Matching in Perfect and Imperfect Worlds

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Abstract

We study frictionless matching models in large production economies. We provide necessary and sufficient conditions for segregation and for positive assortative matching. These conditions focus on the relationship between what we call the segregation payoff — a generalization of the individually rational payoff — and the feasible set for a pair of types. Our approach is useful for clarifying differences in the behavior of models in the literature. It also provides a basis for understanding the effects of changes in technology or in the severity of market imperfections on equilibrium welfare and matching patterns.

1. Introduction

There is a long tradition, dating back at least to Roy [17] and Tinbergen [23], which views the distribution of earnings as an equilibrium outcome of a matching problem.[1] There has been a recent revival of interest in matching models, partly because of their utility in studying the classical problem of income distribution, but also because they provide a useful setting for examining problems as

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[1] Sattinger [18] provides a fine survey of the "classical" literature.

diverse as disease transmission, community formation, education financing, and organizational design ([11], [2], [7], [13], [14]).

Among the main insights of the literature of the 60s was the prediction that in the presence of complementarities there would be positive assortative matching, i.e., that more able individuals are assigned to more productive tasks or to more able individuals. An important consequence of this result is that the distribution of payoffs will tend to be more skewed than that of the underlying distribution of abilities. These classical insights were derived mainly for environments without market imperfections. The work of Becker [1], Sattinger [19] and the recent work by Kremer [10] and Kremer-Maskin [12] are in this line. By contrast, much of the recent literature has been concerned by environments in which market imperfections play a significant role (externalities such as human capital spillovers, moral hazard in production, credit constraints). Unfortunately, though some of the examples in this literature suggest that the equilibrium behavior of matching models with imperfections may be very different from those without imperfections, there seems to be a lack of characterization results that can help us distinguish the various cases.[2]

Our goal in this paper is to help fill this gap. We study frictionless matching models in large production economies without widespread externalities.[3] We provide necessary and sufficient conditions for segregation and for positive assortative matching. These conditions enable us to highlight some fundamental differences in the comparative static behavior of some models in the literature and should be particularly useful in the study of models with imperfections. Another goal of the paper is to clarify the differing effects of changes in technology and changes in the distribution of individual characteristics on the equilibrium welfare and matching

[2] For instance, Legros-Newman [13] study a model of firm formation and find that when capital markets are perfect, matches are segregated, while when capital markets are imperfect, there may be negative assortative matching. In the second case, the outcome depends on the distribution of wealth.

[3] That is, all externalities can be internalized by finite coalitions. This means that the set of payoffs that are feasible for a particular coalition does not depend on what other coalitions exist or what they are doing; of course the equilibrium payoffs will depend on these factors, in general. This restriction may exclude certain types of imperfections from the analysis (e.g. community formation models such as [3] and [7] in which there are congestion effects), but we believe our approach has some relevance to those cases.

patterns in worlds with and without market imperfections.[4,5]

Ideally, for doing comparative statics, we would like to characterize the equilibrium matching map and the corresponding equilibrium utility map. In practice, such a characterization is difficult. We show that a lot of information about the comparative static properties of a model can be obtained by analyzing the changes in the feasible utility set and what we call the segregation payoffs. The segregation payoff for an individual is the equilibrium payoff to this individual in an economy consisting solely of individuals of the same type as himself.

In order to provide a sense of the issues, consider the following model. There is a production process in which two tasks, denoted 1 and 2, have to be simultaneously performed in order for a positive output to be realized. Workers differ in their abilities, which are publicly observable and drawn from $[\underline{a}, \overline{a}]$. A worker can work on one task only. If a worker of ability a_1 is assigned to task 1 and a worker of ability a_2 is assigned to task 2, then the output is $h(a_1, a_2)$; h is increasing in both arguments and has a complementarity property (has positive cross partials). Workers match and share their output. The matching process itself is assumed to be frictionless; competitive equilibrium or the core are typical solution concepts.

This model has been applied widely.[6] For instance, Becker [1] uses this as a model of the marriage market. Individuals are distinguished on the basis of their gender $i = 1, 2$; men perform task 1, women perform task 2. It is well known that the matching in this model is always positive assortative: if one man has a higher ability than another man, his mate will have a higher ability than the other man's mate. It can be shown that for a given distribution of abilities, the equilibrium matching pattern is invariant to the choice of h as long as h has the complementarity property.

Kremer [10] and Kremer and Maskin [12] also use this model to study income distribution. The first paper uses $h(a_1, a_2) = a_1 a_2$ while the second uses

[4] Our results do not rely on optimization methods. In the case of models without imperfections, the latter may be helpful because equilibria are Pareto optimal and can be characterized as the solutions to a planner's problem. But that approach does not seem to extend very well to the case of models with imperfections because there is not generally a convenient optimization problem which corresponds to the equilibrium.

[5] A recent paper by Cole, Mailath and Postlewaite [4] studies some properties of matching models, but their focus is on models in which there is an investment stage before the match and on whether the resulting equilibrium is efficient as the economy gets large.

[6] For instance, the models of Roth [16] (two-sided assignment models), Gale-Shapley [8], Spurr [22] (match lawyers and claims), Roy [17], Sattinger [18] (match workers to tasks or to sectors).

$h(a_1, a_2) = a_1^2 a_2$. In the first case there is perfect segregation: in equilibrium, each firm consists of a single type of worker. In the second case, firms will not be segregated; in particular, if the support of the distribution is small enough, the best worker will match with the median worker and the others will match in a positive assortative way. The change in the matching pattern can only have come from the difference in the production function. As we know, this cannot happen in Becker's model.

Why the dramatic difference in comparative statics? There is an important distinction between the two models. In Becker's case, specialization in tasks is exogenous; individuals cannot choose their gender. In Kremer-Maskin's case, specialization in tasks is endogenous; individuals choose cooperatively after the match on which task to work. This is reflected in the payoff function in the following way. In both models, a worker's willingness to pay for a partner depends on the difference between what he achieves with a partner and his segregation payoff. In Becker's case, if two people of the same gender match together they receive a payoff of zero irrespective of their abilities. Therefore the gains from a heterogeneous match relative to the segregation payoffs are fully described by the output function h. Every man would like to match with the ablest woman, but it is the ablest man who is willing to pay the most. This fact is independent of the specific form of h. By contrast, in Kremer-Maskin, the segregation payoff varies with ability and with the choice of h. Therefore the individuals do not unanimously rank the other individuals and the pattern of matching will be more complex and more sensitive to the specifics of the technology and type distribution.

Our analysis follows a simple economic logic. One normally thinks of an equilibrium as a situation in which individuals' current benefits exceed their outside options. While it is sometimes the case that for some individuals the equilibrium outside option is equal to the segregation payoff, this will not be true in general. Nevertheless, we use the segregation payoff as a lower bound on the outside option and compare it to the utility possibility obtained in different matches. Doing so we have a natural concept of "gains from trade" (more precisely gains from a heterogeneous match). It is the comparative static of these gains from trade that will tell us much about the equilibrium outcome. For example, it helps to indicate situations in which individuals of very different types match together — possibly inefficiently — simply because one type has a very low segregation payoff.

2. The Model

2.1. Notation

Throughout this paper, we will follow much of the literature in restricting attention to matches of size two. We will indicate which results generalize.

The set of individuals is the unit square $I = [0,1]^2$ with Lebesgue measure λ. Individuals are assigned types by a map $\tau : I \longrightarrow T$ which assumes values in the measurable lattice $(T, \mathcal{T}, \succsim)$. We assume that for any $x \in [0,1]$, $\tau(i)$ is constant for all $i \in \{x\} \times [0,1]$.[7]

For any coalition of individuals, the set of feasible payoffs is described by the characteristic function V.[8] Since we are only considering matches of size two, it is enough to specify V for sets of size two and of size one (this automatically rules out "externalities" among coalitions). The feasible set of payoffs depends only on the characteristics of the individuals in the coalition. Therefore, for any two individuals i and j such that $\tau(i) = \tau(j)$, $V(i) = V(j)$ and for any other individual $k \notin \{i,j\}$, $V(i,k) = V(j,k)$. We assume that V is closed, comprehensive and bounded above.[9] We will sometimes abuse notation and denote by $V(t,t')$ the characteristic function of two individuals with types t and t'.

Consider two individuals i and j of the same type t who are matched together. Since V is comprehensive and closed, there exists a payoff $\underline{u}(t)$ such that $(\underline{u}(t), \underline{u}(t))$ is on the Pareto frontier of $V(t,t)$. We call such a payoff the *segregation payoff* of type t.

Figure 1

[7]In other words, all agents in a vertical segment of I have the same type. This does not entail, of course, that there is a continuum of types. We construct the individual space in this way in order to talk meaningfully of segregation.

[8]For instance, in the examples of the Introduction, two agents i and j of abilities a and b have a feasible set $V(i,j) = \{(v_1, v_2) : v_1 + v_2 \leq h(a,b)\}$ and an agent i has a feasible set $V(i) = (-\infty, 0]$ independently of his ability.

[9]V is comprehensive if for any set P, $v \in V(P)$ implies that $v' \in V(P)$ whenever $v'_i \leq v_i$ for all $i \in P$. In economies with imperfect markets, for instance incentive problems, V may fail to be comprehensive. This does not pose problem for the existence of equilibria but might be incompatible with the existence of a segregation payoff for some types. This problem is easily fixed as we discuss in the Appendix.

2.2. Equilibrium

Our focus in this paper is the way individuals are matched to each other, i.e., the way the individuals partition themselves into coalitions. We use the core as the equilibrium concept: a partition can be part of an equilibrium if there exists a payoff structure that is feasible for that partition and such that it is not possible for some individuals to obtain a higher payoff by forming a coalition different than their equilibrium coalition.

For any $v : I \rightarrow \mathbb{R}$, for any subsets P of size one or two, we abuse notation and denote by $v(P)$ the vector $(v(i) : i \in P)$. We denote by $\mathbb{P}$ the set of measure consistent partitions of I into subsets of size two at most.[10] For a given map $v : I \rightarrow \mathbb{R}$ we say that the partition $\mathcal{P}$ is minimal with respect to v if $i \neq j$ and $\{i, j\} \in \mathcal{P}$ imply that $(v(i), v(j)) \notin V(i) \times V(j)$.

Definition 2.1. *An equilibrium is a pair $(\mathcal{P}, v)$ such that $\mathcal{P} \in \mathbb{P}$ is minimal with respect to $v : I \rightarrow \mathbb{R}$ and such that*

(i) For almost all $P \in \mathcal{P}$, $v(P) \in V(P)$.

(ii) For each finite coalition Q, and each $v' \in V(Q)$, there exists $i \in Q$ such that $v'(i) \leq v(i)$.

Our assumptions guarantee that an equilibrium always exists (see Kaneko-Wooders [9] and Wooders [24]; Scotchmer [20] discusses related solution concepts). We first note that all equilibria are constrained Pareto efficient. Indeed, since effective coalitions are finite, the grand coalition cannot achieve anything more than what two person coalitions can achieve. If there were a Pareto improvement, then the grand coalition could block the equilibrium payoff but then a two person coalition could also do it and this would violate the definition of an equilibrium.

We now provide some definitions useful for characterizing equilibria.

Definition 2.2. *An equilibrium $(\mathcal{P}, v)$ satisfies segregation (SEG) if for almost all $P \in \mathcal{P}$, $\tau(i) = \tau(j)$ for all $i, j \in P$.*

[10] Let $\mathcal{P}$ be a partition of I. Let $\mathcal{P}^2$ be the set of elements of $\mathcal{P}$ of size two. List the elements of every $P \in \mathcal{P}^2$ according to the lexicographic order $\succeq_L$on $\mathbb{R}^2$ (hence, write $P = (i, j)$ when $i \succeq_L j$). Let I^1 be the set of agents who are first and I^2 the set of agents who are second. $\mathcal{P}$ is measure consistent if $\lambda(I^1) = \lambda(I^2)$. This restriction rules out partitions in which say, all agents in $[0, 1/3] \times [0, 1]$ are matched one-to-one with all the agents in $(1/3, 1] \times [0, 1]$. See also Wooders [24] and Kaneko-Wooders [9].

Definition 2.3. *An equilibrium* $(\mathcal{P}, v)$ *satisfies essential segregation (ESEG) if there exists another equilibrium* $\left(\widehat{\mathcal{P}}, \widehat{v}\right)$ *satisfying SEG such that* $\widehat{v} = v$ *almost everywhere. An economy is* segregated *if all equilibria are essentially segregated.*

Note that if an economy is segregated, the equilibrium payoff is essentially unique: in equilibrium, almost every individual obtains the segregation payoff for his type. For this reason, the segregation payoff provides a lower bound on the outside option of an individual in any equilibrium match.

In partial equilibrium analyses (bargaining problems, principal-individual models), outside options are exogenously given and are crucial for predicting how gains from cooperation will be allocated across the individuals. In our framework however, the outside option of an individual will usually be his equilibrium payoff. There is therefore no obvious operational concept of outside option that can be used if one wants to understand the structure of equilibria without having to compute them. A theme of this paper is that the segregation payoff is actually an operational concept of outside option: it tells us a lot about the patterns of matching that can arise in equilibrium and from this computation of equilibrium payoffs is greatly simplified.

Segregation is an extreme kind of equilibrium outcome. When the set of types is unidimensional, the literature has used a weaker concept than segregation—the so-called positive assortative matching—which generates a complete order on the set of equilibrium coalitions. Indeed, start first by defining for a coalition (i, j), the ordered vector $\rho(i, j) = (\tau(i) \vee \tau(j), \tau(i) \wedge \tau(j))$. Then, define for any two equilibrium coalitions (i, j) and (k, l), the order corresponding to the natural order on the vectors $\rho(i, j)$ and $\rho(k, l)$, i.e., $(i, j) \gtrapprox (k, l)$ when $\rho(i, j) \geq \rho(k, l)$. Positive matching means that any two equilibrium coalitions can be ordered by the order $\gtrapprox$.

In many applications, the restriction to one dimensional types is undesirable (for instance, if wealth and ability both affect the outcome). A notion of positive assortative matching is therefore needed when T has more than one dimension; it is not immediately obvious how this should be defined. We propose a definition which seems to be the natural one, although, as we point out, it does not necessarily generate a complete order on the set of equilibrium coalitions.[11]

[11] Shimer and Smith [21] propose a different definition of positive assortative matching which requires that the set of matched pairs forms a lattice. This is more restrictive than our definition, which may be an advantage, but it has the drawback that a given set of matched pairs would be considered to be positively matched under certain type distributions but would not be under

Definition 2.4. *An equilibrium $(\mathcal{P}, v)$ satisfies positive assortative matching (PAM) if for any two equilibrium coalitions $P = \{i, j\}$ and $P' = \{k, l\}$, the following are true:*

$$\begin{cases} \tau(i) \vee \tau(j) \succ \tau(k) \vee \tau(l) \Longrightarrow \tau(i) \wedge \tau(j) \succsim \tau(k) \wedge \tau(l) \\ \tau(i) \wedge \tau(j) \succ \tau(k) \wedge \tau(l) \Longrightarrow \tau(i) \vee \tau(j) \succsim \tau(k) \vee \tau(l). \end{cases}$$

A third type of matching is "negative assortative," defined analogously:

Definition 2.5. *An equilibrium $(\mathcal{P}, v)$ satisfies negative assortative matching (NAM) if for any two equilibrium coalitions $P = \{i, j\}$ and $P' = \{k, l\}$, the following are true:*

$$\begin{cases} \tau(i) \vee \tau(j) \succ \tau(k) \vee \tau(l) \Longrightarrow \tau(i) \wedge \tau(j) \precsim \tau(k) \wedge \tau(l) \\ \tau(i) \wedge \tau(j) \succ \tau(k) \wedge \tau(l) \Longrightarrow \tau(i) \vee \tau(j) \precsim \tau(k) \vee \tau(l). \end{cases}$$

PAM generates an order on equilibrium coalitions. Formally, we define $(i, j) \gtrapprox (k, l)$ whenever $\tau(i) \vee \tau(j) \succsim \tau(k) \vee \tau(l)$ and $\tau(i) \wedge \tau(j) \succsim \tau(k) \wedge \tau(l)$. Notice that this definition relies on the order defined on the lattice. As an illustration, in the Becker example, we should define the type space to be the lattice $T = \{1, 2\} \times [\underline{a}, \overline{a}]$ with the lexicographic order, i.e., $(i, a) \succ (j, b)$ whenever $i > j$ or whenever $i = j$ and $a > b$. Using the "natural " order on T would allow matches like that in Figure 2 below to be positive assortative while we know that these matches cannot be part of an equilibrium, and more seriously, do not correspond to the intuitive notion of positive assortative matching.

Figure 2

The order $\gtrapprox$ on coalitions might fail to be complete. In Figure 3a below, the type space is two dimensional with the natural order, i.e., $a \succsim b$ when $a_1 \geq b_1$ and $a_2 \geq b_2$. With an abuse of notation, let $\{a, d\}$and $\{b, c\}$ be two coalitions. These coalitions satisfy PAM since neither the joins nor the meets can be compared. This shows that the order on coalitions induced by PAM is not complete.

Figure 3a and 3b

Notice first that the incompleteness of the order on coalitions stems from the incompleteness of the order on the type space. Indeed, we have the following result which we state without proof:

others.

Observation If the order on the lattice T is complete, then the order on coalitions induced by PAM is complete.

Second, observe that in this example if one looks in only one dimension at a time, the coalitions are ordered by the conventional definition of PAM. Unfortunately this is not generally true as the example in Figure 3b shows. There, coalitions (i, j) and (k, l) satisfy (trivially) the definition of PAM but they are most assuredly not positively matched in either dimension. Observing positive matching or negative matching in either dimension then does not necessarily reveal very much about the underlying economic process generating the match. For instance, we can think of the two dimensions as height and agility and the individuals as basketball players. If what matters is some index of basketball talent that is increasing in both quantities, then as Figure 3c shows, we could even get segregation in basketball talent but negative matching on each dimension.

Definition 2.6. *An equilibrium* $(\mathcal{P}, v)$ *satisfies essential positive assortative matching (EPAM) if there exists an equilibrium* $\left(\widehat{\mathcal{P}}, \widehat{v}\right)$ *satisfying PAM, with* $\widehat{v} = v$ *almost everywhere. An economy is* positively matched *if all equilibria satisfy EPAM.*

2.3. Results

We will provide characterization results for segregated and positively matched economies. They are expressed in terms of two conditions which depend only on the characteristic function and are therefore relatively easy to verify.

Definition 2.7 (Condition S). *Let*

$$X = \{(t, t') \in T^2 : \exists v \in V(t, t'), \, v > (\underline{u}(t), \underline{u}(t'))\}.$$

Condition S is satisfied if X is empty.

X is the set of types for which there are gains from trade, meaning that it is possible for individuals of those types to match and Pareto improve relative to the segregation payoffs.

Figure 4

Going back to Becker, it is clear that $\underline{u}(t) = 0$ for all t, where $T = \{1,2\} \times [\underline{a}, \overline{a}]$. As long as $h(\cdot,\cdot) > 0$, $X = I$, and therefore condition S is not satisfied. In Kremer [10], since $h(a,b) = ab$, $\underline{u}(a) = \frac{a^2}{2}$ and for any $a \neq b$, $h(a,b) < \frac{a^2}{2} + \frac{b^2}{2}$. Therefore $X = \emptyset$ and condition S is satisfied. We know that the economy is segregated in Kremer. In fact, we obtain the following general result.

Proposition 2.8. *(i) An economy is segregated if Condition S is satisfied. (ii) If Condition S is not satisfied, there is a type assignment τ such that the economy is not segregated.*

Proof. (i) Suppose that Condition S holds and that there is an equilibrium which violates ESEG. This means that a positive measure of agents are receiving more than their segregation payoffs. For this to be true, there must be heterogeneous matches of the form (t,t'). In such matches, at least one of the agents is getting more than her segregation payoff; for stability, the other type must be getting at least its segregation payoff. But then there must exist $v \in V(t,t')$ such that $v > (\underline{u}(t), \underline{u}(t'))$, which contradicts Condition S.

(ii) Suppose that Condition S does not hold: there exists a pair of types (t,t') and a $v \in V(t,t')$ such that $v > (\underline{u}(t), \underline{u}(t'))$; clearly this is only possible if $t \neq t'$. Take the type assignment which puts an atom of size $1/2$ at t and an atom of size $1/2$ at t'. There is an equilibrium in which almost every coalition is composed of types (t,t') and the payoffs are given by v. These payoffs cannot be replicated by segregation, hence the economy is not segregated.

■

The result says simply heterogeneous coalitions will form regardless of the type distribution if and only if there are gains from trade relative to the segregation payoffs. The same result is true for general matching problems as long as effective coalitions are finite: one merely has to modify Condition S to say that there is no finite set of heterogeneous types which can Pareto improve relative to the corresponding segregation payoff vector.

Note that if we want all equilibria in an economy to satisfy SEG, we need the segregation payoff vector of any heterogeneous coalition to be outside its feasible set, i.e., that $(\underline{u}(t), \underline{u}(t')) \in V(t,t')$ implies $t = t'$. Note also that in an equilibrium satisfying ESEG, it is not necessary that the members of heterogeneous coalitions have the same segregation payoff. On the other hand, in [13] we have environments in which the economy is segregated and in which equilibria that satisfy ESEG have the property that individuals in equilibrium coalitions have the same segregation payoff, and we conjecture that this is generically the case.

An implication of the foregoing is that the analysis should be made on a modified characteristic function that captures the notion of the potential gains from "trade" (i.e., heterogeneous matching). For convenience, we will define this new characteristic function on the type space. Formally, let $S(t,t') = \{0\} \cup ([V(i,j) - (\underline{u}(t), \underline{u}(t'))] \cap \mathbb{R}^2_+)$ and let $S(t) = \{0\} \cup ([V(i) - \underline{u}(t)] \cap \mathbb{R}_+)$, for any $i \in \tau^{-1}(t)$, $j \in \tau^{-1}(t')$.[12] Note that $S(t,t') = \{0\}$ when $t = \tau(i)$, $t' = \tau(j)$ and $(i,j) \notin X$, with X defined in Condition S. Hence, Condition S can then be restated to say that the set $\{(t,t') : S(t,t') = \{0\}\}$ has full measure.

We now provide a characterization of positive assortative matching.

By construction of the surplus set S, $S(a,b)$ has relative dimension 2 whenever there are gains from trade. Otherwise, S is the zero vector. Let S^P denote the Pareto frontier of S and $S^D = S \backslash S^P$ denote the set of Pareto dominated elements of S.[13]

Definition 2.9. *Condition P is satisfied if for any four elements* $\{a,b,c,d\}$ *of* T, *where* $a \vee d \succ b \vee c \succsim b \wedge c \succ a \wedge d$, *and* $s \in S^P(a,d) \times S^P(b,c)$, *one of the two conditions below is true. Either*

$$s \in S(a,b) \times S(c,d) \text{ or } s \in S(a,c) \times S(b,d) \tag{2.1}$$

or

$$\exists t \in \{a,d\},\ \hat{t} \in \{b,c\} \text{ such that } (s(t), s(\hat{t})) \in S^D(t,\hat{t}) \tag{2.2}$$

Condition P says that for any *negative* match, either it is possible to "rematch" the types in a positive way that keeps all four types (at least) indifferent (A), or the match is not stable (B).

As for the case of segregation, we are looking for conditions on the characteristic function which ensure positive matching regardless of the particular type distribution.

Proposition 2.10. *(i) An economy is positively matched if Condition P is satisfied. (ii) If Condition P is not satisfied, there is a type assignment* τ *such that the economy is not positively matched.*

[12] Therefore, if the segregation payoff vector lies outside the feasible set, we define the surplus set to be $\{0\}$. The fact that the surplus is zero captures the idea that there are no gains from trade.

[13] Hence, $S^P = \{s \in S : \forall \hat{s} \in S, \neg[\hat{s} \gg s]\}$. Note that if there are no gains from trade, $S^D = \emptyset$ since $S = S^P = \{0\}$.

Proof. (i) Suppose that Condition P holds. If an economy is not positively matched, there exist a, b, c, d where $a \vee d \succ b \vee c \succsim b \wedge c \succ a \wedge d$ and payoffs s, where $s \in S^P(a,d) \times S^P(b,c)$,[14] such that the matches (a,d) and (b,c) are part of the equilibrium *and* it is not possible to obtain a positively matched reshuffling of these types which keeps the payoffs the same (A is violated). Since there are no beneficial deviations from the equilibrium payoffs s, (B) is also violated, contradicting Condition P.

(ii) Suppose that Condition P is not satisfied. Since (A) is violated, $s \notin S(a,b) \times S(c,d)$ and $s \notin S(a,c) \times S(b,d)$. Since (B) is violated, for each $\hat{t} \in \{a,d\}$ and each $t \in \{b,c\}$, $\left(s\left(\hat{t}\right), s(t)\right) \notin S^D\left(\hat{t}, t\right)$. Consequently, it is not possible to replicate the payoffs s by a positive match between a, b, c and d. Consider τ such that there are four atoms at a, b, c and d of equal mass. The matches $(a,d), (b,c)$ together with the payoff s constitute an equilibrium and the economy is not positively matched. ■

Proposition 2.10 shows that if the characteristic function satisfies certain properties, the equilibrium matching pattern will (essentially) always assume a positive assortative form. But in economies in which this condition is violated, the outcome will be sensitive to type assignment map: *the equilibrium matching pattern will depend on the distribution of types.* We will illustrate this point in the Applications section below.

We now consider a special case that encompasses most of the complete market examples already present in the literature (but which still allows for multidimensional types). We say that the surplus is transferable if for any S, there exists σ such that the Pareto frontier S^P can be expressed as $S^P = \{s \geq 0 : s_1 + s_2 = \sigma\}$.[15]

For the class of economies with transferable surplus, Condition P can be written in a simpler way, as Condition PT below. While sufficiency of Condition PT will be immediate, necessity is not, mainly because of the restriction that payoffs must be nonnegative.

Definition 2.11. *Condition PT is satisfied if for any four elements $\{a,b,c,d\}$ of T, where $a \vee d \succ b \vee c \succsim b \wedge c \succ a \wedge d$, either*

$$\sigma(a,d) + \sigma(b,c) \leq \sigma(a,c) + \sigma(b,d)$$

[14] If $s \notin S^P(a,d) \times S^P(b,c)$, either (a,d) or (b,c) has an incentive to deviate from the proposed payoffs.

[15] For instance, in the models of Becker and Kremer-Maskin, for each coalition of types (t,t'), the relevant σ is $\sigma(t,t') = \max\left\{0, h(t,t') - \frac{1}{2}\left[h(t,t) + h(t',t')\right]\right\}$, where $h(\cdot,\cdot)$ is the output function.

or

$$\sigma(a,d)+\sigma(b,c)\leq\sigma(a,b)+\sigma(c,d).$$

We observe that Condition PT implies Condition P. To see this, note that if Condition P is not satisfied, there exist four elements $\{a,b,c,d\}$ of T, with $a\vee d\succ b\vee c\succsim b\wedge c\succ a\wedge d$, and $s\in S^P(a,d)\times S^P(b,c)$, i.e.,

$$s(a)+s(d)=\sigma(a,d) \text{ and } s(b)+s(d)=\sigma(b,d) \tag{2.3}$$

such that (A) is not satisfied, i.e.,

$$\begin{array}{c}\forall\ \{t_1,t_2\}=\{b,c\},\exists\hat{t}\in\{a,d\},\exists i\in\{1,2\}\\ s\left(\hat{t}\right)+s\left(t_i\right)>\sigma\left(\hat{t},t_i\right)\end{array} \tag{2.4}$$

and such that (B) is not satisfied either, i.e.,

$$\begin{array}{c}\forall t\in\{a,d\},\ \hat{t}\in\{b,c\}\\ s(t)+s\left(\hat{t}\right)\geq\sigma(t,\hat{t}).\end{array} \tag{2.5}$$

Therefore, by (2.3), (2.5)

$$\begin{array}{c}\forall\{t_1,t_2\}=\{b,c\},\\ \sigma(a,d)+\sigma(b,c)>\sigma(a,t_1)+\sigma(d,t_2).\ ,\end{array} \tag{2.6}$$

which is a violation of PT. Hence, Condition PT implies Condition P. In the Appendix, we show the converse. Hence we have

Proposition 2.12. *When the surplus is transferable: (i) An economy is positively matched if Condition PT is satisfied. (ii) If Condition PT is not satisfied, there is a type assignment τ such that the economy is not positively matched.*

For $a\vee d\succ b\vee c\succsim b\wedge c\succ a\wedge d$, Condition PT can also be written as

either $\sigma(a,c)-\sigma(a,d)\geq\sigma(b,c)-\sigma(b,d)$ **or** $\sigma(a,b)-\sigma(a,d)\geq\sigma(c,b)-\sigma(c,d)$.

This condition resembles the familiar increasing difference (ID) condition discussed for instance in [15]: σ satisfies ID if for all $a\succ b$ and $c\succ d$, $\sigma(a,c)-\sigma(a,d)\geq\sigma(b,c)-\sigma(b,d)$ In the one dimensional case however, Condition PT is weaker since it requires comparison among four ordered elements (while the ID condition would not require that $a > c$).[16]

As is well known, increasing differences is equivalent for smooth functions to non-negative cross partial derivatives. Often, however, σ will not be differentiable everywhere, even if it is derived from a smooth production function. If σ is C^2 and Condition PT is satisfied, then Conditions S is also satisfied. Indeed, it can be shown that PT together with σ smooth implies that the cross partial derivatives are nonnegative, which is equivalent to ID. And ID implies Condition S:

Proposition 2.13. *Let T be a subset of $\mathbb{R}$. If σ satisfies ID, the economy is segregated.*

Proof. Take $a = c$ and $b = d$ and $a > b$ in the definition of ID. Increasing differences implies that $-\sigma(a,b) \geq \sigma(b,a)$ since $\sigma(a,a) = \sigma(b,b) = 0$. However, $\sigma(a,b) = \sigma(b,a)$ implies that $\sigma(a,b) = 0$. Hence, the individuals might as well segregate. ■

As an application of the foregoing discussion, note that the surplus functions in Kremer [10] satisfy increasing differences and result in segregation. As we show below, Kremer-Maskin's [12] model only satisfies Condition PT and does not have segregation. In [12] the surplus function σ is not differentiable on the diagonal while in [10] it is.

Thus, the surplus function may provide more information about the properties of the equilibrium match than the production function. For instance, there are cases in which production functions are neither super- nor submodular and yet from the surplus computation it is easy to see that the economy must be segregated.[17]

Moreover, innocuous-looking restrictions on production functions may be undesirably strong in the context of matching models. For instance if the type

[16] Another way to see this is to note that ID implies that

$$\sigma(a,c) - \sigma(a,d) \geq \sigma(b,c) - \sigma(b,d) \text{ and } \sigma(a,b) - \sigma(a,d) \geq \sigma(c,b) - \sigma(c,d)$$

whenever $a > b \geq c > d$. In the multi-dimensional case, PT and ID are not comparable: condition PT requires the comparison of types that ID does not and vice-versa.

[17] For instance, let $T = [4,5]$ and $f(a,b) = A(\sqrt{a}+\sqrt{b}) - \epsilon \max\{a^3b, b^3a\}$, where $0 < \epsilon < \frac{A}{2000}$. It is straightforward to verify that f_a and f_b are positive wherever they exist (which is everywhere except on the diagonal). And $f_{ab} < 0$ almost everywhere; hence f is not supermodular (neither is it submodular, although this takes slightly more effort to show). But since $f(a,b) - \frac{1}{2}[f(a,a) + f(b,b)] \propto a^4 + b^4 - 2\max\{a^3b, b^3a\} \leq 0$ on T^2, $\sigma(a,b) \equiv 0$, and the economy is segregated.

space is one-dimensional, it is natural to suppose that the production function is symmetric. If it is also supermodular, then heterogeneous matches are ruled out:

Proposition 2.14. *Let the production function $h : \mathbb{R}^2 \to \mathbb{R}$ be symmetric: $h(a,b) = h(b,a)$. If h is supermodular the economy is segregated.*

Proof. If $h(a,b) = h(b,a)$ is supermodular, it satisfies the inequality $h(x \vee y) + h(x \wedge y) \geq h(x) + h(y)$; putting $x = (a,b)$ and $y = (b,a)$ then implies that $h(a,b) - \frac{1}{2}[h(a,a) + h(b,b)] \leq 0$; hence $\sigma(a,b) \equiv 0$ and the economy is segregated. ■

Thus, at least for one-dimensional type spaces, if one is to have (nontrivial) heterogeneous matching, one *must* rule out (symmetric) supermodular production functions, or else introduce imperfections.

Finally, if we index economies by a parameter θ representing some aspect of the production technology or some degree of market imperfection, and do comparative statics on the characteristic function, we will find that $\underline{u}(\cdot)$ and $V(\cdot)$ change differently for different types. For instance, if θ measures a degree of capital market imperfection, increases in θ will typically lower the segregation payoffs of poor agents but may have no effect on those of wealthy ones; at the same time, the production possibilities for mixed coalitions, if one partner is wealthy enough, may also be unaffected. Thus for low θ we could have segregation, while for high θ we would have heterogeneous matching: Conditions S (or P) will be satisfied at some parameter values, while for others it will not. The consequence is that the equilibrium matching pattern for a fixed type distribution will vary across economies and that in some cases the type distribution will play an important role in determining the outcome while in others it will not.[18] We will return to this point in the next section.

3. Applications

We now apply the above theory to the analysis of some recent models. First, we return to the models discussed above, reconcile the differences in their comparative static behaviors and obtain some further results with the apparatus we

[18] Notice that in certain cases, changes in the underlying parameters will not change the equilibrium matching. For instance, for a given distribution of types, if the underlying parameter θ representing technology or imperfections changes the surplus functions in a linear way, i.e., $\sigma(a,b;\theta) = A\sigma(a,b;\theta')$, where $A > 0$ the equilibrium matching will be the same with θ and θ'.

have developed. The next two examples consider economies with imperfections. In both cases the production technology satisfies increasing differences so that the first best version of these economies will display segregation. The first example considers a financial market imperfection which results in the violation of Condition S. We show that the matching configuration will be sensitive to the distribution of types and that in some instances the effects of the imperfections swamp the effects of the production technology. The second example considers production with an incentive problem. There it turns out that Condition S is still satisfied but that segregated matching may be inefficient.

3.1. Perfect-Markets Examples

We return to the models we discussed above. We will generally be interested in looking at the behavior of a set of models parameterized by a single scalar θ. In the present case this will be a technological parameter. As the parameter changes, both the utility possibilities for each pair of types and the segregation payoffs for each type change. As will see, the interaction of these two effects will affect the equilibrium matching pattern.

Let the type space be an interval $[\underline{a}, \bar{a}]$, $\underline{a} > 0$, and $h(a,b) = \max\left\{a^\theta b, b^\theta a\right\}$ (so [10] corresponds to $\theta = 1$ and [12] to $\theta = 2$). Since these are transferable utility models, we will use the surplus function $\sigma(a,b) = \max\left\{0, \max\left\{a^\theta b, b^\theta a\right\} - \frac{1}{2}(a^{\theta+1} + b^{\theta+1})\right\}$ to study these economies.

A quick calculation reveals that Condition PT is satisfied for all $\theta \geq 1$ (in fact, for all $\theta > 0$), so that the economy will always be positively matched. As we mentioned before, a somewhat longer calculation shows that σ satisfies ID if and only if $\theta = 1$. Thus the economy is segregated when $\theta = 1$.

For $\theta > 1$, the analysis is somewhat more complicated. We do know from Proposition 2.8 that there will be heterogeneous matches for some type assignments, and since Condition PT is satisfied, these must be positive assortative. It is useful to graph the surplus for a fixed type a as a function of a potential partner's type $b \leq a$. This is done in Figure 5a for different values of θ.

Figure 5

It is easy to show that the type $M(a) \leq a$ that maximizes the total surplus $\sigma(a,b)$ is first decreasing in θ and then increasing in θ (and tends to a as θ tends to ∞). We can also show that $\sigma(a,b)$ is concave in $b \leq a$ when $\sigma > 0$ and that for $\hat{a} \in (M(a), a)$, the graphs of the surplus functions $\sigma(a,b)_{b \leq a}$ and $\sigma(\hat{a}, b)_{b \leq \hat{a}}$

intersect at $b < M(a)$ (we can think of this as a "single crossing to the left" property). Furthermore, $M(a)$ is decreasing in a. This suggests that for θ close to one, increases in θ will decrease segregation while for larger values of θ segregation will increase. So in general we should expect a change in matching pattern when θ changes.

There is however a case where the matching is invariant to local changes in θ. The condition under which this happens is similar to the condition that yields invariance in Becker's model. Since $M(a)$ is decreasing in a, if $M(\overline{a}) < \underline{a}$, then for any a, $M(a) < \underline{a}$. Because of the single crossing to the left, it follows that any $c < b < \hat{a} < a$, the gains from trade are greater in a match with c than in a match with b for both a and $\hat{a}$. We call this property "unanimous monotone ranking" and we show in the appendix that as long as the surplus function satisfies PT and has this property, the matching always takes the form of the highest type matching with the median type and the other types matching in a positive assortative way while respecting the measure consistency condition. Note that in Becker's model, the surpluses also have this property.

Definition 3.1. *There is unanimous monotone ranking (UMR) on* $[\underline{a}, \overline{a}]$ *if for each* $a \in [\underline{a}, \overline{a}]$ *the surplus function* $\sigma(a, \cdot)$ *is decreasing on* $[\underline{a}, a]$.

When UMR is satisfied and the type distribution is atomless, the equilibrium matching pattern has a simple characterization in which the highest type matches with the median type, and all other match in such a way as to keep the "probability distance" between a type and his partner constant at $1/2$.

Proposition 3.2. *Suppose that there is UMR on* $[\underline{a}, \overline{a}]$, *that* σ *satisfies PT, and that the assignment map* τ *generates a continuous distribution of types* $T(a)$. *Then there exists an essentially unique equilibrium matching in which for* $a \in [a_m, \overline{a}]$, $T(a) - T(m(a)) = \frac{1}{2}$, *where* $m(a)$ *is* a*'s partner and* a_m *is the median type.*

Proof. Appendix. ■

Observe that if we hold θ fixed and instead change the *distribution* of types by "lengthening" the support, an economy can change from satisfying UMR to violating it. The matching pattern will then change from the one described in Proposition 3.2 to one in which some types are more nearly segregated (see Figure 5b). This kind of result is obtained in [12].

3.2. Imperfect-Markets Examples

These examples are based loosely on [13]. The first introduces a financing constraint. The most dramatic change is the differential effect of the imperfection on the segregation payoffs of different types. Put simply, wealthy agents get the same segregation payoff with or without the financing constraint. Poor agents suffer a large decrease in the segregation payoff with sufficiently imperfect financing. The result is possibly a significant change in the matching pattern (as well as in the aggregate output of the economy). The same kind of effect was present in the case of the technological changes studied in the previous subsection, although here the outcome varies more dramatically with the type distribution.

The second example introduces a moral hazard problem into the production process. This reduces the segregation payoff for all types, but again those of the lowest ability are most severely affected. But a second effect now comes into play, which is not present in perfect-markets examples: increasing information costs also reduce the *transferability* of utility. The moral hazard problem requires that payoffs exceed a positive lower bound for each partner. It turns out that this change in the characteristic function offsets the changes in the segregation payoffs in such a way as to keep the matching pattern unchanged. But now aggregate performance will no longer be optimal, even conditional on the information constraints: total surplus could be increased by forcing matches to differ from their equilibrium form. The source of the failure of optimality of equilibrium is the restricted transferability introduced by the incentive problem, on which we comment below.

3.2.1. Production with an Imperfect Financial Market

Consider now a modification of the standard production model in which a fixed amount $k > 0$ of capital is required for production to take place; once this is invested, output depends on the ability of the firm's members according to $h(a, b) = ab$. The cost of a unit of capital is normalized to one. The sum of wealth and output may divided among the individuals in any way. Individuals' types are now two dimensional: they are described by their ability $a \in [\underline{a}, \bar{a}]$ and their initial wealth $w \in [0, \bar{w}]$, $\bar{w} < k$. These have some joint distribution $G(a, w)$. Suppose that production is efficient for all ability levels in the economy, i.e. that $h(\underline{a}, \underline{a}) > k$. Any two individuals my form a firm, possibly pooling their resources (wealth) to purchase capital and then produce as in the examples above (there are no incentive problems, however).

We now consider the family of economies indexed by the parameter $\theta \in [0,1]$. This will represent the degree of imperfection in the capital market. Individuals with wealth levels w and w' will have to satisfy the constraint $w + w' \geq \theta k$. Thus the case $\theta = 0$ corresponds to a perfect capital market since initial wealth places no constraint on what a firm can do; the case $\theta = 1$ represents the case in which the financial market is absent altogether: firms must be entirely self-financed.[19]

The segregation payoff is

$$\underline{u}\left((w,a)\right) = \begin{cases} \frac{a^2-k}{2} & \text{if } \; 2w \geq \theta k \\ 0 & \text{if } \; 2w < \theta k \end{cases}$$

We note first that if $\theta = 0$, the economy will be segregated by ability, since it essentially is the one studied in [10] (The wealth levels of matching partners is indeterminate, however.) This outcome is independent of the initial distribution of types. For now, assume that this distribution is uniform on $[\underline{a}, \bar{a}] \times [0, \bar{w}]$ and that $\theta > 0$. This will create a positive measure of agents (those with wealth less than $\frac{\theta k}{2}$) whose segregation payoff becomes 0. The surplus of a pair agents of types (w,a) and (w',a') is

$$\sigma((w,a),(w',a')) = \begin{cases} aa' - k - \left(\underline{u}\left((w,a)\right) + \underline{u}\left((w',a')\right)\right) & \text{if } w + w' \geq \theta k \\ 0, & \text{if } w + w' < \theta k \end{cases}.$$

We claim that for any value of θ, the equilibrium match will entail segregation by ability. Below we construct an equilibrium and we prove in the Appendix that this equilibrium is essentially unique. The equilibrium payoffs and the way wealth levels are matched depend on the value of θ. If $\theta > 0$ is small enough that $\theta k < \bar{w}$, then for each a, $v(a) = \frac{a^2-k}{2}$ independently of wealth. Intuitively, there is an excess supply of agents with wealth greater than $\frac{\theta k}{2}$ and these agents will bid the payoff to the agents with zero segregation payoff to its maximum. Since there is an excess supply of rich agents, there are many ways to match wealth levels. However, it is clear that two agents of different ability cannot improve on their equilibrium payoff: they each get the segregation payoff of the economy without imperfection, and we know that this economy is segregated.

[19] This kind of capital market imperfection can derived by supposing that the agents, upon having to repay, may renege on their debt and escape a nonmonetary punishment of size Pk ($P > 1$)with probability π. Lenders will make loans of size k only to those firms with aggregate wealth greater than $\theta k \equiv (1 - (1-\pi)P)k$; $\theta = 0$ corresponds to $\pi = (P-1)/P$; with larger values of π escape becomes more likely, until with $\pi = 1$, the market shuts down altogether ($\theta = 1$).

If instead θ is large enough that $\theta k \geq \bar{w}$, then, agents with wealth less than $\theta k - \overline{w}$ cannot find a match for which the total wealth is θk; the matching is therefore between agents of wealth greater than $\theta k - \overline{w}$: those with wealth between this amount and $\frac{\theta k}{2}$ have a zero segregation payoff while those with wealth greater than $\frac{\theta k}{2}$ have a positive segregation payoff. There is a family of equilibrium payoffs: let $c(w)$ be a decreasing (possibly weakly) function of w with $c(\theta k - \overline{w}) \leq \frac{a^2-k}{2}$ and $c\left(\frac{\theta k}{2}\right) = 0$, then the payoff to a type $t = (w, a)$ is $v(t) = \frac{a^2-k}{2} + c(\theta k - w)$ if $w \geq \frac{\theta k}{2}$ and is $v(t) = \frac{a^2-k}{2} - c(\theta k - w)$ if $w \leq \frac{\theta k}{2}$. When $\overline{w} < \theta k$, there is negative matching in wealth; for a given ability level, the richest agents match with the poorest agents. Since $v'(a) = a$ for each a, segregation by ability is indeed the equilibrium situation.

So, depending on the value of θ, the division of the surplus is different, but the equilibrium matching pattern is always segregation by ability and some form of negative matching by wealth level. The fact that the equilibrium match in abilities is the same as in the first best situation is however an artifact of the uniform distribution. To see that for other distributions different patterns of matching happen, consider the following examples.

Figures 6a and 6b

Figure 6a depicts a situation with four atoms $t_i = (w_i, a_i)$. We assume that $a_1 = a_3 = a$ and $a_2 = a_4 = b$, $a > b$, $w_i > \frac{\theta k}{2}$, $i = 1, 2$ and $w_i < \frac{\theta k}{2}$, $i = 3, 4$, $w_1 + w_4 > \theta k$, $w_2 + w_3 > \theta k$ but $w_2 + w_4 < \theta k$. Note that t_2 cannot match with t_4 and generate a positive surplus and that $\underline{u}(t_i) = 0$ for $i = 3, 4$. t_2 will benefit from a match with t_3 as long as $ab - k - v(t_3) \geq \frac{b^2-k}{2}$; hence t_2 is willing to bid the payoff to t_3 up to $ab - \frac{b^2+k}{2}$. t_1 can match with either t_3 or t_4; he will prefer a match with t_4 if $ab - k - v(t_4) \geq \max\left(\frac{a^2-k}{2}, a^2 - k - v(t_3)\right)$. Suppose that the masses on the four atoms are equal, that $a - b$ is not too large and that k is not too large. Then, the unique equilibrium involves matches (t_1, t_4) and (t_2, t_3); for instance, $v(t_3) = ab - \frac{b^2+k}{2}$, $v(t_4) = 0$ are payoffs consistent with this matching and the previous conditions as long as a, b, k are chosen such that $k \leq 4ab - 2a^2 - b^2$. Observe that in this example we cannot really say whether the partners are matched positively or negatively by ability; but if we had perturbed the abilities slightly so that $a_1 > a_3$ and $a_4 < a_2$, the same coalitions would form in equilibrium and the matching would be negative in both ability and wealth.

Figure 6b depicts a situation in which all the probability is put on a diagonal of $[0, \bar{w}] \times [\underline{a}, \bar{a}]$ in a uniform way. By feasibility, the poorest agents need to match with the richest agents; since the poorest agents happen to have the highest ability, this match maximizes the gains from trade for these two types. Note that if the richest agents form a match with other types of agents, they would face competition from other agents and would necessarily end up with a lower surplus. Hence, the unique equilibrium matching is the one in which types (t, t') match when $w_t + w_{t'} = \theta k$ that is to say there is negative matching in wealth and in ability levels. This shows that, at least in some examples, the wealth effects induced by the financial market imperfections overwhelm the forces toward segregation generated by the production function.

Moreover, note that in both examples of Figure 6, redistribution of wealth would increase the surplus. For instance, if we give t_3 the wealth of t_4 and vice versa, the equilibrium match will be now be (t_1, t_3) and (t_2, t_4) and the total surplus is a^2+b^2-2k while it is initially $2ab-2k$. Note however that this increased in surplus must decrease the equilibrium payoff of some type since we know that equilibria are constrained Pareto optimal. Note for instance that in the original equilibrium $v(t_1) + v(t_3) > a^2 - k$, so a Pareto improving redistribution would entail that they were *receiving* wealth as compensation rather than transferring it to the t_4's. (Constrained Pareto optimality is partly an artifact of our restriction to environments in which there are no externalities across coalitions of size two. In more general environments, of course, Pareto improvements may be possible).

The observation that matching can be inefficient has already been made in the literature (e.g., [2], [6], [7], [13]). In some cases policy measures may be tailored specifically to correcting the match itself rather than designed to do so indirectly via redistribution of initial wealth (see [5] on this point). The next example is another illustration of this possibility.

3.2.2. Production with an incentive problem

A somewhat different sort of inefficiency can arise in matching environments when there are restrictions on transferability *within* coalitions. Consider the same production function as before, but now suppose that there is a moral hazard problem: each partner in a match must take some effort in order for output to be produced. The effort levels are low and high, with cost 1 if the high effort is chosen and zero otherwise. In order for partners of ability a and b to produce ab, both must take the high effort. Effort can be monitored at a cost: if $c(\theta, q)$ is invested at the time of the match, the probability of detecting a partner if he takes the low effort is q (this probability is independent across partners, but the same q must be chosen for each partner). We assume that $c(0,q) \equiv 0$, that $c(\cdot,\cdot)$ is increasing in both arguments and convex in q.

Each partner receives a contract which specifies that he receives a payment y if he is not caught taking low effort and 0 if he is.[20] Given the level of monitoring q, incentive compatibility then requires that $y - 1 \geq (1-q)y$, or $y \geq \frac{1}{q}$. The net output generated by a firm with partners a and b and monitoring q is then $ab - c(\theta, q)$; but even though the partners are assumed to be risk neutral they cannot transfer this output to each other arbitrarily: each partner must receive at least $\frac{1}{q}$.

For analyzing this problem it is convenient to consider the maximum payoff that an agent can achieve assuming his partner is incentive compatible, considered as a function of q. This expression, $h - c(\theta, q) - \frac{1}{q} - 1$, with $\theta > 0$, is graphed for different values of h in Figure 7. Also shown is the incentive compatibility constraint $\frac{1}{q} - 1$: if $h = ab$, both a and b must get a payoffs at least this high if they are to be incentive compatible.

Figure 7

For $\theta = 0$, q is optimally set equal to 1. In this case, the first-best allocation with segregation is achieved in equilibrium (We assume it is efficient for all partnerships to produce: if abilities lie in the interval $[\underline{a}, \bar{a}]$, then $\underline{a} > \sqrt{2}$.)

Things can be rather different, however, if $\theta > 0$. Let $q(h)$ be the lower value of q, when it exists, at which the graph of $h - c(\theta, q) - \frac{1}{q} - 1$ intersects the graph of $\frac{1}{q} - 1$. When $q(a^2)$ exists (and lies in $[0,1]$), the segregation payoff of type a is $\frac{a^2 - c(\theta, q(a^2))}{2} - 1$. Clearly, there exists a unique h^0 such that the graphs of $h - c(\theta, q) - \frac{1}{q} - 1$ and $\frac{1}{q} - 1$ are tangent. Hence, when $a < \sqrt{h^0}$, $q(a^2)$ does not exist and the agents have a zero segregation payoff.[21]

[20] If one takes the assumption of two partners literally, this is not likely the optimal contract, since the firm's output would typically serve as a signal of the partner's effort. We have in mind situations, such as those in large firms, where output information reveals little about individual effort and other (costly) signals must be employed instead. See [13] for a more general analysis.

[21] If $c(\theta, 1)$ is finite, the set of types with zero segregation payoffs may be larger than $[\underline{a}, \sqrt{h^0}]$. This hardly affects the analysis, however; when it does, we shall point this out.

Because the choice of q varies with a (when $a \geq \sqrt{h^0}$) and because the segregation payoff will be zero for some types, we might conjecture that there would be gains from trade, and therefore heterogeneous matches in equilibrium, as there were in the examples with imperfect credit markets. However, this is not possible. Indeed, we have

Proposition 3.3. *The economy with moral hazard is segregated for all θ.*

Proof. Suppose instead that there is a heterogeneous match (a, b), with $a > b$. Let q be the level of monitoring they choose. Clearly, a has a positive segregation payoff (if not, then neither does b, and nothing can be gained if they match), and $q > q(a^2)$. Let y_a and y_b be the levels of compensation paid to each of the partners. If b has a positive segregation payoff and $q \geq q(b^2)$, then since for a heterogeneous match to occur we must have

$$y_a + y_b = ab - c(\theta, q) \geq \frac{a^2 - c\left(\theta, q\left(a^2\right)\right)}{2} + \frac{b^2 - c\left(\theta, q\left(b^2\right)\right)}{2},$$

we immediately conclude, since $c(\theta, \cdot)$ is increasing in q, that

$$0 > ab - \frac{a^2 + b^2}{2} \geq c(\theta, q) - \frac{c\left(\theta, q\left(a^2\right)\right) + c\left(\theta, q\left(b^2\right)\right)}{2} > 0,$$

a contradiction.

If instead $q < q(b^2)$ or b has a zero segregation payoff ($q(b^2)$ does not exist), then $b^2 - c\left(\theta, q\right) - \frac{1}{q} < \frac{1}{q}$; heterogeneous matching again requires that

$$ab - c\left(\theta, q\right) - y_b \geq \frac{a^2 - c\left(\theta, q\left(a^2\right)\right)}{2},$$

and, since $y_b \geq \frac{1}{q}$,

$$y_b > \frac{b^2 - c\left(\theta, q\right)}{2};$$

adding these two expressions and rearranging yields

$$0 > ab - \frac{a^2 + b^2}{2} > \frac{c\left(\theta, q\right) - c\left(\theta, q\left(a^2\right)\right)}{2} > 0,$$

a contradiction. We conclude that no heterogeneous matches can occur. ■

Figure 8 shows the feasible sets for three possible coalitions and the segregation payoff vector for the mixed coalition, which clearly lies outside of (the comprehensive extension of) its feasible set.

Figure 8

Even though the matching configuration is unchanged when incentive problems are introduced, there is an important difference between the two cases from a welfare point of view: when θ is large enough, the equilibrium will not always be efficient in the sense of maximizing total output net of monitoring and effort costs. The reason is that some types are "left out" of the economy, and more output could be generated if higher types were forced to match with them.

To see this, suppose that $c(\theta, q) = \theta q$ and $\theta > 2$. Consider two atoms, one at $a > \sqrt{h^0}$ and one at $b < \sqrt{h^0}$ with equal masses.[22] Simple computations show that $\sqrt{h^0} = (8\theta)^{1/4}$ and that the segregation payoffs — which are also the equilibrium payoffs — are $\underline{u}(a) = \frac{a^2+\sqrt{a^4-8\theta}}{4} - 1$ and $\underline{u}(b) = 0$.

Suppose now that a planner forced the a's and b's to match (she might also have to dictate how they share surplus). If they share the surplus equally, they will choose $q(ab) = \frac{ab-\sqrt{a^2b^2-8\theta}}{2\theta}$ and the total payoff in the equal-share ab-firm is equal to $\frac{ab+\sqrt{a^2b^2-8\theta}}{2} - 2$ (note that this is the maximum payoff that such a firm can generate). Therefore the aggregate payoff is $\frac{1}{2}\left(\frac{ab+\sqrt{a^2b^2-8\theta}}{2} - 2\right)$. In the segregated economy, since there were only half as many active firms, the aggregate payoff was $\frac{1}{4}\left(\frac{a^2+\sqrt{a^4-8\theta}}{2} - 2\right)$. Total surplus under this forced match will be higher than under the equilibrium match whenever

$$2ab + 2\sqrt{a^2b^2 - 8\theta} > a^2 + \sqrt{a^4 - 8\theta} + 4. \tag{3.1}$$

Let $b = \delta(8\theta)^{\frac{1}{4}}$ and let $a = \frac{3}{\delta}(\frac{\theta}{8})^{\frac{1}{4}}$, where $0 < \delta < 1$. Clearly $a > \sqrt{h^0} > b$, and $ab = (9\theta)^{\frac{1}{2}}$. Expression (3.1) becomes,

$$\sqrt{\theta}\left[8 - \frac{9}{\delta^2\sqrt{8}} - \sqrt{\frac{81}{8\delta^4} - 8}\right] > 4.$$

As δ approaches 1, the left hand side converges to approximately $3.36\sqrt{\theta}$, which since $\theta > 2$, exceeds 4.75. Hence, for δ close enough to 1 and /or θ large enough, there can be an increase in the aggregate payoff when a and b are forced to match.

[22] Because of the linearity assumption, $c(\theta, q)$ is finite at $q = 1$ and the the set of types who have zero segregation payoffs is actually larger than $[\underline{a}, \sqrt{h^0}]$ when θ is small enough. Specifically, the highest type with a zero segregation payoff is $\sqrt{2+\theta}$ when $\theta < 2$; for $\theta > 2$, it is $\sqrt{h_0}$.

The reason that surplus maximizing matches are not achieved in equilibrium stems from the limited transferability introduced by incentive problems. An a who is forced to match with a b receives less than his segregation payoff and cannot be compensated by the b because that would entail that the b end up with less than an incentive compatible compensation. This would violate feasibility. Thus this simple example illustrates how a conflict between "cake production" (maximizing the surplus generated by matches) and "cake division" (maximizing one's share of a given surplus) can lead to distortions in the pattern of matching.

4. Conclusion

We provide a characterization of equilibria in two person matching environments. We argue that a measure of the gains from trade from a match, which we have called surplus, is a useful concept for understanding this class of models. We believe that this concept will be valuable for analyzing more general matching models. For the purpose of comparative statics, a great deal of insight is obtained from the fact that both the segregation payoffs and the feasible set vary differentially across environments for agents of different characteristics. It seems that future progress in understanding comparative statics in matching models of the type we have considered here will depend on sharper characterization of the comparative static properties of these two quantities. Our examples with imperfect markets illustrate quite clearly that the properties of the production technology are especially insufficient in making welfare evaluations on the basis of an observed matching pattern or in predicting the outcome of the match.

5. Appendix

5.1. A Note on Comprehensiveness

Comprehensiveness is a relatively mild condition to impose when the economy has no imperfections. In the presence of incentive and/or contractibility problems, however, description of the feasible set will often entail that each agent receives a nonnegligible payoff, so comprehensiveness will be violated (see Figure 8 for an example). None of our results depend on comprehensiveness (except in instances where it is already guaranteed by other assumptions), although existence of an equilibrium might. Existence of the segregation payoff is facilitated by comprehensiveness, although in extreme cases in which the Pareto frontier does not intersect the diagonal, the segregation payoff can be found by convexification (this has an appealing interpretation: the segregation payoff represents what an agent could *expect* to get by matching with a partner of his own type).

For existence, the following construction suffices. Define the comprehensive extension of a set $V(\cdot)$ as the smallest comprehensive set containing $V(\cdot)$. The economy in which V is replaced by its comprehensive extension will have a non-empty core. Moreover, there will always exist core allocations in the extended economy in which agents receive utility levels that are on the Pareto frontier of the original feasible set $V(\cdot)$. Such allocations satisfy feasibility, minimality, measure consistency, and the no blocking requirements of an equilibrium of the original economy, and so the original economy has an equilibrium.

When Condition S is satisfied and V is comprehensive, then the segregation payoff vector lies outside (or on the Pareto frontier) of $V(\cdot)$. Violations of this condition entail that the segregation payoff vector lies in the interior of $V(\cdot)$. But if V is not comprehensive, then segregation payoff vector *may* lie outside of $V(\cdot)$ and still entail a violation of Condition S.

5.2. Proof of the Necessity of Condition PT

We continue here the proof of Proposition 2.12. We need only show that Condition P implies Condition PT. To do so, we will show that Condition P is violated when Condition PT is violated. If Condition PT does not hold, there exists $\{a,b,c,d\} \subset T$, where $a \vee d \succ b \vee c \succsim b \wedge c \succ a \wedge d$ such that

$$\begin{gathered} \forall \{t_1, t_2\} = \{b, c\} \\ \sigma(a,d) + \sigma(b,c) > \sigma(a,t_1) + \sigma(d,t_2). \end{gathered} \tag{5.1}$$

If $b = c$, (5.1) can be rewritten as $\sigma(a,d) > \sigma(a,b) + \sigma(b,d)$. In this case, let $s(a) = \sigma(a,b) + \varepsilon$, $s(d) = \sigma(a,d) - \sigma(a,b) - \varepsilon$, with $\varepsilon > 0$, $s(b) = 0$. In an economy with three atoms of equal mass at a, b, d, the matching $\{a,d\}, \{b\}$ and the payoff s constitute an equilibrium. Indeed, if $\{a,b\}$ forms, the most that a can obtain is $\sigma(a,b)$ which is less than $s(a)$. If d and b match, the most that d can obtain is $\sigma(b,d)$ which is less than $s(d)$ as long as $\varepsilon < \sigma(a,d) - \sigma(a,b) - \sigma(b,d)$. Therefore, if σ violates the triangle inequality, Condition P is also violated. Hence we must have[23]

$$\begin{aligned} \sigma(a,b) + \sigma(b,d) \geq \sigma(a,d) \\ \sigma(a,c) + \sigma(c,d) \geq \sigma(a,d) \end{aligned} \tag{5.2}$$

[23] Clearly, $b \vee c \succsim b \succsim b \wedge c$ and $b \vee c \succsim c \succsim b \wedge c$.

Note that we cannot apply this argument to show that $\sigma(b,c) + \sigma(c,d) \geq \sigma(b,d)$ since it is not known that $b \vee d \succ c \succ b \wedge d$.

Suppose now that the triangle inequalities (5.2) are satisfied but that Condition PT is violated. We show that there exists a payoff $s \in S^P(a,d) \times S^P(b,c)$, such that the following holds

$$\begin{cases} s(a) + s(b) > \sigma(a,b) \\ s(a) + s(c) > \sigma(a,c) \\ s(d) + s(b) > \sigma(d,b) \\ s(d) + s(c) > \sigma(d,c) \end{cases} \tag{5.3}$$

Note that since $s \in S^P(a,d) \times S^P(b,c)$, (5.3) can be rewritten as

$$\sigma(a,b) - s(b) < s(a) < \sigma(a,d) + \sigma(b,c) - \sigma(d,c) - s(b) \tag{5.4}$$

$$\sigma(a,c) - \sigma(b,c) + s(b) < s(a) < \sigma(a,d) - \sigma(d,b) + s(b) \tag{5.5}$$

The bounds are consistent by (5.1). Note that by the triangle inequality, $\sigma(a,d) - \sigma(d,b) \leq \sigma(a,b)$, hence that the upper bound in (5.5) is equal to the lower bound in (5.4) when the payoff to b is equal to $s^* = \frac{1}{2}[\sigma(a,b) + \sigma(d,b) - \sigma(a,d)]$. By (5.2), $s^* \geq 0$. Let ε be a small positive number and let $s(b) = s^* + \varepsilon$. Clearly, $s(b) > 0$ since $s^* \geq 0$.

We show that $\sigma(b,c) + \sigma(c,d) \geq \sigma(b,d)$ (recall the discussion after (5.2)). By (5.2), $\sigma(c,d) \geq \sigma(a,d) - \sigma(a,c)$, therefore, $\sigma(b,c) + \sigma(c,d) \geq \sigma(b,c) + \sigma(a,d) - \sigma(a,c)$. Since PT is violated, $\sigma(b,c) + \sigma(a,d) - \sigma(a,c) > \sigma(b,d)$ which proves that

$$\sigma(b,c) + \sigma(c,d) > \sigma(b,d). \tag{5.6}$$

Since PT is violated,

$$\sigma(a,d) + \sigma(b,c) > \sigma(a,b) + \sigma(c,d) \tag{5.7}$$

Hence, after adding (5.6) and (5.7), we obtain $\sigma(a,d) + 2\sigma(b,c) > \sigma(a,b) + \sigma(b,d)$ which proves that $s^* < \sigma(b,c)$ and therefore that $s(b) < \sigma(b,c)$ for $\varepsilon \in (0, \sigma(b,c) - s^*)$.

The result follows if we show that there exists a feasible payoff to a satisfying $\sigma(a,b) - s(b) < s(a) < \sigma(a,d) - \sigma(d,b) + s(b)$. Note that $\sigma(a,d) - \sigma(d,b) \leq \sigma(a,d)$ (trivially) and that $\sigma(a,d) - \sigma(d,b) + \sigma(b,c) > 0$ by (5.1). Therefore there exists ε' small enough and $s(b) \in (0, \sigma(b,c))$ such that by taking $s(a) = \sigma(a,d) - \sigma(d,b) + s(b) - \varepsilon'$, (5.4) and (5.5) are satisfied.

This shows that there exists $s \in S^P(a,d) \times S^P(b,c)$ satisfying (5.3) and therefore that Condition P is violated. Hence, Condition PT is equivalent to Condition P. ■

5.3. Proof of Proposition 3.2

Since UMR holds, the surplus $\sigma(a, a')$ is decreasing in a', if $a' < a$ and is increasing in a', if $a' > a$. We prove the result by a series of claims.

1. The equilibrium surplus is increasing in a. Suppose not. Let a match with b and without loss of generality assume that $b < a$. Let $\hat{a} > a$ and suppose that $s(a) > s(\hat{a})$ (where $s(a) = v(a) - \underline{u}(a)$). Since $s(a) + s(b) = \sigma(a, b)$, UMR implies that $\sigma(\hat{a}, b) > \sigma(a, b)$ hence that $\sigma(\hat{a}, b) - s(b) > s(a) > s(\hat{a})$. This is a contradiction since $\hat{a}$ and b could deviate.

2. Each a matches with at most one type smaller than itself. Suppose instead there are two such types b and c, with $c < b < a$. Then $\sigma(a, b) - s(b) = \sigma(a, c) - s(c)$. By UMR, $\sigma(a, b) < \sigma(a, c)$. Hence, $s(b) < s(c)$ which contradicts the first result. Therefore, a matches with at most one other ability level $b < a$. This proves the existence of a matching map m as in the Proposition.

3. Suppose that $m(\overline{a}) < a_m$. By positive matching, types in $[a_m, \overline{a}]$ match with types less than $m(\overline{a})$. But this violates measure consistency, since the measure of types less than $m(\overline{a})$ is less than $\frac{1}{2}$, while the measure of $[a_m, \overline{a}]$ is $\frac{1}{2}$.

4. Suppose that $m(\overline{a}) > a_m$; let $a_1 = m(\overline{a})$. Measure consistency then implies that there is $\delta > 0$ such that for $a > a_1$, $m(a) - \underline{a} > \delta$. Thus, there exists $a_m < a_2 < a_1$ with $m(a_2) > \underline{a}$. Now, since $(a_2, m(a_2))$ is an equilibrium match,
$$s(a_2) = \sigma(a_2, m(a_2)) - s(m(a_2)) \geq \sigma(a_2, \underline{a}) - s(\underline{a})$$
By UMR, $\sigma(a_2, m(a_2)) < \sigma(a_2, \underline{a})$. Therefore, $s(m(a_2)) < s(\underline{a})$. This contradicts the fact that the equilibrium surplus is increasing in ability.

5. Hence, $m(\overline{a}) = a_m$. Consider any $a \in (a_m, \overline{a})$. If $T(\bar{a}) - T(a) \neq T(a_m) - T(m(a))$, we obtain a contradiction by replicating the arguments in 3 or 4. ■

5.4. Proof that there is segregation by ability in the example of section 3.2.1

Crucial to the proof of segregation is the fact that the payoff function is increasing in wealth and in ability.

Lemma 5.1. $(w,a) > (w',a') \Longrightarrow v((w,a)) \geq v((w',a'))$

Suppose instead that $v((w,a)) < v((w',a'))$. Since $v((w,a)) \geq 0$, it must be that when $(\hat{w},\hat{a})$ is a match of (w',a'), $\hat{w}+w' \geq \theta k$. Therefore, $v((w,a)) + v(((\hat{w},\hat{a}))) < \hat{a}a - k$ and $w+\hat{w} \geq \theta k$. But then, it is possible for the coalition $\{a,\hat{a}\}$ to deviate. A first consequence of Lemma 5.1 is that the payoff function $v(\cdot,\cdot)$ is differentiable almost everywhere.

A second consequence of lemma 5.1 is that an agent (w,a) prefers (weakly) to be matched with an agent of ability $(\theta k - w, a')$ rather than an agent of ability (w',a') : wealth levels greater than $\theta k - w$ only reduce the residual payoff to (w,a) but do not increase the total surplus.

Lemma 5.2. *There is no loss of generality in supposing that whenever an agent of type (w,a) is not segregated, he matches with an agent of wealth $\theta k - w$.*

A second fact crucial for obtaining the result is that "agents of the same wealth" are positively matched in ability levels.

Lemma 5.3. *Let (w,a) and (w,a') be two types that have same wealth component but different ability levels $a > a'$. If (w,a) matches with $(\hat{w},\hat{a})$, then (w,a') must match with $(\hat{w}',\hat{a}')$ where $\hat{a}' \geq \hat{a}$.*

Proof. Let w' be such that $w'+w \geq \theta k$. We show that $\arg\max_b ab - v((w',b))$ is increasing in a (in the strong set order). Let $b(a)$ be an element of the set of maximizers. By a revealed preference argument, for any a and $\hat{a}$,

$$\begin{cases} ab(a) - v((w',b(a))) \geq ab(\hat{a}) - v((w',b(\hat{a}))) \\ \hat{a}b(\hat{a}) - v((w',b(\hat{a}))) \geq \hat{a}b(a) - v((w',b(a))) \end{cases} \tag{5.8}$$

hence, subtracting one inequality from the other, we obtain,

$$(a-\hat{a})\,b(a) \geq (a-\hat{a})\,b(\hat{a})$$

If $a > \hat{a}$, it follows that $b(a) \geq b(\hat{a})$. The result now follows from Lemma 5.2. A consequence of Lemma 5.3 is that if $B(w,a)$ denotes the set of types who prefer to match with (w,a), that whenever $a > a'$, $\inf B(w,a) \geq \sup B(w,a')$. ■

Finally, we establish that $v((w,a))$ is continuous in a.

Lemma 5.4. *For a fixed w, $v((w,a))$ is continuous in a.*

Proof. When $w < \theta k - \overline{w}$, the equilibrium payoff of types (w, a) is equal to zero, which is obviously continuous in a. Assume that $w \geq \theta k - \overline{w}$ and that there exists a such that $\lim_{\varepsilon \downarrow 0} (v((w, a+\varepsilon)) - v((w, a-\varepsilon))) = \delta > 0$. By Lemma 5.1 , and $v \geq 0$, $\lim_{\varepsilon \downarrow 0} v((w, a+\varepsilon)) > 0$. Hence, each type $(w, a+\varepsilon)$ matches with some type $(w(\varepsilon), b(\varepsilon))$ and generate a positive surplus. Now, for each $\varepsilon > 0$, $(w(\varepsilon), b(\varepsilon))$ must prefer to match with $(w, a+\varepsilon)$ rather than with (w, a), i.e.,

$$b(\varepsilon)(a+\varepsilon) - v((w, a+\varepsilon)) \geq b(\varepsilon) a - v((w, a)).$$

This implies that $b(\varepsilon) \geq \frac{\delta}{\varepsilon}$, which is impossible for ε small enough since $b(\varepsilon) \leq \overline{a}$. ■

5.4.1. Case 1 : $\overline{w} > \theta k$

Note that the feasible matches for agents with zero wealth are with those agents with wealth greater than θk. From Lemma 5.2, agents with wealth greater than θk prefer weakly to match with agents with zero wealth.

For a fixed type $(0, a)$, the types $(w, \hat{a})$ who prefer to match with $(0, a)$ satisfy:

$$\begin{aligned} w &\geq \theta k \\ \hat{a} &\leq \frac{v((0,b)) - v((0,a))}{b-a} \text{ if } b > a \\ \hat{a} &\geq \frac{v((0,a)) - v((0,b))}{a-b} \text{ if } b < a. \end{aligned} \tag{5.9}$$

Let $B(0, a)$ be this set. If a is a point of differentiability of $v((0, \cdot))$ and if $a \in (\underline{a}, \overline{a})$,(5.9) is equivalent to $\hat{a} = v_2((0, a))$ (take the limit of the right hand sides of the inequalities as $b \to a$). In particular, if $\underline{a} > v_2((0, a))$, $B(0, a) = \emptyset$ while if $\underline{a} < v_2((0, a)) < \overline{a}$, $B(0, a) = \{(w, v_2((0, a))) : w \geq \theta k\}$.

If for all a, $B(0, a) = \emptyset$, types $(0, a)$ have an equilibrium payoff of zero. But then, $B(0, \overline{a}) = [\theta k, \overline{w}] \times [\underline{a}, \overline{a}]$ which contradicts the assumption that $B(0, \overline{a}) = \emptyset$.

Consider a for which $B(0, a) \neq \emptyset$. Consider a small neighborhood N around $(0, a)$; we can choose this neighborhood such that the Lebesgue measure of N is m. By continuity of v, it follows that the Lebesgue measure of the set $\bigcup_{a' \in N} B(0, a')$ is greater than $m \times (\overline{w} - \theta k)$. This contradicts measure consistency unless each type in $B(0, a)$ is indifferent between matching with $(0, a)$ and matching with himself, i.e., unless $v((0, a))$ is the segregation payoff $\frac{a^2 - k}{2}$. But this implies that $B(0, a) = [\theta k, \overline{w}] \times \{a\}$.

From the differentiability and continuity of v, it follows that for all a, $v((0,a)) = \frac{a^2-k}{2}$. Since $v((w,a))$ is increasing in w, it follows that for all (w,a), $v((w,a)) = \frac{a^2-k}{2}$. The matching in wealth is as in the text.

5.4.2. Case 2: $\overline{w} < \theta k$.

In this case, agents with wealth less than $\theta k - \overline{w}$ cannot find a match with gains of trade. Let $\underline{w} = \theta k - \overline{w}$ and note that agents with types $(\underline{w}, a)$ can feasibly match with types $(\overline{w}, b)$ only. The reasoning of Case 1 can be applied here, but we need to show directly that $v_2((\underline{w}, a)) = a$. (Observe that since $v((0,a))$ is continuous in a and has an increasing derivative, $v((\underline{w}, a))$ is a convex function of a.)

Note that by Lemma 5.3, if $(\overline{w}, \underline{a})$ matches with $(\underline{w}, \hat{a})$, where $\hat{a} > \underline{a}$, types in $\{\underline{w}\} \times [\underline{a}, \hat{a})$ do not match and have a payoff of zero. Because v is convex, measure consistency implies in turn that there exists $a^* < \overline{a}$ such that $a^* = v_2((0, \overline{a}))$ [24] (where v_2 denotes here the left derivative). But then, $B(\underline{w}, \overline{a}) = \{\overline{w}\} \times [a^*, \overline{a}]$. Taking a small neighborhood around $(\underline{w}, \overline{a})$, we obtain a contradiction like in Case 1. Hence, $(\overline{w}, \underline{a})$ matches with $(\underline{w}, \underline{a})$. This is possible only if $v_2((0,a)) = a$ for almost all a. Suppose now that $(\underline{w}, \underline{a})$ matches with $(\overline{w}, \hat{a})$, where $\hat{a} > \underline{a}$. By the same argument as above, measure consistency and convexity of v imply that types in $\{\underline{w}\} \times (\hat{a}, \overline{a}]$ do not match. But then, these types have a zero payoff and this is a contradiction since, in particular, $(\overline{w}, \overline{a})$ would then prefer to match with $(\underline{w}, \overline{a})$ than $(\underline{w}, \overline{a} + \underline{a} - \hat{a})$. Hence, $(\overline{w}, \underline{a})$ matches with $(\underline{w}, \underline{a})$. Measure consistency and continuity of v imply segregation by ability. The payoff structure in the text is possible but, in general, we cannot ensure that the equilibrium payoff is continuous in w (we know however that it is differentiable in w almost everywhere).

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[24] Precisely, $a^* = \overline{a} + \underline{a} - \hat{a}$.

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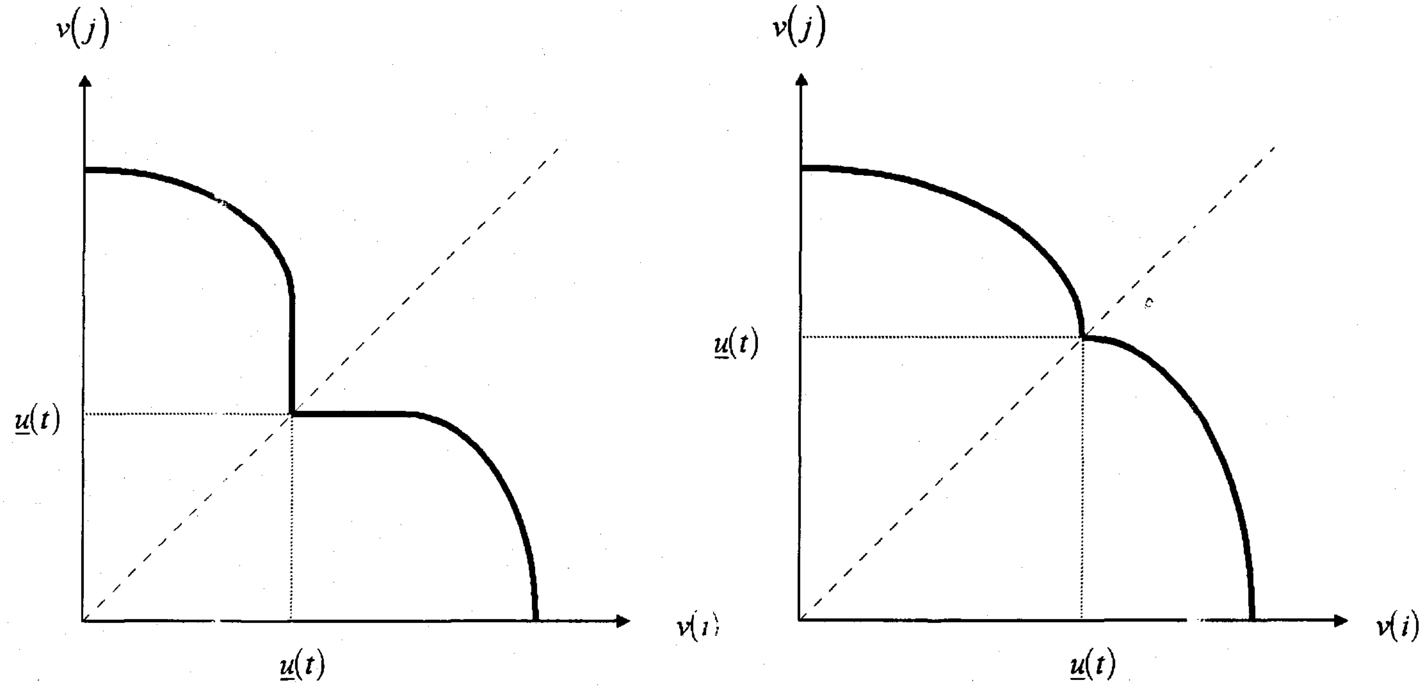
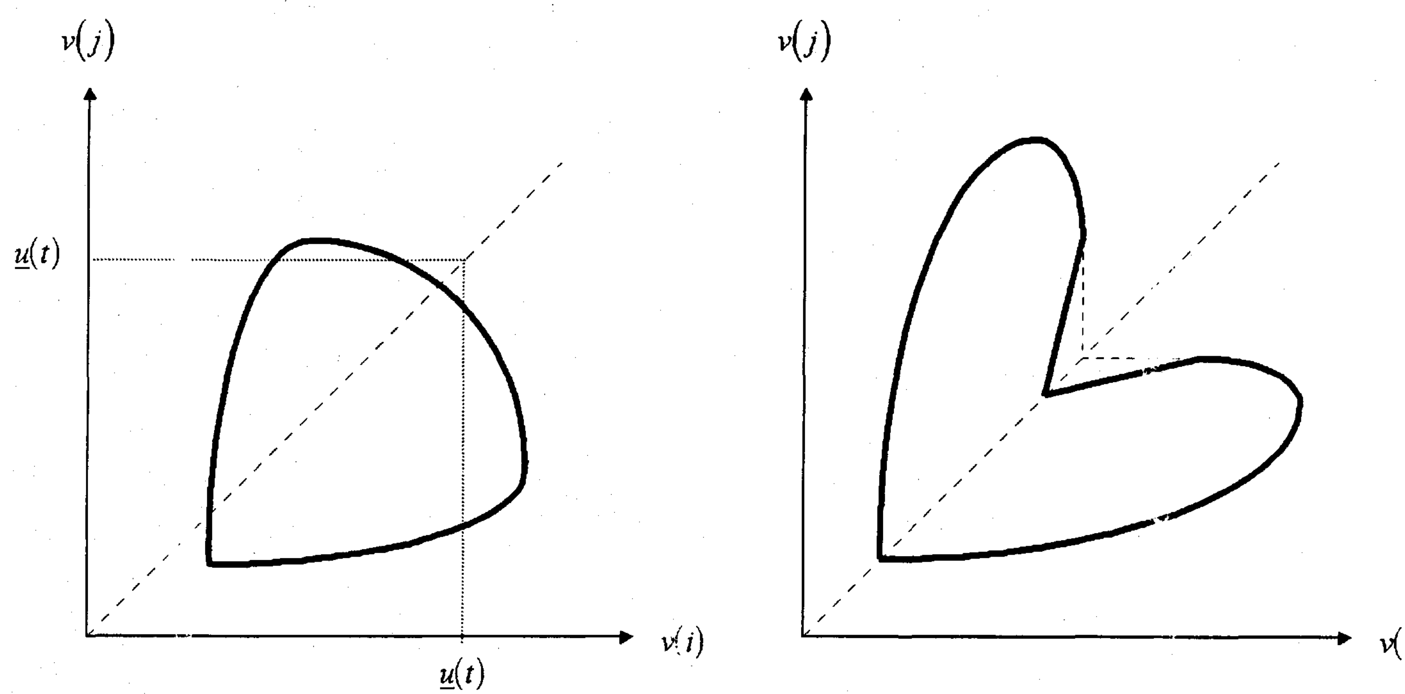


(a) (b)

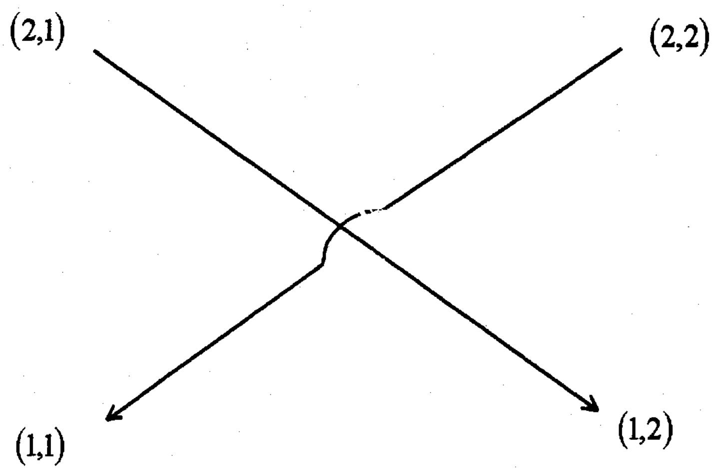
$V(i, j)$ is comprehensive



(c) (d)

$V(i, j)$ is not comprehensive

Figure 1



(a)

$$(2,1) \vee (1,2) = (2,2) \qquad (2,1) \wedge (1,2) = (1,1)$$
$$(2,2) \vee (1,1) = (2,2) \qquad (2,2) \wedge (1,1) = (1,1)$$

PAM is satisfied if $\succeq$ is the vector order

(b)

$$(2,1) \vee (1,2) = (2,1) \qquad (2,1) \wedge (1,2) = (1,2)$$
$$(2,2) \vee (1,1) = (2,2) \qquad (2,2) \wedge (1,1) = (1,1)$$

PAM is not satisfied if $\succeq$ is the lexicographic order

Figure 2

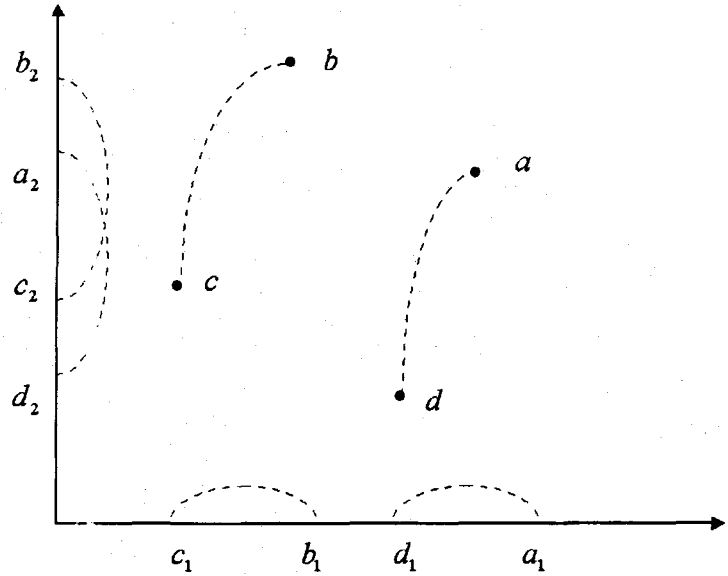


Figure 3a

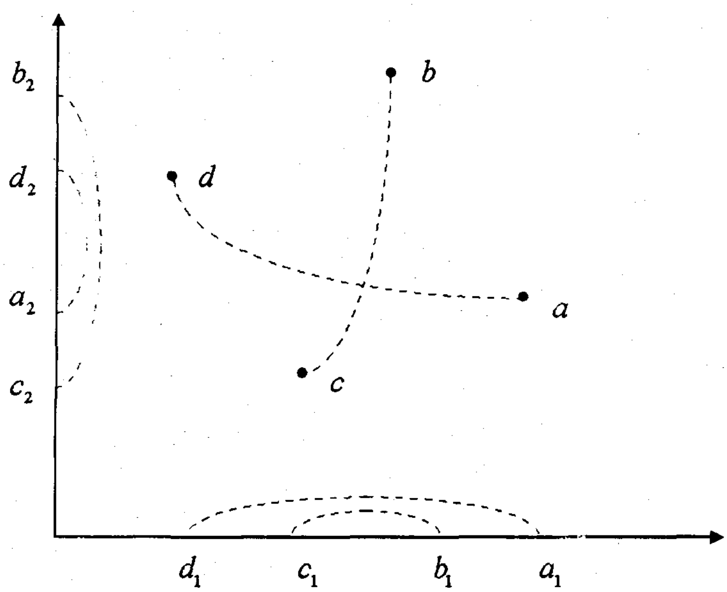


Figure 3b

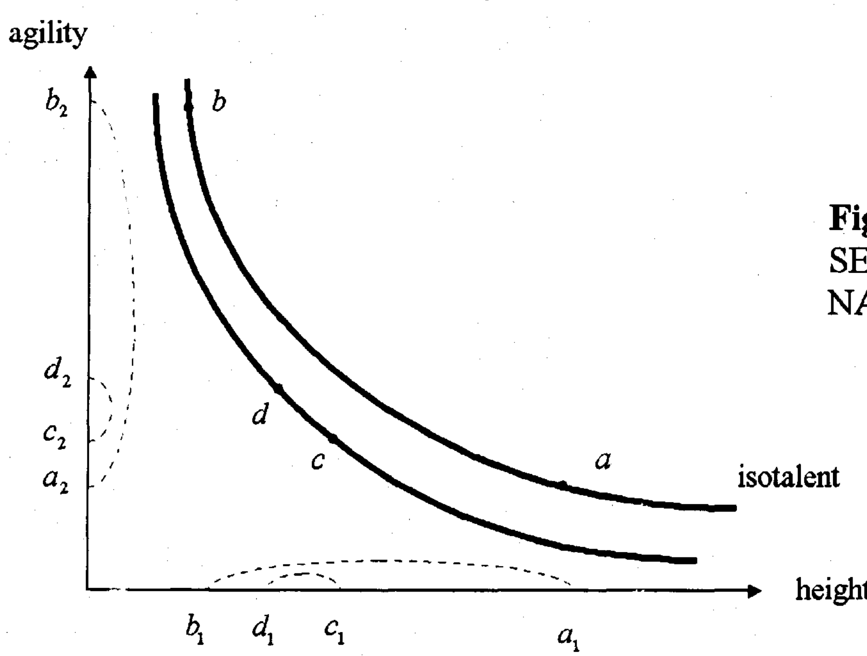


Figure 3c
SEG in talent
NAM in agility and in height

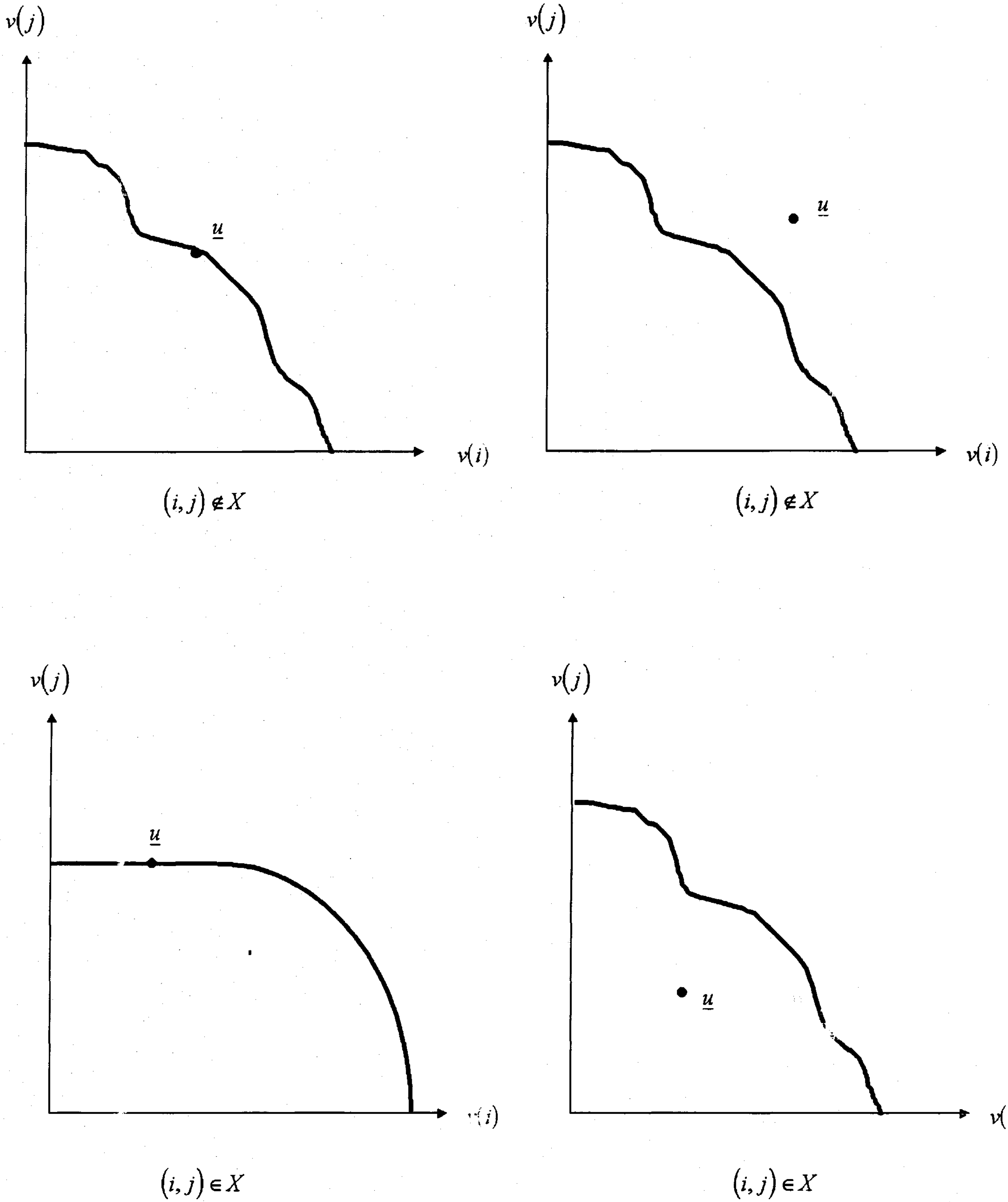


Figure 4

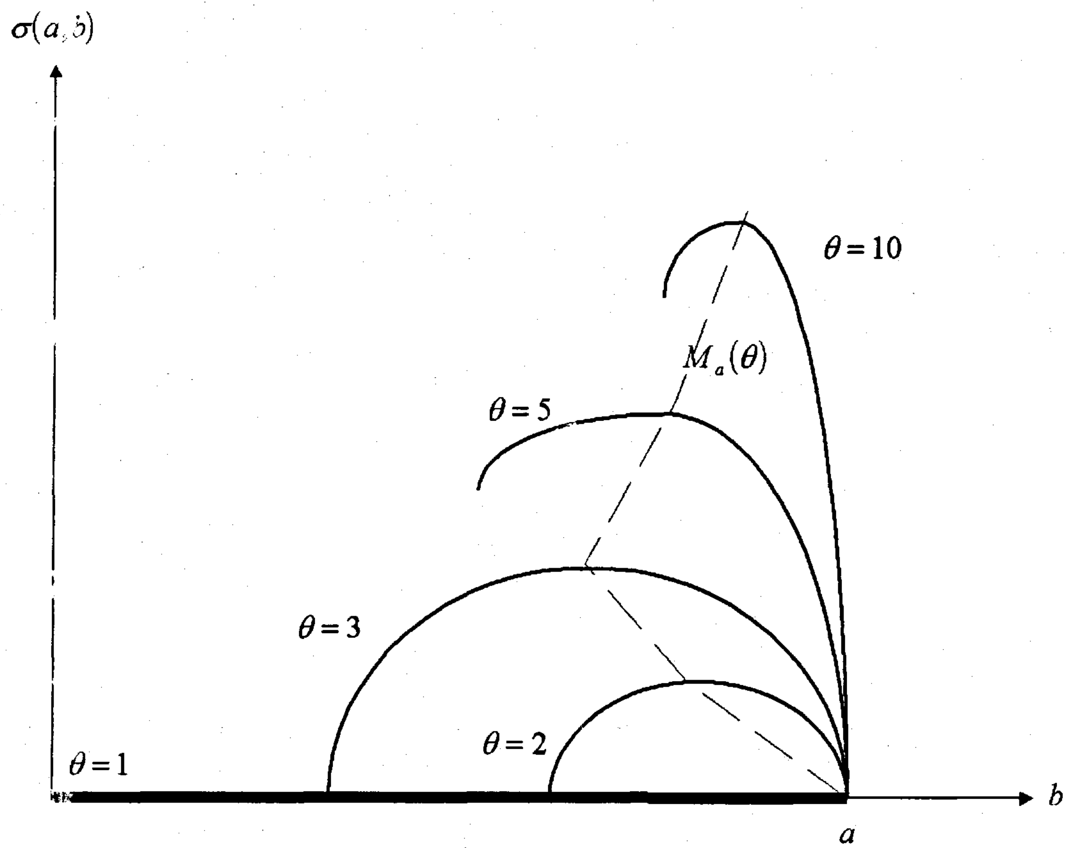


Figure 5

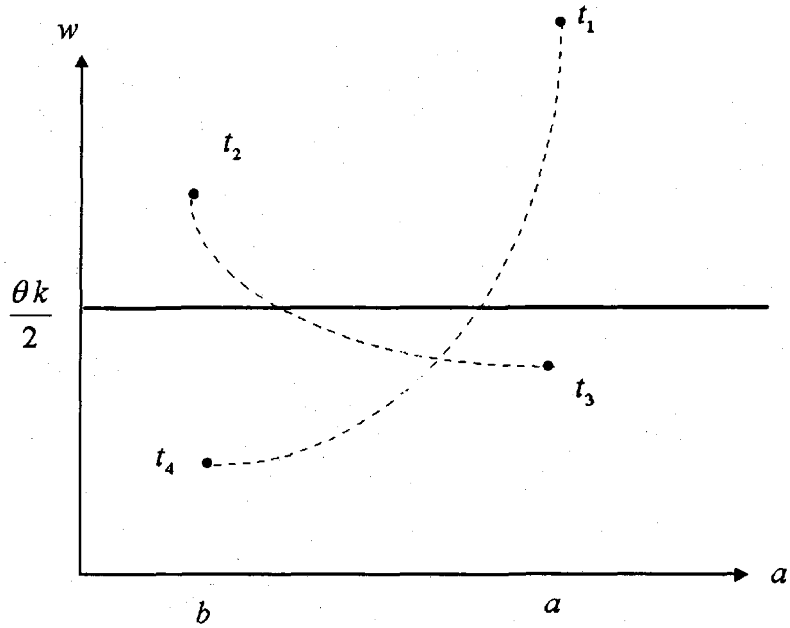


Figure 6a

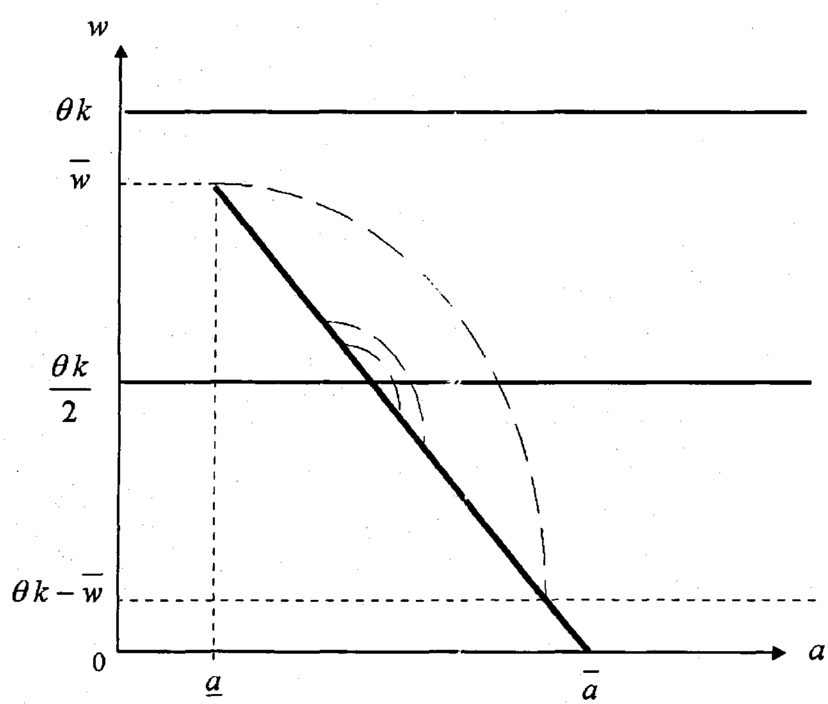


Figure 6b

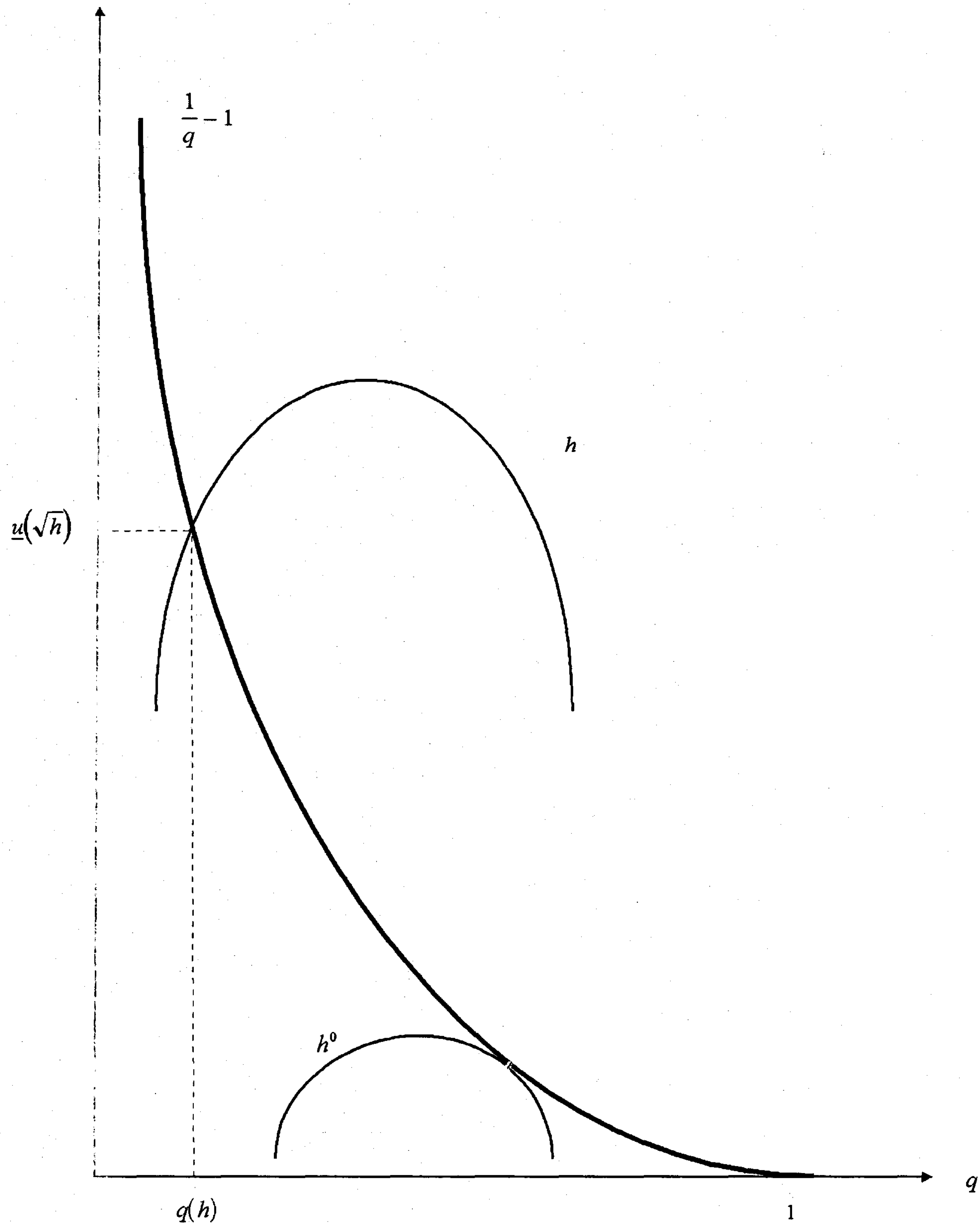


Figure 7

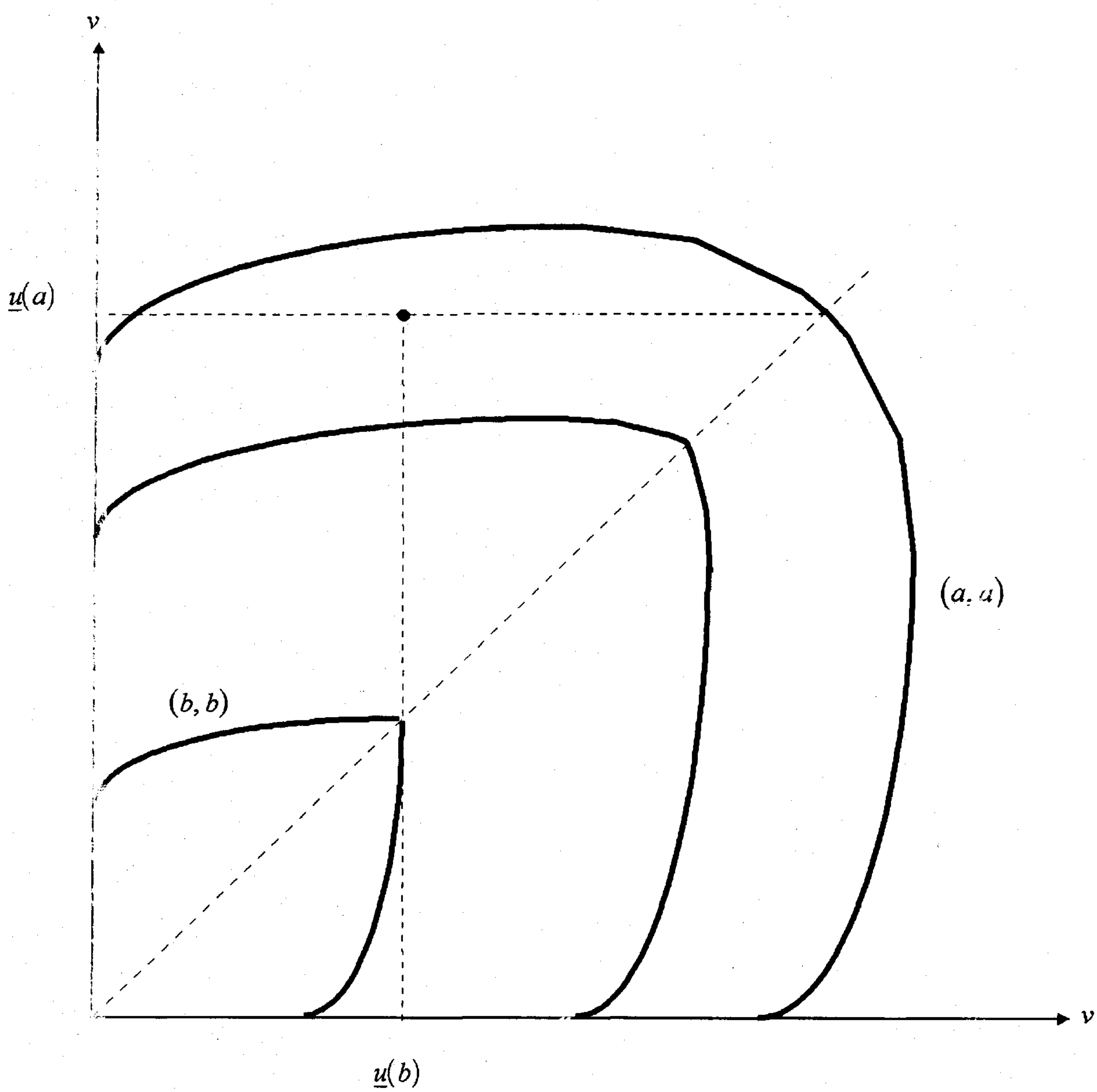


Figure 8

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