

**Arbitrage and Equilibrium in Economies
with Infinitely Many Securities and Commodities**

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1. Introduction

Welfare economics and finance have each evolved their own equilibrium concepts. In welfare economics this is the concept of a competitive general equilibrium: in finance, it is the absence of arbitrage opportunities. These concepts emerged independently and were initially seen as quite distinct. Each plays an absolutely central role in its field, both in the theory and in practical applications. In the 1980s researchers in both fields began to investigate the connections between the two concepts. The first paper to address explicitly the arbitrage-equilibrium relationship was Kreps (1981): subsequently this issue was addressed by Werner (1987), Nielsen (1989) and Brown and Werner (1992). However, Hart's earlier paper on equilibrium in securities markets (1974) in fact contained many of the elements needed to understand the arbitrage-competitive equilibrium relationship. This was developed by Hammond (1983) and Page (1987): Green (1973) covered related topics in a temporary equilibrium framework. This work has established that for finite-dimensional choice spaces the absence of arbitrage opportunities (a no-arbitrage condition) is sufficient for the existence of a competitive equilibrium (Werner (1987)). Recently, Brown and Werner (1992) have extended this to infinite-dimensional choice spaces, though at the cost of a very strong assumption about the structure of the Pareto frontier.

Our paper extends this literature. We work with a weaker condition than no arbitrage,

the concept of limited arbitrage. This concept was introduced in the context of social choice theory by Chichilnisky (1991a). Limited arbitrage, as its name implies, admits arbitrage opportunities that generate bounded increases in utility: it does not rule out all arbitrage opportunities. We show that this concept, limited arbitrage, is fully equivalent to the concept of a competitive general equilibrium. Formally, we show that a competitive general equilibrium exists if and only if arbitrage opportunities are limited. The two equilibrium concepts as used in finance and in welfare economics are therefore logically equivalent. This equivalence disproves the conjecture of Dybvig and Ross (1987) that "absence of arbitrage is more primitive than equilibrium, since only relatively few rational agents are needed to bid away arbitrage opportunities."

In establishing the equivalence of the two equilibrium concepts, competitive equilibrium and limited arbitrage, it is crucial that we relax the condition on arbitrage from "no arbitrage" (as used in Werner (1987), Nielsen (1989)) to "limited arbitrage". "No arbitrage" is sufficient for the existence of competitive equilibrium: "limited arbitrage" is necessary and sufficient.

As a by-product of showing that the equilibrium concepts at the foundations of general equilibrium theory and finance are equivalent, we are able to give necessary and sufficient conditions for the existence of a competitive equilibrium in economies with finitely or infinitely many commodities. Previously we had only sufficient conditions for existence: on the basis of the present results we can say for the first time precisely when a competitive equilibrium will not exist, as well as saying when it will exist.

In order to integrate fully real and financial markets we use a framework similar to that adopted in finance. Therefore we study the equivalence of limited arbitrage and competitive equilibrium in economies with finitely or infinitely many assets and commodities using Hilbert spaces as the commodity space. Financial models often involve continuous trading and use Hilbert spaces as commodity spaces. Hilbert spaces are the closest analog of euclidean spaces in infinite dimensions, and have been used widely in economics and in finance since their

introduction in general equilibrium theory (Chichilnisky (1976)) and in infinite horizon models of economic growth (Chichilnisky (1977)), see e.g. Chamberlin and Rothschild (1983), Connor (1984), Bergstrom (1985). An important example of a Hilbert space is a weighted L_2 space of square integrable functions with a finite measure, which contains another space frequently used in economics, L_∞ , the space of bounded functions (Debreu (1954), Bewley (1972)), as a dense sub space. Our results apply also to other L_p spaces ($1 \leq p < \infty$) and to Sobolev spaces (Appendix).

The results apply to models having as consumption sets:

- (i) general closed convex sets including the positive orthant as in the standard Arrow-Debreu models (Debreu (1954), Bewley (1972)),
- (ii) the whole commodity space as in financial models without bounds on short sales (Hart (1974), Kreps (1981)) trading finitely or infinitely many assets, and
- (iii) more general cases of consumption sets which may be bounded below in some commodities and not in others and which include all the above (Chichilnisky and Heal (1984, 1991)).

Our analysis therefore covers a wide spectrum of market economies which have been used in the literatures of economics and of finance. The most natural interpretation of our model is that some dimensions of the commodity space refer to assets, and others to consumption goods. The former are unbounded below: the latter may or may not be. One can view utility as a function of during - period consumption, and of end - of - period wealth (asset) holdings.

A typical problem which arises in proving the existence of a competitive equilibrium in infinite dimensional spaces is that feasible efficient allocations do not generally exist. While some feasible efficient allocations exist trivially (e.g., give the whole endowment to any one consumer) interesting ones, e.g. with a prescribed ratio of utilities, are not guaranteed to exist. The problem is that the Pareto frontier in utility space may not be closed. Now, the Pareto frontier is always closed in finite dimensions when the consumption sets are the positive

orthants. However, this is no longer true when the consumption set is the whole space even in finite dimensions: Figure 1 below exhibits such a finite dimensional economy where the Pareto frontier is not closed. In infinite dimensions the problem is more acute; even when the consumption set is the positive orthant the Pareto frontier may not be closed, as shown in the example provided by Mas - Colell (1986). The problem is worse when the consumption set is not bounded below, and is infinite dimensional. In this case there are no results in the literature to guarantee that the Pareto frontier is closed. Here is where the Hilbert spaces exhibit their strength: we are able to prove from the properties of preferences, that the Pareto frontier of the economy is closed, Lemmas 2 and 5 in the Appendix. These results hold for all our consumption sets X , finite or infinite dimensional, bounded below or not. In comparison, Brown and Werner (1992) assume that the Pareto frontier is closed, without offering conditions for this to be true. Our arguments are not available in their framework

2. Definitions

There are k *agents*, indexed by i . *Commodities* are indexed by the real numbers. *Consumption bundles* are therefore real valued functions¹ on \mathbb{R} . The space H of commodity bundles is a weighted L_2 space of measurable functions $x(t)$ with the inner product $\langle x, y \rangle = \int_{\mathbb{R}} x(t) \cdot y(t) d\mu(t)$, where $\mu(t)$ is any *finite* and *positive* measure μ on \mathbb{R} (i.e. $\int_{\mathbb{R}} d\mu(t) < \infty$, $\mu(A) > 0$ for all measurable sets A) which is absolutely continuous with respect to the Lebesgue measure. The L_2 norm of a function² x is $\|x\| = \langle x, x \rangle^{1/2}$.

The *consumption set* for the i th agent is $X \subset H$; X is closed and convex, it contains the vector $\{0\}$, and satisfies the following condition: (*) if $x \in X$, and $y > x$, then $y \in X$. Both H

¹ The analysis can easily be extended to real valued functions of \mathbb{R}^n , $n > 1$.

² Chichilnisky (1977) and Chichilnisky (1981 a,b,) gave a straightforward economic interpretation of (weighted) Hilbert space inner products as the value of consumption streams in infinite horizon models of optimal growth; following this, Chamberlin and Rothschild (1984) Connor (1984) and Chichilnisky and Heal (1984, 1991) used Hilbert spaces in financial models and gave a straightforward economic interpretation of L_2 norms in models with infinite dimensional commodity spaces.

and the positive orthant H^+ of H , are included in this definition of consumption sets, as are other subsets of H which are bounded below in some coordinates and not in others.

A *price* p is a real valued function on H giving positive value to positive consumption bundles: this implies that p is continuous on H , that p is itself a function in H , and that the value of the bundle x at price p is given by the inner product $\langle p, x \rangle = \int p(t) \cdot x(t) d\mu(t)$. The *price space* is therefore H^+ , the positive cone of H .

The results given here apply to weighted L_2 and also to the space of real sequences l_2 with a finite measure; the appendix extends the definitions and the results to other L_p ($1 \leq p < \infty$) spaces and to Sobolev spaces H^s , for an integer $s \geq 1$. The order \geq in H is given by $x \geq y$ iff $x(t) \geq y(t)$ a.e., and $x > y$ iff $x(t) > y(t)$ a.e. A sequence (x^n) is said to converge to x in the *weak topology* iff $\langle x^n, h \rangle \rightarrow \langle x, h \rangle$ for all h in H . L_∞ , the space of real valued functions on R which are bounded a.e., is contained in H and is a *dense subset* of H , see Chichilnisky (1977) and Chichilnisky and Heal (1984, 1991). For example, L_∞ is contained in weighted L_2 because if $f \in L_\infty$, then $\int_R f(x)^2 d\mu(x) \leq \sup_{x \in R} |f(x)|^2 \int_R d\mu(x) < \infty$. L_∞ is dense in weighted L_2 because the sequence of "cutoff" functions (f_n) , where $f_n(x) = 0$ for $x > n$, and $f_n(x) = f(x)$ otherwise, converges to f in L_2 .

Society's endowment Ω is the sum $\Omega_1 + \dots + \Omega_k$, where Ω_i is the initial non - negative endowment of individual i . A *feasible consumption bundle* is a vector $v \in H$ which satisfies $v \leq \Omega$. A function $u: R^2 \rightarrow R$ is said to satisfy the *Caratheodory condition* if $u(c, t)$ is continuous with respect to $c \in R$ for almost all $t \in R$, and measurable with respect to t for all values of c . An *allocation* x is a vector $(x_1, \dots, x_k) \in H^k$. A *feasible allocation* x is an allocation such that $\sum_i x_i \leq \Omega$. The *set of feasible allocations* is denoted F .

Each individual i has a utility function W_i defined on a neighborhood of the consumption set X , denoted NX , $W_i: NX \rightarrow R$ which is C^2 (twice continuously differentiable in the norm of H), strictly quasi concave and increasing, i.e. if $u > v$, then $W_i(u) > W_i(v)$, and such that $W_i(0) = 0$. Without loss of generality we assume that $\sup_{x \in X} (W_i(x)) = \infty$; this is simply a normalization of utilities; all the results follow by assuming instead that for each i

there exist a number $W_i^S = \sup_{x \in X} (W_i(x)) < \infty$, and replacing " ∞ " by W_i^S in the notation. The utility level of an allocation x denoted $W(x)$ is the k - dimensional vector $(W_1(x_1), \dots, W_k(x_k))$, also called the *utility vector*. We assume that the ratio of the first and second derivatives of W_i is bounded; this means bounding absolute risk aversion of the agents if the coordinates of the space are states of nature:

(A*) For any $x \in H$, and $i = 1, \dots, k$, $\|D^2W_i(x)\|/\|DW_i(x)\| < \lambda < 1$.³

A utility vector $v = (v_1, \dots, v_k)$ - which may not be the utility vector of a feasible allocation - is called *efficient* if there is no feasible allocation (z_1, \dots, z_k) such that $W_i(z_i) \geq W_i(x_i)$ for all i , $W_i(z_i) > W_i(x_i)$ for some i , and there exists a sequence of feasible allocations $\{z^n\}$ in F such that $v = \lim_{z^n} (W_1(z_1), \dots, W_k(z_k))$. i.e., v is in the closure of the utility vectors corresponding to the set of feasible allocations. The *Pareto frontier* is the set of efficient utility vectors in the positive cone R^{k+} which correspond to feasible allocations. A real number v_i is called an *efficient utility value*, if it is the i th component of an efficient utility vector v . A *feasible efficient allocation* is an efficient allocation which is also feasible. The following condition on preferences will be assumed throughout:

(CL) The set of directions of gradients of an indifference surface is closed, i.e.

$G_{v_i} = \{a = DW_i(x)/\|DW_i(x)\| \text{ s.t. } W_i(x) = v_i\}$ is closed for all utility values v_i .

This condition (CL) rules out the existence of "asymptotic supports" which do not support any

³ $DW_i(x)$ denotes the derivative of W_i , i.e. the gradient of W_i , computed at the point x .

Similarly $D^2W_i(x)$ denotes the second derivative at the point x . Recall that D^2W_i is a bilinear form, so that (A*) means that for all $x, u \in H, u' \in (x, x+u) \exists \lambda > 0$ s.t. $|(u, D^2(W_i(x+u')) \cdot u^T) / (DW_i(x) \cdot u^T)| < \lambda, u^T$ the transpose of u .

allocation but do so asymptotically, see figure 1. Note that this condition does not rule out linear or piecewise linear functions, nor most strictly concave functions.

Figure 1

Society's endowment Ω is said to be *desirable* if $W_i(\alpha\Omega) > W_i(0)$ for all $\alpha > 0$, and all i . This is always satisfied if W_i is strictly increasing, $W_i(0) = 0$, and the initial endowment Ω is positive, which we assume.

Let Δ denote the *unit simplex* in \mathbb{R}^k , $\Delta = \{ y \in \mathbb{R}^{k+} : \sum_i y_i = 1 \}$. A feasible allocation (x_1, \dots, x_k) is a *quasi equilibrium* when there is a price $p \neq 0$ with $\langle p, \Omega_i \rangle = \langle p, x_i \rangle$ and $\langle p, z \rangle \geq \langle p, x_i \rangle$ for any z with $W_i(z) \geq W_i(x_i)$, $i = 1, \dots, k$. A feasible allocation (x_1, \dots, x_k) is an *equilibrium* when it is a quasi equilibrium and $W_i(z) > W_i(x_i) \rightarrow \langle p, z \rangle > \langle p, x_i \rangle$. The latter holds at a quasi equilibrium such that $\langle p, \Omega_i \rangle > 0$ for any i .

We define *the set of undominated sequences for the i -th consumer* as $A_i = \{ \chi = \{x^n\}_{n=1, \dots}, \chi: x^n \in X, \text{ and } \lim_n W_i(x^n) = \infty \}$. The set A_i consists of all sequences in the consumption set X along which individual i 's utility increases without bound⁴. Define now the *asymptotic dual* of A_i , $D_i = \{ h \in H^+ \text{ s.t. if } (x^m) \in A_i \exists n(h) \text{ with } \langle h, x^m \rangle > 0 \text{ for all } m \geq n(h) \}$. D_i is the set of prices giving positive value to all but finitely many terms of any sequence of consumption vectors leading to unbounded utility values.

We now define the condition of *limited arbitrage*: that there exists a vector $h \in H^+$ giving a strictly positive value to all elements of sequences in A_i , $i = 1, \dots, k$, from some allocation onwards,

$$(LA) \quad D = \bigcap_{1 \leq i \leq k} D_i \neq \emptyset.$$

In words: there is a price at which nobody can derive unbounded utility from zero - cost

⁴Debreu (1962) page 258 introduces a cone based on all points preferred to a given vector. Hart (1974) and Brown & Werner (1992) also use cones of directions in which a preference increases. Note that our concept is different because we are looking at sets of sequences along which utility levels go to infinity (or the sup of utility values over the consumption set).

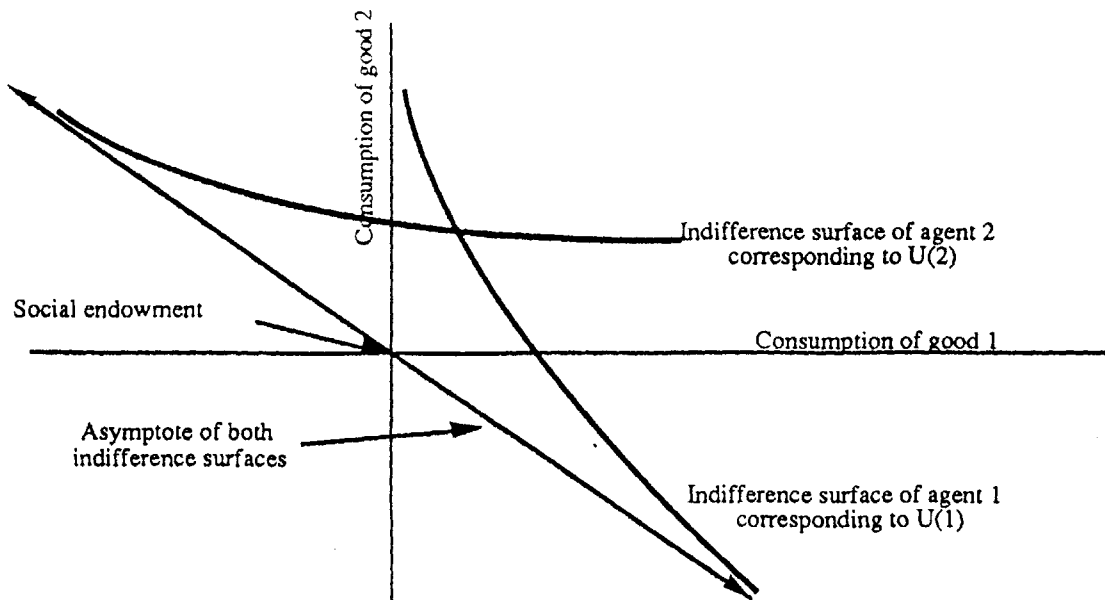


Figure 1: Two agents who have indifference surfaces corresponding to utility values $U(1)$ and $U(2)$ with the line $y = -x$ as asymptote. Consumption sets are the whole space and feasible allocations are those which sum to zero. The utility functions of the agents are

$$U_1 = x_1 + y_1 - e^{-(x_1 - y_1)}, U_2 = x_2 + y_2 - e^{(x_2 - y_2)}$$

The problem:

$$\text{maximize } U_1 \text{ subject to } U_1 = U_2 \text{ and } x_1 + x_2 = 0, y_1 + y_2 = 0,$$

has no solution, but the values of U_1 and U_2 approach 0 from below as $x_1 \rightarrow \infty, y_1 \rightarrow -\infty$.

allocations, and for this reason we call this condition *limited arbitrage*.⁵

Define the set of feasible allocations which give utility values exceeding those of the initial endowments to all agents, $F_{\Omega} = \{ x \in X^k, x = (x_i + \Omega_i)_{i \in [1,k]} \text{ s.t. } \forall i, \sum_i x_i = 0 \text{ and } U^i(x_i + \Omega_i) \geq U^i(\Omega_i) \}$. The set S_r of utility vectors which are co linear with a given element of the unit simplex Δ , $r = (r_1, \dots, r_k) \in \Delta$, and arising from feasible allocations, is defined by $S_r = \{ (u_1, \dots, u_k) \in R^k \text{ s.t. } u_i = W_i(\Omega_i + x_i), \sum_i x_i = 0, \text{ and } \mu(W_i(\Omega_i + x_i)) = r_i \text{ for } i \in [1,k], \text{ for some } \mu > 0 \}$. Define the cone $C(W_i^X \cap X) - x = \{ \lambda v \in X : \lambda > 0, \text{ and } v \text{ satisfies } W^i(v+x) \geq W^i(x) \}$. This is the smallest cone containing the intersection of the set of vectors preferred or indifferent to x for agent i and the consumption set X , translated to the origin. We will work with two *regularity conditions on individual preferences*: condition (R), which is required when $X = H$, and condition (T) required otherwise:

(R) There exists an agent i , $1 \leq i \leq k$, such that $0 \notin$ the weak closure of the union over $x \in H$, of the supporting hyperplanes to the preferred set of individual i at x , W_i^X .⁶

Condition (R) is required when the consumption set of the economy is all of the commodity space, $X = H$; it requires the existence of one individual with marginal utility bounded away from zero.

(T) There exists a vector $w < 0$, $\|w\| = 1$, at distance $\varepsilon > 0$ from the cone $C(W_i^X \cap X) - x$, for all $x \in X$ and $1 \leq i \leq k$.⁷ There is an agent i and $\forall x \in X$ a subset V_i^X supporting the preferred

⁵ Arbitrage possibilities still exist, they are *limited* but not necessarily removed because it may be possible for all agents to raise utility by some positive amount at zero cost at this price.

⁶ W_i^X is the set $\{ y \in H : W_i(y) \geq W_i(x) \}$. The preferences of Example 2 in Chichilnisky and Heal (1991) satisfy condition (R) and (T). In addition, all preferences satisfying a uniform "cone condition" as in Chichilnisky and Kalman (1981), which is the same as Mas - Colell's (1986) uniform properness condition, satisfy both (R) and (T). For further examples, see also Section 3 and footnote 8 below, Chichilnisky (1991) and Chichilnisky and Heal (1991).

⁷ In the Hausdorff distance.

set W_x^i , such that $0 \notin$ the weak closure of $\bigcup_{x \in X} V_i^x$.

Condition (T) is required when $X \neq H$. The first part of condition (T) is the "cone condition" in Chichilnisky and Kalman (1980, Theorem 2, part (C)), where it was shown to be necessary and sufficient for the existence of supporting prices for optimal allocations when consumption sets may have empty interiors. This is a typical situation in Hilbert and L_p spaces ($1 \leq p < \infty$) where the consumption set is the positive orthant, since positive orthants in these spaces have empty interior. This cone condition is not needed when $X = H$ because then there is no problem with empty interiors, as H is its own interior. The second part of condition (T) requires that among the supports for all preferred sets W_n^i there should exist some subset which is weakly bounded away from zero, i.e. the vector 0 is not in the weak closure of this subset. That the vector 0 is not in the weak closure of a set implies that the set has no sequence converging pointwise to zero. Condition (T) is similar to condition (R) but weaker, since we don't require that all supports form a set weakly bounded away from zero, but rather that there should exist some subset of supports with this property.

The difference between the conditions (R) and (T) arises because when $X = H$ and preferences are smooth, there exists a unique supporting hyperplane for each x and i , while in the case $X \neq H$, for example $X = H^+$, and x is in the boundary of H^+ , W_x^i could have a large number of supports even if preferences are smooth. Some of these supports must then be eliminated⁸ to satisfy (T). In both cases, when $X = H$ or when $X \neq H$, the role of the

⁸ For example, condition (R) is satisfied by smooth preferences W_i whose gradient vectors DW_i belong to a weakly closed set disjoint from the origin, or to a norm bounded, closed and convex set disjoint from the origin. Condition (R) is satisfied when for example there exist a vector $v \in H - \{0\}$ such that for all x , the supporting hyperplanes for W_x^i have normals $p_i(x)$ with $\langle p_i(x), v \rangle > \varepsilon > 0$ for all $x \in H$. The main difference between our regularity condition (R) used when $X = H$, and our regularity condition (T) which is used when $X \neq H$, satisfying (*), is that (T) does not imply that the set of all supports to all weakly efficient allocations be weakly bounded away from zero, but rather that there exists some convex set of supports for each weakly efficient allocation within the consumption set, such that the union of these

conditions (R) and (T) is the same: eliminating equilibria with zero prices. Conditions related to (R) and (T) have been used in the literature on equilibrium in infinite dimensional spaces: for example Mas - Colell (1986) uses a *uniform properness* condition. The *properness* condition of Mas - Colell (1986) (without the requirement of uniformity) is the same as the *cone condition* introduced earlier in Chichilnisky and Kalman (1981), see Chichilnisky (1991).

3. Limited Arbitrage and the Existence of an Equilibrium

The following theorem proves that the limited arbitrage (LA) assumption on preferences is necessary and sufficient for the existence of a competitive equilibrium. The proof uses Lemmas 1 to 5 which are found in the Appendix. The main purpose of these Lemmas is to establish two building blocks for Theorem 1: (1) the Pareto frontier in utility space is a closed set under the limited arbitrage assumption (LA), and (2) each Pareto efficient allocation admits a non zero supporting price p in H , a decentralization result. These two results are standard in finite dimensional models with consumption sets which are bounded below (like the positive orthant), but are not generally true when consumption sets are not bounded below, nor in infinite dimensional spaces. The choice of spaces is quite crucial, as is the condition (CL) on preferences. Figure 1 shows that the Pareto frontier may fail to be closed even in finite dimensional models, provided the consumption set is the whole euclidean space. This shows the necessity of condition (CL): when preferences do not satisfy (CL), the Pareto

convex sets over all such allocations is weakly bounded away from zero. The reason for this difference is that if $X = H^+$ the positive orthant of H , (H^+ satisfies (*)) then condition (R) is not satisfied because H^+ has "too many" supports and in particular 0 is in the weak limit of its supports. To see this consider $H = l_2$, then the functions (e^i) , $i = 1, \dots$, defined by $e^i_j = 1$ if $i = j$ and $e^i_j = 0$ otherwise, all support H^+ and 0 is in their weak limit. Note that this is not a problem of lack of existence of supports: it is, rather, a problem of too many supports. The natural solution to this problem, which is the one adopted here, is to eliminate judiciously some of these supports and this is achieved by constructing for every weakly efficient allocation a convex set of supports in such a way that their union over all allocations is weakly bounded away from 0. this is precisely what condition (T) does.

frontier is generally not closed. Even when the consumption set is bounded below, but the commodity space is infinite dimensional, examples can be provided where the Pareto frontier is not closed. Mas - Colell (1986) presents an example of this phenomenon in L_∞ , and, in order to prove existence of an equilibrium, he rules it out by assuming a "closedness condition" which requires the Pareto frontier to be closed, without reference to the primitives of the model such as endowments and preferences. Here is where Hilbert spaces show their strength: from the strict quasi concavity and smoothness of the preferences, the Banach - Saks Theorem, and results of Chichilnisky (1977) and Chichilnisky and Kalman (1981), we prove in Lemma 2 in the Appendix that the Pareto frontier is closed: this is true even when the consumption set is not bounded below provided that (LA) and (CL) are satisfied (Lemmas 1 to 5).

Another issue underlining the importance of the choice of space is the problem of existence of supporting prices to decentralize Pareto efficient allocations. Even when the consumption sets are positive orthants, such prices do not exist in general, as an example in Mas Colell (1986) shows. The existence of supports requires special conditions on preferences. When the consumption set X has an empty interior (e.g. the positive orthant of H) this requires the "cone condition" of Chichilnisky and Kalman (1981), which is always satisfied when the allocation maximizes a continuous utility defined on a neighborhood of the consumption set X (Chichilnisky and Kalman (1981), Theorem 2). To resolve this problem, Mas - Colell (1986) uses the Chichilnisky - Kalman cone condition (which he calls "properness", see Chichilnisky 1991), a condition which is automatically satisfied in our case when the utilities are continuous as shown in Chichilnisky and Heal (1991). In (T) we required the cone condition.

The following Theorem is valid for $H = \text{weighted } L_2$, for $H = \text{weighted } L_p$ ($1 \leq p < \infty$), and for more general Sobolev spaces, as indicated in the Appendix. This theorem is valid when the consumption set X is either all of H or not; when $X \neq H$, then X must satisfy (*) $x \in X, y > x \rightarrow y \in X$. Recall that when the consumption set X is the whole space H , the preferences satisfy the regularity condition (R); while when $X \neq H$, then they satisfy the regularity condition (T). Recall also that utilities are C^2 , strictly increasing, quasi concave and

satisfy (CL), and that the initial endowment $\Omega = (\Omega_1, \dots, \Omega_k)$ is desirable:

Theorem 1. Consider an economy E in which individual utilities $W_i: H \rightarrow R$ satisfy the limited arbitrage condition (LA). Then there exists a Pareto Efficient competitive equilibrium allocation $(x_1^* \dots x_k^*)$ with a supporting price $p \in H^+$, $\|p\| = 1$. Reciprocally, if E has a competitive equilibrium, then the limited arbitrage condition (LA) is satisfied.

Proof:

We establish necessity first. Assume that (p^*, x^*) is a competitive equilibrium. Then if $D = \emptyset$, for all $p \in H^+$ there exists a sequence $\{x^n\}$ s.t. $p \notin D_j$ for some j . Consider now p^* ; if $D = \emptyset$, there exists a j and a sequence $\{x_j^n\}$ in A_j and such that $\langle p^*, x_j^m \rangle \leq 0$ for infinitely many m 's. However, for m large enough, $W_i(x_j^m) > W_i(x_j^*)$ and x_j^m is within j 's budget at prices p^* . This contradicts the fact that (x^*, p^*) is a competitive equilibrium. Therefore, when D is empty, E has no competitive equilibrium.

We now prove sufficiency. Consider first the case where $X = H$, so that (R) is satisfied. We define a correspondence $\phi: \Delta \rightarrow T = \{y \in R^k: \sum_{i=1}^k y_i = 0\}$ with the property that any of its zeros is a quasi equilibrium⁹. For each $r \in \Delta$, let $x(r) = (x_1(r), \dots, x_k(r))$ be the feasible allocation which gives the greatest utility vector co linear with r . Such an allocation defines a nonzero utility vector which depends continuously on r by Lemma 1 in the Appendix. Without loss of generality assume that $\sum_i x_i(r) = \Omega$. Now let $P(r) = \{p \in H^+: \|p\| \leq 1, p \text{ supports } x(r)\}$. $P(r)$ is convex, and is non - empty by Lemmas 2 and 3 in the Appendix. Now define $\phi(r) = \{(\langle p, \Omega_1 - x_1(r) \rangle, \dots, \langle p, \Omega_k - x_k(r) \rangle) : p \in P(r)\}$. $\phi(r)$ is non empty and convex valued, $\sum_i z_i = 0$ for $z \in \phi(r)$ by Walras' Law, and $0 \in \phi(r)$ if and only if $x(r)$ is a quasi equilibrium. The next step is to show that ϕ is upper semi continuous, i.e. if $r^n \rightarrow r$, $z^n \in \phi(r^n)$, $z^n \rightarrow z$ then $z \in \phi(r)$. Consider now the feasible allocation $x(r)$ in H^k , where $r = \lim_n(r^n)$. Let u be any other allocation with $W_i(u_i) > W_i(x_i(r))$, where $x_i(r)$ is the i th component of the vector $x(r)$. Since

⁹ This follows a method which was introduced by Negishi (1960).

$r^n \rightarrow r$, eventually $W_i(u_i) > W_i(x_i(r^n))$, which implies $\langle p^n, u_i \rangle \geq \langle p^n, x_i(r^n) \rangle = \langle p^n, \Omega_i \rangle - z_i^n$, where z_i^n is the i th component of $z^n \in \phi(r)$, and $p^n \in P(r^n)$: this follows from the definitions of z^n and of p^n . Let (p^n) be any such sequence of price vectors in $P(r^n)$. Closed convex bounded sets in H are weakly compact by the Banach Alaoglu theorem (Dunford and Schwartz (1958)) because H is reflexive; thus the set $\{p: \|p\| \leq 1\}$ is weakly compact¹⁰. The weak closure of the set $\cup_r P(r)$ of supporting prices to the preferred sets of the agents is contained within $\{p: \|p\| \leq 1\}$, and is weakly compact as well. There exists therefore a p with $\|p\| \leq 1$ and a subsequence (p^m) of (p^n) such that $\langle p^m, f \rangle \rightarrow \langle p, f \rangle$ for all f in H . Note that each p^m supports the preferred sets of all agents at $x(r)$, so that by condition (R) the weak limit of $p^m = p \neq 0$, so we may take $\|p\| = 1$. In particular such a p exists for $f = u$, i.e. $\langle p^m, u \rangle \rightarrow \langle p, u \rangle$. Therefore in the limit $\langle p, u \rangle \geq \langle p, \Omega_i \rangle - z_i$. Since this is true for all u with $W_i(u_i) > W_i(x_i(r))$, it is also true for u with $W_i(u_i) \geq W_i(x_i(r))$ and in particular for $u = x$, i.e. $\langle p, x_i(r) \rangle \geq \langle p, \Omega_i \rangle - z_i$ for all i so that $\langle p, \sum_i x_i(r) \rangle \geq \langle p, \sum_i \Omega_i \rangle - \sum_i z_i$. Since $\sum_i x_i(r) = \sum_i \Omega_i$ and $\sum_i z_i = 0$, we have $\langle p, x_i(r) \rangle = \langle p, \Omega_i \rangle - z_i$ for all i , implying that $z \in \phi(r)$ as we wanted to prove.

The proof is completed by showing that ϕ has a zero. This is now a standard application of Kakutani's fixed point theorem. Consider the map Γ defined by $\Gamma(r) = r + \phi(r)$. It is upper semi continuous, non empty and convex valued. If r is in the boundary of Δ , $\Gamma(r) \in \Delta$: if $r_i = 0$ for some i , then $x_i(r)$ is indifferent to 0 for i , so that $0 \geq p \cdot x_i(r) \geq 0$ where $p \in P(r)$. This implies that $z_i = \langle p, \Omega_i \rangle - x_i \geq 0$. Since Γ is non - empty, upper semi continuous and convex valued, and it satisfies the appropriate boundary conditions, we may use Kakutani's fixed point theorem: Γ has a fixed point in Δ , which is a zero of ϕ . This completes the proof of existence of a quasi equilibrium.

When all initial endowments Ω_i are strictly positive, the value of Ω_i at the equilibrium prices is also strictly positive, i.e. $\int_{\mathbb{R}} p(t) \Omega_i(t) d\mu(t) > 0$. More generally, in a Hilbert space $p \geq 0$, $\Omega \gg 0$ and $\langle p, \Omega \rangle = 0$ imply $p = 0$. It follows that when all initial endowments Ω_i are

¹⁰ In L_1 , the Banach Alaoglu theorem is that convex, bounded and closed sets are weak* compact, not weakly compact (Dunford and Schwartz (1958)).

strictly positive, then $\langle p, \Omega_i \rangle > 0$ for all i and therefore the quasi equilibrium is a competitive equilibrium. In particular, the value of the society's initial endowment Ω is always positive, i.e. $\langle p, \Omega \rangle > 0$. That the competitive equilibrium is Pareto efficient follows now from standard arguments, see for example Debreu (1954).

Consider now the case where $X \neq H$, so that (T) is satisfied. The proof follows the same lines as above except that Lemmas 2 and 3 are replaced by Lemmas 4 and 5 in the Appendix, and $P(r)$ is replaced by $P'(r) = \{ p \in H^+ : \|p\| \leq 1, p \text{ supports } x(r) \text{ and } \langle p, -w_j \rangle \geq \varepsilon^2 \text{ for a given } w_j(r) \in \Psi \}$. The ε and Ψ are given by condition (T). We shall show that $\forall r, P'(r)$ is not empty. Condition (T) and Theorem 2.1 pp. 25, (a) of Chichilnisky and Kalman [12] imply that for each $r \in \mathbb{R}^{k+}$ there exists a supporting hyperplane $p \in H^+$ for the weakly efficient allocation $x(r)$. We shall show that such a p can be chosen so that $\langle p, -w_j \rangle \geq \varepsilon^2$ for some $w_j = w_j(r)$ in Ω . As the distance between w_j and $C(W_i^{x_i} \cap K) - x_i$ is $d \geq \varepsilon$, and $\|w_j\| = 1$, there exists $y \in D_i = \text{closure of } C(W_i^{x_i} \cap K) - x_i$, such that $\|y - w_j\| = d$. Convexity of $C(W_i^{x_i} \cap K) - x_i$ implies that $\langle y - w_j, y \rangle = 0$ and $\langle y - w_j, z \rangle \geq 0$ for all $z \in C(W_i^{x_i} \cap K) - x_i$. Let $p = (y - w_j)$, then $\|p\|^2 = \|y - w_j\|^2 = \langle y - w_j, y - w_j \rangle = \langle y - w_j, y \rangle + \langle y - w_j, -w_j \rangle = \langle y - w_j, -w_j \rangle = \langle p, -w_j \rangle$ since $\langle p, y \rangle = 0$. Therefore $\langle p, -w_j \rangle = \|y - w_j\|^2 = d^2 \geq \varepsilon^2$ and thus $\forall r$, the set $P'(r)$ is not empty. $P'(r)$ is also convex and closed. Now consider as in the case $X = H$ a sequence (p^m) in $P'(r)$. By construction, (p^m) is contained in a weakly compact set, the unit ball of H . Assumption (T) assures that the weak limit of (p^m) cannot be zero, because for each m , there exists a j such that $\langle p^m, -w_j \rangle \geq \varepsilon^2$ and $w_j \neq 0$. The rest of the proof follows as in the case $X = H$. ♦

4. Appendix

Lemma 1. Consider an economy with consumption set $X = H$, or $X \neq H$ and satisfying (*). The limited arbitrage condition (LA) implies that the set S_r of feasible utility allocations of direction r is bounded for all $r \in \Delta$.

Proof: Consider $r \in \Delta$. Let $S_r = \{(u_1, \dots, u_k) \in R^k \text{ s.t. } u_i = W_i(\Omega_i + x_i), \sum_i x_i = 0 \text{ and } \mu(W_i(\Omega_i + x_i)) = r_i \text{ for } i = [1, k], \text{ some } \mu > 0\}$. Assume S_r is not bounded. Then there exists a sequence of allocations $(x^n)_{n=1, \dots}$ with $\sum_i(x_i^n) = 0$, and a subsequence (x^m) of (x^n) s.t. $\lim_m(W_j(x_j^m)) = \infty$ for some j , call this j' .

First consider the case $r \gg 0$. Then $\lim_m W_j(x_j^m) = \infty$ for all j , for otherwise not all of r 's coordinates could be strictly positive. Therefore, (x^m) is in A_j for all j . But (LA) implies that there exists an $h \in H^+$ and an M such that $\langle h, x_j^m \rangle > 0$ for $m > M$ and all j , which contradicts the fact that $\sum_i(x_i^m) = 0$ for all m . Therefore if $r \gg 0$, S_r is bounded.

Now assume $r \in \partial\Delta$, so that $r_i = 0$ for some i . Let I be the set of indices i s.t. $r_i = 0$. For a given $j' \notin I$, define a sequence (z^m) by $z_{j'}^m = x_{j'}^m + \sum_{i \in I} (x_i^m)$, $z_i^m = 0$ if $i \in I$, and $z_h^m = x_h^m$ otherwise. Then the sequence $(\Omega_{j'} + z_{j'}^m)$ is in $A_{j'}$, and $\sum_{i=1}^k z_i^m = \sum_{i=1}^k x_i^m = 0$. Consider the vector $d \in D$, which exists by condition (LA). For some $i \in I$, a new sequence (v^m) is now defined, which can be described informally as follows: agent h is given the projection of z_h^m on the line defined by d ; agent j' gets 1/2 of the projection of $z_{j'}^m$ on that line, and agent i gets the same. Hence the utilities of j' and i go to infinity along this new sequence of allocations. Formally, let $v_h^m = \langle d, z_h^m \rangle \cdot d$ if $h \neq j', i$, $v_{j'}^m = \langle d, z_{j'}^m / 2 \rangle \cdot d$, and $v_i^m = \langle d, z_{j'}^m / 2 \rangle \cdot d$. Then since $d \in D$, $\lim_{m \rightarrow \infty} W_{j'}(\Omega_{j'} + v_{j'}^m) = \lim_{m \rightarrow \infty} W_i(\Omega_i + v_i^m) = \infty$, and $\sum_{i=1}^k v_i^m = 0$. This contradicts (LA) which requires that for such a sequence (v_i^m) there should be an M s.t. $\langle d, v_i^m \rangle > 0$ for all $m > M$. Therefore (LA) implies that S_r is bounded for all $r \in \Delta$ as we wished to prove. ♦

Using Lemma 1, the following Lemma establishes that, when $X = H$, the Pareto frontier is closed. In establishing that the Pareto frontier is closed, we use heavily the fact that the commodity space is reflexive. We use the Banach-Steinhaus theorem, which requires reflexivity, to establish weak compactness from norm-boundedness, and then use the Banach-Sacks theorem to show that pointwise convergence implies norm convergence.

Lemma 2. Assume that the consumption set of the economy is $X = H$, that society's endowment Ω is desirable, and the utilities of the agents are C^2 , strictly increasing and quasi concave, and satisfy conditions (CL) and (LA). Then on any ray r of the positive cone in \mathbb{R}^{k+} there exists a non - zero feasible efficient utility vector. The Pareto frontier is closed, and the map $v(r) = \sup_{W^j} (W^j)$, $\forall W^j \in S_r$, is continuous in r .

Proof: Since the initial endowment is desirable and each W_i is increasing, for each ray r in \mathbb{R}^{k+} , there exists a feasible allocation (x_1^0, \dots, x_k^0) s.t. $W_1(x_1^0), \dots, W_k(x_k^0)$ is a non zero vector in r . Consider now a sequence of utility vectors (W^m) contained in the set $S_r \in \mathbb{R}^{k+}$ defined in Lemma 1. Without loss of generality we may assume that (W^m) is increasing. By definition, $W^m = (W_1(y_1^m), \dots, W_k(y_k^m))$ for some sequence $(y^m) = (y_1^m, \dots, y_k^m) \in F$. Let $v(r) = (v_1, \dots, v_k) = \sup(W^m)$ in r . We shall prove that $v(r)$ is a utility vector corresponding to some feasible allocation $(\Omega_1 + z_1, \dots, \Omega_k + z_k)$ with $\sum_i z_i = 0$.

Define the vector $s_i^n = DW_i(\Omega_i + y_i^n) / \|DW_i(\Omega_i + y_i^n)\|$. Then by (A*), $\lim_{n,m} (\sup_{i,j} | \langle h, s_i^n - s_j^m \rangle |) = 0$, for every h in the basis of coordinates of H , because otherwise there will exist feasible allocations produced by varying individual allocations along the direction defined by h , which increase everyone's utility and eventually exceed the utility levels (v_1, \dots, v_k) , contradicting the fact that the sequence of utility levels $W_i(\Omega_i + y_i^n)$ converges to its supremum, v . Formally, if $\lim_{n,m} (\sup_{i,j} | \langle h, s_i^n - s_j^m \rangle |) \neq 0$, for some h in the coordinate basis of H , then $\exists \varepsilon > 0$ s.t. for any N , $\exists n, m > N$ with $\langle h, s_i^n - s_j^m \rangle > \varepsilon$. So there exists a number $\lambda > 0$, with $\langle \lambda h, DW_i(\Omega_i + y_i^n) \rangle = \lambda \|DW_i(\Omega_i + y_i^n)\| \langle h, s_i^n \rangle > 0$, $\langle \lambda h, DW_j(\Omega_j + y_j^m) \rangle = \lambda \|DW_j(\Omega_j + y_j^m)\| \langle h, s_j^m \rangle > 0$. We can then write $W_i(\Omega_i + y_i^n + \lambda h) = W_i(\Omega_i + y_i^n) + \langle DW_i(\Omega_i + y_i^n), \lambda h \rangle + 1/2 \lambda h \cdot D^2(\Omega_i + y_i^n + \nu h) (\lambda h)^T$ by Taylor's theorem for some $\nu \in (0, \lambda)$. By (A*), $W_i(\Omega_i + y_i^n + \lambda h) - W_i(\Omega_i + y_i^n) \geq (1-\gamma) \langle DW_i(\Omega_i + y_i^n), \lambda h \rangle$. Now, given any $\delta > 0$, $\exists N(\delta)$ s.t. $\|W_i(\Omega_i + y_i^n - v_i)\| < \delta$ for all i , because $v = \sup W^m$ by assumption. Choose $\delta < (1-\gamma) \langle DW_i(\Omega_i + y_i^n), \lambda h \rangle$ and pick $n > N(\delta)$. We can do the same for $W_j(\Omega_j + y_j^n - \lambda h)$. In particular, we can pick a δ and $n > N(\delta)$ s.t. $W_i(\Omega_i + y_i^n + \lambda h) > v_i$, and $W_j(\Omega_j + y_j^n - \lambda h) > v_j$, contradicting the definition of v as the supremum of attainable utility vectors on the

ray r . Therefore, for all h in the basis of coordinates of H , $\lim_{n,m} (\sup_{i,j} | \langle h, s_i^n - s_j^m \rangle |) = 0$ as we wished to prove.

Now let s^0 denote the weak limit of (s_i^n) with respect to n , for all i . This limit exists by Banach - Steinhaus theorem which establishes that norm bounded sets are weakly precompact, because all vectors s_i^n have norm one. Note that the Banach - Steinhaus theorem requires that the space H be reflexive¹¹, a condition which is satisfied because H is a Hilbert space. Now since $v(r) = \sup W^m$, it follows that for any $\varepsilon > 0$, and $m(\varepsilon)$ large enough, there is a vector denoted z_i^m s.t. $(\Omega_i + z_i^m) \in (W_i)^{-1}(v_i)$ and $\|DW_i(\Omega_i + z_i^m) - DW_i(\Omega_i + y_i^m)\| < \varepsilon$. Therefore the sequence $(t_i^n) = (DW_i(\Omega_i + z_i^n) / \|DW_i(\Omega_i + z_i^n)\|)$ also converges weakly to s^0 for all i . Since (t_i^n) converges weakly to s^0 , it converges pointwise to s^0 . Therefore, we may now apply the Banach - Saks theorem (Dunford and Schwartz (1958)), to establish that the sequence of averages of (t_i^n) , defined by $a_i^n = (1/n)(t_i^1 + \dots + t_i^n)$ converges to s^0 in the norm of H .¹² Strict quasi concavity of W_i implies that a_i^n is the direction of a gradient in $(W_i)^{-1}(v_i)$ as well. We may now apply condition (CL), which implies that s^0 is the direction of the gradient of some vector in the set $(W_i)^{-1}(v_i)$ for all i . In other words, there exists a vector $z_i \in H$, with $(\Omega_i + z_i) \in (W_i)^{-1}(v_i)$ and $\lambda_i \cdot DW_i(\Omega_i + z_i) = s^0$ for all i , some λ_i . By strict concavity, $z_i = \lim z_i^n = \lim y_i^n$; since $\sum_i y_i^n = 0$ for all n , it follows that $\sum_i z_i = 0$, so that the allocation

¹¹The Banach-Steinhaus Theorem (Robertson & Robertson (1973) Chapter 4 Proposition 5 page 72) establishes also that norm-bounded sets are only weakly pre-compact in reflexive spaces. Hence the arguments of this proof are not available in non-reflexive spaces such as L_∞ or L_1 , such as those used e.g. by Mas - Colell (1986) and by Brown and Werner (1992).

¹² The Banach Saks theorem establishes that if a sequence converges pointwise, then the sequence of its averages as defined above converges in the norm; we can apply this theorem here because we know that (t_i^n) converges weakly and therefore pointwise. If, instead, (t_i^n) converged in the weak* topology rather than weakly, the reasoning would not apply since we cannot assure in this latter case that (t_i^n) converges pointwise. The norm boundedness of the vectors s_i^n assures weak precompactness only when H is reflexive: this is the Banach Steinhaus theorem (Robertson and Robertson, 1973). Therefore only with reflexive spaces we can apply this chain of reasoning. For this reason, the use of Hilbert or reflexive spaces is crucial to our results.

$(\Omega_i + z_i)$ is feasible and $W_i(\Omega_i + z_i) = v_i$ as we wished to prove. This completes the proof that along every ray r in \mathbb{R}^k there is a non - zero feasible efficient vector. We now complete the proof that the Pareto frontier is closed in \mathbb{R}^{k+} . If the limit $v \in \mathbb{R}^{k+}$ of a sequence (W^j) in the Pareto frontier is not in this frontier, consider the ray r in \mathbb{R}^{k+} through the origin, passing through this limit point v . There exists a nonzero feasible efficient utility vector $\mu \in \mathbb{R}^{k+}$ on that ray. If $\mu \neq v$ then either μ is not weakly efficient or else for $j \geq N$, some N , W^j is not feasible efficient. In either case we have a contradiction. Thus $\mu = v$ and the Pareto frontier is closed. The last statement of this Lemma is the closed - graph theorem. \blacklozenge

The next result shows that when the consumption set of the economy is $X = H$, every Pareto efficient allocation is supported by an appropriate price in H^+ :

Lemma 3. Consider an economy with consumption set $X = H$, and let $z = (x_1, \dots, x_k)$ be a feasible efficient allocation, and for $1 \leq i \leq k$ let W_i be concave, strictly increasing, and continuous. Then there exists a price $p \in H^+$, $\|p\| = 1$, such that $\langle p, y \rangle \geq \langle p, x \rangle$ for all y satisfying $W_i(y) \geq W_i(x_i)$.

Proof: Let $W_i^{X_i} = \{ x \in H : W_i(x) \geq W_i(x_i) \}$. Let $V = \sum_i W_i^{X_i}$, and $v = x_1 + \dots + x_k \in V$. By Theorem 2.1 of Chichilnisky and Kalman (1981), $\forall i$ there exists a positive price $p_i \in H^+$ which supports $W_i^{X_i}$ at x_i , $\|p_i\| = 1$. Since z is feasible and efficient, and the W_i are strictly increasing, if $v' < v$, then¹³ $v' \notin V$. Furthermore, the set V is convex and has a non empty interior because for each i , the set $W_i^{X_i}$ has a non empty interior by continuity of W_i . It follows from Chichilnisky and Kalman (1981, Theorem 2.1) that there exists a non zero supporting price p for V at v . V contains a translate of the positive orthant because each $W_i^{X_i}$ does. Therefore p is positive and we can take $\|p\| = 1$. It is now easy to check that p supports

¹³ For otherwise there would exist an allocation $z'' = (x''_1, \dots, x''_k)$ s.t. $\sum_i x''_i \leq v$ so that z'' is feasible, and such that $W_i(x''_i) > W_i(x_i)$, contradicting the fact that z is weakly efficient.

W_i^x at x_i . This is because, by construction, p is minimized over V at v . The minimum of a linear function on the sum of sets is equal to the sum of the minima of the linear function on each set, i.e. $p(v) = \sum p(x'_i)$, where x'_i minimizes p over the set W_i^x . Since $\forall i p(x'_i) \leq p(x_i)$, and $\sum_i p(x'_i) = p(v) = \sum_i p(x_i)$, then $p(x'_i) = p(x_i)$ for all i , i.e. $p_i(x_i) = \min p_i(y) \forall y \in W_i^x$ and $\forall i$. Therefore p supports W_i^x at x_i , as we wished to prove. ♦

The following lemmas proves the same results as Lemmas 2 and 3, but when the consumption set X is not H ; X is convex and closed and satisfies: (*) $x \in X, y > x$ implies $x \in X$.

Lemma 4: Consider now an economy with consumption set $X \neq H$ satisfying (*). Let $z = (x_1, \dots, x_k)$ be a feasible efficient allocation in X^k and for $1 \leq i \leq k$, let W_i be a strictly concave and strictly increasing continuous function. Then there exists a price $p \in H^+, \|p\| = 1$, such that $\langle p, y \rangle \geq \langle p, x \rangle$ for all $y \in X$ satisfying $W_i(y) \geq W_i(x_i)$.¹⁴

¹⁴ The parallel to our Lemma 3 in Mas - Colell (1986) requires an additional condition denoted "uniform properness" which is not required here, and which is strictly stronger than our assumptions. We assume H^S continuity of the utilities instead, and it is easy to see that our condition of continuity implies "properness at one point" but it is strictly weaker than "uniform properness". To this end, consider a continuous strictly increasing function $W: H \rightarrow R$. Then $W(x - \lambda h) < W(x)$ implies that there is a ball around $(x - \lambda h)$ where this inequality is also satisfied. Therefore for all x there exists h in H^+ and a neighborhood V of the origin such that $W(x - \lambda h + y) > W(x)$ implies $y \notin \lambda V$. The "uniform properness" condition of Mas - Colell (1986) requires the existence of one $h \in H^+$ and a neighborhood V of $\{0\}$ which are valid for all x in H such that $W(x - \lambda h + y) > W(x)$ implies $y \notin \lambda V$. Our condition of continuity is therefore strictly weaker than the "uniform properness" condition of [27], since in our case the h and δ may vary with x , while "uniform properness" requires instead the existence of one h and one δ which must be the same for all x in H . It is also easy to see that properness at one point is identical to the cone condition introduced in Chichilnisky and Kalman (1981, Theorem 2.1 p. 25) as a necessary and sufficient condition for the existence of a supporting hyperplane for convex sets which may have an empty interior, see also Chichilnisky (1991). The cone condition of Chichilnisky and Kalman (1981) requires that if Y is a convex set, then there exists a $w \geq 0$ which is at a positive distance from the cone defined by Y and $x \in Y, C(Y, x) = \{z = a(y - x) + x, y \in Y, a \geq 0\}$ see Chichilnisky and Kalman (1981, (a) of Theorem 2.1 p. 25). When Y is the convex set of points which are preferred (by \geq_i) to x , this condition is identical to requiring that there exists a vector $w \geq 0$, and an open neighborhood of the origin V such that: $(x - \alpha w) + z \geq_i x$ implies $z \in \alpha V$, which is the "properness" condition for the preference \geq_i (Mas - Colell, 1986, page 1043).

Proof: Consider the feasible efficient allocation $z = (x_1, \dots, x_k)$ in X^k and let $x = \sum_i x_i$. Let $V_i = (W_i^{x_i} \cap X)$. We must show that there exists a $p \in H^+$ with $\|p\| = 1$, supporting $\sum_i V_i$ at x . For all i the function W_i is continuous, it attains a minimum within $W_i^{x_i}$ at x_i , and is increasing so that $y < x \rightarrow y \notin W_i^{x_i}$. By Chichilnisky and Kalman (1981, Theorem 2.1), this implies that there exists a non zero $q_i \in H^+$ supporting the set V_i at x_i . Equivalently, there exists a non zero $p_i \in H^+$ supporting the set $V_i - x_i$ at $\{0\}$. Consider the set $B_i = \{u \in H : \forall z \in (V_i - x_i) \langle u, z \rangle \geq 0\}$, namely B_i is the set of supports to $V_i - x_i$. B_i is convex and closed. Since X satisfies (*) and W_i is increasing, $B_i \subset H^+$. If $\exists v^* \neq 0, v^* \in \cap_i B_i$, then v^* is the desired support for the set $\sum_i V_i$ at x , and thus provides the desired support for the feasible efficient allocation z . Assume, to the contrary, that $\cap_i B_i = \{0\}$. For any given i , define $D_i = \{z \in H : \langle z, y \rangle \geq 0 \forall y \in \sum_{j \neq i} V_j - x_j\}$. This is the closed convex cone of supports of the set $\sum_{j \neq i} V_j - x_j$. Since $\forall j, W_j$ is minimized at x_j over the set $W_j^{x_j}$, by Chichilnisky and Kalman (1981, Theorem 2.1), $D_i \neq \{0\}$. By condition (*) on X and the increasingness of $W_j, D_i \subset H^+$. Note that $\cap_i B_i = \{0\}$ implies $B_i \cap D_i = \{0\}$, which we now assume. Take $w \neq 0$ in D_i . Then $w + D_i \subset D_i$, so that $\forall z \in B_i$ and $y \in D_i, w + y \neq z$, or equivalently $w \neq z - y$. Since this is true for all $z \in B_i$ and $y \in D_i$, then w is not in the cone $B_i - D_i$. Since the set $B_i - D_i$ is closed, this implies $d(w, B_i - D_i) > 0$; by Chichilnisky and Kalman (1981, Theorem 2.1), there is therefore a $p \in H$ s.t. p supports $B_i - D_i$ at $\{0\}$. Equivalently, p separates B_i from D_i , i.e. $\forall x \in B_i, x \neq 0, \langle p, x \rangle \geq 0$, and $\forall y \in D_i, y \neq 0, \langle p, y \rangle \leq 0$. For any such $x, y, d(x, y) > 0$, so that the separation can be made strict:

$$(1) \quad \langle p, x \rangle > 0 \forall x \neq 0 \text{ in } B_i, \text{ and } \langle p, y \rangle < 0 \forall y \neq 0 \text{ in } D_i.$$

Since B_i is the set of all supports of the set $V_i - x_i$, by (1) $\exists \lambda > 0$, such that $\lambda p \in V_i - x_i$. For any $\alpha < \lambda$, since $0 \in V_i - x_i$ and $V_i - x_i$ is convex, αp is in $V_i - x_i$ as well. By strict concavity of $W_i, W_i(\alpha p + x_i) > W_i(x_i)$. Similarly, (1) implies that there exists a $\gamma > 0$, such that $-\gamma p \in \sum_{j \neq i} V_j - x_j$. Let $\beta = \min(\alpha, \gamma)$. Then

$$(2) \quad \beta p \in V_i - x_i, -\beta p \in \sum_{j \neq i} V_j - x_j, \text{ and } W_i(\beta p + x_i) > W_i(x_i).$$

Consider now an allocation that assigns $\beta p + x_i$ to the i -th agent and $-\beta p + \sum_{j \neq i} x_j$ to the rest;

it is feasible because $\beta p + x_i - \beta p + \sum_{j \neq i} x_j = \sum_i x_i$. By (2) such an allocation exists which strictly increases the utility of the i th agent without decreasing that of the others, contradicting the efficiency of z . The contradiction arises from the assumption that $\cap_i B_i = \{0\}$. Therefore, there exists a non zero vector v^* in $\cap_i B_i$. Since X satisfies (*) and W_i is increasing, $v^* \in H^+$, and provides the desired support for the feasible efficient allocation z , as we wished to prove. ♦

Lemma 5: Assume that society's endowment Ω is desirable, the consumption set $X \neq H$ satisfies (*) and the utilities of the agents are continuous, strictly increasing and satisfy condition (LA). Then on any ray r of the positive cone in R^{k+} there exists a non zero feasible efficient utility vector. The Pareto frontier is closed and the map $v(r) = \sup_j (W^j) \forall W^j \in r$, is continuous in r .

Proof: This follows directly from the proof of Lemma 2 by replacing the set S_r in the proof by $S_r^* = \{(u_1, \dots, u_k) \in R^k \text{ s.t. } u_i = W_i(x_i + \Omega_i) \text{ where } x_i \in X, \sum_i x_i = 0 \text{ and } \mu(W_i(x_i + \Omega_i)) = r_i \text{ for } i \in [1, k] \text{ and some positive number } \mu\}$. ♦

Extension of the Existence Results to Sobolev Spaces H^s and to L_p spaces.

Let s and p be integers, $1 \leq s < \infty$, and $1 \leq p < \infty$.

$$H^s = \{ f: R \rightarrow R : f \text{ is } C^s \text{ and } \int (f(t)^2 + \dots + D^s f(t)^2) d\mu(t) < \infty \}$$

where $D^s f$ is the s -th derivative of f , and

$$L_p = \{ f: R \rightarrow R : f \text{ is measurable and } \int |f(t)|^p d\mu(t) < \infty \}.$$

For all $\infty > p \geq 1$, L_p is a Banach (complete, normed) linear space. When $p > 1$ it has the following duality property: the space of continuous linear functions on L_p denoted L_p^* is L_q for $1/p + 1/q = 1$. In particular, $L_p^{**} = L_p$, Dunford and Schwartz [19]. One interesting feature of the Sobolev spaces H^s is that for all $s \geq 1$, H^s is a Hilbert, and in particular self - dual, space with the standard inner product and countable, orthonormal coordinate basis.

Furthermore, by Sobolev's theorem $H^s \subset C^k(\mathbb{R})$ if $s \geq 1/2 + k$, so that H^1 consists entirely of continuous functions, H^2 consists entirely of continuously differentiable functions, and $H^0 = L_2$, see Adams (1975), Nirenberg (1973/4), Chichilnisky (1977). In all these spaces, therefore, prices (which are elements of the dual space H^{s*}) are also continuous or differentiable functions. All the results stated in the paper apply to H^s and L_p spaces with $\infty > p \geq 1$ provided the assumptions are made in the respective norms of these spaces. The main assumption is the continuity of the utility functions. The following results characterize continuous functions in L_p , $1 \leq p < \infty$, in H^1 and H^2 . As before all measures $\mu(t)$ are finite.

Lemma 6. Let $W(c) = \int_{\mathbb{R}} u(c(t), t) d\mu(t)$, with u satisfying the Caratheodory condition. Then W defines a norm continuous functions from L_p to \mathbb{R} ($1 < p < \infty$) for some coordinate system of L_p if and only if $|u(c(t), t)| \leq a(t) + b |c(t)|^p$ where $a(t) \geq 0$, $\int_{\mathbb{R}} a(t) d\mu(t) < \infty$ and $b > 0$.
The proof is the same as given for the case of L_2 in Chichilnisky (1977). ♦

Lemma 7. Given a coordinate system for H^i , $i=1,2$, a function $W(c) = \int_{\mathbb{R}} u(c(t), t) d\mu(t)$ is continuous from H^1 to \mathbb{R} if and only if the conditions in Lemma 7 are satisfied for u and for Du . W is continuous from H^2 to \mathbb{R} if the conditions in Lemma 7 are satisfied for u , Du , and D^2u .

The proof is the same as in Chichilnisky (1977). ♦

5. References

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