How More Taxes Can Be Better Than Less: A Note on Aggregating Deadweight Losses

By

Kevin Lancaster, Columbia University

January 1998

Discussion Paper Series No. 9798-09

How More Taxes Can Be Better Than Less: A Note on Aggregating Deadweight Losses*

Kelvin Lancaster
Columbia University

January
1998

Abstract

There is a deadweight loss from imposing a tax on a single commodity, but there is no such loss from a uniform tax on *all* commodities with lump-sum return, and obviously no loss if no commodity is taxed. The object of this paper is to weave a consistent story relating these three well established propositions, something that has hitherto been lacking. Using a simple general equilibrium model with a CES utility function, it is shown that, as more goods are taxed: (1) The deadweight loss per dollar of revenue falls monotonically, and (2) the aggregate deadweight loss rises to a maximum, then falls to zero. In particular, it is sometimes better to tax more goods than to eliminate taxes on existing ones.

Introduction

We know, from partial equilibrium analysis, that taxing a single commodity and returning the revenue as a lump sum imposes a "deadweight loss" or "excess burden" on consumers. We know that there is no such loss if no goods are taxed. We also know that taxing *all* commodities at the same proportional rate and returning the revenue as a lump sum imposes no deadweight loss, because of the zero-degree homogeneity of demand (a property that always survives aggregation), unless there are production effects. It follows that the aggregate deadweight loss from taxing k commodities cannot be simply k times the loss from taxing a single commodity, at least not for k sufficiently large. While the possibility that taxing a second commodity may reduce the deadweight loss on

*I wish to thank my colleague Stanislaw Wellisz for raising my interest in the problem which is tackled here.

the first was spelt out in a well-known example by Harberger (1974), and is an implication of the theory of second best[1] the analysis was essentially that of partial equilibrium.

The purpose of this paper is to fill a gap in tax theory by investigating the relationship between the deadweight loss from a commodity tax and the number of commodities taxed, using a fully defined, if simple, economic model in which the partial equilibrium concept of the deadweight loss can be placed in a general equilibrium setting but yield explicit solutions. The analysis of models with a high degree of generality, such as those of Diamond and Mirrlees (1971) and Starrett (1988), has been very impressive, but the results typically expressed as first order conditions (often extraordinarily difficult to interpret) do not show clearly what happens for large changes.

We will show here that, as more goods are taxed, the deadweight loss per dollar of revenue falls monotonically, that the aggregate deadweight loss in the economy first rises then falls, reaching a maximum when some but not all goods are taxed, that it is always better to tax more goods at a lower rate than few at a higher rate, and that the Ramsey rule (tax the inelastic goods the most) does not always hold. Numerical computations suggest that actual deadweight losses may be much lower than often estimated.

1 The Model

We want to investigate what happens in a model economy with n goods in which taxes are imposed on a proportion α of these goods and the proceeds returned to consumers in order to close the system in a neutral way.

In order to concentrate on the effect of increasing the *number* of goods being taxed, we seek a model in which the *order* in which the goods are taxed is irrelevant. Thus the model should consist of goods which are symmetrically related in both demand and supply. Since it is the consumer surplus element of loss that is of interest, we want to eliminate production effects by having infinite supply elasticity of each good, and to eliminate distribution effects. Finally, It should be a true general equilibrium model, however simple.

The model which we shall use satisfies the above criteria, and may be the most general from which explicit results can be obtained. It has the following characteristics

[1]See Lipsey and Lancaster (1956). Second best is an old wheel that has been rediscovered several times since.

1. Consumers are identical with identical CES utility functions, who can be treated in the aggregate as a single consumer. The CES function enables us to observe the effects of changing the degree of substitutability between the goods.

2. Production using a single input (possibly a fixed proportions aggregate), with the same constant input/output ratio for every good. This input has no direct value as a consumption good.

3. Ownership of the single input is uniformly distributed and is the only source of income other than distributions of tax receipts. There are no incentive or other effects on the supply of the input to producers.

4. The government does nothing but raise taxes and distribute the proceeds uniformly to the population, this distribution having the properties of a lump sum distribution because of the symmetry of demand and absence of factor supply effects.

CES Utility

Consumers have utility functions of the following form

$$u = \left(\sum x_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \tag{1}$$

where $\sigma \neq 1$ is the elasticity of substitution.

Since consumers are identical and the utility function is homogeneous, we can analyze the problem as though there was a single representative consumer with income m.

For the above utility function, we have the following standard results for the indirect utility function $v(\cdot)$, and the Walrasian demand function $x_j(\cdot)$.

$$v(p,m) = \left(\sum p_i^{1-\sigma}\right)^{\frac{1}{\sigma-1}} m \tag{2}$$

$$x_j(p,m) = \left(\sum p_i^{1-\sigma}\right)^{-1} p_j^{-\sigma} m, \quad j = 1, \ldots, n \tag{3}$$

For $\sigma = 1$ we substitute the corresponding Cobb-Douglas form of the utility function

$$u = \prod_{i=1}^{n} x_i^{1/n} \tag{4}$$

Pre-Tax Equilibrium

We assume identical constant unit costs p for all commodities, so that the competitive equilibrium prices without taxes will be $p_i = p$, the resource being the numeraire.

The total value of the aggregate resource is m, which will also be the value of aggregate income. Demand will be given by

$$x_i = \frac{m}{pn} \quad \text{for all } i$$

giving utility

$$v(p,m) = n^{\frac{1}{\sigma-1}} \frac{m}{p} \tag{5}$$

Effect of a Tax

Now let a proportion α of the n goods (we shall ignore the integer problem) be taxed at the same proportional rate t, the tax revenue being returned to the consumers to give a new money income $m+R$, where R is the tax revenue.

To simplify, write $T = 1+t$, so that the post-tax consumer price is Tp. Then

$$\sum p_i^{1-\sigma} = (\alpha T^{1-\sigma} + 1 - \alpha)\, p^{1-\sigma}\, n = (1-\alpha X)\, p^{1-\sigma} n$$

where

$$X = 1 - T^{1-\sigma} \tag{6}$$

Since $T \geq 1$, $1 \geq X \geq 0$ for all $\sigma > 1$, while $X \leq 0$ for $\sigma < 1$.

For a taxed good, the demand equation (3) becomes

$$x_t = \frac{T^{-\sigma}}{1-\alpha X} \frac{m+R}{np} \tag{7}$$

For this equation to make sense, we require $1 - \alpha X > 0$ for all $\alpha \leq 1$. If $\sigma < 1$, $X < 0$ and this is obviously true. For $\sigma > 1$, $X > 0$ and then $1-\alpha X \geq 1 - X = T^{1-\sigma} > 0$, so the condition is satisfied for $\sigma <> 1$.

Since the revenue is returned

$$\begin{aligned} R &= \alpha\, n\, t\, p x_t = \frac{\alpha t T^{-\sigma}}{1-\alpha X}(m+R) \\ &= \frac{\alpha t T^{-\sigma}}{1-\alpha Y} m \end{aligned} \tag{8}$$

where

$$Y = 1 - T^{-\sigma} = X + tT^{-\sigma} \tag{9}$$

Note that Y, unlike X, is nonnegative for all $\sigma > 0$, and so $1 - \alpha Y \geq 1 - Y = T^{-\sigma} > 0$.

With taxes, $v(\cdot)$ is a function of α, t as well as the basic variables p, m:

$$\begin{aligned} v(\alpha, t; p, m) &= (1 - \alpha X)^{\frac{1}{\sigma-1}} n^{\frac{1}{\sigma-1}} \frac{m + R}{p} \\ &= \frac{(1 - \alpha X)^{\frac{\sigma}{\sigma-1}}}{(1 - \alpha Y)} n^{\frac{1}{\sigma-1}} \frac{m}{p} \end{aligned} \tag{10}$$

2 Measuring the Loss

We shall measure the deadweight loss from taxation by the *equivalent variation* as a proportion of the base income[2]. That is, by the value of $\mu \equiv \mu(\alpha, t)$ such that post-tax consumers are exactly as well off as they would be if there was no tax, but their income was reduced by a proportion μ. The determining equation is

$$v(\alpha, t; p, m) = v(0, 0; p, (1 - \mu)m) \tag{11}$$

In the current model the required relationship is

$$\frac{(1 - \alpha X)^{\frac{\sigma}{\sigma-1}}}{1 - \alpha Y} n^{\frac{1}{\sigma-1}} \frac{m}{p} = (1 - \mu) n^{\frac{1}{\sigma-1}} \frac{m}{p}$$

from which

$$\mu(\alpha, t) = 1 - \frac{(1 - \alpha X)^{\frac{\sigma}{\sigma-1}}}{1 - \alpha Y} \tag{12}$$

Since $1 - X = T^{1-\sigma}$ and $1 - Y = T^{-\sigma}$, it is immediate that $\mu(0, t) = \mu(1, t) = 0$ for all t. It may seem intuitively clear that $\mu(\alpha, t) > 0$ for $0 < \alpha < 1$, and indeed it is true, but this cannot be determined by simple inspection of (12). The proof is given later in the paper.

3 Deadweight Loss Relative to Revenue

Proposition 1 *As the number of taxed goods is increased, the deadweight loss from the tax relative to the revenue from the tax declines monotonically. This is true for all values of the elasticity of substitution and all levels of tax.*

[2] In this simple model, the compensating variation could have been used, but in many cases this measure poses problems. See Pauwels (1986).

Proof

Write r for the ratio of the deadweight loss from the tax to the revenue generated from it. Then, from (12) and (8)

$$r(\alpha) = \frac{\mu m}{R} = \frac{(1-\alpha Y) - (1-\alpha X)^{\frac{\sigma}{\sigma-1}}}{\alpha t T^{-\sigma}} \tag{13}$$

We have $r(1) = 0$, but the value of $r(0)$ must be determined as a limit, from which we find $r(0) = \frac{1}{2}\sigma t > 0$. This is equivalent to the textbook formula for a single good and small values of t, since σ can be identified with the elasticity of demand when there are a large number of goods. Thus $r(\alpha)$ must be declining on balance. To show it is declining monotonically, we determine the slope of $r(\alpha)$, given by

$$\frac{dr}{d\alpha} = -\frac{T^{\sigma}}{\alpha^2 t}\left\{1 - \left[1-\alpha X\right]^{\frac{1}{\sigma-1}}\left[1+\frac{\alpha X}{\sigma-1}\right]\right\} \tag{14}$$

Consider the two factors inside the braces. $dr/d\alpha <> 0$ according as the product of these factors is $<> 1$. Since $X, \sigma-1$ are both positive ($\sigma > 1$) or both negative ($\sigma < 1$), $(1-\alpha X)^{1/(\sigma-1)} < 1$, and $1+\alpha X/(\sigma-1) > 1$ in all cases. Thus simple inspection does not give the answer. We must consider the problem in terms of different ranges of σ.

1. For $\sigma = 2$, (14) becomes

$$\begin{aligned}\frac{dr}{d\alpha} &= -\frac{T^{\sigma}}{\alpha^2 t}\{1-[1-\alpha X][1+\alpha X]\} \\ &= -\frac{T^{\sigma}}{\alpha^2 t}\{1-[1-\alpha^2 X^2]\} \\ &= -t \\ &< 0\end{aligned}$$

   If $\sigma = 2$ the graph of $r(\alpha)$ is that of a straight line sloping from down from $r(0) = t$ to $r(1) = 0$.

2. For $\sigma > 2$, $X > 0$, $1/(\sigma-1) < 1$, $(1-\alpha X)^{1/(\sigma-1)} < 1-\alpha X/(\sigma-1)$. We can use the inequalities $(1+x)^a <> 1+ax$ according as $a <> 1$ and obtain

$$\begin{aligned}\left[1-\alpha X\right]^{\frac{1}{\sigma-1}}\left[1+\frac{\alpha X}{\sigma-1}\right] &< \left[1-\frac{\alpha X}{\sigma-1}\right]\left[1+\frac{\alpha X}{\sigma-1}\right] \\ &= 1-\frac{\alpha^2 X^2}{(\sigma-1)^2} < 1 \\ &\Rightarrow \frac{dr}{d\alpha} < 0\end{aligned}$$

3. For $2 > \sigma > 1$, $1/(\sigma-1) > 1$, $(1+\frac{\alpha X}{\sigma-1}) < (1+\alpha X)^{1/(\sigma-1)}$ and so
$$\begin{aligned}\left[1-\alpha X\right]^{\frac{1}{\sigma-1}}\left[1+\frac{\alpha X}{\sigma-1}\right] &< [1-\alpha X]^{\frac{1}{\sigma-1}}[1+\alpha X]^{\frac{1}{\sigma-1}} \\ &= (1-\alpha^2X^2)^{\frac{1}{\sigma-1}} < 1 \\ &\Rightarrow \frac{dr}{d\alpha} < 0\end{aligned}$$

4. For $\sigma = 1$ we must use Cobb-Douglas utility, from which
$$r(\alpha) = \frac{1+(1-\alpha)t - T^{1-\alpha}}{\alpha t}$$
and
$$\begin{aligned}\frac{dr}{d\alpha} &= -\frac{T}{\alpha^2 t}\left\{1 - T^{-\alpha}(1+\alpha\ln T)\right\} \\ &< 0\end{aligned}$$
after using the inequality $T^{\alpha} = e^{\alpha\ln T} > 1 + \alpha\ln T$.

5. For $1 > \sigma > 0$, $X < 0$, $1/(\sigma-1) < -1$, and $1 < 1-\alpha X < 1+\frac{\alpha X}{\sigma-1}$ so that
$$\begin{aligned}\left[1-\alpha X\right]^{\frac{1}{\sigma-1}}\left[1+\frac{\alpha X}{\sigma-1}\right] &< [1-\alpha X]^{1+\frac{1}{\sigma-1}} < 1 \\ &\Rightarrow \frac{dr}{d\alpha} < 0\end{aligned}$$

As the number of taxed goods increases, the ratio of the deadweight loss to the tax revenue decreases steadily, to reach zero when all goods are taxed.

4 Aggregate Deadweight Loss

Proposition 2 *As the number of taxed goods increases, the aggregate deadweight loss rises monotonically from zero to a strictly positive maximum at some proportion α^* of all goods, and then falls monotonically to reach zero at $\alpha = 1$. For $\alpha^* < \alpha < 1$, removing taxes on some of the goods will actually increase aggregate deadweight loss, while increasing the number of taxed goods will actually decrease it.*

Proof

Differentiating (12) with respect to α
$$\frac{d\mu}{d\alpha} = \frac{(1-\alpha X)^{\frac{1}{\sigma-1}}}{1-\alpha Y}\left(\frac{\sigma}{\sigma-1}X - \frac{1-\alpha X}{1-\alpha Y}Y\right) \tag{15}$$

Then $d\mu/d\alpha = 0$ at

$$\alpha^* = \frac{\sigma}{Y} - \frac{\sigma - 1}{X} = \frac{\sigma}{1 - T^{-\sigma}} - \frac{\sigma - 1}{1 - T^{1-\sigma}} \tag{16}$$

Using the inequality $e^x > 1 + x$ it is easily shown that an expression of the form $x/(1 - A^{-x})$ is increasing in x for $A > 1$, so that $\alpha^* > 0$. It can also be shown that $\alpha^* < 1$ and $\to 1$ as $\sigma \to \infty$. To show that $\mu(\alpha^*) > 0$ we note that, from (15)

$$\begin{aligned} \left(\frac{d\mu}{d\alpha}\right)_{\alpha=0} &= \left(\frac{\sigma}{\sigma - 1} X - Y\right) \\ &= \frac{XY}{\sigma - 1}\left(\frac{\sigma}{Y} - \frac{\sigma - 1}{X}\right) \\ &> 0 \end{aligned}$$

using the argument above and the fact that X, $(\sigma - 1)$ are both positive or both negative, depending on whether $\sigma >< 1$.

Since $\mu(0) = \mu(1) = 0$ and it is obvious from (16) that α^* is unique, μ is rising monotonically from 0 to $\alpha*$, then declining monotonically to 0 again at $\alpha = 1$. To complete the picture, we have the second order condition

$$\left(\frac{d^2\mu(\alpha^*)}{d\alpha^2}\right)_{\alpha=\alpha^*} = -\frac{(1 - \alpha X)^{1/(\sigma-1)}}{1 - \alpha Y} Y T^{-\sigma} t < 0$$

so there is a proper maximum at α^* for all values of t, $\sigma > 0$

If $\sigma = 1$ we use the Cobb-Douglas form (4), from which we find $\mu(\alpha)$ to have properties similar to the CES case, with

$$\alpha^* = 1 + \frac{1}{t} - \frac{1}{\ln T} \tag{17}$$

Some Numerical Values

Running numerical values in the model shows that α^* varies from 0.5 to 0.6 for a 10% tax rate and values of σ between 0.1 and 6.0, which covers the range of all likely real-world elasticity values[3]. The value of α^* increases with both σ and t.

For $\sigma = 0.5$ and $t = 0.25$, the maximum deadweight loss ($\mu(\alpha^*)$) is 0.31% of GDP, and the deadweight loss per dollar of revenue is 2.6%. These are very

[3] The range of price elasticities for commodity *groups* in Houthakker and Taylor (1970) is from about 0.2 to just over 2.0 (absolute values). These are representative of most studies.

much smaller than estimated values for a real economy with similar parameter values given by Honohan and Irvine (1990), and much less than estimates for another small economy by Diewert and Lawrence (1994). The naive computation, multiplying up the single commodity deadweight losses, would give 0.62% for the aggregate deadweight loss, 6.25% for the loss ratio more comparable to the published estimates. There is a *suggestion* here that traditional estimates of deadweight losses from taxes are overstatements.

Effect of σ and t

The expressions for $d\mu/d\sigma, d\mu/dt, dr/d\sigma$, and dr/dt are tedious to derive and equally tedious to interpret. However, since the model has explicit solutions, it is easy to compute solutions for different numerical values of both σ and t. Both the loss ratio r and the aggregate loss μ increase with both σ and t, for given α (except, of course if it is 0 or 1). Since this is in conformity with the standard formula for the single-commodity loss ratio, $\frac{1}{2}\sigma t$, it does not seem necessary to provide formal analytical proofs.

The relationships in the more general model are, however, nonlinear. For example, if $\sigma = 0.5$ and the tax rate doubles from 10% to 20%, the deadweight loss if half the goods are taxed ($\mu(0.5)$) increases by a factor of about 3.5 from 0.06% of GNP to 0.21%, while the loss ratio (loss over revenue) less than doubles, from 1.16% to 2.17%. If $\sigma = 3$, the losses are much larger, the aggregate loss on a 10% tax rate being 0.34% and the loss ratio 7.87%. For a 20% tax, the values rise to 1.21% and 16.45% respectively.

If σ is very large (above about 30 for a 10% tax rate), α^* is very close to 1, and the value of μ falls very steeply as $\alpha \to 1$. At these very high substitution elasticities a tax on a commodity causes a very large shift away from it. Because other commodities are good substitutes, the welfare loss is relatively small, but the revenue from the tax is even smaller, so that $r(\epsilon)$ becomes very large but still declines to $r(1) = 0$.

5 The Optimal Tax

Proposition 3 *For a given revenue, there is always less deadweight loss from a smaller tax on more commodities than from a larger tax on fewer commodities, if the elasticity of substitution is the same for all. In particular, there is no deadweight loss from a uniform tax on all commodities, whatever the level of revenue raised. If there are subgroups with different elasticities of substitution, it may be optimal to tax the group with the higher elasticity, contrary to the*

Ramsey rule, because of the declining loss ratio effect.

This proposition follows immediately from the properties of r, which is decreasing in α from Proposition 1, and increasing in t.

The uniformity of the elasticity of substitution (and thus the demand elasticities) prevents a direct comparison with the Ramsey efficient tax formula[4], according to which taxes should be inversely proportional to elasticities. We can consider the following modified version of our model, however, in which there are two groups of goods and an additively separable utility function of the form

$$u = \left(\sum_{1}^{n_1} x_i^{\frac{\sigma_1 - 1}{\sigma_1}} \right)^{\frac{\sigma_1}{\sigma_1 - 1}} + \left(\sum_{n_1+1}^{n_1+n_2} x_i^{\frac{\sigma_2 - 1}{\sigma_2}} \right)^{\frac{\sigma_2}{\sigma_2 - 1}} \qquad \sigma_1 \neq \sigma_2$$

If we can tax *all* goods, the first best solution (zero deadweight loss) is to tax every good at the same rate, whatever its group. This follows from the homogeneity property.

But if we place a constraint that only one group can be taxed, then we have a second best problem. If $n_1 \approx n_2$, so that the groups are comparable in size, then it will better to raise the revenue from the group with the lower value of σ since r is increasing in σ, a result compatible with the Ramsey rule. If the groups are not comparable in size, the choice is not so straightforward, since r declines with the size of the group. If the group with the higher value of σ is larger, it may be optimal to tax that group because of the size effect, contrary to the Ramsey rule.

References

Atkinson, A. B., and Stiglitz, J. E. 1980. *Lectures on Public Economics.* New York, McGraw-Hill.

Debreu, G. 1951. The coefficient of resource utilization. *Econometrica* **19**, 273-92.

Diamond, P., and Mirrlees, J. 1971. Optimal taxation and public production II: Tax rules. *American Economic Review* **61**, 261-78.

Diewert, W. E., and Lawrence, D. A. 1994. Working Paper 94/32. University of British Columbia, Department of Economics.

[4]See Ramsey (1927). Ramsey made his problem a *de facto* second best problem by having an untaxed good which also entered the utility function. Both Ramsey and Hotelling (1928), who produced related results, were confused about the numeraire good.

Harberger, A. C. 1974. Taxation, resource allocation, and welfare. In Harberger, A. C. (ed.) *Taxation and Welfare.* Boston, Little Brown.

Honohan, P., and Irvine, I. 1990. Deadweight loss measurement with generalized Gorman polar form. *Public Finance* **45**, 26067.

Hotelling, H. 1938. The general welfare in relation to problems of taxation and of railway and utility rates. *Econometrica* **6**, 242-69.

Houthakker, H. S., and Taylor, L. D. 1970. *Consumer Demand in the United States.* Cambridge, Mass., Harvard University Press.

Lipsey, R. G., and Lancaster, K. J. 1956. The general theory of second best. *Review of Economic Studies* **24**, 11-32.

Laffont, J-J. 1988. *Fundamentals of Public Economics.* Cambridge, Mass., MIT Press.

Pauwels, W. 1986. Correct and incorrect measures of the deadweight loss of taxation. *Public Finance* **41**, 267-76.

Ramsey, F. P. 1927. A contribution to the theory of taxation. *Economic Journal* **37**, 47-61.

Starrett, D. A. 1988. *Foundations of Public Economics.* Cambridge University Press.

1997-1998 Discussion Paper Series

Department of Economics
Columbia University
1022 International Affairs Bldg.
420 West 118th Street
New York, N.Y., 10027

The following papers are published in the 1997-98 Columbia University Discussion Paper series which runs from early November to October 31 of the following year (Academic Year).

The following is a website address where viewers may download some discussion papers. It is a project that is in progress, and therefore the site may not contain all recent working papers.

http://www.columbia.edu/dlc/wp/econ/index.html.

For Ordering Hardcopies:

Domestic orders for discussion papers are available for purchase at the cost of $8.00 (U.S.) Per paper and $140.00 (US) for the series.

Foreign orders cost $10.00 (US) per paper and $185.00 for the series.

To order discussion papers, please write to the Discussion Paper Coordinator at the above address along with a check for the appropriate amount, made payable to Department of Economics, Columbia University. Please be sure to include the series number of the requested paper when you place an order.

1997-98 Discussion Papers
Economics Department, Columbia University

Series no.	Title	Authors
9798-01	Optimal Regulatory Transparency	Herk, L.F.
9798-02	Russia's Market Transition: Who Has Lost and How?	Desai, P. Idson, T.
9798-03	The Wage Arrears Crisis in Russia	Desai, P. Idson, T.
9798-04	Taxes Versus Legal Rules as Instruments for Equity	Sanchirico, C.W.
9798-05	Enforcement by Hearing: An Integrated Model of Evidence Production (first draft: 12/95)	Sanchirico, C.W.
9798-06	Evidence Production, Adversarial Process and the Private Instigation of Suits (first draft: 12/95)	Sanchirico, C.W.
9798-07	Structural VAR, MARMA and Open Economy Models	Dhrymes, P.J. Thomakos, D.
9798-08	Small Sample Properties of Certain Cointegration Test Statistics: A Monte Carlo Study (Revd. from 7/96)	Dhrymes, P.J. Thomakos, D.
9798-09	How More Taxes Can Be Better Than Less: A Note On Aggregating Deadweight Losses	Lancaster, K.