Believing in Multiple Equilibria

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BELIEVING IN MULTIPLE EQUILIBRIA

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Abstract: If agents have common priors concerning the probability with which an equilibrium is selected, they have an incentive to trade beforehand. If their trading process satisfies a certain notion of individual rationality, these trades will reduce and ultimately remove all uncertainty concerning equilibrium selection. In this sense, multiple equilibria engender institutions which can uniquely determine an equilibrium.

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1. Introduction

Consider an economy which is known to have more than one competitive equilibrium. The agents in such an economy face uncertainty about which of the possible equilibria will occur. This type of uncertainty is called endogenous uncertainty (Kurz, 1974), to distinguish it from the normal concept of exogenous uncertainty which can appear in the economy's specification of preferences, technologies, and/or endowments.

This paper makes two related points about endogenous uncertainty. First, given standard concavity assumptions, agents having common priors about the endogenous uncertainty have cause to make contingent trades before the equilibrium is realized. Second, we identify certain trading processes that inevitably reduce and ultimately eliminate the endogenous uncertainty. This suggests that *while there is nothing inherently illogical about an economy which admits multiple equilibria, common priors about the equilibrium to be selected generate gains from trade which may eventually remove the multiplicity.*

The intuition behind these results is straightforward. Consider an economic system in which there is uncertainty about which of several possible equilibria will be chosen. The ex-ante utility of risk averse agents is reduced by this uncertainty. Further, this uncertainty is endogenous: it is not forced on the system by the randomness of nature, but arises only because agents are unsure which equilibrium will be chosen by the system. It is therefore possible for the agents to make deals amongst themselves which reduces this uncertainty. If agent A is well-off at equilibrium a and badly-off at equilibrium b, while agent B is well-off at equilibrium b but badly-off at equilibrium a, then each can partially insure the other against his adverse equilibrium. Such mutual insurance deals can ultimately remove all of the endogenous uncertainty, ensuring a unique allocation of resources in the system which obtains whatever the equilibrium selected.

Broadly speaking, our result suggests that any market structure which generates endogenous uncertainty carries within it "the seeds of its own destruction," i.e., incentives for agents to make changes in that market structure. Uniqueness of equilibrium may thus arise not as a result of the underlying tastes and technology in an economy, but rather as the result of endogenous changes to the market structure.

We prove this result with comparatively little structure upon the way that the agents realize gains from trade. In particular, we do not assume that these new trades can be accomplished without introducing still more endogenous uncertainty. Thus we subsume the price-contingent securities which are developed in Chichilnisky, Dutta, and Heal (1991) and Hahn (1991) and the endogenous uncertainty about default which arises in Chichilnisky and Wu (1992). In spite of the fact that our analysis is more general than these other papers, our arguments are more transparent, cutting through the layers of endogenous uncertainty that can be generated as agents make contingent trades in the face of endogenous uncertainty.

Implications of the analysis for bargaining theory are explored in the example of Section 3.2.

2. General Results

2.1. Weak Efficiency

Let **X** be the set of feasible allocations, and let **x** denote a generic element of this set. We assume that **X** is a convex and compact subset of some topological vector space. Let there be n individuals, and let $U = (U^1, U^2, \ldots U^n)$ list their utility functions. We assume that each U^i: $\mathbf{X} \rightarrow \mathbb{R}$ is continuous and concave.

Our theorem considers a sequence of probability measures $\{m_t\}_{t=1}^{\infty}$ on **X**. This sequence is interpreted as a trading process in the pure-exchange economies of Section 3.1, where m_t reflects "expectations" about the allocations which would result were the process to end in period t. The sequence reflects uncertainty about a negotiation process in the bargaining problems of Section 3.2. At stage t of the process, m_t gives the distribution over allocations We assume that the utility which individual i receives from probability measure m_t is $u_t^i \equiv \int U^i \, dm_t$ (implicitly, we assume that each m_t is defined over a σ-algebra which contains the Borel subsets of **X**, and thus the continuity of each U^i implies its measurability with respect to each m_t). Because each utility function U^i is thus employed as a von Neumann-Morgenstern index, the assumed concavity of U^i implies that agent i is either risk-neutral or risk-averse.

An allocation $\mathbf{x} \in \mathbf{X}$ is <u>individually rational</u> given m_{t-1} if $U(\mathbf{x}) \geq \mathbf{u}_{t-1} \equiv \int U \, dm_{t-1}$. An allocation $\mathbf{x} \in \mathbf{X}$ is <u>weakly efficient</u> if there is no alternative allocation $\mathbf{x}' \in \mathbf{X}$ such that $U(\mathbf{x}') >> U(\mathbf{x})$. A utility vector $\mathbf{u} \in \text{conv } U(\mathbf{X})$ is <u>weakly efficient</u> if there is no alternative utility vector $\mathbf{u}' \in \text{conv } U(\mathbf{X})$ such that $\mathbf{u}' >> \mathbf{u}$ (conv $U(\mathbf{X})$ denotes the convex hull of $U(\mathbf{X})$).

<u>Theorem 1</u>: Let $\{m_t\}_{t=1}^{\infty}$ be such that the support of each m_t consists of weakly efficient allocations which are individually rational given m_{t-1}. Then $\lim_{t\rightarrow\infty} \mathbf{u}_t$ exists and is a weakly efficient element of conv $U(\mathbf{X})$.

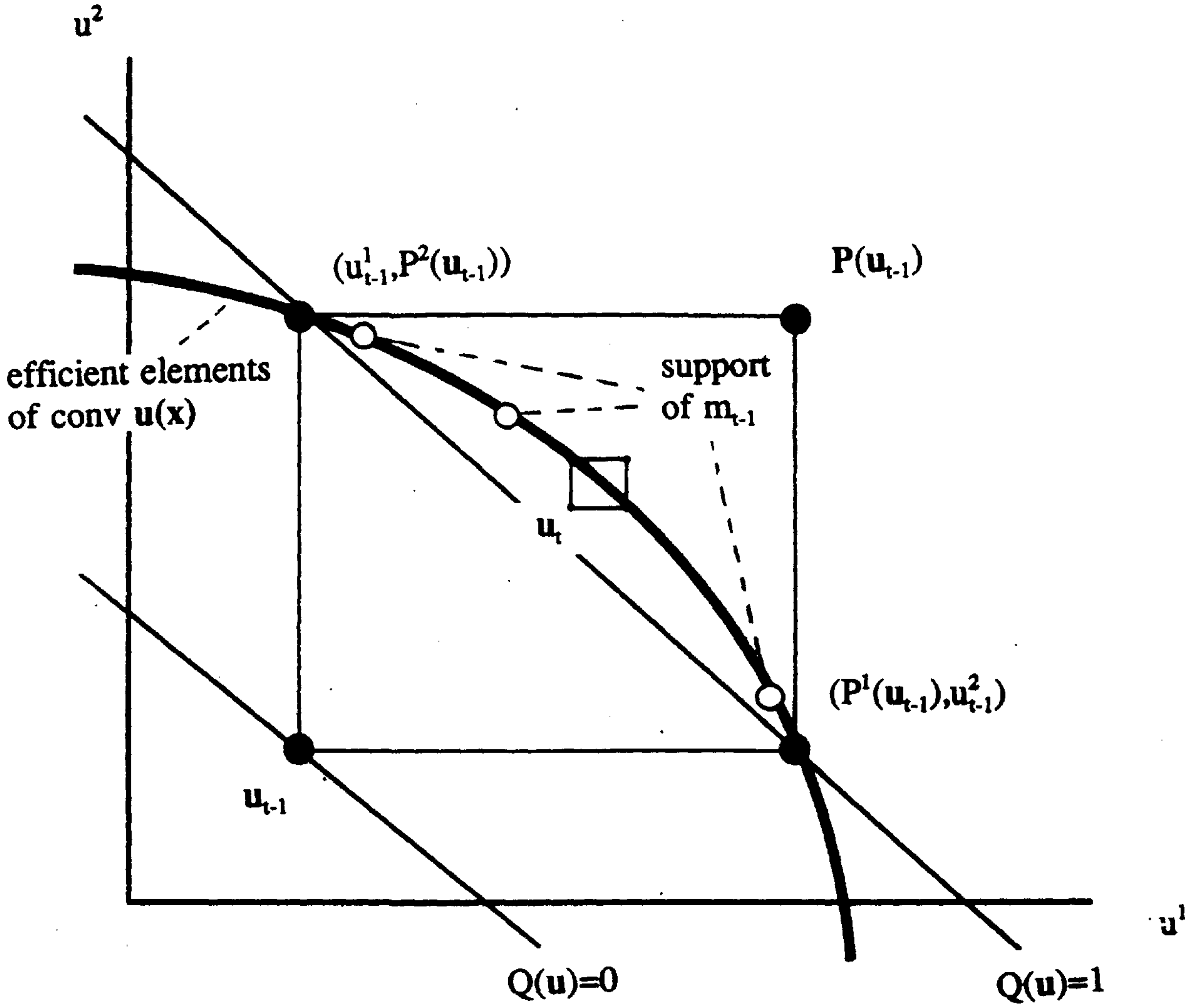

Figure 1: The "distance" between expected utility u_t and the efficient frontier declines exponentially with t. (See the intuition and proof following Theorem 1.)

Intuition: See Figure 1, ignoring the functions **P** and **Q** (these functions are used by the proof). For any t, consider the area (under the standard Lebesgue measure) of the set consisting of those utility vectors in conv **U**(**X**) which dominate $\mathbf{u}_t$. This area provides a measure of the "distance" between $\mathbf{u}_t$ and the efficient frontier. This distance is reduced when agents trade away m_t for m_{t+1}. Specifically, the individual rationality and efficiency of each allocation **x** in the support of m_{t+1} imply that **U**(**x**) lies in the half-moon-shaped region above the diagonal line, and consequently, the new expected utility $\mathbf{u}_{t+1}$ must also lie in this region. This observation implies that the area of the set of utility vectors which dominate $\mathbf{u}_{t+1}$ is no greater than one-half the area of the set of utility vectors which dominated $\mathbf{u}_t$. Thus the distance between expected utility and the efficient frontier declines exponentially with t. When Figure 1 is redrawn for n rather than 2 individuals, each iteration reduces the distance between expected utility and the efficient frontier by a factor of (n!-1)/n! rather than 1/2.

Proof: Note that each $\mathbf{u}_t$ is in conv **U**(**X**) (i.e., the convex hull of **U**(**X**)), which is compact by the compactness of **X** and the continuity of **U**. Also note that each $\mathbf{u}_{t+1} \geq \mathbf{u}_t$ by individual rationality. These two observations immediately imply that $\lim_{t\to\infty}\mathbf{u}_t$ exists. Efficiency remains to be proven.

Define **P**: conv **U**(**X**) $\to \mathbb{R}^n$ by

$$(\forall i)\ P^i(\mathbf{u}) = \max\ \{v^i \mid \mathbf{v} \in \text{conv } \mathbf{U}(\mathbf{X}) \text{ and } \mathbf{v} \geq \mathbf{u}\}.$$

Since $\mathbf{P}(\mathbf{u}) \geq \mathbf{u}$, we may usefully consider $\mu([\mathbf{u},\mathbf{P}(\mathbf{u})])$, which is the volume of the box $[\mathbf{u},\mathbf{P}(\mathbf{u})]$ whose opposing vertices are **u** and **P**(**u**). Since **P** is continuous (by the Maximum Theorem as stated in Debreu (1959), p. 19), and since $\mu([\mathbf{u}_t,\mathbf{P}(\mathbf{u}_t)])$ declines exponentially (by Lemma 4 (Appendix)),

$$\begin{aligned}
&\mu(\ [\ \lim_{t\to\infty}\mathbf{u}_t,\ \mathbf{P}(\lim_{t\to\infty}\mathbf{u}_t)\ ]\ )\\
&= \lim_{t\to\infty}\ \mu([\mathbf{u}_t,\mathbf{P}(\mathbf{u}_t)])\\
&\leq \lim_{t\to\infty}\ ((n!-1)/n!)^{t-1}\ \mu([\mathbf{u}_1,\mathbf{P}(\mathbf{u}_1)])\\
&= 0.
\end{aligned}$$

By Lemma 2 (Appendix), this is equivalent to the weak efficiency of $\lim_{t\to\infty}\mathbf{u}_t$. ∎

2.2. Strong Efficiency

U is strictly concave if

$$(\forall\lambda \in (0,1))(\forall \mathbf{x}^0,\mathbf{x}^1 \in \mathbf{X})\ \ \mathbf{x}^0 \neq \mathbf{x}^1 \Rightarrow \mathbf{U}((1-\lambda)\mathbf{x}^0+\lambda\mathbf{x}^1) >> (1-\lambda)\mathbf{U}(\mathbf{x}^0) + \lambda\mathbf{U}(\mathbf{x}^1).$$

Strict concavity is reasonable when there are two persons dividing a fixed allocation of resources. Yet it is difficult to defend when there are three or more individuals. For example, imagine that three (selfish) persons are dividing a pie and that **x** and **x'** are two distinct allocations that give person 1 the same share. There is no reason to expect that person 1 will strictly prefer a convex combination of **x** and **x'** over either **x** alone or **x'** alone. As a reasonable alternative to strict concavity when $n \geq 3$, we say that **U** is semi-strictly concave if

$$(\forall\lambda \in (0,1))(\forall \mathbf{x}^0,\mathbf{x}^1 \in \mathbf{X})\ \ \mathbf{x}^0 \neq \mathbf{x}^1 \Rightarrow \mathbf{U}((1-\lambda)\mathbf{x}^0+\lambda\mathbf{x}^1) > (1-\lambda)\mathbf{U}(\mathbf{x}^0) + \lambda\mathbf{U}(\mathbf{x}^1),$$

where the vector inequality > denotes a strict inequality in one coordinate and weak inequalities in all other coordinates. Because each U^i serves as a von Neumann-Morgenstern utility index, these two varieties of strict concavity imply that some or all of the persons are risk averse.

An allocation $\mathbf{x} \in \mathbf{X}$ is strongly efficient if there is no alternative allocation $\mathbf{x}' \in \mathbf{X}$ such that $\mathbf{U}(\mathbf{x}') > \mathbf{U}(\mathbf{x})$. A utility vector $\mathbf{u} \in$ conv $\mathbf{U}(\mathbf{X})$ is strongly efficient if there exists no alternative $\mathbf{u}' \in$ conv $\mathbf{U}(\mathbf{X})$ such that $\mathbf{u}' > \mathbf{u}$.

Theorem 2 assumes the equivalence of strong and weak efficiency. This equivalence follows from monotonicity by standard arguments made within any Walrasian model (Section 3.1). It also follows from strict concavity (Lemma 5 (Appendix)).

Theorem 2: As in Theorem 1, let $\{m_t\}_{t=1}^{\infty}$ be such that the support of each m_t consists of weakly efficient allocations which are individually rational given m_{t-1}. In addition, assume that **U** is semi-strictly concave and that any utility vector $\mathbf{u} \in$ conv $\mathbf{U}(\mathbf{X})$ is strongly efficient iff it is weakly efficient. Then $\lim_{t\to\infty}\mathbf{u}_t$ is strongly efficient and there exists a unique $\mathbf{x}^* \in \mathbf{X}$ such that $\mathbf{U}(\mathbf{x}^*) = \lim_{t\to\infty}\mathbf{u}_t$.

Proof: Strong Efficiency. By Theorem 1 and the assumed equivalence of strong and weak efficiency, $\lim_{t\to\infty}\mathbf{u}_t$ is strongly efficient.

Existence. Since $\lim_{t\to\infty}\mathbf{u}_t \in$ conv $\mathbf{U}(\mathbf{X})$ by Theorem 1, Carathéodory's Theorem (Rockafellar (1970), p. 155) implies the existence of n+1 allocations $\{\mathbf{x}_j\}_{j=0}^{n}$ and n+1 nonnegative scalars $\{\lambda_j\}_{j=0}^{n}$ such that $\Sigma_{j=0}^{n}\lambda_j = 1$ and $\Sigma_{j=0}^{n}\lambda_j\mathbf{U}(\mathbf{x}_j) = \lim_{t\to\infty}\mathbf{u}_t$. If any two of the allocations receiving positive weight are distinct, semi-strict concavity implies that $\mathbf{U}(\Sigma_{j=0}^{n}\lambda_j\mathbf{x}_j) > \Sigma_{j=0}^{n}\lambda_j\mathbf{U}(\mathbf{x}_j) = \lim_{t\to\infty}\mathbf{u}_t$, which contradicts the strong efficiency of $\lim_{t\to\infty}\mathbf{u}_t$. Thus the allocations receiving positive weight are identical and we may set $\mathbf{x}^*$ equal to any of them.

Uniqueness. If $\mathbf{x} \neq \mathbf{x}'$ and $\mathbf{U}(\mathbf{x}) = \mathbf{U}(\mathbf{x}') = \lim_{t\to\infty}\mathbf{u}_t$, semi-strict concavity implies that $\mathbf{U}((\mathbf{x}+\mathbf{x}')/2) > (\mathbf{U}(\mathbf{x})+\mathbf{U}(\mathbf{x}'))/2 = \lim_{t\to\infty}\mathbf{u}_t$, which contradicts the strong efficiency of $\lim_{t\to\infty}\mathbf{u}_t$. ∎

One might conjecture that the strong efficiency of $\lim_{t\to\infty} \mathbf{u}_t$ could also be obtained without the assumed equivalence of weak and strong efficiency if one assumed instead that every allocation in the support of every m_t is strongly (rather than weakly) efficient. This reasonable conjecture is proven false by the following example.

Example: Before constructing the pertinent three-person example, we develop a simpler two-person example. Consider Figure 2a, noting that U(X) is the heavy curve constituting the northeast edge of the figure [this could be constructed by defining $\mathbf{X} = \{ (x^1,x^2) \in \mathbb{R}^2_+ \mid x^1+x^2 = 2 \}$ and by defining $\mathbf{U}$ by $U^1(x) = (x^1)^{1/2}$ and $U^2(x) = \min \{ (x^2)^{1/2}, 1 \}$]. Define m_1 to be the probability measure whose support consists of two allocations yielding the utility vectors $\mathbf{v}_1$ and $\mathbf{w}_1$ and whose expected utility is $\mathbf{u}_1$. Given $\mathbf{u}_1$, define $\mathbf{v}_2$ and $\mathbf{w}_2$ as in Figure 2a. Then define m_2 to be the probability measure whose two-element support consists of the two utility vectors $\mathbf{v}_2$ and $\mathbf{w}_2$ and whose expected utility is $\mathbf{u}_2$. Repeat this process indefinitely to define $\{m_t\}_{t=3}^{\infty}$. The sequence $\{m_t\}_{t=1}^{\infty}$ satisfies the assumptions of Theorem 1, and as Theorem 1 requires, $\lim_{t\to\infty} \mathbf{u}_t$ exists and is weakly (but not strongly) efficient.

Now consider Figure 2b, which is a top view of the three-dimensional object constructed by pivoting Figure 2a on its vertical axis through a 90-degree angle. This provides a top view of U(X) in the pertinent three-person example [it can be constructed by defining $\mathbf{X} = \{ (x^{1a},x^{1b},x^2) \in \mathbb{R}^3_+ \mid x^1+x^2+x^3 = 2 \}$, and by defining $\mathbf{U}$ by $(\forall i=1a,1b)$ $U^i(x) = (x^i)^{1/2}$ and $U^2(x) = \min \{ (x^2)^{1/2}, 1 \}$.]

Imbed Figure 2a into Figure 2b by aligning the horizontal axis $\{ \mathbf{u} \mid u^2 = 0 \text{ and } u^1 \geq 0 \}$ in Figure 2a with the diagonal $\{\mathbf{u} \mid u^2 = 0 \text{ and } u^{1a} = u^{1b} \geq 0\}$ in Figure 2b. Note that $\mathbf{v}_1$ can be expressed as a mixture of the strongly efficient utility vectors $\mathbf{v}_{1a}$ and $\mathbf{v}_{1b}$. Define m_1 to be the probability measure whose support consists of three allocations yielding the utility vectors $\mathbf{v}_{1a}$, $\mathbf{v}_{1b}$, and $\mathbf{w}_1$, and whose expected utility is $\mathbf{u}_1$. Then for t = 2, express $\mathbf{v}_2$ as a mixture of two strongly efficient utility vectors $\mathbf{v}_{2a}$ and $\mathbf{v}_{2b}$, and define m_2 to be the probability measure whose three-element support yields the utility vectors $\mathbf{v}_{2a}$, $\mathbf{v}_{2b}$, and $\mathbf{w}_2$ and whose expected utility is $\mathbf{u}_2$. Repeat this process indefinitely to obtain $\{m_t\}_{t=3}^{\infty}$.

The sequence $\{m_t\}_{t=1}^{\infty}$ satisfies the assumptions of Theorem 1, and as Theorem 1 requires, $\lim_{t\to\infty} \mathbf{u}_t$ exists and is weakly efficient. However, it is not strongly efficient in spite of the fact that the support of each m_t consists only of strongly efficient allocations.

Figure 2a

Figure 2b

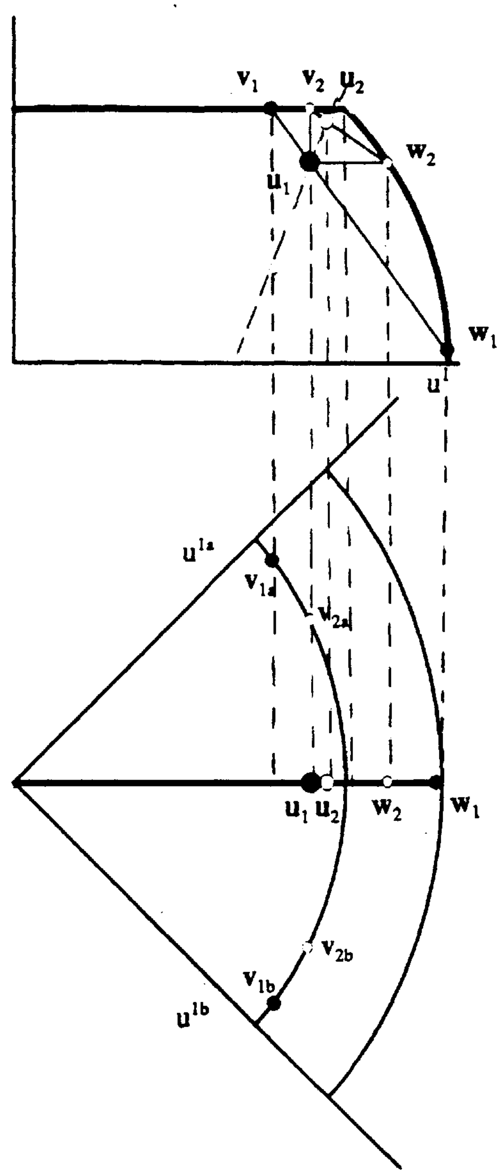

Figure 2: The heavy line in Figure 2a illustrates the utility frontier for a two person example. Figure 2b provides a top view of the utility frontier obtained from 2a by pivoting 2a on its vertical axis to obtain a three person example. Each u_t is obtained as a mixture of strongly efficient allocations. Nonetheless, each u_t lies below the utility frontier, and $\lim_{t\to\infty} u_t$ is only weakly efficient.

3. APPLICATIONS

3.1. Pure-Exchange Economies

Consider a pure-exchange economy in which the aggregate endowment of the k goods is $\omega \in \mathbf{R}_+^k$ and the utility function of each of the n individuals is U^i: $\mathbf{R}_+^k \to \mathbf{R}$. Assume that each U^i is strictly monotonic in each coordinate, strictly concave, and continuous. Let $\mathbf{X} = \{\mathbf{x} = [\mathbf{x}^1\ \mathbf{x}^2 \ldots \mathbf{x}^n] \in \mathbf{R}_+^{kn} \mid \Sigma_{i=1}^n \mathbf{x}^i = \omega\}$ be the set of all feasible consumption allocations (i.e., the Edgeworth box). By standard arguments (e.g., Varian (1984), p. 198), the assumed continuity and monotonicity are sufficient to imply the equivalence of strong and weak efficiency. Furthermore, the assumed strict concavity of each U^i implies that $\mathbf{U} = (U^1, U^2, \ldots U^n)$ is semi-strictly concave (also note that the strict concavity of U^i is equivalent to i's risk aversion).

Let W: $\mathbf{X} \to \mathbf{X}$ denote the Walrasian equilibrium correspondence (i.e., for any endowment allocation $\mathbf{x} \in \mathbf{X}$, let W(x) denote the collection of all equilibrium allocations). Further suppose that for any endowment x in X, all n persons agree on a probability measure over the equilibrium allocations W(x). Call this probability measure M(x). Let x_0 be an exogenously given endowment, and let $m_1 = M(x_0)$ be the probability measure which the persons assign to equilibrium allocations in $W(x_0)$.

We now define a straightforward trading process $\{m_t\}_{t=2}^{\infty}$ such that $\{m_t\}_{t=1}^{\infty}$ satisfies our assumptions of efficiency and individual rationality. We construct this example recursively: for all t ≥ 2 we define $m_t = M(E[m_{t-1}])$. In other words, we let each m_t be the commonly held probability measure assigned to the equilibrium allocations that result from an endowment equal to the expectation of m_{t-1}. At each t, every allocation in the support of m_t is efficient (since it is an equilibrium allocation) and individually rational (since its utility vector is bounded from below by $\mathbf{U}(E[m_{t-1}])$, which is in turn bounded from below by $\mathbf{u}_1$ because of the concavity of U). Therefore, because of the semi-strict concavity of U and the equivalence of strong and weak efficiency (see the first paragraph of this section), Theorem 2 implies that $\lim_{t\to\infty}\mathbf{u}_t$ is strongly efficient and that there exists a unique allocation $\mathbf{x}^*$ such that $\lim_{t\to\infty}\mathbf{u}_t = \mathbf{U}(\mathbf{x}^*)$. Thus the trading process $\{m_t\}_{t=2}^{\infty}$ eliminates all endogenous uncertainty.

The trading process $\{m_t\}_{t=2}^{\infty}$ defined in the previous paragraph is a rather arbitrary example of a trading process that utilizes efficient allocations and is individually rational at each stage. Many other trading processes, such as those considered by Chichilnisky, Dutta, and Heal (1991) and Hahn (1991), also satisfy these basic assumptions. These other trading processes can be quite realistic in that the gains from trade against endogenous uncertainty can be realized through contingent securities which have counterparts in actual financial markets. They can also be quite complicated in that the number of markets (and hence the "dimension" of endogenous uncertainty) can explode geometrically as the trading process evolves. It remains to be debated which of these many trading processes is the best descriptive tool. Our contribution is to note that endogenous uncertainty is eliminated by any trading process that satisfies the assumptions of efficiency and individual rationality.

It must be recognized that our assumption of individual rationality is strong. For example, Chichilnisky, Dutta, and Heal (1991) construct a trading process in which the securities market at any stage assumes that the individuals are endowed with the equilibrium consumption allocation obtained at the previous stage. This endowment structure implies individual rationality. More generally, however, it seems that markets might develop in ways that need not satisfy individual rationality. For example, consider a k-good market whose endogenous uncertainty gives rise to a price-contingent security. It is not immediately clear that everyone must be better off in the new market where k+1 prices are determined *simultaneously*. Modelling and understanding such markets strikes us as a fundamental direction for future research.

3.2. Bargaining Problems

Consider a bargaining problem in which the set of feasible alternatives is $\mathbf{X}$ and the utility function of each of the n individuals is U^i: $\mathbf{X} \to \mathbf{R}$. Assume that $\mathbf{X}$ is a compact and convex subset of some topological vector space, that $\mathbf{U} = (U^1,U^2,...U^n)$ is continuous and semi-strictly concave (Section 2.2), and that weak and strong efficiency are equivalent (Section 2.2). By allowing lotteries, the set of feasible utility vectors is conv U(X).

Suppose that there is a random arbitrator who, given any threat point $\mathbf{u} \in$ conv U(X), selects an efficient allocation which is individually rational given $\mathbf{u}$. Further suppose that the n persons agree upon the probability measure M(u) which governs the random arbitrator's selection. Let $\mathbf{u}_0$ be an exogenously given threat point, and define $m_1 = M(\mathbf{u}_0)$.

We now define a simple negotiation process $\{m_t\}_{t=2}^{\infty}$ such that the support of each m_t consists of efficient allocations that are individually rational given m_{t-1}. We construct this example recursively: at each $t \geq 2$, we define $m_t = M(\mathbf{u}_{t-1})$. In other words, we let each m_t be the probability measure that the arbitrator would use to select an efficient allocation which is individually rational given the threat point $\mathbf{u}_{t-1}$. Thus the agents' negotiation process is governed at each stage by imagining what the arbitrator would do. By Theorem 2, $\lim_{t\to\infty} u_t$ is efficient and there exists a unique x^* such that $\lim_{t\to\infty} u_t = U(x^*)$.

Each stage of this negotiation process is easily interpreted. If the support of m_{t-1} consists of more than one allocation, semi-strict concavity (i.e., risk aversion) implies that u_{t-1} is not efficient. In other words, the endogenous uncertainty introduced by following the imaginary random arbitrator at the previous stage leads to potential gains from risk sharing among the agents. Since m_{t-1} defines a new bargaining problem whose threat point is u_{t-1}, the agents can once again govern their negotiation process by what the arbitrator would do: they agree (for the moment) to select an allocation according to $m_t = M(u_{t-1})$. Theorem 2 implies that this negotiation process of repeated imaginary arbitration ultimately eliminates the endogenous uncertainty regardless of the endogenous uncertainty which might be introduced by following the imaginary random arbitrator at each stage. The result is to settle upon the strongly efficient allocation x^*.

This example suggests that while there is nothing inherently illogical about a random arbitrator, common knowledge about the arbitrator's probability measure generates incentives for risk sharing that ultimately remove the endogenous uncertainty introduced by the arbitrator. The particular negotiation process $\{m_t\}_{t=2}^{\infty}$ defined above is one example of how the agents might realize these gains. Our results show that every such process eliminates endogenous uncertainty provided that it is efficient and individually rational at each step.

Finally, our results show that actual arbitration rather than pre-arbitration settlement occurs only when the agents disagree about the behavior of the arbitrator (given the model's other assumptions). In particular, each agent must expect, relative to the expectations of the others, that the arbitrator will tend to favor their cause. Although such inconsistent expectations are anathema to theoretical economics (Aumann (1976)), they can be easily fostered by opposing lawyers in an actual dispute. These ideas are developed further by Chichilnisky, Dalvi, and Heal (1992).

APPENDIX

Lemma 1: $(\forall i)\ P^i(u) = \max\{w^i \mid w \in U(X) \text{ and } w \geq u\}$.

Proof: Take any i. Since $U(X) \subseteq \text{conv } U(X)$, it is obvious that $P^i(u) \geq \max\{w^i \mid w \in U(X)$ and $w \geq u\}$. To show the reverse inequality, let $v \in \text{conv } U(X)$ be such that $v^i = P^i(u)$ and $v \geq u$. By Carathéodory's Theorem (Rockafellar (1970), p. 155), there exist n+1 allocations $\{x_j\}_{j=0}^n$ and n+1 nonnegative scalars $\{\lambda_j\}_{j=0}^n$ such that $\Sigma_{j=0}^n\lambda_j = 1$ and $\Sigma_{j=0}^n\lambda_j U(x_j) = v$. Set $w = U(\Sigma_{j=0}^n\lambda_j x_j)$. The convexity of X implies that $w \in U(X)$, and the concavity of U implies that $w = U(\Sigma_{j=0}^n\lambda_j x_j) \geq \Sigma_{j=0}^n\lambda_j U(x_j) = v$. This and $v \geq u$ imply that $\max\{w^i \mid w \in U(X) \text{ and } w \geq u\} \geq v^i = P^i(u)$. ∎

Remark: We conjecture but do not prove that the weak efficiency of $u \in \text{conv } U(X)$ implies $u \in U(X)$. We only know that $u \in \text{conv } U(X)$ implies the existence of n+1 allocations $\{x_i\}_{i=0}^n$ such that u is a convex combination of $\{U(x_i)\}_{i=0}^n$ (see Carathéodory's Theorem in Rockafellar (1970), p. 155). The number of allocations in such a convex combination is important only to the extent that it is finite.

Lemma 2: $\mu([u,P(u)]) = 0$ iff u is weakly efficient.

Proof: $\Leftarrow$. $\mu([u,P(u)]) > 0$ implies $P(u) \gg u$. By Lemma 1, this implies that for each i, there exists an $x_i \in X$ such that $U^i(x_i) > u^i$ and $U(x_i) \geq u$. The convexity of X implies that $\Sigma_{i=1}^n(1/n)x_i \in X$, and the concavity of each U^h implies that $U^h(\Sigma_{i=1}^n(1/n)x_i) \geq \Sigma_{i=1}^n(1/n)U^h(x_i) = (1/n)\{\ U^h(x_h) + \Sigma_{i\neq h}U^h(x_i)\ \} > u^h$. Hence, u is not weakly efficient.

$\Rightarrow$. If u is not weakly efficient, there exists $v \in U(X)$ such that $v \gg u$. Hence $P(u) \geq v \gg u$, which implies $\mu([u,P(u)]) > 0$. ∎

Lemma 3: If n is a positive integer, and if $a_1, a_2, \ldots, a_n$ and r are positive reals, then

$$\mu\left(\left\{x \geq 0 \,\middle|\, \sum_{i=1}^n \frac{x_i}{a_i} \leq r\right\}\right) = \frac{r^n}{n!}\prod_{i=1}^n a_i.$$

Proof: The result holds at n = 1 since

$$\mu\left(\left\{x \geq 0 \,\middle|\, \frac{x_1}{a_1} \leq r\right\}\right) = \int_0^{r a_1} dx_1 = ra_1.$$

The remaining are proven by induction:

$$\mu\left(\left\{x\geq 0 \,\middle|\, \sum_{i=1}^{n} \frac{x_i}{a_i} \leq r\right\}\right)$$

$$= \int_0^{ra_n} \mu\left(\left\{x\geq 0 \,\middle|\, \sum_{i=1}^{n-1} \frac{x_i}{a_i} \leq r - \frac{x_n}{a_n}\right\}\right) dx_n$$

$$= \int_0^{ra_n} \left(\frac{\left(r - \frac{x_n}{a_n}\right)^{n-1}}{(n-1)!}\right) \prod_{i=1}^{n-1} a_i \, dx_n$$

$$= \frac{\prod_{i=1}^{n-1} a_i}{(n-1)!} \left[\left(\frac{-a_n}{n}\right)\left(r - \frac{x_n}{a_n}\right)^n\right]\Bigg|_0^{ra_n}$$

$$= \frac{r^n}{n!} \prod_{i=1}^{n} a_i .$$

■

Lemma 4: $(\forall t \geq 1)$ $\mu([u_{t+1}, P(u_{t+1})]) \leq (n!-1)/n!\ \mu([u_t, P(u_t)])$.

Proof: Take any t. Since $u_{t+1} \geq u_t$ by individual rationality, $P(u_{t+1}) \leq P(u_t)$ and thus

$$[u_{t+1}, P(u_{t+1})] \subseteq [u_t, P(u_t)]. \tag{1}$$

Hence if $\mu([u_t, P(u_t)]) = 0$, the result holds. Otherwise, $P(u_t) >> u_t$ and we may define Q: $\mathbb{R}^n \to \mathbb{R}$ by

$$Q(u) = q(u-u_t),$$

where $(\forall i)$ $q^i = (P^i(u_t) - u_t^i)^{-1}$. In terms of the box $[u_t, P(u_t)]$, Q is the linear functional which assigns a value of 0 at the vertex u_t and assigns a value of 1 at each vertex of the form $(u_t^{-i}, P^i(u_t))$.

In this paragraph, we show that

$$(\forall u \geq u_t)\ \ Q(u) < 1 \Rightarrow u \text{ is not weakly efficient.} \tag{2}$$

Assume $Q(u) < 1$. Since $q \neq 0$ is the gradient of the linear functional Q, there must be an $\alpha > 0$ such that $Q(u+\alpha q) = 1$. Since $u \geq u_t$ and $q >> 0$, this implies that $u+\alpha q$ lies in the simplex defined by the n vertices $\{(u^{-i}, P^i(u)\}_{i=1}^n$. Thus there exist nonnegative scalars (i.e., barycentric coordinates) $\{\lambda_i\}_{i=1}^n$ such that $\Sigma_{i=1}^n \lambda_i = 1$ and

$$u+\alpha q. = \Sigma_{i=1}^n \lambda_i(u^{-i}, P^i(u)). \tag{3}$$

By Lemma 1, we have for each i the existence of an $x_i \in \mathbf{X}$ such that

$$\mathbf{u}(x_i) \geq (\mathbf{u}^{-i}, P^i(\mathbf{u})) \tag{4}$$

Therefore, $\mathbf{u}$ is not weakly efficient since it is dominated by $\Sigma_{i=1}^n \lambda_i x_i$:

$$\begin{aligned} &\mathbf{U}(\Sigma_{i=1}^n \lambda_i \mathbf{x}_i) \\ &\geq \Sigma_{i=1}^n \lambda_i \mathbf{U}(\mathbf{x}_i) \\ &\geq \Sigma_{i=1}^n \lambda_i (\mathbf{u}^{-i}, P^i(\mathbf{u})) \\ &= \mathbf{u} + \alpha \mathbf{q} \\ &\gg \mathbf{u}. \end{aligned}$$

The first inequality follows from the concavity of $\mathbf{U}$, the second inequality follows from (4), the equality follows from (3), and the final inequality follows from $\mathbf{q} \gg \mathbf{0}$.

Equation (2) implies that every allocation $\mathbf{x}$ in the support of m_{t+1} satisfies $Q(\mathbf{U}(\mathbf{x})) \geq 1$ by virtue of the assumption that every such $\mathbf{x}$ is weakly efficient and individually rational. Hence, the linearity of Q implies

$$Q(\mathbf{u}_{t+1}) = Q(\textstyle\int \mathbf{U}\, dm_{t+1}) = \int Q \circ \mathbf{U}\, dm_{t+1} \geq 1. \tag{5}$$

Finally, we obtain

$$\begin{aligned} &\mu([\mathbf{u}_{t+1}, \mathbf{P}(\mathbf{u}_{t+1})]) \\ &\leq \mu(\{\mathbf{u} \in [\mathbf{u}_t, \mathbf{P}(\mathbf{u}_t)] \mid Q(\mathbf{u}) \geq 1\}) \\ &= \mu([\mathbf{u}_t, \mathbf{P}(\mathbf{u}_t)]) - \mu(\{ \mathbf{u} \in [\mathbf{u}_t, \mathbf{P}(\mathbf{u}_t)] \mid Q(\mathbf{u}) < 1\}) \\ &= \mu([\mathbf{u}_t, \mathbf{P}(\mathbf{u}_t)]) - 1/n!\ \mu([\mathbf{u}_t, \mathbf{P}(\mathbf{u}_t)]) \\ &= (n!-1)/n!\ \mu([\mathbf{u}_t, \mathbf{P}(\mathbf{u}_t)]). \end{aligned}$$

The inequality follows from (1) and (5), and the second equality follows from Lemma 3 (applied at $\mathbf{x} = \mathbf{u} - \mathbf{u}_t$, $\mathbf{a} = \mathbf{q}$, and $r = 1$). ∎

<u>Lemma 5</u>: Given strict concavity, any vector $\mathbf{u} \in \operatorname{conv} \mathbf{U}(\mathbf{X})$ is strongly efficient iff it is weakly efficient.

<u>Proof</u>: Suppose $\mathbf{u} \in \operatorname{conv} \mathbf{U}(\mathbf{X})$ is not strongly efficient. Then there exists $\mathbf{u}' \in \operatorname{conv} \mathbf{U}(\mathbf{X})$ such that $\mathbf{u}' > \mathbf{u}$. By Carathéodory's Theorem, there exists n+1 allocations $\{\mathbf{x}_j\}_{j=0}^n$ and n+1 nonnegative scalars $\{\lambda_j\}_{j=0}^n$ such that $\Sigma_{j=0}^n \lambda_j = 1$ and $\Sigma_{j=0}^n \lambda_j \mathbf{U}(\mathbf{x}_j) = \mathbf{u}$. Similarly, there are $\{\mathbf{x}_j'\}_{j=0}^n$ and $\{\lambda_j'\}_{j=0}^n$ such that $\Sigma_{j=0}^n \lambda_j' = 1$ and $\Sigma_{j=0}^n \lambda_j' \mathbf{U}(\mathbf{x}_j') = \mathbf{u}'$. By the distinctness of $\mathbf{u}'$ and $\mathbf{u}$, the set $\{\{\mathbf{x}_j\}_{j=0}^n, \{\mathbf{x}_j'\}_{j=0}^n\}$ contains at least two (distinct) elements. Thus strict concavity implies

$$\mathbf{U}((\Sigma_{j=0}^n \lambda_j \mathbf{x}_j + \Sigma_{j=0}^n \lambda_j' \mathbf{x}_j')/2)$$

$$\gg (\Sigma_{j=0}^{n}\lambda_j U(\mathbf{x}_j) + \Sigma_{j=0}^{n}\lambda_j' U(\mathbf{x}_j'))/2$$

$$= (\mathbf{u}'+\mathbf{u})/2$$

$$> \mathbf{u}.$$

Thus **u** is not weakly efficient. The converse is obvious. ∎

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