SOME PROPERTIES OF C-COMPACT SETS IN p-ADIC SPACES

by

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In this note we shall prove some properties of c-compact sets that may or may not be part of the 'folklore'. The concept of c-compactness, introduced by Springer [7], takes over the role played by convex-compact sets in Functional Analysis over $\mathbb{R}$ or $\mathbb{C}$ (or, any locally compact valued field).

Throughout, let K be a nonarchimedean nontrivially valued field with valuation $|\ |$. We assume K to be maximally (= spherically) complete. A subset A of a K-linear space E is absolutely convex if it is a submodule of E, considered as a module over the valuation ring $\{\lambda \in K : |\lambda| \leq 1\}$. A set $C \subset E$ is convex if it is either empty or an additive coset of an absolutely convex set. For a set $X \subset E$ we denote by co X its absolutely convex hull, by [X] its K-linear span.

From now on in this paper E is a locally convex space over K ([8]). We assume E to be Hausdorff.

§ 1. DEFINITION AND FIRST PROPERTIES

DEFINITION 1.1. ([7]) Let $C \subset E$ be a nonempty convex set. A *convex filter on* C is a filter of subsets of C that has a basis consisting of convex sets. C is *c-compact* if each convex filter on C has a cluster point in C.

In other words, C is c-compact if and only if the following is true. Let $\mathcal{C}$ be a family of nonempty relatively closed convex subsets of C such that $C_1, C_2 \in \mathcal{C}$ implies $C_1 \cap C_2 \in \mathcal{C}$. Then $\cap\, \mathcal{C} \neq \emptyset$.

We quote the following properties, proved in [7].

PROPOSITION 1.2.

(i) *K is c-compact.*

(ii) *A c-compact set is complete.*

(iii) *A nonempty closed convex subset of a c-compact set is c-compact.*

(iv) *Let $(E_i)_{i \in I}$ be a family of Hausdorff locally convex spaces over K. Suppose, for each i, C_i is c-compact in E_i. Then $\prod_{i \in I} C_i$ is c-compact in $\prod_{i \in I} E_i$.*

(v) *The image of a c-compact set under a continuous linear map is c-compact.*

In [1] we find the following.

PROPOSITION 1.3.

(i) *If K is locally compact then a bounded nonempty convex set $C \subset E$ is c-compact if and only if it is convex and compact.*

(ii) *E is c-compact if and only if E is linearly homeomorphic to a* *closed subspace of some power of* K.

In § 3 (Theorem 3.3) we shall characterize arbitrary c-compact sets in the spirit of Proposition 1.3 (ii). But we conclude this first section with two statements that have nothing to do with the sequel. I just want to get rid of them.

PROPOSITION 1.4.

A c-compact set is a Baire space.

Proof.

Let $U_1, U_2, \dots$ be (relatively) open dense subsets of a c-compact set $C \subset E$. We prove that $\bigcap_n U_n \neq \emptyset$. There exists a nonempty open convex subset $B_1 \subset U_1$. As U_2 is dense we can find a nonempty open convex set $B_2 \subset B_1 \cap U_2$. Continuing this way we find nonempty open convex sets $B_1 \supset B_2 \supset \dots$ with $B_n \subset \bigcap_{i=1}^{n} U_i$ for each n. The open sets B_n are cosets of an additive group, hence closed. By c-compactness, $\bigcap B_n \neq \emptyset$. It follows that $\bigcap_n U_n \neq \emptyset$.

PROPOSITION 1.5.

Let $X \subset E$ be closed, let $C \subset E$ be c-compact. Then X+C is closed.

Proof.

Let $z \in \overline{X+C}$ (the closure of X+C), let $\mathcal{U}$ be the collection of all absolutely convex neighbourhoods of 0. For each $U \in \mathcal{U}$ the set z+U intersects X+C so

$$C_U := \{c \in C : z-c \in X+U\}$$

is not empty. X+U is a union of cosets of U, so is its complement.

Therefore, X+U is closed and C_U is closed in C. Further we have

$$C_U \cap C_V \supset C_{U \cap V} \qquad (U,V \in \mathcal{U})$$

By c-compactness there exists a $c \in C$ such that

$$z-c \in \bigcap_{U \in \mathcal{U}} (X+U) = \overline{X} = X$$

i.e., $z \in X+c \subset X+C$.

Remark.

If the base field is not spherically complete there exist a complete absolutely convex compactoid $C \subset c_0$ and an element $a \in c_0$ such that $C+\mathrm{co}\{a\}$ is not closed ([3], 6.25).

§ 2. LOCAL COMPACTOIDITY

DEFINITION 2.1. ([3], (6.7)) A subset X of E is a <u>local compactoid</u> if for each neighbourhood U of 0 in E there exists a finite dimensional K-linear subspace D of E such that $X \subset U+D$.

PROPOSITION 2.2.

<u>Let A be an absolutely convex subset of E. A is c-compact if and only if A is a complete local compactoid.</u>

<u>Proof</u>.

For E a Banach space this is proved in [3], 6.15. Now let E be a locally convex.

(i) Assume A is c-compact. By Proposition 1.2 (ii), A is complete. To prove local compactoidity let U be an absolutely convex neighbourhood of 0 in E. There is a continuous seminorm p such that $\{x \in E : p(x) \leq 1\} \subset U$. Let $\pi_p : E \to E_p$ be the quotient map where E_p is the canonically normed space E/Kerp. Now $\pi_p(A)$ is c-compact (Proposition 1.2 (v)) so by the above it is a local compactoid in the completion $\hat{E_p}$ of E_p. By Corollary 6.15 of [3] we have $\pi_p(A) = R+T$ where R is a compactoid and T a finite dimensional subspace of $\hat{E_p}$. Then $T \subset E_p$. Now $\pi_p(U)$ is open in E_p and by Katsaras' Theorem ([5], Lemma 8.1) there exist $x_1,\dots,x_n \in [R]$ such that $R \subset \pi_p(U)+co\{x_1,\dots,x_n\}$. Combining our knowledge on R and T we find a finite dimensional space $F \subset [\pi_p(A)]$ such that $\pi_p(A) \subset \pi_p(U)+F$. Choose a finite dimensional space $D \subset [A]$ such that $\pi_p(D) = F$. Then

$$A \subset U + D + \text{Ker}\,\pi_p \subset U + D.$$

(ii) Let A be a complete local compactoid. Let Γ be the collection of all continuous seminorms on E. For each $p \in \Gamma$ we have that $\pi_p(A)$, and also $\overline{\pi_p(A)}$, is a local compactoid in $\hat{E_p}$.

As $E_p^{\sim}$ is a Banach space we know that $\overline{\pi_p(A)}$ is c-compact. Then also $A_0 := \prod_{p \in \Gamma} \overline{\pi_p(A)}$ is a c-compact subset of $\prod_{p \in \Gamma} E_p^{\sim}$ (Proposition 1.1 (iv)). The canonical map $E \to \prod_{p \in \Gamma} E_p^{\sim}$ sends A homeomorphically and linearly into A_0. Its image is closed in A_0 because A is complete. Then A is c-compact (Proposition 1.2 (iii)).

The following Proposition may look innocent.

PROPOSITION 2.3.

Let $A \subset E$ be absolutely convex and c-compact. For each neighbourhood U of 0 there exists a finite dimensional absolutely convex set $F \subset A$ such that $A \subset U+F$.

(The crucial part is the phrase '$F \subset A$'.) For the proof we use a lemma.

LEMMA 2.4.

Let A,U be absolutely convex subsets of E, where U is closed, A is c-compact. Let $x \in E$ be such that $A \subset U+Kx$. Then there exists an $y \in E$ and an absolutely convex $C \subset K$ such that $Cy \subset A$ and $A \subset U+Cy$.

Proof.

Let $C := \{c \in K : (U+cx) \cap A \neq \emptyset\}$. We have $A \subset U+Cx$, $C = \{c \in K : cx \in A+U\}$, so C is absolutely convex. If $C = (0)$ then $A \subset U$ and we choose $y := 0$. So assume $C \neq (0)$. For each $c \in C$, $c \neq 0$ define

$$H_c := c^{-1}(A \cap (cx + U)).$$

Each H_c is a convex, closed, nonempty subset of $c^{-1}A$ hence c-compact. Further, if $c,d \in C$, $0 < |c| \le |d|$ then $H_d \subset H_c$. (Proof. Let $z \in H_d$. Then $dz \in A \cap (dx+U)$. By absolute convexity of A and U,

$$cz = \frac{c}{d} \cdot dz \in A$$

$$cz \in \frac{c}{d}(dx+U) \subset cx + \frac{c}{d} U \subset cx+U.$$

It follows that $cz \in A \cap (cx+U)$ i.e. $z \in H_c$.) By c-compactness there exists an $y \in \bigcap_{\substack{c \in C \\ c \neq 0}} H_c$. Let $c \in C$, $c \neq 0$. Then

$$cy \in cH_c \subset A \cap (cx+U) \subset A.$$

Also, $cy \in cx+U$ so that $cx-cy \in U$. Let $a \in A$. Then $a = u+cx$ for some $u \in U$, $c \in C$. We see that $a = u+cy+cx-cy \in cy+U$. It follows that $A \subset U+Cy$.

<u>Proof of Proposition</u> 2.3.

We may assume that U is absolutely convex. By Proposition 2.2 A is a local compactoid so there exist $x_1,\ldots,x_n \in E$ such that $A \subset U+Kx_1+\ldots+Kx_n$. By the Lemma, applied to $U+Kx_2+\ldots+Kx_n$ in place of U, there exist a $y_1 \in E$ and an absolutely convex $C_1 \in K$ such that $C_1y_1 \subset A$ and

$$\begin{aligned} A &\subset U + C_1y_1 + Kx_2 + \ldots + Kx_n \\ &= (U + C_1y_1 + Kx_3 + \ldots, Kx_n) + Kx_2 \end{aligned}$$

and we can continue. After n of these procedures we arrive at $y_1,\ldots,y_n \in E$, absolutely convex $C_1,\ldots,C_n \subset K$ such that $C_iy_i \subset A$ for each i and $A \subset U+C_1y_1+\ldots+C_ny_n$.

<u>Warning</u>.

The property of Proposition 2.3 is not shared by <u>all</u> absolutely convex local compactoids even when we require them to be closed ! In fact we have:

EXAMPLE 2.5.

Let the valuation of K be dense. Set

$$A = \{x \in c_0 : \|x\| \le 1\}.$$

([4], p.47).

(i) A is a closed (local) compactoid for the weak topology of c_0.

(ii) There exists a weak neighbourhood U of 0 such that for any finite dimensional set $F \subset A$

$$A \not\subset U+F.$$

Proof.

(i) Let U be a weak neighbourhood of 0. There exists a weakly continuous seminorm p such that $\{x \in c_0 : p(x) \le 1\} \subset U$. Then Kerp has finite codimension. Choose a finite dimensional space $D \subset c_0$ with $\pi_p(D) = E_p$ (where as previously, $E_p := c_0/\text{Kerp}$ and $\pi_p : c_0 \to E_p$ is the quotient map). We have $A \subset \text{Kerp}+D \subset U+D$ (in fact, we have shown that each subset of c_0 is a local compactoid for the weak topology). To prove weak closedness of A, let $(x_i)_{i \in I}$ be a net in A converging weakly to $x \in c_0$. By [4], Lemma 4.35 (i) there exists an $f \in c_0'$ $f \neq 0$ for which $|f(x)| = \|f\| \, \|x\|$. We have

$$\|f\| \, \|x\| = |f(x)| = \lim|f(x_i)| \le \lim\sup \|f\| \, \|x_i\| \le \|f\|$$

so that $\|x\| \le 1$.

(ii) Choose $\tau_1, \tau_2, \ldots, \in K$, $0 < |\tau_1| < |\tau_2| < \ldots$, $\lim_{n\to\infty}|\tau_n| = 1$. The formula

$$f(a_1, a_2, \ldots) = \sum_{i=1}^{\infty} a_i \tau_i$$

defines an element $f \in c_0'$. Observe that $\sup_A|f| = 1$ but $|f(x)| < 1$ for each $x \in A$. Set $U := \{x : |f(x)| \le \tfrac{1}{2}\}$, let F be any finite dimensional set in A. We shall arrive at $A \not\subset U+F$ by showing that $\sup_{U+F}|f| < 1$. To this end it suffices to prove $\sup_{F}|f| < 1$. [F] is a finite dimensional subspace of c_0 and therefore ([4], Theorem 5.9) has an orthonormal base $x_1,\ldots,x_n$. It is easily seen that

$$F' := \mathrm{co}\{x_1,\ldots,x_n\} \supset F$$

and $\sup_{F}|f| \leq \sup_{F'}|f| = \max(|f(x_1)|,\ldots,|f(x_n)|) < 1$.

Remark.

The above construction works also for the case where the base field is not spherically complete. Then A is even weakly complete ! ([5], Theorem 9.6 and [4], Theorem 4.17)

§ 3. A REPRESENTATION THEOREM FOR C-COMPACT SETS

LEMMA 3.1.

Let $\lambda \in K$, $|\lambda| > 1$. *Let* $G \subset E$ *be closed, absolutely convex, and let* $F \subset [G]$ *be a finite dimensional set. If* $(x_i)_{i \in I}$ *is a net in* $G+F$ *converging to* 0 *then* $x_i \in \lambda G$ *for large* i.

Proof.

[6], Lemma 1.3.

PROPOSITION 3.2.

(See also [2], Proposition 4, p. 93.) *Let* $A \subset E$ *be absolutely convex, c-compact. Let* τ' *be a Hausdorff locally convex topology on* E, *weaker than the initial topology* τ. *Then* $\tau = \tau'$ *on* A.

Proof.

Let $(x_i)_{i \in I}$ be a net in A converging to 0 for τ'. Let $\lambda \in K$, $|\lambda| > 1$, let U be an absolutely convex neighbourhood of 0 for τ. Then $(\lambda^{-1}U) \cap A$ is c-compact in (E,τ) hence in (E,τ'), so that $(\lambda^{-1}U) \cap A$ is τ'-closed. There is (Proposition 2.3) a finite dimensional $F \subset A$ with $A \subset \lambda^{-1}U+F$. Then $A = (\lambda^{-1}U) \cap A + F$. Lemma 3.1 applies. It follows that $x_i \in \lambda(\lambda^{-1}U) \cap A \subset U$ for large i, so $\lim x_i = 0$ in the sense of τ.

THEOREM 3.3.

Let $A \subset E$ *be absolutely convex. The following are equivalent.*

(α) A *is* c-*compact*.

(β) A *is isomorphic (as a topological module over* $\{\lambda \in K : |\lambda| \le 1\}$) *to a closed submodule of some power of* K.

Proof.

$(\beta) \Rightarrow (\alpha)$. This follows from Proposition 1.2, (i), (iv), (iii). Now suppose (α). The map

$$x \mapsto (f(x))_{f \in E'}$$

is a continuous linear injection $E \to K^{E'}$ (Hahn-Banach Theorem). According to Proposition 3.2 it is a homeomorphism, if restricted to A, and (β) follows.

REFERENCES

[1] N. de Grande de Kimpe: C-compactness in locally K-convex spaces. Indag. Math. 33 (1971), 176-180.

[2] L. Gruson: Théorie de Fredholm p-adique. Bull. Soc. Math. France 94 (1966), 67-95.

[3] A.C.M. van Rooij: Notes on p-adic Banach Spaces. Report 7725. Mathematisch Instituut, Katholieke Universiteit Nijmegen, the Netherlands (1977), 1-52.

[4] A.C.M. van Rooij: Non-Archimedean Fuctional Analysis. Marcel Dekker, New York (1978).

[5] W.H. Schikhof: Locally convex spaces over nonspherically complete valued fields. Groupe d'étude d'Analyse ultramétrique 12 (1984/85), no 24, 1-33.

[6] W.H. Schikhof: Topological stability of p-adic compactoids under continuous injections. To appear.

[7] T.A. Springer: Une notion de compacité dans la théorie des espaces vectoriels topologiques. Indag. Math. 27 (1965), 182-189.

[8] J. van Tiel: Espaces localement K-convexes. Indag. Math. 27 (1965), 249-289.