Ordinary and Generalized Circulation Algebras for Regular Matroids

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Let E be a finite set, and let $R(E)$ denote the algebra of polynomials in indeterminates $(x_e)_{e\in E}$, modulo the squares of these indeterminates. Subalgebras of $R(E)$ generated by homogeneous elements of degree 1 have been studied by many authors [1, 16, 22, 27, 28] and can be understood combinatorially in terms of the matroid represented by the linear equations satisfied by these generators. Such an algebra is related to algebras associated to deletions and contractions of the matroid by a short exact sequence, and can also be written as the quotient of a polynomial algebra by certain powers of linear forms.

We study such algebras in the case that the matroid is regular, which we term *circulation algebras* following Wagner [27]. In addition to surveying the existing results on these algebras, we give a new proof of Wagner's result that the structure of the algebra determines the matroid, and construct an explicit basis in terms of basis activities in the matroid. We then consider *generalized circulation algebras* in which we mod out by a fixed power of each variable, not necessarily equal to 2. We show that such an algebra is isomorphic to the circulation algebra of a "subdivided" matroid, a variation on a result of Nenashev [16], and derive from this generalized versions of many of the results on ordinary circulation algebras, including our basis result. We also construct a family of short exact sequences generalizing the deletion-contraction decomposition.

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$\langle a_1, \ldots, a_k \rangle$	ideal generated by $a_1, \ldots, a_k$
$A[S]$	submatrix of A consisting of columns with indices in S, page 15
$\mathcal{B}(M)$	bases of the matroid M, page 6
$\beta_{B,e}$	basic flow of e with respect to B, page 15
$\mathbb{C}$	complex numbers
$\mathcal{C}(M)$	circuits of the matroid M, page 6
$\mathcal{C}^*(M)$	cocircuits of the matroid M, page 6
$C_{B,e}$	fundamental circuit of e with respect to B, page 8
$C^*_{B,e}$	fundamental cocircuit of e with respect to B, page 8
$\mathrm{EA}(B)$	externally active elements with respect to B, page 8
$\mathrm{EP}(B)$	externally passive elements with respect to B, page 8
$\mathrm{Flow}(M)$	rational flow space of the regular matroid M, page 14
$\leq_{\mathrm{G}}$	Gale ordering, page 6
$\mathrm{IA}(B)$	internally active elements of B, page 8
$\mathrm{IP}(B)$	internally passive elements of B, page 8
$M^{(\sigma)}$	σ-subdivision of the matroid M, page 36
$\mathbb{N}$	natural numbers (including zero)
ν	support functional, page 17
$\mathrm{null}(S)$	nullity of S, page 31
$P(V;t)$	Poincaré series of the graded algebra A, page 10
∂_e	partial derivative map $\Phi(M) \to \Phi(M \,/\, e)$, page 22

$\Phi(M)$	ordinary circulation algebra of the regular matroid M, page 20
$\Phi^{(\sigma)}(M)$	generalized circulation algebra of M with parameter σ, page 35
$\varphi_{B,S}$	element of the standard basis for an ordinary circulation algebra, page 27
$\tilde{\varphi}_{B,S}$	element of the nullity-homogeneous basis for an ordinary circulation algebra, page 33
$\mathbb{Q}$	rational numbers
$r_M(S)$	rank of the set S in the matroid M, page 6
$\mathbb{R}$	real numbers
$R(E)$	polynomial algebra in indeterminates indexed by E modulo squares, page 20
$R^{(\sigma)}(E)$	polynomial algebra in indeterminates indexed by E modulo powers given by σ, page 35
$\operatorname{Supp}\theta$	support of the function θ, page 16
$\operatorname{Sym} V$	symmetric algebra of the vector space V, page 10
$T(M; x, y)$	Tutte polynomial of the matroid M, page 7
$U_{r,m}$	uniform matroid of rank r on m elements
$V(d)$	dth twist of the graded module V, page 10
$\mathbb{Z}$	integers
$\tilde{Z}(M; q, \mathbf{v})$	multivariate Tutte polynomial of the matroid M, page 8

Chapter 1

Introduction

1.1 Background

A *flow* on a graph Γ is a function defined on the edge set $E(\Gamma)$ which satisfies a "conservation law" at each vertex. The study of flows has its origins in the theory of electrical networks: interpreting the values of the flow function as measuring electric current through the corresponding edges, the conservation constraint is precisely Kirchhoff's current law. In this guise, Kirchhoff himself developed much of the elementary theory of flows in [14]. Though phrased in terms of electricity, much of this work applies in a far more general context: for instance, while electric current is real-valued, the circuit laws are linear equations with integer coefficients and can thus be interpreted in any abelian group.

Flows on graphs have been studied in many different contexts. In physics, conservation laws appear in many contexts besides electrical current: for instance, we may think of the edges as paths through spacetime of particles moving at constant velocity, with the vertices as positions at which they interact. Then by conservation of momentum, the function assigning each edge the momentum vector of the corresponding particle is an $\mathbb{R}^4$-valued flow. This is essentially a simplified version of the idea of a *Feynman diagram*, and the *Feynman integrals* of quantum field theory can be written as integrals over flow spaces.

Of course, these concepts are not restricted to physics. Mathematically, flows can be studied from the perspective of algebraic topology: A-valued flows on Γ can be identified with the elements of the homology group $H_1(\Gamma; A)$. A consequence of this, together with the fact that the sphere is simply connected, is that for graphs embedded in the sphere, every flow can be obtained as the boundary of a 2-chain; in other words, a flow can be written as a difference in "potential" between the two incident faces. Planar duality, then, interchanges the roles of current and voltage in electrical networks.

This duality gives a connection to the map colouring problem: a flow which is nowhere zero implies the existence of a 2-chain in which no two adjacent faces have equal potential; in other words, a proper face colouring, with the elements of A as colours. In particular, the four-colour theorem is equivalent to the existence of flows on planar graphs valued in an abelian group of order 4. Stemming from this, flows valued in finite abelian groups are of great importance in structural graph theory, giving on one hand an algebraic perspective on face colourings of planar graphs and on the other, a notion of "face colouring" for non-planar graphs, which unlike the planar case is not equivalent to vertex colouring on a dual graph. See chapter 6 of Diestel's book [8] for an exposition of this perspective on flows.

The introduction of matroids by Whitney [30] allowed for vast generalizations of these ideas. A matroid can be associated to any system of linear equations whatsoever, and solutions to the system can still be loosely viewed as "flows" on the matroid. Surprisingly, much of Kirchhoff's work still applies in this context. This is especially true in the special case of *regular* matroids, a class which includes those derived from either of Kirchhoff's laws. For instance, regular matroids satisfy a natural generalization of Kirchhoff's matrix-tree theorem. Moreover, regular matroids have a unique representation property which ensures that, like graphs, all interesting algebraic properties of flows can be derived from the combinatorial properties of the matroid.

Matroid duality generalizes planar duality in precisely the correct way to maintain the duality between current and voltage. Flows on graphic matroids behave like current, while flows on cographic matroids behave like voltage. Kirchhoff's current and voltage laws are thus two examples of the same mathematical phenomenon: regular matroids and their flow spaces. Indeed, the decomposition theorem of Seymour [19, Theorem 13.1.1] says that all regular matroids can be built out of graphic and cographic matroids, plus a single other minimal example, so in this sense the theory of regular matroids is really just a mild generalization of the theory of electrical networks.

In [27], Wagner introduced the *circulation algebra* $\Phi(\Gamma)$ of a graph Γ as a tool to study flows. This is a graded commutative algebra with the property that the homogeneous elements of degree 1 can be identified with flows on Γ. The algebra can be described in several different ways:

- As dual to a certain coalgebra, the "Kirchhoff group" $K(\Gamma)$.
- As a subalgebra of the "squarefree algebra" $\mathbb{Q}[x_e \colon e \in E(\Gamma)] / \langle x_e^2 \colon e \in E(\Gamma)\rangle$ generated by the space of flows.
- As a quotient of the symmetric algebra of the flow space by certain powers of linear forms.

We will take the second formulation as primary. Wagner showed that for e an edge, $\Phi(\Gamma)$ is related to $\Phi(\Gamma \setminus e)$ and $\Phi(\Gamma \,/\, e)$ by a short exact sequence, and from this derived the equivalence of the second and third descriptions above. The short exact sequence also allows one to express the Poincaré polynomial of $\Phi(\Gamma)$ in terms of its Tutte polynomial:

$$P(\Phi(\Gamma); t) = t^{r(\Gamma)} T(\Gamma; t^{-1}, 1 + t).$$

(Here $r(\Gamma)$ denotes the rank of the associated graphic matroid, i.e. the number of vertices minus the number of connected components.)

Though initially described for graphs, circulation algebras – like flows – are really much more general. Viewing a matrix as defining a space of "flows" on the matroid it represents, we can view the corresponding subalgebra of the squarefree algebra as a circulation algebra of the corresponding matroid. In [28], Wagner studied arbitrary subalgebras of the squarefree algebra generated by degree-1 elements, and extended most of his results to this context. In particular, all such algebras have an exact sequence and corresponding Tutte polynomial formula, as well as a presentation as a quotient of a polynomial algebra by powers of linear forms.

These same ideas were also discovered independently in a very different context. Let B be the group of $n \times n$ complex upper triangular matrices with determinant 1, the *Borel subgroup* of the special linear group $\mathrm{SL}_n(\mathbb{C})$. The coset space $\mathrm{Fl}_n = \mathrm{SL}_n(\mathbb{C})/B$ is the *complete flag variety* (of type A) parametrizing flags

$$V_1 \subset \cdots \subset V_n$$

of subspaces of $\mathbb{C}^n$, with $\dim V_i = i$. A classical result gives a presentation for its cohomology ring:

$$H^\bullet(\mathrm{Fl}_n; \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_n] / \langle e_1(\mathbf{x}), \ldots, e_n(\mathbf{x}) \rangle$$

where $e_i(\mathbf{x})$ denotes the ith elementary symmetric polynomial. Shapiro and Shapiro [23] observed that there is a quite natural way to choose differential 2-forms $\omega_1, \ldots, \omega_n$ representing the cohomology classes corresponding to the indeterminates $x_1, \ldots, x_n$, and proceeded to study the ring generated by these forms, of which the cohomology is a quotient. Based on computations for small n, they conjectured that this algebra's graded dimension had a combinatorial interpretation related to forests.

This conjecture was confirmed by Postnikov, Shapiro, and Shapiro in [21] and extended by the same authors to flag varieties of other types in [22]. The key observation of this latter paper is simply that the forms ω_i can be expressed as complex linear combinations of another collection of 2-forms $\varphi_1, \ldots, \varphi_m$ which are linearly independent, pairwise commute, and square to zero. In other words, the algebra they are interested in is a subalgebra of a squarefree algebra, generated by degree-1 elements. The remainder of their results are applicable to any algebra of this kind, and overlap heavily with Wagner's: they show that any such algebra can be presented as a quotient of a polynomial algebra by powers of linear forms, and that its graded dimension can be expressed in terms of basis activity, a result equivalent to the Tutte polynomial formula. The connection to forests is explained by the fact that their algebra, in the original case of Fl_n, is generated by the *cut space* of the complete graph K_n. Thus it is a "cocirculation algebra", related to voltage in the same way that $\Phi(K_n)$ relates to current. Extending the notion of circulation algebra to regular matroids therefore encompasses this algebra as well.

1.2 Summary of this thesis

In Chapter 2 we review the background material from matroid theory and from commutative algebra that we will make use of.

In Chapter 3, we discuss the theory of flows on regular matroids. The main result is Theorem 3.7, which states that the structure of a regular matroid is uniquely determined by equipping the flow space with a "black box" that can determine the number of edges on which a flow is nonzero. This result is not entirely new, but we give a new proof which is more combinatorial than previous ones. The multiplicative structure of the circulation algebra gives an implementation of such a black box, motivating its study.

In Chapter 4 we detail the theory of circulation algebras of regular matroids. We first show (Theorem 4.2) that the isomorphism type of the algebra determines the matroid, which follows easily from the results of the previous chapter. We then study various structural properties of these algebras. These results are not new, being a special case of the general theory in [22] and [28]. However, we present it in a somewhat different way, closer to how the graphic version was handled in [27] and emphasizing the special properties of regular matroids. Highlights include the deletion-contraction exact sequence (Theorem 4.5) and presentation as a quotient by powers of linear forms (Theorem 4.7). The one totally new result of the chapter, Theorem 4.11, gives an explicit basis for the circulation algebra in terms of basis activities in the matroid.

Section 4.5, the final section of the chapter, details a filtration that can be placed on the algebra, adapting a result of Berget [3]. This gives a two-variable version of the Poincaré polynomial, which is essentially equivalent to the full Tutte polynomial. We also study quotients of the algebra by the ideals of the filtration, in particular one we call $\widetilde{\Phi}(M)$ which has a theory somewhat parallel to the circulation algebra.

In Chapter 5, we introduce "generalized circulation algebras", in which we replace the squarefree algebra with the quotient of $\mathbb{Q}[\mathbf{x}]$ by the monomials $x_e^{1+\sigma(e)}$ where the weight $\sigma(e)$ may be any natural number. We show this algebra is isomorphic to the circulation algebra of a "subdivided" matroid, a result which had previously appeared in the work of Nenashev [16]. However, the main results of this chapter are new and do not follow directly from this fact. Theorem 5.15 gives a family of exact sequences generalizing Theorem 4.5. From one of these we derive an expression for the graded dimension of generalized circulation algebras in terms of Sokal's multivariate Tutte polynomial. Theorem 5.9 generalizes Theorem 4.11 to give explicit bases for all generalized circulation algebras.

Finally, in Chapter 6 we discuss the conjecture of Wagner [27, 28] that the graded dimension of circulation algebras is logarithmically concave. Proving this was the original goal of this project, but unfortunately very little progress was made. We give overviews of two potential approaches to solving this problem.

1.3 Related work

Before beginning, we mention some other related work. As previously discussed, the results of [22] and [28] show that any subalgebra of the squarefree algebra generated by linear forms can be seen as a "circulation algebra", and in particular has a presentation as a quotient of a polynomial by an ideal generated by powers of linear forms. Such an ideal is called a *power ideal*. A natural question is to what extent the theory of circulation algebras can be extended more generally to algebras with such a presentation. While *arbitrary* power ideals are likely too general, Ardila and Postnikov [1] were able to do this for a larger class of power ideals associated to (represented) matroids, showing that the associated algebras – or rather their duals – satisfy a deletion-contraction exact sequence generalizing the one for circulation algebras. Their work can be thought of as generalizing the power ideal description of circulation algebras in a way analogous to how we generalize the squarefree algebra description.

One particularly interesting algebra that falls into the above class is the one considered by Postnikov and Shapiro in [20]. Analogously to the multiple descriptions of the circulation algebra, this algebra can described, in addition to the power ideal presentation, as the subalgebra generated by the cut space of a graph in the quotient of the squarefree algebra by those squarefree monomials corresponding to cuts of the graph. This is precisely the algebra $\widetilde{\Phi}(M)$ for M the cographic matroid. However, the main results of that paper involve relating the algebra to a different algebra which is more specifically "graphic" and thus do not seem likely to generalize to matroids.

The final section of Wagner's paper [27] deals with another interesting topic related to flows, namely the lattice of integer-valued flows within the euclidean space of real-valued flows. Like the circulation algebra, this naturally generalizes to regular matroids. Su and Wagner [25] proved that a regular matroid can be reconstructed from its lattice of integer flows, a result which can be seen as a cousin of our Theorems 3.7 and 4.2. Dancso and Garoufalidis [7] recently gave another proof which is algorithmic in nature (stated only for graphic matroids, but plausibly generalizable). Their methods resemble our proof of Theorem 3.7, though there does not appear to be a direct connection.

Finally, we mention that there are many other algebra-valued matroid invariants that do not appear to be part of the circulation algebra "story". Some of these, nonetheless, share similar properties. Perhaps the best-known is the *Orlik–Solomon algebra* $\mathrm{OS}(M)$, introduced by Orlik and Solomon in [17]. It can be defined combinatorially as the quotient of the exterior algebra with a generator for each element of the ground set by the "boundaries" of dependent sets in the matroid. The Orlik–Solomon algebra has a deletion-contraction exact sequence which is formally identical to that for the circulation algebra, giving its graded dimension the same recurrence relation but with different initial conditions. The corresponding Tutte polynomial formula is

$$P(\mathrm{OS}(M);t) = t^{r(M)}T(M;1+t^{-1},0)$$

which since the second variable is set to zero is usually thought of as a transformation of the *characteristic polynomial* $\chi(M;t) = (-1)^{r(M)}T(M;1-t,0)$. For matroids represented over $\mathbb{C}$, the Orlik–Solomon algebra is isomorphic to the cohomology of the complement of the corresponding arrangement of hyperplanes.

Chapter 2

Preliminaries

2.1 Matroids

We assume basic familiarity with matroids; see Oxley's book [19] for a reference. We review some basic results and notation in this section, but no attempt is made at completeness.

Matroids can be defined in many ways. The most convenient for us will be the following: a matroid is a finite set E (the *ground set*) equipped with a collection $\mathcal{C}(M)$ of subsets called *circuits*, such that

1. The empty set is not a circuit.

2. No circuit properly contains another.

3. If C_1 and C_2 are distinct circuits and $e \in C_1 \cap C_2$, then $(C_1 \cup C_2) \setminus e$ contains a circuit.

A subset of E is said to be *dependent* if it contains a circuit, and *independent* otherwise. A maximal independent set is a *basis*. A set which has a nonempty intersection with every basis is *codependent*; a minimal codependent set is a *cocircuit*. The sets of all bases and co circuits of M are denoted $\mathcal{B}(M)$ and $\mathcal{C}^*(M)$ respectively. The *rank* of a set $S \subseteq E$ is the size of the largest independent set contained in S. This is denoted $r_M(S)$ or just $r(S)$ when there is no chance of confusion. We will also write $r(M)$ for $r_M(E)$.

Another characterization of matroids involves a certain partial order on sets. Let E be a totally ordered finite set and $1 \leq k \leq |E|$. The *Gale order* on k-subsets of E is given as follows: let

$$S = \{s_1 < \cdots < s_k\}$$

and

$$T = \{t_1 < \cdots < t_k\}$$

be subsets of E. Then write $S \leq_{\mathrm{G}} T$ if $s_i \leq t_i$ for $1 \leq i \leq k$.

Proposition 2.1 (Gale [10]). *Let E be a finite set, and $\mathcal{A}$ a collection of subsets of E. The following are equivalent:*

- *For every total ordering on E, there exists a unique maximal element of $\mathcal{A}$ with respect to Gale order.*

- *There exists a matroid M on ground set E such that $\mathcal{A} = \mathcal{B}(M)$.*

This is essentially equivalent to the *greedy algorithm* characterization discussed in [19, Section 1.8]. In particular, the Gale-maximal basis can be found as follows:

- Start with $X = \emptyset$.
- Let $(e_1, \ldots, e_m)$ be the elements of E listed in descending order.
- For i from 1 to m: if $X \cup \{e_i\}$ is independent, set $X = X \cup \{e_i\}$.

At the end of the loop, X is the Gale-largest basis (see [19, Lemma 1.8.3]).

2.2 The Tutte polynomial

The *Tutte polynomial* $T(M; x, y)$ of a matroid M of is a polynomial invariant introduced by Crapo [6] generalizing a construction of Tutte [26] for graphs. It is given by

$$T(M; x, y) = \sum_{S \subseteq E} (x-1)^{r(M) - r_M(S)} (y-1)^{|S| - r_M(S)} \tag{2.1}$$

where E is the ground set. We refer to [5] for the basic theory of this polynomial.

Let R be a commutative ring and f be an R-valued matroid invariant; that is, a function that assigns each matroid an element of R, such that isomorphic matroids are assigned the same element. We say f is a *Tutte–Grothendieck invariant* if it for any matroids M and N we have

$$f(M \oplus N) = f(M) f(N) \tag{2.2}$$

and for any matroid M and element e which is not a loop or coloop,

$$f(M) = f(M \setminus e) + f(M \,/\, e). \tag{2.3}$$

Lemma 2.2. *The Tutte polynomial is a Tutte–Grothendieck invariant.*

Proof. See [5, Lemma 6.2.1]. □

More generally, suppose $a, b \in R$ are not zero-divisors. We say f is a *generalized Tutte–Grothendieck invariant* with coefficients a and b if it satisfies (2.2), and for e not a loop or coloop,

$$f(M) = a f(M \setminus e) + b f(M \,/\, e). \tag{2.4}$$

The Tutte polynomial is the *universal* Tutte–Grothendieck invariant in the following sense.

Proposition 2.3. *Let f be a generalized Tutte–Grothendieck invariant satisfying (2.2) and (2.4). Then for any matroid M on ground set E,*

$$f(M) = a^{|E| - r(M)} b^{r(M)} T(M; b^{-1} f(U_{1,1}), a^{-1} f(U_{0,1})).$$

Proof. See [5, Corollary 6.2.6]. □

Another formula for the Tutte polynomial is given in terms of *basis activity*. Let E be a totally ordered set and M be a matroid on ground set E. For a basis B of M and an element $e \in B$, we define the *fundamental cocircuit* of e with respect to B to be the unique cocircuit $C^*_{B,e}$ such that $C^*_{B,e} \cap B = \{e\}$. Dually, for $e \notin B$ the *fundamental circuit* of e with respect to B is the unique circuit $C_{B,e}$ contained in $B \cup \{e\}$.

For an element $e \in E$, we say that e is

- *internally active* (with respect to B) if $e \in B$ and e is the smallest element in $C^*_{B,e}$
- *internally passive* if $e \in B$ and e is not internally active,
- *externally active* if $e \notin B$ and e is the smallest element in $C_{B,e}$, and
- *externally passive* if $e \notin B$ and e is not externally active.

The sets of internally active, internally passive, externally active, and externally passive elements with respect to B are denoted $\mathrm{IA}(B)$, $\mathrm{IP}(B)$, $\mathrm{EA}(B)$, and $\mathrm{EP}(B)$ respectively.

Basis activity gives a way to decompose the collection of all subsets of E into "structureless intervals", which contain exactly one basis of M.

Proposition 2.4 (Crapo [6, Proposition 12]). *Let M be a matroid on totally ordered ground set E. For any $S \subseteq E$, there is a unique basis B of M such that* $\mathrm{IP}(B) \subseteq S \subseteq B \cup \mathrm{EA}(B)$.

From this we can derive the *activity formula* for the Tutte polynomial. This is essentially given by rewriting (2.1) as a sum over bases using Proposition 2.4.

Proposition 2.5 (Tutte [26], Crapo [6]). *For any matroid M and any total order on its ground set,*

$$T(M; x, y) = \sum_B x^{|\mathrm{IA}(B)|} y^{|\mathrm{EA}(B)|}.$$

2.3 The multivariate Tutte polynomial

Let E be a finite set and let $\mathbf{v} = (v_e)_{e\in E}$ be a sequence of indeterminates. Introduced by Sokal in [24], the *multivariate Tutte polynomial* of a matroid M on E is given by

$$\tilde{Z}(M; q, \mathbf{v}) = \sum_{S\subseteq E} q^{-r_M(S)} \mathbf{v}^S.$$

Two remarks are in order. The first is that despite the name, $\tilde{Z}(M; q, \mathbf{v})$ is not a many-variable version of the Tutte polynomial in the most obvious way. Rather, setting all variables to the same value v gives

$$\tilde{Z}(M; q, v) = (q^{-1}v)^{r(M)} T(M; 1 + qv^{-1}, 1 + v)$$

as is easily seen from (2.1). Thus the multivariate Tutte polynomial is really a many-variable version of the polynomial on the right. The ordinary Tutte polynomial can be recovered as

$$T(M; x, y) = (x - 1)^{r(M)} \tilde{Z}(M; (x - 1)(y - 1), y - 1).$$

The second remark is that since a matroid is determined by its rank function, the multivariate Tutte polynomial of a matroid is really just an algebraic encoding of its entire combinatorial structure. Thus, in principle, any property of M can be recovered from $\tilde{Z}(M; q, \mathbf{v})$. However, it is still an interesting question to ask which properties can be computed from it algebraically by making substitutions for the variables.

Like the Tutte polynomial, the multivariate Tutte polynomial has a deletion-contraction formula.

Proposition 2.6. *For M a matroid and any element e which is not a loop, the multivariate Tutte polynomial satisfies the recurrence*

$$\tilde{Z}(M; q, \mathbf{v}) = \tilde{Z}(M \setminus e; q, \mathbf{v}) + q^{-1} v_e \tilde{Z}(M / e; q, \mathbf{v}).$$

Proof. See [24, Section 4.3]. □

The base cases of this recurrence can easily be computed from (2.3):

$$\tilde{Z}(U_{0,1}; q, v_e) = 1 + v_e$$

and

$$\tilde{Z}(U_{1,1}; q, v_e) = 1 + q^{-1} v_e.$$

Proposition 2.7. *For matroids M and N,*

$$\tilde{Z}(M \oplus N; q, \mathbf{v}) = \tilde{Z}(M; q, \mathbf{v}) \tilde{Z}(N; q, \mathbf{v}).$$

Proof. Immediate from the definition. □

2.4 Graded and filtered algebras

Throughout this thesis, the noun *algebra* without qualifiers will always refer to a unital, associative, commutative algebra over a field. A *graded algebra* is an algebra A with a decomposition

$$A = \bigoplus_{i \in \mathbb{N}} A_i$$

into finite-dimensional linear subspaces, such that $A_i A_j \subseteq A_{i+j}$. For a concise and self-contained introduction to the subject of graded algebras, chapter 4 of Hibi's book [11] is highly recommended. (Of course, any standard commutative algebra book, such as Eisenbud [9], will also contain the relevant results, implicitly or explicitly.) We will on occasion refer to standard results without proof; in this section we simply establish some notation.

The elements of A_i are said to be *homogeneous of degree i*. A subalgebra generated by homogeneous elements is a *graded subalgebra*; an ideal generated by homogeneous elements is a *homogeneous ideal*. A graded subalgebra is indeed a graded algebra in its own right, as is the quotient by a graded ideal. For any fixed $n \in \mathbb{N}$ let

$$A_{\geq n} = \bigoplus_{i \geq n} A_i.$$

This is a homogeneous ideal in A. We will also write A_+ for $A_{\geq 1}$; this is known as the *positive ideal* or *irrelevant ideal*.

A *graded A-module* is an A-module V equipped with a decomposition

$$V = \bigoplus_{j \in \mathbb{Z}} V_j$$

into finite-dimensional linear subspaces, such that $A_i V_j \subseteq V_{i+j}$. Naturally, A itself is a graded A-module, as is any homogeneous ideal. For V a graded module and $d \in \mathbb{Z}$, the dth *twist*, denoted $V(d)$, has the same underlying module, and grading given by

$$V(d)_n = V_{n+d}.$$

If U and V are graded A modules, an A-linear map $P \colon U \to V$ is said to be *grade-preserving*, or a *map of graded modules*, if $P(U_i) \subseteq V_i$. A map $U \to V$ which does not preserve degree but instead shifts it by a constant amount d is the same as a grade-preserving map $U \to V(d)$.

The *Poincaré series* or *Hilbert series* of a graded algebra is the space A is

$$P(A;t) = \sum_{i \in \mathbb{N}} \dim(A_i) t^i.$$

When A is finite-dimensional this is a polynomial, and we call it the *Poincaré polynomial.*[1] Naturally, we may extend this notion to graded modules, in which case it is in general a *Laurent* series or polynomial since we allow $\mathbb{Z}$-grading. Clearly, we have

$$P(V(d);t) = t^{-d} P(V;t).$$

An important example of a graded algebra is the symmetric algebra of a finite-dimensional vector space V. To define this, we first construct a graded vector space $T(V)$ from the tensor powers of V:

$$T(V) = \bigoplus_{i \in \mathbb{N}} V^{\otimes i}.$$

This has a *noncommutative* multiplication given by "removing brackets", i.e.

$$(u_1 \otimes \cdots \otimes u_i)(v_1 \otimes \cdots \otimes v_j) = u_1 \otimes \cdots \otimes u_i \otimes v_1 \otimes \cdots \otimes v_j$$

extended linearly. The *symmetric algebra* or *free commutative algebra*, denoted $\operatorname{Sym} V$, is the quotient of $T(V)$ given by forcing this multiplication to be commutative, i.e. we mod out by the two-sided ideal generated by all degree-2 elements of the form $u \otimes v - v \otimes u$. Since the generators of this ideal are homogeneous of degree 2, $\operatorname{Sym} V$ is a graded algebra and its degree-1 piece is naturally identified with V. If V is the vector space spanned by the indeterminates $x_1, \dots, x_m$ then $\operatorname{Sym} V$ is the polynomial ring in m variables. Of course, any m-dimensional vector space is isomorphic to any other, so $\operatorname{Sym} V$ is always isomorphic to a polynomial algebra, but the former does not require choosing coordinates. For more details on symmetric algebras see [9, Section A2.3].

[1] To make life interesting, the *Hilbert polynomial* is something different.

We will in one section make use of the notion of a *filtered algebra*; this is an algebra A equipped with a descending chain

$$A = I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$$

of ideals (the *filtration*), such that $I_i I_j \subseteq I_{i+j}$, and the quotients I_{i+1}/I_i are finite-dimensional. If A is graded, we can take $I_i = A_{\geq i}$. The *Poincaré series* of a filtered algebra A is

$$P(A;t) = \sum_{i \in \mathbb{N}} \dim(I_i/I_{i+1}) t^i$$

which generalizes the notion for graded algebras.

2.5 Exact sequences

A sequence of vector spaces and linear maps[2]

$$V_1 \xrightarrow{P_1} V_2 \xrightarrow{P_2} \cdots \xrightarrow{P_{k-1}} V_k$$

is said to be *exact* if the image of P_i equals the kernel of P_{i+1}, for each i. A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow U \xrightarrow{P} V \xrightarrow{Q} W \longrightarrow 0.$$

The exactness condition in the short case amounts to saying the map P is injective, the map Q is surjective, and $QP = 0$. The first isomorphism theorem for vector spaces then says that Q descends to an isomorphism $V/P(U) \xrightarrow{\sim} W$.

The vector spaces in our exact sequences will often be (graded) modules over some (graded) algebra, with the maps preserving this structure. We will simply refer to an "exact sequence of graded A-modules" or similar to mean this.

Homological algebra has many things to say about exact sequences (see Mac Lane's book [15]) but the relevant part to us will boil down to the following classic pair of diagram lemmas.

Lemma 2.8 (Short Five Lemma)**.** *Suppose the diagram*

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r] \arrow[d, "P"] & V \arrow[r] \arrow[d, "Q"] & W \arrow[r] \arrow[d, "R"] & 0 \\
0 \arrow[r] & U' \arrow[r] & V' \arrow[r] & W' \arrow[r] & 0
\end{tikzcd}$$

commutes, and both rows are exact. If P and R are isomorphisms, then so is Q.

Proof. See [15, Lemma 3.1]. □

[2] Of course, these results are true in more generality than this.

Lemma 2.9 (Nine Lemma)**.** *Suppose the diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & U' \arrow[r] \arrow[d] & V' \arrow[r] \arrow[d] & W' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U'' \arrow[r] \arrow[d] & V'' \arrow[r] \arrow[d] & W'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

commutes. If all three columns and the middle row are exact, then the first row is exact if and only if the last row is exact.

Proof. See [15, Lemma 5.1]. □

Chapter 3

Flows on Regular Matroids

3.1 Regular matroids

An integer matrix A is *totally unimodular* if the determinant of every square submatrix lies in $\{0, 1, -1\}$. (In particular, this includes the entries of the matrix.) A matroid is *regular* if it can be represented in characteristic 0 by a totally unimodular matrix. Since these subdeterminants are nonzero modulo any prime, it follows that the same matrix represents the matroid over any field of any characteristic. Indeed, a well-known result [19, Theorem 6.6.3] characterizes regular matroids as precisely those that are representable over every field.

Let A and A' be matrices over some field K, with columns labelled by a finite set E. We say A and A' are *row-equivalent* if their row spaces are equal, and *projectively equivalent* if their row spaces are related by a diagonal automorphism of K^E. Projectively equivalent matrices represent the same matroid, but in general the converse need not hold.

Proposition 3.1. *Let K be a field and M be a matroid. If M is regular, all matrices representing M over K are projectively equivalent.*

Proof. Follows from [19, Proposition 6.6.5]. □

This property is usually referred to as "unique representability", though it is worth noting that over any field larger than $\mathbb{F}_2$, there is no *canonical* choice of diagonal automorphism to witness the projective equivalence.

If M is represented by A and e is an element of the ground set, the matroid $M \setminus e$ is represented by the matrix obtained by deleting column e of M, and the matroid M / e is obtained by pivoting on some entry in column e (i.e. applying the "obvious" row operations to make that the only nonzero entry) and then deleting both the column and the corresponding row. We will refer to these as the *canonical representations*[1] of $M \setminus e$ and M / e respectively, and will always assume that these are the representations used for them.

It is clear that deleting a row or column preserves total unimodularity, and by it can be shown [19, Lemma 2.2.20] that pivoting does as well. Thus if we begin with a totally unimodular representation, all canonical representations of minors will be totally unimodular as well.

[1] In the case of the contraction, it is only "canonical" up to row-equivalence, but this will be good enough since we really only care about the row space and nullspace rather than the entries.

The signed incidence matrix of an oriented graph is totally unimodular. Thus all graphic matroids, and by duality all cographic matroids, are regular. Not all regular matroids fall into one of these classes; the matroid R_{10} represented by the totally unimodular matrix

$$A_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}$$

is a counterexample of smallest possible size. (See [19, Section 6.6] for details.)

3.2 Flows

Let M be a regular matroid on ground set E of size m. Let A be a $k \times m$ totally unimodular integer matrix (with columns indexed by E) representing M. Then we can think of A as a linear map $\mathbb{Q}^E \to \mathbb{Q}^k$. The *(rational) flow space*[2] of M, which we denote as $\mathrm{Flow}(M)$, is the kernel of this map. We refer to its elements as *flows*. These are $\mathbb{Q}$-valued functions on E satisfying a system of linear equations which we can think of as the "circuit law" of M.

The flow space, of course, strictly speaking depends on the representing matrix A, not just the matroid M. However, the unique representation property of regular matroids means we do not get into too much trouble by thinking of it as a matroid invariant: a diagonal automorphism of $\mathbb{Q}^E$ which preserves the row space will also preserve the kernel, and all properties we care about are invariant under such automorphisms.

Example 3.1. Suppose M is the cycle matroid of a graph Γ. In this case, we can choose A as the signed incidence matrix of Γ with respect to some orientation, and flows on M are flows on Γ in the usual graph-theoretic sense. The circuit law of the cycle matroid is *Kirchhoff's current law* for an electrical network with underlying graph Γ.

Recall that when a matrix A represents a matroid M, a representation for the dual matroid M^* is given by choosing any matrix A' with the property that the row space of A' is the kernel of A or equivalently, the kernel of A' is the row space of A. Thus $\mathrm{Flow}(M^*)$ is simply the orthogonal complement of $\mathrm{Flow}(M)$, or equivalently the row space of the matrix A.

Example 3.2. Continuing from the previous example, the dual M^* is the bond matroid of Γ. Flows on M^* are functions on the edges which are linear combinations of the rows of the incidence matrix. These are *coboundaries* on Γ in the sense of algebraic topology.

The circuit law of the bond matroid is *Kirchhoff's voltage law* for an electrical network with underlying graph Γ. As in the electrical case, an element $\theta \in \mathrm{Flow}(M^*)$ can be represented as the "difference in potential" between its two endpoints, where the potential of a vertex is simply the coefficient on the corresponding row of the incidence matrix in the expansion of θ. This is not uniquely defined, since the rows of the incidence matrix are not linearly independent, reflecting the fact (familiar from physics) that electric potential is defined only relative to an arbitrarily chosen "ground" reference point.

[2] Also known as the rational cycle space.

Thus the study of flows is interesting for both graphic and cographic matroids, making regular matroids a natural setting in which to study them.

Proposition 3.2. *Let A be a matrix with columns indexed by E and M the matroid it represents. The minimal subsets of E supported by nonzero elements of* $\operatorname{Ker} A$ *are precisely the circuits of M.*

Proof. See [19, Proposition 9.2.4], which is equivalent by duality to our statement. □

The space of flows supported on a *particular* circuit is 1-dimensional: if we could find two linearly independent flows supported on the same circuit we could find a linear combination supported on a proper subset, contradicting minimality. In fact, since the matroid is regular we can do slightly better than this.

Proposition 3.3. *For any circuit C of M, there exists a flow ζ supported on C such that $\zeta(e) = \pm 1$ for all $e \in C$.*

Proof. Choose $e_0 \in C$ arbitrarily. Let ζ be the unique flow supported on C such that $\zeta(e_0) = 1$.

The set $C \setminus \{e_0\}$ is independent, so there exists a basis B containing it. For $S \subseteq E$ write $A[S]$ for the submatrix of A with all rows and only the columns indexed by S. The restriction of ζ is the unique solution in $\mathbb{Q}^{B\cup\{e_0\}}$ to the system of linear equations

$$\begin{aligned} A[B \cup \{e_0\}]\zeta &= 0 \\ \zeta(e_0) &= 1 \end{aligned}$$

and hence by Cramer's rule, for $e \in C$ we have

$$\zeta(e) = \frac{\det A[B \cup \{e_0\} \setminus \{e\}]}{\det A[B]}$$

which is ± 1 by the total unimodularity of A. □

We will call a flow supported on a circuit which takes only the values ± 1 on that circuit *simple*. Clearly there are exactly two simple flows supported on any given circuit, which are negatives of one another.

For a basis B of M and an element $e \notin B$, the *basic flow* of e with respect to B is the unique (simple) flow $\beta_{B,e}$ supported on the fundamental circuit $C_{B,e}$ such that $\beta_{B,e}(e) = 1$. These flows can be used to construct a basis for $\operatorname{Flow}(M)$. This result is classical, the graphic case appearing in Kirchhoff's work on electrical networks and the general case in Whitney's original paper [30] introducing matroids.[3]

Proposition 3.4. *Let M be a regular matroid and B a basis of M. The flows $\beta_{B,e}$ for $e \notin B$ form a basis for* $\operatorname{Flow}(M)$.

Proof. By the rank-nullity equation, $\operatorname{Flow}(M)$ has dimension $|E| - r(M) = |E \setminus B|$. For $e, f \notin B$ we have $\beta_{B,e}(f) = \delta_{e,f}$ by construction, so these flows are linearly independent. Thus they also span the space. □

[3] It is obtained by combining his theorems 9 and 29.

For an element E, we can think of a function on $E \setminus e$ as a function on E which vanishes when evaluated at e; this gives an inclusion $\mathbb{Q}^{E\setminus e} \hookrightarrow \mathbb{Q}^E$. On the other hand, we can also take a function on E and restrict it to $E \setminus e$, which gives a projection map $\mathbb{Q}^E \twoheadrightarrow \mathbb{Q}^{E\setminus e}$. These operations correspond to deletion and contraction.

Proposition 3.5. *Let e be an element of M.*

(i) The inclusion $\mathbb{Q}^{E\setminus e} \hookrightarrow \mathbb{Q}^E$ maps $\mathrm{Flow}(M \setminus e)$ into $\mathrm{Flow}(M)$. This is an isomorphism if and only if e is a coloop.

(ii) The projection $\mathbb{Q}^E \twoheadrightarrow \mathbb{Q}^{E\setminus e}$ maps $\mathrm{Flow}(M)$ surjectively onto $\mathrm{Flow}(M \,/\, e)$. This is an isomorphism if and only if e is not a loop.

Proof. Let A' be the canonical representation of $M \setminus e$ obtained by deleting column e. For $\theta \in \mathbb{Q}^{E\setminus e}$, we clearly have $A\theta = A'\theta$, since column e simply does not contribute. In particular the one vanishes if and only if the other does. The map is an isomorphism precisely when every flow vanishes at e; by Proposition 3.2 this is the same as e being a coloop.

Let A'' be the canonical representation of $M \,/\, e$. Suppose $\theta \in \mathrm{Flow}(M)$ and let $\overline{\theta}$ be the image in $\mathbb{Q}^{E\setminus e}$. The construction of A'' makes it clear that $A''\overline{\theta} = 0$, as $A\xi = 0$ and the deleted column contributes only to the value in the deleted row. So the image of $\mathrm{Flow}(M)$ lies in $\mathrm{Flow}(M/e)$. Now suppose $\xi \in \mathrm{Flow}(M/e)$. Let $r \in \mathbb{Q}^E$ be the row which was deleted to obtain A''; then we can define

$$\theta(f) = \begin{cases} \xi(f), & e \neq f \\ -\frac{1}{r(e)} \sum_{g\in E\setminus e} r(g)\xi(g), & e = f \end{cases}$$

and it is clear that this is a flow. Thus the map is surjective.

Finally, we have

$$\begin{aligned} \dim \mathrm{Flow}(M \,/\, e) &= |E \setminus e| - r(M \setminus e) \\ &= m - 1 - r(M) + r_M(\{e\}) \\ &= \dim \mathrm{Flow}(M \,/\, e) + r_M(\{e\}) - 1 \end{aligned}$$

so the dimensions match if and only if e is not a loop. □

3.3 Reconstructing a matroid from its space of flows

Proposition 3.2 shows exactly how the structure of the matroid M is encoded in the structure of $\mathrm{Flow}(M)$ as a space of functions on E. On the other hand, clearly $\mathrm{Flow}(M)$ as an abstract vector space does not determine M, since the only invariant of a vector space is its dimension. The next question we address is: *how much* extra structure on $\mathrm{Flow}(M)$ is necessary to reconstruct M?

We denote the support of a function θ by $\mathrm{Supp}\,\theta$, and write

$$\nu(\theta) = |\,\mathrm{Supp}\,\theta|.$$

The (nonlinear) map $\nu\colon \mathrm{Flow}(M) \to \mathbb{Q}$ will be referred to as the *support functional*. It is not obviously clear that ν carries much useful information, but in fact, we will see that it is indeed sufficient to reconstruct the matroid M provided it has no coloops. The fact that the support functional determines the matroid up to isomorphism is not a new result: it is implicit in the proof of [28, Theorem 2.9]. However we give a new proof which is somewhat more constructive.

Lemma 3.6. *Suppose $\theta, \xi \in \mathbb{Q}^E$. For all $q \in \mathbb{Q}$,*

$$\mathrm{Supp}(q\theta + \xi) \subseteq \mathrm{Supp}\,\theta \cup \mathrm{Supp}\,\xi$$

with equality holding for all but finitely many values of q.

Proof. We clearly have

$$\mathrm{Supp}(q\theta + \xi) \subseteq \mathrm{Supp}\,\theta \cup \mathrm{Supp}\,\xi$$

for any q, since if $\theta(e) = \xi(e) = 0$ we certainly have $q\theta(e) + \xi(e) = 0$. Moreover, if $e \in \mathrm{Supp}\,\xi$ but $e \notin \mathrm{Supp}\,\theta$, then we certainly have $e \in \mathrm{Supp}(q\theta + \xi)$ since no cancellation is possible. Finally, for $e \in \mathrm{Supp}\,\theta$, we have $e \in \mathrm{Supp}(q\theta + \xi)$ unless

$$q = -\frac{\xi(e)}{\theta(e)}$$

so there are at most $\nu(\theta)$ values of q for which $\mathrm{Supp}(q\theta + \xi) \neq \mathrm{Supp}\,\theta \cup \mathrm{Supp}\,\xi$. □

A consequence of this is that any set which is a union of circuits is the support of some flow. This also appears as [19, Corollary 9.2.5]. We can now prove the reconstruction result.

Theorem 3.7. *A regular matroid M without coloops can be reconstructed from the data of the vector space $\mathrm{Flow}(M)$ equipped with the support functional ν.*

Proof. Define a relation $\prec$ on $\mathrm{Flow}(M)$ as follows: $\theta \prec \xi$ if $\nu(q\theta + \xi) \leq \nu(\xi)$ for all $q \in \mathbb{Q}$. We make the following observation: if θ and ξ are flows, and if $\mathrm{Supp}\,\theta \subseteq \mathrm{Supp}\,\xi$, then $\mathrm{Supp}(q\theta + \xi) \subseteq \mathrm{Supp}\,\xi$ for all q, so $\theta \prec \xi$. On the other hand, by Lemma 3.6, if $\theta \prec \xi$ then

$$\begin{aligned} |\,\mathrm{Supp}\,\theta \cup \mathrm{Supp}\,\xi| &= \nu(q\theta + \xi) && \text{for some } q \\ &\leq \nu(\xi) \\ &= |\,\mathrm{Supp}\,\xi| \end{aligned}$$

which is possible only if $\mathrm{Supp}\,\theta \subseteq \mathrm{Supp}\,\xi$. Thus this relation, defined only in terms of ν, encodes the inclusions between supports of flows. An immediate consequence is that $\prec$ is a reflexive and transitive relation. Write $\theta \sim \xi$ if $\theta \prec \xi$ and $\xi \prec \theta$; this is an equivalence relation, and $\prec$ descends to a partial order on the set of equivalence classes. Denote this poset by P. By the preceding argument, P is isomorphic to the poset of subsets of E which are unions of circuits. In particular, P has a unique maximal element, corresponding to the union of all circuits; since M has no coloops this is the full ground set E. Thus we have

$$|E| = \nu(\theta)$$

for any θ in this maximal equivalence class.

The poset P contains much information about the matroid structure. Recall that a subset of E is co-closed (i.e. closed in M^*) if and only if it can be written as an intersection of complements of circuits. Thus the opposite poset P^{op} is isomorphic to the lattice of flats of M^*. If M lacks not only coloops but also "coparallel elements" (i.e. cocircuits of size 2), then M^* is simple and hence can be reconstructed from its lattice of flats. (See [19, Section 1.7].) Of course, M and M^* determine one another. So under this additional hypothesis we can reconstruct M from the poset P alone.

Thus it remains to show that we can recover the sizes of coparallel classes. These correspond to the minimal nonempty elements of P^{op}, equivalently the maximal nonfull elements of P. Since the correspondence is by taking complements, it follows that the size of this coparallel class is precisely $|E| - \nu(\xi)$ where ξ is any flow in the corresponding equivalence class. □

Nothing about this argument was really special to regular matroids; indeed, what we really showed is that a matroid can be reconstructed from the kernel of any representing matrix over $\mathbb{Q}$ (or, indeed, any infinite field) as a vector space equipped with the support functional. What is special about regular matroids is that unique representability allows us to state this as an "if and only if" result: diagonal automorphisms of $\mathbb{Q}^E$ preserve the support of any function, and hence preserve ν. We summarize this as follows.

Corollary 3.8. *Let M and M' be regular matroids without coloops. Then $M \cong M'$ if and only if there exists a ν-preserving linear isomorphism* $\mathrm{Flow}(M) \xrightarrow{\sim} \mathrm{Flow}(M')$.

Corollary 3.8 does not nicely generalize to non-regular matroids. For instance, consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

and

$$A' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

which both represent $U_{2,4}$ over $\mathbb{Q}$. These representations are inequivalent, and indeed, we have the following.

Proposition 3.9. *There is no ν-preserving isomorphism between* $\operatorname{Ker} A$ *and* $\operatorname{Ker} A'$.

Proof. Write e_i for the ith standard basis vector of $\mathbb{Q}^4$ and let

$$\begin{aligned} \theta &= e_1 + e_2 - e_3, \\ \xi &= e_1 - e_2 - e_4, \\ \xi' &= e_1 + 3e_2 - e_4. \end{aligned}$$

Then $\operatorname{Ker} A$ is spanned by θ and ξ, and $\operatorname{Ker} A'$ is spanned by θ and ξ'. Suppose $P\colon \operatorname{Ker} A \to \operatorname{Ker} A'$ is a linear isomorphism. If P is to preserve ν, we must have

$$\nu(P\theta) = \nu(P\xi) = \nu(P(\theta + \xi)) = \nu(P(\theta - \xi)) = 3.$$

Up to scaling, the set of vectors of support size 3 in $\operatorname{Ker} A'$ is $\{\theta, \xi', \theta - \xi', 3\theta - \xi'\}$. Without loss of generality $P\theta$ is exactly equal to one of these four, and $P\xi$ is a scalar multiple of one of the other three. Suppose $P\theta = \theta$ and $P\xi = q\xi'$. If P is ν-preserving we must have that $P(\theta + \xi)$ and $P(\theta - \xi)$ are scalar multiples of the other two. Replacing q with its negative will swap the two, so we may assume

$$\theta + q\xi' = r_1(\theta - \xi')$$

and

$$\theta - q\xi' = r_2(3\theta - \xi')$$

for some scalars r_1 and r_2. Since θ and ξ' are linearly independent, we must have $r_1 = 1$ and $r_2 = 1/3$. Then solving the former equation gives $q = -1$ and the latter gives $q = 1/3$, so P cannot be ν-preserving. The other eleven cases can be ruled out by the same method; we omit the details. □

Chapter 4

Ordinary Circulation Algebras

4.1 Introduction

Theorem 3.7 gives a nice answer to the question of how much structure on the flow space is necessary to encode the matroid structure, but has the deficiency that the support functional is a somewhat strange invariant, most obviously seen in its nonlinear nature. We thus seek a more "natural" object that encodes the same information. The circulation algebra is just such a thing.

Let $R(E)$ denote the quotient of the polynomial algebra $\mathbb{Q}[x_e \colon e \in E]$ by the squares of the indeterminates. As a quotient by a homogeneous ideal, $R(E)$ is a graded $\mathbb{Q}$-algebra. We naturally identify an element $\theta \in \mathbb{Q}^E$ with the linear form

$$\sum_{e \in E} \theta(e) x_e$$

and hence can think of $\mathrm{Flow}(M)$ as a linear subspace of the degree-1 piece $R_1(E)$. The *(ordinary) circulation algebra* $\Phi(M)$ is the (graded) subalgebra of $R(E)$ generated by flows. We refer to its homogeneous elements (of degree k) as *k-circulations* on M. In the next section we will study the structure of this algebra in more detail, but for now we make only the following straightforward observation.[1]

Proposition 4.1. *Let θ be a flow on M, thought of as a 1-circulation. Then $\theta^k \neq 0$ if and only if $k \leq \nu(\theta)$.*

Proof. By the multinomial theorem and the fact that squares vanish, we have

$$\theta^k = k! \sum_S \left(\prod_{e \in S} \theta(e) \right) \mathbf{x}^S$$

where S ranges over k-element subsets of E. The term for S is nonzero precisely when $S \subseteq \operatorname{Supp} \theta$, so $\theta^k = 0$ when $k > \nu(\theta)$. On the other hand, since the monomials $\mathbf{x}^S$ for distinct S are linearly independent, the sum is nonzero as long as at least one of the

[1] This holds only in characteristic zero, making this the first (but not the last) time that the choice to work over $\mathbb{Q}$ is really essential.

coefficients is nonzero, which happens as long as there are k elements on which θ does not vanish, i.e. $\nu(\theta) \leq k$. $\square$

We can therefore reconstruct the support functional from the structure of $\Phi(M)$ as a graded algebra. In fact, its structure as an abstract algebra ignoring grading is good enough; see Lemma A.1 in the appendix.

Theorem 4.2 ([28, Theorem 2.9]). *Let M and M' be regular matroids without coloops. Then M and M' are isomorphic if and only if $\Phi(M)$ and $\Phi(M')$ are isomorphic as algebras.*

Proof. Suppose $\Phi(M) \cong \Phi(M')$. Let P be a grade-preserving isomorphism. Such an isomorphism descends to an isomorphism $\mathrm{Flow}(M) \xrightarrow{\sim} \mathrm{Flow}(M')$, and by Proposition 4.1, it preserves ν. The result follows from Corollary 3.8. $\square$

We close this section with a few remarks. First, we note that yet again, the essential use of regularity in Theorem 4.2 is to allow $\Phi(M)$ to be a matroid invariant at all: a matroid which is non-uniquely representable over $\mathbb{Q}$ will have several different "circulation algebras" that need not be isomorphic. (Indeed, [28] works in this more general context, and hence the theorem stated there is in terms of equivalence of representations, rather than isomorphism of matroids.)

Second, we observe that the algebra $\Phi(M)$ is equivalent information to the support functional. This is of course trivially true in the sense that $\Phi(M)$ determines ν, which determines the matroid structure of M, which determines $\Phi(M)$. In Section 4.3, after studying the structure of circulation algebras in more detail, we will show that it is true in the more interesting sense that Proposition 4.1 gives a complete set of relations among a natural set of generators for $\Phi(M)$. Thus one can construct $\Phi(M)$ "directly" from $\mathrm{Flow}(M)$ equipped with the support functional, without going through Theorem 3.7.

4.2 Structure of circulation algebras

We now study the algebraic structure of $\Phi(M)$ in more detail. As a first observation, we note that since $\Phi(M)$ is defined in terms of the flow space, coloops (which are not supported by any flow) are irrelevant: the circulation algebra is unaffected by deleting (equivalently, contracting) all coloops. Thus our examples will largely be coloopless.

Example 4.1. Suppose M consists only of loops. Then any function at all is a flow, so we simply have $\Phi(M) = R(E)$.

Example 4.2. Suppose $M = U_{m-1,m}$, the matroid consisting of a single circuit of size m. The flow space is one-dimensional, spanned by either of the two simple flows. This flow is supported on the entire ground set, so its mth power is nonzero and its $(m+1)$st power is zero. Thus $\Phi(M) \cong \mathbb{Q}[z]/(z^{m+1})$.

Example 4.3. Suppose $M = U_{1,m}$, the matroid consisting of m parallel elements. It has rank 1, so its flow space has codimension 1 in $\mathbb{Q}^E$. Since any two-element set in this matroid is a circuit, it follows that for distinct elements $e, f \in E$, either $x_e - x_f$ or $x_e + x_f$ is a 1-circulation. Thus $x_e x_f$ is a 2-circulation, and similarly any squarefree monomial of degree greater than 1 is in $\Phi(M)$. So $\Phi(M)$ has codimension 1 in $R(E)$, with the only "missing piece" being in degree 1.

If M is a matroid on E and $S \subseteq E$, we can consider the restriction $M|S$. If A is a matrix representing M then $M|S$ is represented by the matrix $A[S]$. Thus any flow on $M|S$ extends to a flow on M by simply having it vanish on the complement of S. It follows that the linear inclusion $\mathbb{Q}^S \hookrightarrow \mathbb{Q}^E$ restricts to an inclusion $\mathrm{Flow}(M|S) \hookrightarrow \mathrm{Flow}(M)$, and thus the subalgebra inclusion $R(S) \hookrightarrow R(E)$ restricts to a subalgebra inclusion $\Phi(M|A) \hookrightarrow \Phi(M)$.

If a matroid decomposes as a direct sum, the algebra has a corresponding decomposition.

Theorem 4.3. *Let M and N be regular matroids. Then $\Phi(M \oplus N) \cong \Phi(M) \otimes \Phi(N)$.*

Proof. M and N are restrictions of $M \oplus N$, so $\Phi(M)$ and $\Phi(N)$ are subalgebras of $\Phi(M \oplus N)$. Thus there is a multiplication map $\Phi(M) \otimes \Phi(N) \to \Phi(M \oplus N)$. The image of this map is the subalgebra generated by $\Phi(M)$ and $\Phi(N)$. Since $\Phi(M \oplus N)$ is generated by $\mathrm{Flow}(M \oplus N) = \mathrm{Flow}(M) \oplus \mathrm{Flow}(N)$, this subalgebra is the entirety of $\Phi(M \oplus N)$, so the multiplication map is surjective. To show it is an isomorphism we must show that $\varphi\psi \neq 0$ when $\varphi \in \Phi(M)$ and $\psi \in \Phi(N)$ are nonzero. This follows from the fact that since the ground sets are disjoint, no variable x_e can appear in the expansion of both φ and ψ into squarefree monomials. Thus no x_e^2 can appear in the product. □

We can also decompose $\Phi(M)$ by deletion and contraction. Let e be an element of M which is not a coloop. We will write $R(E \,/\, e)$ for the quotient $R(E)/\langle x_e \rangle$. (This is the same as $R(E \setminus e)$ as a $\mathbb{Q}$-algebra, but expressing it as a quotient makes it into an $R(E)$-algebra.) We define a map $\partial_e \colon R(E) \to R(E \,/\, e)$ by

$$\partial_e \rho = \frac{\partial \rho}{\partial x_e} + \langle x_e \rangle \,.$$

We remark that this is a slight abuse of notation as the partial derivative is not well-defined as an operator on $R(E)$, since it sends x_e^2 to $2x_e$ which is nonzero in $R(E)$. Since the quotient map annihilates x_e, however, the composite ∂_e has no such definition problem. The map ∂_e is an $R(E \setminus e)$-linear derivation which reduces degree by 1. We also observe that for a flow θ, we simply have $\partial_e \theta = \theta(e)$.

Proposition 4.4. *For any regular matroid M and element e which is not a coloop, $\partial_e \Phi(M) = \Phi(M \,/\, e)$. In particular, $\Phi(M \,/\, e)$ is a $\Phi(M \setminus e)$-submodule of $R(E \,/\, e)$.*

Proof. Since e is not a coloop, there exists a basis B of M with $e \notin B$. For brevity, let $\{f_1, \dots, f_d\} = E \setminus B$ with $f_d = e$; and let $\theta_i = \beta_{B,f_i}$ for each i. By Proposition 3.4, these elements generate $\Phi(M)$, and by Proposition 3.5 their classes in $R(E \,/\, e)$ generate $\Phi(M \,/\, e)$.

Suppose $\varphi \in \Phi(M)$. We can write

$$\varphi = \sum_{k=0}^{m} f_k(\theta_1, \dots, \theta_{d-1})\theta_d^k$$

for some polynomials f_i, so

$$\begin{aligned}\partial_e\varphi &= \sum_{k=0}^{m} f_k(\theta_1,\ldots,\theta_{d-1})\partial_e(\theta_d^k)\\ &= \sum_{k=1}^{m} k f_k(\theta_1,\ldots,\theta_{d-1})\theta_d^{k-1} + \langle x_e\rangle\end{aligned}$$

which is in $\Phi(M\,/\,e)$. Thus we have $\partial_e\Phi(M) \subseteq \Phi(M\,/\,e)$.

Conversely, for any $\psi \in \Phi(M\,/\,e)$, we can write

$$\psi = \sum_{k=0}^{m-1} g_k(\theta_1,\ldots,\theta_{d-1})\theta_d^k + \langle x_e\rangle.$$

Let

$$\varphi = \sum_{k=0}^{m-1} \frac{1}{k+1} g_k(\theta_1,\ldots,\theta_{d-1})\theta_d^{k+1} \in \Phi(M).$$

Then by the computation above we have $\partial_e\varphi = \psi$. Therefore $\partial_e\Phi(M) \supseteq \Phi(M\,/\,e)$ also and we are done. $\square$

Using Proposition 4.4 gives a way to interpret the circulation algebra: a 2-circulation on M determines a flow on $M\,/\,e$ for each element e. More generally a k-circulation determines a $(k-1)$-circulation on $M\,/\,e$ for each e. Thus a circulation is something like a family of flows on contractions of M.

Theorem 4.5 (Wagner [27, 28])**.** *Let M be a regular matroid and e an element of M which is not a coloop. The sequence*

$$0 \to \Phi(M \setminus e) \hookrightarrow \Phi(M) \xrightarrow{\partial_e} \Phi(M\,/\,e)(-1) \to 0$$

of graded $\Phi(M \setminus e)$-modules is exact.

Proof. Exactness at $\Phi(M \setminus e)$ is trivial as the map $\Phi(M \setminus E) \to \Phi(M)$ is just the inclusion of a subalgebra. Exactness at $\Phi(M\,/\,e)(-1)$ follows from Proposition 4.4. Thus we need only show exactness in the middle.

For $\varphi \in \Phi(M)$ we can write

$$\varphi = \sum_{S\subseteq E} a_S \mathbf{x}^S$$

for some coefficients a_S. Then

$$\partial_e\varphi = \sum_{S\ni e} a_S \mathbf{x}^{S\setminus e} + \langle x_e\rangle$$

which vanishes if and only if $[\mathbf{x}^S]\varphi = 0$ for all sets S containing e. But this is the same as saying $\varphi \in \Phi(M \setminus e)$. $\square$

Corollary 4.6. *The Poincaré polynomial of* $\Phi(M)$ *is given by*

$$P(\Phi(M);t) = t^{r(M)}T(M;t^{-1},1+t).$$

Proof. It follows from Theorem 4.3 that the Poincaré polynomial satisfies

$$P(\Phi(M \oplus N);t) = P(\Phi(M);t)P(\Phi(N);t)$$

and from the exact sequence that it satisfies

$$P(\Phi(M);t) = P(\Phi(M \setminus e);t) + tP(\Phi(M \,/\, e);t)$$

whenever e is not a coloop. Thus it is a generalized Tutte–Grothendieck invariant. The base cases

$$P(\Phi(U_{0,1});t) = 1+t$$

and

$$P(\Phi(U_{1,1});t) = 1$$

follow from the discussion about loops and coloops at the beginning of this section. The result follows from the universal property (Proposition 2.3) of the Tutte polynomial. $\square$

4.3 Generators and relations

We now give a presentation of $\Phi(M)$ as the quotient of a polynomial algebra by powers of linear forms. As indicated earlier, the relations in this presentation will be those given by Proposition 4.1. Explicitly, let M be a regular matroid and for brevity write $\mathcal{S}(M) = \operatorname{Sym}\operatorname{Flow}(M)$. Note that, for this section only, multiplication of flows should be interpreted as taking place in $\mathcal{S}(M)$ rather than $\Phi(M)$.

Let $I(M)$ be the ideal

$$I(M) = \langle \theta^{1+\nu(\theta)} \colon \theta \text{ a simple flow on } M \rangle$$

in $\mathcal{S}(M)$. Since $\Phi(M)$ is generated by $\operatorname{Flow}(M)$, there is an obvious surjective map $\mathcal{S}(M) \to \Phi(M)$. Moreover, by Proposition 4.1, the kernel of this map contains $I(M)$. We will ultimately prove the following.

Theorem 4.7 (Wagner [27, 28], Postnikov–Shapiro–Shapiro [22])**.** *The natural map from* $\mathcal{S}(M)/I(M)$ *to* $\Phi(M)$ *is an isomorphism.*

For the moment, we write $\mathcal{A}(M)$ for $\mathcal{S}(M)/I(M)$. To show $\mathcal{A}(M) \cong \Phi(M)$, we will make use of the exact sequence of Theorem 4.5 to give an inductive proof. To do this, we first need to see how to relate $\mathcal{A}(M)$ to $\mathcal{A}(M \setminus e)$ and $\mathcal{A}(M \,/\, e)$ for an element e. Deletion is straightforward: the inclusion $\operatorname{Flow}(M \setminus e) \hookrightarrow \operatorname{Flow}(M)$ gives an inclusion $\mathcal{S}(M \setminus e) \hookrightarrow \mathcal{S}(M)$. Since the inclusion takes simple flows to simple flows and does not alter the size of the support, we see that $I(M \setminus e) = \mathcal{S}(M \setminus e) \cap I(M)$, so this descends to an inclusion $\mathcal{A}(M \setminus e) \hookrightarrow \mathcal{A}(M)$.

For contraction, we wish to find a map $\mathcal{A}(M) \to \mathcal{A}(M \,/\, e)$ that corresponds to ∂_e. To do this, observe that the inclusion $\mathrm{Flow}(M) \hookrightarrow \mathbb{Q}^E$ gives an embedding $\mathcal{S}(M) \hookrightarrow \mathbb{Q}[x_e \colon e \in E]$. Since $\mathcal{S}(M)$ is a subalgebra generated by linear forms, it is closed under differentiation: if $\theta_1, \ldots, \theta_N$ are flows then

$$\frac{\partial}{\partial x_e}(\theta_1 \cdots \theta_N) = \sum_{i=1}^{N} \theta_i(e)\theta_1 \cdots \widehat{\theta_i} \cdots \theta_N$$

by the product rule. Thus any linear combination of products of flows is taken to a linear combination of products of flows. Write D_e for the operator on $\mathcal{S}(M)$ given by restricting this partial derivative operator.

Lemma 4.8. *The map $D_e \colon \mathcal{S}(M) \to \mathcal{S}(M)$ is surjective, and its kernel is precisely $\mathcal{S}(M \setminus e)$.*

Proof. Clearly, $\mathcal{S}(M \setminus e)$ is in the kernel of D_e, as the variable x_e will not appear in the expansion of any element. Since D_e is a derivation, this implies that it is $\mathcal{S}(M \,/\, e)$-linear.

Since $\mathrm{Flow}(M)$ has dimension 1 higher than $\mathrm{Flow}(M \setminus e)$, it follows that $\mathcal{S}(M)$ is isomorphic (as a graded algebra) to $\mathcal{S}(M \setminus e)[y]$, where the isomorphism is given by sending y to any flow that does not vanish on e.

Standard results from commutative algebra (see for instance [9, Proposition 16.1]) imply that the space of A-linear derivations on $A[y]$ is generated as an A-module by $\frac{\partial}{\partial y}$. Moreover, since $\frac{\partial}{\partial y}$ and D_e both decrease degree by 1, D_e must be a *scalar* multiple of this operator, as multiplying by a positive-degree element would destroy this property. Since the partial derivative operator has the required properties, so does D_e. $\square$

Let $P_e \colon \mathcal{S}(M) \to \mathcal{S}(M \,/\, e)$ be the extension of the natural surjection $\mathrm{Flow}(M) \to \mathrm{Flow}(M \,/\, e)$ given by Proposition 3.5.

Lemma 4.9. *$P_e D_e I(M) = I(M \,/\, e)$.*

Proof. First we show $P_e D_e I(M) \subseteq I(M/e)$. By additivity, it is sufficient to show $P_e D_e(\theta^{1+\nu(\theta)} f) \in I(M \,/\, e)$ when θ is a simple flow and f is an arbitrary element of $\mathcal{S}(M)$. We have

$$\begin{aligned} D_e(\theta^{1+\nu(\theta)} f) &= D_e(\theta^{1+\nu(\theta)}) \cdot f + \theta^{1+\nu(\theta)} \cdot D_e f \\ &= (1+\nu(\theta))\theta^{\nu(\theta)} D_e\theta \cdot f + \theta^{1+\nu(\theta)} \cdot D_e f. \end{aligned}$$

Since P_e takes simple flows to simple flows and does not increase the size of the support, we have

$$P_e(\theta^{1+\nu(\theta)} \cdot D_e f) = (P_e\theta)^{1+\nu(\theta)} \cdot P_e D_e f \in I(M \,/\, e).$$

On the other hand, since D_e is $\mathcal{S}(M \setminus e)$-linear, we have $D_e\theta = 0$ unless $\theta(e) \neq 0$. In this case, we have $\nu(P_e\theta) = \nu(\theta) - 1$, so

$$P_e((1+\nu(\theta))\theta^{\nu(\theta)} D_e\theta \cdot f) = (1+\nu(\theta))(P_e\theta)^{1+\nu(P_e\theta)} P_e(D_e\theta \cdot f) \in I(M \,/\, e).$$

Next we show $P_e D_e I(M) \supseteq I(M \,/\, e)$. Again it is sufficient to show that the image contains elements of the form $\xi^{1+\nu(\xi)} g$ for simple flows ξ and arbitrary g. Given such an element, choose θ such that $P_e\theta = \xi$. First, we consider the case $\theta(e) = 0$. In this case, $\nu(\theta) = \nu(\xi)$. Choose some f such that $P_eD_ef = g$; then

$$P_eD_e(\theta^{1+\nu(\theta)}f) = \xi^{1+\nu(\xi)}g.$$

On the other hand, if $\theta(e) \neq 0$, we have $\nu(\theta) = 1 + \nu(\xi)$. Choose some f' such that $P_ef' = g$. Then, viewing $\mathcal{S}(M)$ as a subalgebra of $\mathbb{Q}[x_e\colon e \in E]$, we let f be the antiderivative of $\theta^{\nu(\theta)}f'$ with zero constant term, so $P_eD_ef = \xi^{1+\nu(\xi)}g$. Integrating by parts gives

$$f = \int \theta^{\nu(\theta)}f'\,dx_e = \frac{1}{(1+\nu(\theta))\theta(e)}\left(\theta^{1+\nu(\theta)}f' - \int \theta^{1+\nu(\theta)}D_e(f')\,dx_e\right)$$

(where integrals are always taken with zero constant term) and repeatedly integrating by parts will give all terms of the form $\theta^{1+\nu(\theta)}$ multiplied by some other element. $\square$

It follows from this that P_eD_e descends to a derivation $\tilde{D}_e\colon \mathcal{A}(M) \to \mathcal{A}(M \,/\, e)$.

Theorem 4.10. *For any edge e which is not a coloop,*

$$0 \to \mathcal{A}(M \setminus e) \hookrightarrow \mathcal{A}(M) \xrightarrow{\tilde{D}_e} \mathcal{A}(M \,/\, e)(-1) \to 0$$

is an exact sequence of graded vector spaces.

Proof. The diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I(M \setminus e) \arrow[r, hook] \arrow[d] & I(M) \arrow[r, "P_eD_e"] \arrow[d] & I(M \,/\, e)(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{S}(M \setminus e) \arrow[r, hook] \arrow[d] & \mathcal{S}(M) \arrow[r, "P_eD_e"] \arrow[d] & \mathcal{S}(M \,/\, e)(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{A}(M \setminus e) \arrow[r, hook] \arrow[d] & \mathcal{A}(M) \arrow[r, "\tilde{D}_e"] \arrow[d] & \mathcal{A}(M \,/\, e)(-1) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

of graded vector spaces clearly commutes. The columns are exact by definition. By Lemma 4.8, the middle row is exact. By the fact that $I(M \setminus e) = I(M) \cap \mathcal{S}(M \setminus e)$, the top row is exact as well. By the nine lemma (Lemma 2.9), the bottom row is exact. $\square$

Proof of Theorem 4.7. If M consists only of coloops, both algebras are 1-dimensional and the result is trivial. Otherwise, let e be an element which is not a coloop and suppose the result holds for $M \setminus e$ and $M \,/\, e$. Let Q_M denote the map $\mathcal{A}(M) \to \Phi(M)$ and consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}(M \setminus e) \arrow[r, hook] \arrow[d, "Q_{M\setminus e}"] & \mathcal{A}(M) \arrow[r, "\tilde{D}_e"] \arrow[d, "Q_M"] & \mathcal{A}(M \,/\, e)(-1) \arrow[r] \arrow[d, "Q_{M/e}"] & 0 \\
0 \arrow[r] & \Phi(M \setminus e) \arrow[r, hook] & \Phi(M) \arrow[r, "\partial_e"] & \Phi(M \,/\, e)(-1) \arrow[r] & 0
\end{tikzcd}$$

of graded vector spaces. The left square clearly commutes, as in both spaces the subalgebra corresponding to $M \setminus e$ is simply the one generated by $\mathrm{Flow}(M \setminus e)$, and the downward maps restrict to the identity on flow spaces. To see the right square commutes, let ζ be a flow such that $\zeta(e) = 1$. Then $\mathcal{A}(M / e)$ is generated as an $\mathcal{A}(M \setminus e)$-module by the powers of ζ, so it is sufficient to check commutativity for these. We compute:

$$\begin{aligned} Q_{M/e}\tilde{D}_e(\zeta^k) &= Q_{M/e}(k(\zeta / e)^{k-1}) \\ &= k\left(\sum_{f\in E/e} \zeta(f)x_f\right)^{k-1} \end{aligned}$$

and

$$\begin{aligned} \partial_e Q_M(\zeta^k) &= \partial_e\left(\sum_{f\in E} \zeta(e)x_e\right)^{k} \\ &= k\left(\sum_{f\in E/e} \zeta(f)x_f\right)^{k-1} \end{aligned}$$

as wanted.

The rows are exact, by Theorem 4.10 and Theorem 4.5 respectively. The result follows from the induction hypothesis and the five lemma (Lemma 2.8). □

4.4 A basis

Suppose M is a matroid on a totally ordered ground set. Using Proposition 2.5 along with Corollary 4.6, we can write the Poincaré polynomial of $\Phi(M)$ as a sum over bases of the matroid M:

$$\begin{aligned} P(\Phi(M); t) &= t^{r(M)} \sum_{B} t^{-|\mathrm{IA}(B)|}(1+t)^{|\mathrm{EA}(B)|} \\ &= \sum_{B} \sum_{S\subseteq \mathrm{EA}(B)} t^{|\mathrm{IP}(B)|+|S|}. \end{aligned}$$

This suggests we ought to be able to find, for each pair (B, S), a homogeneous element of degree $|\mathrm{IP}(B)| + |S|$, such that the collection of these elements form a $\mathbb{Q}$-basis for $\Phi(M)$. Such a basis does indeed exist; the construction is due to Wagner but has not previously been published. Let B be a basis of M and $e \in \mathrm{IP}(B)$. We define a flow $\alpha_{B,e}$ as follows:

$$\alpha_{B,e} = \beta_{B,\min C^*_{B,e}}.$$

Then for B a basis of M and $S \subseteq \mathrm{EA}(B)$ define

$$\varphi_{B,S} = \left(\prod_{e\in \mathrm{IP}(B)} \alpha_{B,e}\right)\left(\prod_{e\in S} \beta_{B,e}\right). \tag{4.1}$$

We will ultimately show the following.

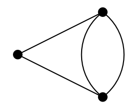

Figure 4.1: Bugle graph.

Theorem 4.11. *Let M be a matroid on totally ordered ground set E. The elements $\varphi_{B,S}$, for B a basis of M and $S \subseteq \mathrm{EA}(B)$, form a $\mathbb{Q}$-basis for $\Phi(M)$.*

Before doing so, we give some examples of this construction. Let M be the matroid on ground set $\{1,2,3,4\}$ represented by the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

which is equivalently the cycle matroid of the three-vertex “bugle” graph. In Table 4.1, the ten basis vectors of $\Phi(M)$ are listed together with the data from which they were computed.

As a second example, we look at the contraction $M \,/\, 4$ of the previous example. This has canonical representation

$$\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$$

and is isomorphic to $U_{1,2} \oplus U_{0,1}$. Its basis is listed in Table 4.2. We observe that this demonstrates that the basis does not behave the way one might naïvely hope with respect to contraction: ∂_e does not in general take the basis elements for M to those for $M \,/\, e$, at least in the case that e is last in the total ordering. Comparing Tables 4.1 and 4.2, we see a number of cases where differentiating with respect to x_4 takes a basis element in the former to a scalar multiple of one in the latter. This would not be so bad, but in the case of the $\varphi_{\{1,4\},\{3\}}$ and $\varphi_{\{2,4\},\{3\}}$ even this does not happen and the images are linear combinations of multiple basis elements.

This counterexample dashes the hope that Theorem 4.11 might follow easily from Theorem 4.5; we instead proceed in a different fashion. We will need a couple of lemmas.

B	$\mathrm{IP}(B)$	S	$\varphi_{B,S}$
$\{1,2\}$	$\emptyset$	$\emptyset$	1
$\{1,3\}$	$\{3\}$	$\emptyset$	$x_1 + x_2 + x_3$
$\{1,4\}$	$\{4\}$	$\emptyset$	$x_1 + x_2 + x_4$
		$\{3\}$	$x_1x_3 - x_1x_4 + x_2x_3 - x_2x_4 + x_3x_4$
$\{2,3\}$	$\{2,3\}$	$\emptyset$	$2x_1x_2 + 2x_1x_3 + 2x_2x_3$
		$\{1\}$	$6x_1x_2x_3$
$\{2,4\}$	$\{2,4\}$	$\emptyset$	$2x_1x_2 + 2x_1x_4 + 2x_2x_4$
		$\{1\}$	$6x_1x_2x_4$
		$\{3\}$	$2x_1x_2x_3 - 2x_1x_2x_4 - 2x_1x_3x_4 - 2x_2x_3x_4$
		$\{1,3\}$	$6x_1x_2x_3x_4$

Table 4.1: Basis for the ordinary circulation algebra of the bugle graph.

B	$\mathrm{IP}(B)$	S	$\varphi_{B,S}$
$\{1\}$	$\emptyset$	$\emptyset$	1
		$\{3\}$	x_3
$\{2\}$	$\{2\}$	$\emptyset$	$x_1 + x_2$
		$\{1\}$	$2x_1x_2$
		$\{3\}$	$x_1x_3 + x_2x_3$
		$\{1,3\}$	$2x_1x_2x_3$

Table 4.2: Basis for the ordinary circulation algebra of $U_{1,2} \oplus U_{0,1}$.

Lemma 4.12. *The squarefree monomial* $\mathbf{x}^{\mathrm{IP}(B)\cup S}$ *appears with a nonzero coefficient in* $\varphi_{B,S}$.

Proof. We show this by induction on $|S|$. For $S = \emptyset$, we can write the coefficient as a sum over permutations w of $\mathrm{IP}(B)$:

$$[\mathbf{x}^{\mathrm{IP}(B)}]\varphi_{B,\emptyset} = \sum_{w} \prod_{e \in \mathrm{IP}(B)} \alpha_{B,w(e)}(e). \tag{4.2}$$

The product is equal to ± 1 if w is the identity permutation, so it is sufficient to show there is no cancellation in the sum.

Let w be a permutation and let $(e_1, \ldots, e_k)$ be a cycle of w. Write $f_i = \min C^*_{B,e_i}$. If the f_i are not all equal, there must be a descent, i.e. there must be some i such that $f_i > f_{i+1}$ where the subscript is taken mod k. Then $f_{i+1} \notin C^*_{B,e_i}$ since f_i is the minimum, so $e_i \notin C_{B,f_i}$. Thus

$$\alpha_{B,w(e_i)}(e_i) = \beta_{B,f_{i+1}}(e_i) = 0$$

and so the term corresponding to w in (4.2) vanishes.

Thus if w is a permutation such that the corresponding term is nonzero, we have $\min C^*_{B,w(e)} = \min C^*_{B,e}$ and hence $\alpha_{B,w(e)} = \alpha_{B,e}$ for all $e \in \mathrm{IP}(B)$. Thus the term for w is equal to the term for the identity. It follows that the coefficient is, up to sign, the number of such permutations.

Now suppose $S \neq \emptyset$ and $e \in S$. Let $S' = S \setminus e$ and suppose the result holds for S'. Since (4.1) expresses the basis elements as products of basic flows, we can write

$$\varphi_{B,S} = \beta^k_{B,e}\psi$$

for some $k \geq 1$ and some circulation $\psi \in \Phi(M \setminus e)$. Thus

$$\varphi_{B,S'} = \beta^{k-1}_{B,e}\psi$$

so

$$[\mathbf{x}^{\mathrm{IP}(B)\cup S}]\varphi_{B,S} = k[\mathbf{x}^{\mathrm{IP}(B)\cup S'}]\varphi_{B,S'} \neq 0$$

as wanted. □

Our goal is to show that this "canonical" monomial is the largest one which appears, in some total ordering on the monomials. We will do this combinatorially. Suppose $\{A_i\}_{i\in I}$ is a family of sets indexed by some set I. By a *partial transversal* we mean a set $T \subseteq \bigcup_{i\in I} A_i$ such that there exists a $X \subseteq I$ and a bijection $b\colon X \to T$ with $b(i) \in A_i$ for all $i \in X$. If this exists with $X = I$ then T is a *(full) transversal*. By [19, Theorem 1.6.2] the partial transversals are the independent sets of a matroid; if a full transversal exists then the full transversals are the bases of this matroid.

Fixing some B and $S \subseteq \mathrm{EA}(B)$, for $e \in \mathrm{IP}(B) \cup S$ let

$$A_e = \begin{cases} \operatorname{Supp} \alpha_{B,e}, & e \in \mathrm{IP}(B) \\ \operatorname{Supp} \beta_{B,e}, & e \in S. \end{cases}$$

Clearly, $\mathrm{IP}(B) \cup S$ is itself a transversal of this set family. Moreover, any monomial which appears in the expansion of $\varphi_{B,S}$ is of the form $\mathbf{x}^T$ where T is a transversal of the set family $\{A_e\}_{e\in \mathrm{IP}(B)\cup S}$.

Lemma 4.13. $\mathrm{IP}(B) \cup S \geq_{\mathrm{G}} T$ *for any transversal* T*.*

Proof. Since a transversal exists, the transversals form a bases of a matroid, so by Proposition 2.1 there exists a unique Gale-maximal transversal $T_\star$. To show $T_\star = \mathrm{IP}(B) \cup S$ it is sufficient to show $T_\star \supseteq \mathrm{IP}(B) \cup S$, since they must have the same cardinality.

Suppose $e \in \mathrm{IP}(B) \cup S$, and let

$$U = \{f \in T_\star \colon f > e\}.$$

Clearly U is a partial transversal. By the greedy algorithm, $e \in T_\star$ if and only if $U \cup \{e\}$ is a partial transversal. Since U is a partial transversal, there exists a set $X \subset \mathrm{IP}(B) \cup S$ and a bijection $b\colon X \to U$ such that $b(f) \in A_f$ for all $f \in X$. We must find a set X' and a bijection $b'\colon X' \to U \cup \{e\}$ with the same property. Clearly if $e \notin X$ we can do this.

Suppose $e \in X$. Then we have $b(e) \in A_e$ and $b(e) > e$. We observe that this implies $b(e) \in \mathrm{IP}(B)$. To see this, note that if $e \in \mathrm{IP}(B)$ we have $A_e = C_{B,f}$ where $f < e$. Since $b(e) > e > f$ we have $b(e) \in B$, and $f \in C^*_{B,b(e)}$. If $e \in S$ then $A_e = C_{B,e}$ and thus $b(e) \in B$ and $e \in C^*_{B,b(e)}$. Either way, $b(e)$ is not the smallest element of its fundamental cut. Thus we can again ask the question of whether $b(e) \in X$. Repeating this process, we obtain a sequence $e = e_1, \dots, e_k$ of elements of X where $b(e_i) = e_{i+1}$ for $1 \leq i < k$ and $b(e_k) \notin X$. There are two cases, depending on whether or not $b(e_k) \in \mathrm{IP}(B) \cup S$.

First suppose $b(e_k) \notin \mathrm{IP}(B) \cup S$. For $1 \leq i \leq k$ let

$$f_i = \begin{cases} \min C^*_{B,e_i}, & e_i \in \mathrm{IP}(B) \\ e_i, & e_i \in S \end{cases}$$

so $A_{e_i} = C_{B,f_i}$. Thus we have $e_{i+1} \in C_{B,f_i}$ for $1 \leq i < k$. If $e_{i+1} \in \mathrm{IP}(B)$, this implies $f_i \in C^*_{B,e_{i+1}}$ and hence $f_{i+1} \leq f_i$. On the other hand, if $e_{i+1} \in S$ we must have $e_{i+1} = f_{i+1} = f_i$, as f_i is the only element C_{B,f_i} which is not in B. Thus

$$e \geq f_1 \geq \cdots \geq f_k.$$

Now, we have $b(e_k) \in A_{e_k} = C_{B,f_k}$. Since $b(e_k) \notin \mathrm{IP}(B)$, we either have $b(e_k) = f_k$ or $b(e_k) \in \mathrm{IA}(B)$. In the latter case we have $f_k \in C^*_{B,b(e_k)}$ so $b(e_k) < f_k$. Thus in either case $b(e_k) \leq f_k$, but this is a contradiction, as by hypothesis $b(e_k) \in U$ and hence $b(e_k) > e \geq f_k$.

Thus we must have $b(e_k) \in \mathrm{IP}(B)$. Then we may take

$$X' = X \cup \{b(e_k)\}$$

and define b' by

$$b'(f) = \begin{cases} f, & f \in \{e_1, \dots, e_k, b(e_k)\} \\ b(f), & \text{otherwise} \end{cases}$$

which is clearly a bijection $U \cup \{e\} \to X'$ with the property that $b'(f) \in A_f$ for all f. Thus $U \cup \{e\}$ is indeed a partial transversal, so $e \in T_\star$. Since this holds for all $e \in \mathrm{IP}(B) \cup S$ we have $T_\star = \mathrm{IP}(B) \cup S$ as wanted. □

Combining these essentially gives the desired result.

Proof of Theorem 4.11. We have already observed that we have the right number of elements to form a basis, so it is sufficient to show that they are linearly independent. List the elements $\varphi_{B,S}$ according to $\mathrm{IP}(B) \cup S$, first by size and then by some total extension of Gale order. By Proposition 2.4, the monomials $\mathbf{x}^{\mathrm{IP}(B)\cup S}$ are distinct (and hence linearly independent) for different choices of B and S, and by Lemmas 4.12 and 4.13 this monomial first appears in in $\varphi_{B,S}$. Thus the latter are also linearly independent. □

The results of this section allow us to easily re-prove part of a result of Ardila and Postnikov.

Corollary 4.14 ([1, Proposition 4.21])**.** *The coefficient extraction operators* $[\mathbf{x}^{\mathrm{IP}(B)\cup S}]$ *for* $B \in \mathcal{B}(M)$ *and* $S \subseteq \mathrm{EA}(B)$ *form a basis for the linear dual of* $\Phi(M)$*.*

Proof. Again order pairs (B, S) first by size and then by a total extension of Gale order on $\mathrm{IP}(B) \cup S$. By Lemmas 4.12 and 4.13 the expansion of these coefficient-extraction operators in terms of the dual basis to the $\varphi_{B,S}$ is lower triangular with nonzero diagonal entries. □

4.5 Filtration by nullity

Corollary 4.6 has a two-variable refinement. This is a variation[2] of a construction of Berget [3], which is itself a variation on one due to Orlik and Terao [18]. For a subset S of a matroid M, the *nullity* of S is

$$\mathrm{null}(S) = |S| - r(S).$$

Fixing a matroid M on ground set E, we put a bigrading on $R(E)$ by setting

$$R_{i,j} = \mathrm{Span}\{\mathbf{x}^A \colon |A| = i, \mathrm{null}(A) = j\}.$$

Clearly we have

$$R(E) = \bigoplus_{i,j \geq 0} R_{i,j}.$$

[2]The equivalence between his result and ours is given by applying each of the following commuting involutions: matroid duality, linear duality, turning both gradings upside down, reversing the order of subscripts, and finally inverting the polynomial identity.

Note that this does not however make $R(E)$ into a bigraded algebra. The submodular inequality for rank implies that $\mathrm{null}(S \cup T) \geq \mathrm{null}(S) + \mathrm{null}(T)$ when S and T are disjoint, but this inequality can certainly be strict. If S has nullity j and T has nullity j', we thus have $\mathbf{x}^S \in R_{i,j}$ and $\mathbf{x}^T \in R_{i',j'}$ but we need not have $\mathbf{x}^S\mathbf{x}^T \in R_{i+i',j+j'}$. However, the inequality does imply that multiplication respects the *filtration* by nullity.

Define $\Phi_{i,j}(M) = \Phi(M) \cap R_{i,j}$. We would like this to give a decomposition of $\Phi(M)$; the discussion above means this is not completely obvious. Fortunately, it is nonetheless true.

Theorem 4.15 (Berget [3, Theorem 1.1])**.** *As a vector space, $\Phi(M)$ decomposes as a direct sum*

$$\Phi(M) = \bigoplus_{i,j \geq 0} \Phi_{i,j}(A).$$

With respect to this decomposition, the Poincaré polynomial of $\Phi(M)$ is given by

$$P(\Phi(M); t, u) = t^{r(M)} T(M; t^{-1}, 1 + tu).$$

Proof. If M consists of only loops and coloops we have $\Phi_i(M) = \Phi_{i,i}(M)$. Thus the decomposition part is obviously true, and we have

$$P(\Phi(M); t, u) = P(\Phi(M); tu) = (1 + tu)^{\ell}$$

where ℓ is the number of loops, which agrees with the formula.

Otherwise, suppose e is an element which is neither a loop nor a coloop. Recall the formula [19, Proposition 3.1.6] for the rank function of a contraction. Since e is not a loop, it gives for all $S \subseteq E \setminus e$,

$$r_{M/e}(S) = r_M(S \cup \{e\}) - 1.$$

Thus

$$\mathrm{null}_{M/e}(S) = \mathrm{null}_M(S \cup \{e\}).$$

Since ∂_e annihilates monomials corresponding to subsets which don't contain e, and simply removes e from the ones that do, it follows that ∂_e preserves the grading by nullity: $\partial_e R_{i,j} \subseteq R_{i-1,j}$. Thus Theorem 4.5 breaks up into graded pieces

$$0 \to \Phi_{i,j}(M \setminus e) \hookrightarrow \Phi_{i,j}(M) \xrightarrow{\partial_e} \Phi_{i-1,j}(M \,/\, e) \to 0 \tag{4.3}$$

which remain exact. Adding these up gives exactness of the first row of the commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{i,j \geq 0} \Phi_{i,j}(M \setminus e) & \hookrightarrow & \bigoplus_{i,j \geq 0} \Phi_{i,j}(M) & \xrightarrow{\partial_e} & \bigoplus_{i,j \geq 0} \Phi_{i,j}(M \,/\, e) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Phi(M \setminus e) & \hookrightarrow & \Phi(M) & \xrightarrow{\partial_e} & \Phi(M \,/\, e) & \longrightarrow & 0
\end{array}$$

(where the downward maps are the obvious inclusions). Thus by the Lemma 2.8, if the decomposition result holds for $M \setminus e$ and $M \,/\, e$ it holds for M. The exact sequences (4.3) give the recurrence

$$P(\Phi(M); t, u) = P(\Phi(M \setminus e); t, u) + tP(\Phi(M \,/\, e), t, u)$$

and the formula follows from Proposition 2.3. $\square$

As observed by Berget, one interesting property of the expression in Theorem 4.15 is that it is invertible; we can rearrange it to get

$$T(M; x, y) = x^{r(M)} P(\Phi(M); x^{-1}, xy - x).$$

Thus, while Theorem 4.2 showed that the structure of $\Phi(M)$ as an *algebra* essentially encodes all information about the matroid M, Theorem 4.15 says that its structure as a *bigraded vector space* encodes "Tutte–Grothendieck" information.

The basis of Theorem 4.11 is *not* in general homogeneous with respect to the nullity filtration. The example of the bugle graph demonstrates this; referring to Table 4.1, consider the basis element

$$\varphi_{\{1,4\},\{2\}} = x_1x_3 - x_1x_4 + x_2x_3 - x_2x_4 + x_3x_4.$$

The edges 3 and 4 are parallel, so the set $\{3, 4\}$ has nullity 1. On the other hand, all other 2-element sets in this matroid are independent, as there is no other pair of parallel edges. Thus this circulation is not nullity-homogeneous. To remedy this, we can modify the basis to get one which is homogeneous with respect to both gradings. Define $\tilde{\varphi}_{B,S}$ to be the sum of the monomials in $\varphi_{B,S}$ with nullity $|S|$.

Theorem 4.16. *The elements $\tilde{\varphi}_{B,S}$ for $B \in \mathcal{B}(M)$ and $S \subseteq \mathrm{EA}(B)$ with $|\mathrm{IP}(B)| = i - j$ and $|S| = j$ form a $\mathbb{Q}$-basis for $\Phi_{i,j}(M)$.*

Proof. By Theorem 4.15, we do in fact have $\tilde{\varphi}_{B,S} \in \Phi(M)$. We have

$$\mathrm{null}(\mathrm{IP}(B) \cup S) = |S|$$

so it follows from Lemma 4.12 that $\mathbf{x}^{\mathrm{IP}(B)\cup S}$ appears with a nonzero coefficient in $\tilde{\varphi}_{B,S}$. Since the monomials which appear in $\tilde{\varphi}_{B,S}$ are a subset of those that appear in $\varphi_{B,S}$, Lemma 4.13 implies that this monomial is the largest in Gale order; the result follows by the same argument used to prove Theorem 4.11. □

Since nullity gives a filtration, we can consider the ideals

$$I_{\mathrm{null}}^{[k]}(M) = \bigoplus_{i \geq 0} \bigoplus_{j \geq k} \Phi_{i,j}(M).$$

The quotient algebras $\Phi(M)/I_{\mathrm{null}}^{[k]}(M)$ share some properties with the circulation algebra itself. For instance, since ∂_e preserves nullity when e is not a loop, we automatically get an analogue of Theorem 4.5. We have also done the necessary work to get an analogue of Theorem 4.11.

Theorem 4.17. *The images of the elements $\varphi_{B,S}$ for $B \in \mathcal{B}(M)$ and $S \subseteq \mathrm{IP}(B)$ with $|S| \leq k$ give a $\mathbb{Q}$-basis for $\Phi(M)/I_{\mathrm{null}}^{[k]}(M)$.*

Proof. This is obviously true for if $\varphi_{B,S}$ is replaced with $\tilde{\varphi}_{B,S}$, since the latter are homogeneous and the ones with $|S| \leq k$ are precisely those that don't vanish modulo $I_{\mathrm{null}}^{[k]}(M)$. But since multiplication respects the filtration by nullity, we have

$$\varphi_{B,S} \equiv \tilde{\varphi}_{B,S} \pmod{I_{\mathrm{null}}^{[k]}(M)}$$

and so the stated result is also true. □

The case $k = 1$ is especially interesting. Denote the quotient $\Phi(M)/I^{[1]}_{\mathrm{null}}(M)$ by $\widetilde{\Phi}(M)$. An immediate consequence of Theorem 4.15 is that

$$P(\widetilde{\Phi}(M); t) = t^{r(M)} T(M; t^{-1}, 1).$$

Of course, the graded dimension can be computed from the Tutte polynomial for any value of k, but only in this case can it be done by simply making a substitution for the variables. We can also generalize Theorem 4.7 to this algebra in a natural way.

Theorem 4.18. $\widetilde{\Phi}(M)$ *is naturally isomorphic to the quotient*

$$\operatorname{Sym}\operatorname{Flow}(M) \,/\, \left\langle \theta^{\nu(\theta)} \colon \theta \text{ a simple flow on } M \right\rangle .$$

Proof. The minimal sets of nonzero nullity are precisely the circuits. For C a circuit, the monomial $\mathbf{x}^C$ is, up to a scalar, equal to $\zeta^{|C|}$ where ζ is a flow supported on C. Thus $I^{[1]}_{\mathrm{null}}(M)$ is precisely the ideal in $\Phi(M)$ generated by elements of the form $\theta^{\nu(\theta)}$. By Theorem 4.7, $\widetilde{\Phi}(M)$ is isomorphic to the quotient of $\operatorname{Sym}\operatorname{Flow}(M)$ first by elements of the form $\theta^{1+\nu(\theta)}$ and then by elements of the form $\theta^{\nu(\theta)}$. Since the latter divide the former, we can skip the first step. $\square$

The analogy between $\Phi(M)$ and $\widetilde{\Phi}(M)$ is a particular case of the theory of power ideals developed by Ardila and Postnikov. They are respectively dual to the $k = 0$ and $k = -1$ cases of the objects considered in [1, Section 4].

Chapter 5

Generalized Circulation Algebras

5.1 Introduction

We now introduce a family of algebras associated to a regular matroid M, of which $\Phi(M)$ is one member. For a finite set E and $\sigma\colon E \to \mathbb{N}$, consider the algebra

$$R^{(\sigma)}(E) = \mathbb{Q}[x_e\colon e \in E] / \left\langle x_e^{1+\sigma(e)}\colon e \in E \right\rangle .$$

As in the construction of $\Phi(M)$, we think of $\mathbb{Q}^E$ as the degree-1 part of this graded algebra, and consider the subalgebra generated by $\mathrm{Flow}(M)$. Denote this subalgebra by $\Phi^{(\sigma)}(M)$. We will call such an algebra a *generalized circulation algebra* of M. Of course, in the case that σ identically equals 1, this is simply $\Phi(M)$. In general, when σ identically equals some constant s, we will denote the generalized circulation algebra by $\Phi^{(s)}(M)$. We note that algebras of the form $\Phi^{(s)}(M)$ for M a cographic matroid have previously been studied by Nenashev [16, Section 3].

In fact, as a class of algebras, generalized circulation algebras are not really any more general than ordinary circulation algebras: $\Phi^{(\sigma)}(M)$ is isomorphic the ordinary circulation algebra of a matroid constructed from M by an operation we will call *σ-subdivision*.[1] In Section 5.2 we introduce this operation and prove its relationship to generalized circulation algebras. In Section 5.3, we derive a number of generalizations of results from Chapter 4 using the subdivision isomorphism. Finally in Section 5.4 we give a family of short exact sequences generalizing Theorem 4.5, which do not directly follow from the subdivision construction.

5.2 Subdivisions

For S a finite set and $\sigma\colon S \to \mathbb{N}$, write

$$S^{(\sigma)} = \bigcup_{e \in S} \{e\} \times \{1, \ldots, \sigma(e)\}.$$

Proposition 5.1. *Let E be a finite set and M a matroid on ground set E. For any $\sigma\colon E \to \mathbb{N}$, there exists a matroid $M^{(\sigma)}$ on $E^{(\sigma)}$ whose circuits are precisely those sets of the form $C^{(\sigma)}$ for $C \in \mathcal{C}(M)$.*

[1] This result also essentially appears in Nenashev's work, as part of the proof of [16, Theorem 3.1].

Proof. We must show that the set $\{C^{(\sigma)}\colon C \in \mathcal{C}(M)\}$ satisfies the definition given in Section 2.1. Clearly we have $C^{(\sigma)} \neq \emptyset$ if $C \neq \emptyset$, and $C_1^{(\sigma)} \subset C_2^{(\sigma)}$ if and only if $C_1 \subset C_2$, so the first two properties for $M^{(\sigma)}$ follow from those for M. For the third property, let C_1 and C_2 be circuits of M, and $(e,i) \in C_1^{(\sigma)} \cap C_2^{(\sigma)}$. Then $e \in C_1 \cap C_2$, so there is a circuit $C \subseteq (C_1 \cap C_2) \setminus e$. Clearly we have $C^{(\sigma)} \subseteq C_1^{(\sigma)} \cap C_2^{(\sigma)} \setminus (e,i)$. □

We call the matroid $M^{(\sigma)}$ the *σ-subdivision* of M. If M is the cycle matroid of a graph then $M^{(\sigma)}$ is the cycle matroid of the graph obtained by replacing each edge e with a path of length $\sigma(e)$. This is a subdivision in the usual graph-theoretic sense, at least if $\sigma(e) > 0$ for all e. We also observe any matroid, if σ takes on only the values 0 and 1, then $S^{(\sigma)}$ simply looks like the set obtained by discarding the elements e such that $\sigma(e) = 0$. Thus there is an isomorphism

$$M^{(\sigma)} \cong M/\{e\colon \sigma(e) = 0\}$$

by the usual characterization of circuits in contractions. We can also generalize the usual characterization of bases in contractions. We will say σ is *proper* if the set $\{e \in E\colon \sigma(e) = 0\}$ is independent.

Lemma 5.2. *Let M be a matroid on ground set E and $\sigma\colon E \to \mathbb{N}$ be proper. Suppose B is a basis of M and $\mu\colon E \setminus B \to \mathbb{N}$ is a function such that $1 \leq \mu(e) \leq \sigma(e)$ for each $e \notin B$. Let*

$$B_\mu = \{(e,i)\colon e \in B \text{ or } i \neq \mu(e)\} \subseteq E^{(\sigma)}.$$

Then B_μ is a basis of $M^{(\sigma)}$. Moreover, every basis is of this form.

Proof. For any $C \in \mathcal{C}(M)$, there is some $e \in C$ with $e \notin B$. Then $(e,\mu(e)) \notin B_\mu$, so $C^{(\sigma)} \not\subseteq B_\mu$. Thus B_μ is independent. On the other hand, adding any element $(e,\mu(e))$ would complete the circuit $C_{B,e}^{(\sigma)}$, so no proper superset of B_μ is independent. Note that since σ is proper, there exist a B and μ satisfying the hypotheses. Thus every basis of $M^{(\sigma)}$ excludes exactly $|E| - r(M)$ elements.

To show every basis is of this form, let B' be any basis. We choose

$$B = \{e\colon (e,i) \in B' \text{ for } 1 \leq i \leq \sigma(e)\}.$$

In other words, B is the largest subset of E such that $B^{(\sigma)} \subseteq B'$. Clearly B is independent in M. Since the number of elements of $E^{(\sigma)}$ that are not in B' is $|E| - r(M)$, it follows that there are at most this many elements of E that are not in B. Since B is independent and contains at least $r(M)$ elements, B is a basis of M. Thus there are exactly $|E| - r(M)$ elements which are not in B. For each $e \notin E \setminus B$, there is some i such that $(e,i) \notin B'$. Since $|E \setminus B| = |B^{(\sigma)} \setminus B'|$, it follows that there is exactly one such i for each $e \in E \setminus B$. Defining $\mu(e)$ to be this value of i for each such e, we have $B' = B_\mu$. □

Corollary 5.3. *If σ is proper,*

$$r(M^{(\sigma)}) = r(M) - |E| + \sum_{e \in E} \sigma(e).$$

To relate subdivisions to circulation algebras, we will need to understand representations of subdivisions. We will assume that $\sigma(e) > 0$ for all e; if not, we can first contract the edges on which σ vanishes. Let A be a matrix with k rows representing A. For $e \in E$, let A'_e be the block matrix

$$A'_e = \begin{bmatrix} \mathbf{a}_e & 0 & \cdots & 0 \end{bmatrix}$$

where $\mathbf{a}_e$ is the column of A corresponding to e. For $n \in \mathbb{N}$ let Δ_n be the $(n-1) \times n$ matrix

$$\Delta_n = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}.$$

Proposition 5.4. *Let $E = \{e_1, \ldots, e_m\}$, and $\sigma \colon E \to \mathbb{N}$ be a function such that $\sigma(e) > 0$ for all e. The matroid $M^{(\sigma)}$ is represented by the block matrix*

$$A^{(\sigma)} = \begin{bmatrix} A'_{e_1} & \cdots & A'_{e_m} \\ \Delta_{\sigma(e_1)} & & \\ & \ddots & \\ & & \Delta_{\sigma(e_m)} \end{bmatrix}.$$

Proof. Suppose $\theta \in \mathbb{Q}^{E^{(\sigma)}}$. Let $\tilde{\theta} \in \mathbb{Q}^E$ be given by

$$\tilde{\theta}(e) = \theta(e, 1)$$

which is well-defined since $\sigma(e) > 1$. For $1 \leq i \leq m$ let $\theta_i \in \mathbb{Q}^{\sigma(e_i)}$ be given by

$$\theta_i(j) = \theta(e_i, j).$$

Then, thinking of θ as a column vector, we can write $A^{(\sigma)}\theta$ in block form as

$$A^{(\sigma)}\theta = \begin{bmatrix} A\tilde{\theta} \\ \Delta_{\sigma(e_1)}\theta_1 \\ \vdots \\ \Delta_{\sigma(e_m)}\theta_m \end{bmatrix}.$$

This vanishes precisely when $\tilde{\theta} \in \operatorname{Ker} A$ and θ_i is constant for each i. Thus if $\theta \in \operatorname{Ker} A^{(\sigma)}$ we have

$$\operatorname{Supp} \theta = \left(\operatorname{Supp} \tilde{\theta}\right)^{(\sigma)}$$

and in particular, by Proposition 3.2, the circuits of the matroid represented by $A^{(\sigma)}$ are precisely the sets $C^{(\sigma)}$ for C a circuit of M, as wanted. □

Theorem 5.5. *For any regular matroid M, $\Phi^{(\sigma)}(M) \cong \Phi(M^{(\sigma)})$.*

Proof. We compute that in $R(E^{(\sigma)})$, for any $e \in E$ we have

$$\left(\sum_{i=1}^{\sigma(e)} x_{e,i}\right)^{1+\sigma(e)} = 0.$$

Thus there is a well-defined algebra homomorphism $P\colon R^{(\sigma)}(E) \to R(E^{(\sigma)})$ given by

$$P(x_e) = \sum_{i=1}^{\sigma(e)} x_{e,i}.$$

Using the matrix representation of Proposition 5.4, flows on $M^{(\sigma)}$ are the same as flows on M, with each subdivided element being given the same value. Thus P maps $\mathrm{Flow}(M)$ isomorphically to $\mathrm{Flow}(M^{(\sigma)})$ and hence maps $\Phi^{(\sigma)}(M)$ onto $\Phi(M^{(\sigma)})$. □

5.3 Consequences of Theorem 5.5

Theorem 5.5 means that every theorem about ordinary circulation algebras also gives a theorem about generalized circulation algebras. In some cases (in particular, Theorem 4.5) one does not obtain the most natural generalization of the result this way. However, in many cases one *does*; we examine some of these in this section.

Theorem 5.6. *Let M and M' be regular matroids without coloops. For any $s > 0$, $M \cong M'$ if and only if $\Phi^{(s)}(M) \cong \Phi^{(s)}(M')$.*

Proof. It is clearly the case that $M^{(s)} \cong (M')^{(s)}$ if and only if $M \cong M'$. The result is immediate from Theorem 4.2 and Theorem 5.5. □

Note that no such reconstruction result exists for non-constant σ. Of course, a bijection between the ground sets of the matroids must be specified in order to even make sense of using "the same" σ for two different matroids. Once this has been done, however, one can find isomorphic matroids with non-isomorphic σ-subdivisions. For instance, the two graphs of Fig. 5.1 have non-isomorphic cycle matroids, but the graphs (and hence also the matroids) become isomorphic when the edge labelled e is subdivided in each of them.

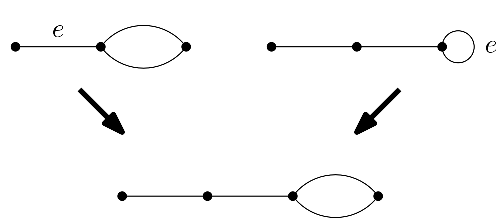

Figure 5.1: A pair of non-isomorphic graphs with an isomorphic subdivision.

Theorem 5.7. *Let M and N be regular matroids and E be the disjoint union of their ground sets. For any function $\sigma\colon E \to \mathbb{N}$,*

$$\Phi^{(\sigma)}(M \oplus N) \cong \Phi^{(\sigma)}(M) \otimes \Phi^{(\sigma)}(N)$$

Proof. We clearly have $(M \oplus N)^{(\sigma)} = M^{(\sigma)} \oplus N^{(\sigma)}$. The result follows from Theorem 4.3 and Theorem 5.5. □

To generalize Theorem 4.7 this way, fix a regular matroid M on ground set E and a function $\sigma\colon E \to \mathbb{N}$. For $\theta \in \operatorname{Flow}(M)$, write

$$\nu(\theta, \sigma) = \sum_{e \in \operatorname{Supp}\theta} \sigma(e).$$

Of course, if σ is identically equal to 1 then $\nu(\theta, \sigma) = \nu(\theta)$. Let $I(M, \sigma)$ be the ideal in $\operatorname{Sym}\operatorname{Flow}(M)$ given by

$$I(M, \sigma) = \left\langle \theta^{1+\nu(\theta,\sigma)} \colon \theta \text{ a simple flow on } M \right\rangle.$$

Theorem 5.8. *$\Phi^{(\sigma)}(M)$ is naturally isomorphic to the quotient of* $\operatorname{Sym}\operatorname{Flow}(M)$ *by $I(M, \sigma)$.*

Proof. Let $P\colon \operatorname{Flow}(M) \xrightarrow{\sim} \operatorname{Flow}(M^{(\sigma)})$ be given by

$$(P\theta)(e, i) = \theta(e).$$

Thus for each $e \in \operatorname{Supp}\theta$, we have $\sigma(e)$ elements in $\operatorname{Supp} P\theta$ corresponding to it. Thus

$$\nu(P\theta) = \nu(\theta, \sigma)$$

so P takes $I(M, \sigma)$ isomorphically to the ideal $I(M^{(\sigma)})$ considered in Section 4.3. The result then follows from Theorem 4.7. □

Next we construct a basis for $\Phi^{(\sigma)}(M)$, analogous to the basis from Theorem 4.11 for ordinary circulation algebras. Fix a total order on E, and for a basis B of M say a function $\lambda\colon E \setminus B \to \mathbb{N}$ is *admissible* if $\lambda(e) \leq \sigma(e)$ for $e \in \operatorname{EA}(B)$ and $\lambda(e) < \sigma(e)$ for $e \in \operatorname{EP}(B)$. For an admissible function λ let

$$\varphi_{B,\lambda} = \left(\prod_{e \in \operatorname{IP}(B)} \alpha_{B,e}^{\sigma(e)} \right) \left(\prod_{e \in E \setminus B} \beta_{B,e}^{\lambda(e)} \right).$$

Note that in the case that σ is identically 1, an admissible function is simply the characteristic function of a subset of $\operatorname{EA}(B)$, so this does generalize the earlier result.

Theorem 5.9. *Let M be a matroid on totally ordered ground set E. The elements $\varphi_{B,\lambda}$ for $B \in \mathcal{B}(M)$ and λ admissible, form a $\mathbb{Q}$-basis for $\Phi^{(\sigma)}(M)$.*

Proof. Suppose $B \in \mathcal{B}(M)$ and λ is admissible. Define $\mu\colon E \setminus B \to \mathbb{N}$ by

$$\mu(e) = \max(\sigma(e) - \lambda(e), 1)$$

and consider the basis B_μ of $M^{(\sigma)}$ from Lemma 5.2. For any $f \notin B$ we clearly have

$$C_{B_\mu,f,\mu(f)} = C^{(\sigma)}_{B,f}.$$

From this it follows that

$$\begin{aligned} C^*_{B_\mu,e,i} &= \{(e,i)\} \cup \{(f,\mu(f))\colon (e,i) \in C_{B_\mu,f,\mu(f)}\} \\ &= \{(e,i)\} \cup \{(f,\mu(f))\colon e \in C_{B,f}\}. \end{aligned}$$

Thus if $e \in B$ we have

$$C^*_{B_\mu,e,i} = \{(e,i)\} \cup \{(f,\mu(f))\colon f \in C^*_{B,e}\}$$

and if $e \notin B$ then

$$C^*_{B_\mu,e,i} = \{(e,i),(e,\mu(e))\}$$

since e is not contained in the fundamental cycle of any element except itself.

Order the elements of $E^{(\sigma)}$ lexicographically, i.e. $(e,i) < (e',i')$ if $e < e'$ or $e = e'$ and $i < i'$. With respect to this ordering, the above description of the fundamental circuits and cocircuits clearly gives

$$\mathrm{IP}(B_\mu) = \{(e,i)\colon e \in \mathrm{IP}(B) \text{ or } e \notin B \text{ and } \mu(e) < i\}$$

and

$$\mathrm{EA}(B_\mu) = \{(e,1)\colon e \in \mathrm{EA}(B) \text{ and } \mu(e) = 1\}.$$

since $(e,1) \in C_{B_\mu,e,\mu(e)}$ always.

Note that we may have $\mu(e) = 1$ either when $\lambda(e) = \sigma(e)$ or $\lambda(e) = \sigma(e) - 1$. Let

$$S = \{(e,1)\colon \lambda(e) = \sigma(e)\} \subseteq \mathrm{EA}(B_\mu).$$

Let $P\colon \Phi^{(\sigma)}(M) \to \Phi(M^{(\sigma)})$ be the isomorphism of Theorem 5.5. Then we have for $e \in \mathrm{IP}(B)$ and $1 \le i \le \sigma(e)$

$$P(\alpha_{B,e}) = \alpha_{B_\mu,e,i}$$

and for $e \notin B$

$$P(\beta_{B,e}) = \beta_{B_\mu,e,\mu(e)} = \alpha_{B_\mu,e,j}$$

for any $j > \mu(e)$. Thus

$$\begin{aligned} P(\varphi_{B,\lambda}) &= \left(\prod_{e\in\mathrm{IP}(B)} P(\alpha_{B,e})^{\sigma(e)}\right)\left(\prod_{e\in E\setminus B} P(\beta_{B,e})^{\lambda}(e)\right) \\ &= \left(\prod_{e\in\mathrm{IP}(B)}\prod_{i=1}^{\sigma(e)} \alpha_{B_\mu,e,i}\right)\left(\prod_{e\in E\setminus B}\prod_{j=\mu(e)}^{\sigma(e)} \beta_{B_\mu,e,j}\right)\left(\prod_{e\in S}\beta_{B_\mu,e,1}\right) \\ &= \varphi_{B_\mu,S}. \end{aligned}$$

By Lemma 5.2, every basis is of the form B_μ for some μ. Given a pair (B_μ, S) for $S \subseteq \mathrm{EA}(B_\mu)$, write χ_S for the characteristic function and define

$$\lambda = \sigma - \mu + \chi_S.$$

Then we have $P(\varphi_{B,\lambda}) = \varphi_{B_\mu,S}$. Thus the image of the $\varphi_{B,\lambda}$ is the entirety of $\Phi(M^{(\sigma)})$, and since P is an isomorphism this implies that they form a basis. $\square$

Summing over all basis vectors immediately gives a formula for the Poincaré polynomial. This is a slight generalization of [16, Theorem 3.2].

Corollary 5.10. *The Poincaré polynomial of $\Phi^{(\sigma)}(M)$ is given by*

$$P(\Phi^{(\sigma)}(M); t) = \sum_{B \in \mathcal{B}(M)} t^{|\mathrm{IP}(B)|} \left(\prod_{e \in \mathrm{EP}(B)} \frac{1 - t^{\sigma(e)}}{1 - t} \right) \left(\prod_{e \in \mathrm{EA}(B)} \frac{1 - t^{1+\sigma(e)}}{1 - t} \right).$$

In the case of constant s, this can be expressed in terms of the Tutte polynomial.

Corollary 5.11. *For $s > 0$, the Poincaré polynomial of $\Phi^{(s)}(M)$ is given by*

$$P(\Phi^{(s)}(M); t) = t^{sr(M)} \left(\frac{1 - t^s}{1 - t} \right)^{|M| - r(M)} T\left(M; \frac{1}{t^s}, \frac{1 - t^{s+1}}{1 - t^s} \right).$$

Proof. Immediate from Proposition 2.5. $\square$

This Tutte polynomial formula implies that the Poincaré polynomial, at least when σ is a constant, satisfies a deletion-contraction recurrence. It is natural to expect that this recurrence reflects a short exact sequence analogous to Theorem 4.5. This we do in the next section.

5.4 Exact sequences for generalized circulation algebras

The formula of Corollary 5.11 gives a recurrence

$$P(\Phi^{(s)}(M); t) = (1 + \cdots + t^{s-1}) P(\Phi^{(s)}(M \setminus e); t) + t^s P(\Phi^{(s)}(M / e); t)$$

for which we would like to find a corresponding exact sequence. The formula suggests that

$$0 \to \bigoplus_{i=0}^{s-1} \Phi^{(s)}(M \setminus e)(-i) \to \Phi^{(s)}(M) \to \Phi^{(s)}(M)(-s) \to 0$$

should be exact for some choice of maps. We would expect that for $s = 1$, these should reduce to the maps that appear in Theorem 4.5.

The first term of the exact sequence is, ignoring the grading, simply the free $\Phi^{(s)}(M \setminus e)$-module of rank s. Thus if we expect the maps to be $\Phi^{(s)}(M \setminus e)$-linear as they are in the ordinary case, then ι is determined by a list $(\psi_0, \dots, \psi_{s-1})$ of elements of $\Phi^{(s)}(M)$. To make the grading work, we want ψ_i to be homogeneous of degree i. The most straightforward thing to do then is to take $\psi_i = \xi^i$ for some flow ξ. The map will certainly not be injective if $\xi \in \Phi^{(s)}(M \setminus e)$, so we insist that $\xi(e) \neq 0$. There is no obvious canonical choice of ξ, so we will remain flexible and avoid making any further demands. Explicitly we work with the map ι_ξ^s given by

$$\iota_\xi^s(\varphi_0, \dots, \varphi_{s-1}) = \sum_{j=0}^{s-1} \varphi_j \xi^j.$$

We then need the appropriate substitute for the ∂_e map. For any $e \in E$, define $\delta_e \colon E \to \mathbb{N}$ by

$$\delta_e(f) = \begin{cases} 1, & e = f \\ 0, & \text{otherwise.} \end{cases}$$

Then, so long as $\sigma(e) > 0$, we can consider the generalized circulation algebra $\Phi^{(\sigma-\delta_e)}(M)$. We can naturally identify $R^{(\sigma-\delta_e)}(E)$ with $R^{(\sigma)}(E)/\langle x_e^{\sigma(e)} \rangle$. We define a map $\partial_e \colon R^{(\sigma)}(E) \to R^{(\sigma-\delta_e)}(E)$ by

$$\frac{\partial \rho}{\partial x_e} + \left\langle x_e^{\sigma(e)} \right\rangle.$$

This is a $R^{(\sigma)}(E \setminus e)$-linear derivation.

Lemma 5.12. *For any regular matroid M and any element e which is not a coloop, $\partial_e \Phi^{(\sigma)}(M) = \Phi^{(\sigma-\delta_e)}(M)$. In particular, $\Phi^{(\sigma-\delta_e)}(M)$ is a $\Phi^{(\sigma)}(M \setminus e)$-submodule of $R^{(\sigma-\delta_e)}(E)$.*

Proof. Analogous to Proposition 4.4. □

More generally, for $1 \leq k \leq \sigma(e)$, we define

$$\partial_e^k = \frac{\partial^k \rho}{\partial x_e^k} + \left\langle x_e^{1+\sigma(e)-k} \right\rangle.$$

Proposition 5.13. *For any regular matroid M, non-coloop element e, and $1 \leq k \leq \sigma(e)$, $\partial_e^k \Phi^{(\sigma)}(M) = \Phi^{(\sigma-k\delta_e)}(M)$. In particular, $\Phi^{(\sigma-k\delta_e)}(M)$ is a $\Phi^{(\sigma)}(M \setminus e)$-submodule of $R^{(\sigma-k\delta_e)}(E)$.*

Proof. Follows by applying Lemma 5.12 repeatedly. □

In the case that $k = \sigma(e)$, we identify $\Phi^{(\sigma-k\delta_e)}(M)$ with $\Phi^{(\sigma)}(M \,/\, e)$. This allows us to finally write down the appropriate exact sequence.

Theorem 5.14. *Let M be a regular matroid, let e be an element of M which is not a coloop, and let ξ be a flow on M with $\xi(e) \neq 0$. The sequence*

$$0 \to \bigoplus_{i=0}^{s-1} \Phi^{(s)}(M \setminus e)(-j) \xrightarrow{\iota_\xi^s} \Phi^{(s)}(M) \xrightarrow{\partial_e^s} \Phi^{(s)}(M \,/\, e)(-s) \to 0$$

of graded $\Phi^{(s)}(M \setminus e)$-modules is exact.

In fact, this is a special case of the following more general result, which also encompasses the exact sequence obtained by applying the subdivision construction to Theorem 4.5. We extend the ι^s notation slightly by defining, for any fixed σ, element e, and $1 \le k \le \sigma(e)$, the map

$$\iota_\xi^k \colon \bigoplus_{i=1}^{k-1} \Phi^{(\sigma)}(M \,/\, e)(-i) \to \Phi^{(\sigma)}(M)$$

by

$$\iota^k(\varphi_0, \ldots, \varphi_{k-1}) = \sum_{j=0}^{k-1} \varphi_j \xi^j.$$

We have the following very general family of short exact sequences.

Theorem 5.15. *Let M be a regular matroid and let e an element of M which is not a coloop. Suppose $1 \le k \le \sigma(e)$. The sequence*

$$0 \to \bigoplus_{j=0}^{k-1} \Phi^{(\sigma)}(M \setminus e)(-j) \xrightarrow{\iota_\xi^k} \Phi^{(\sigma)}(M) \xrightarrow{\partial_e^k} \Phi^{(\sigma - k\delta_e)}(M)(-k) \to 0$$

of graded $\Phi^{(s)}(M \setminus e)$-modules is exact.

Proof. Injectivity of ι_ξ^k is equivalent to the set $\{1, \xi, \ldots, \xi^{k-1}\}$ being linearly independent over $\Phi^{(s)}(M \setminus e)$, which follows immediately from the independence of $\{1, x_e, \ldots, x_e^{k-1}\}$. Surjectivity of ∂_e^s follows from Proposition 5.13, so we need only show exactness in the middle.

Let $\{\theta_1, \ldots, \theta_d\}$ be a basis for $\mathrm{Flow}(M \setminus e)$, so $\{\theta_1, \ldots, \theta_d, \xi\}$ is a basis for $\mathrm{Flow}(M)$. Any element $\psi \in \Phi^{(\sigma)}(M)$ can be written as

$$\psi = F(\theta_1, \ldots, \theta_d, \xi)$$

for some polynomial F. Splitting this up by the exponent on the last variable, we get

$$\psi = \sum_{i=0}^{\sigma(e)} \varphi_i \xi^i$$

for some coefficients $\varphi_i \in \Phi^{(s)}(M \setminus e)$. Then

$$\partial_e^s = s! \xi(e) \varphi_s$$

which vanishes if and only if $\varphi_s = 0$, since $\xi(e) \neq 0$. Thus ψ is in the kernel of ∂_e^s if and only if it is a $\Phi^{(s)}(M \setminus e)$-linear combination of $1, \xi, \ldots, \xi^{s-1}$, i.e. it is in the image of ι_ξ. $\square$

Taking $k = \sigma(e)$ recovers a deletion-contraction sequence like Theorem 5.14, while taking $k = 1$ gives the sequence obtained by combining Theorem 4.5 with Theorem 5.5.

This sequence gives a formula for the Poincaré polynomial in terms of the multivariate Tutte polynomial.

Theorem 5.16. *The Poincaré polynomial of $\Phi^{(\sigma)}(M)$ is given by*

$$P(\Phi^{(\sigma)}(M);t) = (1-t)^{r(M)} \left(\prod_{e \in E} \frac{1-t^{\sigma(e)}}{1-t} \right) \tilde{Z}\left(M; 1-t, \left(\frac{t^{\sigma(e)} - t^{1+\sigma(e)}}{1-t^{\sigma(e)}} \right)_{e \in E} \right).$$

Proof. We will show both sides satisfy the same recurrence relation with the same initial conditions. The base case is when M consists of a single element e with $\sigma(e) = s$. The element is either a loop or a coloop. If it is a loop then

$$P(\Phi^{(\sigma)}(M);t) = 1 + \cdots + t^s$$

(easily seen by Theorem 5.5 for instance) and using the initial conditions for the multivariate Tutte polynomial given after Proposition 2.6, the right-hand side is

$$\begin{aligned}\frac{1-t^s}{1-t}\tilde{Z}\left(M; 1-t, \frac{t^s - t^{1+s}}{1-t^s}\right) &= \frac{1-t^s}{1-t}\left(1 + \frac{t^s - t^{1+s}}{1-t^s}\right) \\ &= \frac{1-t^{s+1}}{1-t}\end{aligned}$$

which matches.

If e is instead a coloop then

$$P(\Phi^{(\sigma)}(M);t) = 1$$

and the right-hand side is

$$\begin{aligned}(1-t)\frac{1-t^s}{1-t}\tilde{Z}\left(M; 1-t, \frac{t^s - t^{1+s}}{1-t^s}\right) &= (1-t^s)\left(1 + \frac{t^s - t^{1+s}}{(1-t)(1-t^s)}\right) \\ &= (1-t^s)\left(1 + \frac{t^s}{1-t^s}\right) \\ &= 1.\end{aligned}$$

By Theorem 5.7 and Proposition 2.7 both sides multiply over direct sums, so they agree for all matroids consisting of only loops and coloops.

When e is neither a loop nor a coloop, Theorem 5.15 gives

$$P(\Phi^{(\sigma)}(M);t) = \frac{1-t^{\sigma(e)}}{1-t} P(\Phi^{(\sigma)}(M \setminus e);t) + t^{\sigma(e)} P(\Phi^{(\sigma)}(M / e);t)$$

and Proposition 2.6 gives the same recurrence for the right-hand side. Thus the two are equal. □

Chapter 6

Open Problem: Logarithmic Concavity

Recall that a sequence $(a_0, a_1, \ldots, a_m)$ of real numbers is *logarithmically concave* if

$$a_i^2 \geq a_{i-1}a_{i+1}$$

for $0 < i < m$.

In [27], Wagner conjectured that the Poincaré polynomial for the ordinary circulation algebra of any graphic matroid has logarithmically concave coefficients. Ghislain McKay has verified that logarithmic concavity holds for both the cycle and bond matroids of all graphs on up to nine vertices. Less calculation has been done for regular matroids which are neither graphic nor cographic, but no counterexamples are known, and in [28] the conjecture was made that logarithmic concavity holds for this specialization of the Tutte polynomial for all matroids representable over a field of characteristic zero.

Of course, by Theorem 5.5, the result for arbitrary ordinary circulation algebras would also imply it for generalized circulation algebras.

Conjecture 6.1. *$P(\Phi^{(\sigma)}(M); t)$ has logarithmically concave coefficients for all regular matroids M and all σ.*

6.1 One possible approach

One approach to solving this problem, outlined in [29], uses the representation theory of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. In brief, to prove the logarithmic concavity of the graded dimension of a graded complex vector space A, it suffices to construct linear operators X and Y on $A \otimes A$ such that for all i and j,

- $X(A_i \otimes A_j) \subseteq A_{i-1} \otimes A_{j+1}$,
- $Y(A_i \otimes A_j) \subseteq A_{i+1} \otimes A_{j-1}$, and
- $XY - YX$ acts on $A_i \otimes A_j$ as scaling by $j - i$.

Thus we could prove Conjecture 6.1 by constructing a pair of such operators on $A = \Phi(M) \otimes \mathbb{C}$, for an arbitrary regular matroid M.

Some very weak partial progress in this direction has been made. Let E be a finite set and equip $R(E) \otimes \mathbb{C}$ with the inner product for which the square-free monomials form an orthonormal basis. Let M be a matroid on E, and $A = \Phi(M) \otimes \mathbb{C}$. Then for each $e \in E$, we can define an operator μ_e on A by defining $\mu_e(\varphi)$ to be the orthogonal projection of $x_e\varphi$ onto $\Phi(M)$. If we set

$$X = \sum_{e \in E} \mu_e^\vee \otimes \mu_e,$$
$$Y = \sum_{e \in E} \mu_e \otimes \mu_e^\vee,$$

where $\mu_e^\vee$ is the adjoint operator to μ_e, then it is clear that X and Y satisfy the first two of the needed three properties. It can be verified that in the case $M = U_{n-1,n}$, the matroid consisting of a single circuit, the third property holds as well. (Of course, the graded dimension of $\Phi(U_{n-1,n})$ is constant, so trivially log-concave.) Unfortunately, this fails miserably in the next-simplest case of $U_{1,3}$, with $XY - YX$ failing even to be diagonal in a homogeneous basis.

6.2 Another possible approach

Another method of proving logarithmic concavity results, which has been successfully applied to a number of graph and matroid polynomials, is to make use of homological identities from algebraic geometry, or combinatorial analogues thereof. (See [12] for a survey.) For our purposes, one particularly relevant example due to Huh [13] is the following: the *h-vector* of a d-dimensional simplicial complex Δ is defined by

$$h_i(\Delta) = [t^{d+1-i}]f(\Delta; t-1)$$

where $f(\Delta; t)$ is the polynomial with the property that $[t^{d-i}]f(\Delta; t)$ counts the number of i-dimensional simplices in Δ. In the case Δ is the simplicial complex whose simplices are the independent subsets of a matroid M, it can be shown [4] that

$$\sum_i h_i(\Delta)t^i = t^{r(M)}T(M; t^{-1}, 1).$$

We observe that the right-hand side is the Poincaré polynomial of the algebra $\widetilde{\Phi}(M)$ considered in Section 4.5. Huh [13, Theorem 3] proved logarithmic concavity of the h-vector for matroids representable over fields of characteristic zero, which implies in particular the following.

Theorem 6.2. *$P(\widetilde{\Phi}(M); t)$ has logarithmically concave coefficients for all regular matroids M.*

It is not clear whether Huh's methods could be extended to $\Phi(M)$. To make the analogy stronger, we offer the following construction of a simplicial complex whose h-vector coincides with the graded dimension of $\Phi(M)$. Given a matroid M on ground set E, let

$$E' = E \sqcup \{c_1, \ldots, c_{|E|-r(M)}\}$$

and let Δ' be the simplicial complex on E' with facets all sets of the form

$$F_S = S \sqcup \{c_1, \ldots, c_{|E|-|S|}\}$$

for S a spanning set of M, i.e. $r(S) = r(M)$. Then each face consists of a subset of E together with some of the c_i. A minimal spanning set containing a set $X \subseteq E$ has size $|X| + r(M) - r(S)$, with corresponding facet containing $\{c_1, \ldots, c_{|E|-|X|-r(M)+r(S)}\}$ so these are the ones that can be included with X. Thus we have

$$\begin{aligned} f(\Delta'; t) &= \sum_{X \subseteq E} t^{|E|-|X|}(1+t^{-1})^{|E|-|X|-r(M)+r(S)} \\ &= (1+t)^{|E|-r(M)} T(M; 1+t, 1+(1+t)^{-1}) \end{aligned}$$

by (2.1). Thus

$$\begin{aligned} h_i(\Delta') &= [t^{|E|-i}] f(\Delta'; t-1) \\ &= [t^{|E|-i}] T(M; t, 1+t^{-1}) \\ &= [t^i] P(\Phi(M); t). \end{aligned}$$

In general, Δ' is not of the kind for which Huh proved logarithmic concavity of the h-vector.[1] It could be hoped that it is "close enough" to attempt to apply similar methods, though to directly apply these methods one would need to find an algebraic variety that relates to this complex in a way analogous to the variety of critical points considered by Huh. It is unclear how this could be done.

[1]In the case of $U_{n-1,n}$ it *is* of the correct kind, giving yet another proof that constant sequences are logarithmically concave.

Appendix A

Isomorphisms of Graded Algebras

We now give a proof of the lemma necessary to fully justify the statement of Theorem 4.2. While straightforward, this result does seem to appear in the commutative algebra literature. A more general version appears in the relatively recent paper [2] but the case that we actually use is easier to prove.

Lemma A.1. *Let A and B be finite-dimensional graded algebras which are generated by homogeneous elements of degree 1 (i.e. no proper subalgebra of A contains the graded piece A_1, and analogously for B). Then A and B are isomorphic as algebras if and only if they are isomorphic as graded algebras.*

Proof. Suppose $\{a_1, \dots, a_d\}$ is a basis for A_1. Then by hypothesis it is also a generating set for A as an algebra. Indeed, it must be a *minimal* generating set for A, as any proper subset does not span A_1, and there is no way to obtain elements of A_1 as products in a nontrivial way. Let $P\colon A \to B$ be an isomorphism. We wish to modify P in order to construct an isomorphism that preserves the grading.

First, observe that $P(a_i) \in B_+$ for each i. This is because, since A is finite-dimensional, a_i must be nilpotent. Thus $P(a_i)$ is nilpotent also. Since B is generated by degree-1 elements, it follows that $B_0 = K$, so all nilpotent elements lie in B_+.

Now for each i, let b_i be the projection of $P(a_i)$ onto B_1. We observe that these elements must span B_1: any element of B can be written as some polynomial in the $P(a_i)$, and since $P(a_i) \in B_+$ only the degree-1 term can contribute to a degree-1 element.

For any $k_1, \dots, k_d$ we have

$$P(a_1^{k_1} \cdots a_d^{k_d}) = b_1^{k_1} \cdots b_d^{k_d} + (\text{higher-degree terms}).$$

Thus if F is a homogeneous polynomial, we have

$$P(F(a_1, \dots, a_d)) = F(b_1, \dots, b_d) + (\text{higher-degree terms})$$

and hence if $F(a_1, \dots, a_d) = 0$ then $F(b_1, \dots, b_d) = 0$. It follows that there is a well-defined algebra homomorphism $\hat{P}\colon A \to B$ given by $a_i \mapsto b_i$. Since its image clearly contains B_1, it is surjective. Since A and B are isomorphic, they have the same dimension, so a surjective map is necessarily an isomorphism. Finally, since $\hat{P}$ maps A_1 to B_1 it is grade-preserving. $\square$

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