Markets, Arbitrage and Social Choices

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Abstract

The paper establishes a clear connection between equilibrium theory and social choice theory by showing that, for a well defined social choice problem, the conditions which are necessary and sufficient to solve this problem are the same as the conditions which are necessary and sufficient to establish existence of a competitive equilibrium. We define a condition of *limited arbitrage* on the preferences and the endowments of an Arrow-Debreu economy. This bounds the utility gains that the traders can afford from their initial endowments. Theorem 2 proves that limited arbitrage is necessary and sufficient for the existence of a social choice rule which allocates society's resources among individuals in a manner which depends continuously and anonymously on their preferences over allocations, and which respects unanimity. *Limited arbitrage* is also necessary and sufficient for the existence of a competitive equilibrium in the Arrow - Debreu economy, with or without bounds on short sales, Theorem 7. Theorem 4 proves that any market allocation can be achieved as a social choice allocation, i.e. an allocation which is maximal among all feasible allocations according to a social preference defined via a social choice rule which is continuous, anonymous and respects unanimity.

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1 Introduction

In a world with finite resources, there will generally be conflicting opinions about how resources should be allocated among the members of society. This leads to the classical resource allocation problem, perhaps the most basic and challenging problem in economics. An acceptable, agreed solution to the resource allocation problem holds a society together, providing a "social contract".

An *allocation* is a distribution of endowments across individuals. It consists of H vectors x_i in R^N, one for each individual. A *feasible allocation* is one adding up to the total resources available to the economy, $\sum_{i=1}^{H} x_i = \Omega$. The *resource allocation problem* consists of finding a feasible allocation with desirable or acceptable properties.

One widely used solution to the resource allocation problem is provided by markets. The *market solution* finds a feasible allocation which is individually optimal and which clears the markets. Each individual initially owns a vector of commodities $\Omega_i \in R^N$, $i = 1...H$, and has a preference ρ_i over his/her private consumption. Society's resources are the sum $\Omega = \sum_i \Omega_i$. The allocation chosen is one where individuals maximize their preferences within their budgets and all markets clear. When markets are competitive, as in the Arrow-Debreu specification, then a market allocation is Pareto efficient under classical assumptions. Efficiency is a major virtue of market allocations, and is principally what makes them desirable.

A different solution to the resource allocation problem is provided by *social choice rules*. These are rules for deriving a social preference as a function of individual preferences, for example through voting. They are "universal" in the sense of Kant [28]: the principle followed to derive the social preference must apply consistently *a priori* to all societies with all sorts of endowments and individuals. The social preference ranks allocations by their social desirability. An allocation which is optimal among all feasible allocations according to a social preference derived via a social choice rule which satisfies desired axioms, is called a *social allocation*.

This paper establishes a clear connection between equilibrium theory and social choice theory by showing that, for a well defined social choice problem, a condition which is necessary and sufficient to solve this problem - *limited arbitrage* - is the same as the condition which is necessary and sufficient to establish the existence of an equilibrium (Theorems 2 and 7)[1]. Theorem 4 strengthens this connection by establishing that a market allocation can always be realized as a social allocation.

The first problem of resource allocation is to establish the existence of solutions: the existence of social choice rules and the existence of a market equilibrium. Each market trader has a preference over private consumption, leading naturally to a preference over allocations of resources in society. We consider a universe consisting of all preferences which are similar to those of the traders, and

[1] Comments and suggestions from Andreu Mas Colell, Geoff Heal, Eric Maskin, Myrna Wooders, three anonymous referees and the participants of several seminars in Europe and the U.S. are gratefully acknowlegded

ask whether a social choice rule in this universe exists satisfying adequate axioms. Preferences are similar when they have similar views on choices of large utility values: a preference ρ is *similar* to another σ when ρ increases in all those directions of large utility gains for σ, and only in those directions.

Theorem 2 shows that a social choice rule exists on spaces of preferences similar to those of the traders, if and only if the preferences and the endowments of the traders satisfy a *limited arbitrage* condition. Furthermore, this same condition is necessary and sufficient for ensuring the existence of a competitive equilibrium of the economy, Theorem 7, Chichilnisky [17]. Having connected the two problems of existence of resource allocations, it remains to connect the allocations themselves. This is achieved in Theorem 4, which shows that the two forms of resource allocation are closely related: any market allocation can be achieved as a social allocation.

The role of the *limited arbitrage* condition is to restrict the diversity of the traders in the economy. In economic terms, it bounds the potential gains from trade, by defining a price at which only limited increases in utility are affordable from initial endowments. In mathematical terms, *limited arbitrage* is the non-empty intersection of a family of cones, and, as shown in Theorem 5 in the Appendix, it is in fact identical to a topological condition: the *contractibility* of spaces of preferences. But the contractibility of spaces of preferences is a restriction on the diversity of the preferences, no more and no less. Furthermore, contractibility was shown elsewhere to be necessary and sufficient for the existence of continuous anonymous social choice rules respecting unanimity, Chichilnisky and Heal [20]. Therefore limited arbitrage provides a clear connection between equilibrium theory and social choice. By restricting social diversity, it ensures the existence of a social choice rule and also of a competitive equilibrium.

While a market allocation is desirable by virtue of its efficiency, a social allocation is desirable by virtue of its ethical properties. To ensure such properties, the rule that derives social preferences from individual preferences must satisfy certain ethical axioms. Equal treatment, or *anonymity*, is one axiom. *Respecting unanimity* is another: it means that when all individuals agree on the ranking of all possible allocations, society will adopt this common preference. These two axioms, plus a condition of *continuity* to assure statistical tractability, were first proposed in Chichilnisky [11], [12]. These axioms are an alternative to Arrow's [2] classic axioms of social choice, and, as shown in Section 8.2, are not comparable with his. Arrow's axioms work best for finite sets of choices. Our resource allocation problem, however, requires choosing among an infinite number of choices, indeed a Euclidean set of choices: the space of all possible allocations. Here the axioms of [12] seem better suited. These axioms, which have been developed in several directions in the last decade [5],[6],[33],[13], have a useful property: there are simple necessary and sufficient conditions on the domain of preferences which ensure the existence of a social choice rule, Chichilnisky and Heal [20]. Such results, which are not available with Arrow's axioms, are key for the understanding of the connection between market allocations and social allocations.

1.1 Efficiency and Social Ethics

Between the two forms of resource allocation, by markets and by social choice, stand two wedges: one is intellectual and the other practical. The intellectual wedge is based on a separation of efficiency and of ethical concerns. Economics treats these two issues separately. It regards markets as a practical representation of *positive economics*, the way things are. Social choice, which is associated with ethical concerns, falls into the rubric of *normative economics*, the way we may want things to be. This wedge has succeeded in keeping the literature on these two forms of allocation quite separate, and almost antagonistic to each other.

The second wedge is practical, although practical concerns are sometimes a reflection of intellectual beliefs. Market allocations have always been considered more practical than social allocations, although in reality all modern democracies use both side by side at all times. One reason for this is that markets are viewed as having equilibria very generally, while social choice theory has always stressed paradoxes and non-existence results. Kenneth Arrow's work, which developed fundamental insights into the two theories, appeared to provide fuel for this viewpoint. Arrow's impossibility theorem for social choice [2] led to a large literature focusing on the difficulties of finding acceptable social allocations. Instead, Arrow's result on existence of a market equilibrium with Debreu [4] led to more and more general existence theorems which reinforced our view of market allocations as being always available. In all fairness, Arrow and Debreu discussed the problems of non-existence of a competitive equilibrium created by the discontinuity of uncompensated demand when some prices are zero ([4], Sections 4 and 5) and Arrow and Hahn ([3], Chapter 4, 1) provided examples of standard market economies with no competitive equilibrium. Nevertheless, the results on existence of an equilibrium took precedence in the literature.

1.2 Social Diversity and Resource Allocation

What is interesting about Arrow and Hahn's example [3] of the non-existence of a competitive equilibrium is that it arises due to sharp interpersonal differences between the traders, differences in their endowments and in their preferences. Such differences emerge, for example, when some traders have zero endowments of some goods, a situation which Arrow and Hahn find realistic ([3], Chapter 4, p. 80). It is notable that the social choice literature has also focused on interpersonal diversity as a reason for the non-existence of social choice rules. Prominent examples are the work of Black [7][8], Pattanaik and Sen [34], and Chichilnisky and Heal [20]. These works offer conditions for resolving social choice problems by limiting the diversity of the individual preferences, which is usually called a "domain restriction" on preferences. Domain restrictions are simply a way of limiting the diversity of individuals. One is turning the issue of existence of universal social choice rules - a problem which in its more general form has no solution - into the question: for what societies can the social choice problem be resolved? Or: how much diversity can a society function with?

Black's singlepeakedness condition restricts diversity and solves the problem proposed by Condorcet's [21] paradox of majority voting; Pattanaik and Sen [34] do the same, finding domain restrictions which assure the existence of majority rules satisfying Arrow's axioms of social choice, and Chichilnisky and Heal [20] find domain restrictions which are necessary and sufficient for the existence of social choice rules which satisfy the axioms of [11]. Although these works deal with somewhat different axioms, they all find the same type of solution: a restriction of individuals' diversity.

As already pointed out, similar restrictions of individual diversity are implicit in the conditions for existence of a competitive equilibrium which developed in order to resolve Arrow's example of non-existence of a competitive equilibrium: Arrow and Debreu's conditions on endowments of any household being desired, indirectly or directly, by others, so that their incomes cannot fall to zero [4], preferences with indifference surfaces which never which never meet the axis, Debreu [22], MacKenzie's *irreducibility* condition [30][31][32], and the *resource relatedness* condition in Arrow and Hahn [3]. These conditions are somewhat different, but they all have the same effect: to restrict the diversity of individuals' preferences and endowments. This is discussed further in 6.1 and 8.1 below.

1.3 Unifying Two Approaches to Resource Allocation

The aim of this paper is to show that a restriction on individual diversity which is necessary and sufficient to secure the existence of a market allocation, is the same as that which is necessary and sufficient to secure the existence of a social allocation. The condition of *limited arbitrage* introduced in Chichilnisky [17], which is defined on individual preferences at their initial endowments, unifies the two problems.

Limited arbitrage is the non-empty intersection of a family of cones; it admits also an interpretation as a contractibility condition of spaces of preferences over allocations which are similar to those of the traders (Theorem 5). The term similarity of preferences refers to a form of agreement on choices of large utility values. Formally, a space of preferences similar to those in the market consists of preferences which increase in those directions which give unbounded utility gains to some trader in the market, and only those directions. Lemma 1 establishes that the gradients of similar preferences are in the union of a family of cones, each cone consisting of vectors having positive inner products with directions of unbounded utility increases. If the union of these cones is contractible, then the social choice problem on spaces of similar preferences has a solution. Indeed the social choice problem has a solution when, and only when, this union is contractible: this is Theorem 1 in Chichilnisky and Heal [13]. But the union of these cones is contractible if and only if their intersection is non empty: this is Theorem 5 in the Appendix. Furthermore, the non-empty intersection of the dual cones is *limited arbitrage, by definition.* Therefore social choice rules exist if and only if the limited arbitrage condition is satisfied (Theorem 2). Limited arbitrage is also necessary and sufficient for the existence of a competitive equilibrium (Theorem 7). *Limited arbitrage* provides therefore a clear link between the problem of existence of social choice rules and that of existence of a competitive equilibrium. In this sense, the two problems of resource allocation, the existence of market allocations and of social choice rules, are one and the same. One problem admits a solution when and only when the other one does. Theorem 3 proves that the condition of limited arbitrage need only be required on subsets of traders with cardinality smaller than the dimension of the commodity space. Theorem 4 proves that a market allocation is always a social allocation.

The following sections provide definitions, a formal statement of the theorems, and their proofs. The conclusions summarize the results, and an Appendix provides background results.

2 Definitions and Examples

Consider an Arrow-Debreu pure exchange private market economy E. There are $N \geq 1$ commodities, and the consumption space X is either the positive orthant $X = R^N_+ = \{y = (y_1 ... y_N) \in R^N : \forall i, y_i \geq 0\}$, or all of the Euclidean space $X = R^N$. For vectors $x, y \in R^N$ we use the standard notation: $x \geq y \Leftrightarrow \forall i, x_i \geq y_i$, $x > y \Leftrightarrow x \geq y$ and for some $i, x_i > y_i$, and $x >> y \Leftrightarrow \forall i, x_i > y_i$. $R^N_{++} = \{y \in R^N : \forall i, y_i >> 0\}$ The economy has $H \geq 2$ traders indexed by $i = 1 ... H$. Each has a non-zero initial endowment vector $\Omega_i \in R^N_+$, where Ω_i may have some coordinates equal to zero. Each trader has a preference ρ_i over private consumption in X, which is concave and monotonic. When ρ_i prefers x to y we write $x \succeq_{\rho_i} y$; $x \succ_{\rho_i} y$ means that x is strictly preferred to y by ρ_i. The preferences of the traders are monotonic: $\forall x, y \in X$, if $x \geq y$ then $\forall i$, $x \succeq_{\rho_i} y$.

$\sum_{i=1}^{H} \Omega_i = \Omega$ are the *total endowments* of the economy.

The *space of allocations* is $X^H = \{(x_1 ... x_H) \in R^{NH} : x_i \in X\}$.

A *market economy* is therefore defined by $E = \{X, \Omega_i, \rho_i, i = 1 ... H\}$.

A $k-$ *trader sub-economy of* E is an economy consisting of a subset of $k \leq H$ traders in E, each with their endowments and preferences:

$$F = \{X, \rho_j, \Omega_j, j \in J \subset \{1 ... H\}, \mathit{cardinality}(J) = k\}.$$

The *space of resource allocations* or *feasible allocations* is $\Upsilon = \{x_1 ... x_H \in X^H : \sum_{i=1}^{H} x_i = \Omega\}$.

In order to compare market allocations and social allocations we shall consider preferences which are either defined over private consumption, namely over the consumption set X, or preferences which are defined over resource allocations for the H individuals of the economy, namely over the set X^H. The preferences in a market economy E are defined over private consumption, over the space X. Instead, the preferences which are the arguments of social choice rules are defined over allocations, of commodity vectors across the individuals in the economy, i.e. over X^H.

We consider smooth preferences, as defined in Debreu [24], see also Chichilnisky [11]. A *smooth preference* defined on private consumption, i.e. on X , is one which can be represented by a smooth utility function. When $X = R^N_+$ the utilities are defined either in the interior of X, $Int(X)$ or on a neighborhood of X. Without loss of generality we assume that the utility value of zero consumption is zero. In Debreu [24] a smooth preference is identified by the normal directions to its indifference surfaces. At each point $x \in X$ the normal direction to a preference ρ is defined by the gradient vector of the utility function u which represents the preference, normalized to have length one, denoted $\rho(x) = Du(x)/||Du(x)||$. A standard topology for spaces of smooth preferences defines the proximity of two preferences ρ and σ by the uniform proximity of the preferences' normals, i.e. by the sup norm on the normals, see Chichilnisky [11], $Sup_{x \in X}||\rho(x) - \sigma(x)||$. Such a topology will be utilized here. A similar definition is given for smooth preferences over allocations.

When $X = R^N_+$ our preferences may have all indifference surfaces corresponding to positive utility values contained in R^N_{++}, or they may instead have indifference surfaces which intersect the boundary of the positive orthant. In the former case the preferences are smooth in R^N_{++}, and in the latter they are smoothly defined on a neighborhood of R^N_+. The former is a standard specification of preferences which includes Cobb-Douglas utilities, CES utilities with elasticity of substitution $\sigma < 1$ and the classical preferences considered in Debreu [22] and in Arrow and Debreu [4]. The latter case includes preferences whose indifference surfaces intersect the boundary of the positive orthant, such as CES utilities with elasticity of substitution $\sigma > 1$, and the type of preferences considered in Arrow and Hahn [3]. In this latter case, we assume that the utilities representing the preferences are transversal to the boundary of the positive orthant so that if $u(y) > 0$ for $y \in \partial X$, then $\forall x \in \partial X$, $< Du(x), x >> \in > 0$, where $< .,. >$ indicates the standard inner product in R^N, see e.g. Smale [35]. A simple interpretation of this transversality condition is that when a boundary vector $x \in \partial X$ has a positive utility, $u(x) > 0$, then the ray of boundary vectors it defines achieves all utility levels higher than $u(x)$.

When $X = R^N$ we assume that the set of gradients of an indifference surface is closed in R^N. This is not required when $X = R^N_+$.

Examples of market economies E without a competitive equilibrium. The endowments and preferences in our market economy E follow a general specification and therefore, without further conditions, E may have no competitive equilibrium. An example is a two-person economy with two different linear preferences, each defined over the consumption set $X = R^2$ as illustrated in Figure 5 below; such an economy has no competitive equilibrium. The economy of Figure 5 has no lower bounds for its consumption set X, which means that there are no bounds on short sales. Figure 6 provides a different example: a two-person economy with consumption set $X = R^2_+$, and with endowments and preferences as illustrated. Even though the consumption set in the example of Figure 6 is $X = R^2_+$ and therefore bounded below, so that there are bounds on short sales, this economy has no competitive equilibrium. This is discussed further in Section 6.1.

Two topological spaces X and Y are ***homeomorphic***, denoted $X \approx Y$, if there exists a one to one map $F : X \to Y$ with a continuous inverse $F^{-1} : Y \to X$, i.e. $F^{-1}(F(x)) = x \; \forall x \in X$.

A *connected topological space* Y is a topological space which cannot be expressed as the union $Y = A \bigcup B$ of two subsets, A and B, each of which is simultaneously open and closed in Y.

This formalizes the notion that any element $y \in Y$ may be connected to any other $z \in Y$ through a continuous path in the space Y.

A topological space Y is called *contractible* when there exists a continuous deformation of the space through itself into one of its points. Formally, Y is contractible when there exists a continuous map

$$F : Y \times [0.1] \to Y, \textit{ and a point } z_0 \in Y \textit{ s.t.}$$
$$\forall z \in Y, \; F(z, 0) = z \textit{ and } F(z, 1) = z_0.$$

Examples of contractible and non-contractible spaces. **If a space X is homeomorphic to a contractible space Y, then X is contractible. All convex sets are contractible spaces, but contractible spaces are generally not convex. Examples of contractible spaces which are not convex are the "star shaped" spaces, or non-convex spaces which are homeomorphic to convex sets, see Figure 1:**

Figure 1
A star shaped space, a contractible non-convex space
and the torus $S^1 \times S^1$ which is not contractible.

A disconnected space is not contractible. Any contractible space X is connected, but the converse is not true: the space of all non-zero linear preferences on R^2 is the unit circle S^1 (Chichilnisky [11],[12]), which is a typical example of a connected space which is not contractible. Another example of a space which is not contractible is the space of preference profiles of two individuals with non-zero linear preferences on R^2 each: this is the "torus" , which is the product of the unit circle with itself $S^1 \times S^1$ (Chichilnisky [11],[12]). A product of contractible spaces is contractible. A product of spaces which are not contractible, is not contractible. A discussion of the role of contractibility in public decision making is in Heal [26]. It was shown in Chichilnisky [11][12] that the space of all smooth preferences defined on Euclidean choice spaces is not contractible. The space of all linear preferences defined on Euclidean space R^N is not contractible either: this space is the union of the sphere S^{N-1} and the vector $\{0\}$,Chichilnisky[11],[12]. The space of all smooth preferences defined on Euclidean space for which there exists a one dimensional subspace which intersects transversely all the indifference surfaces of each preference, is contractible; this was proved in Chichilnisky [10]. This result will be used later in this paper.

It turns out that contractibility is a crucial condition for the existence of social choice maps: it has been proved that a social choice rule $\Phi : P^k \to P$ satisfying desirable axioms exists $\forall k \geq 2$ if and only if P is contractible, Chichilnisky and Heal [13].

The set of *supports to individually rational efficient resource allocations* of the economy $E = \{X, \Omega_i, \rho_i, i = 1...H\}$ is:

$$S(E) = \{v \in R^N : \exists (x_1...x_H) \in \Upsilon \text{ with } x_i \geq_{\rho_i} \Omega_i,\ i = 1...H, \\ \text{and } \forall x \in X,\ x \geq_{\rho_i} x_i \Rightarrow \ \langle v, x - x_i \rangle \geq 0\}. \tag{1}$$

This is the set of prices which support those feasible and efficient allocations which all individuals prefer to their initial endowments.

2.1 Asymptotic Preferred Cones and Dual Cones

Consider an economy $E = \{X, \rho_i, \Omega_i, i = 1...H\}$.

2.1.1 The case $X = R^N$

Consider a given preference ρ_i in E and an initial endowment vector $\Omega_i \in X$.

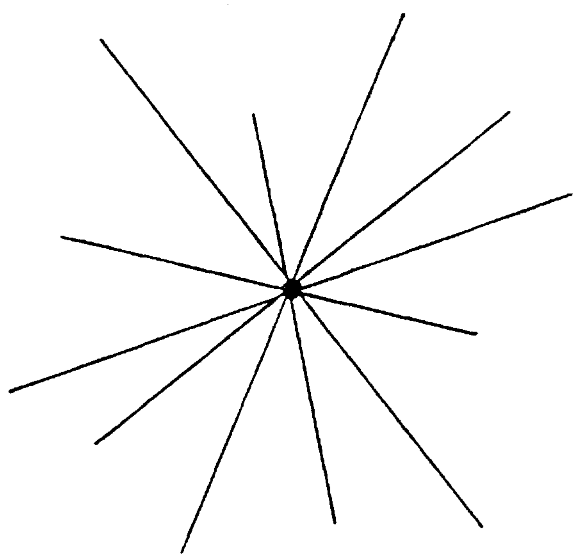
The *asymptotic preferred cone of $\rho_i \in E$ at the initial endowment* Ω_i, $A(\rho_i, \Omega_i)$,is:

$$A(\rho_i, \Omega_i) = \{y \in R^N : \forall \lambda > 0,\ (\Omega_i + \lambda y) \succ_{\rho_i} \Omega_i, \\ \text{and } \forall z \in X, \exists \lambda > 0 \ s.t.\ (\Omega_i + \lambda y) \succ_{\rho_i} z\} \tag{2}$$

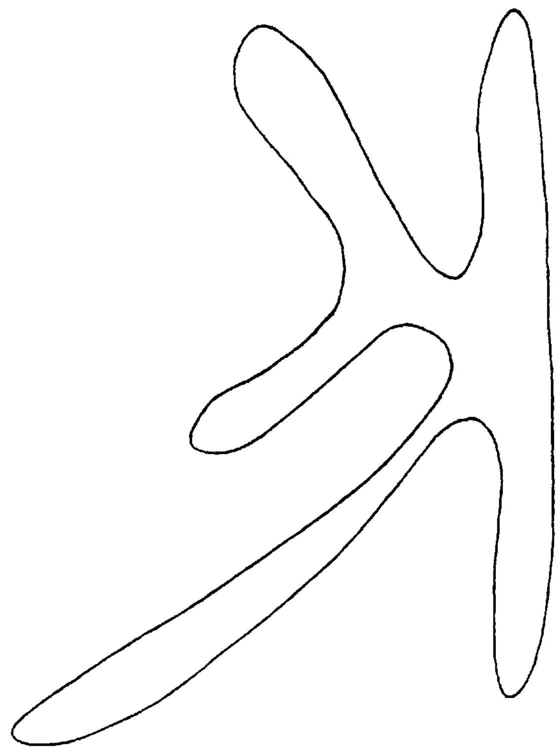
Figure 2
Example : The asymptotic preferred cone $A(\rho_i, \Omega_i)$ of a
preference ρ_i in the market economy E translated to the initial endowment Ω_i.
The consumption set is $X = R^2$.

Figure 1

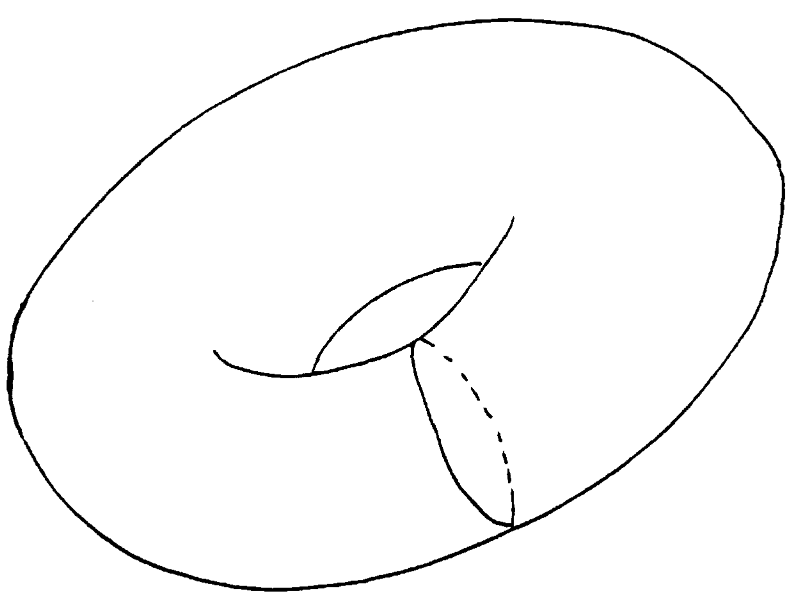
Examples of contractible and non-contractible sets



Star Shaped: contractible



Contractible set
homeomorphic to a
convex set.



The torus $T = S^1 x S^1$ is not contractible.

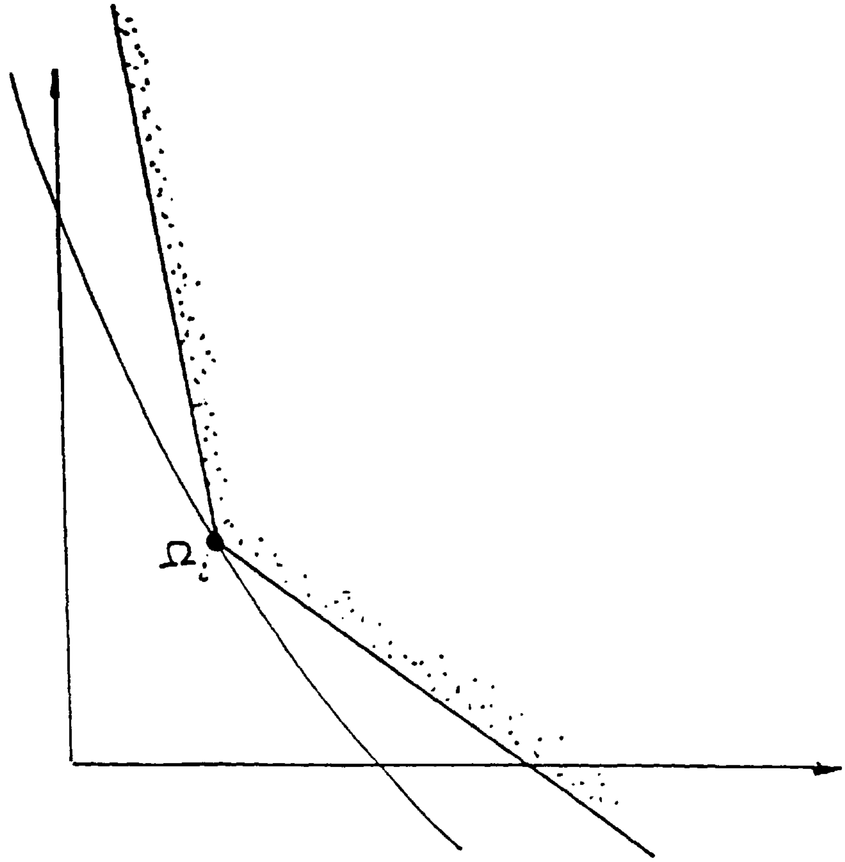


Figure 2

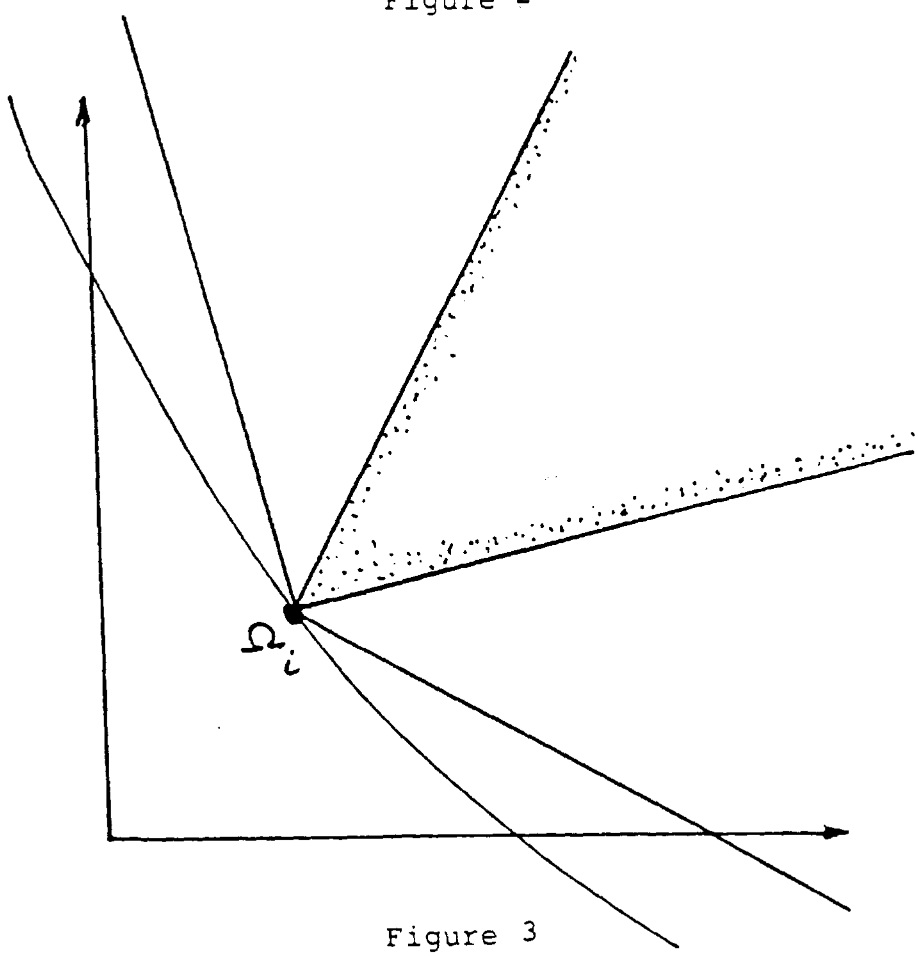


Figure 3

Relation with other asymptotic and recession cones: Our asymptotic preferred cone $A(\rho_i, \Omega_i)$ is related to Debreu's [22] definition of the "asymptotic cone" corresponding to the preferred set of ρ_i at the initial endowment Ω_i, in that along any of its rays utility always increases. $A(\rho_i, \Omega_i)$ has also a similarity with the "recession" cone introduced by Rockafeller and used e.g. in Werner [37]. However, the similarity with those cones ends here, because along the rays in $A(\rho_i, \Omega_i)$ not only does utility increase forever, but it increases beyond the utility level of any other vector in the consumption space X. In ordinal terms, ***the rays of the asymptotic preferred cone $A(\rho_i, \Omega_i)$ must intersect all indifference surfaces of the preference ρ_i corresponding to bundles preferred to Ω_i***. This condition is not necessarily satisfied by Debreu's asymptotic cones [22], nor by Werner's "recession" cones [37]. For example, there are increasing preferences on R^2 in which the cone $A(\rho_i, \Omega_i)$ is different from Debreu's: consider a strictly concave and increasing preference having indifference surfaces which asymptote to a fixed line. Related conditions appear in Chichilnisky [9][10]; otherwise there appear to be no precedents in the literature for cones such as $A(\rho_i, \Omega_i)$.

Note also that ***the asymptotic preferred cone $A(\rho_i, \Omega_i)$ depends on the initial endowments as well as on the preferences.*** It is defined by rays or "directions" from the initial endowment vector Ω_i. As the endowment Ω_i changes, the cone $A(\rho_i, \Omega_i)$ varies. This differs from cones used in other works which are assumed to be the same at all vectors in the consumption set (e.g. Werner[37], Assumption A3 and Proposition 1).

The *dual cone* of $A(\rho_i, \Omega_i)$ is defined by

$$D(\rho_i, \Omega_i) = \{z \in X : \forall y \in A(\rho_i, \Omega_i), < z, y >> 0.\} \tag{3}$$

Figure 3
Example : The dual cone $D(\rho_i, \Omega_i)$ of the preference $\rho_i \in E$ of Figure 2 translated to the initial endowment Ω_i. The consumption set is $X = R^2$.

To simplify notation we shall also denote:

$$A_i = A(\rho_i, \Omega_i)$$

and

$$D_i = D(\rho_i, \Omega_i).$$

2.1.2 The case $X = R^N_+$.

The *asymptotic preferred cone* of the i-th individual in the economy E in this case is similar to the case $X = R^N$, as defined as in (2):

$$\begin{aligned} A(\rho_i, \Omega_i) = \{y \in X : \forall\lambda > 0,\ \lambda y \in X,\ (\Omega_i + \lambda y) \succ_{\rho_i} \Omega_i, \\ and\ \forall z \in X,\ \exists\lambda > 0\ s.t.\ (\Omega_i + \lambda y) \succ_{\rho_i} z\}. \end{aligned} \tag{4}$$

In this case, the cone $A(\rho_i, \Omega_i)$ can only contain strictly positive vectors, or positive vectors with some zero coordinates, i.e. $A(\rho_i, \Omega_i) \subset R^N_+$.

When $X = R^N_+$ the *boundary dual cone* is defined as:

$$\begin{aligned} \partial D(\rho_i, \Omega_i) = \{q \in X :\ \text{if}\ \forall p \in S(E),\ \ < p, \Omega_i >= 0, \\ \text{then}\ q \in S(E)\ \text{and} < q, v >> 0\ \forall v \in A(\rho_i, \Omega_i)\}. \end{aligned} \tag{5}$$

Examples of boundary dual cones: The boundary dual cone ∂D_i is the whole consumption set X when there is at least one support in $S(E)$ which assigns non-zero income to some individual j. But if all supports in $S(E)$ assign some j zero income, then ∂D_i consists of all those prices which assign individual i zero income and at which allocations in A_i have strictly positive cost. In this latter case, therefore $\partial D(\rho_i, \Omega_i) = S(E) \bigcap D(\rho_i, \Omega_i)$.

To simplify notation we shall also denote

$$A_i = A(\rho_i, \Omega_i)$$

and

$$\partial D_i = \partial D(\rho_i, \Omega_i).$$

A preference ρ_i in the market economy E is called ***private*** because it is defined over individual *i's* (private) consumption set X. However, for any number $r \geq 2$ of individuals, ρ_i also defines a ***public preference***, one which is defined over the space X^r of allocations for r individuals: we say that an allocation $(x_1...x_r) \in X^r$ is preferred by individual i in position $j \in \{1...r\}$ to another $(y_1...y_r) \in X^r$ if and only if *i's* preference ρ_i prefers $x_j \in X$ to $y_j \in X$. We may therefore consider a preference over private consumption ρ_i as a preference defined over all allocations in X^r and positions j, using for it the same notation when the meaning is clear from the context.

Let Y be a topological space, [29]. If $Z \subset Y$ the interior of Z, denoted $Int(Z)$ is the largest open subset of Y which is contained in Y. $Cl(Y)$ denotes the closure of Y which is the smallest closed set containing Y. A continuous map $f : Y^k \to Y$ is called ***symmetric*** if it is invariant under the permutations of its arguments, i.e. $f(y_1...y_k) = f(y_{\pi_1}...f_{\pi_k})$ where π is a permutation of the set $\{1...k\}$. Assume that Y is a Hausdorff space [29]. If U, V are two open neighborhoods of $x \in Y$, $U \subset V, U \neq V$, a continuous map $\upsilon : Y \to R$ is called a ***partition of unity*** for Y, U, V, if: $\forall y \in Y - V$, $\upsilon(y) = 0, \forall y \in U$, $\upsilon(y) = 1$, and $\forall y \in Y$, $1 \geq \upsilon(y) \geq 0$. Such partitions of unity always exist in Hausdorff spaces, [29]. Using a partition of unity we may construct new maps from given maps, having specific values, as follows. For any map $f : Y \to Z$ and for any $y \in Y$, $z \in Z$, and open neighborhood U of $y \in Y$, there exists a continuous map $g : Y \to Z$ such that $g(y) = z$ and $g(x) = f(x)\ \forall x \in X - U$. If the map f is symmetric, g can be constructed to be symmetric also.

2.2 Competitive Equilibrium and Market Allocations

A *competitive equilibrium* of the economy $E = \{X, \Omega_i, \rho_i, i = 1...H)$ is a price vector $p^* \in R_+^N$ and a *feasible resource allocation* $(x_1^*...x_H^*) \in \Upsilon$ such that x_i^* optimizes ρ_i over the budget set

$$B_i(p^*) = \{x \in X :< x, p^* >=< \Omega_i, p^* >\}.$$

When our economy E has a market equilibrium, a ***market allocation*** is the allocation $(x_1^*...x_H^*)$ defined by a competitive equilibrium of E, and is therefore Pareto efficient (Arrow [1], Debreu [23]).

2.3 Social Choice and Social Allocations

We shall first define the *social choice problem* in general terms, and then specialize it to the case of social allocations in a market economy E.

2.3.1 Social Choice

Consider a general topological space χ consisting of preferences over the space of allocations X^K for $K \geq 2$ individuals. The preferences in χ need not be those of a market economy. Each individual has a preference $\kappa_i \in \chi$, $i = 1...K$. A *profile of individual preferences* is a list of preferences for the K individuals, i.e. a $K-$tuple of preferences in the space χ, denoted $(\kappa_1...\kappa_K) \in \chi^K$.

The *social choice problem* is defined on any space of preferences χ as the problem of finding for all $K \geq 2$ a map $\phi : \chi^K \to \chi$ such that

$(A1)$ ϕ is *continuous*

$(A2)$ ϕ is ***anonymous***, i.e. $\phi(\kappa_1...\kappa_K) = \phi(\pi(\kappa_1)...\pi(\kappa_K))$ for any permutation π of the set $\{1...K\}$, and

$(A3)$ ϕ ***respects unanimity***, i.e. $\phi(\kappa_1...\kappa_K) = \kappa$ if $\kappa_i = \kappa_j = \kappa$ for all $i, j \in \{1...K\}$.

The axioms for social choice $(A1), (A2), (A3)$ were introduced in Chichilnisky [11][12]. A comparison between these axioms and Arrow's classic axioms of social choice is in Section 8.2.

It was proven in Theorem 1 of Chichilnisky [11] that these three axioms are generally inconsistent, in the sense that when χ is ***the space of all smooth preferences*** defined on Euclidean choice spaces, ***there exists no map ϕ satisfying these three axioms.*** The result is valid whether or not the preferences admit satiation, but extreme cases such as the case where the social preference $\phi(\kappa_1...\kappa_K)$ is indifferent among all possible resource allocations are eliminated on the grounds that they do not resolve the allocation problem, Chichilnisky [12]. Note, however, that when respect of unanimity (A3) is replaced by a Pareto condition, the impossibility result of Chichilnisky (1980) holds even when the total indifference between all choices is allowed as the social preference. The *Pareto condition* is that if all preferences in a profile prefer one choice y to a second choice x, then so should the social choice map.

2.3.2 Social Allocations

We may now define a *social allocation*. Consider any space χ of preferences over the space of allocations X^r; there are $r \geq 2$ individuals, each with a preference $\kappa_i \in \chi$, $i = 1...r$. The preferences in χ may or may not be part of a market economy. When a social choice map $\phi : \chi^r \to \chi$ exists satisfying the three axioms: continuity $(A1)$, anonymity $(A2)$, and respect of unanimity $(A3)$ it defines *a social allocation for the space of preferences* χ as follows:

For each profile of individual preferences $(\kappa_1...\kappa_r) \in \chi^r$ a *social allocation* is a resource allocation in X^r which is optimal within the set of feasible allocations Υ according to the social preference $\phi(\kappa_1...\kappa_r) \in \chi$. Such a resource allocation is located by a social choice rule which satisfies the three ethical axioms $(A1)(A2), (A3)$.

3 Limited Arbitrage: Definition and Examples

Our link between the existence of ***market allocations*** and ***social allocations*** will be provided by the condition of *limited arbitrage*. Our next step is therefore to define this condition, to discuss its meaning, and to provide examples. We shall consider two cases separately. Case 1 is when the consumption set is $X = R^N$, and Case 2 is when the consumption set $X = R^N_+$. The limited arbitrage condition is somewhat different in these two cases, although in both cases it involves the non-empty intersection of dual cones. We shall also discuss more general consumption spaces, and provide an interpretation of the limited arbitrage condition in such cases.

3.1 Limited Arbitrage without bounds on short sales: $X = R^N$

Consider a market economy $E = \{X, \Omega_i, \rho_i, i = 1...H\}$, where $X = R^N$. E has *limited arbitrage* iff

$$(LA) \bigcap_{i=1}^{H} D(\rho_i, \Omega_i) \neq \emptyset. \qquad (6)$$

This condition can be interpreted as follows: ***there exists a price p at which only limited (or bounded) increases in utility are affordable from initial endowments.*** This interpretation illustrates the connection between limited arbitrage and the concept of no-arbitrage which is used frequently in the finance literature.

Examples: **economies which do not satisfy the limited arbitrage condition in Case 1 are those where the different individuals have different linear preferences, Figure 4, and also those represented in Figure 5. In both examples, the economy has no competitive equilibrium.**

Figure 4
Two linear preferences $i = 1, 2$ over the consumption set $X = R^2$.
The asymptotic preferred cones A_i are half spaces.
The dual cones D_i are the two gradients.
The limited arbitrage condition is not satisfied,
and the economy has no competitive equilibrium.

Figure 5
Three preferences over $X = R^3$.
The dual cones are D_1, D_2 and D_3 as indicated.
Each two person subeconomy satisfies limited arbitrage
but the three person economy does not because the three
dual cones do not intersect.

3.2 Limited arbitrage with bounds on short sales: $X = R^N_+$

Consider now a market economy $E = \{X, \Omega_i, \rho_i, i = 1...H\}$, where now $X = R^N_+$.

The *limited arbitrage* condition when $X = R^N_+$ is:

$$(\partial LA) \quad \bigcap_{i=1}^{H} \partial D(\rho_i, \Omega_i) \neq \emptyset \tag{7}$$

where the $\partial - dual$ cones $\partial D(\rho_i, \Omega_i)$ are defined in Section 2.

The interpretation of this condition is that if *all supporting prices in $S(E)$ assign some trader zero income, then there is one at which only limited (or bounded) increases in utility are affordable from initial endowments.*

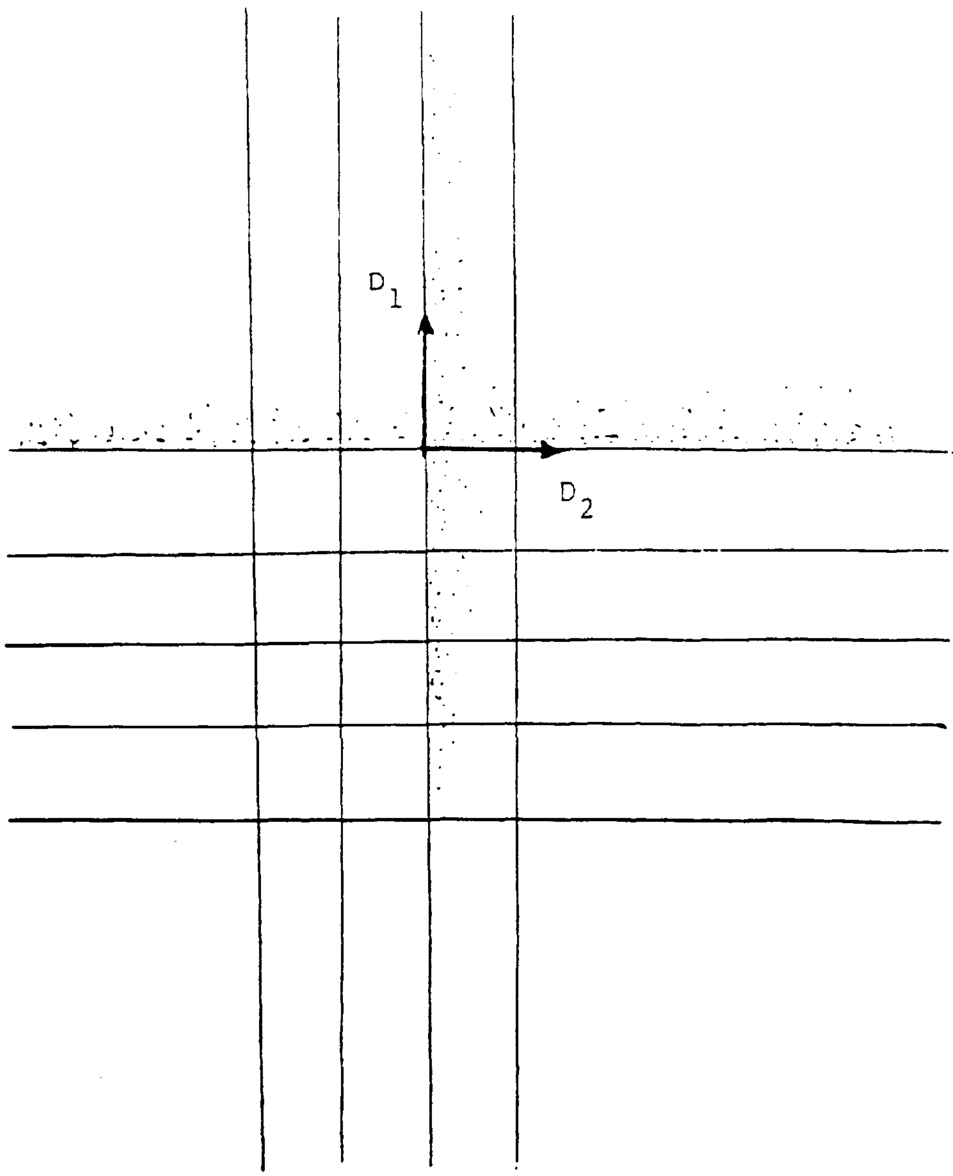
Examples: An example of an economy which does not satisfy limited arbitrage in Case 2 is illustrated in Figure 6.

Figure 6
A two person economy with consumption set $X = R^2_+$. $S(E) = \{v\}$.
All supporting prices in $S(E)$ assign individual one zero income, $< v, \Omega_1 >= 0$.
There exists no price in $S(E)$ having a positive inner product
with the asymptotic preferred cones of both traders,
because the cone $A(\rho_1, \Omega_1)$ contains the vertical axis, since ρ_1 increases in the second coordinate. Limited arbitrage is not satisfied, and there is no competitive equilibrium.

Examples: In Case 2, when the consumption set is $X = R^N_+$, limited arbitrage is always satisfied by preferences whose indifference surfaces corresponding to positive consumption bundles never intersect the boundary of X. Examples of such preferences are Cobb-Douglas utilities or CES utilities with elasticity of substitution $\sigma < 1$. All such preferences have the same asymptotic preferred cone, namely the positive orthant, and therefore their dual cones always intersect. Since their asymptotic preferred cones are identical, these preferences are very close indeed on choices involving large utility levels. Similarly, economies where the individuals' initial endowments are strictly interior always satisfy this condition too, since in this case $\partial D(\rho_i, \Omega_i) = X = R^N_+$ for all $i = 1...H$.

The limited arbitrage condition may fail when $\forall p \in S(E)$ some trader's endowment vector Ω_i is in the boundary of the consumption space $X = R^N_+$, and this trader has zero income $< p, \Omega_i >= 0$.

Figure 4



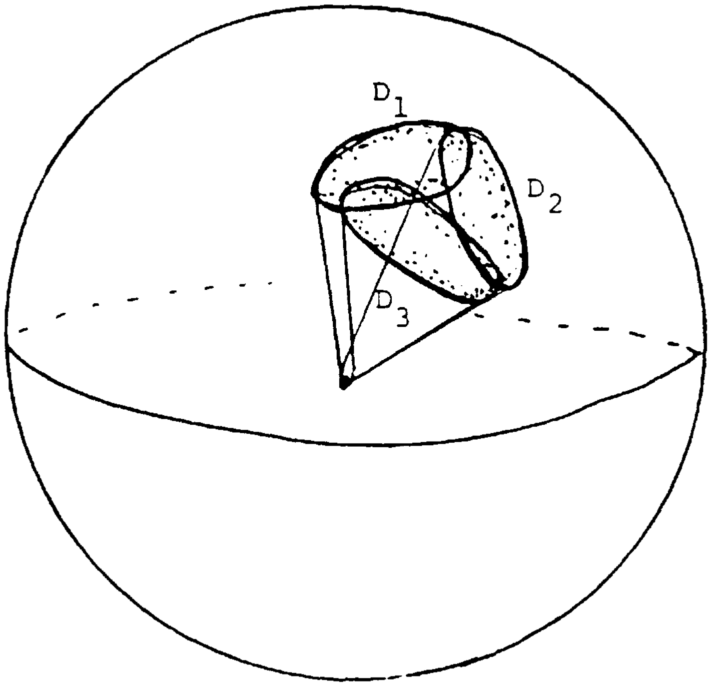


Figure 5

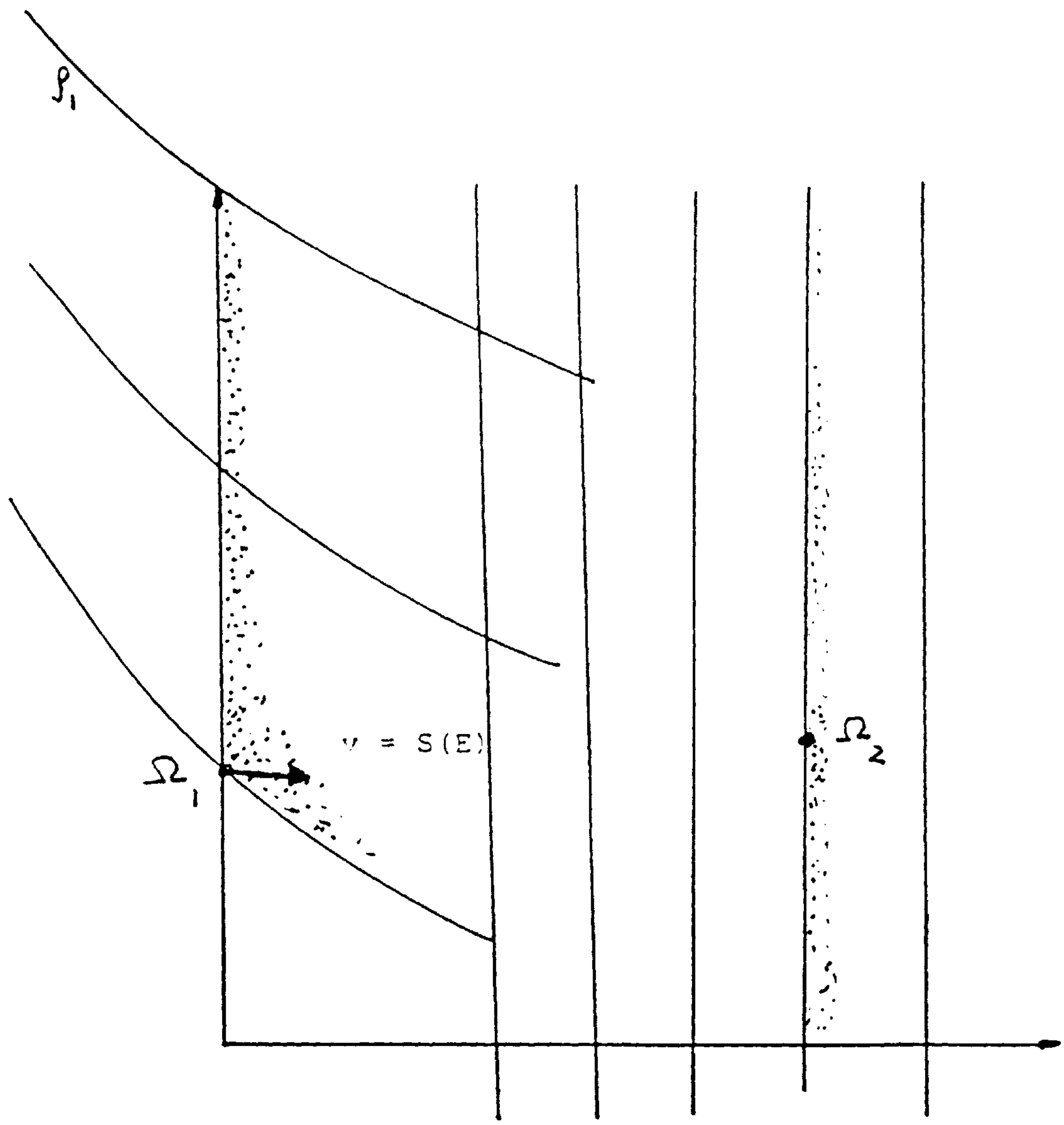


Figure 6

Otherwise, limited arbitrage is always satisfied since by definition $\partial D(\rho_i, \Omega_i) = X = R^N_+$ for all $i = 1...H$.

We shall now define limited arbitrage for subsets of traders in the market E.

The economy E *satisfies limited arbitrage for any subset of k traders*, when for any subset $K \subset \{1...H\}$ of cardinality $k \leq H$,

Case 1 ($X = R^N$) :

$$(LA) \bigcap_{j \in K} D(\rho_j, \Omega_j) \neq \emptyset. \tag{8}$$

Case 2 ($X = R^N_+$) :

$$(LA) \bigcap_{j \in K} \partial D(\rho_j, \Omega_j) \neq \emptyset. \tag{9}$$

Theorem 6 in the Appendix establishes that an economy satisfies limited arbitrage if and only if it satisfies limited arbitrage for any subset of at most $N + 1$ traders, where N is the number of commodities in the economy. This implies that the economy E has a competitive equilibrium if and only if every subeconomy of $N + 1$ traders does, [18].

3.3 Limited Arbitrage with more general consumption sets

We may consider also more general cases of economies where there exist lower bounds on the consumption sets, but these bounds are not zero. For example, we may consider consumption sets which are translates of positive orthants in commodity space,

$$X = \{v \in R^N : \exists w \in R^N \text{ with } v \geq w\}.$$

In this case the limited arbitrage condition has a similar interpretation but is not restricted to prices in $S(E)$ which assign some individual's endowment zero value: it applies instead to cases where $\forall p \in S(E)$ some individual has endowments of minimal value. The interpretation of limited arbitrage is now as follows: among those prices in $S(E)$ which assign some individual minimal value, there must exist one at which only bounded increases of utility are affordable from initial endowments.

4 Similarity and Social Diversity

Consider a market economy $E = \{X, \Omega_i, \rho_i, i = 1...H\}$. Each preference ρ_i in E is defined over i's private consumption set X. However, as we saw in Section 2, ρ_i also defines a preference over all allocations in X^r for any number of individuals $r \geq 1$: an allocation $(x_1...x_r) \in X^r$ is preferred by individual i in position j to another $(y_1...y_r)$ if and only if ρ_i prefers x_j to y_j. We may therefore consider ρ_i as a preference defined over allocations, using for it the same notation when the meaning is clear from the context. Our next task is to define spaces of preferences which are *similar* to those of the market's traders.

We aim to define precisely what is meant by preferences being similar, and to show how to define populations of individuals having preferences similar to those in the market E. The purpose is to formalize the concepts of *similarity and diversity* by taking the preferences and endowments in the economy E as a benchmark.

An essential characteristic of the agents is their "asymptotic preferred cones" over allocations, as defined in Section 2. These cones are made of rays along which the individuals' utilities increase without bound from the initial endowments. A geometric interpretation of similarity is that a preference σ is *similar* to another preference ρ when σ increases along those directions which give ρ unlimited utility gains, and only along those directions. In particular, a preference σ is not similar to another ρ if - and only if - there exist a direction along which ρ can achieve unbounded increases in utility, yet along the same direction, $\sigma's$ utility decreases. Intuitively the preferences ρ and σ should not be considered similar in this latter case.We shall therefore consider a universe of preferences in which individuals' preferred directions are similar to those of the individuals in the market in the sense that their gradients have positive inner products with the asymptotic cones of some trader in the economy E.

Consider first the case $X = R^N$. Let η be any concave, smooth preference defined over allocations in X^r, $r \geq 1$. Note that, in general, a preference over allocations may not be monotonic. We seek to formalize the notion that η is *similar* to the preference of trader i in the market E if it agrees with the preference ρ_i of i on important choices. A geometric interpretation of this concept of similarity is as follows: the preference η must increase in the directions of individual $i's$ asymptotic cone. This is formalized in the following definition of *similarity to the preference of trader i*.

Let the normal to the indifference surface of the preference η at the allocation $\xi \in X^r$ be denoted $G\eta(\xi)$; it is an $N \times r$ vector indicating the direction of increase of preference η at the social allocation $\xi \in R^r$. This normal always exists and is unique because the preference η is smooth (Debreu [24])

When $X = R^N$, an individual preference η over resource allocations in X^r is said to be similar to the preference of trader i of the economy $E = \{X, \rho_i, \Omega_i, i = 1...H\}$ in position j when:

$$\forall \xi \in X^r \quad G\eta^j(\xi) \in D(\rho_i, \Omega_i) \tag{10}$$

where $D(\rho_i, \Omega_i)$ is the dual cone of individual i defined in 2, Section 2. This means that the gradient of η has a strictly positive inner product with the vectors of the asymptotic cone $A(\rho_i, \Omega_i)$ of individual i in position j, as defined in Section 2.

When $X = R^N_+$ the definition of *similarity to the preference of trader i* is the same as in (10), but replacing the cone $D(\rho_i, j)$ in (10) by the boundary dual cone $\partial D(\rho_i, j)$ defined in Section 2. The cone $\partial D(\rho_i, j)$ is the analog of the dual cone $D(\rho_i, j)$ for the case $X = R^N_+$. Formally,

When $X = R^N_+$, a preference η over resource allocations in X^r is said to be similar to the preference of trader i of the economy $E = \{X, \rho_i, \Omega_i, i = 1...H\}$ in position j when:

$$\forall \xi \in X^r \quad G\eta^j(\xi) \in \partial D(\rho_i, \Omega_i). \tag{11}$$

Note that a preference which is similar to an increasing preference, need not be itself increasing: all that it is required for similarity in (10) is that the gradients be in the dual cone D_i, and the vectors in the dual cone $D(\rho_i, \Omega_i)$ of an increasing preference ρ_i need not be positive.

4.1 Private and Public Preferences

We shall say that an individual's preference η is *private* when it is indifferent to the consumption of anyone else, and that it is *public* otherwise. When a preference is private and the individual occupies position $j \in \{1...r\}$, then its normal $G\eta(\xi) \in R^N$ has only N non-zero components, those in position j, indicated $G\eta^j(\xi)$. A private preference is therefore similar to that of an individual i in the economy E in one position j. By contrast, if a preference is not private, it may in principle be similar to those of different individuals i in different positions $j = 1...r$. The results of this paper apply equally well when we consider solely private individual preferences, or when individual preferences are either private or public. When the space of preferences consists solely of private preferences, the concept of respect of unanimity must be modified slightly. We consider here spaces consisting of preferences which are either private or public.

We may now define *similarity of a preference* with respect to a set K of traders in the economy E (as opposed to similarity to the preference of a trader i in position j, which was defined in (11) and (10):

A private preference η over resource allocations in X^r is similar to those of a set of traders $K \subset \{1...H\}$ in the market economy E if:

$$\forall \xi \in X^r,\ \exists i \in K \ \textit{and}\ \exists\ j \in \{1...r\}\ \textit{s.t.}\ \eta \textit{ is similar to } \rho_i \textit{ in position } j. \tag{12}$$

When the preference η is public, we have:

$$\forall \xi \in X^r \ \textit{and}\ \forall j \in \{1...r\},\ \exists i \in K\ \textit{s.t.}\ \eta \textit{ is similar to } \rho_i \textit{ in position } j. \tag{13}$$

The next step is to define *spaces of preferences* P_E which consist of preferences similar to those of a subset of traders in the market E. The intuitive notion is that of a class of preferences P_E defined over allocations in X^H where each preference in P_E is similar to the preference of some trader i within a subset of traders of E. Formally, consider the space of allocations for $r \geq 2$ individuals, X^r, where $X = R^N$ or $X = R^N_+$. Let j denote possible positions, $j = 1...r$:

A space of preferences P_E over allocations X^r consists of preferences similar to those preferences of a set of traders $K \subset \{1...H\}$ in the market economy $E = \{X, \rho_i, \Omega_i, i = 1..H\}$, for $X = R^N$ or $X = R^N_+$ when:

$$\eta \in P_E \Longleftrightarrow \eta \textit{ is a preference similar to those of the set of traders } K. \tag{14}$$

The space of preferences P_E consists therefore of either private or public preferences over allocations in X^r; in either case, its preferences are similar to those of the subset K of traders in the market economy E.

Note that the preferences in a space P_E need not be increasing even when all the preferences in the market are increasing. Similarity only requires that the gradients of preferences in P_E be in the dual cones of some preference in the set of traders K, and the dual cones of increasing preferences may contain vectors which are not positive.

5 Social Allocations in a Market Economy

We shall now specialize the social choice problem described in Section 2.3 to the case of social allocations in a market economy.

Consider a market economy $E = \{X, \rho_i, \Omega_i, i = 1..H\}$. Let P_E be a space of preferences over resource allocations in X^r, $r \geq 2$ consisting of preferences which are similar to those of K traders in the economy E as defined in (14). A social choice map ϕ assigns to each profile of r individual preferences in X^r another preference in P_E, the social preference over allocations, $\phi(\kappa_1...\kappa_r) = \theta$. Each κ_j is a preference similar to the preference of an individual i in E and so is $\theta = \phi(\kappa_1...\kappa_r)$.

5.1 Example: A Classical Social Welfare Function

An example of the type of social choice map we consider here is provided by the classical "social welfare function" which assigns to each profile of individual utilities over private consumption a utility over allocations as described in the following. In this standard example, each individual $i = 1...H$ has a utility function $u_i : X \rightarrow R$, where $X = R^N$ or $X = R^N_+$. The social preference W over allocations in X^H is defined for each allocation $(x_1...x_H) \in X^H$ by:

$$W(x_1...x_H) = \sum_{i=1}^{H} u_i(x_i). \tag{15}$$

Note that the gradient of the function $W : R^{NH} \rightarrow R$ at an allocation $(x_1 ... x_H)$ is a vector in R^{NH}, while each individual utility u_i has a gradient in R^H at the vector x_i. Indeed, the gradient of the function W is a vector of H $N-$dimensional gradients, those of the H individuals. In particular, the gradient of the function W is *not* the sum or any convex combination of the gradients of the individuals $i = 1...H$.

The construction of the social welfare function W of (15) is cardinal, in the sense that it is defined over profiles of individual *utilities*, and yields a social *utility* function $W(x_1 ... x_H)$. However, the social welfare function of (15) can be used to define a social choice map in ordinal terms, namely a map from profiles of individual preferences to social preferences. Consider a profile of private preferences $(\kappa_1 ... \kappa_H)$, where κ_i is a private preference over allocations in X^H. Then $\forall i$ κ_i is induced by a utility function over private consumption $u_i(\kappa_i) : X \rightarrow R$. We may now define the social preference over allocations in X^H corresponding to the profile of preferences $(\kappa_1 ... \kappa_H)$, the preference $\Theta(\kappa_1 ... \kappa_H)$, as follows. The normal of $\Theta(\kappa_1 ... \kappa_H)$ at an allocation $\xi = (x_1 ... x_H) \in X^H$ is defined as the vector

$$\begin{gathered} N\Theta(\kappa_1 ... \kappa_H)(\xi) = \\ (\lambda_1 G_j W(u_1(\kappa_i) ... u_H(\kappa_H))(\xi), ..., \lambda_H G_H W(u_1(\kappa_i) ... u_H(\kappa_H))(\xi)), \end{gathered} \tag{16}$$

where $G_j W$ is the normalized gradient of $W(x_1 ... x_H)$ with respect to x_j, i.e.

$$G_j W(\xi) = (\partial W / \partial x_j)(\xi) \in R^N,$$

and $\forall j$, $\lambda_j \in R_+$. The role of the real numbers λ_j in (16) is to normalize the right hand side of the expression (16) so that it defines a vector of length one as it corresponds to the normal of a smooth preference defined in Section 2 (Debreu [22]). Expressions (15) and (16) define a map from profiles of private preferences over allocations in X^H, into preferences over allocations in X^H:

$$(\kappa_1 ... \kappa_H) \rightarrow \Theta(\kappa_1 ... \kappa_H) \tag{17}$$

Clearly, the definition in (17) is the ordinal version of the cardinal construction in (15).

The interest of the social welfare function Θ defined in (17) is that it provides an example of the type of structure for social choice with which we work in this paper. The similarities with our framework are as follows: Θ is defined over profiles of *private* preferences over allocations, and it assigns to such a profile another preference over allocations which is typically a *public* preference. If the preferences in the profile $(\kappa_1 ... \kappa_H)$ are those of a market economy $E = \{X, \kappa_i, \Omega_i, i = 1...H\}$, then the social preference assigned to this profile $\Theta(\kappa_1 ... \kappa_H)$ is similar to those of the traders in E as defined in (14) above. Furthermore, the map Θ is continuous on its arguments and it respects unanimity. Therefore the map Θ satisfies many of the properties required of our social choice functions.

The differences with our framework are as follows: firstly, Θ is defined solely on private preferences, while we allow social choice maps which are more general, defined over both private and public preferences over allocations. Secondly, the map Θ, by its construction, is not anonymous. Indeed, the map Θ assigns individual i a dictatorial power over the ith position in the allocation.

5.2 Example: The Convex Addition of Gradients

As pointed out in 5.1, the gradient of the classical welfare function W defined in (15), (16) is neither the sum nor the convex combination of the individual utilities' gradients. A natural question is whether a construction based on the sum or a convex combination of the individual utilities's gradients could yield an adequate social preference. Such construction would have the advantage that it is defined generally, without reference to additional conditions such as limited arbitrage. The following example shows that such a construction, although appearing to be natural, typically does not work. Indeed we argue below that it only works properly when the limited arbitrage condition is satisfied.

Consider for example an economy with two agents each with a linear utility defined on the consumption set $X = R^2$. The two utilities are different. Figure 7 illustrates. The asymptotic preferred cones A_1, A_2 of the two linear preferences are the two half spaces as indicated in Figure 7, and the corresponding dual cones D_1, D_2 are the two gradient vectors. Since the two vectors are different, the dual cones do not intersect, and limited arbitrage is violated. Now consider the linear preference ρ over allocations in $R^2 \times R^2$ with gradient (D_3, D_3), where D_3 is a convex combination of D_1 and D_2 as shown in Figure 7,

$$D_3 = \lambda D_1 + (1 - \lambda) D_2, \ 0 < \lambda < 1.$$

A problem arises because even though the initial endowment allocation (Ω_1, Ω_2) is strictly preferred by both individuals to the allocation denoted (θ_1, θ_2) in Figure 7, the social preference ρ prefers, instead, (θ_1, θ_2) strictly to (Ω_1, Ω_2).

Figure 7
Both individuals prefer allocation (Ω_1, Ω_2) to (θ_1, θ_2)
but the social preference with gradients equal to the convex combination
of the gradients D_1 and D_2 prefers (θ_1, θ_2) to (Ω_1, Ω_2) instead.

This property of ρ contradicts the standard Pareto condition which requires that when everyone strictly prefers a given allocation to another, so should the social preference. The reason for this contradiction is that the two dual cones do not intersect. When the dual cones intersect, then a vector in this intersection defines the gradient of a linear preference which respects the Pareto condition: this follows directly from the definition of dual cones. Otherwise, no preference exists respecting the Pareto condition.

Another way of looking at the same problem is that the social preference defined by the addition of the two gradients is not *similar* to the preferences with gradients D_1 and D_2, as defined in Section 4. Therefore the addition of gradients cannot define a map from profiles of similar preferences into similar preferences as required

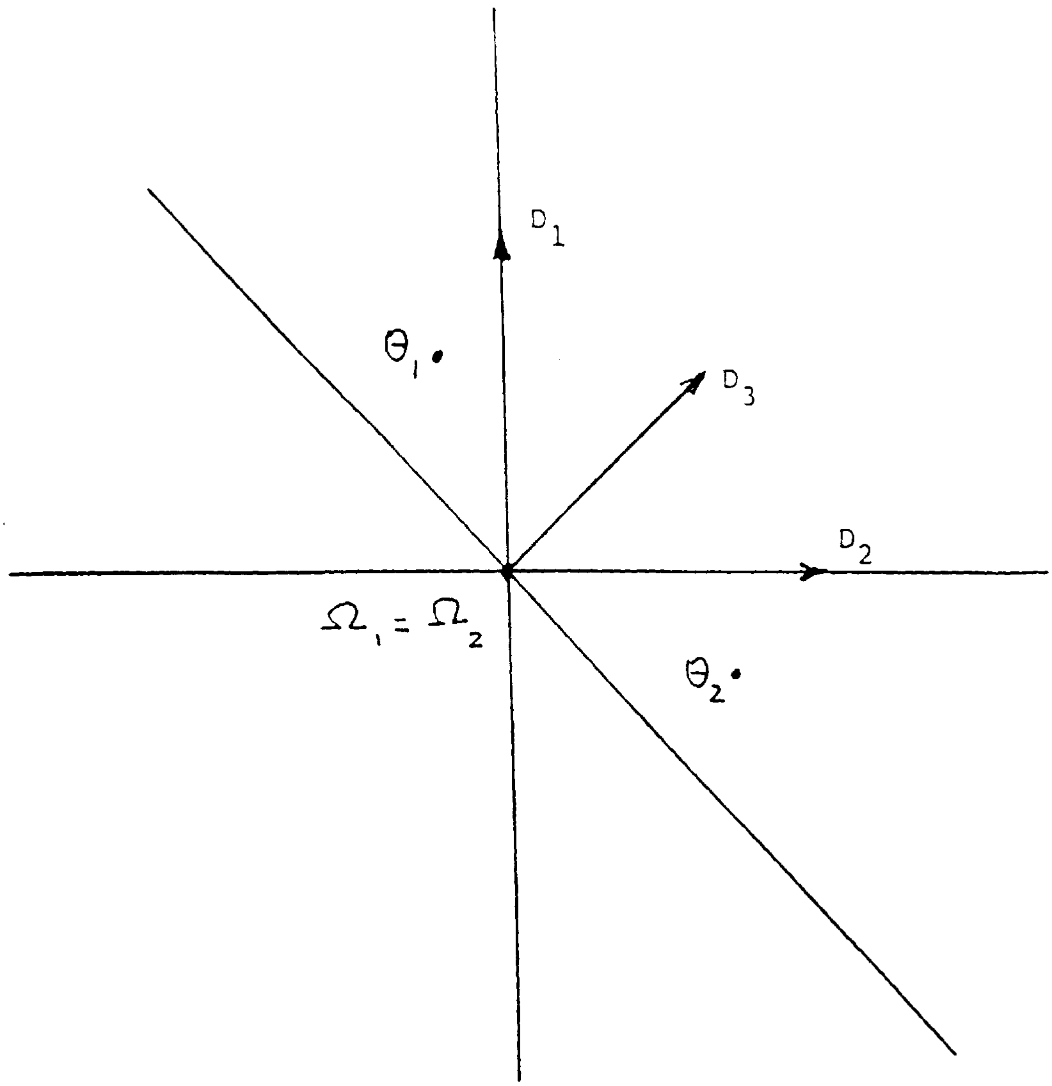
This example shows that the convex sum of gradients cannot be used generally as a way of defining appropriate social choice rules: the preference with gradients $D_3 = \lambda D_1 + (1 - \lambda) D_2$, $0 < \lambda < 1$, is not an adequate social preference for the profile (D_1, D_2).

The only possibility of respecting the similarity of preferences or the Pareto condition, would be to assign the social preference with gradient equal to either D_1 or to D_2 to the profile (D_1, D_2). This would respect the similarly condition, and would define a continuous, anonymous social choice map respecting unanimity on the space of preferences $P_E = \{D_1, D_2\}$. But this is possible only because the space P_E is a discrete case consisting only of two elements, the two linear preferences. In the general case in which the space P_E is connected, a continuous anonymous social choice rule satisfying unanimity exists if and only if P_E is contractible, as proved in Chichilnisky and Heal [13]. And, as shown in the Appendix Theorem 5, limited arbitrage is actually equivalent to the contractibility of the spaces of preferences.

Our next step is to extend and refine the classical welfare function W defined in (15). The welfare function W is a cardinal construct, while we wish to provide an ordinal one: a social choice map defined over profiles of private and public preferences over allocations, preferences which are similar to those of a market economy E. We seek an anonymous social choice map, a requirement that neither W in (15) nor its ordinal counterpart Θ in (17), satisfy. Theorem 2 establishes that this task can be accomplished when the preferences in the market economy E satisfy limited arbitrage, and therefore when the market E has a competitive equilibrium, and only then.

Assume now that a social choice map ϕ with the desired properties $(A1)(A2)(A3)$ exists. The existence of a social choice map ϕ solves the resource allocation problem from the point of view of social choice:

Figure 7



We say that ***the social allocation problem is resolved for preferences similar to those of the market*** **$E = \{X, \rho_i, \Omega_i, i = 1..H\}$ when for any space of preferences P_E over allocations in X^r similar to a set of traders $K \subset \{1...H\}$ in E there exists a continuous anonymous social choice map $\phi : (P_E)^r \rightarrow P_E$ respecting unanimity.**

Consider a market economy $E = \{X, \rho_i, \Omega_i, i = 1..H\}$ having a social choice map $\phi : (P_E)^H \rightarrow P_E$ satisfying the axioms $(A1)(A2)(A3)$ on the space of preferences P_E similar to those of the economy E.

A *social allocation* for E is a feasible allocation of E which is optimal in the space of feasible allocations $\Upsilon = \{x_1...x_H : \sum x_i = \Omega\}$ according to the social preference $\phi(\rho_1...\rho_H)$.

6 Comparing the two resource allocations problems

Given a market economy $E = \{X, \Omega_i, \rho_i, i = 1...H\}$ the *market allocation problem* is solved whenever there exists a market equilibrium for E. To resolve the market allocation problem we must therefore give conditions which assure the existence of competitive equilibrium: when the competitive equilibrium does not exist, the social contract based on market allocations fails to deliver acceptable resource allocations.

6.1 Existence of Market Allocations

We know that not all Arrow-Debreu exchange economies have a competitive equilibrium, even when all individual preferences are smooth, concave and increasing, and even when the consumption sets are positive orthants. The simplest example of a market with continuous concave and increasing preferences over $X = R^N_+$ which has no competitive equilibrium is provided in Arrow and Hahn [3], Chapter 4, p. 80, in a market with two goods and two individuals, but the problem is quite general and it occurs in economies with any number of individuals and of goods. The example in [3] is based on the diversity of endowments and preferences of the individuals in the economy, which leads to a failure of continuity of the demand function. With a discontinuous demand, a competitive equilibrium generally fails to exist. Arrow and Hahn say: "*This discontinuity will necessarily occur in some part of the price space, except in the unrealistic case in which the household has a positive initial endowment of all goods*" (Chapter 4, p. 80, [3]).

Of course, other concepts of market equilibrium could be utilized to define a market allocation, such as for example *quasi-equilibrium* which was first introduced by Debreu [25], or *compensated equilibrium,* as defined in Arrow and Hahn [3]. These are closely related but different definitions; they define allocations which could have excess supply, or where individuals minimize cost rather than maximizing utility. When individuals' incomes are strictly positive, these concepts agree with the competitive market equilibrium (Arrow and Hahn [3], Chapter 4); these allocations have the advantage that they always exist with continuous concave preferences and with positive orthants as consumption sets.

However, as Arrow and Hahn point out, the condition that all individuals should have strictly positive endowments of all goods is unrealistic ([3], Chapter 4, p. 80., para. 4), so that quasi-equilibrium or compensated equilibrium allocations will not be competitive equilibrium allocations in general. But the competitive equilibrium stands alone in terms of its welfare properties. Quasi-equilibrium or compensated equilibrium allocations are not generally Pareto efficient. Therefore the main justification for using market allocations, which is efficiency, would be lost unless we remain within the confines of a competitive equilibrium. For this reason we concentrate here on competitive equilibrium allocations.

The problems of non-existence of a competitive equilibrium are somewhat different when the consumption set X is the whole Euclidean space - i.e. when there are no bounds on short sales -

than when the consumption set X is the positive orthant. Consider first the case when $X = R^N$. In this case, the *limited arbitrage* condition defined in the next Section ensures that individuals, who must typically be diverse in order for gains of trade to exist, are not too diverse in the sense that a competitive equilibrium still exists. This is Theorem 7 below; indeed this theorem proves that limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium. For example, there must exist a degree of consistency between individuals' asymptotic cones at the initial endowments: a price hyper plane must exist which leaves all these cones to one side, so that from initial endowments, no individual can afford allocations which lead to unbounded utility at these prices. Figures 4, 5 and 6 in Section 3 illustrate this point, and show when this condition of limited arbitrage fails.

The case when the consumption space is $X = R^N_+$ is somewhat different. Figure 6 in Section 3 illustrates a failure of the limited arbitrage condition in this case. None of the economies of Figures 4, 5 or 6 have a competitive equilibrium. Conditions are therefore needed to rule out such economies. The conditions should ideally be on the exogenous data which identify the economy, namely on the endowments Ω_i and on the preferences ρ_i of the agents $i = 1...H$. The *limited arbitrage* condition of Chichilnisky [18] - defined in Section 3 - is therefore a good candidate, because it provides a necessary and sufficient condition for existence of a competitive equilibrium which is defined on these exogenous data of the economy E: endowments Ω_i and preferences ρ_i. Limited arbitrage limits precisely the degree of diversity among the agents of the economy to one at which market equilibrium will exist. This is Theorem 7.

Indeed, for economies with consumption bounded below, $X = R^N_+$, such limits on diversity are implicit in Arrow's resource relatedness [3] and in McKenzie's *irreducibility* condition [30][31][32]. All these conditions ensure that the endowments of any household are desired, directly or indirectly, by others, so that their incomes cannot fall to zero. In this case our limited arbitrage condition is always satisfied.

Irreducibility and *resource relatedness* conditions work by assuring that at a quasi-equilibrium or at a compensated equilibrium, all individuals' incomes are strictly positive. When individuals' incomes are all positive, all the notions of equilibrium coincide. The problem of maximizing utility subject to a budget constraint, which is the competitive equilibrium condition, is then identical to that of minimizing the cost of an allocation with a certain utility level, which is the condition which defines a compensated equilibrium. Thus a compensated equilibrium, which always exists when preferences are concave and continuous and the commodity space is the positive orthant, is also a competitive equilibrium.

The key to the conditions of Arrow, Debreu and McKenzie is to eliminate zero income allocations. Yet zero income does not rule out the existence of a competitive equilibrium; there may be economies with a competitive equilibrium in which some consumers have zero income. Furthermore an allocation where some individuals have zero income reflects a real problem: the fact that some individuals are considered worthless, that they have nothing to offer that others want. Such a situation could clearly arise at a competitive equilibrium. Indeed, it seems realistic that markets could lead to such allocations; we observe them all the time in city ghettos. Our condition of *limited arbitrage* brings out the issue of diversity by focusing on the problem of zero income individuals. It does not attempt to rule out the case of individuals with zero income; instead, it seeks to determine if society's evaluation of their worthlessness is widely shared. Individuals are diverse in the sense of limited arbitrage when they have sufficiently different endowments that someone's income could be zero - which requires that some individuals have zero quantities of some goods - and in addition when there is no agreement about the worthlessness of those who have zero income. With more general consumption sets as discussed above, the issue is the worthlessness of those having minimum income, which may or not be zero.

6.2 Existence of Social Allocations

Within the same market economy $E = \{X, \Omega_i, \rho_i, i = 1...H\}$, we may resolve the *social allocation problem* when we can find a social choice rule ϕ satisfying the required axioms $(A1)(A2)(A3)$. ϕ must be defined on a space of preferences which contains those of the market E and is otherwise large enough to satisfy Kant's universality criterion. The problem has generally no solution because a social choice rule satisfying the required axioms may not exist. Therefore a social contract promising a solution which satisfies the ethical principles agreed and having a completely universal domain of preferences, may not deliver. Again, conditions are needed. Here, as in the case of the market equilibrium, the conditions should be on the exogenous data which identify the economy E, namely on the individuals' endowments Ω_i and their preferences ρ_i.

We know that the space of all smooth preferences over allocations in X^H is too large: the three axioms of continuity, anonymity and respect of unanimity are inconsistent in that case and no social choice rule satisfying them exists, Chichilnisky [11]. We must therefore search for a universe of preferences and endowments in which a social choice rule ϕ does exist. Such a universe must contain the preferences of our economy E, and we shall require that it should consist of individuals who share the essential characteristics of the economy E. The motivation is to compare the types of restrictions needed for existence of a competitive equilibrium with those required for the existence of a social choice map.

Consider a space P_E of preferences which are similar to those of the traders in E. From Chichilnisky and Heal [13] we know that a social choice rule on this space of preferences exists if and only if the space of preferences P_E is *contractible*, as defined in Section 2. This condition of contractibility simply means that there exists a continuous way of deforming the preferences through the space P_E, so that at the end of this process we have complete unanimity. With contractibility of the space χ we are assured of the existence of a social choice map, and therefore we are assured of a resolution to our resource allocation problem.

But contractibility is a restriction on social diversity, no more and no less. It tests whether there is a way of deforming continuously our space of individual preferences into itself so that at the end of this deformation all the individuals have identical preferences. A discussion of the role of contractibility in limiting social diversity and in public decision making is in Heal [26]. Thus we are back at the source of the problem of resource allocation in markets: individual diversity. As before, we shall focus on the degree of diversity which allows a solution to the resource allocation problem to exist, this time, at the social choice level.

Theorem 2 in Section 7 shows that the degree of diversity which is necessary and sufficient to solve the allocation problem in markets - *limited arbitrage* - is the same as that needed to solve the allocation problem with social choice rules. In other words, Theorem 2 shows that necessary and sufficient conditions for the existence of a competitive equilibrium - *limited arbitrage* - are also necessary and sufficient for the existence of social choice rules. The former conditions are on the diversity of preferences at the initial endowments. The latter restrict our universe of preferences to be similar to those of the market traders, at their initial endowments. Theorem 2 proves that when the market has an equilibrium, the social choice map exists and when the market does not have a competitive equilibrium, the social choice rule does not exist. As formulated here, the two problems of resource allocation, by markets and by social choice, are therefore equivalent. Theorem 4 proves that market allocations are always social allocations.

7 Linking Markets, Arbitrage and Social Choice

This section provides the main results linking the two forms of resource allocation: by markets and by social choices. The following preparatory lemma describes the geometrical structure of spaces of preferences P_E which are similar to those in a set K of traders of the economy E.

Lemma 1 *Consider a market economy $E = \{X, \rho_i, \Omega_i, i = 1..H\}$, where $X = R^N$ or $X = R^N_+$. Let P_E be a space of preferences over allocations for r individuals X^r which are similar to those of a set $K \subset \{1...H\}$ of traders in E. Then when $X = R^N$ at each allocation $\xi \in X^r$ and for each position $j = 1...r$, the normals to the indifference surfaces of all preferences in P_E define the set*

$$N^K(P_E) = \{\bigcup_{i \in K} D(\rho_i, \Omega_i)\} \cup \{0\} = N_K \cup \{0\} \subset R^N. \tag{18}$$

Furthermore, when every two-trader sub economy of E satisfies limited arbitrage, then $\forall K \subset \{1...H\}$ the set N_K is connected. The set N_K is contractible $\forall K \subset \{1...H\}$ if and only if the market economy E satisfies limited arbitrage. When $X = R^N_+$ all the above statements hold replacing N_K by the set

$$M_K = \{\bigcup_{i=1}^{K} \partial D(\rho_i, \Omega_i)\} \cup \{0\}.$$

Proof: Since P_E consists of preferences over allocations in X^r which are similar to those of the subset K of traders in the economy E, and these preferences may be private or public, then by definition (14)(12)(13)(10), at each $\xi \in X^r$ the gradients in the set $N^K(P_E)$ define the set

$$\{\bigcup_{i \in K} D(\rho_i, \Omega_i)\} \bigcup \{0\} \subset R^N.$$

The next step is to establish that the set N_K is connected. The condition of *limited arbitrage* for any subset of two traders in the statement of this Lemma, is that is that for any two traders l, j in E

$$D(\rho_l, \Omega_l) \bigcap D(\rho_j, \Omega_j) \neq \emptyset.$$

This implies that the set

$$N_K = \bigcup_{i=1}^{r} D_i(\rho_i, \Omega_i)$$

is connected.

Finally we study the contractibility of the set N_K. Theorem 2 in Chichilnisky [18], see also Theorem 5 in the Appendix, establishes that if $\{C_j\}_{j=1...J}$ is a family of convex sets and $L \subset J$ then

$$\begin{array}{c} \bigcup_{j \in L} C_j \text{ is contractible } \forall L \subset J \\ \Leftrightarrow \forall L \subset J, \ \bigcap_{j \in L} C_j \neq \emptyset. \end{array} \tag{19}$$

Since the dual cone $\forall j$ $D(\rho_j, \Omega_j)$ is a convex set, the second statement in (19) is equivalent to the condition that E has limited arbitrage. The first statement, in turn, states that the set N_K is contractible. Therefore Theorem 5 in the Appendix implies that limited arbitrage is satisfied if and only if $\forall K \subset \{1...H\}$, the set N_K is contractible.◇

The following result links the resolution of the social allocation problem with the resolution of the market allocation problem. A minimal restriction, that every two traders have limited arbitrage, is now imposed in the market economy E to eliminate somewhat pathological economies where no two traders can reach a competitive equilibrium, or those economies where the spaces of preferences similar to those of the traders are discrete or very disconnected:

Assumption (C1): every two-trader sub economy of E has limited arbitrage.

The role of this condition is to ensure that the set of gradients N_K is connected: this was shown in Lemma 1. Note that ($C1$) is rather mild: it certainly does not imply that N_K is contractible, nor that the economy E satisfies limited arbitrage, nor that E has a competitive equilibrium. For example, Figure 4 illustrates an economy where every two dual cones intersect and thus ($C1$) is satisfied, but the economy of Figure 4 does not satisfy limited arbitrage, and does not have a competitive equilibrium.

We shall explain intuitively the role of the *limited arbitrage* condition in the existence of social choice rules. A condition which is necessary and sufficient for the existence of social choice rules is that the space of preferences should be contractible, Chichilnisky and Heal [13]. Therefore to show the existence of social choice rules on preferences similar to those of the market E we must show that the space of such preferences is contractible. This we do using the condition of limited arbitrage and Theorem 5 in the Appendix.

The condition of limited arbitrage is the non-empty intersection of dual cones, as defined in Section 3. However, Theorem 5 in the Appendix proves that the dual cones intersect if and only if their union is contractible. And, by Lemma 1, the union of the dual cones is precisely the space where the gradients of the preferences similar to those of the market "live". In other words: the proof that limited arbitrage is necessary for the existence of a social choice rule derives from the results of Lemma 1 above, from Theorem 1 of Chichilnisky and Heal [13], and from Theorem 5 in the Appendix. It may be worth mentioning here that preferences which are similar to those in the market E are not necessarily increasing, and that, furthermore, a space of preferences consisting of increasing preferences may not admit social choice rules because it may have "holes" and therefore may fail to be contractible. For a geometrical example see Figure 5 in Section 3.For the sufficiency part we need to use additional results: in particular those in Chichilnisky [10]. The result that a competitive equilibrium exists if and only if E satisfies limited arbitrage is Theorem 7 in the Appendix; note that condition $(C1)$ is not necessary for the equivalence of limited arbitrage and the existence of a competitive equilibrium.

Theorem 2 *Consider an economy $E = \{X, \rho_i, \Omega_i, i = 1...H\}$, $H \geq 2$, $X = R^N$ or $X = R^N_+, N \geq 1$ satisfying $(C1)$.*

Then the following three properties are equivalent:

(a) the economy E has limited arbitrage

(b) the economy E has a competitive equilibrium

(c) there exists a continuous anonymous social choice map $\phi : (P_E)^r \to P_E$ respecting unanimity on any space P_E of preferences over allocations which are similar to those of a set $K \subset \{1...H\}$ of traders in the market E, for all $K \subset \{1...H\}$ and all $r \geq 2$.

Proof: The equivalence between (a) and (b) is in the Appendix, Theorem 6, see also [18].

We shall now establish the equivalence between (a) and (c).

Consider Case 1, when the consumption set $X = R^N$. We first show that limited arbitrage is a necessary condition for the existence of the social choice map ϕ for all $K \subset \{1...H\}$ and all $k \geq 2$ satisfying the three axioms $(A1)(A2)(A3)$.

Assume that such a social choice map ϕ exists for all $r \geq 2$. Let LP_E be the subspace of P_E consisting of all its linear preferences, i.e. those preferences within the space P_E which are representable by linear utility functions having their gradients, in each position $j = 1...r$, contained in the set $N_K \cup \{0\} \subset R^N$. Let $in : LP_E \to P_E$ denote the inclusion map. Note that each preference $\nu \in LP_E$ is uniquely identified by the normal $N\nu \in R^{Nr}$ to one of its indifference surfaces: by linearity all such normals are the same. Therefore

$$LP_E \approx (N_K \cup \{0\})^r, \tag{20}$$

i.e. the space of linear preferences LP_E is homeomorphic to the product space $(N_K \cup \{0\})^r$. Now consider an allocation $\xi \in X^K$ and define the map $\pi : P_E \to LP_E$ so that

$$\pi(\kappa) = \kappa_\xi,$$

where $\kappa_\xi \in LP_E$ is the linear preference over allocations in X^r having as its gradient in R^{Nr} the vector $G\kappa(\xi)$ which is the normal to the indifference surface of the preference κ at ξ. Both maps *in* and π are continuous in their domains.

By assumption, there exists a social choice map $\phi : (P_E)^r \to P_E$ for any $r \geq 2$, satisfying the three axioms. Now consider for any $r \geq 2$ the map induced by the composition of in and ϕ, defined by:

$$\psi : (LP_E)^r \to LP_E,$$

$$\psi(\nu_1 ... \nu_r) = \pi(\phi(in[\nu_1] ... in[\nu_r])).$$

The map $\psi : (LP_E)^r \to LP_E$, is continuous, anonymous and respects unanimity by construction, because ϕ satisfies these three properties. Since by (20) $\forall j$, $LP_E \approx (N_K \cup \{0\})^r$, ψ defines a map $\Lambda : [(N_K \cup \{0\})^r]^r \to (N_K \cup \{0\})^r$ for all $r \geq 2$ satisfying the three axioms $(A1)(A2)(A3)$. Since Λ is continuous and respects unanimity, it maps each connected component of the space $(N_K \cup \{0\})^r$ into itself. In particular, the restriction of the map Λ to the connected component $[(N_K)^r]^r$ of $[(N_K \cup \{0\})^r]^r$, denoted $\Lambda/(N_K)^r$, maps $[(N_K)^r]^r$ into $(N_K)^r$, i.e. $\Lambda/(N_K)^r : [(N_K)^r]^r \to (N_K)^r$, and it satisfies the three axioms $(A1)(A2)(A3)$. By Theorem 1 of Chichilnisky and Heal [13] such a map Λ exists if and only if the space $(N_K)^r$ is contractible for all $r \geq 2$; this in turn is true if and only if the space N_K is contractible - see Section 2. Therefore the contractibility of the space N_K is necessary for the existence of the social choice map ϕ. But Theorem 5 in the Appendix proves that N_K is contractible for all $K \subset \{1 ... H\}$ if and only if the *limited arbitrage* condition (6) is satisfied. This completes the proof of necessity of limited arbitrage.

We now turn to the proof of sufficiency of limited arbitrage. By definition, limited arbitrage implies the existence of a non-zero vector $v \in R^N$ in the intersection of all the dual cones:

$$v \in \bigcap_{i=1}^{H} D(\rho_i, \Omega_i).$$

Furthermore, by definition of the space P_E, all the indifference surfaces of any preference in P_E must intersect the ray defined by the vector $(v ... v)$ in the space of allocations X^{Nr}. Therefore the conditions required in Chichilnisky [10] for the existence of a social choice map $\phi : (P_E)^r \to P_E$ satisfying the three axioms $(A1)(A2)(A3)$ are satisfied for the space P_E. This completes the proof of sufficiency of limited arbitrage.

Case 2. Here $X = R^N_+$. The proof in this case is the same replacing the set of gradients N_K by the set

$$M_K = \bigcup_{i=1}^{K} \partial D(\rho_i, \Omega_i). \ \diamond$$

The following result establishes that the existence of an equilibrium and of a social choice map in Theorem 2 requires only that the limited arbitrage condition be satisfied on subsets of at most $N + 1$ traders, where N is the number of commodities in the economy:

Theorem 3 *Consider an economy E as in Theorem 2. The following four properties are equivalent:*

(a) The market economy E has a competitive equilibrium

(b) Every sub economy of E with at most $N + 1$ traders has a competitive equilibrium

(c) E has limited arbitrage

(d) E has limited arbitrage for any subset of individuals with no more than $N + 1$ members.

(e) For any space of preferences P_E similar to those of a subset K of market traders, there exists a continuous anonymous social choice map $\phi : (P_E)^r \to P_E$ respecting unanimity, for all $r \leq N + 1$.

Proof: The proof that (a)⇔(c) and (b)⇔(d) follows directly from Theorem 7 in the Appendix. That (c)⇔(d) follows from Theorem 6 in the Appendix. The proof that (c)⇔(e) follows from Theorem 2.◇

Theorem 4 *Consider a market economy $E = \{X, \rho_i, \Omega_i, i = 1 ... H\}$. Any market allocation $(x_1^* ... x_H^*) \in Int(X^H)$ of E is also a social allocation for E.*

Proof: When a market allocation for E exists, E has a competitive equilibrium described by a price vector p^* and an allocation $(x_1^*...x_H^*) \in X^H$ which is individually optimal within the budget sets and which clears all the markets. We shall show that if this market allocation $(x_1^*...x_H^*)$ is interior to X^H, then it is also a *social allocation*. This means that any space of preferences P_E defined over allocations for the H traders, namely on X^H, and consisting of preferences which are similar to those of the H traders in the economy $E = \{X, \rho_i, \Omega_i, i = 1...H\}$, admits a social choice map Ψ: $(P_E)^H \rightarrow P_E$ satisfying the axioms $(A1)(A2)(A3)$, and such that $(x_1^*...x_H^*)$ optimizes the social preference $\Psi(\rho_1...\rho_H)$ over the set of all feasible allocations Υ of the economy E.

Since a market equilibrium exists, by Theorem 2, there exists a social choice map $\phi : (P_E)^H \rightarrow P_E$ satisfying the three axioms $(A1)(A2)$ and $(A3)$, for the space of preferences P_E similar to the H traders in the economy E. By Theorem 2 we also know that limited arbitrage is satisfied, i.e. $\bigcap_{i=1}^{H} D(\rho_i, \Omega_i) \neq \emptyset$, since by assumption E has a competitive equilibrium.

We now use a partition of unity on the space P_E as defined in Section 2 in order to define a modification of the social choice map ϕ, called Ψ, which also satisfies the three axioms, and according to which the equilibrium allocation is optimal within all feasible allocations for the social preference $\Psi(\rho_1...\rho_H)$. Note that by the results of Chichilnisky [10] the space P_E is contained in a manifold which is the inverse image under a smooth retraction of a linear space, the space spanned by the vector $(v...v) \in R^{NH}$, where $v \in \bigcap_{i=1}^{H} D(\rho_i, \Omega_i)$. Therefore the space P_E is Hausdorff and we can therefore apply a partition of unity.

Let U be the set of preference profiles in P_E consisting of the profile $(\rho_1...\rho_H)$ and of all its permutations,

$$U = \bigcup_{\pi} \{(\rho_{\pi_1}...\rho_{\pi_H})\}, \text{ for all permutations } \pi \text{ of the set } \{1...H\}\}.$$

The set U consists of finitely many points in $(P_E)^H$; U is disjoint from the diagonal $\Delta(P^E)^H = \{(\kappa_1...\kappa_H) \in (P^E)^H : \forall i, j \ \kappa_i = \kappa_j\}$, because the profiles in U consist of private preferences each defined over a different position. Since $(P_E)^H$ is contained in a manifold, it is a Hausdorff space.

We shall now construct a new social choice map Ψ with the desired properties by using a partition of unity for $(P_E)^H$, as defined in Section 2. Using this partition of unity we modify the map $\phi : (P_E)^H \rightarrow P_E$ to obtain another continuous anonymous map $\psi : (P_E)^H \rightarrow P_E$, which differs from ϕ only in an open neighborhood $\theta(U)$ of the set U, and is otherwise identical to ϕ. Within the set U the new map Ψ satisfies:

$$\Psi(\rho_1...\rho_H) = \Psi(\rho_{\pi_1}...\rho_{\pi_H}) = \kappa_{p^*} \tag{21}$$

for all permutations π of the set of indices $\{1...H\}$, where in (21) κ_{p^*} is the linear preference over allocations in X^H with gradient vector $(\lambda_1 Du_1(x_1^*)...\lambda_H Du_H(x_H^*))$, for some vector $(\lambda_1...\lambda_H) \in R_+^H$, $Du_i(x_i^*)$ is the gradient of u_i at (x_i^*) for a utility u_i which represents ρ_i, and where

$$\Psi(\kappa_1...\kappa_H) = \phi(\kappa_1...\kappa_H)$$

if $(\kappa_1...\kappa_H) \notin \theta(U)$. The map

$$\Psi : (P_E)^H \rightarrow P_E$$

satisfies the three axioms by construction. We shall now show that the competitive equilibrium allocation $(x_1^*...x_H^*) \in X^H$ is a social allocation for the economy E with the social choice map Ψ. Since the allocation $(x_1^*...x_H^*) \in Int(X^H)$ by the assumptions of the Theorem and is the equilibrium allocation corresponding to the price vector p^*, there exists a vector $(\lambda_1^*...\lambda_H^*) \in R_+^H$, such that:

$$W(x_1^*...x_H^*) = Max_{(x_1...x_H)\in\Upsilon} \ W(x_1...x_H) \tag{22}$$

$$\text{where } W(x_1...x_H) = \sum_{i=1}^{H} \lambda_i^* u_i(x_i),$$

and where the utility $u_i : X \rightarrow R$ represents the preference ρ_i. Now choose the vector $(\lambda_1 ... \lambda_H)$ in the definition of Ψ following (21) to be the vector $(\lambda_1^* ... \lambda_H^*)$ in (22). Then, with this definition of $(\lambda_1 ... \lambda_H)$, the allocation $(x_1^* ... x_H^*)$ maximizes the social preference $\Psi(\rho_1 ... \rho_H)$ over the feasible set Υ. The allocation $(x_1^* ... x_H^*) \in X^H$ is therefore a social allocation for the social choice map Ψ and the economy E, as we wished to prove.$\diamond$

8 Related Literature

8.1 Irreducibility, Resource Relatedness and No Arbitrage

It may be useful to situate our conditions in the context of the literature. We shall refer to three other main conditions which have been used in various ways to prove the existence of a competitive equilibrium: *resource relatedness*, *irreducibility* and *no-arbitrage*. None of these conditions has been used to show the existence of social allocations. Indeed, to our knowledge, no conditions exist in the literature linking the properties of endowments and preferences in a market economy to the problem of social choice.

While our condition of *limited arbitrage* - defined in Section 3 - bounds the extent of diversity among the market's traders, it does so in a different way than *irreducibility* and *resource relatedness* do. The latter are only applicable to economies where the consumption is bounded below, or where there is a bound on short sales. Instead, *limited arbitrage* is applicable both to this case and also to the case where neither consumption nor short sales are bounded below. Irreducibility and resource relatedness are sufficient conditions for the existence of a competitive equilibrium, while limited arbitrage is necessary as well as sufficient for the existence of a competitive equilibrium [18]. Limited arbitrage is necessary as well as sufficient for the existence of a competitive equilibrium, either when the markets have bounds on short sales or when they do not, while irreducibility and resource relatedness apply only when consumption is bounded below.

When there is a bound on short sales, irreducibility and resource relatedness work by ensuring that all individuals' endowments are desired by others so that none will have zero income, see e.g. Arrow and Hahn [3], Chapter 4. Limited arbitrage works differently: by ensuring sufficient similarity of preferences that even when some individuals may have zero endowments a competitive equilibrium still exists.

Werner [37] defines a condition of *no-arbitrage*, which requires the existence of a price at which no increases in utility are possible at zero costs. His condition is different from our condition of *limited arbitrage*. There is a formal difference in the definitions of both conditions which has major practical implications, as follows. The cones defining *limited arbitrage* consist of rays which intersect every indifference surface of an individual's preference, while the "recession" cones used by Werner to define no-arbitrage generally do not satisfy this condition [37]. This formal difference leads to substantial differences in results, as is discussed in the following. In addition, the definition of our condition depends on endowments as well as preferences: with the same preferences, our economy will have a competitive equilibrium at some individual endowments and not at others. Indeed, one should expect that the existence of a competitive equilibrium should depend not only on individuals' preferences but also on their endowments, and this is precisely what limited arbitrage shows. Instead, Werner's cones are assumed to be the same at every allocation ([37], Assumption A3 and Proposition 1), so that his condition of no-arbitrage is independent from the initial endowments. *No-arbitrage* requires the existence of a price at which no individual can make positive utility increases at zero costs, a condition which must be verified in principle at all allocations. Our condition of limited arbitrage requires, instead, that there should exist a price at which only finite utility increases are achievable at zero cost from initial allocations: limited arbitrage needs only be satisfied at one allocation, the initial endowment.

Another difference is that Werner's condition is binding only when consumption sets are not bounded below and it is always satisfied otherwise ([37], Section 6, p. 1414) while, as already pointed out, our limited arbitrage condition is binding whether consumption sets are bounded below or not. The difference in conditions allows to obtain stronger results on the existence of a competitive equilibrium in Chichilnisky [18]. The *no-arbitrage* condition in [37] is sufficient, but it is not necessary for the existence of a competitive equilibrium. The necessity requires that preferences have no linear half-subspaces in their indifference surfaces. This eliminates linear and piecewise linear preferences, see Theorem 1 [37]; by contrast such preferences are included in our framework. Furthermore, no-arbitrage is neither necessary nor sufficient for the existence of a competitive equilibrium when the consumption sets are bounded below; in such cases it is always satisfied. Instead, *limited arbitrage* is binding, and necessary as well as sufficient, in all cases: when consumption sets are bounded below and when they are not.

8.2 Comparison with Arrow's axioms

It seems worth comparing our axioms with Arrow's [2] classic axioms of social choice. Arrow's axioms are more suitable for finite set of choices, such as voting among a finite set of n candidates. Instead, we are choosing here among an infinite set of choices, namely among the set Υ of all feasible allocations in Euclidean space.

The three axioms $(A1)(A2)(A3)$ are different from Arrow's [2]; indeed, they are not comparable with his axioms. The *anonymity* condition $(A2)$ is stronger than Arrow's *non-dictatorship* axiom, because the former requires equal treatment while the latter eliminates only extreme inequality of treatment. *Respect of unanimity* $(A3)$ is strictly weaker than his *Pareto condition*, since respect of unanimity is only binding when all preferences within a profile are identical. Instead, the Pareto condition applies to any profile of preferences which, equal or not, prefer a given choice x to another y. Finally, Arrow does not consider continuity $(A1)$ as we do, and we do not require Arrow's axiom of *independence from irrelevant alternatives*, an axiom which has been somewhat controversial. In other words: neither set of axioms implies the other.

Continuity is required here on the grounds of statistical tractability: it implies that the sampling of populations' preferences will approach the true distribution provided the grid of observations in the sample is fine enough. The continuity axiom makes this formulation of social choice better suited to continuous sets of choices and to connected sets of preferences. This is because when the space of preferences is discrete or finite - as it would be when there are finitely many choices - then the space of preferences is itself finite and therefore continuity is a vacuous requirement. Recent work by Baigent [5][6] and by Nitzan [33] has extended the axiom $(A1)$ of continuity to one of "proximity" of preferences, a concept which is appropriate for discrete spaces of preferences and for preferences over finitely many choices. Using "proximity" instead of "continuity", and preserving the other two axioms - anonymity and respect of unanimity - Baigent [5][6] proved the impossibility results of Chichilnisky [11] for the case of finitely many choices. Chichilnisky [14] has recently shown the connection between the existence of social choice rules and the manipulation of non cooperative games.

What makes the axioms used in this paper particularly well suited for our problem is that they lend themselves naturally to the study of preferences and choices similar to those which are studied in market economics. Furthermore, with these axioms there exist simple necessary and sufficient conditions for resolving the social choice paradox, Chichilnisky and Heal [13]; this is not true for Arrow's axioms.

9 Conclusions

Theorem 2 established that limited arbitrage is a necessary and sufficient condition for the existence of social choice maps on spaces of preferences similar to those of a market economy. The same condition - *limited arbitrage* - is shown in Theorem 7 to be necessary and sufficient for the existence of a competitive equilibrium in a market economy. Furthermore, Theorem 3 established that a market allocation is always a social allocation. In this sense, the results of this paper unify two forms of resource allocation, by markets and by social choice, which have developed and remained separate until now.

We have chosen competitive equilibrium allocations - rather than other concepts of market equilibrium - because of the Pareto efficiency of competitive equilibrium. This property is generally lost in weaker forms of market equilibrium, such as quasi-equilibrium and compensated equilibrium.

Similarly we have concentrated on three axioms of social choice: continuity, anonymity and respect of unanimity, introduced in [11], which are particularly well suited for problems where the choices are elements in Euclidean space. Arrow's [2] classic axioms appear to be better suited for problems with finitely many choices.

This choice of axioms has allowed us to use the necessary and sufficient "domain" restrictions for a resolution of the social choice paradox proven in Chichilnisky and Heal [13], namely the contractibility of the space of preferences. We have also utilized the necessary and sufficient conditions for the non-empty intersection of a family of sets in Theorem 5, Chichilnisky [15][16]: these require the contractibility of the union of all subfamilies of these sets. The circle closes because the *limited arbitrage* condition, which is necessary and sufficient for the existence of a competitive equilibrium, is defined as the non-empty intersection of a family of convex sets, the "dual cones" of the traders of a market economy. This requires, in turn, contractibility. Thus we have linked *limited arbitrage,* which is needed for the existence of a market equilibrium with the *contractibility of the space of preferences* which is needed for a resolution of the social choice problem.

The results presented here apply equally to the case of Arrow-Debreu economies having as consumption sets the positive orthant, and to those having as consumption sets the whole Euclidean space, but they have a somewhat different interpretation in these two cases. In the case where there are no bounds on short sales, so that the consumption space is the whole Euclidean space, the condition of *limited arbitrage* is described precisely by the existence of a price which makes unaffordable for all players those allocations which could bring them unbounded utility increases from their initial endowments. This limits the diversity of preferences and endowments in the economy. This direction, in turn, is used as in Chichilnisky [10], to define a social choice map from individual preferences into social preferences which can rank all feasible allocations and yield a socially optimal allocation.

In the second case, when consumption is bounded below so that the consumption set is the positive orthant, limited arbitrage limits the diversity of endowments and preferences in a modified form. In this case, the results relate to the conditions of irreducibility and of resource relatedness which Arrow and Debreu [4] and McKenzie [30] introduced to prove the existence of a competitive equilibrium in economies where consumption is bounded below. Their conditions place also limits of diversity and eliminate the possibility that some individuals have zero or minimal income. Limited arbitrage means, instead, that among all supporting prices which assign zero income to some individuals, there is always one which makes un affordable those allocations leading to unbounded utility from the initial allocation. This price defines also a direction which intersects transversely all the indifference surfaces of all the traders in the market, and yields a way to construct a social choice map.

The limited arbitrage condition is somewhat different when there are bounds on short sales than when such bounds do not exist, although mathematically it is very similar. When consumption sets are positive orthants, limited arbitrage measures social agreement about allocating zero values to the endowments of certain members of society. This agreement must include those same individuals to whom society assigns zero value. It may seem surprising that such an agreement could exist. In the case that it does not, both forms of resource allocation break down: the competitive market has no competitive equilibrium, and the social choice map as defined here does not exist.

When short sales are bounded below, but the consumption sets have more general specifications, such as translates of the positive orthant, the limited arbitrage condition is no longer restricted to individuals with zero value. It applies to individuals having endowments of minimal value. In this case, the limited arbitrage condition requires that within the set of prices assigning an individual's endowment a minimum value, there should exist one price at which only bounded utility should be attainable from initial endowments.

An extension of the results provided here could elucidate the connection between the existence of a competitive equilibrium and the manipulation of social games. Such results could follow from the connection between the existence of social choice maps and the manipulation of games, Chichilnisky [14]. It seems also possible to extend the results of this paper to economies with production. Issues of survival and underemployment in market economies are obvious directions in which to extend the inquiry of this paper. Finally, recent results show that it is possible to develop useful algorithms for computing a competitive equilibrium from the limited arbitrage condition (Chichilnisky and Eaves [20]).

10 Appendix

Theorem 5 *Consider a family* $\bigsqcup = \{U_i\}_{i=1...n}$ *of convex sets in* R^m $n, m \geq 1$. *Then*

$$\bigcap_{i=1}^{n} U_i \neq \emptyset \text{ if and only if } \bigcup_{j \in J} U_j \text{ is contractible}$$

$\forall J \subset \{1...n\}$

This theorem was proved in Chichilnisky [15]; see also Chichilnisky[16]. Theorem 5 implies the Helly's theorem [27] and the Knaster-Kuratowski-Marzukiewicz theorem.[16]-the latter of which implies in turn the Brouwer's fixed point theorem.

Theorem 6 *Consider a family* $\bigsqcup = \{U_i\}_{i=1...n}$ *of convex sets in* R^m $n, m \geq 1$. *Then*

$$\bigcap_{i=1}^{n} U_i \neq \emptyset \text{ if and only if } \bigcap_{j \in J} U_i \neq \emptyset$$

for any subset of indices $J \subset \{1...n\}$ *having at most* $m+1$ *elements. In particular, a market economy* E *as defined in Section 2 with* n *traders and* m *commodities satisfies* limited arbitrage *if and only if every sub economy of* E *with at most* $m+1$ *traders satisfies limited arbitrage.*

For a proof see Chichilnisky [16].◇

The following theorem links the existence of a competitive equilibrium with the condition of limited arbitrage, see also Chichilnisky [18].

Theorem 7 *Consider an economy* $E = \{X, \rho_h, \Omega_h, h = 1...H\}$, *where* $H \geq 2$, $X = R^N$ *or* $X = R^N_+$ *and* $N \geq 1$.

Then the following two properties are equivalent:

(a) the economy E *has limited arbitrage*

(b) the economy E *has a competitive equilibrium*

Proof: The strategy of the proof is as follows. First we prove that limited arbitrage is necessary for the existence of a competitive equilibrium. Next we establish sufficiency. The proof of sufficiency has two parts. The first part is the proof of existence of a pseudo equilibrium; for this we use a fixed point argument on the Pareto frontier of the economy. This requires in turn to prove that the Pareto frontier of the economy is homeomorphic to a simplex, a property which we establish using limited arbitrage. Finally, using limited arbitrage we prove that the pseudo equilibrium is also a competitive equilibrium.

We prove first the necessity of limited arbitrage. Let the utility function $u_h : X \rightarrow R$ represent the preference $\rho_h \in E$, i.e.$\forall x, y \in X$, $u_h(x) > u_h(y) \Leftrightarrow x \succ_{\rho_h} y$. By appropriate renormalization we can assume without loss of generality that $u_h(0) = 0$ so that $u_h(\Omega_h) \geq 0$, and that $Sup_{x \in X}(u_h(x)) = \infty$. Now assume that (a) is not true, and consider the case $X = R^N$ first. Then

$$\bigcap_{h=1}^{H} D(\rho_h, \Omega_h) = \emptyset, \tag{23}$$

which implies that for all $y \in R^N$, there exists an $h \in \{1...H\}$ and a vector $v(y) \in A(\rho_h, \Omega_h)$ such that:

$$\begin{array}{c} < y, \lambda v(y) > \leq 0, \text{ and} \\ \lim_{\lambda \rightarrow \infty}(u_h(\Omega_h + \lambda v(y)) = \infty. \end{array} \tag{24}$$

Consider now a competitive equilibrium described by a price p^* and an allocation $(x_1^* ... x_H^*)$. By (24) for some $\lambda > 0$,

$$u_h(\Omega_h + \lambda v(y)) > u_h(x_h^*) \text{ and } < p^*, \lambda v(y) > \leq 0,$$

contradicting the fact that x^* is an equilibrium allocation. Therefore no competitive equilibrium exists when (24) is true: limited arbitrage is necessary for the existence of a competitive equilibrium when $X = R^N$. Consider next the case $X = R_+^N$. Assume first that $\forall q \in S(E)\ \exists\ h \in \{1...H\}$ s.t. $< q, \Omega_h >= 0$. Then if limited arbitrage is not satisfied

$$\bigcap_{h=1}^{H} \partial D(\rho_h, \Omega_h) = \emptyset, \tag{25}$$

which implies that

$$\begin{array}{c} \forall q \in R^N, \exists h \text{ and } v(q) \in A(\rho_h, \Omega_h) : \\ < q, \Omega_h >= 0, \text{ and } \forall \lambda > 0, < q, \lambda v(q) > \leq 0. \end{array} \tag{26}$$

Since $v(q) \in A(\rho_h, \Omega_h)$

$$\lim_{\lambda \rightarrow \infty} (u_h(\Omega_h + \lambda v(y)) = \infty. \tag{27}$$

Consider now a competitive equilibrium price p^* and the corresponding allocation $(x_1^* ... x_H^*)$. Then $p^* \in S(E)$, and (26) and (27) imply that $\exists h$ s.t. for some $\lambda > 0$,

$$u_h(\Omega_h + \lambda v(y)) > u_h(x_h^*) \text{ and } < p^*, \lambda v(y) > \leq 0,$$

contradicting the assumption that p^* and $(x_1^* ... x_H^*)$ define a competitive equilibrium.

It remains to consider the case where $\exists q \in S(E)$ such that $\forall h \in \{1...H\} < q, \Omega_h > \neq 0$. But in this case by definition $\bigcap_{h=1}^{H} \partial D(\rho_h, \Omega_h) \neq \emptyset$ since $\forall h \in \{1...H\}\ \partial D(\rho_h, \Omega_h) = R_+^N$, so that limited arbitrage is always satisfied. This completes the proof that limited arbitrage is *necessary* for the existence of a competitive equilibrium, when $X = R^N$ and when $X = R_+^N$.

The next step is to prove that limited arbitrage is sufficient for the existence of a competitive equilibrium. For this we will utilize the standard method - introduced by Negishi - of proving first the existence of a *quasi-equilibrium* as defined in Section 2, using a fixed point theorem on the Pareto frontier

$$\begin{array}{c} P(E) = \{(U_1 ... U_H) \in R^H : U_h = u_h(x_h) \text{ where } (x_1 ... x_H) \in \Upsilon \\ \text{and} \sim \exists (V_1 ... V_H) : V_h = u_h(y_h),\ (y_1 ... y_H) \in \Upsilon \text{ with } \forall h,\ V_h \geq U_h,\ V_h > U_h \text{ for some } h\}. \end{array}$$

The quasi-equilibrium is subsequently shown to be a competitive equilibrium, thus completing the proof. The proof must address two practical difficulties in applying this strategy, one when the consumption set $X = R^N$, and a different one when $X = R_+^N$. Both difficulties are resolved by the limited arbitrage condition. The problem is as follows: when $X = R^N$ the Pareto frontier

$P(E)$ may fail to be bounded and closed, because the utility obtained by the traders from their initial endowments may not attain a maximum over feasible allocations when there are no bounds on short sales. This failure leads to the non-existence of a competitive equilibrium in well known cases; this problem of existence appears also in economies with infinitely many commodities, but when commodity spaces are infinite dimensional it can appear even if the consumption set is the positive orthant, see the examples in Chichilnisky and Heal [19]. In practical terms, the problem is that the Pareto frontier may not be homeomorphic to a unit simplex, a property which is essential in the proof of existence of a quasiequilibrium. The role of the limited arbitrage condition in this case is to ensure that the Pareto frontier is bounded and closed; together with the quasi concavity of preferences this implies that the Pareto frontier is homeomorphic to a unit simplex so that standard existence arguments can be invoked.

A more standard difficulty arises when the consumption set is $X=R^N_+$. Here the Pareto frontier is always closed and bounded and a quasi-equilibrium exists. However, in this case the quasi-equilibrium may fail to be a competitive equilibrium. This is the type of problem which the conditions of resource relatedness and of irreducibility are meant to circumvent. The problem arises only when some individual has zero income at the quasi-equilibrium allocation and is illustrated in Figure 8 above. In this case, minimizing costs may not imply maximizing utility so that a quasi-equilibrium may fail to be a competitive equilibrium. This second potential failure of existence is also ruled out by the condition of limited arbitrage.

Consider first the case $X = R^N$. We shall show that the Pareto frontier is a closed bounded set in R^H. Define the set F_Ω of feasible and individually rational allocations:

$$F_\Omega = \{z \in R^{NH} : z = (x_1 + \Omega_1 ... x_H + \Omega_H\},$$
$$\text{where} \sum_{h=1}^{H} x_h \leq 0 \text{ and } \forall h,\ u_h(x_h + \Omega_h) \geq u_h(\Omega_h) \geq 0\}$$

Let Δ denote the unit simplex in R^H, and define the set S_r of utility vectors which are collinear with a given element $r = (r_1 ... r_H) \in \Delta$,

$$S_r = \{(U_1 ... U_H) \in R^H : \forall h = 1...H,\ U_h = u_h(z_h), \text{ where } z = (z_1 ... z_H) \in F_\Omega$$
$$\text{and } \exists \nu > 0 \text{ s.t. } \nu.U_h = r_h\}$$

We shall now prove that limited arbitrage implies that S_r is bounded for all $r \in \Delta$. Consider first the case where $r >> 0$. Assume that S_r is not bounded. Then there exists a sequence denoted $\{x^n\}_{n=1,2...} = \{(x_1^n ... x_H^n)\}_{n=1,2} \subset F_\Omega$ such that $(u_1(x_1^n) ... u_H(x_H^n))$ is in S_r and for some $h \in \{1...H\}, \lim_{n\to\infty} u_h(x_h^n) = \infty$. Let $z_h^n = x_h^n - \Omega_h$. We may assume that $\forall n \sum_{h=1}^{H}(x_h^n - \Omega_h) = \sum_{h=1}^{H} z_h^n = 0$. Since $r >> 0$, if $\exists h : \lim_{n\to\infty} u_h(x_h^n) = \infty$, then $\forall h$=1...H $\lim_{n\to\infty} u_h(z_h^n) = \infty$. For all $h \in \{1...H\}$, let α_h be a point of accumulation of the sequence of vectors $\{(z_h^n/\|z_h^n\|)\}$; this sequence has such a point because it is contained in the unit sphere $S^{N-1} \subset R^N$. Note that $\alpha_h \in A(\rho_h, \Omega_h)$. Now let α_h^n denote the projection of the vector z_h^n on the line in R^N defined by the vector α_h, and consider a subsequence $\{z^m\}$ of $\{z^n\}$ satisfying

$$\lim_{m\to\infty} \|z_h^m - \alpha_h^m\| = 0.$$

Then

$$0 = \lim_{m\to\infty} \sum_{h=1}^{H} z_h^m = \lim_{m\to\infty} \sum_{h=1}^{H} \alpha_h^m = \sum_{h=1}^{H} \alpha_h.$$

In particular $\forall q \in R^N < q, \alpha_h >= \lim_{m\to\infty} < q, \sum_{h=1}^{H} \alpha_h^m >= 0$. Since $\alpha_h \in A(\rho_h, \Omega)$, this implies that there exists no $q \in R^N$ such that $< q, y >> 0$ for all $y \in A(\rho_h, \Omega_h)$, so that

$$\bigcap_{h=1}^{H} D(\rho_h, \Omega_h) = \emptyset,$$

contradicting the limited arbitrage condition. Therefore limited arbitrage implies that S_r must be bounded $\forall\ r >> 0$.

Now assume that $r \in \partial\Delta$, so that $r_h = 0$ for some $h = 1...H$.. We may assume that $r_h \neq 0$ for some h, for otherwise the ray S_r is clearly bounded. Then we may approximate r by a sequence of rays $\{r^k\}_{k=1,2...} \subset \Delta$, such that $\forall k$, $r^k >> 0$, and for which a proof similar to the previous case applies. There exist a sequence $\{x^k\}_{k=1,2...} \subset F_\Omega$ such that $\forall k$, $(u_1(x_1^k)...u_H(x_H^k))$ is maximal in S_{r^k}. In particular, if $z_h^n = x_h^n - \Omega_h$, then $\forall k$ $\sum_{h=1}^H z_h^k = \sum_{h=1}^H (x_h^n - \Omega_h) = 0$. For any h let α_h be a point of accumulation of the sequence of vectors $(z_h^k/||z_h^k||) \subset S^{N-1} \subset R^N$. By definition, $\alpha_h \in A(\rho_h, \Omega_h)$. Let α_h^n denote the projection of the vector z_h^k on the line in R^N containing the vector α_j. Consider now a subsequence $\{z^m\}$ of $\{z^k\}$ satisfying

$$\lim_{m\to\infty} ||z_h^m - \alpha_{hj}^m|| = 0.$$

Then

$$0 = \lim_{m\to\infty} \sum_{h=1}^H z_h^m = \lim_{m\to\infty} \sum_{h=1}^H \alpha_h^m = \sum_{h=1}^H \alpha_h.$$

Since $\alpha_h \in A(\rho_h, \Omega_h)$ this implies that there exists no $q \in R^N$ such that $< q, y >> 0$ for all $y \in A(\rho_h, \Omega_h)$, i.e.

$$\bigcap_{h=1}^H D(\rho_h, \Omega_h) = \emptyset$$

contradicting limited arbitrage. Therefore limited arbitrage implies that S_r must be bounded for all $r \in \Delta$. This in turn implies that the utility possibility set $U(E) \subset R^H$ is bounded.

We now complete the proof that the Pareto frontier $P(E)$ is bounded and closed when $X = R^N$ by proving that limited arbitrage implies that $P(E)$ is closed. For any $r \in \Delta$, let $v = (v_1...v_H) \in R_+^H$ satisfy $v = Sup_{y\in S_r} y_i$. we know that such a v exists because the utility possibility set $U(E)$ is bounded. To prove that $P(E)$ is closed it suffices to show that there exists an allocation $(z_1...z_H) \in F_\Omega$ such that $v = (u_1(z_1)...u_H(z_H))$. Consider a sequence $\{z^n\} \subset F_\Omega$ such that: $U^n = (u_1(z_1^n)...u_H(z_H^n)) \in S_{r^n}$, U^n is maximal in the set S_r, $\lim_n\{r^n\} = r$, and $\lim_n U^n = v$. Since $U(E)$ is bounded, and each utility u_h is monotonic, there exists a vector of utility values $(U^1...U^H) = (u_1(y_1)...u_H(y_H)) \in R^H$, where $(y_1...y_H)$ may or not be a feasible allocation, such that $\lim_{n\to\infty} U^n = v$. It is straightforward to see that since $\lim_n U^n = v$ and v is optimal in S_r, the directions of the gradients of the sequence of the utilities define a Cauchy sequence, i.e.

$$\forall h = 1...H,\ \lim_{n,m}\left(\frac{Gu_h(z_h^n)}{||Gu_h(z_h^n)||}\right) - \frac{Gu_h(z_h^m)}{||Gu_h(z_h^m)||}) = 0.$$

Define now the sequence $\{s_h^n\}_{n=1,2...}$ where $s_h^n = Gu_h(z_h^n)/||u_h(z_h^n)|| \in S^{N-1} \subset R^N$. Since S^{N-1} is compact, $\forall h$ there exists a point of accumulation of $\{s_h^n\}_{n=1,2...}$, denoted s_h. Since for all h, $u_h(z_h^n) \to u_h(y_h)$, then $\forall\ \epsilon > 0, \exists T$ and $\exists\ w_h^n \in R^N$ such that $u_h(w_h^n) = v_h$ and

$$||\frac{Gu_h(z_h^n)}{||Gu_h(z_h^n)||} - \frac{Gu_h(w_h^n)}{||Gu_h(z_h^m)||}|| < \epsilon \text{ for } n > T.$$

The sequence $\{Gu_1(z_1^n)...Gu_H(z_H^n)\}$ consists of gradients of efficient utility levels and it converges to $\{s_1...s_H\}$, so by the assumptions on the utilities u_h $\forall h$ there exists a vector $z_h \in u_h^{-1}(v_h) \in R^N$ such that $Gu_h(z_h) = \lambda_h s_h$ for some $\lambda_h > 0$. Furthermore, $\sum_{h=1}^H z_h^n = \sum_{h=1}^H \Omega_h$, so that $(z_1...z_H) \in F_\Omega$. Since $(v_1...v_H) = (u_1(z_1)...u_H(z_H))$ we have completed the proof that the Pareto frontier $P(E)$ of the economy E is closed, as we wished to prove. We have shown that limited arbitrage implies that when $X = R^N$ the set $P(E)$ is closed and bounded. Therefore the proof that $P(E)$ is homeomorphic to the unit simplex $\Delta \in R^H$ is now standard from the quasi concavity of the preferences, see for example Arrow and Hahn [3]. For the case $X = R^N_+$, their proof establishes directly that this Pareto frontier is always homeomorphic to the unit simplex. It is now standard to establish that a *quasi-equilibrium* always exists, either when $X = R^N$ or $X = R^N_+$: for completeness we provide now a formal proof of existence of a quasi-equilibrium which works equally for these two cases next:

Define the set

$$T = \{y \in R^H : \sum_{h=1}^{H} y_h = 0\}.$$

For each $r >> 0$ in Δ let $(x_1(r)...x_H(r)) \in F_\Omega$ now denote the feasible allocation which gives the greatest utility vector collinear with r :

$$(u_1(x_1(r))...u_H(x_H(r))) = Sup_{w \in S_r}(u_1(w_1(r))...u_H(w_H(r))),.$$

in the vector order of R^H, and $\sum_{i=1}^{H}(x_i(r) - \Omega_i) = 0$. Such an allocation always exists because $\forall r \in \Delta$, S_r is bounded and closed; it defines a non-zero utility vector which depends continuously on r. Now let

$$P = \{p \in R^N : ||p|| = 1\} \text{ and}$$
$$P(r) = \{p \in P : p \text{ supports } x(r)\}.$$

By standard arguments, $P(r)$ is not empty, see e.g. Chichilnisky and Heal [19], Lemma 3. Define now a map $\varphi : \Delta \to T$:

$$\varphi(r) = \{< p, \Omega_1 - x_1(r) > ... < p, \Omega_H - x_H(r) >: p \in P(r)\}$$

$\varphi(r)$ is a non-empty convex valued correspondence, $\sum_{h=1}^{H} z_h = 0$ if $z \in \varphi(r)$, and

$$0 \in \varphi(r) \Leftrightarrow (x^*, p^*) \text{ is a quasi-equilibrium of } E,$$
$$\text{where } r = r(x^*) \text{ and } p^* \in P(r).$$

The next step is to show that φ is upper semi-continuous, i.e. if $r^n \to r$, $z^n \in \varphi(r^n)$, $z^n \to z$ then $z \in P(r)$. Consider the feasible allocation $x(r)$, where $r = \lim_n(r^n)$. Let v be any other allocation satisfying $u_h(v_h) > u_h(x_h(r))$, where $x_h(r)$ is the h-th coordinate of the vector $x(r)$ and v_h is the h-th coordinate of the vector v. Let $z^n \in \varphi(r^n)$ and $p^n \in P(r^n)$. Since $r^n \to r$, eventually $u_h(v_h) > u_h(x_h(r^n))$ so that $< p^n, v_h > \geq < p^n, x_h(r^n) > = < p^n, \Omega_h > -z_h^n$, where z_h^n is the h-th coordinate of z^n : this follows from the definitions of z^n and p^n. Let $\{p^n\}$ be a sequence of vectors such that $p^n \in P(r^n)$. The set P is compact and $\bigcup_r P(r)$ is closed; therefore $\bigcup_r P(r)$ is compact as well. There exists therefore a vector $p \in P$ and a subsequence $\{p^m\}$ of $\{p^n\}$ such that $< p^m, v_h > \to < p, v_h >$, so that in the limit $< p, v_h > \geq < p, \Omega_h > -z_h$. Since this is true for all such v, it is also true for v satisfying $u_h(v_h) \geq u_h(x_h(r))$ and in particular for $v = x$ so that $< p, x_h > \geq < p, \Omega_h > -z_h$ implying that $z \in \varphi(r)$ as we wished to prove. The proof of existence of a quasi-equilibrium is completed by showing that φ has a zero. For all $r \in \Delta$ define $\theta(r) = r + \varphi(r)$. The map $\theta : \Delta \subset \Delta$ is non-empty, upper semi continuous, convex-valued correspondence and it satisfies appropriate boundary conditions. By Kakutani's fixed point theorem, θ must have a fixed point r^* which is a zero of the map φ. The allocation $x^* = x^*(r^*)$ and a price $p^* \in P(r^*)$ define a quasi-equilibrium of the economy E.

The proof of existence of a quasi-equilibrium just provided is equally valid when $X = R^N$ or when $X = R^N_+$. Therefore to complete the proof of the theorem it remains only to show that the quasi-equilibrium is a competitive equilibrium.

Consider first the case $X = R^N$. Then $\forall h = 1...H$ there exists an allocation in X of strictly lower value than x_i^* at the price p^*. Therefore by Lemma 3, Chapter 4, page 81 of Arrow and Hahn [3], the quasi-equilibrium is also a competitive equilibrium. This establishes the existence of a competitive equilibrium when limited arbitrage is satisfied and $X = R^N$.

Now consider the case $X = R^N_+$. We have shown that when limited arbitrage is satisfied the economy E has a quasi-equilibrium consisting of a price p^* and an allocation x^*. It remains to show that the quasi-equilibrium is also a competitive equilibrium.

First note that if at the quasi-equilibrium (p^*, x^*) every individual has a positive income, i.e. $\forall h = 1...H < p^*, \Omega_h >> 0$, then by Lemma 3, Chapter 4 of Arrow and Hahn [3] the quasi-equilibrium is also a competitive equilibrium. Furthermore, since the quasi equilibrium $p^* \in S(E)$, then the set $S(E) \neq \emptyset$. To prove existence we consider two cases: first, the case where $\forall h, < q, v >> 0$. In this case, by the above remarks, (p^*, x^*) is a competitive equilibrium.

The second case is when $\forall q \in S(E)\ \exists\ h \in \{1...H\}$ s.t. $< p^*, \Omega_h >= 0$, a case where the vectors p^* and Ω_h must have some zero coordinates. The limited arbitrage condition in this case implies

$$\exists q^* \in S(E): \ \forall h, \ < q^*, v >> 0 \text{ for all } v \in A(\rho_h, \Omega_h). \tag{28}$$

Let $x^* = x^*_1...x^*_H$ be the allocation in Υ supported by the vector q^* defined in (28). Then by definition, $\forall h$, $x^*_h \geq_{\rho_h} \Omega_h$ and q^* supports x^*.

Recall that any h minimizes costs at x^*_h because q^* is a support. Now, (q^*, x^*) can fail to be a competitive equilibrium only when for some $h < q^*, x^*_h >= 0$, for otherwise the cost minimizing allocation is also utility maximizing in the budget set $B_h(q^*) = \{w \in X :< q^*, w >=< q^*, \Omega_h >\}$. It remains therefore to prove existence when $< q^*, x^*_h >= 0$ for some h. Since by the definition of $S(E)$, x^* is individually rational, i.e. $u_h(x^*_h) \geq u_h(\Omega_h)$, it follows that when $< q^*, x^*_h >= 0$, then $< q^*, \Omega_h >= 0$, because q^* is a supporting price for x^*. If $\forall h$, $u_h(x^*_h) = 0$ then $x^*_h \in \partial R^N_+$, and by the monotonicity and quasi-concavity of u_h, any vector $y \in B_h(q^*)$ must also satisfy $u_h(y) = 0$, so that x^*_h maximizes utility in $B_h(q^*)$, which implies that (q^*, x^*) is a competitive equilibrium. Therefore (q^*, x^*) is a competitive equilibrium unless for some h, $u_h(x^*_h) \neq 0$,. Assume then that (q^*, x^*) is not a competitive equilibrium. Then for some h, $u_h(x^*_h) \neq 0$, and therefore an indifference surface of a positive commodity bundle of u_h intersects ∂X at $x^*_h \in \partial X$. Let r be the ray in ∂X containing x^*_h. If $w \in r$ then $< q^*, w >= 0$, because $< q^*, x^*_h >= 0$. Since $u_h(x^*_h) > 0$, by the assumptions on u_h, all other indifference surfaces of u_h with higher utility intersect r, so that $r \subset A(\rho_r, x^*_h)$. Define now the ray $s = \{v : \exists w \in r : v = (x^*_h - \Omega_r) + w\}$. The ray $s \subset \partial X$; $s \subset A(\rho_h, \Omega_h)$ and $\forall v \in s$ $< q^*, v >=< q^*, (x^*_h - \Omega_r) + w >= 0$. But this contradicts the choice of q^* as a supporting price satisfying (28) since

$$\exists h \text{ and } y \in A(\rho_h, \Omega_h) \text{ such that } < q^*, y >= 0. \tag{29}$$

Since the contradiction arises from the assumption that (q^*, x^*) is not a competitive equilibrium, (q^*, x^*) must be a competitive equilibrium, and the proof of the theorem is complete.◇

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