if a social choice correspondence $\varphi\colon \mathcal{R} \twoheadrightarrow A_{nt}$ is Maskin monotonic with respect to A_{nt} and is a sub-correspondence of the Pareto correspondence, then for all $R \in \mathcal{R}$ and all $a \in \varphi(R)$,*

$$a_i \, R_i \, (\omega_i, 0) \quad \textit{for all } i \in N_r. \tag{2}$$

Proof. We prove the following:

Claim. For all $R \in \mathcal{R}$ and all $a \in P(R) \cap A_{nt}$, if $a \in P(R')$ for all R' that is a Maskin monotonic transformation of R at a with respect to A_{nt}, then a satisfies (2).

Suppose, by way of contradiction, that $(\omega_i, 0) \, P_i \, a_i \equiv (x_i, t_i)$ holds for some $i \in N_r$. Since $a \in A_{nt}$, $x_i \neq \omega_i$. Since the richness condition is satisfied for

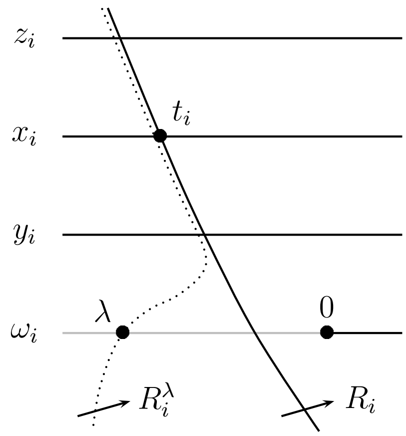

Figure 1:

agent i, it follows that for all $\lambda < 0$, there exists a (quasi-linear) preference relation $R_i^\lambda \in \mathcal{R}_i$ such that

(i) $a_i \; I_i^\lambda \; (\omega_i, \lambda)$,

(ii) for all $x_i' \in X_i \setminus \{\omega_i\}$ and all $t_i' \in \mathbb{R}$, if $a_i \; I_i \; (x_i', t_i')$, then $a_i \; I_i^\lambda \; (x_i', t_i')$.

See Figure 1. In words, the indifference set of R_i^λ associated with (x_i, t_i) is identical to that of R_i except that the former contains (ω_i, λ). It is easy to check that R_i^λ is a Maskin monotonic transformation of R_i at (x_i, t_i) with respect to $Proj_i(A_{nt})$.[6][7] Thus, by the hypothesis of the claim, a is Pareto efficient for (R_i^λ, R_{-i}). Now, by F1, there exists a feasible allocation $a' = (x_j', t_j')_{j \in N}$ such that $a_i' = (\omega_i, 0)$. Let us define another allocation a^λ by

$$\begin{aligned} a_i^\lambda &= (\omega_i, \lambda), \\ a_j^\lambda &= (x_j', t_j' - \lambda/(n-1)) \quad \text{ for all } j \in N \setminus \{i\}. \end{aligned}$$

F2 ensures that a^λ is a feasible allocation. By P1* and P2*, all agents $j \neq i$ strictly prefer a_j^λ to a_j if $\lambda < 0$ is sufficiently small. Since agent i is indifferent between a_i^λ and a_i under R_i^λ, we conclude that a is not Pareto efficient for (R_i^λ, R_{-i}) for a sufficiently small λ, a contradiction. This proves the claim.

[6]To see this, let $(\hat{x}_i, \hat{t}_i) \in L(R_i, a_i) \cap Proj_i(A_{nt})$. If $\hat{x}_i = \omega_i$, then $(\omega_i, 0) \, P_i \, a_i \, R_i \, (\omega_i, \hat{t}_i)$, which implies $\hat{t}_i < 0$ in violation of no-tax-for-endowments. Thus it must be the case that $\hat{x}_i \neq \omega_i$. It then follows from (ii) that $a_i \; R_i^\lambda \; (\hat{x}_i, \hat{t}_i)$.

[7]It is easy to see that the converse holds as well; that is, R_i is also a Maskin monotonic transformation of R_i^λ with respect to $Proj_i(A_{nt})$.

The rest of the proof is straightforward. Take φ, R, and $a \in \varphi(R)$ as in the statement of the theorem. Then $a \in P(R) \cap A_{nt}$. Since φ is Maskin monotonic with respect to A_{nt}, it follows that for all R' that is a Maskin monotonic transformation of R at a with respect to A_{nt}, we have $a \in \varphi(R')$ and so $a \in P(R')$. Hence the claim just proved tells us that a satisfies (2). Q.E.D.

We introduce another piece of notation. For all $i \in N$, all $R_i \in \mathcal{R}_i$, all $a_i = (x_i, t_i) \in X_i \times \mathbb{R}$, and all $\lambda \in \mathbb{R}$, if $x_i \neq \omega_i$, then let $T_i(R_i, a_i, \lambda)$ denote the set of preferences $R_i^\lambda \in \mathcal{R}_i$ that satisfy (i) and (ii) in the above proof. Note that $T_i(R_i, a_i, \lambda) \neq \emptyset$ for all $i \in N_r$. While $\lambda < 0$ in the above proof, we define $T_i(R_i, a_i, \lambda)$ for all $\lambda \in \mathbb{R}$.

Our next theorem, which is our main result, states that Maskin monotonicity with respect to A_{ns} together with Pareto efficiency implies that the social choice is not blocked by any coalition that contains all agents who violate the richness condition.

Theorem 2. *Assume F3 (no negative externality). Then, if a social choice correspondence $\varphi\colon \mathcal{R} \twoheadrightarrow A_{ns}$ is Maskin monotonic with respect to A_{ns} and is a sub-correspondence of the Pareto correspondence, then for all $R \in \mathcal{R}$, no allocation $a \in \varphi(R)$ can be improved upon by a coalition $S \supseteq N \setminus N_r$.*

Proof. We prove the following:

Claim. For all $R \in \mathcal{R}$ and all $a \in P(R) \cap A_{ns}$, if $a \in P(R')$ for all R' that is a Maskin monotonic transformation of R at a with respect to A_{ns}, then a cannot be improved upon by any $S \supseteq N \setminus N_r$.

To prove the claim, denote $a = (x_i, t_i)_{i \in N}$ and let $T = \{i \notin S : a_i \, P_i \, (\omega_i, 0)\}$. If $T = \emptyset$, we can skip the remaining part of this paragraph and proceed to the next paragraph. Thus we assume $T \neq \emptyset$. Since $a \in A_{ns}$, we have $x_i \neq \omega_i$ for all $i \in T$. For all $i \in T$, let $R'_i \in T_i(R_i, a_i, 0)$. See Figure 2. In words, the indifference set of R'_i at a_i is identical to that of R_i except that the former contains his initial consumption bundle. The existence of R'_i is ensured by the fact that $i \notin S$ implies $i \in N_r$. It is easy to check that R'_i is a Maskin monotonic transformation of R_i at a_i with respect to $\mathit{Proj}_i(A_{ns})$.[8] Thus, by the assumption of the claim, a is Pareto efficient for $(R_{N \setminus T}, R'_T)$.

[8]To see this, let $(\hat{x}_i, \hat{t}_i) \in L(R_i, a_i) \cap \mathit{Proj}_i(A_{ns})$. If $\hat{x}_i = \omega_i$, then no-subsidy-for-endowments implies $\hat{t}_i \leq 0$. Then by Condition (i) of $T_i(R_i, a_i, 0)$, it follows that $a_i \, R'_i \, (\hat{x}_i, \hat{t}_i)$. If $\hat{x}_i \neq \omega_i$, then Condition (ii) of $T_i(R_i, a_i, 0)$ implies $a_i \, R'_i \, (\hat{x}_i, \hat{t}_i)$.

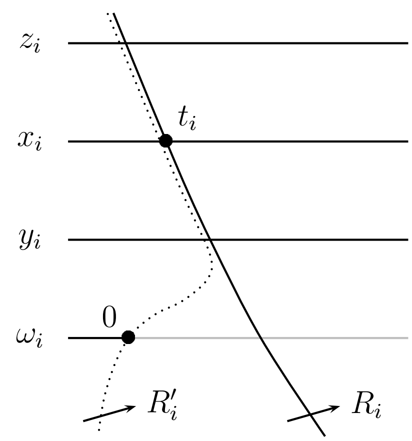

Figure 2:

Now, suppose, by way of contradiction, that S can improve upon a. This means that there exists $a'_S \in A_S$ such that $a'_i \; R_i \; a_i$ for all $i \in S$ with strict preference holding for some $i \in S$. By F3, $(a'_S, (\omega_i, 0)_{i \in N \setminus S})$ is a feasible allocation. Furthermore, it Pareto dominates a under $(R_{N \setminus T}, R'_T)$ since agents $i \in T$ are indifferent between a_i and $(\omega_i, 0)$, and agents $i \in N \setminus (S \cup T)$ weakly prefer $(\omega_i, 0)$ to a_i by the definition of T. This contradiction establishes the claim.

To complete the proof, take φ as in the theorem and let $R \in \mathcal{R}$ and $a \in \varphi(R)$. Then $a \in P(R) \cap A_{ns}$. Since φ is Maskin monotonic with respect A_{ns}, it follows that for all R' that is a Maskin monotonic transformation of R at a with respect to A_{ns}, $a \in \varphi(R')$ and so $a \in P(R')$. Hence, the claim just proved tells us that a cannot be improved upon by any $S \supseteq N \setminus N_r$. Q.E.D.

Setting $N_r = N$ in Theorems 1 and 2 together with Fact 1 gives us the following corollaries.

Corollary 1. *Assume F1 and F2. Assume also that R1 holds for all agents. Then, if a mechanism satisfies no-tax-for-endowments and implements a social choice correspondence $\varphi \subseteq P$ in Nash equilibrium, then $\varphi \subseteq I$.*

Corollary 2. *Assume F3. Assume also that R1 holds for all agents. Then, if a mechanism satisfies no-subsidy-for-endowments and implements a social choice correspondence $\varphi \subseteq P$ in Nash equilibrium, then $\varphi \subseteq C$.*

Remark 2. In most of the applications, $C \subseteq I$. However, this is not necessarily the case in our general framework because we do not assume $(\omega_i, 0) \in A_{\{i\}}$. For example, if $A_S = \emptyset$ for all $S \subsetneq N$, then $C = P \not\subseteq I$.

Corollary 2 is somewhat surprising because neither our equilibrium concept nor our axiom suggests coalitional stability. However, it should be clear from the proof of Theorem 2 that the key source of coalitional stability is Pareto efficiency, which is a cooperative notion. The proof says that since allocation a is Pareto efficient, $(a'_S, (\omega_i, 0)_{i \in N \setminus S})$ does not Pareto dominate a, but since each i in $N \setminus S$ weakly prefers $(\omega_i, 0)$ to a_i, it follows that a'_S does not dominate a_S for coalition S. What no-subsidy-for-endowments does is to derive the desired preference of agents $N \setminus S$.

Our results apply to mechanisms in which equilibria do not exist for some preference profiles. The reason is that for all the results above, it is immaterial whether φ is nonempty-valued. Thus, what Corollary 2, for example, says is that for any mechanism that satisfies no-subsidy-for-endowments, if the equilibrium allocations are Pareto efficient whenever equilibria exist, then the equilibrium allocations, when they exist, are in the core.

Given the generality of our model, the core is empty for some preference profiles for some specifications of A and A_S. For allocation problems for which $C(R) = \emptyset$ for some R, what Corollary 2 shows is that there exists *no* mechanism that satisfies no-subsidy-for-endowments and achieves Pareto efficiency for all preference profiles. That is, to achieve Pareto efficiency for all preference profiles, the planner has to use a mechanism that sometimes gives a positive amount to an agent who keeps his endowments.

Remark 3. The above corollaries hold for other notions of implementation for which Maskin monotonicity is a necessary condition. An example is implementation in strong Nash equilibrium (Maskin, 1979). Another example is when a generalized notion of mechanism is considered; e.g., Greenberg (1990) and Miyagawa (2001). On the other hand, our results do not extend to subgame-perfect equilibrium; see Section 6.1.2 for a counter-example.

There are similar results in the literature. Hurwicz (1979) shows, in the context of classical exchange economies, that if a sub-correspondence φ of P and I can be implemented by a game form whose "attainable set" is convex, then φ is a sub-correspondence of the Walrasian correspondence. The attainable set is the set of consumption bundles that an agent can obtain by changing his strategy given the others' strategies. Hurwicz considers a game form for which this set is convex for any agent at any strategy profile.

Hurwicz (1979) also shows that if a sub-correspondence φ of P and I is Maskin monotonic in the usual sense and upper-hemi continuous, then φ is a *super-*

correspondence of the Walrasian correspondence. A similar result is obtained by Gevers (1986). In the context of marriage problems (without transfers), Kara and Sönmez (1996) show that if a sub-correspondence φ of P and I is Maskin monotonic, then φ is a super-correspondence of C.

Remark 4. Theorems 1 and 2 do not use the full force of R1. It should be evident from the proof that in Theorem 1, R1 can be replaced with the following condition (stated for a particular $i \in N$):

R2. For all $(x_j, t_j)_{j \in N} \in A$, all $R_i \in \mathcal{R}_i$, and all $\lambda < 0$, if $x_i \neq \omega_i$ and $(\omega_i, 0)\, P_i\, (x_i, t_i)$, then $T_i(R_i, (x_i, t_i), \lambda) \neq \emptyset$.

For Theorem 2, it suffices to assume a variant of R2 where λ is set to zero. Specifically, in Theorem 2, R1 can be replaced with the following condition:

R3. For all $(x_j, t_j)_{j \in N} \in A$, and all $R_i \in \mathcal{R}_i$, if $x_i \neq \omega_i$ and $(x_i, t_i)\, P_i\, (\omega_i, 0)$, then $T_i(R_i, (x_i, t_i), 0) \neq \emptyset$.

Remark 5. Similarly, Theorems 1 and 2 do not use the full force of Maskin monotonicity. In Theorem 1, Maskin monotonicity with respect to A_{nt} can be replaced with the following condition:

M1. For all $R \in \mathcal{R}$, all $a \in \varphi(R) \cap A_{nt}$, all $i \in N$, all $R'_i \in \mathcal{R}_i$, and all $\lambda < 0$, if $(\omega_i, 0)\, P_i\, a_i$ and $R'_i \in T_i(R_i, a_i, \lambda)$, then $a \in \varphi(R'_i, R_{-i})$.

Similarly, in Theorem 2, Maskin monotonicity with respect to A_{ns} can be replaced with the following condition:

M2. For all $R \in \mathcal{R}$, all $a \in \varphi(R) \cap A_{ns}$, all $i \in N$, and all $R'_i \in \mathcal{R}_i$, if $a_i\, P_i\, (\omega_i, 0)$ and $R'_i \in T_i(R_i, a_i, 0)$, then $a \in \varphi(R'_i, R_{-i})$.

We close this section by establishing that under weak assumptions on A and A_S, core allocations satisfy no-transfer-for-endowments and the core correspondence is Maskin monotonic with respect to A_n.

We start with assumptions under which core allocations satisfy no-transfer-for-endowments.

F4. For all $i \in N$, $(\omega_i, 0) \in A_{\{i\}}$.

F5. For all $a = (x_i, t_i)_{i \in N} \in A$ and all $k \in N$, if $x_k = \omega_k$, then

$$(x_i, t_i + t_k/(n-1))_{i \in N \setminus \{k\}} \in A_{N \setminus \{k\}}.$$

F4 simply says that the initial bundle is feasible for each agent. F5 says that if an agent k consumes his endowments ($x_k = \omega_k$) in some feasible allocation, then the other agents can obtain their non-monetary bundles without k's participation. If $t_k > 0$, then $N \setminus \{k\}$ can increase their aggregate amount of money by t_k by excluding k. If $t_k < 0$, then their aggregate amount of money decreases by t_k without k's participation.

Fact 2. *If F4 (resp. F5) holds, then $C(R) \subseteq A_{nt}$ (resp. $C(R) \subseteq A_{ns}$) for all $R \in \mathcal{R}$.*

Proof. F4 obviously implies $C(R) \subseteq I(R) \subseteq A_{nt}$. To prove the other part, suppose F5 holds and let $a \in A \setminus A_{ns}$. This means that there exists $k \in N$ such that $a_k = (\omega_k, t_k)$ and $t_k > 0$. By F5,

$$(x_i, t_i + t_k/(n-1))_{i \in N \setminus \{k\}}$$

is feasible for $N \setminus \{k\}$ and dominates a for $N \setminus \{k\}$. Q.E.D.

We now give assumptions under which the core correspondence is Maskin monotonic with respect to A_n. The first condition is that F5 holds not only for A but for A_S for all $S \subseteq N$:

F6. For all $S \subseteq N$, all $a_S = (x_i, t_i)_{i \in S} \in A_S$ and all $k \in S$, if $x_k = \omega_k$, then

$$(x_i, t_i + t_k/(|S|-1))_{i \in S \setminus \{k\}} \in A_{S \setminus \{k\}}.$$

That is, if a feasible allocation for a coalition S is such that a member $k \in S$ receives his endowments, then the other members of S can obtain the same allocation of non-monetary goods without k's participation.

F7. For all $i \in N$ and all $(x_i, t_i) \in X_i \times \mathbb{R}$, if either $x_i \neq \omega_i$ or $t_i = 0$, then there exists $a \in A_n$ such that $a_i = (x_i, t_i)$.

This says that any consumption bundle that satisfies no-transfer-for-endowments is a part of some feasible allocation in which no-transfer-for-endowments is satisfied for all agents. F7 is equivalent to the requirement that for all $i \in N$,

$$Proj_i(A_n) = \{(x_i, t_i) \in X_i \times \mathbb{R} : \text{either } x_i \neq \omega_i \text{ or } t_i = 0\}.$$

This is easily satisfied in applications. For example, consider single-object auctions and take any bundle (x_i, t_i) for buyer i where he wins the object (thus

$x_i \neq \omega_i$). Then we can take $a \in A$ where the budget is balanced between i and the seller, i.e., $t_0 = -t_i$, and other buyers do not pay any amount.

The next condition states that for any feasible allocation for any coalition, the total amount that the members receive is not positive.

F8. For all $S \subseteq N$ and all $(x_i, t_i)_{i \in S} \in A_S$, $\sum_{i \in S} t_i \leq 0$.

This is a standard condition of budget balance.

Fact 3. *If F4–F8 hold, then the core correspondence is Maskin monotonic with respect to A_n.*

Proof. See Appendix B.

By Maskin (1999) and Saijo (1988), Maskin monotonicity with respect to A_n is almost sufficient for a correspondence to be implementable by a normal mechanism. Precisely, we say that a social choice correspondence φ satisfies ***no veto power*** with respect to A_n if for all $R \in \mathcal{R}$ and all $a \in A_n$, if $[a_i \ R_i \ a'_i$ for all $a' \in A_n]$ for at least $|N| - 1$ agents, then $a \in \varphi(R)$. This condition is usually vacuous in applications when monetary transfers are feasible. By a simple application of the classical result of Maskin (1999) and Saijo (1988), we obtain that if a social choice correspondence φ satisfies Maskin monotonicity and no veto power with respect to A_n and $|N| \geq 3$, then φ can be implemented by a normal mechanism in Nash equilibrium.

6 Applications

This section presents a few specific allocation problems that satisfy our assumptions and discusses issues associated with the examples.

6.1 Auctions

Let us return to auction problems and describe them formally using our notation. Denote by K a non-empty finite set of objects. Then $X_i = 2^K$ for all agents. There is one agent (called the seller) who initially owns these objects. If we call the seller agent 0 (thus we set $N = \{0, 1, \ldots, n\}$), then $w_0 = K$ and $\omega_i = \emptyset$ for all $i \neq 0$. For all buyers i, $\mathcal{R}_i$ is the set of all quasi-linear preferences. On the other hand, the seller is a revenue-maximizer and thus $|\mathcal{R}_0| = 1$. Given $S \subseteq N$, a list

$(x_i, t_i)_{i \in S}$ is a feasible S-allocation if and only if $(x_i)_{i \in S}$ is a partition of $\cup_{i \in S} w_i$ and $\sum_{i \in N} t_i = 0$. F1–F8 are then satisfied.

Since the seller is a revenue-maximizer, he does not satisfy any of the richness conditions R1–R3. Then $N_r = N \setminus \{0\}$ and Theorem 2 says only that the social choice is not blocked by any coalition that contains the seller, which does not imply individual rationality for buyers. Thus, Theorem 2 alone does not give us the core. However, we can use Theorem 1 to establish that the social choice is individually rational for all buyers. That is, Theorems 1 and 2 together imply that if a mechanism satisfies no-transfer-for-endowments and implements $\varphi \subseteq P$ in Nash equilibrium, then $\varphi \subseteq C$. This example illustrates the usefulness of Theorem 1.

6.1.1 First-Price Auctions

A mechanism that satisfies all of our requirements in this context is the first-price mechanism, which satisfies no-transfer-for-endowments and achieves Pareto efficiency in Nash equilibrium. To avoid a problem with ties and nonexistence of equilibrium, consider the following variant of the first-price mechanism with an "integer game." Each buyer announces $m_i = (b_i, k_i) \in \mathbb{R} \times \{0, 1, 2, \ldots\}$, where b_i is his bid and k_i is an integer. As usual, the buyer with the highest non-negative bid wins the object and pays the amount equal to his bid. When there are multiple buyers with the highest non-negative bid, the winner is the one who announced the highest integer. When there is a tie in the "integer game" as well, the winner is the one with the smallest index among those who announced the highest bid and integer. Formally, let $N' = \{i \in N \setminus \{0\} : b_i \geq \max\{b_j, 0\} \text{ for all } j \in N \setminus \{0\}\}$ be the set of buyers who announce the highest non-negative bid. When $N' = \{i\}$, then i is the winner. When N' is not a singleton, the winner is given by

$$\min\{i \in N' : k_i \geq k_j \text{ for all } j \in N'\}.$$

Then it is easy to check that the mechanism has a Nash equilibrium for all preference profiles and the equilibrium allocations are Pareto efficient. Thus our results (Theorems 1 and 2) show that the equilibrium allocations are in the core, which is easy to check directly.

6.1.2 Sequential First-Price Auctions

Another interesting mechanism is a sequential version of the above mechanism. The mechanism is an extensive game form with perfect information in which there are $(|N| - 1)$ stages. In stage $i \in N \setminus \{0\}$ of the mechanism, buyer i announces $(b_i, k_i) \in \mathbb{R} \times \{0, 1, 2, \ldots\}$ knowing what buyers $j < i$ have announced in the previous stages. Given the announcements of the buyers, the allocation is determined in the same way as in the simultaneous version of the mechanism.

Fact 4. *Let φ be the correspondence that the sequential first-price mechanism implements in subgame-perfect equilibrium. Then $\varphi \subseteq P$ but $\varphi \not\subseteq C$.*

We first show that the subgame-perfect equilibria of the mechanism are Pareto efficient. To see this, take any valuation profile $v \in \mathbb{R}^{N \setminus \{0\}}$ for buyers, while $v_0 = 0$ by our assumption. Let $H = \{i \in N : v_i \geq v_k \text{ for all } k \in N\}$ be the set of agents with the highest valuation. Assume that the highest valuation is positive since the other case is trivial. Suppose, by way of contradiction, that in some subgame-perfect equilibrium, the winner is agent $w \notin H$. Let $h = \max H$ be the buyer who moves last among H. Let $v^* = \max\{v_i : i > h\}$ be the highest valuation of buyers who move after h; if $h = n$ (so he moves in the last stage), then set $v^* = 0$. Then $v^* < v_h$. Thus, for a sufficiently small $\varepsilon > 0$, we have $v_h - \varepsilon > \max\{v^*, v_w, 0\}$. Since h is a loser, his utility is zero in the equilibrium. However, if he announces $b_h = v_h - \varepsilon$, he wins the auction because $v_h - \varepsilon$ is higher than the valuation of any buyer i who moves after h. This is a desired contradiction.

We now show that the subgame-perfect equilibria of the sequential first-price mechanism are not necessarily core allocations. To see this, suppose that there are two buyers (1 and 2) and their valuations are $(v_1, v_2) = (3, 5)$. The following is an equilibrium:

$$(b_1, k_1) = (0, 0),$$
$$(b_2, k_2) = \begin{cases} (b_1, k_1 + 1) & \text{if } b_1 \leq 5, \\ (0, 0) & \text{if } b_1 > 5. \end{cases}$$

That is, 2's strategy is to match his bid with 1's and announce a higher integer whenever 1's bid does not exceed 2's valuation. When 1's bid exceeds 2's valuation, 2 announces $(0, 0)$. Given 2's strategy, 1 knows that he can never win the auction, so he announces $(0, 0)$. This is a subgame-perfect equilibrium, but the equilibrium outcome is that 2 wins the object without paying any amount, which is not in the core.

6.2 Exchange Economies with Indivisibilities

We can generalize the above example by allowing more than one agent to have non-empty endowments. For all $i \in N$, let ω_i be the set of objects that agent i initially owns. The set of objects in the economy is $K \equiv \cup_{i \in N} \omega_i$ and the consumption space is $X_i \times \mathbb{R} = 2^K \times \mathbb{R}$. The sets A and A_S are defined as in the previous section. Then F1–F8 are all satisfied.

An important solution concept in this context is Walrasian equilibrium. A ***Walrasian equilibrium*** for $R \in \mathcal{R}$ is a list $((x_i, t_i)_{i \in N}, (p_k)_{k \in K}) \in A \times \mathbb{R}^K$ such that for all $i \in N$,

$$(x_i, t_i) = (x_i, \sum_{k \in \omega_i} p_k - \sum_{k \in x_i} p_k) \; R_i \; (Y, \sum_{k \in \omega_i} p_k - \sum_{k \in Y} p_k) \quad \text{for all } Y \subseteq K.$$

An obvious but important observation is that the budget set contains the initial bundle $(\omega_i, 0)$. Let $W(R)$ denote the set of Walrasian equilibrium allocations when the preference profile is R. Then it is easy to check that the correspondence W, i.e., the Walrasian correspondence, is Maskin monotonic with respect to A_n. Furthermore, $W \subseteq P$ by the First Welfare Theorem (Bikhchandani and Mamer, 1997). Then, what Corollary 2 tells us is that $W \subseteq C$.

6.3 Public Goods

We have remarked that our allocation problems allow for public goods to some extent. The following is an example.

There is a finite set K of indivisible public goods and society can choose at most one from the set. The cost of good $k \in K$ is given by $c(k) \in \mathbb{R}_+$. A coalition S such that $|S| > n/2$ is entitled to choose any good in K and all agents in the society can benefit from the good. We assume, however, that an agent is not forced to consume the public good. That is, goods are non-rival but *refusable*. For example, suppose that the public good is the speaker that an economics department invites to its annual public seminar. The set K is the set of candidates and $c(k)$ is the cost of inviting speaker k. Then our assumption simply means that one does not have to attend the seminar.

Formally, $X_i = K \cup \{0\}$ where 0 means that one does not attend the seminar. We set $c(0) = 0$. The initial bundle is $(\omega_i, 0) = (0, 0)$. For coalitions $S \subseteq N$ such that $|S| > n/2$, an S-allocation $(x_i, t_i)_{i \in S}$ is feasible if and only if there exists $y \in K \cup \{0\}$ such that for all $i \in S$,

$$x_i \in \{0, y\},$$
$$c(y) + \sum_{i \in S} t_i = 0.$$

The first condition captures the assumption that attendance is not mandatory. For coalitions $S \subseteq N$ such that $|S| \leq n/2$, an S-allocation $(x_i, t_i)_{i \in S}$ is feasible if and only if $x_i = 0$ for all $i \in S$ and $\sum_{i \in S} t_i = 0$. Then F1–F3 are satisfied. F3 is satisfied simply because when a coalition S in majority determines the speaker and pays the cost, it is feasible that other agents neither attend the seminar nor pay a fee. In this context, no-tax-for-endowments, for example, means that a fee is charged only to people who attend the seminar, which is not completely realistic but reasonable.

On the other hand, this example violates F6. To see this, take any coalition S such that $|S| > n/2$ and $|S| - 1 \leq n/2$. This coalition is entitled to propose a speaker even when some member $k \in S$ does not attend the seminar. That is, the coalition is effective for $(x_i, t_i)_{i \in S}$ such that $x_j \neq 0$ for some $j \in S$ while $x_k = 0$ for some $k \in S$. However, $S \setminus \{k\}$ is no longer entitled to propose a speaker. Indeed, the core correspondence is not Maskin monotonic with respect to A_n.[9]

6.4 Marriage Problems

Our next example is marriage problems (Gale and Shapley, 1962) where monetary transfers are feasible. There are two finite disjoint sets M and W, where M denotes the set of men and W denotes the set of women. For each man $m \in M$, $X_m = W \cup \{m\}$. For each woman $w \in W$, $X_w = M \cup \{w\}$. The initial consumption is $(\omega_i, 0) = (i, 0)$ for all $i \in N \equiv M \cup W$. For each coalition $S \subseteq N$,

[9] To see this, suppose that there are two agents (1 and 2) and two candidates ($K = \{\alpha, \beta\}$), and that inviting a speaker is costless. Preferences are quasi-linear and the valuation functions are given by $(v_1(\alpha), v_1(\beta), v_1(0)) = (3, 0, 0)$ and $(v_2(\alpha), v_2(\beta), v_2(0)) = (0, 2, 0)$. Then a core allocation is $a_1 = a_2 = (\alpha, 0)$. Now, change 1's valuation function to $(w_1(\alpha), w_1(\beta), w_1(0)) = (0, -3, 0)$, which is a Maskin monotonic transformation of v_1 at a_1 with respect to $Proj_1(A_n)$. However, for valuation profile (w_1, v_2), allocation (a_1, a_2) is not Pareto efficient and hence not in the core.

an S-allocation $(x_i, t_i)_{i \in S}$ is feasible if and only if for all $i \in S$,

$$\begin{aligned} x_i &\in S, \\ x_i &= j \text{ if and only if } x_j = i, \\ \sum_{i \in S} t_i &= 0. \end{aligned}$$

Then, F1–F8 are all satisfied. When preferences are quasi-linear, the allocation problems are equivalent to the *assignment problems* of Shapley and Shubik (1972). It is easy to see that our model also covers *college admissions problems* (Gale and Shapley, 1962) with monetary transfers.

6.5 Coalition Formation

Consider coalition formation problems where any coalition can form. Each agent starts as a singleton and cares only about the coalition to which he belongs. Then we set $X_i = \{S \subseteq N : i \in S\}$ and $\omega_i = \{i\}$. For all $S \subseteq N$, an S-allocation $(x_i, t_i)_{i \in S}$ is feasible if and only if

$$\begin{aligned} x_i &\subseteq S & &\text{for all } i \in S, \\ x_i &= x_j \text{ or } x_i \cap x_j = \emptyset & &\text{for all } i, j \in S, \\ \sum_{i \in S} t_i &= 0. \end{aligned}$$

The first two conditions simply say that S is partitioned. F1–F8 are then all satisfied. This class of problems is studied by Banerjee *et al.* (2001) and Bogomolnaia and Jackson (2002).

On the other hand, a violation of F3 occurs if an agent cares about the entire partition, i.e., X_i is the set of coalition structures. In this case, ω_i would be the initial coalition structure (i.e., $(\{1\}, \{2\}, \ldots, \{n\})$) and so $(\omega_i, 0)$ is not available for i when a non-trivial coalition forms outside of i.

7 When Monetary Transfers Are Infeasible

Theorem 2 does not use F2, which implies that the theorem also applies to allocation problems in which monetary transfers are not admissible. When transfers are not admissible, the axiom of no-subsidy-for-endowments becomes vacuous. Thus, the theorem tells us the relation between the core and the standard version of Maskin monotonicity. This section clarifies what Theorem 2 implies in this context.

We assume that for any coalition, monetary transfers are not feasible:

F9 (No Transfers). For all $S \subseteq N$, all $(x_i, t_i)_{i \in S} \in A_S$, and all $i \in S$, $t_i = 0$.

Under this assumption, the relevant information about preferences is the ranking between bundles of the form $(x_i, 0)$. We let $\succcurlyeq_{R_i}$ denote the preference relation over X_i that is induced from R_i, i.e., for all $x_i, y_i \in X_i$,

$$x_i \succcurlyeq_{R_i} y_i \iff (x_i, 0)\ R_i\ (y_i, 0).$$

The strict preference and indifference relations associated with $\succcurlyeq_{R_i}$ are denoted by $\succ_{R_i}$ and $\sim_{R_i}$, respectively.

Richness condition R1 in this context is equivalent to the assumption of unrestricted domain:

R4 (Unrestricted Domain). For any complete and transitive binary relation $\succcurlyeq$ defined over X_i, there exists $R_i \in \mathcal{R}_i$ such that $\succcurlyeq_{R_i}$ is equivalent to $\succcurlyeq$.

This is stronger than necessary. The following condition suffices for our result:

R5. For all $R_i \in \mathcal{R}_i$, and all $x_i \in X_i$, there exists $R'_i \in \mathcal{R}_i$ such that

$$\begin{aligned} &y_i \succcurlyeq_{R'_i} z_i \iff y_i \succcurlyeq_{R_i} z_i \quad \text{for all } y_i, z_i \in X_i \setminus \{\omega_i\}, \\ &\omega_i \sim_{R'_i} x_i. \end{aligned}$$

This says that for any $x_i \in X_i$ and any admissible ranking $\succcurlyeq_i$ defined over X_i, there exists another admissible ranking $\succcurlyeq'_i$ that coincides with $\succcurlyeq_i$ over $X_i \setminus \{\omega_i\}$ but such that ω_i is indifferent to x_i. For example, in the context of marriage problems, R'_i and R_i have the same ranking over the agents on the opposite side of the market but differ in the set of acceptable agents.[10]

Theorem 2 concerns Maskin monotonicity with respect to A_{ns}. When monetary transfers are not admissible, no-subsidy-for-endowments is trivially satisfied by any mechanism. Thus, Maskin monotonicity with respect to A_{ns} reduces to Maskin monotonicity with respect to A.

Furthermore, in view of Remark 5, we can weaken Maskin monotonicity. The following condition suffices for our result.

[10] If x_i is individually rational for R_i, then R'_i is similar to what Roth and Rothblum (1999) call a *truncation* of R_i.

Definition. A social choice correspondence $\varphi\colon \mathcal{R} \twoheadrightarrow A$ satisfies ***M3*** if for all $R \in \mathcal{R}$, all $(x_j, 0)_{j\in N} \in \varphi(R)$, all $i \in N$, and all $R'_i \in \mathcal{R}_i$, if

$$y_i \succcurlyeq_{R_i} z_i \iff y_i \succcurlyeq_{R'_i} z_i \quad \text{for all } y_i, z_i \in X_i \setminus \{\omega_i\}, \tag{3}$$

$$\omega_i \sim_{R'_i} x_i, \tag{4}$$

then $(x_j, 0)_{j\in N} \in \varphi(R'_i, R_{-i})$.

Then we obtain the following corollary.

Corollary 3. *Assume F3 and F9. Assume also that R5 (or R4) holds for all agents. Then, if a social choice correspondence $\varphi \subseteq P$ satisfies M3 (or Maskin monotonicity with respect to A), then $\varphi \subseteq C$.*

Proof. It follows from Theorem 2 and Remarks 4 and 5. Q.E.D.

Fact 2 remains true when transfers are not feasible since F4 and F5 do not rely on the feasibility of transfers. On the other hand, Fact 3 does not hold without transfers because F7 does not hold.[11] In fact, the core correspondence often violates Maskin monotonicity in important applications when preferences $\succcurlyeq_{R_i}$ are not strict since our notion of the core allows for indifference for some members of the blocking coalition. Furthermore, the richness conditions do not allow us to exclude indifference. The following is an example in the context of marriage problems.

Example 1. Suppose there are two men (m_1 and m_2) and one woman (w_1). Their preferences are given by

$$m_1 : w_1 \succ m_1 \qquad\qquad w_1 : m_2 \succ m_1 \succ w_1,$$
$$m_2 : m_2 \succ w_1.$$

For this preference profile, the matching where m_1 and w_1 are matched is in the core. However, this is no longer the case if m_2's preferences change to

$$m_2 : m_2 \sim w_1.$$

Since m_2's lower-contour set at m_2 remains the same, this change is a Maskin monotonic transformation.

[11] To see why F7 does not hold, note that if we take (x_i, t_i) such that $t_i \neq 0$, then trivially there exists no feasible allocation a such that $a_i = (x_i, t_i)$.

This example suggests that we may want to use the weak core and weak Pareto efficiency ruling out indifference in the dominating coalition. However, Corollary 3 does not hold for this case since the weak Pareto correspondence itself is Maskin monotonic with respect to A.

On the other hand, M3 is significantly easier to satisfy, as the following fact shows.

Fact 5. *If F6 and F9 hold, then the core correspondence satisfies M3.*

Proof. Suppose F6 and F9 hold and let $R \in \mathcal{R}$ and $a = (x_j, 0)_{j \in N} \in C(R)$. Take any $i \in N$ and any $R'_i \in \mathcal{R}_i$ that satisfies (3) and (4). Take any $S \subseteq N$ and $a'_S = (x'_j, 0)_{j \in S} \in A_S$. We would like to show that a'_S does not dominate a for coalition S when the preference profile is (R'_i, R_{-i}). This is trivial if $i \notin S$, so we assume $i \in S$. We distinguish three cases.

Case 1: Suppose $\omega_i \notin \{x_i, x'_i\}$. Then by (3), $(x'_i \succsim_{R'_i} x_i \iff x'_i \succsim_{R_i} x_i)$ and $(x'_i \succ_{R'_i} x_i \iff x'_i \succ_{R_i} x_i)$. Then, since a'_S does not dominate a for S under R, the domination does not hold under (R'_i, R_{-i}) either.

Case 2: Suppose $\omega_i = x_i$. Then R'_i and R_i are identical, so the desired result follows.

Case 3: Suppose $\omega_i = x'_i$. Then (4) implies

$$x_i \sim_{R'_i} x'_i. \tag{5}$$

Furthermore, since $x'_i = \omega_i$, F6 implies

$$a'_{S \setminus \{i\}} \in A_{S \setminus \{i\}}.$$

This together with $a \in C(R)$ implies that $a'_{S \setminus \{i\}}$ does not dominate a for $S \setminus \{i\}$. This and (5) imply that a'_S does not dominate a for S under (R'_i, R_{-i}). Q.E.D.

8 Concluding Remarks

A downside of implementation theory is that it relies heavily on mechanisms that are enormously complex and unrealistic. Given that not all mechanisms are reasonable, it makes sense to identify important properties of mechanisms and characterize implementability when these properties are required.[12] Our result

[12] Studies along this line include Jackson (1992), Dutta *et al.* (1995), Saijo *et al.* (1996), and Sjöström (1996).

shows that a practical but seemingly trivial restriction on admissible mechanisms can significantly reduce the class of implementable correspondences.

Another insight that our result gives us is the importance of the core. The importance of the core is not clear from its definition when noncooperative behavior is dominant among agents. Furthermore, the core has been criticized and alternative cooperative solution concepts have been proposed. However, our result suggests that the core is relevant even when agents' behavior is noncooperative and regardless of the deficiencies in the definition of the core.

One of the most useful corollaries of our results is that for allocation problems for which the core is empty for some preference profiles, Pareto efficiency can be achieved in Nash equilibrium only by a mechanism that sometimes pays agents who keep their endowments.

An important weakness of this paper is the assumption of richness. It limits the application of our results considerably, although there exist a number of allocation problems for which our results apply. Another weakness is the use of Nash equilibrium with complete information, which also limits the importance of our results for certain applications.

A Appendix: Strategy-Proofness

This section shows that similar results hold when we replace Maskin monotonicity with strategy-proofness and non-bossiness. These results are less significant because strategy-proofness and non-bossiness are rather demanding and would be incompatible with efficiency in many of the interesting applications of our model (Green and Laffont, 1979; Holmström, 1979). However, we find it instructive to prove parallel results for strategy-proofness.

A single-valued social choice correspondence $\varphi\colon \mathcal{R} \to A$ is called a ***social choice function***. We denote by $\varphi_i(R)$ the consumption bundle assigned to agent i when the preference profile is R. A social choice function φ is ***strategy-proof*** if for all $R \in \mathcal{R}$, all $i \in N$, and all $R'_i \in \mathcal{R}_i$, $\varphi_i(R)\; R_i\; \varphi_i(R'_i, R_{-i})$.

We modify the richness conditions slightly. As before, the following condition is stated for a given agent $i \in N$:

R6. For all $a = (x_j, t_j)_{j\in N} \in A$, all $R_i \in \mathcal{R}_i$, and all $\lambda \in \mathbb{R}$, if $x_i \neq \omega_i$, then there exists $R'_i \in \mathcal{R}_i$ such that

(i) $a_i\; I'_i\; (\omega_i, \lambda)$,

(ii) for all $x'_i \in X_i \setminus \{x_i, \omega_i\}$ and all $t'_i \in \mathbb{R}$, if $a_i\; R_i\; (x'_i, t'_i)$, then $a_i\; P'_i\; (x'_i, t'_i)$.

This condition differs from R2 and R3 in two respects. First, λ is nonnegative in the previous conditions while the restriction is not imposed in R6. Second, in (ii), R'_i is allowed to be indifferent between (x'_i, t'_i) and a_i in the previous conditions while R6 requires that a_i should be strictly preferred for R'_i. We let $N'_r \subseteq N$ denote the set of agents for whom R6 is satisfied. Let $T'_i(R_i, a_i, \lambda)$ denote the set of preferences $R'_i \in \mathcal{R}_i$ that satisfy (i) and (ii) in the definition of R6.

Proposition 1. *Assume F1 and F2. Assume also that for all $i \in N$, $\mathcal{R}_i$ contains only quasi-linear preferences and X_i is finite. Then, if a social choice function $\varphi\colon \mathcal{R} \to A_{nt}$ is strategy-proof and is a selection*[13] *from P, then for all $R \in \mathcal{R}$ and all $i \in N'_r$, $\varphi_i(R)\; R_i\; (\omega_i, 0)$.*

Proof. Suppose, by way of contradiction, that $(\omega_i, 0)\; P_i\; \varphi_i(R) \equiv (x_i, t_i)$ for some $i \in N'_r$. Since $\varphi(R) \in A_{nt}$, $x_i \neq \omega_i$. For a given $\lambda < 0$, let $R^\lambda_i \in T'_i(R_i, (x_i, t_i), \lambda)$. Strategy-proofness then implies $(x_i, t_i)\, R_i\, \varphi_i(R^\lambda_i, R_{-i})\, R^\lambda_i\, (x_i, t_i)$. This together with no-tax-for-endowments implies $\varphi_i(R^\lambda_i, R_{-i}) = (x_i, t_i)$. By F1,

[13] A social choice function φ is a *selection* from a social choice correspondence F if $\varphi(R) \in F(R)$ for all $R \in \mathcal{R}$.

there exists a feasible allocation $a^* = (x_i^*, t_i^*)_{i \in N}$ such that $(x_i^*, t_i^*) = (\omega_i, 0)$. Since φ is a selection from P, $(x_j^\lambda, t_j^\lambda)_{j \in N} \equiv \varphi(R_i^\lambda, R_{-i})$ is Pareto efficient for (R_i^λ, R_{-i}). By quasi-linearity and F2, we obtain

$$\begin{aligned} v_i^\lambda(\omega_i) + \sum_{j \neq i} v_j(x_j^*) &\leq v_i^\lambda(x_i^\lambda) + \sum_{j \neq i} v_j(x_j^\lambda), \\ &\leq v_i^\lambda(x_i) + \sum_{j \neq i} \max_{y \in X_j} v_j(y), \end{aligned}$$

where v_j denote the valuation function of agent $j \neq i$ and v_i^λ denote agent i's valuation function associated with R_i^λ. However, the inequality does not hold for all $\lambda < 0$ since $v_i^\lambda(x_i) - v_i^\lambda(\omega_i) = \lambda - t_i$. Q.E.D.

This result corresponds to Theorem 1. A parallel result that corresponds to Theorem 2 does not hold, even with the assumption of quasi-linear preferences. A counter-example is the Vickrey–Clarke–Groves mechanism, which does not necessarily produce a core allocation in auctions with multiple objects (Bikhchandani and Ostroy, 1999).

However, we can obtain a counterpart of Theorem 2 if we strengthen strategy-proofness by adding non-bossiness. This is not very surprising given the well-known similarity between Maskin monotonicity and strategy-proofness with non-bossiness. However, it may be worth knowing that a parallel result does hold. It is not the case that one result follows from the other.

We first give the definition of non-bossiness.

Definition. A social choice function φ is ***non-bossy*** if for all $R \in \mathcal{R}$, all $i \in N$, and all $R_i' \in \mathcal{R}_i$, if $\varphi_i(R_i', R_{-i}) = \varphi_i(R)$, then $\varphi_j(R_i', R_{-i}) \, I_j \, \varphi_j(R)$ for all $j \in N$.

In words, non-bossiness says that no one can affect other agents' welfare without changing his consumption. To understand the meaning, suppose that it is violated, i.e., $\varphi_i(R_i', R_{-i}) = \varphi_i(R)$ and not $\varphi_j(R_i', R_{-i}) \, I_j \, \varphi_j(R)$ for some $j \neq i$. Then agent i can affect agent j's welfare with no cost to himself. This suggests a possibility of collusion where agent i reports a preference relation that is favorable for agent j in exchange for a transfer from agent j.[14]

[14]Non-bossiness has several versions. A version used often in the literature is that no one can change other agents' *consumption* without changing his own *consumption*. Another reasonable version is that no one can change other agents' *welfare* without changing his own *welfare*. These versions are stronger than the one we use.

Proposition 2. *Assume F1 and F2. If a social choice function* $\varphi\colon \mathcal{R} \to A_{nt}$ *is strategy-proof, non-bossy, and is a selection from* P*, then*

$$\varphi_i(R)\ R_i\ (\omega_i, 0) \quad \text{for all } i \in N'_r \text{ and all } R \in \mathcal{R}.$$

Proof. Suppose that $(\omega_i, 0)\ P_i\ \varphi_i(R) \equiv (x_i, t_i)$ for some $i \in N'_r$. For any $\lambda < 0$, let $R_i^\lambda \in T'_i(R_i, (x_i, t_i), \lambda)$ as in the proof of Proposition 1. Then as before, strategy-proofness implies $\varphi_i(R_i^\lambda, R_{-i}) = (x_i, t_i)$. Non-bossiness then implies that for all $j \neq i$, $\varphi_j(R_i^\lambda, R_{-i})\, I_j\, \varphi_j(R)$. By the argument in the proof of Theorem 2, we can show that for small λ, there exists a feasible allocation a^λ that Pareto dominates $\varphi(R)$ under (R_i^λ, R_{-i}). Since $\varphi(R_i^\lambda, R_{-i})$ is Pareto indifferent to $\varphi(R)$ under (R_i^λ, R_{-i}), the Pareto domination is in contradiction with φ being a selection from the Pareto correspondence. Q.E.D.

Proposition 3. *Assume F2 and F3. Then, if a social choice function* $\varphi\colon \mathcal{R} \to A_{ns}$ *is strategy-proof, non-bossy, and is a selection from* P*, then there exists no* $R \in \mathcal{R}$ *and no coalition* $S \supseteq N \setminus N'_r$ *such that* S *improves upon* $\varphi(R)$ *under* R*.*

Proof. Suppose that there exist $R \in \mathcal{R}$ and $S \supseteq N \setminus N'_r$ such that S can improve upon $\varphi(R)$ under $R \in \mathcal{R}$. This means that there exists a feasible S-allocation $a'_S = (x'_i, t'_i)_{i \in S}$ such that $a'_i\, R_i\, \varphi_i(R)$ for all $i \in S$ with strict preference holding for some $m \in S$. P1* and P2* imply that for a sufficiently small $\varepsilon \geq 0$, we have $(x'_m, t'_m - \varepsilon)\ P_m\ \varphi_m(R)$.

Let $K = \{i \in N \setminus S : \varphi_i(R)\ P_i\ (\omega_i, 0)\}$. If this set is empty, we can proceed directly to the last paragraph of the proof. So, we assume $K \neq \emptyset$. Without loss of generality, we rename the agents so that $K = \{1, 2, \ldots, k\}$. Let $\lambda > 0$ be a small number such that $\varphi_i(R)\ P_i\ (\omega_i, \lambda)$ for all $i \in K$.

We now define a list of preference profiles $(R^\ell)_{\ell=0}^k$ recursively. Let $R^0 = R$. Having defined $R^0, \ldots, R^{\ell-1}$, let $R'_\ell \in T'_\ell(R_\ell^{\ell-1}, \varphi_\ell(R^{\ell-l}), \lambda)$ and then $R^\ell = (R'_\ell, R_{-\ell}^{\ell-1})$. Note that agent ℓ satisfies R6 since $\ell \notin S$ implies $\ell \in N'_r$. For R'_ℓ to be well defined, it has to be the case that he does not get ω_ℓ in $\varphi_\ell(R^{\ell-1})$, which we will see shortly.

Claim. For all $\ell \in \{0, 1, \ldots, k\}$ and all $i \in N$,

$$\begin{aligned} &\varphi_i(R^\ell)\ I'_i\ (\omega_i, \lambda) && \text{if } i \leq \ell, \\ &\varphi_i(R^\ell)\ I_i\ \varphi_i(R) && \text{if } i > \ell. \end{aligned}$$

We prove the claim by induction on ℓ. The case of $\ell = 0$ is trivial since $R^0 = R$. So suppose that the claim is proved for some $\ell < k$. Since the claim holds for ℓ, we have $\varphi_{\ell+1}(R^\ell)\ I_{\ell+1}\ \varphi_{\ell+1}(R)\ P_{\ell+1}\ (\omega_{\ell+1}, \lambda)$ where the last relation comes from $\ell + 1 \in K$. This together with $\varphi\colon \mathcal{R} \to A_{ns}$ implies that agent $\ell + 1$ does not obtain $\omega_{\ell+1}$ in allocation $\varphi(R^\ell)$. This ensures that $R'_{\ell+1}$ is well defined. Then since φ is strategy-proof and maps into A_{ns}, we obtain

$$\varphi_{\ell+1}(R^{\ell+1}) = \varphi_{\ell+1}(R^\ell). \tag{6}$$

Non-bossiness then implies

$$\varphi_i(R^{\ell+1})\ I_i^{\ell+1}\ \varphi_i(R^\ell) \quad \text{for all } i \in N. \tag{7}$$

We now check that the claim holds for $\ell+1$. For agents $i \leq \ell$, we have $R_i^{\ell+1} = R'_i$. Thus (7) together with the induction hypothesis implies $\varphi_i(R^{\ell+1}) I'_i \varphi_i(R^\ell) I'_i (\omega_i, \lambda)$. For agents $i > \ell + 1$, we have $R_i^{\ell+1} = R_i$, and thus $\varphi_i(R^{\ell+1})\ I_i\ \varphi_i(R^\ell)\ I_i\ \varphi_i(R)$. For agent ℓ, (6) and the construction of $R'_{\ell+1}$ imply $\varphi_{\ell+1}(R^{\ell+1}) = \varphi_{\ell+1}(R^\ell)\ I'_{\ell+1}$ $(\omega_{\ell+1}, \lambda)$. This proves the claim.

We now construct an allocation a^* by

$$a_i^* = \begin{cases} (x'_m, t'_m - \varepsilon) & \text{if } i = m \\ (x'_i, t'_i) & \text{if } i \in S \setminus \{m\} \\ (\omega_i, \lambda) & \text{if } i \in K \\ (\omega_i, 0) & \text{if } i \notin S \cup K. \end{cases}$$

Since ε and λ can be arbitrarily small, we choose them so that $\varepsilon = |K|\lambda$; if $K = \emptyset$, then we set $\varepsilon = 0$. By F2 and F3, a^* is a feasible allocation. However, a^* Pareto dominates $\varphi(R^k)$ under R^k, which gives us a desired contradiction. To see the Pareto domination, note first that for agents in K, the claim just proved for $\ell = k$ implies that they are indifferent between (ω_i, λ) and $\varphi_i(R^k)$. For the other agents, the claim implies that they are indifferent between $\varphi(R^k)$ and $\varphi(R)$. Moreover, $(x'_i, t'_i)_{i \in S}$ dominates $\varphi(R)$ for coalition S, and the definition of K implies that the agents outside of $S \cup K$ weakly prefer $(\omega_i, 0)$ to $\varphi_i(R)$. Finally, the choice of $m \in S$ and $\varepsilon \geq 0$ ensures that the preference is strict for agent m. Q.E.D.

Corollary 4. *Assume F1 and F2. Assume also that R6 is satisfied for all agents. Then, if a social choice function $\varphi\colon \mathcal{R} \to A_{nt}$ is strategy-proof, non-bossy, and is a selection from P, then φ is a selection from I.*

Corollary 5. *Assume F2 and F3. Assume also that R6 is satisfied for all agents. Then, if a social choice function* $\varphi\colon \mathcal{R} \to A_{ns}$ *is strategy-proof, non-bossy, and is a selection from* P*, then* φ *is a selection from* C*.*

The last corollary resembles the result of Sönmez (1999), who shows, for a general class of allocation problems in which monetary transfers are not feasible, that if a social choice function φ is strategy-proof, Pareto efficient, and individually rational, then φ is a selection from C. There are at least two differences between his result and ours. First, Sönmez uses individual rationality while we use non-bossiness. Second, Sönmez assumes that transfers are not feasible, while we assume that transfers are feasible. Our result and Sönmez's both show that there exists a strong and robust link between strategy-proofness and the core.

B Appendix: Proof of Fact 3

Let $R \in \mathcal{R}$ and $a \in C(R) \cap A_n$. By F4, $a \in I(R)$. Let R' be a Maskin monotonic transformation of R at a with respect to A_n. Suppose, by way of contradiction, that $a \notin C(R')$, so there exist $S \subseteq N$ and $a'_S \in A_S$ such that $a'_i \equiv (x'_i, t'_i)\, R'_i\, a_i$ for all $i \in S$ with strict preference holding for some $i \in S$. Let

$$T = \{i \in S : x'_i = \omega_i\},$$

which may be empty. By F7, $(\omega_i, 0) \in \mathit{Proj}_i(A_n)$ for all $i \in N$. Since R'_i is a Maskin monotonic transformation of R_i at a_i with respect to $\mathit{Proj}_i(A_n)$, it follows that for all $i \in T$,

$$(\omega_i, t'_i) = a'_i\, R'_i\, a_i\, R'_i\, (\omega_i, 0),$$

which implies $t'_i \geq 0$ for all $i \in T$. The inequality is strict for all $i \in T$ such that $a'_i\, P'_i\, a_i$.

We have $T \subsetneq S$, since if $T = S$, then the result just proved implies $\sum_{i \in S} t'_i > 0$, which is in violation of F8.

Now, F6 implies

$$(a''_i)_{i \in S \setminus T} \equiv (x'_i, t'_i + \sum_{j \in T} t'_j / |S \setminus T|)_{i \in S \setminus T} \in A_{S \setminus T}.$$

We show that this allocation dominates a for coalition $S \setminus T$ under R, which gives a desired contradiction. So, take any $i \in S \setminus T$. Let $t_i^* \in \mathbb{R}$ be such that $(x_i', t_i^*)\ I_i\ a_i$. Since $x_i' \neq \omega_i$, F7 implies $(x_i', t_i^*) \in Proj_i(A_n)$. Since R_i' is a Maskin monotonic transformation of R_i at a_i with respect $Proj_i(A_n)$, we have

$$a_i'\ R_i'\ a_i\ R_i'\ (x_i', t_i^*), \tag{8}$$

which implies $t_i' \geq t_i^*$. Thus

$$a_i''\ R_i\ a_i'\ R_i\ (x_i', t_i^*)\ I_i\ a_i, \tag{9}$$

which holds for all $i \in S \setminus T$.

It remains to show that there exists at least one $i \in S \setminus T$ such that $a_i''\ P_i\ a_i$. We know that there exists $k \in S$ such that $a_k'\ P_k'\ a_k$. If $k \notin T$, then the first preference in (8) holds strictly for k, so the second preference in (9) holds strictly for him. If $k \in T$, then $t_k' > 0$, so the first preference in (9) holds strictly for all $i \in S \setminus T$. Q.E.D.

References

Banerjee, S., Konishi, H., and Sönmez, T. (2001). "Core in a simple coalition formation game," *Social Choice and Welfare* **18**, 135–53.

Bikhchandani, S. and Mamer, J. W. (1997). "Competitive equilibrium in an exchange economy with indivisibilities," *Journal of Economic Theory* **74**, 385–413.

Bikhchandani, S. and Ostroy, J. M. (1999). "The package assignment model," mimeo, University of California, Los Angeles.

Bogomolnaia, A. and Jackson, M. O. (2002). "The stability of hedonic coalition structures," *Games and Economic Behavior* **38**, 201–30.

Corchon, L. C. (1996). *The Theory of Implementation of Socially Optimal Decisions in Economics*, New York: St. Martin's Press.

Dutta, B., Sen, A., and Vohra, R. (1995). "Nash implementation through elementary mechanisms in economic environments," *Economic Design* **1**, 173–203.

Gale, D. and Shapley, L. (1962). "College admissions and the stability of marriage," *American Methematical Monthly* **69**, 9–15.

Gevers, L. (1986). "Walrasian social choice: Some simple axiomatic approaches," in "Social Choice and Public Decision Making, Essays in Honor of K. J. Arrow," (W. P. Heller, R. M. Starr, and D. A. Starrett, eds.), Cambridge: Campridge University Press.

Green, J. and Laffont, J. J. (1979). *Incentives in Public Decision-Making*, Amsterdam: North-Holland.

Greenberg, J. (1990). *The Theory of Social Situations: An Alternative Game-Theoretic Approach*, Cambridge: Cambridge University Press.

Holmström, B. (1979). "Groves schemes on restricted domains," *Econometrica* **47**, 1137–44.

Hurwicz, L. (1979). "On allocations attainable through Nash equilibria," *Journal of Economic Theory* **21**, 140–165.

Jackson, M. (1992). "Implementation in undominated strategies: A look at bounded mechanisms," *Review of Economic Studies* **59**, 757–775.

Jackson, M. O. (2001). “A crash course in implementation theory,” *Social Choice and Welfare* **18**, 655–708.

Kara, T. and Sönmez, T. (1996). “Nash implementation of matching rules,” *Journal of Economic Theory* **68**, 425–439.

Maskin, E. (1979). “Implementation and strong Nash equilibrium,” in “Aggregation and Revelation of Preferences,” (J.-J. Laffont, ed.), Amsterdam: North-Holland.

Maskin, E. (1999). “Nash equilibrium and welfare optimality,” *Review of Economic Studies* **66**, 23–38.

Miyagawa, E. (2001). “Reduced-form implementation,” mimeo, Columbia University.

Moore, J. (1991). “Implementation, contracts and renegotiation in environments with complete information,” in “Advances in Economic Theory: Proceedings of the Sixth World Congress of the Econometric Society,” (J.-J. Laffont, ed.), Cambridge: Cambridge University Press.

Palfrey, T. R. (2000). “Implementation theory,” mimeo, forthcoming in *Handbook of Game Theory, Vol. III*, edited by Robert Aumann and Sergiu Hart.

Roth, A. E. and Rothblum, U. G. (1999). “Truncation strategies in matching markets—In search of advice for participants,” *Econometrica* **67**, 21–43.

Saijo, T. (1988). “Strategy space reductions in Maskin’s theorem: Sufficient conditions for Nash implementation,” *Econometrica* **56**, 693–700.

Saijo, T., Tatamitani, Y., and Yamato, T. (1996). “Toward natural implementation,” *International Economic Review* **37**, 949–980.

Shapley, L. and Shubik, M. (1972). “The assignment game I: the core,” *International Journal of Game Theory* **1**, 111–30.

Sjöström, T. (1996). “Implementation by demand mechanisms,” *Economic Design* **1**, 343–354.

Sönmez, T. (1999). “Strategy-proofness and essentially single-valued cores,” *Econometrica* **67**, 677–689.