Limited Arbitrage is Necessary and Sufficient for the Existence of a Competitive Equilibrium

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Abstract

A condition of *limited arbitrage* is defined on the endowments and the preferences of the traders in an Arrow-Debreu economy. It bounds the diversity of the traders in the economy, and the gains from trade which they can afford from initial endowments. Theorem 1 shows that *limited arbitrage* is necessary and sufficient for the existence of a competitive equilibrium, when consumption sets are either positive orthants or the whole euclidean space. The results apply therefore to market economies with or without bounds on short sales. Theorem 6 establishes that an Arrow - Debreu economy has a competitive equilibrium if and only if every subeconomy with $N+1$ traders does, where N is the number of commodities. Limited arbitrage has been shown elsewhere to be equivalent to the contractibility of spaces of preferences, and therefore, by the results of Chichilnisky and Heal [12], to be necessary and sufficient for the existence of social choice rules defined on individual preferences over allocations, rules which are continuous, anonymous and respect unanimity.

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1 Introduction

In a world with finite resources, there will generally be conflicting opinions about how the resources should be allocated among different individuals, leading to the classical resource allocation problem.

One widely used solution is provided by markets. A *market solution* assigns one commodity vector $x_i \in R^N$ to each individual, and distributes the available resources among the H traders in a way which is individually optimal and which clears all the markets, $\sum x_i = \Omega$. When markets are competitive as in the Arrow and Debreu specification [4], then a market allocation is called a *competitive equilibrium* and is Pareto efficient under classical assumptions, Arrow [1], Debreu [23]. Efficiency is a major virtue of such market allocations, and is principally what makes them desirable. The efficiency of a competitive equilibrium is quite general. For example, it does not depend on the concavity of preferences nor on the specification of the consumption or the production sets.

In order to allocate resources following market forces, a necessary precondition is the existence of a competitive equilibrium. The classical theorems of Arrow and Debreu [4] and of McKenzie [33], [35] established formal sufficient condition for the existence of a competitive equilibrium. This led to a large and productive literature dedicated to extending and refining the conditions under which a competitive equilibrium exists, reviewed for example in Arrow and Hahn [3] and more recently in McKenzie [34].

The conditions which are sufficient for the existence of a competitive equilibrium can be restrictive, such as for instance the requirement that each trader should own a strictly positive amount of each good in the economy. Arrow and Hahn have described this condition as "unrealistic" [3], Chapter 4, page 80. Yet without this or similar conditions an otherwise well behaved economy, with continuous and concave preferences, positive endowments and positive orthants as consumption sets, may fail to have a competitive equilibrium. The simplest example of this failure was provided in Arrow and Hahn [3], Chapter 4, page 80, for a two good, two person pure exchange economy, but the problem is quite general applying to economies with production and with any number of individuals and of goods.

Other concepts of market equilibrium have been considered to solve the problems of non-existence of a competitive equilibrium, such as that of a *quasi-equilibrium*, introduced in Debreu [21], or of a *compensated equilibrium* studied in Arrow and Hahn [3]. These are related but different concepts, which differ from the competitive equilibrium in that they relax one of main conditions: either the condition of utility maximization by the agents, or else the market clearing condition. In a quasi-equilibrium allocation the traders minimize costs rather than maximizing utility. At a compensated equilibrium allocation there may be excess supply in some markets. From a practical point of view, these concepts of equilibrium have the advantage that their existence is ensured under general conditions. However, from the point of view of resource allocation they are less satisfactory: quasi-equilibrium or compensated equilibrium allocations are not generally Pareto efficient. This limits their value from the point of view of welfare. For this reason in this paper we concentrate on competitive equilibrium allocations.

Classical formalizations of markets assume that the consumption sets of the individuals are bounded below, an assumption motivated by the inability of humans to provide more than a fixed number of hours of labor per day. However, it can be argued that such restrictions on trading should not be imposed exogenously but should, instead, be derived from the individuals' characteristics and behavior, for example from the characteristics of endowments and preferences. For otherwise, the market equilibrium will inevitably depend on the chosen bounds, and there is often little reason to choose one bound over another. Furthermore, the trading of financial instruments does not involve naturally exogenous limits on the quantities of the assets traded. Short trading, for example, involves contracts to deliver assets in quantities which may exceed initial endowments. Actually, in the Arrow-Debreu [4] specification of a pure exchange market, the demand function of a trader may involve contracts to deliver amounts in excess of his/her initial endowments, or even in excess of the total endowments of the economy. The same is true in Arrow-Debreu economies with production where the contracts in equilibrium typically exceed the initial endowments of the economy. Despite all this, the Arrow-Debreu formulation limits the quantities to be traded in each market by bounding them below by an exogenously chosen bound. With such bounds, individual consumption sets are then positive orthants, or translates of positive orthants. This has the technical advantage that the demand function is always well defined when preferences are concave and all prices are positive. In such cases, the consumption vectors which are within the budget sets of a trader define a compact set. This compactness ensures the existence of a demand vector at all positive prices, which maximizes utility within the budget set, the existence of which is questionable without compactness.

Despite the technical advantages of defining trading bounds exogenously, there are at least two substantial reasons for removing these bounds, as already mentioned. One is that any exogenously given limit on the quantities traded is artificial and in many cases suffices to determine the equilibrium. For example, when preferences are linear, such bounds determine by themselves the market equilibrium. Choosing the bounds is then tantamount to choosing the market allocation exogenously. This is not satisfactory, since the aim of the theory is to explain prices endogenously, from the functioning of markets. The second reason is that in the trading of financial assets, physical limitations are less natural than those imposed by the scarcity of human time. The literature on financial markets therefore allows any amount of short trading quite generally, examples are Hart [28], Werner [42] and Chichilnisky and Heal [17]. For these reasons, here we shall include such cases along with the standard cases. The consumption sets of the traders in our Arrow-Debreu markets will be either positive orthants as in the classical theory, or the whole Euclidean space. The latter case thus allows any short trading, without exogenous bounds. In both cases the condition of *limited arbitrage* is shown to be necessary and sufficient for the existence of a competitive equilibrium.

The condition of *limited arbitrage* has somewhat different economic interpretations when

any short trade is allowed than when they are not. But in both cases it involves the non-empty intersection of "dual cones" defined from the traders' preferences at their initial endowments, and it has the same mathematical formulation. From the economic viewpoint, as already mentioned, limited arbitrage limits the diversity of the traders and their potential gains from trade. It seems useful to mention another interpretation of limited arbitrage, from the mathematical viewpoint, an interpretation which is not necessary for the results presented in this paper, but which provides additional motivation. The non-empty intersection of the cones is equivalent to a *contractibility* condition on the spaces of preferences (Appendix, Theorem 7): this is a topological condition which ensures that all traders can be continuously deformed into one. Contractibility is therefore a form of similarity of preferences, this time in a topological formulation, as was pointed out in Heal [29]. The connection between non-empty intersection of asymptotic cones, and the contractibility property of spaces of preferences is what allows us to connect the existence of a competitive equilibrium with the existence of social choice rules: this is because is has been already established that the contractibility of the space of preferences is necessary and sufficient for the existence of social choice rules, Chichilnisky and Heal, [17].

1.1 Arbitrage and Equilibrium

Welfare economics and finance have each evolved their own equilibrium concepts. In welfare economics, this is the competitive equilibrium: in finance, it is the absence of arbitrage opportunities. These concepts emerged independently and were intially seen as quite distinct. However, in the 1980s researchers in both areas began to investigate the connections between the two concepts. The first explicit study of the arbitrage-equilibrium relationship was Kreps [31]: subsequently Werner [42], and Nielsen [37] developed this theme further. In fact Hart's 1974 paper on securities markets [28] contained many of the elements needed to understand the arbitrage-equilibrium relationship. His approach was developed by Hammond [27] and Page [38]. Green covered related topics in a temporary equilibrium framework [26].

The absence of arbitrage opportunities (a no-arbitrage condition) is clearly necessary for the existence of a competitive equilibrium. If arbitrage opportunities remained, the traders could not be maximizing their utility at the equilibrium allocations. The condition of no-arbitrage is an equilibrium condition, one which must be satisfied at an equilibrium allocation. This condition does not help to evaluate the economy's ability to reach a competitive equilibrium, which is the problem we study here, in the sense that only after an equilibrium allocation is found one can verify this condition. It remains therefore to give conditions on the *primitives* of the economy, such as the traders' endowments and preferences, which are both necessary and sufficient for the existence of a competitive equilibrium, and which hold both for economies with and without bounds on short sales. This is accomplished in this paper: we provide a geometric condition on initial endowments and preferences - limited arbitrage - which is necessary and sufficient for the existence of a competitive equilibrium. The condition is valid with and without bounds on short sales, and has an additional useful property: it has been linked with the existence of social choice rules in Chichilnisky [18], thus providing a clear connection between equilibrium theory and social choice theory.

Results in this direction are in Werner [42], and in Chichilnisky and Heal [17], who obtained sufficient conditions for existence of a market equilibrium which are related to no-arbitrage. Their conditions are however too strong to be necessary in general; these works are compared with the results of this paper in Section 5.2 below. Chichilnisky and Heal deal with finite and infinite dimensional economies, with or without bounds on short sales. Werner's work is instead only finite dimensional, and is restricted to economies without bounds on short sales, for otherwise his conditions are redundant. However, economies with bounds on short sales can behave very differently from those without such bounds. One important difference between the two types of economies is that the Pareto frontier of an economy without bounds on short sales, which describes the Pareto efficient utility values for the traders, may not be a bounded closed set. Instead, with bounds on short sales the Pareto frontier is always closed and bounded, Arrow and Hahn [3]. In other words, without bounds on short sales it may be possible to obtain infinite utility even if the total resources of the economy are bounded. Typically this failure prevents the existence of a competitive equilibrium. There exist many examples of economies where this problem arises, some of which are discussed below in Section 3.1, see also Chichilnisky and Heal [17]. In Theorem 1 we establish that our limited arbitrage conditions ensure *inter alia* that the Pareto frontier is closed and bounded. A second difference between the two types of economies is that a pseudo equilibrium is always a competitive equilibrium in an economy without bounds on short sales, but this is typically not true in economies with bounds on short sales. In the latter case, the boundary behavior of the economies is very important to decide the existence of a competitive equilibrium, as discussed in Sections 3.2 and 5.1 below.

Our paper extends this literature. We work with a weaker condition than no arbitrage, the condition of *limited arbitrage*, a condition introduced in the context of social choice theory by Chichilnisky [18]. This is a condition on the primitives of the economy: the initial endowments and preferences of the traders. Limited arbitrage limits, but does not rule out, arbitrage opportunities. We prove that a competitive equilibrium exists if and only if arbitrage opportunities at the initial endowments are, in a precise sense, limited. The equilibrium concepts used in economics and finance are therefore fully equivalent in the context of limited arbitrage. In our context, therefore, this equivalence contrasts with a conjecture of Dybvig and Ross [25] to the effect that "absence of arbitrage is more primitive than equilibrium, since only relatively few rational agents are needed to bid away arbitrage opportunities".

1.2 The Existence of a Market Equilibrium

Market allocations have always been considered a practical solution for the resource allocation problem. One reason for this is that markets are viewed as having equilibria very generally, while other forms of allocation, such as that provided by social choice theory, has always stressed paradoxes and non-existence results. Kenneth Arrow's work, which developed fundamental insights into the two theories, appeared to provide fuel for this viewpoint. Arrow's impossibility theorem for social choice [2] led to a large literature focusing on the difficulties of finding acceptable social allocations. Instead, Arrow's result on existence of a market equilibrium with Debreu [4] led to more and more general existence theorems which reinforced our view of market equilibrium allocations as being always available. In all fairness, Arrow and Debreu did discuss the problems of non-existence of a competitive equilibrium created for example by the discontinuity of uncompensated demand when some prices are zero (Arrow and Debreu [4], Sections 4 and 5, and Arrow and Hahn [3], Chapter 4, 1) provided examples of standard market economies with no competitive equilibrium. Nevertheless, the results on existence of an equilibrium took precedence in the literature.

1.3 Social Diversity and the Non-Existence of a Competitive Equilibrium

What is interesting about Arrow and Hahn's example [3] of the non-existence of a competitive equilibrium is that it arises due to sharp interpersonal differences between the traders, measured in terms of their endowments and their preferences. Such differences emerge, for example, when some traders have zero endowments of some goods, a situation which Arrow and Hahn find quite realistic ([3], Chapter 4, p. 80). It is notable that the social choice literature has also focused on interpersonal diversity as a reason for the non-existence of social choice rules. Prominent examples are the work of Black [5][6], Pattanaik and Sen [39], and Chichilnisky and Heal [12]. These works offer conditions for resolving social choice problems by limiting the diversity of the individual preferences, which is usually called a "domain restriction" on preferences. Domain restrictions are simply a way of limiting the diversity of individuals. The issue of existence of universal social choice rules - a problem which in its more general form has no solution - is turned into the question: for what societies can the social choice problem be resolved? Or: how much diversity can a society function with?

Black's singlepeakedness condition restricts diversity and solves the problem proposed by Condorcet's [20] paradox of majority voting; Pattanaik and Sen [39] do the same, finding domain restrictions which assure the existence of majority rules satisfying Arrow's axioms of social choice, and Chichilnisky and Heal [12] find domain restrictions which are necessary and sufficient for the existence of social choice rules which satisfy the axioms of [10]. Although these works deal with somewhat different axioms, they all find similar solutions involving the restriction of individuals' diversity.

As already pointed out, different but related restrictions of individual diversity are implicit in the conditions for existence of a competitive equilibrium which developed in order to resolve Arrow's example of non-existence of a competitive equilibrium: Arrow and Debreu's conditions require the endowments of any household being desired, indirectly or directly, by others, so that their incomes cannot fall to zero [4]. Other such conditions include the requirement that preferences should have indifference surfaces which never meet the axis, Debreu [21], a condition which, as we show in Section 5, requires a form of similarity of preferences, McKenzie's *irreducibility* condition [33][34][35], and the *resource relatedness* condition in Arrow and Hahn [3], both of which require explicitly that the agents should desire the goods held by others in the economy, again a requirement of similarity of preferences. These conditions are somewhat different, but they all have the same effect: to restrict the diversity of individuals' preferences and endowments. They are discussed in more detail in Section 5.1. The issue of the universal existence of market equilibrium - a problem which in its more general form has no solution - is turned into the question: for what societies can market equilibrium allocations be found? Or: how much diversity can a society function with?

The aim of this paper is to introduce a restriction on individual diversity, *limited arbi-*

***trage*, and to prove that it is necessary and sufficient to ensure the existence of a competitive equilibrium in Arrow-Debreu exchange economies (Theorem 1). Theorem 6 proves that limited arbitrage only needs to be satisfied for subsets of $N+1$ traders, where N is the number of commodities in the economy: an Arrow-Debreu economy has a competitive equilibrium if and only if every sub economy with $N+1$ traders does. The condition of limited arbitrage has an additional interest: as already mentioned it can be interpreted as a contractibility condition (Theorem 7 in the Appendix), a topological condition which, when applied to spaces of preferences, has been shown to be necessary and sufficient for the existence of social choice rules, Chichilnisky and Heal [12].**

Limited arbitrage is defined on the exogenous parameters of the economy: the individuals' preferences at their initial endowments. This limited arbitrage assumption also links the concept of an equilibrium used in the general equilibrium literature to that of non-arbitrage used in the finance literature, and in this sense it unifies three forms of resource allocation: the allocation provided by a competitive equilibrium, the allocation provided by social choice and the allocation and prices provided by no-arbitrage conditions.

The following sections provide definitions, a formal statement of the theorems, and their proofs. The conclusions summarize the results, and an Appendix provides background results.

2 Definitions and Examples

Consider an Arrow-Debreu pure exchange private market economy E. There are $N \geq 1$ commodities, and the consumption space is X. X is either the positive orthant R^N_+ or all of the Euclidean space R^N. R^N_{++} denotes the interior of R^N_+. For any two vectors $x, y \in R^N$, denote $x \geq y \Leftrightarrow \forall i\ x_i \geq y_i$, $x > y \Leftrightarrow x \geq y$ and for some $i, x_i > y_i$, and $x >> y \Leftrightarrow \forall i$, $x_i > y_i$. The economy E has $H \geq 2$ individuals indexed by $h = 1...H$; each has a non-zero initial endowment in R^N, $\Omega_h \geq 0$, where $\Omega = \sum_{h=1}^{H} \Omega_h >> 0$ is the total endowment of the economy. Some individuals may have zero endowments of some goods. Each individual has a preference ρ_h over private consumption, which is continuous, quasiconcave and monotonically increasing: if $x \geq y$ then $x \succeq_{\rho_h} y$.

We may also consider more general specifications of the consumption set X. For example, we will discuss consumption sets X which are translates of the positive orthant,

$$X = \{v \in R^N : v \geq w \text{ for some } w \in R^N\},$$

and convex sets $X \subset R^N$ which are bounded below and satisfying $x \in X$, $y \geq x \Rightarrow y \in X$, see Chichilnisky and Heal [17].

In the case $X = R^N_+$, we require that if an indifference surface corresponding to a positive consumption bundle x intersects a boundary ray $r \subset \partial X$, all indifference surfaces of other bundles preferred to x intersect r. A boundary ray r in R^N_+ is a set which contains all the positive multiples of a vector $v \in \partial R^N_+ : r = \{w \in R^N_+ : \exists \lambda > 0 \text{ s.t. } w = \lambda v\}$. Included here are therefore all standard preferences used in the literature such as: Cobb-Douglas, CES utilities and all other preferences with the indifference surfaces corresponding to positive consumption contained in the interior of X, $Int(X)$, (Debreu [21]); it includes also linear preferences, piecewise linear preferences, Leontief preferences without substitution, continuous increasing and quasi-concave utilities with indifference surfaces which intersect the boundary of the positive orthant (Arrow and Hahn [3]), as well as all smooth utilities defined on a neighborhood of X which are transversal to its boundary ∂X, Smale [40].

When $X = R^N$ the preferences ρ_h are represented by a smooth (C^2) utility functions $u_h : R^N \to R$ [23], $\exists\ \epsilon, K > 0 : \|Du_h(x)\| > \epsilon$ and $\|D^2u_h(x)\| < K$ for all $x \in R^n$, and the directions of the gradients of each indifference surface which is not bounded below define a closed set[1]. This includes preferences having indifference surfaces which are contained in the interior of a translate of the positive orthant, as well as preferences whose indifference surfaces are not contained in the interior of any translation of the positive orthant, such as for example, linear preferences or preferences which have partially linear indifference surfaces; it includes preferences which are extensions to R^n of Cobb-Douglas or CES utilities defined on a closed subset of the strictly positive orthant, and preferences which may or not be transversal to the boundary of the positive orthant.

A *market economy* is defined by $E = \{X, \Omega_h, \rho_h, h = 1...H\}$, $X = R^N$ or $X = R^N_+$.

The *space of allocations* is $X^H = \{(x_1...x_H) \in R^{NH} : x_h \in X\}$.

The *space of feasible allocations* is $\Upsilon = \{(x_1...x_H) \in X^H : \sum_{h=1}^{H} x_h = \Omega\}$.

A *$k-$ trader sub-economy of E* is an economy consisting of a subset of $k \leq H$ traders in E, each with the endowments and preferences as in E:

$$F = \{X, \rho_h, \Omega_h, h \in J \subset \{1...H\}, cardinality(J) = k\}.$$

Two topological spaces X and Y are *homeomorphic*, denoted $X \approx Y$, if there exists a one to one map $F : X \to Y$ with a continuous inverse $F^{-1} : Y \to X$, i.e. $F^{-1}(F(x)) = x$ $\forall x \in X$.

A *connected topological space* Y is a topological space which cannot be expressed as the union $Y = A \bigcup B$ of two subsets, A and B, each of which is simultaneously open and closed in Y.

This formalizes the notion that any element $y \in Y$ may be connected to any other $z \in Y$ through a continuous path in the space Y.

A topological space Y is called *contractible* when there exists a continuous deformation of the space through itself into one of its points. Formally, Y is contractible when there exists a continuous map

$$F : Y \times [0,1] \to Y, \textit{ and a point } z_0 \in Y \textit{ s.t.}$$
$$\forall z \in Y,\ F(z,0) = z \textit{ and } F(z,1) = z_0.$$

Examples of contractible and non-contractible spaces. If a space X is homeomorphic to a contractible space Y, then X is contractible. All convex sets are contractible spaces, but

[1] A set is bounded below if all its elements are larger than a given vector.

contractible spaces may not be convex, such as a "star shaped" set in R^N, or a non-convex set which is homeomorphic to a convex set see Figure 1.

Figure 1
A contractible star shaped set, a contractible non-convex set, and the "torus" $T = S^1 \times S^1$, which is not contractible.

A disconnected space is not contractible. Any contractible space X is connected, but the converse is not true: the unit circle S^1 is a typical example of a connected space which is not contractible, another is the "torus" in Figure 1, which is the product of the unit circle with itself $T = S^1 \times S^1$. A product of contractible spaces is contractible. A product of spaces which are not contractible, is not contractible. A discussion of the role of contractibility in public decision making is in Heal [29]. It was shown in Chichilnisky [10][11] that the space of all smooth preferences defined on Euclidean choice spaces is not contractible. The space of all linear preferences defined on Euclidean space R^N is not contractible either: this space is the union of the sphere S^{N-1} and the vector $\{0\}$. The space of all smooth preferences defined on Euclidean space for which there exists a one dimensional subspace which intersects transversely all the indifference surfaces of each preference, is contractible (Chichilnisky [9]). It turns out that contractibility is a crucial condition for the existence of social choice maps: for example, if P is any manifold of preferences, then a social choice rule $\Phi : P^k \to P$ satisfying desirable axioms exists $\forall k \geq 2$ if and only if P is contractible (Chichilnisky and Heal [12]). We shall show here that contractibility, which is closely connected to our condition of *limited arbitrage*, is also a crucial condition for the existence of a competitive equilibrium, see also Appendix Theorem 3.

The set of *supports to individually rational efficient resource allocations* of the economy $E = \{X, \Omega_h, \rho_h, h = 1...H\}$ is:

$$S(E) = \{v \in R^N : \exists\ (x_1...x_H) \in \Upsilon \text{ with } x_h \geq_{\rho_h} \Omega_h\ \forall h = 1,...H, \\ \text{and } \forall z_h \in X,\ z_h \geq_{\rho_h} x_h \Rightarrow < v, z - x_h > \geq 0\}. \tag{1}$$

This is the set of prices which support those feasible allocations which all individuals prefer to their initial endowments; the allocations are efficient because they the vectors $z - x_h$ are supported by the same v. An element v of $S(E)$ is called a *support for the allocation* $x = (x_1...x_H) \in \Upsilon$.

Consider now a utility representation u_h for each preference ρ_h, with $u_h(0) = 0$. The *utility possibility set* of the economy E is the set of all possibility utility values which individuals can obtain from feasible allocations:

$$U(E) = \{(U_1...U_H) \in R^H : U_h = u_h(x_h), \text{ where } (x_1...x_H) \in \Upsilon \text{ and} \\ \forall h = 1...H\ u_h : X \to R \text{ represents the preference } \rho_h\}. \tag{2}$$

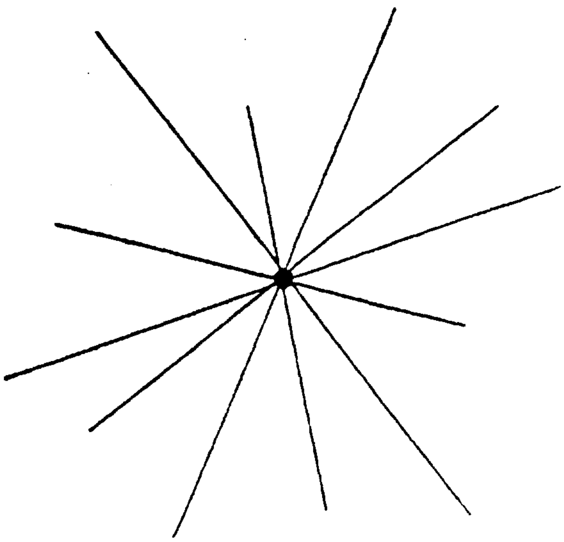
The *Pareto frontier* of the economy E is the subset of vectors in the utility possibility set which are not dominated in the order of R^H :

$$P(E) = \{(U_1...U_H) \in U(E) : \ \sim \exists (W_1...W_H) \in U(E) : \\ \forall h = 1...H,\ W_h \geq U_h \text{ and } W_h > U_h \text{ for some } h \in \{1...H\}\}. \tag{3}$$

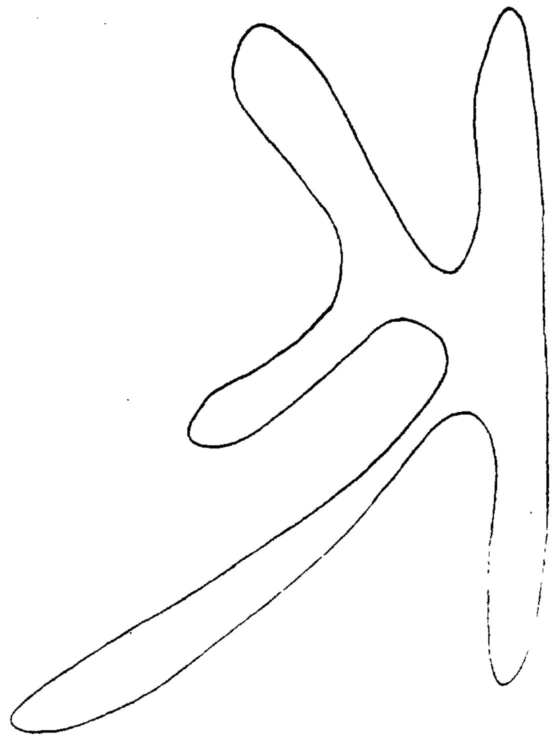
Consider an economy $E = \{X, \rho_h, \Omega_h, h = 1...H\}$.

Figure 1

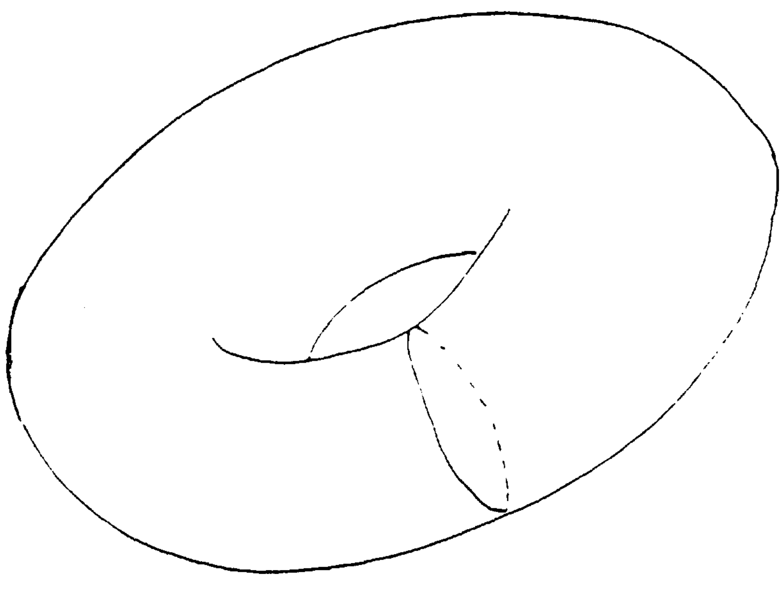
Examples of contractible and non-contractible sets



Star Shaped: contractible



Contractible set homeomorphic to a convex set.



The torus $T = S^1 x S^1$ is not contractible.

2.1 Case 1: $X = R^N$

Consider a preference ρ_h in E and an initial endowment vector $\Omega_h \in X$.

The ***asymptotic preferred cone of $\rho_h \in E$ at the initial endowment Ω_h*** is:

$$A(\rho_h, \Omega_h) = \{y \in R^N : \forall \lambda > 0,\ (\Omega_h + \lambda y) \succ_{\rho_h} \Omega_h, \\ and\ \forall z \in X, \exists \lambda > 0\ s.t.\ (\Omega_h + \lambda y) \succ_{\rho_h} z\} \tag{4}$$

Figure 2
The asymptotic preferred cone $A(\rho_h, \Omega_h)$ of a preference ρ_h over $X = R^2$ translated to the endowment Ω_h

Relationship of $A(\rho_h, \Omega_h)$ with other cones in the literature. The cone $A(\rho_h, \Omega_h)$ is related to Debreu's [21] "asymptotic cone" corresponding to the preferred set of ρ_h at the initial endowment Ω_h, in that along any of the rays of $A(\rho_h, \Omega_h)$ utility always increases. This cone has also a similarity with the "recession" cone introduced by Rockafeller and used for example in Werner [42]. However, the similarity with those cones ends here, because along the rays in $A(\rho_h, \Omega_h)$ not only does utility increase forever, but it increases beyond the utility level of any other vector in the consumption space X. In ordinal terms, *the rays of the asymptotic cone $A(\rho_h, \Omega_h)$ intersect all indifference surfaces corresponding to bundles preferred by ρ_h to Ω_h.* This condition is not necessarily satisfied by Debreu's asymptotic cones [21], nor by Werner's "recession" cones [42]. Related conditions appear in Chichilnisky [8][9]; otherwise there appear to be no precedents in the literature for such cones.

Note that the cone $A(\rho_h, \Omega_h)$ depends on the initial endowments as well as on the preferences. It is defined by rays or "directions" from the initial endowment vector Ω_h. As the endowment Ω_h varies, the cone $A(\rho_h, \Omega_h)$ may also vary. This differs from the cones used in other works, such as e.g. Werner, which are assumed to be the same at all vectors in the consumption set ([42], Assumption A3 and Proposition 1).

The *dual cone* of $A(\rho_h, \Omega_h)$ is defined by

$$D(\rho_h, \Omega_h) = \{z \in X : \forall y \in A(\rho_h, \Omega_h), < z, y >> 0.\} \tag{5}$$

Figure 3
The dual cone $D(\rho_h, \Omega_h)$ of the preference ρ_h in Figure 2, translated to the initial endowment Ω_h

Examples of dual cones. The dual cone $D(\rho_h, \Omega_h)$ of a linear preference ρ_h which is defined by its gradient vector, $G \in R^N$, is the vector G itself. The dual cone of a preference ρ_h with asymptotic preferred cone $A(\rho_h, \Omega_h) = X_+^N$, is the same as its asymptotic preferred cone, i.e. $D(\rho_h, \Omega_h) = X_+^N$. The dual cones of an increasing preference may contain vectors with some negative coordinates, but will not contain strictly negative vectors. In general, the larger is the asymptotic cone, the smaller the dual cone, and reciprocally the smaller the asymptotic preferred cone, the larger the dual cone.

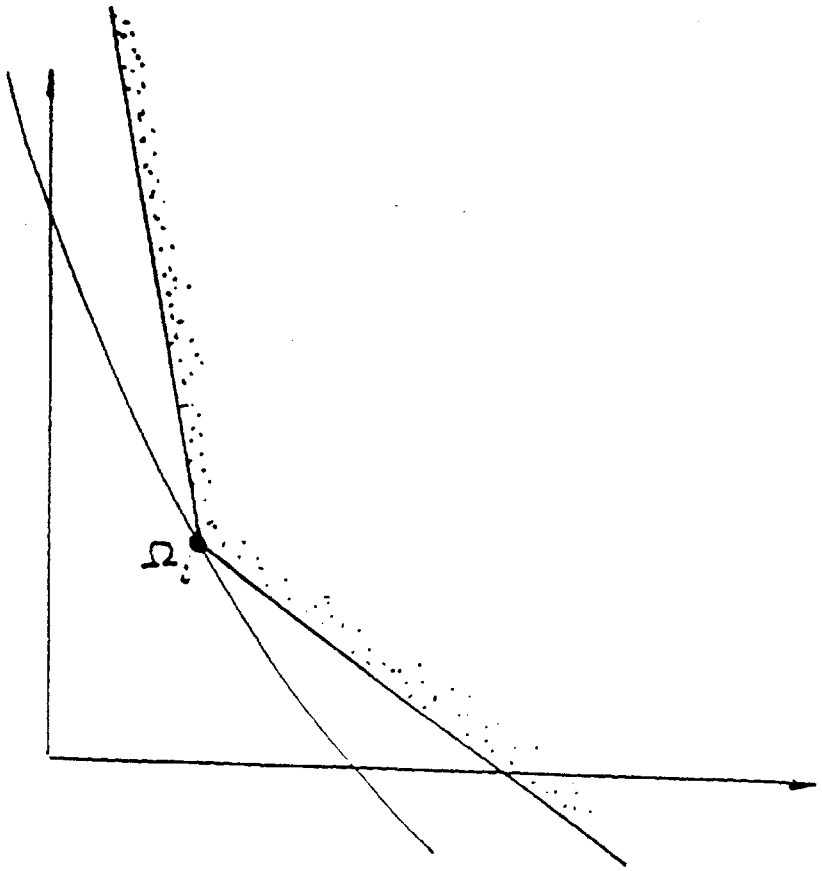


Figure 2

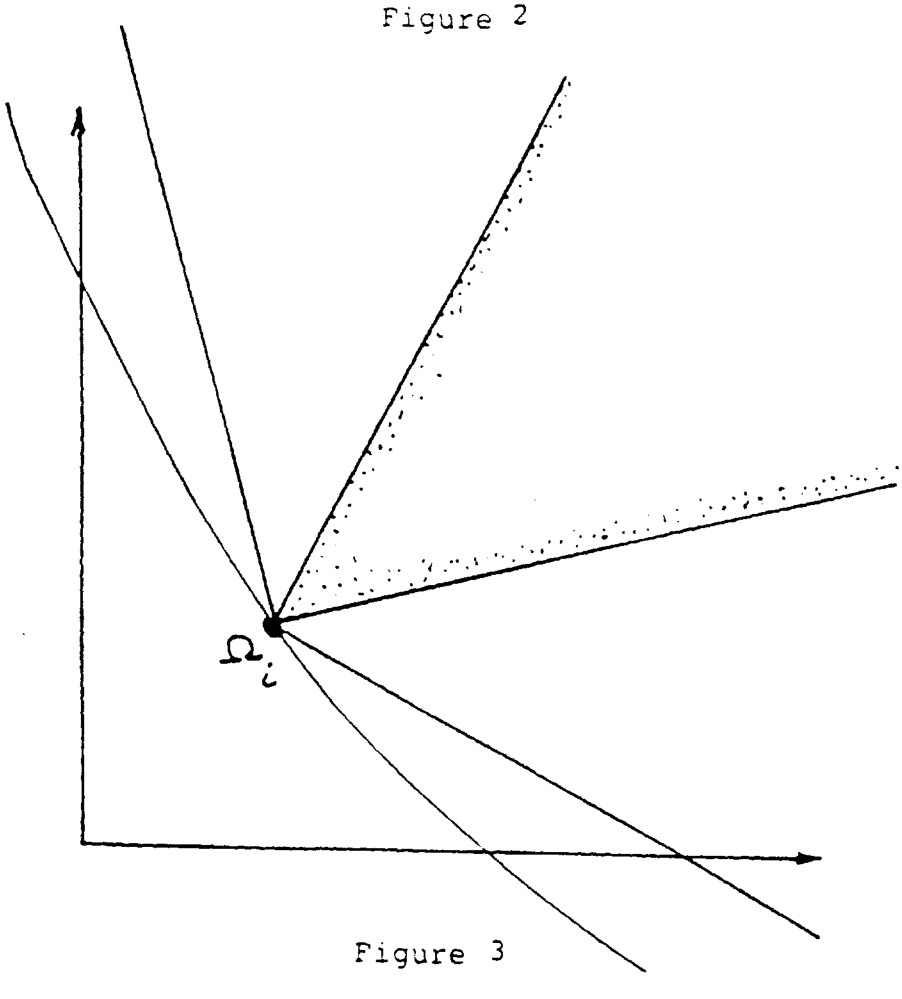


Figure 3

Figure 4

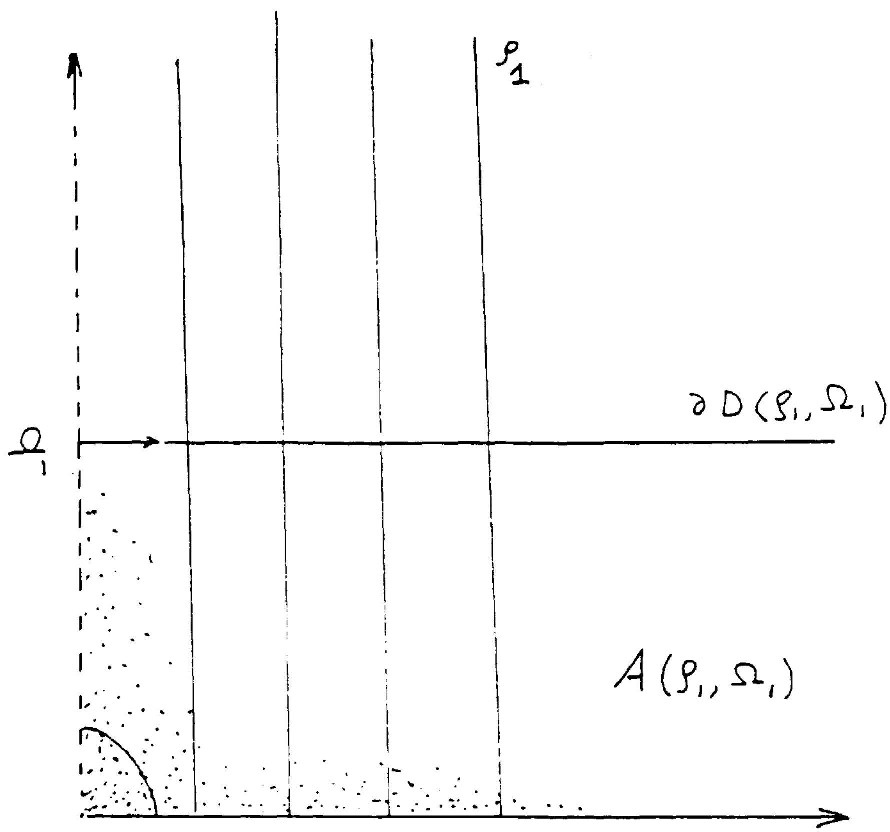


Figure 5a

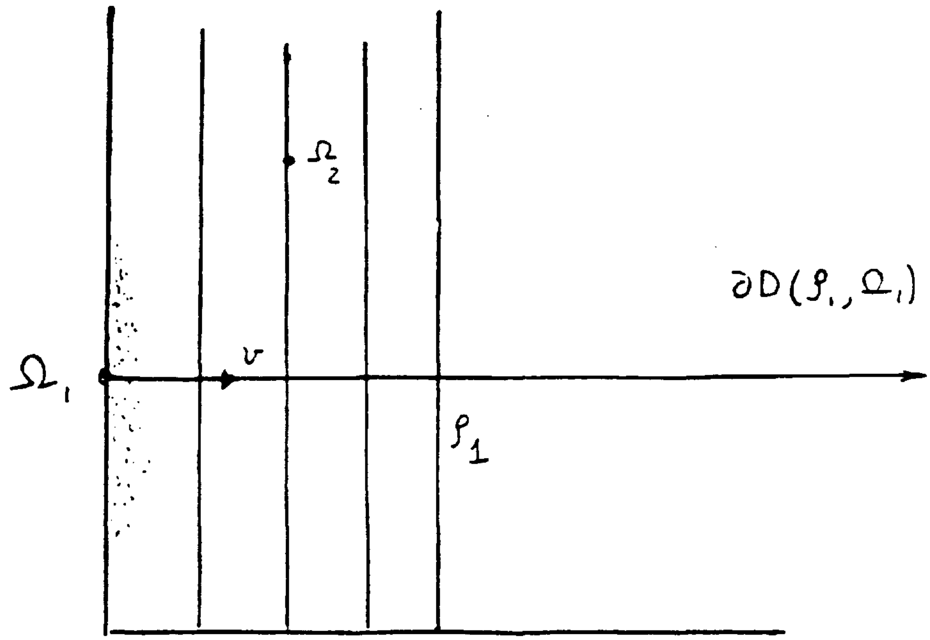
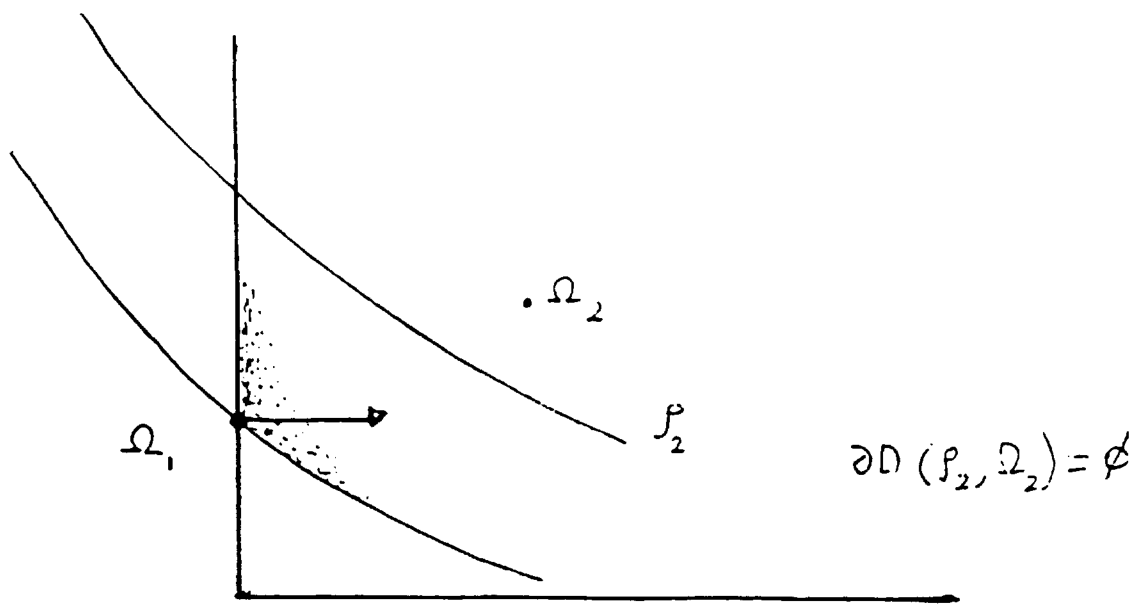


Figure 5b



2.2 Case 2: $X = R^N_+$

The ***asymptotic preferred cone*** $A(\rho_h, \Omega_h)$ of the hth individual in the economy E, is defined as in (4) above:

$$A(\rho_h, \Omega_h) = \{y \in X : \forall \lambda > 0,\ (\Omega_h + \lambda y) \in X,\ (\Omega_h + \lambda y) \succ_{\rho_h} \Omega_h, \\ and\ \forall z \in X,\ \exists \lambda > 0\ s.t.\ (\Omega_h + \lambda y) \succ_{\rho_h} z\}. \quad (6)$$

Figure 4
The asymptotic cone $A(\rho_1, \Omega_1)$ of the preference ρ_1
is the positive orthant R^2_+ minus the positive part of the vertical axis.

When $X = R^N_+$ the *boundary dual cone* is defined as:

$$\partial D(\rho_h, \Omega_h) = \{q \in R^N_+ : \text{ if } \forall p \in S(E)\ \exists h \text{ s.t. } < p, \Omega_h >= 0, \\ \text{then } q \in S(E), \text{ and } \forall v \in A(\rho_h, \Omega_h), < q, v >> 0\}. \quad (7)$$

Figure 5
Figure 5a illustrates the boundary dual cone $\partial D(\rho_1, \Omega_1)$
of the preference ρ_1 on $X = R^2_+$ in Figure 4, translated to Ω_1.
In this economy $\forall p \in S(E) : \exists h :\ < p, \Omega_1 >= 0$, because all preferences are as ρ_1 ,
and $\partial D(\rho_1, \Omega_1)$ is the one-dimensional cone defined by the vector v.
Figure 5b illustrates another economy, with a different preference
ρ_2 which increases in the second coordinate. Here $\partial D(\rho_2, \Omega_2) = \emptyset$.

The interpretation of ∂D_h is as follows. *If all supports in $S(E)$ assign some trader h zero income, then ∂D_h consists of all those supporting prices at which only limited increases in utility can be afforded from initial endowments.*

The boundary dual cone ∂D_h is the whole consumption set X when $S(E)$ has a support assigning strictly positive income to all individuals. Note that if for all prices in $S(E)$ some trader has zero income, then this trader must have a boundary endowment.

Figure 5a illustrates the boundary dual cone of the preference in Figure 4, in an economy where for all $i = 1...H$, the preferences satisfy $\rho_i = \rho_1$ which is indifferent in the second good, and the endowments are as illustrated. Since Ω_i is in the interior of X and ρ_i is indifferent in the second good, then the only possible price in $S(E)$ is the vector v. Since Ω_1 only owns the second good, then for every price in $S(E)$, there exits one individual with zero income, namely trader 1. The asymptotic cone $A(\rho_i, \Omega_i)$ is the positive orthant minus the vertical axis of coordinates, for all i. In this economy, for all i the boundary dual cone $\partial D(\rho_i, \Omega_i)$ is the half-line spanned by the vector v, because v has strictly positive inner product with all positive vectors including those with second coordinate equal to zero. Figure 5b illustrates a different economy. It has the same number of traders as the economy in 5a. The endowments and preferences of its traders are the same as those in Figure 5a except for the preference of trader ρ_2, which is now strictly increasing in the second coordinate as illustrated. Here the asymptotic cone $A(\rho_2, \Omega_2)$ is the whole positive orthant R^2_+, and the dual cone $\partial D(\rho_2, \Omega_2) = \emptyset$, because v is the only vector in $S(E)$, and $< v, y >= 0, \forall y = (y_1, 0) \in A(\rho_2, \Omega_2)$.

A *competitive equilibrium* of the economy $E = \{X, \Omega_h, \rho_h, h = 1...H\}$ consists of a price vector $p^* \in R^N_+$ and a *feasible allocation* $(x_1^*...x_H^*) \in \Upsilon$ such that x_h^* optimizes ρ_h over the budget set

$$B_h(p^*) = \{x \in X :< x, p^* >=< \Omega_h, p^* >\}.$$

A *pseudo-equilibrium* of the economy $E = \{X, \Omega_h, \rho_h, h = 1...H\}$ consists of a price vector $p^* \in R^N_+$ and a *feasible resource allocation* $(x_1^*...x_H^*) \in \Upsilon$ such that $\forall h = 1...H$, $y \geq_{\rho_h} x_h^* \Rightarrow < p^*, y > \geq < p^*, x_h^* >$. This implies that the allocation $(x_1^*...x_H^*)$ minimizes costs at p^*.

A pseudo-equilibrium need not be a competitive equilibrium, because a cost minimizing allocation may not maximize utility within the corresponding budget set. However, when $\forall h...H$, $< p^*, \Omega_h >> 0$, then a pseudo-equilibrium is also a competitive equilibrium, Arrow and Hahn [3].

3 Limited Arbitrage: Definition and Examples

Our next step is to define *limited arbitrage*, to discuss its meaning, and provide examples. We shall consider two cases separately. Case 1 is when the consumption set of the market economy $E = \{X, \rho_h, \Omega_h, h = 1...H\}$ is $X = R^N$, so that there are no bounds on short sales, and Case 2 is when the consumption set $X = R^N_+$. The limited arbitrage condition is somewhat different in these two cases, although in both cases it involves the non-empty intersection of dual cones. In addition, we discuss the interpretation of limited arbitrage for more general consumption sets and provide a geometric interpretation as a transversality condition.

3.1 Limited arbitrage without bounds on short sales, $X = R^N$

Consider a market economy $E = \{X, \Omega_h, \rho_h, h = 1...H\}$, where $X = R^N$. E satisfies *limited arbitrage* iff

$$(LA) \quad \bigcap_{h=1}^{H} D(\rho_h, \Omega_h) \neq \emptyset.$$

This condition can be interpreted as follows: *there exists a price p at which only limited (or bounded) increases in utility are affordable from initial endowments for all traders.*

Examples: economies which do not satisfy the limited arbitrage condition in this case are those where the individuals have different linear preferences, Figure 6, and also those represented in Figure 7. In Figure 6 the asymptotic preferred cones of the preferences are half spaces, and the dual cones are the two gradient vectors defining the preferences. Clearly, if the preferences are different these dual cones do not intersect. In Figure 7, each two dual cones intersect, but the three dual cones do not intersect, and the economy violates limited arbitrage. This figure illustrates the fact that the union of the dual cones may fail to be contractible: indeed, this failure corresponds to the failure of the dual cones to intersect, as proven in Theorem 7 in the Appendix.

Figure 6

$X = R^2$. Two different individuals with different linear preferences.
Limited arbitrage fails, and the economy has no competitive equilibrium.

Figure 7

$X = R^3$. Every two trader subeconomy satisfies limited arbitrage, but
in the economy as a whole limited arbitrage fails.
There is no competitive equilibrium.

3.2 Limited Arbitrage with bounds on short sales $X = R^N_+$

Consider now a market economy $E = \{X, \Omega_h, \rho_h, h = 1...H\}$, where $X = R^N_+$.

The *limited arbitrage* condition is now

$$(\partial LA) \quad \bigcap_{h=1}^{H} \partial D(\rho_h, \Omega_h) \neq \emptyset \qquad (8)$$

where the boundary-dual cones $\partial D(\rho_h, \Omega_h)$ are defined in Section 2.

This condition ensures that *if all supporting prices in $S(E)$ assign zero income to some trader, there is one at which only limited (or bounded) increases in utility are affordable from initial endowments.*

Examples: An example of an economy with $X = R^N_+$ which does not satisfy limited arbitrage in this case is illustrated in Figure 8. As shown in Figure 5b above, in this economy the dual cone of the first trader is empty, $\partial D(\rho_1, \Omega_1) = \emptyset$, so limited arbitrage as defined in (8) is violated. This economy has no competitive equilibrium.

Figure 8

$X = R^2_+$.The boundary dual cones do not intersect
and limited arbitrage fails.
The economy has no competitive equilibrium.
This is similar to an example in Arrow and Hahn [3]

Examples of economies which satisfy limited arbitrage and of preferences which do not. When the consumption set is $X = R^N_+$, limited arbitrage is always satisfied when all indifference surfaces through positive consumption bundles are contained in the interior of X, R^N_{++} (Debreu [21]). Examples of such preferences are those given by Cobb-Douglas utilities or by CES utilities with elasticity of substitution $\sigma < 1$. This is because all such preferences have the same asymptotic preferred cone, namely the positive orthant, and therefore their dual cones always intersect. Since their asymptotic preferred cones are identical, these preferences are very similar to each other on choices involving large utility levels. This is a form of similarity of preferences.

Economies where the individual's initial endowments are strictly interior to the consumption set X always satisfy the limited arbitrage condition in the case $X = R^N_+$, since

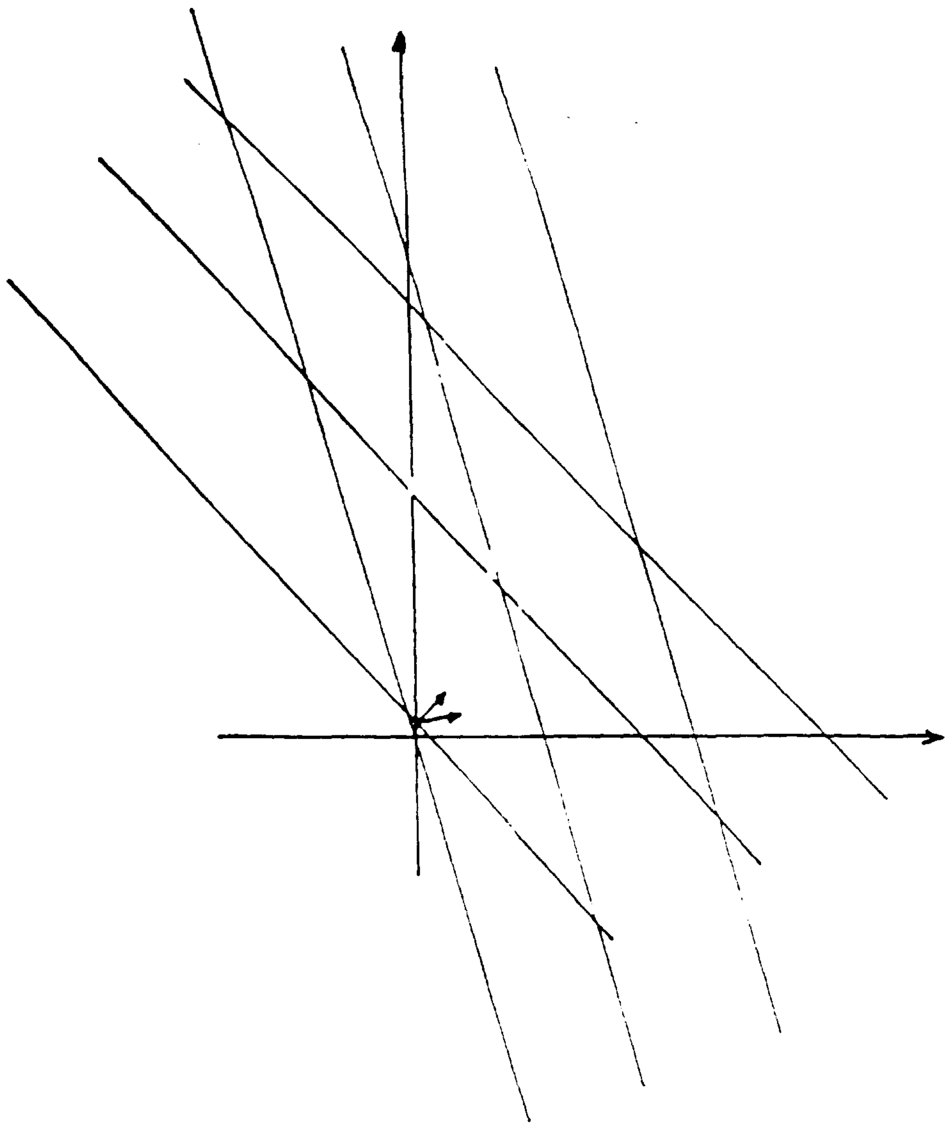


Figure 6

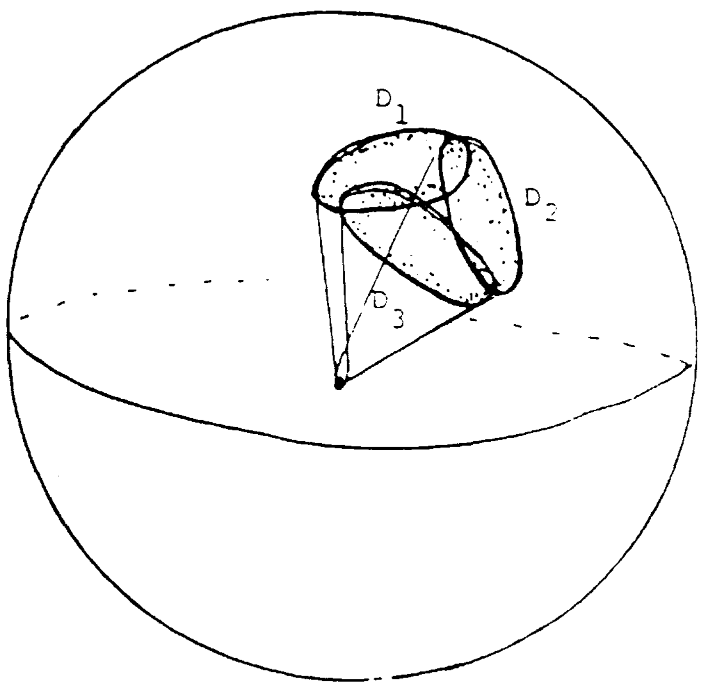


Figure 7

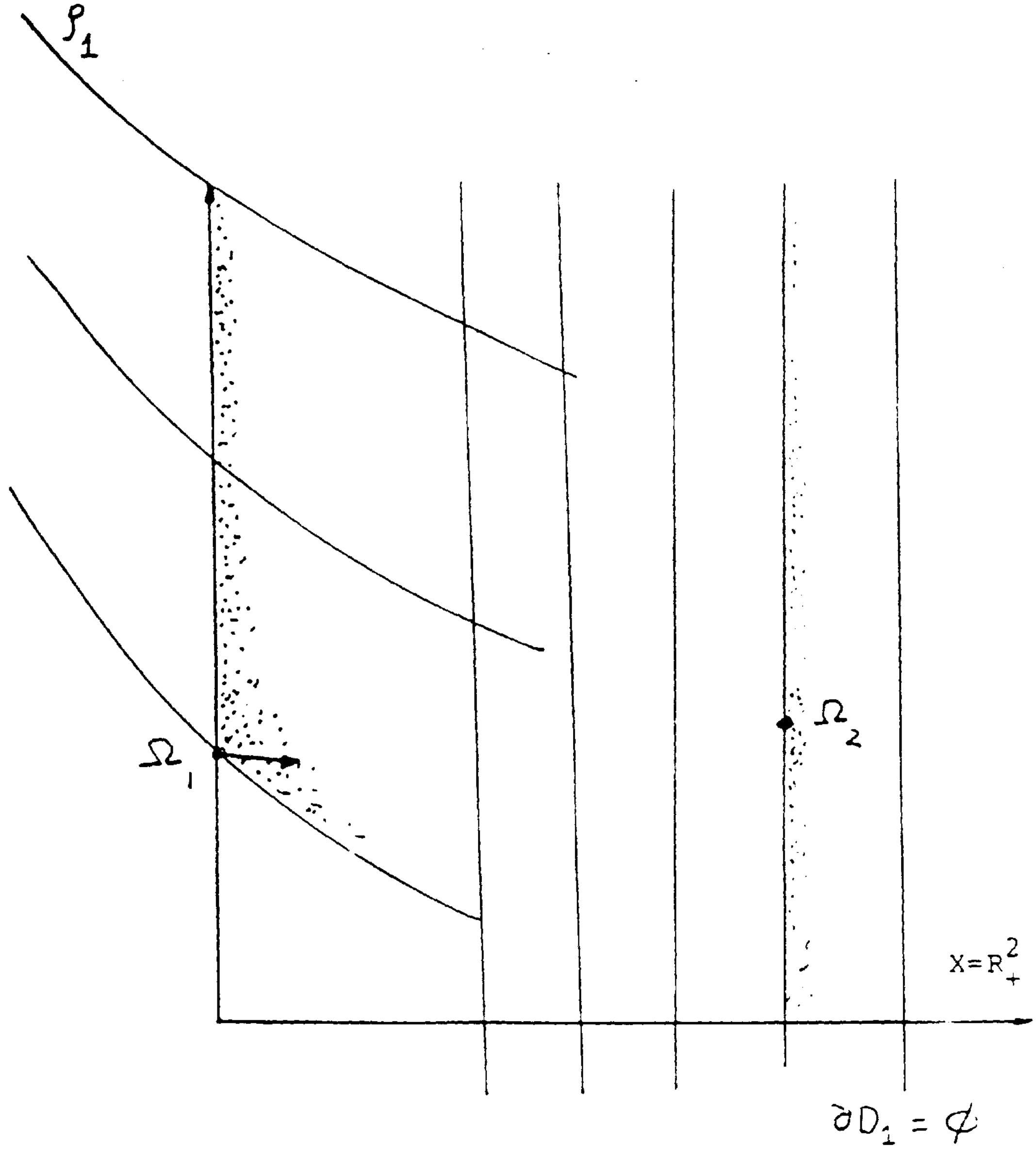


Figure 8

in this case $\forall h$, $\partial D(\rho_h, \Omega_h) = R^N_+$ for all $h = 1...H$. This is because all individuals have non-zero income at any supporting price in $S(E)$.

When $X = R^N_+$ the limited arbitrage condition may fail to be satisfied when some trader's endowment vector Ω_h is in the boundary of the consumption space, ∂X, and at all supporting prices some trader has zero income: $\forall p \in S(E)\ \exists h$ such that $< p, \Omega_h >= 0$. This case is illustrated in Figure 8; it is a rather general case which may occur in economies with many individuals and with many commodities. When all individuals have positive income at some price $p \in S(E)$, then limited arbitrage is always satisfied since by definition in this case $\forall h$, $\partial D(\rho_h, \Omega_i) = R^N_+$ for all $h = 1...H$.

We shall now define limited arbitrage for *subsets of traders* in the economy E. When $X = R^N$, we say that the economy E satisfies *limited arbitrage for any subset of k traders*, when for any subset $K \subset \{1...H\}$ of cardinality $k \leq H$

$$(LA) \bigcap_{h \in K} D(\rho_h, \Omega_h) \neq \emptyset. \tag{9}$$

When $X = R^N_+$ the definition is

$$(LA) \bigcap_{h \in K} \partial D(\rho_h, \Omega_h) \neq \emptyset. \tag{10}$$

Theorem 8 in the Appendix establishes that a market economy E has limited arbitrage if and only if it has limited arbitrage for any subset of k traders, where $k \leq N + 1$, where N is the dimension of the commodity space.

3.3 Limited Arbitrage as a Transversality Condition

We now discuss more general consumption sets and how to interpret the limited arbitrage condition geometrically as a transversality condition.

For example, consider the consumption set X which is a translate of a positive orthant, as defined in Section 2. In this case, the dual cones ∂D are no longer defined in terms of individuals with zero income at supporting prices in $S(E)$, but rather in terms of individuals whose endowments have *minimal income at those prices*, i.e. the minimal income possible in the economy. Limited arbitrage then requires that within the set of supporting prices in $S(E)$ giving an individual's endowment minimal value there exists a price at which only bounded increases of utility are possible from initial endowments for all traders.

More generally the condition of limited arbitrage applies to any convex consumption set $X \subset R^N$ which is bounded below and has the property that $y \in X$ and $z \geq y \Rightarrow z \in X$, used for example in Chichilnisky and Heal [17]. For simplicity, assume that either X is a translate of a positive orthant or else that the boundary of X, ∂X, is a manifold of dimension $N - 1$. For example, ∂X could be defined locally by a smooth function $f : R^N \to R$. Consider the gradient vector $G\partial X(y)$ of the function f which defines ∂X in a neighborhood of $y \in \partial X$, and let $H(G\partial X(y))$ be the line in R^N defined by the vector $G\partial X(y)$. We say that a vector v is *transversal* to $H(G\partial X(y))$, denoted $v \top H\ (G\partial X(y))$ when the vector v is linearly independent of the subspace H; otherwise v is *not transversal* to H, denoted $v \dashv H$. We say that all supports $v \in S(E)$ are not transversal to an individual endowment Ω_h, when $\forall v \in S(E)$, $v \dashv H\ (G\partial X(\Omega_h))$. The *limited arbitrage* condition is now defined as follows.

(LA) *If* $\forall q \in S(E)$, $q \dashv H(G\partial X(\Omega_h))$, **then there exists a price** $p \in S(E)$ **such that** $<p, v>> 0$ $\forall v \in A(\rho_h, \Omega_h)$, **for all** $h \in \{1...H\}$.

The interpretation of this condition is that if every support in $S(E)$ fails to be transversal to some individual endowment, there is a supporting price at which only bounded increases in utility are affordable for all traders from initial endowments. Limited arbitrage can therefore be viewed as a transversality condition on the economy E.

Figure 9
A more general consumption space in R^2.
Limited arbitrage as a transversality condition.

4 Competitive Equilibrium and Limited Arbitrage

This section provides the main results linking the existence of a competitive equilibrium with the condition of limited arbitrage. Theorem 1 is proven in several steps, formalized in lemmas 2 to 5.

Theorem 1 *Consider an economy* $E = \{X, \rho_h, \Omega_h, h = 1...H\}$, *where* $H \geq 2$, $X = R^N$ *or* $X = R^N_+$ *and* $N \geq 1$.
Then the following two properties are equivalent:
(a) the economy E has limited arbitrage
(b) the economy E has a competitive equilibrium

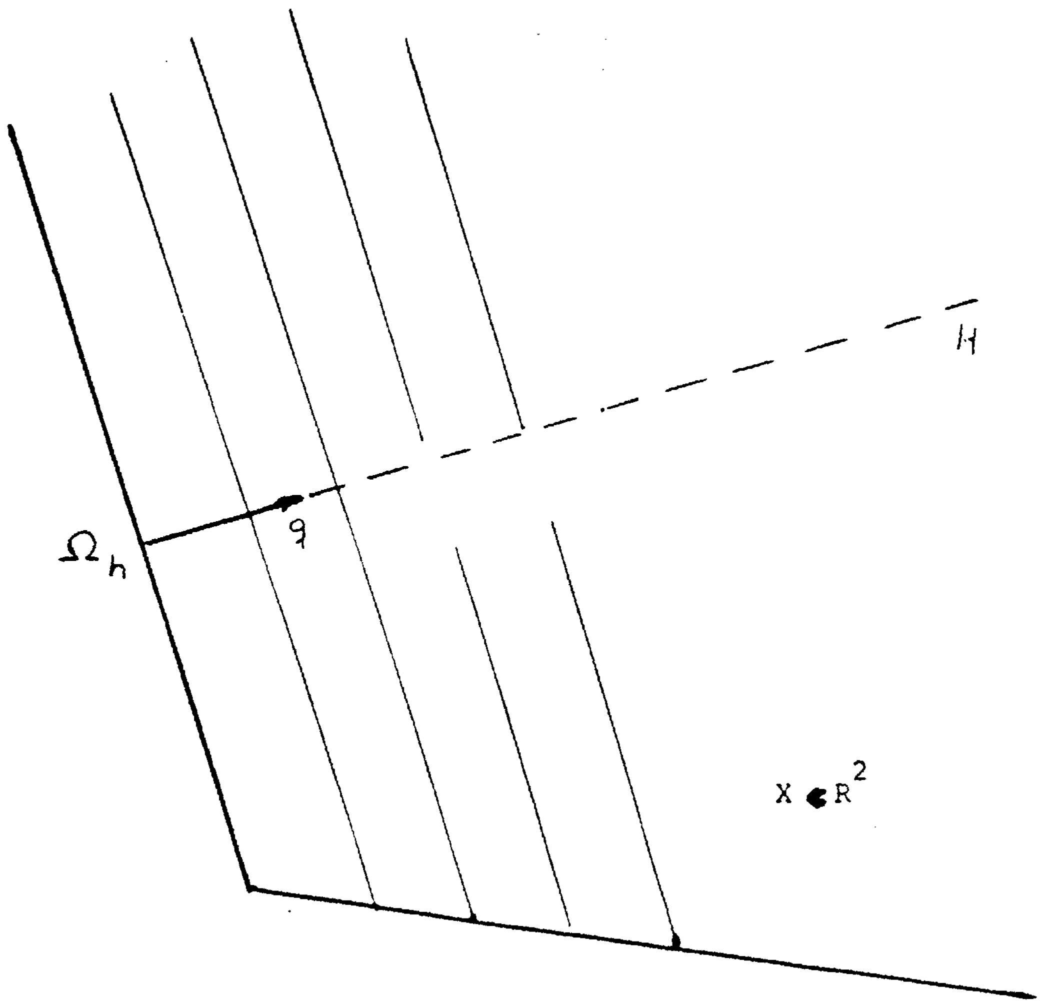
Proof: The strategy of the proof is as follows. First we prove that limited arbitrage is necessary for the existence of a competitive equilibrium. This is Lemma 2.. Next we establish that limited arbitrage is sufficient for the existence of a competitive equilibrium. The proof of sufficiency has two parts. The first is the proof of existence of a *pseudo equilibrium*; for this we use a fixed point argument on the Pareto frontier of the economy. This requires in turn to prove that the Pareto frontier of the economy is homeomorphic to a simplex, a property which may fail to be satisfied in general. Lemma 3 establishes that limited arbitrage implies this property of the Pareto frontier. Lemma 4 then establishes the existence of a pseudoequilibrium. Finally, using limited arbitrage we prove in Lemma 5 that the pseudo equilibrium is also a competitive equilibrium.

Lemma 2 *Limited arbitrage is necessary for the existence of a competitive equilibrium in the economy E of Theorem 1.*

Proof: Let the utility function $u_h : X \rightarrow R$ represent the preference $\rho_h \in E$, i.e.$\forall x, y \in X$, $u_h(x) > u_h(y) \Leftrightarrow x \succ_{\rho_h} y$. By appropriate renormalization we can assume without loss of generality that $u_h(0) = 0$ so that $u_h(\Omega_h) \geq 0$, and that $Sup_{x \in X}(u_h(x)) = \infty$. Now assume that (a) is not true, and consider the case $X = R^N$ first. Then

$$\bigcap_{h=1}^{H} D(\rho_h, \Omega_h) = \emptyset, \tag{11}$$

Figure 9



which implies that for all $y \in R^N$, there exists an $h \in \{1...H\}$ and a vector $v(y) \in A(\rho_h, \Omega_h)$ such that:

$$\begin{array}{c} < y, \lambda v(y) > \leq 0, \text{ and} \\ \lim_{\lambda \to \infty}(u_h(\Omega_h + \lambda v(y)) = \infty. \end{array} \tag{12}$$

Consider now a competitive equilibrium described by a price p^* and an allocation $(x_1^*...x_H^*)$. By (12) for some $\lambda > 0$,

$$u_h(\Omega_h + \lambda v(y)) > u_h(x_h^*) \text{ and } < p^*, \lambda v(y) > \leq 0,$$

contradicting the fact that x^* is an equilibrium allocation. Therefore no competitive equilibrium exists when (12) is true: limited arbitrage is necessary for the existence of a competitive equilibrium when $X = R^N$. Consider next the case $X = R_+^N$. Assume first that $\forall q \in S(E)$ $\exists\, h \in \{1...H\}$ s.t. $< q, \Omega_h >= 0$. Then if limited arbitrage is not satisfied

$$\bigcap_{h=1}^{H} \partial D(\rho_h, \Omega_h) = \emptyset, \tag{13}$$

which implies that

$$\begin{array}{c} \forall q \in R^N,\ \exists h \text{ and } v(q) \in\ A(\rho_h, \Omega_h): \\ < q, \Omega_h >= 0, \text{ and } \forall \lambda > 0, < q, \lambda v(q) > \leq 0. \end{array} \tag{14}$$

Since $v(q) \in A(\rho_h, \Omega_h)$

$$\lim_{\lambda \to \infty} (u_h(\Omega_h + \lambda v(y)) = \infty. \tag{15}$$

Consider now a competitive equilibrium price p^* and the corresponding allocation $(x_1^*...x_H^*)$. Then $p^* \in S(E)$, and (14) and (15) imply that $\exists h$ s.t. for some $\lambda > 0$,

$$u_h(\Omega_h + \lambda v(y)) > u_h(x_h^*) \text{ and } < p^*, \lambda v(y) > \leq 0,$$

contradicting the assumption that p^* and $(x_1^*...x_H^*)$ define a competitive equilibrium.

It remains to consider the case where $\exists q \in S(E)$ such that $\forall h \in \{1...H\}$ $< q, \Omega_h > \neq 0$. But in this case by definition $\bigcap_{h=1}^{H} \partial D(\rho_h, \Omega_h) \neq \emptyset$ since $\forall h \in \{1...H\}$ $\partial D(\rho_h, \Omega_h) = R_+^N$, so that limited arbitrage is always satisfied. This completes the proof that limited arbitrage is *necessary* for the existence of a competitive equilibrium, when $X = R^N$ and when $X = R_+^N$.

Limited arbitrage is sufficient for the existence of a competitive equilibrium. For this result we will utilize the standard method - introduced by Negishi [36] - of proving first the existence of a *quasi-equilibrium* as defined in Section 2, using a fixed point theorem on the Pareto frontier $P(E)$. The quasi-equilibrium is subsequently shown to be a competitive equilibrium, thus completing the proof. The proof must address two practical difficulties in applying this strategy, one when the consumption set $X = R^N$, and a different one when $X = R_+^N$. Both difficulties are resolved by the limited arbitrage condition. The problem is as follows: when $X = R^N$ the Pareto frontier $P(E)$ may fail to be bounded and closed, because the utility obtained by the traders from their initial endowments may not attain a maximum over feasible allocations when there are no bounds on short sales. This failure leads to the non-existence of a competitive equilibrium in well known cases; this problem of existence appears also in economies with infinitely many commodities, but when commodity spaces are infinite dimensional it can appear even if the consumption set is the positive orthant, see the examples in Chichilnisky and Heal [17]. In practical terms, the problem is that the Pareto frontier may not be homeomorphic to a unit simplex, a property which is essential in the proof of existence of a quasiequilibrium. The role of the limited arbitrage condition in this case is to ensure that the Pareto frontier is bounded and closed; together with the quasi concavity of preferences this implies that the Pareto frontier is homeomorphic to a unit simplex so that standard existence arguments can be invoked.

A more standard difficulty arises when the consumption set is $X = R^N_+$. Here the Pareto frontier is always closed and bounded and a quasi-equilibrium exists. However, in this case the quasi-equilibrium may fail to be a competitive equilibrium. This is the type of problem which the conditions of resource relatedness and of irreducibility are meant to circumvent. The problem arises only when some individual has zero income at the quasi-equilibrium allocation and is illustrated in Figure 8 above. In this case, minimizing costs may not imply maximizing utility so that a quasi-equilibrium may fail to be a competitive equilibrium. This second potential failure of existence is also ruled out by the condition of limited arbitrage.

Lemma 3 *The Pareto frontier of the economy E in Theorem 1, $P(E)$, is homeomorphic to a unit simplex, when $X = R^N$ and when $X = R^N_+$.*

Proof: Consider first the case $X = R^N$. We shall show that the Pareto frontier is a closed bounded set in R^H. Define the set F_Ω of feasible and individually rational allocations:

$$F_\Omega = \{z \in R^{NH} : z = (x_1 + \Omega_1 ... x_H + \Omega_H\},$$
$$\text{where} \sum_{h=1}^{H} x_h \leq 0 \text{ and } \forall h,\ u_h(x_h + \Omega_h) \geq u_h(\Omega_h) \geq 0\}$$

Let Δ denote the unit simplex in R^H, and define the set S_r of utility vectors which are collinear with a given element $r = (r_1 ... r_H) \in \Delta$,

$$S_r = \{(U_1 ... U_H) \in R^H : \forall h = 1 ... H,\ U_h = u_h(z_h), \text{ where } z = (z_1 ... z_H) \in F_\Omega$$
$$\text{and } \exists \nu > 0 \text{ s.t. } \nu . U_h = r_h\}$$

Figure 10 illustrates the set S_r:

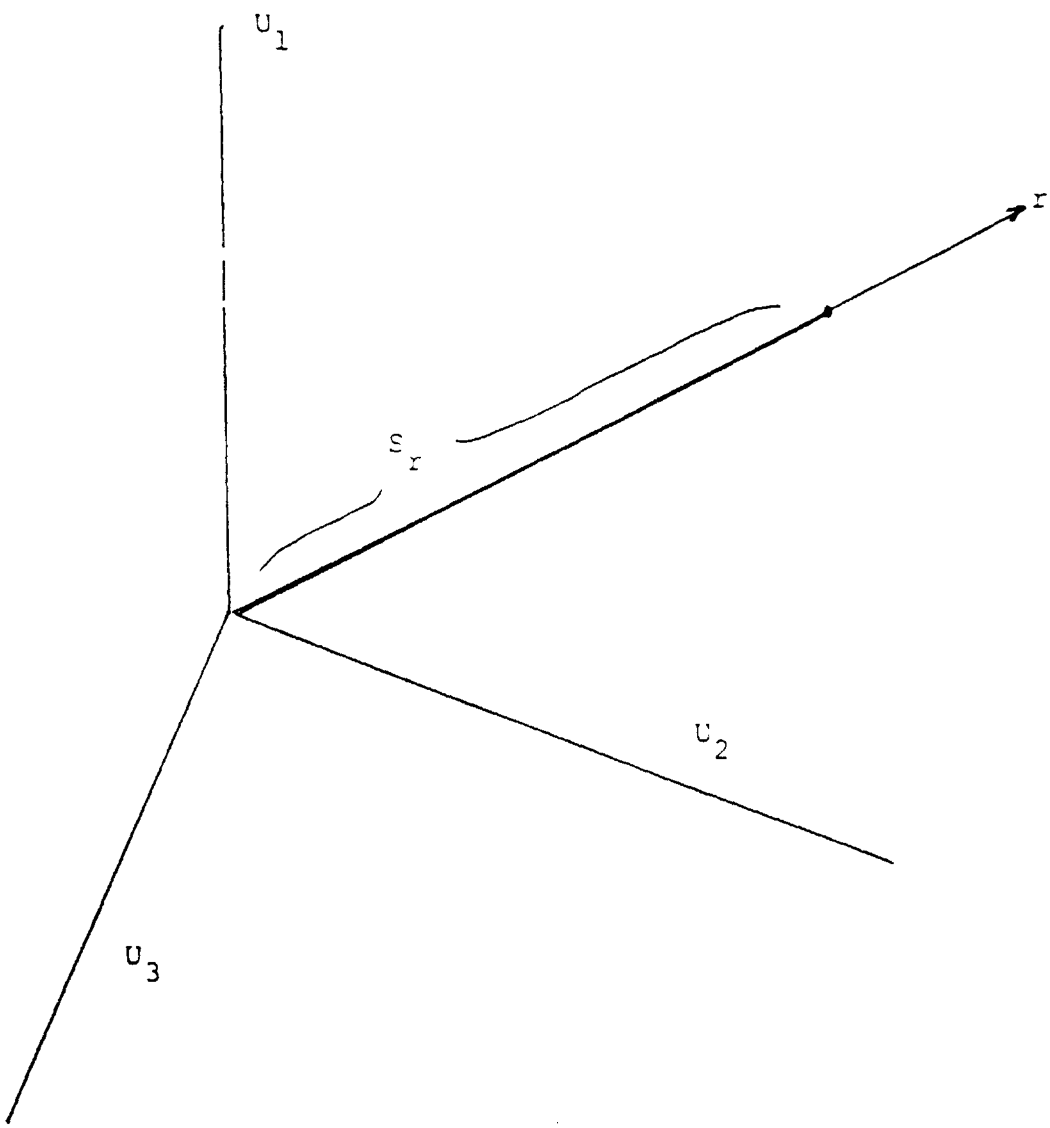
Figure 10
The set S_r

We shall now prove that limited arbitrage implies that S_r is bounded for all $r \in \Delta$. Consider first the case where $r >> 0$. Assume that S_r is not bounded. Then there exists a sequence denoted $\{x^n\}_{n=1,2...} = \{(x_1^n ... x_H^n)\}_{n=1,2...} \subset F_\Omega$ such that $(u_1(x_1^n) ... u_H(x_H^n))$ is in S_r and for some $h \in \{1...H\}, \lim_{n \to \infty} u_h(x_h^n) = \infty$. Let $z_h^n = x_h^n - \Omega_h$. We may assume that $\forall n$ $\sum_{h=1}^{H}(x_h^n - \Omega_h) = \sum_{h=1}^{H} z_h^n = 0$. Since $r >> 0$, if $\exists h : \lim_{n \to \infty} u_h(x_h^n) = \infty$, then $\forall h$=1...H $\lim_{n \to \infty} u_h(z_h^n) = \infty$. For all $h \in \{1...H\}$, let α_h be a point of accumulation of the sequence of vectors $\{(z_h^n / \|z_h^n\|)\}$; this sequence has such a point because it is contained in the unit sphere $S^{N-1} \subset R^N$. Note that $\alpha_h \in A(\rho_h, \Omega_h.)$. Now let α_h^n denote the projection of the vector z_h^n on the line in R^N defined by the vector α_h, and consider a subsequence $\{z^m\}$ of $\{z^n\}$ satisfying

$$\lim_{m \to \infty} \|z_h^m - \alpha_h^m\| = 0.$$

Figure 10

The set S_r in utility space



Then

$$0 = \lim_{m\to\infty} \sum_{h=1}^{H} z_h^m = \lim_{m\to\infty} \sum_{h=1}^{H} \alpha_h^m = \sum_{h=1}^{H} \alpha_h.$$

In particular $\forall q \in R^N < q, \alpha_h >= \lim_{m\to\infty} < q, \sum_{h=1}^{H} \alpha_h^m >= 0$. Since $\alpha_h \in A(\rho_h, \Omega)$, this implies that there exists no $q \in R^N$ such that $< q, y >> 0$ for all $y \in A(\rho_h, \Omega_h)$, so that

$$\bigcap_{h=1}^{H} D(\rho_h, \Omega_h) = \emptyset,$$

contradicting the limited arbitrage condition. Therefore limited arbitrage implies that S_r must be bounded $\forall\ r >> 0$.

Now assume that $r \in \partial\Delta$, so that $r_h = 0$ for some $h = 1...H$. We may assume that $r_h \neq 0$ for some h, for otherwise the ray S_r is clearly bounded. Then we may approximate r by a sequence of rays $\{r^k\}_{k=1,2...} \subset \Delta$, such that $\forall k$, $r^k >> 0$, and for which a proof similar to the previous case applies. There exist a sequence $\{x^k\}_{k=1,2...} \subset F_\Omega$ such that $\forall k$, $(u_1(x_1^k)...u_H(x_H^k))$ is maximal in S_{r^k}. In particular, if $z_h^n = x_h^n - \Omega_h$, then $\forall k \sum_{h=1}^{H} z_h^k = \sum_{h=1}^{H}(x_h^n - \Omega_h) = 0$. For any h let α_h be a point of accumulation of the sequence of vectors $(z_h^k/\|z_h^k\|) \subset S^{N-1} \subset R^N$. By definition, $\alpha_h \in A(\rho_h, \Omega_h)$. Let α_h^k denote the projection of the vector z_h^k on the line in R^N containing the vector α_h. Consider now a subsequence $\{z^m\}$ of $\{z^k\}$ satisfying

$$\lim_{m\to\infty} \|z_h^m - \alpha_h^m\| = 0.$$

Then

$$0 = \lim_{m\to\infty} \sum_{h=1}^{H} z_h^m = \lim_{m\to\infty} \sum_{h=1}^{H} \alpha_h^m = \sum_{h=1}^{H} \alpha_h.$$

Since $\alpha_h \in A(\rho_h, \Omega_h)$ this implies that there exists no $q \in R^N$ such that $< q, y >> 0$ for all $y \in A(\rho_h, \Omega_h)$ and all h, i.e.

$$\bigcap_{h=1}^{H} D(\rho_h, \Omega_h) = \emptyset$$

contradicting limited arbitrage. Therefore limited arbitrage implies that S_r must be bounded for all $r \in \Delta$. This in turn implies that the utility possibility set $U(E) \subset R^H$ is bounded.

We now complete the proof that the Pareto frontier $P(E)$ is bounded and closed when $X = R^N$ by proving that limited arbitrage implies that $P(E)$ is closed. For any $r \in \Delta$, let $v = (v_1...v_H) \in R_+^H$ satisfy $v = Sup_{y \in S_r} y$; we know that such a v exists because the utility possibility set $U(E)$ is bounded. To prove that $P(E)$ is closed it suffices to show that there exists an allocation $(z_1...z_H) \in F_\Omega$ such that $v = (u_1(z_1)...u_H(z_H))$. Consider a sequence $\{z^n\} \subset F_\Omega$ such that: $U^n = (u_1(z_1^n)...u_H(z_H^n)) \in S_{r^n}$, U^n is maximal in the set S_r, $\lim_n\{r^n\} = r$, and $\lim_n U^n = v$. Since $U(E)$ is bounded, and each utility u_h is monotonic, there exists a vector of utility values $(U^1...U^H) = (u_1(y_1)...u_H(y_H)) \in R^H$, where $(y_1...y_H)$ may or not be a feasible allocation, such that $\lim_{n\to\infty} U^n = v$. It is straightforward to see that since $\lim_n U^n = v$ and v is optimal in S_r, the directions of the gradients of the sequence of the utilities define a Cauchy sequence, i.e.

$$\forall h = 1...H,\ \lim_{n,m}\left(\frac{Gu_h(z_h^n)}{\|Gu_h(z_h^n)\|}\right) - \frac{Gu_h(z_h^m)}{\|Gu_h(z_h^m)\|}) = 0.$$

Define now the sequence $\{s_h^n\}_{n=1,2...}$ where $s_h^n = Gu_h(z_h^n)/||u_h(z_h^n)|| \in S^{N-1} \subset R^N$. Since S^{N-1} is compact, $\forall h$ there exists a point of accumulation of $\{s_h^n\}_{n=1,2...}$, denoted s_h. Since for all h, $u_h(z_h^n) \to u_h(y_h)$, then $\forall \epsilon > 0, \exists T$ and $\exists\ w_h^n \in R^N$ such that $u_h(w_h^n) = v_h$ and

$$||\frac{Gu_h(z_h^n)}{||Gu_h(z_h^n)||} - \frac{Gu_h(w_h^n)}{||Gu_h(z_h^m)||}|| < \epsilon \quad \text{for } n > T.$$

The sequence $\{Gu_1(z_1^n)...Gu_H(z_H^n)\}$ consists of gradients of efficient utility levels and it converges to $\{s_1...s_H\}$, so by the assumptions on the utilities u_h $\forall h$ there exists a vector $z_h \in u_h^{-1}(v_h) \in R^N$ such that $Gu_h(z_h) = \lambda_h s_h$ for some $\lambda_h > 0$. Furthermore, $\sum_{h=1}^H z_h^n = \sum_{h=1}^H \Omega_h$, so that $(z_1...z_H) \in F_\Omega$. Since $(v_1...v_H) = (u_1(z_1)...u_H(z_H))$ we have completed the proof that the Pareto frontier $P(E)$ of the economy E is closed, as we wished to prove. We have shown that limited arbitrage implies that when $X = R^N$ the set $P(E)$ is closed and bounded. Therefore the proof that $P(E)$ is homeomorphic to the unit simplex $\Delta \in R^H$ is now standard from the quasi concavity of the preferences, see for example Arrow and Hahn [3]. For the case $X = R^N_+$, their proof establishes directly that this Pareto frontier is always homeomorphic to the unit simplex.

Lemma 4 *Limited arbitrage implies the existence of a pseudo equilibrium in the economy E of Theorem 1.*

Proof: In view of Lemma 3, it is now standard to establish that a *quasi-equilibrium* always exists, either when $X = R^N$ or $X = R^N_+$: for completeness we provide now a formal proof of existence of a quasi-equilibrium which works equally for these two cases next:

Define the set

$$T = \{y \in R^H : \sum_{h=1}^H y_h = 0\}.$$

For each $r >> 0$ in Δ let $(x_1(r)...x_H(r)) \in F_\Omega$ now denote the feasible allocation which gives the greatest utility vector collinear with r :

$$(u_1(x_1(r))...u_H(x_H(r))) = Sup_{w \in S_r}(u_1(w_1(r))...u_H(w_H(r))),.$$

in the vector order of R^H, and $\sum_{i=1}^H (x_i(r) - \Omega_i) = 0$. Such an allocation always exists because $\forall r \Delta\ S_r$ is bounded and closed; it defines a non-zero utility vector which depends continuously on r. Now let

$$P = \{p \in R^N : ||p|| = 1\} \text{ and}$$
$$P(r) = \{p \in P : p \text{ supports } x(r)\}.$$

By standard arguments, $P(r)$ is not empty, see e.g. Chichilnisky and Heal [17], Lemma 3. Define now a map $\varphi : \Delta \to T$:

$$\varphi(r) = \{< p, \Omega_1 - x_1(r) > ... < p, \Omega_H - x_H(r) >: p \in P(r)\}$$

$\varphi(r)$ is a non-empty convex valued correspondence, $\sum_{h=1}^H z_h = 0$ if $z \in \varphi(r)$, and

$$0 \in \varphi(r) \Leftrightarrow (x^*, p^*) \text{ is a quasi-equilibrium of } E,$$
$$\text{where } r = r(x^*) \text{ and } p^* \in P(r).$$

The next step is to show that φ is upper semi-continuous, i.e. if $r^n \to r$, $z^n \in \varphi(r^n)$, $z^n \to z$ then $z \in P(r)$. Consider the feasible allocation $x(r)$, where $r = \lim_n(r^n)$. Let v be any other allocation satisfying $u_h(v_h) > u_h(x_h(r))$, where $x_h(r)$ is the h-th coordinate of the vector $x(r)$ and v_h is the h-th coordinate of the vector v. Let $z^n \in \varphi(r^n)$ and $p^n \in P(r^n)$. Since $r^n \to r$, eventually $u_h(v_h) > u_h(x_h(r^n))$ so that $< p^n, v_h > \geq < p^n, x_h(r^n) > = < p^n, \Omega_h > -z^n_h$, where z^n_h is the h-th coordinate of z^n : this follows from the definitions of z^n and p^n. Let $\{p^n\}$ be a sequence of vectors such that $p^n \in P(r^n)$. The set P is compact and $\bigcup_r P(r)$ is closed; therefore $\bigcup_r P(r)$ is compact as well. There exists therefore a vector $p \in P$ and a subsequence $\{p^m\}$ of $\{p^n\}$ such that $< p^m, v_h > \to < p, v_h >$, so that in the limit $< p, v_h > \geq < p, \Omega_h > -z_h$. Since this is true for all such v, it is also true for v satisfying $u_h(v_h) \geq u_h(x_h(r))$ and in particular for $v = x$ so that $< p, x_h > \geq < p, \Omega_h > -z_h$ implying that $z \in \varphi(r)$ as we wished to prove. The proof of existence of a quasi-equilibrium is completed by showing that φ has a zero. For all $r \in \Delta$ define $\theta(r) = r + \varphi(r)$. The map $\theta : \Delta \subset \Delta$ is non-empty, upper semi continuous, convex-valued correspondence and it satisfies appropriate boundary conditions. By Kakutani's fixed point theorem, θ must have a fixed point r^* which is a zero of the map φ. The allocation $x^* = x^*(r^*)$ and a price $p^* \in P(r^*)$ define a quasi-equilibrium of the economy E.

The proof of existence of a quasi-equilibrium just provided is equally valid when $X = R^N$ or when $X = R^N_+$. Therefore to complete the proof of the theorem it remains only to show that the quasi-equilibrium is a competitive equilibrium.

Lemma 5 *Limited arbitrage implies the existence of a competitive equilibrium in the economy E of Theorem 1.*

Proof: In view of Lemma 4 it suffices to prove that a quasi equilibrium is a competitive equilibrium. Consider first the case $X = R^N$. Then $\forall h = 1...H$ there exists an allocation in X of strictly lower value than x^*_i at the price p^*. Therefore by Lemma 3, Chapter 4, page 81 of Arrow and Hahn [3], the quasi-equilibrium is also a competitive equilibrium. This establishes the existence of a competitive equilibrium when limited arbitrage is satisfied and $X = R^N$.

Now consider the case $X = R^N_+$. We have shown that when limited arbitrage is satisfied the economy E has a quasi-equilibrium consisting of a price p^* and an allocation x^*. It remains to show that the quasi-equilibrium is also a competitive equilibrium.

First note that if at the quasi-equilibrium (p^*, x^*) every individual has a positive income, i.e. $\forall h = 1...H < p^*, \Omega_h >> 0$, then by Lemma 3, Chapter 4 of Arrow and Hahn [3] the quasi-equilibrium is also a competitive equilibrium. Furthermore, since the quasi equilibrium $p^* \in S(E)$, then the set $S(E) \neq \emptyset$. To prove existence we consider two cases: first, the case where $\forall h, < q, v >> 0$. In this case, by the above remarks, (p^*, x^*) is a competitive equilibrium.

The second case is when $\forall q \in S(E)\ \exists\ h \in \{1...H\}$ s.t. $< p^*, \Omega_h >= 0$, a case where the vectors p^* and Ω_h must have some zero coordinates. The limited arbitrage condition in this case implies

$$\exists q^* \in S(E): \ \forall h, \ < q^*, v >> 0 \text{ for all } v \in A(\rho_h, \Omega_h). \tag{16}$$

Let $x^* = x^*_1...x^*_H$ be the allocation in Υ supported by the vector q^* defined in (16). Then by definition, $\forall h$, $x^*_h \geq_{\rho_h} \Omega_h$ and q^* supports x^*.

Recall that any h minimizes costs at x_h^* because q^* is a support. Now, (q^*, x^*) can fail to be a competitive equilibrium only when for some $h < q^*, x_h^* >= 0$, for otherwise the cost minimizing allocation is also utility maximizing in the budget set $B_h(q^*) = \{w \in X :< q^*, w >=< q^*, \Omega_h >\}$. It remains therefore to prove existence when $< q^*, x_h^* >= 0$ for some h. Since by the definition of $S(E)$, x^* is individually rational, i.e. $u_h(x_h^*) \geq u_h(\Omega_h)$, it follows that when $< q^*, x_h^* >= 0$, then $< q^*, \Omega_h >= 0$, because q^* is a supporting price for x^*. If $\forall h$, $u_h(x_h^*) = 0$ then $x_h^* \in \partial R_+^N$, and by the monotonicity and quasi-concavity of u_h, any vector $y \in B_h(q^*)$ must also satisfy $u_h(y) = 0$, so that x_h^* maximizes utility in $B_h(q^*)$, which implies that (q^*, x^*) is a competitive equilibrium. Therefore (q^*, x^*) is a competitive equilibrium unless for some h, $u_h(x_h^*) \neq 0$,. Assume then that (q^*, x^*) is not a competitive equilibrium. Then for some h, $u_h(x_h^*) \neq 0$, and therefore an indifference surface of a positive commodity bundle of u_h intersects ∂X at $x_h^* \in \partial X$. Let r be the ray in ∂X containing x_h^*. If $w \in r$ then $< q^*, w >= 0$, because $< q^*, x_h^* >= 0$. Since $u_h(x_h^*) > 0$, by the assumptions on u_h, all other indifference surfaces of u_h with higher utility intersect r, so that $r \subset A(\rho_r, x_h^*)$. Define now the ray $s = \{v : \exists w \in r : v = (x_h^* - \Omega_r) + w\}$. The ray $s \subset \partial X$; $s \subset A(\rho_h, \Omega_h)$ and $\forall v \in s < q^*, v >=< q^*, (x_h^* - \Omega_r) + w >= 0$. But this contradicts the choice of q^* as a supporting price satisfying (16) since

$$\exists h \text{ and } y \in A(\rho_h, \Omega_h) \text{ such that } < q^*, y >= 0. \tag{17}$$

The contradiction between (17) and (16) arises from the assumption that (q^*, x^*) is not a competitive equilibrium. Therefore (q^*, x^*) must be a competitive equilibrium, and the proof of the theorem is complete.◊

4.1 Subeconomies and Similar Preferences

Having completed the proof of the main result which establishes that limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium, it seems useful to point out that the condition of limited arbitrage need only be satisfied on subeconomies with no more traders than the number of commodities plus one. This is the next theorem, which also requires a definition of spaces of preferences similar to those in the market E.

Consider a market economy E as in Theorem 1, $X = R^N$. Then a smooth preference σ defined over allocations for $k \leq H$ traders, i.e. over X^k, is called *similar* to a preference of trader h in position j, when $\forall x \in X^r$, the $j-th$ coordinates of the gradient of σ, denoted $D\sigma_j$ satisfy

$$D\sigma_j(x) \in D_h(\rho_h, \Omega_h).$$

The interpretation of this condition of similarity is that the preference σ increases in the direction of that of the trader h in position j for choices with large utility values. We say that P_θ is *a space of preferences similar to those in the subset of traders* $\theta \subset 1...H$ in the market E when it consists of preferences which are similar to those of some trader i in the set θ, in some position j :

$$\sigma \in P_\theta \Leftrightarrow \forall x \in X^k, \forall j = 1...k,\ D\sigma_j(x) \in \bigcup_{h\in\theta} D_h(\rho_h, \Omega_h).$$

Theorem 6 *Consider a market economy E as in Theorem 1. The following five properties are equivalent:*

(a) E has a competitive equilibrium

(b) Every sub economy of E with at most $N+1$ traders has a competitive equilibrium

(c) E has limited arbitrage

(d) E has limited arbitrage for any subset of traders with no more that $N+1$ members.

(e) For any space of preferences P_θ similar to those of a subset θ of market traders in E, there exists a continuous anonymous social choice map $\phi : (P_\theta)^r \rightarrow P_\theta$ respecting unanimity, for any $r \leq N+1$.

Proof: The proofs that (a)⇔(c) and that (b)⇔(d) follow directly from Theorem 1. That (c)⇔(d) follows from Theorem 7 in the Appendix. Finally (c)⇔(e) follows from Theorem 2 of Chichilnisky ([18]).◇

5 The Problem of Existence of Competitive Equilibrium

It seems useful to situate the results of the previous section in the context of the literature on the existence of a competitive equilibrium, and to discuss how the limited arbitrage assumption resolves the problem of non-existence.

5.1 Related Literature with Bounds on Short Sales

As already pointed out, not all Arrow-Debreu exchange economies have a competitive equilibrium, even when all individual preferences are smooth, concave and increasing, and even when the consumption sets are positive orthants. We have illustrated in Figure 8 of Section 3 the simplest example of a market with continuous concave and increasing preferences over $X = R^N_+$ which has no competitive equilibrium as provided originally in Arrow and Hahn [3], Chapter 4, p. 80, in a market with two goods and two individuals. The problem is however quite general and it occurs in economies with any number of individuals and of goods. All that is required is that some individual j with an interior endowment Ω_j should have no taste for one of the goods, say good N, and that another trader i should only own such a good, i.e. $\Omega_i = (0...0, x_N)$. Then any supporting price for the good N in $S(E)$ must be zero, and the individual i will always have zero income at all supporting prices in $S(E)$. Then, if any individual in the economy strictly prefers N no competitive equilibrium exists.

We discussed the example in [3] because it is based on the diversity of endowments and preferences of the individuals in the economy: this diversity leads to a failure of continuity of the demand function. With a discontinuous demand, a competitive equilibrium generally fails to exist. "*This discontinuity will necessarily occur in some part of the price space, except in the unrealistic case in which the household has a positive initial endowment of all goods*" (quote from Arrow and Hahn, Chapter 4, p. 80, [3]).

While discussing related literature with bounds on short sales, however, it seems desirable to indicate why we have chosen the concept of a competitive equilibrium over and above all other related equilibrium concepts which appear in this literature, some of which yield existence of an equilibrium under more general conditions. Of course, other concepts of market equilibrium could be utilized to define a market allocation, such as for example *quasi-equilibrium* which was first introduced by Debreu [24], or *compensated equilibrium,* as defined in Arrow and Hahn [3]. These are closely related, but different, definitions.

They define allocations where individuals minimize cost rather than maximizing utility as in a quasi-equilibrium, or which could have excess supply, as in a compensated equilibrium When prices and all individuals' incomes are strictly positive, these concepts agree with the competitive market equilibrium (Arrow and Hahn [3], Chapter 4). These allocations have the advantage that they always exist when preferences are continuous and concave and the individuals' consumption sets are positive orthants, a property that the competitive equilibrium does not share.

However, as Arrow and Hahn point out, the conditions that all prices are strictly positive, or that all individuals should have strictly positive endowments of all goods is unrealistic ([3], Chapter 4, p.80, para.4), so that quasi-equilibrium or compensated equilibrium allocations will not be competitive equilibrium allocations in general. This technical issue has major welfare implications. Economies with a competitive equilibrium stand alone in terms of their welfare properties. This is because quasi-equilibrium or compensated equilibrium allocations do not generally have the Pareto efficiency property that the competitive equilibrium has. Therefore the main justification for using market allocations, which is efficiency, would be lost unless we remain within the confines of a competitive equilibrium. For this reason we concentrate here on competitive equilibrium allocations.

The problems of non-existence of a competitive equilibrium are somewhat different when the consumption set X is the whole Euclidean space - i.e. when there are no bounds on short sales - than when the consumption set X is the positive orthant. Consider first the case when $X = R^N$. In this case, the *limited arbitrage* condition ensures that individuals, who must typically be diverse in order to achieve gains of trade, are not too diverse. For example, there must exist a degree of consistency between individuals' asymptotic cones at the initial endowments: a price hyper plane must exist leaving all these cones to one side, the same for all traders, so that from initial endowments, no individual can afford allocations which lead to unbounded utility at these prices. Figures 6, 7 and 8 illustrate this point, and show when this condition of limited arbitrage fails. The case when the consumption space is $X = R^N_+$ is somewhat different. Figure 8 illustrates a failure of the limited arbitrage condition in this case.

As already indicated none of the economies of Figures 6, 7 or 8 have a competitive equilibrium. Conditions are therefore needed to rule out such economies. The conditions should be defined on the exogenous data which identify the economy, namely on the endowments Ω_h and on the preferences ρ_h of the individuals $h = 1...H$. The *limited arbitrage* condition defined above is therefore a good candidate, because it provides a necessary and sufficient condition for existence of a competitive equilibrium which is defined on these exogenous data of the economy E, namely on endowments Ω_h and on preferences ρ_h. What limited arbitrage does is to limit precisely the degree of diversity among the agents of the economy so that market equilibrium will exist.

Indeed, as already indicated, for economies with consumption bounded below, i.e. $X = R^N_+$, such limits on diversity are implicit in Arrow's resource relatedness [3] and in McKenzie's irreducibility condition [33][34][35]. All these conditions ensure that the endowments of any household are desired, directly or indirectly, by others, so that their incomes cannot fall to zero. In this case our limited arbitrage condition is always satisfied.

Irreducibility and *resource relatedness* conditions work by ensuring that at a quasi-equilibrium or at a compensated equilibrium, all individuals' incomes are strictly positive, or strictly larger than the minimum possible income. When individuals' incomes are all positive, or not minimal, all the notions of equilibrium coincide. The problem of maximizing utility subject to a budget constraint, which is the competitive equilibrium condition, is then identical to that of minimizing the cost of an allocation with a certain utility level, which is the condition which defines a compensated equilibrium. Thus a quasi equilibrium, which always exists when preferences are concave and continuous and the commodity space is the positive orthant, is also a competitive equilibrium.

The key to the conditions of Arrow, Debreu and McKenzie is to eliminate minimum income allocations. Yet traders with zero or minimum income do not by themselves rule out the existence of a competitive equilibrium. An allocation where some individuals have zero, or the minimum possible, income, reflects a real problem: the fact that some individuals are considered worthless, they have nothing to offer that others want. Such a situation could be a competitive equilibrium. It seems realistic that markets could lead to such allocations: one observes them all the time in city ghettos. Our condition of *limited arbitrage* brings out the issue of diversity by focusing on the problem of zero or minimal income individuals. It does not attempt to rule out individuals with minimum income; instead, it seeks to determine if society's evaluation of their worthlessness is shared. Individuals are diverse in the sense of not satisfying limited arbitrage, when someone has minimal income - which requires in turn that some individuals have minimal quantities of some goods - and, in addition, when there is no agreement about the value of those who have minimal income.

In sum: our condition of *limited arbitrage* is geometric in nature: it admits an interpretation as a transversality condition. It bounds the extent of diversity among the market's traders, but it does so in a different way than *irreducibility* [33][34] [35] and *resource relatedness* [3]. The latter two are only applicable to economies where the consumption is bounded below, or where there is a bound on short sales. Instead, *limited arbitrage* is applicable both to this case and also to the case where neither consumption nor short sales are bounded below. Irreducibility and resource relatedness are sufficient conditions for the existence of a competitive equilibrium but not necessary, while limited arbitrage is necessary as well as sufficient for the existence of a competitive equilibrium, Theorem 1. Limited arbitrage is necessary as well as sufficient either when the markets have bounds on short sales or when they do not. Irreducibility and resource relatedness work for the case of bounded consumption sets by ensuring that all individuals' endowments are desired by others so that none will have zero income. Limited arbitrage works differently: in the bounded case, by ensuring sufficient similarity of preferences that even when some individuals may have endowments of zero or minimal value, a competitive equilibrium still exists. We have extended this to more general consumption spaces, and given it a geometrical interpretation as a transversality condition.

Finally we consider the condition that indifference surfaces of preferences of positive consumption bundles should be in the interior of the positive orthant, Debreu [21]. This condition has also a simple geometric interpretation: it implies that the set of directions along which the utilities increase without bound from initial endowments is the same for all traders. This condition implies that all individuals agree on choices with large utility values, again a form of similarity of preferences.

5.2 Related Literature Without Bounds on Short Sales

It may be useful to situate our conditions in the context of other conditions which exist in the literature. We have already referred to three such conditions, *resource relatedness, irreducibility* and the condition that the indifference surfaces of non-zero consumption vectors are in the interior of the positive orthant. All these conditions are sufficient for the existence of a competitive equilibrium, but only in economies with bounds on short sales, i.e. when the consumption set is the positive orthant. We shall add here two more conditions, which apply to economies with no bounds on short sales, i.e. when the consumption set is the whole Euclidean space: *no-arbitrage* and *Condition C.* These two latter conditions help to clarify the connection of our results with those in the finance literature, which defines equilibrium mainly from non-arbitrage conditions.

The first three conditions and Condition C of Chichilnisky and Heal [17] are sufficient for the existence of a competitive equilibrium, but not necessary. Condition C requires that, if along a sequence of allocations the utility of one of the traders increases beyond bound, then there exists another trader whose utility eventually decreases below the level of this trader's initial endowment along this sequence. The condition of no arbitrage is necessary for an equilibrium but it is not generally sufficient. None of these conditions has been used to show the existence of social choice rules. Indeed, to our knowledge, no other condition exists on preferred sets at initial endowments which is both necessary *and* sufficient for the existence of a competitive equilibrium, or which is necessary and sufficient for the existence of social choice rules.

Condition C defined in Chichilnisky and Heal [17] applies to economies without bounds on short sales, where the consumption set is the whole Euclidean space; instead our limited arbitrage condition applies to economies with or without bounds on short sales. Condition C is sufficient for the existence of a competitive equilibrium but it is not necessary. Formally, the asymptotic preferred cone used here to define limited arbitrage is strictly contained in general in the set of unbounded feasible allocations which appears in Condition C. This makes our condition of limited arbitrage strictly weaker than Condition C.

Werner [42] uses a condition of *no-arbitrage*, which requires the existence of a price at which no increases in utility are possible at zero costs. His condition is related to, but quite different from, our condition of *limited arbitrage*. There is a geometric difference which has major practical implications. Formally, our cones defining *limited arbitrage* consist of rays which intersect every indifference surface of an individual's preference corresponding to utility values above that of the initial endowment. Instead, the "recession" cones used by Werner to define no-arbitrage generally do not satisfy this condition [42]. This formal difference leads to substantial differences in results, and it allows ours to be considerably stronger. As shown above, our condition depends on endowments as well as preferences: with the same preferences, our economy will satisfy limited arbitrage for certain initial endowments of the traders and not for others. Therefore, for the same individual preferences, our market economy will have a competitive equilibrium for some individual endowments and not for others: this seems natural. Indeed, one expects that the similarity of individuals should be defined in terms of their endowments as well as in terms of their preferences. The existence of a competitive equilibrium should also generally depend not only on individuals' preferences, but also on their endowments, and this is precisely what limited arbitrage shows.

In contrast, Werner's cones are assumed to be the same at every allocation ([42], Assumption A3 and Proposition 1), so that in particular his condition depends on preferences but does not depend on endowments. *No-arbitrage* requires the existence of a price at which no individual can make positive utility increases at zero costs, a condition which must be verified in principle at all allocations [42]. Our condition of limited arbitrage requires, instead, that there should exist a price at which only finite utility increases are achievable at zero cost from initial endowments: limited arbitrage needs only be satisfied at one allocation, the initial endowment. Werner's condition is binding only when consumption sets are not bounded below and it is always satisfied otherwise ([42], Section 6, p. 1414) while, as already pointed out, limited arbitrage is binding whether consumption sets are bounded below or not. Furthermore, the *no-arbitrage* condition is sufficient, but it is not necessary, for the existence of a competitive equilibrium unless preferences have no linear half-subspaces in their indifference surfaces: this eliminates linear and piecewise linear preferences, see Theorem 1 [42]. In contrast, such preferences are included in our framework. No-arbitrage is neither necessary nor sufficient for the existence of a competitive equilibrium when the consumption sets are bounded below; in such cases it is always satisfied. Instead, *limited arbitrage* is binding, and necessary as well as sufficient, in all cases: when consumption sets are bounded below and when they are not.

6 Limited Arbitrage and Social Choice

It seems desirable to explain at this point the link between the existence of social choice rules and the existence of a competitive equilibrium. Consider a space P_E of preferences which are similar to those of the traders in E, as defined in Chichilnisky [18]. These are preferences which increase in the directions defined by the asymptotic preferred cone of some trader in the economy E, see Theorem 6 above. From Chichilnisky and Heal [12] we know that a social choice rule on this space of preferences exists if and only if the space of preferences P_E is *contractible* - as defined in Section 2. This condition of contractibility means that there exists a continuous way of deforming the preferences through the space P_E, so that at the end of this process we have complete unanimity. With contractibility of the space P_E we are assured of the existence of a social choice map satisfying the required axioms[2] and therefore we are assured of a resolution to the resource allocation problem from the point of view of social choice. But contractibility is a restriction on diversity, no more and no less. It tests whether there is a way of deforming continuously our space of individual preferences into itself so that at the end of this deformation all the individuals have identical preferences. For a discussion of the role of contractibility in public decision making see e.g. Heal [29]. Thus we are back at the source of the problem of resource allocation in market economies: individual diversity.

Theorem 1 establishes precisely the degree of diversity which is necessary and sufficient to solve the allocation problem in markets - *limited arbitrage.* Elsewhere [18] we have shown that this same degree of diversity is needed to solve the allocation problem with social choice rules. In other words, necessary and sufficient conditions for the existence of a competitive equilibrium - *limited arbitrage* - are also necessary and sufficient for the existence of social

[2]The axioms used are continuity, anonymity and respect of unanimity.

choice rules. As formulated here, the two problems of resource allocation, by markets and by social choice, are therefore equivalent. Indeed, in Chichilnisky [18], Theorem 3, we have also shown that market allocations are always social allocations.[3] Theorem 6 above proves that we only need to restrict the degree of diversity to subsets of at most $N+1$ traders, where N is the number of commodities in the economy.

7 Conclusions

We have shown that *limited arbitrage* is a necessary and sufficient condition for the existence of a competitive equilibrium in Arrow-Debreu economies with or without bounds on short sales. The same condition - limited arbitrage - was shown elsewhere [18] to be necessary and sufficient for the existence of a continuous anonymous social choice map respecting unanimity on the space of all preferences which are similar to those of the traders in the economy E. In this particular sense, the results of this paper and of [18] unify two forms of resource allocation, by markets and by social choices, which have developed separately and remained quite separate until now.

We have chosen competitive equilibrium allocations - and no other forms of equilibrium - because of the Pareto efficiency of competitive equilibrium, a property which is generally lost in weaker forms of market equilibrium, such as quasi-equilibrium and compensated equilibrium.

The interpretation of the limited arbitrage condition is somewhat different when short trades are bounded and when they are not, although mathematically they are very similar. In the former case it measures social agreement about allocating minimal value to the endowments of certain members of society, and this agreement, clearly, must include those same individuals to which society assigns minimal value. It may seem surprising that such an agreement could exist. In the case that it does not, both forms of resource allocation break down: the competitive market has no competitive equilibrium, and the social choice map does not exist.

The connection between the existence of a competitive equilibrium and the manipulation of market games would be a natural extension of these results. This could follow from the connection between the existence of social choice maps and the manipulation of games, Chichilnisky [15]. It also seems possible to extend the results of this paper to economies with production. Issues of survival and underemployment in market economies are also directions in which to extend the inquiry of this paper. Finally, recent results show that it is possible to develop algorithms for computing a competitive equilibrium from the limited arbitrage condition (Chichilnisky and Eaves [19]).

8 Appendix

The following results, Theorems 7 and 8, have not been used, and are not needed, in establishing that limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium, which is Theorem 1. They have an auxiliary role. Theorem 7 below is used to

[3] I.e. allocations that maximize a social ordering derived from individual preferences by a social choice rule, over the set of feasible allocations.

link the condition of limited arbitrage to the contractibility of spaces of similar preferences, a condition which is necessary and sufficient in the existence of social choice functions, see Chichilnisky and Heal [12] and which is used in Theorem 6. Theorem 8 is used in Theorem 6 in proving that for the economy E to have a competitive equilibrium limited arbitrage must only be satisfied on subsets of at most $N+1$ traders, where N is the number of commodities in the economy. This condition simplifies the requirements of verifying limited arbitrage, restricting this to subsets of at most $N+1$ trades in the economy.

Theorem 7 *Consider a family* $\bigsqcup = \{U_i\}_{i=1...H}$ *of convex sets in* R^N, $H, N \geq 1$.

$$\text{Then } \bigcap_{i=1}^{H} U_i \neq \emptyset \Leftrightarrow \bigcup_{i \in J} U_j \text{ is contractible}, \forall J \subset \{1...H\}.$$

This is also true of acyclic families, which are those families $\bigsqcup = \{U_i\}_{i=1...H}$ *consisting of acyclic sets (i.e. the sets* U_i *have zero homology), such that the intersection of any subfamily is either acyclic or empty.*

Proof: This theorem was originally proved in Chichilnisky [13]; see also Chichilnisky [14] for the proof. Theorem 3 implies Helly's theorem [30] and the Knaster-Kuratowski-Marzukiewicz theorem [14]- the latter of which implies in turn the Brouwer's fixed point theorem.◊

Theorem 8 *Consider a family* $\bigsqcup = \{U_i\}_{i=1...H}$ *of convex sets in* R^N $H, N \geq 1$. *Then*

$$\bigcap_{i=1}^{H} U_i \neq \emptyset \text{ if and only if } \bigcap_{j \in J} U_i \neq \emptyset$$

for any subset of indices $J \subset \{1...H\}$ having at most $N+1$ elements.

In particular, an economy E as defined in Section 2 satisfies limited arbitrage, if and only if it satisfies limited arbitrage for any subset of $k \leq N+1$ traders, where N is the number of commodities in E.

Proof: See Chichilnisky [14].◊

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