C^*-ALGEBRAS ASSOCIATED TO BOOLEAN DYNAMICAL SYSTEMS

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Dedicated to the memory of Uffe Haagerup

ABSTRACT. The goal of these notes is to present the C^*-algebra $C^*(\mathcal{B},\mathcal{L},\theta)$ of a Boolean dynamical system $(\mathcal{B},\mathcal{L},\theta))$, that generalizes the C^*-algebra associated to Labelled graphs introduced by Bates and Pask, and to determine its simplicity, its gauge invariant ideals, as well as compute its K-Theory.

1. INTRODUCTION

In 1980 Cuntz and Krieger [9] associated a C^*-algebra $\mathcal{O}_A$ to a shift of finite type with transition matrix A. Various authors –including Bates, Fowler, Kumjian, Laca, Pask and Raeburn– extended the original construction to more general subshifts associated to oriented graphs, giving origin to the graph C^*-algebra $C^*(E)$ associated to E (see e.g. [17, 24]). Using a different approach, Exel and Laca [14] generalize Cuntz-Krieger algebras, by associating a C^*-algebra to an infinite matrix which 0 and 1 entries. After that, with the goal of unifying Exel-Laca algebras and graph C^*-algebras, Tomforde [30] introduced the class of ultragraph algebras. Also, motivated by Cuntz-Krieger construction, Matsumoto [27] introduced a C^*-algebra associated to a general two-sided subshift over a finite alphabet. Later, the first named author [7] extended Matsumoto's construction, by constructing the C^*-algebra $\mathcal{O}_\Lambda$ associated to a general one-sided subshift Λ over a finite alphabet.

The underlying idea of associating a C^*-algebra to a dynamical system comes from the Franks classification of irreducible shifts of finite type up to flow equivalence [16]. This classification use the Bowen-Franks group of the shift space, that turns out to be the K_0 group of the associated Cuntz-Krieger algebra [9]. Therefore, the point was to state a connection between classification of shift spaces and classification of C^*-algebras. In this line, the recent results of Matsumoto and Matui [28] characterize continuous orbit equivalence of shifts of finite type by using K-theoretical invariants of the associated C^*-algebra. Next step was to extend the scope of this strategy to classify shift space over a countable alphabet. By adapting the left-Krieger cover construction given in [25], any shift space over a countable alphabet may be presented by a left-resolving labelled graph. Thus, in the same spirit of the previous constructions, labelled graph algebras, introduced by Bates and Pask in [1], provided a method for associating a C^*-algebra to a shift space over a countable alphabet. The class of

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labelled graph C^*-algebras contains, in particular, all the above C^*-algebra classes. Properties like simplicity, ideal structure and purely infinity was studied in [2, 20] and the computation of the K-theory was achieved in [3].

The original goal of the present paper was to continue the study of the labelled graph C^*-algebras, by characterizing them as 0-dimensional topological graphs [21]. However, the topological graph E associated to the data of the labelled graph is just a realization as a Boolean algebra of a family of subsets of vertices of E, plus some partial actions given by the arrows of E. Thus, we adapt the labelled graph C^*-algebra construction, as well as our topological graph characterization, to the context of a C^*-algebra associated to a general family of partial actions over a fixed Boolean algebra (we call it a Boolean dynamical system). This class of C^*-algebras, that we call Boolean Cuntz-Krieger algebras associated to Boolean dynamical systems, includes labelled graph C^*-algebras, homeomorphism C^*-algebras over 0-dimensional compact spaces, and graph C^*-algebras, among others. Essentially, it is not a new class of C^*-algebras, since they are (0-dimensional) algebras over topological graphs, a class deeply studied by Katsura [21, 22]. However, the advantage of our approach is that we can skip to deal with the topology of the graph, and concentrate only in combinatorial properties of actions over a Boolean algebra. In particular, we can benefit of a different picture when studying C^*-algebras associated to combinatorial objects, by using groupoid C^*-algebras. This is a classical approach, as shown by Kumjian, Pask, Raeburn and Renault [24] when studying graph C^*-algebras. This approach attained a new level of efficiency when Exel [11] developed a huge machinery that helps to represent any "combinatorial" C^*-algebra as a full groupoid C^*-algebra. The strategy is to associate to the C^*-algebra an $*$-inverse semigroup (see e.g. [26]) and a "tight" representation (i.e. a representations preserving additive identities on pairwise orthogonal idempotents). When this situation holds, there is a standard way of producing a étale, second countable topological groupoid which full C^*-algebra is isomorphic to the original C^*-algebra under consideration. In the case of Boolean Cuntz-Krieger algebras associated to Boolean dynamical system this strategy works, and so we can use all the machinery developed by Exel [11, 12] for analyze the structure of the algebras under study. A recent example of application of such an strategy is [15].

The contents of this paper can be summarized as follows: In Section 2 we recall Boolean algebra Theory. In particular, we summarize some well-known results about the topology of the space of characters (Stone's spectrum) of a Boolean algebra. In Section 3 we define Boolean dynamical systems, that are families of partial actions on a Boolean algebra, and their representations in a C^*-algebra; the C^*-algebra associated to the universal representation will be the Boolean Cuntz-Krieger algebra. We state the existence of a universal representation and the gauge uniqueness theorem, that will be proved later. In Section 4 we recall the definition of Katsura's topological graph. When E is a 0-dimensional space, i.e. both the vertex and edge spaces are 0-dimensional and compactly supported (definition 4.4), we construct a Boolean dynamical system that can be represented in the associated topological graph C^*-algebra $\mathcal{O}(E)$. In Section 5 we focus on finding a universal representation of a given Boolean dynamical system. This is achieved by constructing a compactly supported 0-dimensional topological graph with the data of the Boolean dynamical system, and defining a representation of the Boolean dynamical system in the topological graph C^*-algebra. We conclude proving that the Boolean Cuntz-Krieger algebras are isomorphic to a 0-dimensional topological graph C^*-algebra, and using this characterization to compute its K-Theory. In Sections 6,7 and 8 we apply Exel's machinery to Boolean Cuntz-Krieger algebras. To this end, we first define an $*$-inverse semigroup associated to a Boolean dynamical system, and then we prove that the C^*-algebra associated to the universal tight representation of this $*$-inverse semigroup is isomorphic to our Boolean Cuntz-Krieger algebra. Finally, we define the groupoid of germs of the partial actions of the $*$-inverse semigroup on the space of tight filters defined over its semilattice of idempotents. Thus, by using Exel's results, we can see that the Boolean Cuntz-Krieger algebra is the full C^*-algebra of this groupoid. This allows us to work in the realm of groupoid C^*-algebra, and to use the known results on this class to characterize properties of Boolean Cuntz-Krieger algebras. In particular, we use the groupoid characterization of the Boolean Cuntz-Krieger algebras in Section 9 to characterize its simplicity in terms of intrinsic properties of the associated Boolean dynamical system. A similar approach was used by Marrero and Muhly for ultragraph C^*-algebras [29], although the way they constructed the groupoid is quite different to ours; also, after the final version of the present paper was ready, we were aware of Boava, de Castro and Mortari's work for labelled graph C^*-algebras [4], were they constructed an inverse semigroup in the same mood as our $\mathcal{S}$ (see Section 6), but they concentrated their attention in understanding the nature of the tight spectra, and do not work out either an associated groupoid or a groupoid picture of labelled C^*-algebras associated to it. In Section 10 we define an admissible pair for a Boolean dynamical system, and we state an order lattice bijection between the admissible pairs and the gauge invariant ideals of the Boolean Cuntz-Krieger algebras. Finally, we realize the quotient of a Boolean Cuntz-Krieger algebra modulo a gauge invariant ideal as the Boolean Cuntz-Krieger algebra of another induced Boolean dynamical system.

2. Boolean C^*-algebras

The main object we will use in this paper is a Boolean algebra and its associated C^*-algebras. We will first introduce the basic definitions and results, mostly well-known, and then we will focus on finding a representation of a Boolean algebra as the set of open subsets of a topological space (Stone's representation). It turns out that the points of this topological space are the set of the ultrafilters of the elements of the Boolean algebra.

Definition 2.1. A *Boolean algebra* is a quadruple $(\mathcal{B}, \cap, \cup, \setminus)$, where $\mathcal{B}$ is a set with a distinguished element $\emptyset \in \mathcal{B}$, that we called *empty*, and maps $\cup : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, $\cap : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and $\setminus : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ that we call the *union*, *intersection* and *relative complement* maps, satisfying the standard axioms (see [18, Chapter 2]). The Boolean algebra $\mathcal{B}$ is *unital* if does exist $\mathbf{1} \in \mathcal{B}$ such that $\mathbf{1} \cup A = \mathbf{1}$ and $\mathbf{1} \cap A = A$ for every $A \in \mathcal{B}$.

Remark 2.2. What we call a Boolean algebra is sometimes called a Boolean ring, and that what we call a unital Boolean algebra is sometimes simple called a Boolean algebra. The theories of Boolean algebras and Boolean rings are very closely related; in fact, they are just different ways of looking at the same subject. See [18] for further explanation.

A subset $\mathcal{B}' \subseteq \mathcal{B}$ is called a *Boolean subalgebra* if $\mathcal{B}'$ is closed by the union, intersection and the relative complement operations.

Given a Boolean algebra $\mathcal{B}$, we can define the following partial order: given $A, B \in \mathcal{B}$

$$A \subseteq B \qquad \text{if and only if} \qquad A \cap B = A\,.$$

Then $(\mathcal{B}, \subseteq)$ is a partially ordered set.

Definition 2.3. Let $\mathcal{B}$ be a Boolean algebra. A subset $\mathcal{C} \subseteq \mathcal{B}$ is called a *filter of* $\mathcal{B}$ if satisfies:
F0: $\emptyset \notin \mathcal{C}$,
F1: given $B \in \mathcal{B}$ and $A \in \mathcal{C}$ with $A \subseteq B$ then $B \in \mathcal{C}$,
F2: given $A, B \in \mathcal{C}$ then $A \cap B \in \mathcal{C}$.
If moreover $\mathcal{C}$ satisfies:
F3: given $A \in \mathcal{C}$ and $B, B' \in \mathcal{B}$ with $A = B \cup B'$ then either $B \in \mathcal{C}$ or $B' \in \mathcal{C}$,
then it is called an *ultrafilter of* $\mathcal{B}$.

Given two filters $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{B}$, we say that $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if every $A_1 \in \mathcal{F}_1$ is also in $\mathcal{F}_2$. This defines a partial order on the set of filters of $\mathcal{B}$. Then, an easy application of the Zorn's Lemma shows that an ultrafilter as a maximal filter.

We will denote by $\mathcal{D}(\mathcal{B})$ the set of ultrafilters of $\mathcal{B}$. Given any $A \in \mathcal{B}$, we define the *cylinder set of* A as $Z(A) := \{\mathcal{C} \in \mathcal{D}(\mathcal{B}) : A \in \mathcal{C}\}$. The following (straightforward) result shows that the family $\{Z(A) : A \in \mathcal{B}\}$ defines a topology of $\mathcal{D}(\mathcal{B})$, in which the sets $Z(A)$ are clopen and compact. We will call $\mathcal{D}(\mathcal{B})$ the *Stone's spectrum of* $\mathcal{B}$.

Lemma 2.4. *Let $\mathcal{B}$ be a Boolean algebra. Let $A, B \in \mathcal{B}$ then:*
(1) $Z(A) \cap Z(B) = Z(A \cap B)$,
(2) $Z(A) \cup Z(B) = Z(A \cup B)$,
(3) if $A \subseteq B$ then $Z(B \setminus A) = Z(B) \setminus Z(A)$.

Proof. (1) Let $\mathcal{C} \in Z(A \cap B)$, then $A \cap B \in \mathcal{C}$. Therefore by **F1** we have that $A, B \in \mathcal{C}$, and hence $\mathcal{C} \in Z(A) \cap Z(B)$. Conversely, let $\mathcal{C} \in Z(A) \cap Z(B)$, so $A, B \in \mathcal{C}$. Therefore by **F2** we have that $A \cap B \in \mathcal{C}$, so $\mathcal{C} \in Z(A \cap B)$.

(2) Let $\mathcal{C} \in Z(A \cup B)$, so $A \cup B \in \mathcal{C}$. Then by **F3** either $A \in \mathcal{C}$ or $B \in \mathcal{C}$, so either $\mathcal{C} \in Z(A)$ or $\mathcal{C} \in Z(B)$. Thus, $\mathcal{C} \in Z(A) \cup Z(B)$. In the other hand, let $\mathcal{C} \in Z(A) \cup Z(B)$. Then either $\mathcal{C} \in Z(A)$ or $\mathcal{C} \in Z(B)$, so $A \cup B \in \mathcal{C}$ by **F1**. Therefore, $\mathcal{C} \in Z(A \cup B)$.

(3) Let $\mathcal{C} \in Z(B) \setminus Z(A)$, i.e., $B \in \mathcal{C}$ but $A \notin \mathcal{C}$. Then $(B \setminus A) \cup A \in \mathcal{C}$, but by **F3** either $B \setminus A$ or A belongs to $\mathcal{C}$. But by hypothesis, $A \notin \mathcal{C}$, we have that $B \setminus A \in \mathcal{C}$. Thus, $\mathcal{C} \in Z(B \setminus A)$. Conversely, let $B \setminus A \in \mathcal{C}$. Then by **F1** $B \in \mathcal{C}$, but A cannot belong to $\mathcal{C}$ because otherwise $\emptyset = A \cap (B \setminus A) \in \mathcal{C}$ by **F2**, but this contradicts **F0**. Therefore, $\mathcal{C} \in Z(B) \setminus Z(A)$. □

And as consequence it follows:

Lemma 2.5. *Let $\mathcal{B}$ be a Boolean algebra and let $\mathcal{D}(\mathcal{B})$ the Stone's spectrum of $\mathcal{B}$. If $A \in \mathcal{B}$, then $Z(A) \subseteq \mathcal{D}(\mathcal{B})$ is a clopen set.*

Example 2.6. Let $X = \mathbb{N}$ and let $\mathcal{B} := \{F \subseteq \mathbb{N} : F \text{ finite }\} \cup \{\mathbb{N} \setminus F : F \text{ finite }\}$. Clearly, $\mathcal{B}$ is a Boolean algebra. Given $i \in \mathbb{N}$ we have that $\mathcal{C}_i = \{A \in \mathcal{B} : i \in A\}$. We will see that there exists an ultrafilter $\mathcal{C}$ of $\mathcal{B}$ that is not of the form $\mathcal{C}_i$ for some $i \in \mathbb{N}$. Indeed, let us define

$$\mathcal{C}_\infty := \{A \in \mathcal{B} : \exists N \in \mathbb{N} \text{ such that } k \in A \ \forall k \geq N\},$$

that is clearly an ultrafilter of $\mathcal{B}$. Now, let $\mathcal{C}$ be an ultrafilter of $\mathcal{B}$ such that $\bigcap\limits_{A \in \mathcal{C}} A = \emptyset$. Given $k \in \mathbb{N}$, let us denote by $[k, \infty)$ the set $\mathbb{N} \setminus \{1, \ldots, k-1\} \in \mathcal{B}$. Observe that, since $\bigcap\limits_{A \in \mathcal{C}} A = \emptyset$, given any $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ and $A_1, \dots, A_{n_k} \in \mathcal{C}$ such that $A_1 \cap \cdots \cap A_{n_k} \subseteq [k, \infty)$. Therefore, by **F1**, $[k, \infty) \in \mathcal{C}$ for every $k \in \mathbb{N}$.

Now, given any $A \in \mathcal{C}_\infty$, there exists $k \in \mathbb{N}$ such that $[k, \infty) \subseteq A$, whence $A \in \mathcal{C}$ by **F1**. On the other side, given any $A \in \mathcal{C}$, we claim that $|A| = \infty$. Otherwise, if $|A| = n < \infty$, then there exist $A_1, \dots, A_n \in \mathcal{C}$ such that $A \cap A_1 \cap \cdots \cap A_n = \emptyset$, contradicting condition **F2**. Thus, $|A| = \infty$. Therefore, since $A \in \mathcal{B}$, we have that $A = \mathbb{N} \setminus F$ for some finite set F of $\mathbb{N}$. Then, there exists $k \in \mathbb{N}$ such that $[k, \infty) \subseteq A$. So, since $[k, \infty) \in \mathcal{C}_\infty$, condition **F1** says that $A \in \mathcal{C}_\infty$ too. Thus $\mathcal{C} = \mathcal{C}_\infty$.

Therefore, we have that $\mathcal{D}(\mathcal{B}) = \{\mathcal{C}_i : i \in \mathbb{N} \cup \{\mathcal{C}_\infty\}\}$. Finally observe that, with the induced topology, we have that $\mathcal{D}(\mathcal{B})$ is the one point compactification of $\mathbb{N}$.

Definition 2.7. An element $B \in \mathcal{B}$ is called a *least upper-bound* for $\{A_\lambda\}_{\lambda \in \Lambda}$ with $A_\lambda \in \mathcal{B}$ if it is the least element of $\mathcal{B}$ satisfying $A_\lambda \subseteq B$ for every $\lambda \in \Lambda$. We will write the unique least upper-bound as $\bigcup\limits_{\lambda \in \Lambda} A_\lambda$.

Observe that least upper-bound do not necessarily exist, but if $|\Lambda| < \infty$ then the least upper-bound of $\{A_\lambda\}_{\lambda \in \Lambda}$ is $\bigcup\limits_{\lambda \in \Lambda} A_\lambda$.

Definition 2.8. Let $\mathcal{B}$ be a Boolean algebra. We say that a subset $\mathcal{I}$ of $\mathcal{B}$ is an *ideal* if given $A, B \in \mathcal{B}$, then:

(1) if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$,
(2) if $A \in \mathcal{I}$ then $A \cap B \in \mathcal{I}$.

An ideal $\mathcal{I}$ of a Boolean algebra $\mathcal{B}$ is itself a Boolean algebra.

Example 2.9. Given a collection $\{A_\lambda\}_{\lambda \in \Lambda}$ of elements $A_\lambda \in \mathcal{B}$, the subset

$$\mathcal{I}_{\bigcup\limits_{\lambda \in \Lambda} A_\lambda} := \left\{ A \in \mathcal{B} : \exists \lambda_1, \dots, \lambda_n \text{ such that } A \subseteq \bigcup_{i=1}^{n} A_{\lambda_i} \right\}$$

is an ideal of $\mathcal{B}$. Observe that every ideal $\mathcal{I}$ of $\mathcal{B}$ is of this form.

Definition 2.10. Let $\mathcal{B}$ be the Boolean algebra and let $\mathcal{I}$ be an ideal of $\mathcal{B}$. Given $A, B \in \mathcal{B}$, we define the following equivalent relation: $A \sim B$ if and only if there exists $A', B' \in \mathcal{I}$ such that $A \cup A' = B \cup B'$. We define by $[A]$ the set of all the elements of $\mathcal{B}$ equivalent to A, and we denote by $\mathcal{B}/\mathcal{I}$ the set of all equivalent classes of $\mathcal{B}$. Moreover, we say that $[A] \subseteq [B]$ if and only if there exists $H \in \mathcal{I}$ such that $A \subseteq B \cup H$.

Let $\mathcal{B}$ be a Boolean algebra, and let $\mathcal{I}$ be an ideal of $\mathcal{B}$. Then, the map $\iota : \mathcal{D}(\mathcal{I}) \longrightarrow \mathcal{D}(\mathcal{B})$ defined by $\iota(\mathcal{C}) = \{A \in \mathcal{B} : B \subseteq A \text{ for some } B \in \mathcal{C}\}$ is injective. So, given $A \in \mathcal{B}$, we have that $Z(A) = \iota(\mathcal{D}(\mathcal{I}_A))$.

Moreover, there exists a bijection between the ultrafilters of $\mathcal{B}/\mathcal{I}$ and the ultrafilters of $\mathcal{B}$ that do not contain any element of $\mathcal{I}$. Therefore, the natural map $\pi : \mathcal{B} \to \mathcal{B}/\mathcal{I}$ is surjective, and it induces an injective map $\iota : \mathcal{D}(\mathcal{B}/\mathcal{I}) \longrightarrow \mathcal{D}(\mathcal{B})$ given by $[\mathcal{C}] \to \pi^{-1}([\mathcal{C}]) = \{A \in \mathcal{B} : [A] \in [\mathcal{C}]\}$ for every $[\mathcal{C}] \in \mathcal{D}(\mathcal{B}/\mathcal{I})$.

So, we will identify $\mathcal{D}(\mathcal{I})$ and $\mathcal{D}(\mathcal{B}/\mathcal{I})$ with the corresponding subspaces of $\mathcal{D}(\mathcal{B})$.

Lemma 2.11. *Let $\mathcal{B}$ be a Boolean algebra and let $\mathcal{I}$ be an ideal of $\mathcal{B}$.*

(1) If $U \subseteq \mathcal{D}(\mathcal{B})$, then U is an open subset of $\mathcal{B}$ if and only if $U = \mathcal{D}(\mathcal{I})$.

(2) $\mathcal{D}(\mathcal{B}) = \mathcal{D}(\mathcal{I}) \cup \mathcal{D}(\mathcal{B}/\mathcal{I})$ *and* $\mathcal{D}(\mathcal{I}) \cap \mathcal{D}(\mathcal{B}/\mathcal{I}) = \emptyset$. *Thus, for every closed subset* V *of* $\mathcal{D}(\mathcal{B})$, *there exists an ideal* $\mathcal{I}$ *of* $\mathcal{B}$ *such that* $V = \mathcal{D}(\mathcal{B}/\mathcal{I})$.

Proof. (1) Let U be an open subset of $\mathcal{D}(\mathcal{B})$. Given $\mathcal{C} \in U$, let us pick one $A_{\mathcal{C}} \in \mathcal{B}$ such that $Z(A_{\mathcal{C}}) \subseteq U$ and $\mathcal{C} \in Z(A_{\mathcal{C}})$. If we define $\mathcal{I} = \mathcal{I}_{\bigcup_{\mathcal{C}\in U} A_{\mathcal{C}}}$, then $U = \mathcal{D}(\mathcal{I})$. Conversely, if $U = \mathcal{D}(\mathcal{I})$, let us define $\Lambda_{\mathcal{I}} = \{A \in \mathcal{B} : A \in \mathcal{I}\}$. Then, $U = \bigcup_{A\in\Lambda_{\mathcal{I}}} Z(A)$, so U is open.

(2) Let $\mathcal{C} \in \mathcal{D}(\mathcal{B})$, and suppose that $\mathcal{C} \cap \mathcal{I} = \emptyset$. Let $[\mathcal{C}] = \{[A] : A \in \mathcal{C}\}$ be a subset of $\mathcal{B}/\mathcal{I}$. It is routine to check that $[\mathcal{C}]$ is an ultrafilter of $\mathcal{B}/\mathcal{I}$, and that $\iota([\mathcal{C}]) = \mathcal{C}$. Finally, let $\mathcal{C} \in \mathcal{D}(\mathcal{I}) \cap \mathcal{D}(\mathcal{B}/\mathcal{I})$. Then, there exists $A \in \mathcal{C}$ with $A \in \mathcal{I}$. But then $[A] \in [\mathcal{C}]$, contradicting the fact that $[A] = [\emptyset]$. □

Now, we will describe the associated topological space that represents the Boolean algebra, the so-called Stone's representation..

Given a Boolean algebra $\mathcal{B}$ and given $A \in \mathcal{B}$ we let χ_A denote the function defined on $\mathcal{B}$ by

$$\chi_A(B) = \begin{cases} 1 & \text{if } A \cap B \neq \emptyset \\ 0 & \text{otherwise} \end{cases} .$$

We will regard χ_A as an element of the C^*-algebra of bounded operators on $\ell^2(\mathcal{B})$.

Definition 2.12. Let $\mathcal{B}$ be a Boolean algebra. Then we define the *Boolean* C^**-algebra of* $\mathcal{B}$ as the sub-C^*-algebra of the $B(\ell^2(\mathcal{B}))$ generated by $\{\chi_A : A \in \mathcal{B}\}$. We denote it as $C^*(\mathcal{B})$.

$C^*(\mathcal{B})$ is a commutative C^*-algebra, and given $A, B \in \mathcal{B}$ we have that

$$\chi_A \cdot \chi_B = \chi_{A\cap B} \qquad \text{and} \qquad \chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B} ,$$

where $\chi_\emptyset = 0$. Thus, $C^*(\mathcal{B}) = \overline{\text{span}}\{\chi_A : A \in \mathcal{B}\}$.

First, recall that the spectrum of $C^*(\mathcal{B})$, denoted by $\widehat{C^*(\mathcal{B})}$, is the set of characters of $C^*(\mathcal{B})$. Observe that an additive map $\eta : C^*(\mathcal{B}) \longrightarrow \mathbb{C}$ is a $*$-homomorphism if and only if given $A, B \in \mathcal{B}$

$$\begin{array}{ll} (C1) & \eta(\chi_A)\eta(\chi_B) = \eta(\chi_{A\cap B}) \\ (C2) & \eta(\chi_{A\cup B}) = \eta(\chi_A) + \eta(\chi_B) - \eta(\chi_{A\cap B}) . \end{array}$$

If η is a character of $C^*(\mathcal{B})$, then we define

$$\mathcal{C}_\eta := \{A \in \mathcal{B} : \eta(\chi_A) = 1\} .$$

Recall that, since χ_A is a projection for every $A \in \mathcal{B}$ and η is a $*$-homomorphism, $\eta(\chi_A)$ is either 0 or 1.

Lemma 2.13. *If* η *is a character of* $C^*(\mathcal{B})$, *then* $\mathcal{C}_\eta$ *is an ultrafilter of* $\mathcal{B}$.

Proof. We must check **F0** − **F3**. For **F0**, recall that by definition $\chi_\emptyset = 0$, and thus $\eta(\chi_\emptyset) = 0$, so $\emptyset \notin \mathcal{C}_\eta$. For **F1**, let $A, B \in \mathcal{B}$ with $A \subseteq B$ and $\eta(\chi_A) = 1$. Since $\chi_A = \chi_A\chi_B$, it follows that $1 = \eta(\chi_A) = \eta(\chi_A)\eta(\chi_B) = \eta(\chi_B)$, so $B \in \mathcal{C}_\eta$ as desired. For **F2**, let $A, B \in \mathcal{C}_\eta$. Then, using $(C1)$, we have that $1 = \eta(\chi_A)\eta(\chi_B) = \eta(\chi_{A\cap B})$, so $A \cap B \in \mathcal{C}_\eta$. Finally, for **F3**, let $A \in \mathcal{C}_\eta$ and $B, B' \in \mathcal{B}$ with $A = B \cup B'$. Then, using $(C2)$, it follows that

$$1 = \eta(\chi_A) = \eta(\chi_{B\cup B'}) = \eta(\chi_B) + \eta(\chi_{B'}) - \eta(\chi_{B\cap B'}) .$$

Therefore, either $B \in \mathcal{C}_\eta$ or $B' \in \mathcal{C}_\eta$, as desired. □

Given an ultrafilter $\mathcal{C}$ of $\mathcal{B}$, we define the unique additive map $\eta_{\mathcal{C}} : C^*(\mathcal{B}) \longrightarrow \mathbb{C}$ such that

$$\eta_{\mathcal{C}}(\chi_A) = \begin{cases} 1 & \text{if } A \in \mathcal{C} \\ 0 & \text{if } A \notin \mathcal{C} \end{cases}$$

Lemma 2.14. *$\eta_{\mathcal{C}}$ is a character of $C^*(\mathcal{B})$.*

Proof. We must check that $\eta_{\mathcal{C}}$ satisfies $C1$ and $C2$. For $C1$, let $A, B \in \mathcal{B}$, and recall that $\chi_A \cdot \chi_B = \chi_{A\cap B}$. First, suppose that $\eta_{\mathcal{C}}(\chi_{A\cap B}) = 0$. Therefore, $A \cap B \notin \mathcal{C}$ and hence, by **F2**, either A or B are not in $\mathcal{C}$. Thus, $\eta_{\mathcal{C}}(A)\eta_{\mathcal{C}}(A) = 0 = \eta_{\mathcal{C}}(\chi_{A\cap B})$, as desired. Now, suppose that $\eta_{\mathcal{C}}(\chi_{A\cap B}) = 1$, so $A \cap B \in \mathcal{C}$. Therefore, by **F1**, it follows that $A, B \in \mathcal{C}$ too, and hence $\eta_{\mathcal{C}}(A)\eta_{\mathcal{C}}(A) = 1 = \eta_{\mathcal{C}}(\chi_{A\cap B})$, as desired. Thus, $C1$ is verified.

For $C2$, let $A, B \in \mathcal{B}$. First, suppose that $\eta_{\mathcal{C}}(\chi_{A\cup B}) = 0$. So, $A \cup B \notin \mathcal{C}$, and since $A, B, A \cap B \subseteq A \cup B$, it follows from **F1** that $A, B, A \cap B \notin \mathcal{C}$. Therefore,

$$\eta_{\mathcal{C}}(\chi_{A\cup B}) = 0 = \eta_{\mathcal{C}}(\chi_A) + \eta_{\mathcal{C}}(\chi_B) - \eta(\chi_{A\cap B})\,.$$

Finally, suppose that $A \cup B \in \mathcal{C}$. Hence, by **F3**, either A or B belongs to $\mathcal{C}$. First suppose that $A, B \in \mathcal{C}$. Then, by **F2** so does $A \cap B$. Therefore,

$$\eta_{\mathcal{C}}(\chi_{A\cup B}) = 1 + 1 - 1 = \eta_{\mathcal{C}}(\chi_A) + \eta_{\mathcal{C}}(\chi_B) - \eta(\chi_{A\cap B})\,,$$

as desired. Now, suppose that $A \in \mathcal{C}$ but $B \notin \mathcal{C}$. By **F2**, we have that $A \cap B \notin \mathcal{C}$, so

$$\eta_{\mathcal{C}}(\chi_{A\cup B}) = 1 + 0 - 0 = \eta_{\mathcal{C}}(\chi_A) + \eta_{\mathcal{C}}(\chi_B) - \eta(\chi_{A\cap B})\,,$$

as desired. □

The following result follows directly from the definitions.

Proposition 2.15. *Let $\mathcal{C}$ be an ultrafilter of $\mathcal{B}$ and let η a character of $\mathcal{A}$. Then $\mathcal{C}_{\eta_{\mathcal{C}}} = \mathcal{C}$ and $\eta_{\mathcal{C}_\eta} = \eta$. Therefore, there is a bijection between the ultrafilters of $\mathcal{B}$ and the characters of $\mathcal{A}$.*

By Proposition 2.15 there is a bijection between $\mathcal{D}(\mathcal{B})$ and the set of characters of $C^*(\mathcal{B})$. Now, we will endow $\mathcal{D}(\mathcal{B})$ with a topology such that it become homeomorphic to the spectrum of $C^*(\mathcal{B})$. Recall that by the Gelfand-Naimark Theorem $C^*(\mathcal{B}) \cong C_0(\widehat{C^*(\mathcal{B})})$, where $\widehat{C^*(\mathcal{B})}$ has the Jacobson topology. Recall that, given a subset of Y of $\widehat{C^*(\mathcal{B})}$, we define the closure of Y as $\{\eta \in \widehat{C^*(\mathcal{B})} : \text{Ker } \eta \supseteq \bigcap\limits_{\rho\in Y} \text{Ker } \rho\}$.

Proposition 2.16 (Stone's Representation Theorem)**.** *Let $\mathcal{B}$ be a Boolean algebra and let $\mathcal{D}(\mathcal{B})$ be the Stone's spectrum of $\mathcal{B}$. Then $\widehat{C^*(\mathcal{B})}$ and $\mathcal{D}(\mathcal{B})$ are homeomorphic topological spaces. Therefore, $C^*(\mathcal{B}) \cong C_0(\mathcal{D}(\mathcal{B}))$.*

Proof. First recall that, using Proposition 2.15, we identify a character η of $C^*(\mathcal{B})$ with its associated ultrafilter $\mathcal{C}_\eta$. Observe that, given $\mathcal{C} \in \mathcal{D}(\mathcal{B})$, we have $\text{Ker } \eta_{\mathcal{C}} = \{\chi_B : B \notin \mathcal{C}\}$. Then, given a set $Y \subseteq \mathcal{D}(\mathcal{B})$, we define

$$I_Y := \bigcap_{\mathcal{C}\in Y} \text{Ker } \eta_{\mathcal{C}} = \overline{\text{span}}\{\chi_B : B \notin \mathcal{C},\, \forall \mathcal{C} \in Y\}\,.$$

Using the definitions, it is straightforward to check that $I_Y = \overline{\text{span}}\{\chi_B : B \in \mathcal{B},\, Y \cap Z(B) = \emptyset\}$.

Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of elements of $\mathcal{B}$ and let us consider $V := \bigcup\limits_{\lambda\in\Lambda} Z(A_\lambda)$. We will prove that $Y := \mathcal{D}(\mathcal{B}) \setminus V$ is closed in the Jacobson topology, whence every closed subset of $\mathcal{D}(\mathcal{B})$ with respect to the induced topology $\mathcal{T}$ is also closed with respect to the Jacobson topology. Hence, $I_Y = \overline{\text{span}}\{\chi_B : B \in \mathcal{B},\, Z(B) \subseteq V\}$. Then, the closure of Y with respect the Jacobson topology is the set

$$\{\mathcal{C} \in \mathcal{D}(\mathcal{B}) : \text{Ker } \eta_{\mathcal{C}} \supseteq I_Y\} = \{\mathcal{C} \in \mathcal{D}(\mathcal{B}) : \text{ if } B \in \mathcal{C} \text{ then } Z(B) \nsubseteq V\}\,.$$

Let $\mathcal{C} \notin Y$ but in the closure of Y with respect to the Jacobson topology. Then, $\mathcal{C} \in V = \bigcup\limits_{\lambda\in\Lambda} Z(A_\lambda)$. So, there exists $\lambda' \in \Lambda$ such that $\mathcal{C} \in Z(A_{\lambda'})$. But since $Z(A_{\lambda'}) \subseteq V$, this contradicts that $A_{\lambda'} \in \mathcal{C}$. Therefore, Y is closed with respect to the Jacobson topology, as desired. So, every closed subset of $\mathcal{D}(\mathcal{B})$ is also closed with the Jacobson topology.

Now, let Y be a closed subset of $\mathcal{D}(\mathcal{B})$ with respect the Jacobson topology, and let $\mathcal{C}$ be an ultrafilter that does not belong to Y. Therefore, we have that Ker $\eta_{\mathcal{C}} \not\supseteq I_Y$. This is equivalent to say that there exists $B_{\mathcal{C}} \in \mathcal{C}$ such that $Z(B_{\mathcal{C}}) \cap Y = \emptyset$. Thus, for every $\mathcal{C} \in \mathcal{U} \setminus Y$ we can find $B_{\mathcal{C}} \in \mathcal{B}$ such that $Z(B_{\mathcal{C}}) \cap Y = \emptyset$. Then, we have that $\mathcal{D}(\mathcal{B}) \setminus Y = \bigcup\limits_{\mathcal{C}\in\mathcal{D}(\mathcal{B})\setminus Y} Z(B_{\mathcal{C}})$. Hence, $\mathcal{D}(\mathcal{B}) \setminus Y$ is an open set because it is a union of open subsets. Therefore, Y is a closed subset of $\mathcal{D}(\mathcal{B})$ □

Corollary 2.17. *Let $\mathcal{B}$ be a Boolean algebra and let $\mathcal{D}(\mathcal{B})$ be the Stone's spectrum of $\mathcal{B}$. Then, given any $A \in \mathcal{B}$, we have that $Z(A)$ is a compact subspace of $\mathcal{D}(\mathcal{B})$.*

Proof. We will use Proposition 2.16, that says that $C^*(\mathcal{B}) \cong C_0(\mathcal{D}(\mathcal{B}))$. Given $A \in \mathcal{B}$, we have that $Z(A)$ is an open subset of $\mathcal{D}(\mathcal{B})$. Consider the ideal $I = C_0(Z(A)) \lhd C_0(\mathcal{D}(\mathcal{B}))$, and observe that I is the ideal generated by the projection χ_A, so $I = \overline{\text{span}}\{\chi_B : B \in \mathcal{B},\, B \subseteq A\}$. Then, I is a unital ideal, and hence $C_0(Z(A)) = C(Z(A))$. Thus, $Z(A)$ must be compact. □

3. Actions on Boolean spaces and crossed products

By the previous results, it is possible to define a partial action on the Boolean C^*-algebra by describing a partial action on the Boolean algebra. This gives a more intuitive way to understand the actions at the level of the C^*-algebra, and to extract information of this action by understanding the dynamics of the elements of the Boolean algebra. In this section, we will introduce dynamical systems on a Boolean algebra, and define what is a Cuntz-Krieger representation of this dynamical system on a C^*-algebra. Essentially, this is a generalization of a Cuntz-Krieger representation of directed graphs, considering the set of vertices the Boolean algebra, and the set of edges the partially defined actions on the vertices.

Definition 3.1. Let $\mathcal{B}$ be a Boolean algebra, we say that a map $\theta : \mathcal{B} \longrightarrow \mathcal{B}$ is an *action on* $\mathcal{B}$ if given $A, B \in \mathcal{B}$ we have that:

A1: $\theta(A \cap B) = \theta(A) \cap \theta(B)$,
A2: $\theta(A \cup B) = \theta(A) \cup \theta(B)$,

Observe that these two above conditions imply

A3: $\theta(A \setminus B) = \theta(A) \setminus \theta(B)$.

We say that the action has *compact range* if $\{\theta(A)\}_{A\in\mathcal{B}}$ has least upper-bound, that we will denote $\mathcal{R}_\theta$. Moreover, we say that the action has *closed domain* if there exists $\mathcal{D}_\theta\in\mathcal{B}$ such that $\theta(\mathcal{D}_\theta)=\mathcal{R}_\theta$.

Remark 3.2. Observe that given an action θ with compact range and closed domain, there is not necessarily a unique $\mathcal{D}_\theta$ with $\theta(\mathcal{D}_\alpha)=\mathcal{R}_\theta$, but we will assume that in the definition there is a fixed one.

Given a set $\mathcal{L}$, and given any $n\in\mathbb{N}$, we define $\mathcal{L}^n=\{(\alpha_1,\dots,\alpha_n):\alpha_i\in\mathcal{L})\}$, and $\mathcal{L}^*=\bigcup\limits_{n=0}^{\infty}\mathcal{L}^n$, where $\mathcal{L}^0=\{\emptyset\}$. Given $\alpha\in\mathcal{L}^n$ for $n\geq 1$, we will write it as $\alpha=\alpha_1\cdots\alpha_n$ where $\alpha_i\in\mathcal{L}$. Given $1\leq l\leq k\leq n$, we define $\alpha_{[l,k]}:=\alpha_l\cdots\alpha_k$. We can also endow an order on $\mathcal{L}^*$ as follows: given $\alpha\in\mathcal{L}^n$ and $\beta\in\mathcal{L}^m$,

$$\alpha\leq\beta\qquad\text{if and only if}\qquad n\leq m\text{ and }\alpha=\beta_{[1,n]}\,.$$

In case that $\alpha\leq\beta$, we define $\beta\setminus\alpha:=\beta_{[n+1,m]}$ if $n<m$ and $\emptyset$ otherwise.

Definition 3.3. A *Boolean dynamical system* on a Boolean algebra $\mathcal{B}$ is a triple $(\mathcal{B},\mathcal{L},\theta)$ such that $\mathcal{L}$ is a set, and $\{\theta_\alpha\}_{\alpha\in\mathcal{L}}$ is a set of actions on $\mathcal{B}$. Moreover, given $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathcal{L}^{\geq 1}$ the action $\theta_\alpha:\mathcal{B}\longrightarrow\mathcal{B}$ defined as $\theta_\alpha=\theta_{\alpha_n}\circ\cdots\circ\theta_{\alpha_1}$ has compact range and closed domain.

Notation 3.4. Given any $\alpha\in\mathcal{L}^*$, we will write $\mathcal{D}_\alpha:=\mathcal{D}_{\theta_\alpha}$ and $\mathcal{R}_\alpha:=\mathcal{R}_{\theta_\alpha}$. Also, when $\alpha=\emptyset$, we will define $\theta_\emptyset=\mathrm{Id}$, and we will formally assume that $\mathcal{R}_\emptyset=\mathcal{D}_\emptyset:=\bigcup\limits_{A\in\mathcal{B}}A$, in order to guarantee that $A\subseteq\mathcal{R}_\emptyset$ for every $A\in\mathcal{B}$.

Definition 3.5. Let $(\mathcal{B},\mathcal{L},\theta)$ be a Boolean dynamical system. Given $B\in\mathcal{B}$ we define

$$\Delta_B:=\{\alpha\in\mathcal{L}:\theta_\alpha(B)\neq\emptyset\}\qquad\text{and}\qquad\lambda_B:=|\Delta_B|\,.$$

We say that $A\in\mathcal{B}$ is a *regular set* if given any $\emptyset\neq B\in\mathcal{B}$ with $B\subseteq A$ we have that $0<\lambda_B<\infty$, otherwise is called a *singular set*. We denote by $\mathcal{B}_{reg}$ the set of all regular sets, and $\mathcal{B}_{sg}$ the set of all singular sets.

Definition 3.6. A Boolean dynamical system $(\mathcal{B},\mathcal{L},\theta)$ is *locally finite* if given an ultrafilter $\mathcal{C}$ of $\mathcal{B}$ do not exist infinite $\{\alpha_j\}_{j=1}^{\infty}\subseteq\mathcal{L}$ such that $\theta_{\alpha_j}(A)\neq\emptyset$ for every $A\in\mathcal{C}$.

Observe that if $|\mathcal{L}|<\infty$ then $(\mathcal{B},\mathcal{L},\theta)$ is locally finite.

Definition 3.7. A *Cuntz-Krieger representation of the Boolean dynamical system* $(\mathcal{B},\mathcal{L},\theta)$ in a C^*-algebra $\mathcal{A}$ consists of a family of projections $\{P_A:A\in\mathcal{B}\}$ and partial isometries $\{S_\alpha:\alpha\in\mathcal{L}\}$ in $\mathcal{A}$, with the properties that:

(1) If $A,B\in\mathcal{B}$, then $P_A\cdot P_B=P_{A\cap B}$ and $P_{A\cup B}=P_A+P_B-P_{A\cap B}$, where $P_\emptyset=0$.
(2) If $\alpha\in\mathcal{L}$ and $A\in\mathcal{B}$, then $P_A\cdot S_\alpha=S_\alpha\cdot P_{\theta_\alpha(A)}$.
(3) If $\alpha,\beta\in\mathcal{L}$ then $S_\alpha^*\cdot S_\beta=\delta_{\alpha,\beta}\cdot P_{\mathcal{R}_\alpha}$.
(4) Given $A\in\mathcal{B}_{reg}$ we have that

$$P_A=\sum_{\alpha\in\Delta_A}S_\alpha\cdot P_{\theta_\alpha(A)}\cdot S_\alpha^*\,.$$

Given a representation $\{P_A, S_\alpha\}$ of a Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$ in a C^*-algebra $\mathcal{A}$, we define $C^*(P_A, S_\alpha)$ to be the minimum sub-C^*-algebra of $\mathcal{A}$ containing $\{P_A, S_\alpha : A \in \mathcal{B},\ \alpha \in \mathcal{L}\}$.

A universal representation $\{p_A, s_\alpha\}$ of a Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$ is a representation satisfying the following universal property: given a representation $\{P_A, S_\alpha\}$ of $(\mathcal{B}, \mathcal{L}, \theta)$ in a C^*-algebra $\mathcal{A}$, there exists a non-degenerate $*$-homomorphism $\pi_{S,P} : C^*(p_A, s_\alpha) \longrightarrow \mathcal{A}$ such that $\pi_{S,P}(p_A) = P_A$ and $\pi_{S,P}(s_\alpha) = S_\alpha$ for $A \in \mathcal{B}$ and $\alpha \in \mathcal{L}$. We will set $C^*(\mathcal{B}, \mathcal{L}, \theta) := C^*(p_A, s_\alpha)$. The existence of the universal representation can be found in [2], but we will show it in a different way in Section 5: given a Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$, we will construct a topological graph E [21], and we will prove that there exists a one to one correspondence between Cuntz-Krieger representations of $(\mathcal{B}, \mathcal{L}, \theta)$ and Cuntz-Krieger representations of E. Hence, the universal C^*-algebra $C^*(\mathcal{B}, \mathcal{L}, \theta)$ is isomorphic to the universal C^*-algebra $\mathcal{O}(E)$ associated to the topological graph E.

Theorem 3.8 (Existence of a Universal representation)**.** *Given a Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$ there exists a unique universal representation of $(\mathcal{B}, \mathcal{L}, \theta)$. If $C^*(\mathcal{B}, \mathcal{L}, \theta)$ is the associated C^*-algebra, we will call $C^*(\mathcal{B}, \mathcal{L}, \theta)$ the Cuntz-Krieger Boolean algebra of the Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$.*

By the universality of $C^*(\mathcal{B}, \mathcal{L}, \theta)$, there exists a strongly continuous action $\beta : \mathbb{T} \curvearrowright \text{Aut}\ (C^*(\mathcal{B}, \mathcal{L}, \theta))$ such that $\beta_z(p_A) = p_A$ and $\beta_z(s_\alpha) = z s_\alpha$ for every $A \in \mathcal{B}$, $\alpha \in \mathcal{L}$ and $z \in \mathbb{T}$. The action β is called the *gauge action*

Therefore, we can use the representation of $C^*(\mathcal{B}, \mathcal{L}, \theta)$ as a topological graph C^*-algebra to obtain a gauge uniqueness theorem [21, Theorem 4.5].

Theorem 3.9 (Gauge Uniqueness Theorem)**.** *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system and let $\{P_A, S_\alpha\}$ be a representation of $(\mathcal{B}, \mathcal{L}, \theta)$ in $\mathcal{A}$. Suppose that $P_A \neq 0$ whenever $A \neq \emptyset$, and that there is a strongly continuous action γ of $\mathbb{T}$ on $C^*(P_A, S_\alpha) \subseteq \mathcal{A}$, such that for all $z \in \mathbb{T}$ we have that $\gamma_z \circ \pi_{S,P} = \pi_{S,P} \circ \beta_z$. Then, $\pi_{S,T}$ is injective.*

4. 0-dimensional topological graphs

Our goal in this section is to use a topological graph $E = (E^0, E^1, d, r)$ with E^0 and E^1 being locally compact 0-dimensional spaces (i.e., Hausdorff, totally disconnected and having a basis consisting of clopen sets) to construct a Boolean dynamical system.

First, we should recall the definition of topological graph given in [21].

Definition 4.1. Let E^0 and E^1 be locally compact spaces, let $d : E^1 \to E^0$ be a local homeomorphism, and let $r : E^1 \to E^0$ be a continuous map. Then, the quadruple $E = (E^0, E^1, d, r)$ is called a *topological graph*.

Let us denote $C_d(E^1)$ the set of continuous functions on E^1 such that

$$\langle \xi | \xi \rangle (v) := \sum_{e \in d^{-1}(v)} |\xi(e)|^2 < \infty$$

for any $v \in E^0$ and $\langle \xi | \xi \rangle \in C_0(E^0)$. For $\xi, \zeta \in C_d(E^1)$ and $f \in C_0(E^0)$, we define $\xi f \in C_d(E^1)$ and $\langle \xi | \zeta \rangle \in C_0(E^1)$ by

$$(\xi f)(e) = \xi(e) f(d(e)) \qquad \text{for } e \in E^1$$

$$\langle \xi|\zeta\rangle(v) = \sum_{e\in d^{-1}(v)} \overline{\xi(e)}\zeta(e) \qquad \text{for } v \in \overline{E}^0 .$$

With these operations, $C_d(E^1)$ is a right Hilbert $C_0(E^0)$-module. We define a left action π_r of $C_0(E^0)$ on $C_d(E^1)$ by $(\pi_r(f)\xi)(e) = f(r(e))\xi(e)$ for $e \in E^1$, $\xi \in C_d(E^1)$ and $f \in C_0(E^0)$. In this way, we define a C^*-correspondence $C_d(E^1)$ over $C_0(E^0)$.

Definition 4.2. A *Toeplitz E-pair* on a C^*-algebra $\mathcal{A}$ is a pair of maps $T = (T^0, T^1)$, where $T^0 : C_0(E^0) \longrightarrow \mathcal{A}$ is a $*$-homomorphism and $T^1 : C_d(E^1) \longrightarrow \mathcal{A}$ is a linear map, satisfying:

(1) $T^1(\xi)^*T^1(\zeta) = T^0(\langle\xi|\zeta\rangle)$ for $\xi, \zeta \in C_d(E^1)$,
(2) $T^0(f)T^1(\xi) = T^1(\pi_r(f)\xi)$ for $f \in C_0(E^0)$ and $\xi \in C_d(E^1)$.

We will denote by $C^*(T^0, T^1)$ the sub-C^*-algebra of $\mathcal{A}$ generated by the Toeplitz E-pair $(T0, T^1)$.

Given a topological graph E, we define the following 3 open subsets of E^0:

$$E_{sce} := E^0 \setminus \overline{r(E^0)}\,,$$
$$E^0_{fin} := \{v \in E^0 : \exists V \text{neighborhood of } v \text{ such that } r^{-1}(V) \text{ is compact}\}, \text{ and}$$
$$E^0_{rg} := E^0_{fin} \setminus \overline{E^0_{sce}}\,.$$

We have that $\pi_r^{-1}(\mathcal{K}(C_d(E^1))) = C_0(E^0_{fin})$ and $\operatorname{Ker}\pi_r = C_0(E^0_{sce})$. For a Toeplitz E-pair $T = (T^0, T^1)$, we define a $*$-homomorphism $\Phi : \mathcal{K}(C_d(E^1)) \longrightarrow \mathcal{A}$ by $\Phi(\theta_{\xi,\zeta}) = T^1(\xi)T^1(\zeta)^*$ for $\xi, \zeta \in C_d(E^1)$.

Definition 4.3. A Toeplitz E-pair $T = (T^0, T^1)$ is called a *Cuntz-Krieger E-pair* if $T^0(f) = \Phi(\pi_r(f))$ for any $f \in C_0(E^0_{rg})$. We denote by $\mathcal{O}(E)$ the C^*-algebra is generated by the universal Cuntz-Krieger E-pair $t = (t^0, t^1)$.

Therefore, $\mathcal{O}(E)$ is generated by $\{t^0(f) : f \in C_0(E^0)\}$ and $\{t^1(\xi) : \xi \in C_d(E^1)\}$, where (t^0, t^1) is a universal Cuntz-Krieger pair of E.

Now, we suppose that E^0 and E^1 are locally compact and 0-dimensional spaces. Since d is a local homeomorphism, there exist $\{U_\alpha\}_{\alpha\in\mathcal{L}}$ and $\{V_\alpha\}_{\alpha\in\mathcal{L}}$ (for some index set $\mathcal{L}$) clopen and compact subsets of E^0 and E^1 respectively, such that $E^1 = \bigcup_{\alpha\in\mathcal{L}} V_\alpha$ with $V_\alpha \cap V_\beta = \emptyset$ when $\alpha \neq \beta$, and the restriction $d_{|V_\alpha}$ is a homeomorphism for every $\alpha \in \mathcal{L}$. Then, we define $\mathcal{B}$ as the Boolean algebra of all the clopen and compact subsets of E^0. Given $\alpha \in \mathcal{L}$, the action θ_α is defined by $\theta_\alpha(A) := d(r^{-1}(A) \cap V_\alpha)$ for every $A \in \mathcal{B}$. Observe that θ_α has compact range $\mathcal{R}_\alpha := U_\alpha$, but not necessarily there exists $D_\alpha \in \mathcal{B}$ such that $\theta_\alpha(D_\alpha) = U_\alpha$. Thus, the existence of these D_α's should be included in the hypotheses.

Definition 4.4. Let $E = (E^0, E^1, d, r)$ be a topological graph. Then, E is said to be 0-dimensional if E^0 and E^1 are 0-dimensional spaces, (i.e., they have covering dimension equal to 0). Moreover, E is said to be *compactly supported* if there exist $\{V_\alpha\}_{\alpha\in\mathcal{L}} \subseteq E^1$ and $\{D_\alpha\}_{\alpha\in\mathcal{L}} \subseteq E^0$ such that:

(1) V_α and D_α are clopen and compact sets for every $\alpha \in \mathcal{L}$,
(2) $V_\alpha \cap V_\beta = \emptyset$ when $\alpha \neq \beta$,
(3) $E^1 = \bigcup_{\alpha\in\mathcal{L}} V_\alpha$,
(4) the restrictions $d_{|V_\alpha}$ are homeomorphisms for every $\alpha \in \mathcal{L}$,

(5) $V_\alpha \subseteq r^{-1}(D_\alpha)$.

Remark 4.5. Observe that if E is a compactly supported 0-dimensional topological graph, then we can construct a Boolean dynamical system. However, it is not unique, because it could exist several $\{V_\alpha\}_{\alpha\in\mathcal{L}} \subseteq E^1$ and $\{D_\alpha\}_{\alpha\in\mathcal{L}} \subseteq E^0$ satisfying the conditions of Definition 4.4. We will see that, despite of the choice of the above pairs of sets, the C^*-algebras of the associated Boolean dynamical systems are isomorphic.

Now, let E be a compactly supported 0-dimensional topological space, and choose a pair $\{V_\alpha\}_{\alpha\in\mathcal{L}} \subseteq E^1$ and $\{D_\alpha\}_{\alpha\in\mathcal{L}} \subseteq E^0$ satisfying Definition 4.4. We set $\mathcal{R}_\alpha := d(V_\alpha)$ and $\mathcal{D}_\alpha := D_\alpha$ for every $\alpha \in \mathcal{L}$. Given $\alpha \in \mathcal{L}$, we define $\theta_\alpha(A) := d(r^{-1}(A) \cap V_\alpha)$ for every $A \in \mathcal{B}$.

Now, we write E^1 as

$$E^1 = \bigsqcup_{\alpha\in\mathcal{L}} V_\alpha \,,$$

and given $A \in \mathcal{B}$ and $\alpha \in \mathcal{L}$, we set

$$\mathcal{N}_A := A \subseteq E^0 \qquad \text{and} \qquad \mathcal{M}^\alpha_A := d^{-1}(U_\alpha \cap A) \subseteq E^1$$

compact and clopen subsets.

Remark 4.6. Observe that, given $A \in \mathcal{B}$

$$\begin{aligned}
\mathcal{N}_A \subseteq E^0_{sce} &\Leftrightarrow \mathcal{N}_A \cap r(E^1) = \emptyset \Leftrightarrow \mathcal{N}_A \cap r(V_\alpha) = \emptyset \text{ for every } \alpha \in \mathcal{L} \\
&\Leftrightarrow r^{-1}(A) \cap V_\alpha = \emptyset \text{ for every } \alpha \in \mathcal{L} \\
&\Leftrightarrow \theta_\alpha(A) = d(r^{-1}(A) \cap V_\alpha) = \emptyset \text{ for every } \alpha \in \mathcal{L} \\
\mathcal{N}_A \subseteq E^0_{fin} &\Leftrightarrow r^{-1}(\mathcal{N}_B) = \bigcup_{\alpha\in\mathcal{L}} \mathcal{M}^\alpha_{r^{-1}(B)} \text{ is compact for all } \emptyset \neq B \subseteq A \\
&\Leftrightarrow V_\alpha \cap r^{-1}(B) \neq \emptyset \text{ for at most a finite number of } \alpha, \text{ for all } \emptyset \neq B \subseteq A \\
\mathcal{N}_A \subseteq E^0_{rg} &\Leftrightarrow V_\alpha \cap r^{-1}(B) \neq \emptyset \text{ for at most a finite and non-zero number of } \alpha, \\
&\quad \text{for all } \emptyset \neq B \subseteq A \\
&\Leftrightarrow \theta_\alpha(B) \neq \emptyset \text{ for at most a finite and non-zero number of } \alpha, \text{ for all } \emptyset \neq B \subseteq A \\
&\Leftrightarrow A \in \mathcal{B}_{reg} \,.
\end{aligned}$$

Proposition 4.7. *Let E be a 0-dimensional and compact supported topological graph, and let $\{V_\alpha\}_{\alpha\in\mathcal{L}}$ and $\{D_\alpha\}$ be a sets as Definition 4.4. If $(\mathcal{B}, \mathcal{L}, \theta)$ is the associated Boolean dynamical system defined as*

$$\mathcal{B} := \{A \subseteq E^0 : A \textit{ is a compact clopen}\}$$

and for every $\alpha \in \mathcal{L}$ and $A \in \mathcal{B}$ the action

$$\theta_\alpha(A) := d(r^{-1}(A) \cap V_\alpha) \textit{ with range } \mathcal{R}_\alpha := d(V_\alpha) \,,$$

then, given any Cunt-Krieger E-representation (T^0, T^1) on $\mathcal{A}$, the family of elements of $\mathcal{A}$ defined by

$$P_A := T^0(\chi_{\mathcal{N}_A}) \textit{ and } S_\alpha := T^1(\chi_{\mathcal{M}^\alpha_{V_\alpha}}) \,.$$

for every $A \in \mathcal{B}$ and $\alpha \in \mathcal{L}$, is a representation of $(\mathcal{B}, \mathcal{L}, \theta)$ on $\mathcal{A}$, i.e.,

(1) If $A, B \in \mathcal{B}$ then $P_A P_B = P_{A\cap B}$ and $P_{A\cup B} = P_A + P_B - P_{A\cap B}$, where $P_\emptyset = 0$.
(2) If $\alpha \in \mathcal{L}$ and $A \in \mathcal{B}$ then $P_A S_\alpha = S_\alpha P_{\theta_\alpha(A)}$.

(3) If $\alpha,\beta\in\mathcal{L}$ then $S_\alpha^*S_\alpha=P_{\mathcal{R}_\alpha}$, and $S_\alpha^*S_\beta=0$ unless $\alpha=\beta$.
(4) For $A\in\mathcal{B}_{reg}$, we have

$$P_A=\sum_{\alpha\in\Delta_A}S_\alpha P_{\theta_\alpha(A)}S_\alpha^*\,.$$

Proof. For (1), observe that $\{P_A\}_{A\in\mathcal{B}}$ is a family of commuting projections. Then, $P_{A\cap B}=P_AP_B$ and $P_{A\cup B}=P_A+P_B-P_{A\cap B}$ for every $A,B\in\mathcal{B}$ follows from the fact that T^0 is a homomorphism. For (2), given $A\in\mathcal{B}$ and $\alpha\in\mathcal{L}$, we have that

$$\begin{aligned}P_AS_\alpha&=T^0(\chi_{\mathcal{N}_A})T^1(\chi_{\mathcal{M}^\alpha_{V_\alpha}})=T^1(\pi_r(\chi_{\mathcal{N}_A})\chi_{\mathcal{M}^\alpha_{V_\alpha}})=T^1((\chi_{\mathcal{N}_A}\circ r)\chi_{\mathcal{M}^\alpha_{V_\alpha}})\\&=T^1(\chi_{\mathcal{M}^\alpha_{r^{-1}(A)}}\chi_{\mathcal{M}^\alpha_{V_\alpha}})=T^1(\chi_{\mathcal{M}^\alpha_{r^{-1}(A)\cap V_\alpha}})\\&=T^1(\chi_{\mathcal{M}^\alpha_{V_\alpha}})T^0(\chi_{\mathcal{N}_{d(r^{-1}(A)\cap V_\alpha)}})=T^1(\chi_{\mathcal{M}^\alpha_{V_\alpha}})T^0(\chi_{\mathcal{N}_{\theta_\alpha(A)}})=S_\alpha P_{\theta_\alpha(A)}\,.\end{aligned}$$

For (3), we look at the equality

$$S_\alpha^*S_\beta=T^1(\chi_{\mathcal{M}^\alpha_{V_\alpha}})^*T^1(\chi_{\mathcal{M}^\beta_{V_\beta}})=T^0(\langle\chi_{\mathcal{M}^\alpha_{V_\alpha}}|\chi_{\mathcal{M}^\beta_{V_\beta}}\rangle)\,.$$

By the definition,

$$\langle\chi_{\mathcal{M}^\alpha_{V_\alpha}}|\chi_{\mathcal{M}^\beta_{V_\beta}}\rangle(v)=\sum_{e\in d^{-1}(v)}\overline{\chi_{\mathcal{M}^\alpha_{V_\alpha}}(e)}\chi_{\mathcal{M}^\beta_{V_\beta}}(e)\,,$$

for any $v\in E^0$. Since $\mathcal{M}^\alpha_{V_\alpha}$ and $\mathcal{M}^\beta_{V_\beta}$ are disjoint subsets of E^1 whenever $\alpha\neq\beta$, we get that this expression will sum 0 if $\alpha\neq\beta$. Now, note that d is a homeomorphism when restricted to V_α. So, it follows that

$$\sum_{e\in d^{-1}(v)}|\chi_{\mathcal{M}^\alpha_{V_\alpha}}(e)|^2=|\{e\in\mathcal{M}^\alpha_{V_\alpha}:d(e)=v\}=\chi_{\mathcal{N}_{d(V_\alpha)}}(v)=\chi_{\mathcal{N}_{\mathcal{R}_\alpha}}(v)\,.$$

For (4), we will use the Cuntz-Krieger relation

$$T^0(f)=\Phi(\pi_r(f)),$$

which holds whenever $f\in C_0(E^0_{rg})$. Since $A\in\mathcal{B}_{reg}$, by the Remark 4.6 we have that $\mathcal{N}_A\subseteq E^0_{rg}$. So, it is enough to show that

$$\pi_r(\chi_{\mathcal{N}_A})=\sum_{\alpha\in\Delta_A}\theta_{\chi_{\mathcal{M}^\alpha_{V_\alpha}},\chi_{\mathcal{M}^\alpha_{V_\alpha}}\cdot\chi_{\mathcal{N}_{\theta_\alpha(A)}}}\,.$$

Evaluating at $\xi\in C_d(E^1)$ and $e\in E^1$, we have that

$$\sum_{\alpha\in\Delta_A}\theta_{\chi_{\mathcal{M}^\alpha_{V_\alpha}},\chi_{\mathcal{M}^\alpha_{V_\alpha}}\cdot\chi_{\mathcal{N}_{\theta_\alpha(A)}}}(\xi)(e)=$$

$$\sum_{\alpha\in\Delta_A}\chi_{\mathcal{M}^\alpha_{V_\alpha}}(e)\langle\chi_{\mathcal{M}^\alpha_{V_\alpha}}\cdot\chi_{\mathcal{N}_{\theta_\alpha(A)}}|\xi\rangle(d(e))=$$

$$\sum_{\alpha\in\Delta_A}\chi_{\mathcal{M}^\alpha_{V_\alpha}}(e)\left(\sum_{d(e')=d(e)}\chi_{\mathcal{M}^\alpha_{V_\alpha}}(e')\chi_{\mathcal{N}_{\theta_\alpha(A)}}(d(e'))\xi(e')\right)\,.$$

Whenever $e, e' \in \mathcal{M}^{\alpha}_{V_\alpha}$ for some $\alpha \in \mathcal{L}$, since $d(e) = d(e')$ if and only if $e = e'$, this reduces to

$$\sum_{\alpha\in\Delta_A} \chi_{\mathcal{M}^{\alpha}_{\mathcal{R}_\alpha}}(e)\chi_{\mathcal{N}_{\theta_\alpha(A)}}(d(e))\xi(e) = \begin{cases} \chi_{\mathcal{N}_{\theta_\alpha(A)}}(d(e))\xi(e) & \text{whenever } e \in \mathcal{M}^{\alpha}_{V_\alpha} \text{ for } \alpha \in \Delta_A \\ 0 & \text{otherwise} \end{cases}.$$

In addition, $\theta_\alpha(A) = \emptyset$ when $\alpha \notin \Delta_A$. Thus, we can omit the case clause. What remains is $\chi_{\mathcal{N}_{\theta_\alpha(A)}}(d(e))\xi(e)$ when $e \in \mathcal{M}^{\alpha}_{V_\alpha}$ for any $\alpha \in \mathcal{L}$. On the other hand,

$$(\pi_r(\chi_{\mathcal{N}_A})\xi)(e) = \chi_{\mathcal{N}_A}(r(e))\xi(e)\,.$$

Now, when $e \in \mathcal{M}^{\alpha}_{V_\alpha}$ for some $\alpha \in \mathcal{L}$, we get that $\chi_{\mathcal{N}_A}(r(e)) = \chi_{\mathcal{N}_{d(r^{-1}(A)\cap V_\alpha)}}(d(e)) = \chi_{\mathcal{N}_{\theta_\alpha(A)}}(d(e))$, so we are done. $\square$

5. A faithful representation of $(\mathcal{B}, \mathcal{L}, \theta)$.

Now, given a Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$, we will construct a faithful representation of $(\mathcal{B}, \mathcal{L}, \theta)$ in $\mathcal{O}(E)$, where E is a compactly supported 0-dimensional topological graph.

Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system. We define E^0 to be the Stone's spectrum $\mathcal{D}(\mathcal{B})$ of $\mathcal{B}$, and E^1 to be the disjoint union

$$E^1 = \bigsqcup_{\alpha\in\mathcal{L}} \mathcal{D}(\mathcal{I}_{\mathcal{R}_\alpha})\,,$$

of Stone's spectrums of the principal ideals of $\mathcal{B}$ generated by the range subsets $\mathcal{R}_\alpha$ of the actions θ_α. Since $\mathcal{D}(\mathcal{B})$ and each $\mathcal{D}(\mathcal{I}_{\mathcal{R}_\alpha})$ have a basis of clopen sets, they are 0-dimensional spaces, and since they are totally disconnected spaces they are locally compact Hausdorff spaces too. These properties are transfered to arbitrary unions of such spaces, so E^0 and E^1 are also locally compact Hausdorff 0-dimensional spaces. Also observe that, given any $\alpha \in \mathcal{L}$, then $\mathcal{D}(\mathcal{I}_{\mathcal{R}_\alpha})$ is a clopen and compact subset of $\mathcal{D}(\mathcal{B})$.

Notation 5.1. To distinguish the edge and the vertex space of the topological graph E, we will denote

$$E^0 = \{v_{\mathcal{C}} : \mathcal{C} \in \mathcal{D}(\mathcal{B})\} \text{ and } E^1 = \bigsqcup_{\alpha\in\mathcal{L}} E^1_\alpha\,,$$

where $E^1_\alpha = \{e^{\alpha}_{\mathcal{C}} : \mathcal{C} \in \mathcal{D}(\mathcal{I}_{\mathcal{R}_\alpha})\}$. Given $\alpha \in \mathcal{L}$ and $A, B \in \mathcal{B}$ with $B \subseteq \mathcal{R}_\alpha$, we define the clopen and compact subsets

$$\mathcal{N}_A := \{v_{\mathcal{C}} : A \in \mathcal{C}\} \subseteq E^0 \qquad \text{and} \qquad \mathcal{M}^{\alpha}_B := \{e^{\alpha}_{\mathcal{C}} : B \in \mathcal{C}\} \subseteq E^1_\alpha\,.$$

Lemma 5.2. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system Then, given $\alpha \in \mathcal{L}$ and $A \in \mathcal{I}_{\mathcal{D}_\alpha}$, we have that*

$$\{\mathcal{C} \in \mathcal{D}(\mathcal{I}_{\mathcal{R}_\alpha}) : \theta_\alpha(A) \in \mathcal{C}\} = \{\mathcal{C} \in \mathcal{D}(\mathcal{I}_{\mathcal{R}_\alpha}) : \exists \mathcal{C}' \in \mathcal{D}(\mathcal{I}_A) \text{ such that } \theta_\alpha(B) \in \mathcal{C}\,,\ \forall B \in \mathcal{C}'\}$$

Proof. The inclusion $\supseteq$ it is clear, since every $\mathcal{C}' \in \mathcal{D}(\mathcal{I}_A)$ contains A. For the inclusion $\subseteq$, let us define the set $\mathcal{F} = \{B \in \mathcal{I}_A : \theta_\alpha(B) \in \mathcal{C}\}$. By hypothesis, we have that $A \in \mathcal{F}$, so **F0** is satisfied. **F1** and **F2** follows because of conditions **F1** and **F2** of $\mathcal{C}$, and the fact that θ_α preserves intersections. Let Γ be the set of all the filters $\mathcal{F}$ of $\mathcal{I}_A$ such that $\theta_\alpha(B) \in \mathcal{C}$ $B \in \mathcal{F}$. Given any ascending sequence of filters $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ of Γ, we have that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is an upper-bound that is contained in Γ. By Zorn's Lemma, there exists a maximal element $\mathcal{C}'$ in Γ. We claim that $\mathcal{C}'$ is an ultrafilter of $\mathcal{I}_A$. To prove that claim, we only need to check condition **F3**. Let $B_1, B_2 \in \mathcal{I}_A$ such that $B_1 \cup B_2 \in \mathcal{C}'$, and suppose that neither B_1 nor B_2 belong to $\mathcal{C}'$. Since $\theta_\alpha(B_1) \cup \theta_\alpha(B_2) \in \mathcal{C}$, and condition **F3** of $\mathcal{C}$ holds, we have that either $\theta_\alpha(B_1)$ or $\theta_\alpha(B_2)$ belong to $\mathcal{C}$. Let us suppose that $\theta_\alpha(B_1) \in \mathcal{C}$. Then, the set $\mathcal{C}'' := \{D \in \mathcal{I}_A : C \cap B_1 \subseteq D \text{ for some } C \in \mathcal{C}'\}$ strictly contains $\mathcal{C}'$. Given any $D \in \mathcal{I}_A$ with $C \cap B_1 \subseteq D$ for some $C \in \mathcal{C}'$, we have that

$$\theta_\alpha(C) \cap \theta_\alpha(B_1) = \theta_\alpha(C \cap B_1) \subseteq \theta_\alpha(D) \in \mathcal{C}\,,$$

by condition **F1** and **F2** of $\mathcal{C}$. Then, it is easy to verify that $\mathcal{C}''$ is a filter of $\mathcal{I}_A$, and hence that $\mathcal{C}'' \in \Gamma$. But this contradicts the maximality of $\mathcal{C}$. Thus, condition **F3** is satisfied, whence $\mathcal{C}'$ is an ultrafilter of $\mathcal{I}_A$ such that $\theta_\alpha(B) \in \mathcal{C}$ for every $B \in \mathcal{C}'$. So we are done. $\square$

Proposition 5.3. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system, and let the maps $d, r : E^1 \longrightarrow E^0$ be defined by*

$$d(e^\alpha_{\mathcal{C}}) = v_{\mathcal{C}} \qquad \textit{and} \qquad r(e^\alpha_{\mathcal{C}}) = v_{\varphi_\alpha(\mathcal{C})}\,,$$

where $\varphi_\alpha : \mathcal{D}(\mathcal{I}_{\mathcal{R}_\alpha}) \longrightarrow \mathcal{D}(\mathcal{I}_{\mathcal{D}_\alpha})$ is a map given by $\varphi_\alpha(\mathcal{C}) = \{A \in \mathcal{I}_{\mathcal{D}_\alpha} : \theta_\alpha(A) \in \mathcal{C}\}$. Then, the quadruple (E^0, E^1, d, r) is a topological graph.

Proof. First, by the above arguments, we have that E^0 and E^1 are locally compact Hausdorff spaces. Let $d : E^1 \longrightarrow E^0$ be the map defined by $d(e^\alpha_{\mathcal{C}}) = v_{\mathcal{C}}$ for some $e^\alpha_{\mathcal{C}} \in E^1_\alpha$. Every point of E^1 belongs to a component E^1_α for some $\alpha \in \mathcal{L}$, and clearly we have that $d_{|E^1_\alpha}$ is an homeomorphism. Thus, d is a local homeomorphism.

Let $\varphi_\alpha : \mathcal{D}(\mathcal{I}_{\mathcal{R}_\alpha}) \longrightarrow \mathcal{D}(\mathcal{I}_{\mathcal{D}_\alpha})$ be the map given by $\varphi_\alpha(\mathcal{C}) = \{A \in \mathcal{I}_{\mathcal{D}_\alpha} : \theta_\alpha(A) \in \mathcal{C}\}$. It is routine to check that $\{A \in \mathcal{I}_{\mathcal{D}_\alpha} : \theta_\alpha(A) \in \mathcal{C}\}$ is an ultrafilter of $\mathcal{I}_{\mathcal{D}_\alpha}$. Thus, φ_α is a well-defined map. If $A \in \mathcal{I}_{\mathcal{D}_\alpha}$, then by Lemma 5.2 $\varphi_\alpha^{-1}(Z(A)) = \{\mathcal{C} \in \mathcal{I}_{\mathcal{R}_\alpha} : \theta_\alpha(A) \in \mathcal{C}\} = Z(\theta_\alpha(A))$, so φ_α is a continuous map. $\square$

Corollary 5.4. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system, let E be the associated topological graph defined in Proposition 5.3, and let (t^0, t^1) the universal Cuntz-Krieger E-pair. Then,*

$$p_A := t^0(\chi_{\mathcal{N}_A}) \qquad \textit{and} \qquad s_\alpha := t^1(\chi_{E^1_\alpha})$$

for $A \in \mathcal{B}$ and $\alpha \in \mathcal{L}$, defines a faithful representation of $(\mathcal{B}, \mathcal{L}, \theta)$ in $\mathcal{O}(E)$.

Proof. Let $E = (E^0, E^1, d, r)$ be the topological graph defined in Proposition 5.3. Observe that the pair $\{E^1_\alpha\}_{\alpha\in\mathcal{L}}$ and $\{\mathcal{N}_{\mathcal{D}_\alpha}\}_{\alpha\in\mathcal{L}}$ satisfies Definition 4.4, so that E is compactly supported. It is easy to check that the Boolean dynamical system associated to E is $(\mathcal{B}, \mathcal{L}, \theta)$ again. Now, using Proposition 4.7 with the universal faithful representation (t^0, t^1) of $\mathcal{O}(E)$, we conclude the proof. $\square$

Our next step is to prove that the faithful representation constructed in Corollary 5.4 is the universal one. To do that, we first have to look closer at the topological graph E associated to a Boolean dynamical system.

Lemma 5.5. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system, and let $\alpha \in \mathcal{L}$ and $\mathcal{C} \in \mathcal{D}(\mathcal{I}_{\mathcal{D}_\alpha})$. Then, given any $\mathcal{C}' \in \mathcal{D}(\mathcal{I}_{\mathcal{R}_\alpha})$ such that $\theta_\alpha(A) \in \mathcal{C}'$ for every $A \in \mathcal{C}$, we have that $\mathcal{C} = \{B \in \mathcal{I}_{\mathcal{D}_\alpha} : \theta_\alpha(B) \in \mathcal{C}'\}$.*

Proof. The first inclusion is clear because $\mathcal{C}'$ contains $\theta_\alpha(A)$ for every $A \in \mathcal{C}$. Now, let $B \in \mathcal{B}$ such that $\theta_\alpha(B) \in \mathcal{C}'$. Then, given any $A \in \mathcal{C}$ we have that $\theta_\alpha(A) \in \mathcal{C}'$. So, we have that

$$\emptyset \neq \theta_\alpha(A) \cap \theta_\alpha(B) = \theta_\alpha(A \cap B) \in \mathcal{C}'\,.$$

Thus, $A \cap B \neq \emptyset$. Then, $A = (A \cap B) \cup (A \setminus (A \cap B))$, but by condition **F3** it follows that either $A \cap B$ or $A \setminus (A \cap B)$ belongs to $\mathcal{C}$. Observe that $A \setminus (A \cap B)$ cannot belong to $\mathcal{C}$, as otherwise

$$\theta_\alpha(A \cap B) \cap \theta_\alpha(A \setminus (A \cap B)) = \emptyset,$$

contradicting condition **F2** of the ultrafilter $\mathcal{C}'$. Therefore, $A \cap B \in \mathcal{C}$, whence so does B by condition **F1**. $\square$

Lemma 5.6. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system. If $\alpha \in \mathcal{L}$, then $\varphi_\alpha : \mathcal{D}(\mathcal{I}_{\mathcal{R}_\alpha}) \longrightarrow \mathcal{D}(\mathcal{I}_{\mathcal{D}_\alpha})$ is injective if and only if given any $B \in \mathcal{I}_{\mathcal{R}_\alpha}$ there exists $A \in \mathcal{I}_{\mathcal{D}_\alpha}$ such that $\theta_\alpha(A) = B$.*

Proof. Observe that given $\alpha \in \mathcal{L}$, the action θ_α induces a $*$-homomorphism $\widehat{\theta}_\alpha : C^*(\mathcal{I}_{\mathcal{D}_\alpha}) \to C^*(\mathcal{I}_{\mathcal{R}_\alpha})$ defined by $\chi_A \mapsto \chi_{\theta_\alpha(A)}$ for every $A \in \mathcal{B}$. Then, using the Stone's Representation Theorem, we have that $\widehat{\theta}_\alpha : C(\mathcal{I}_{\mathcal{D}_\alpha}) \to C(\mathcal{I}_{\mathcal{R}_\alpha})$ is defined by $f \mapsto f \circ \varphi_\alpha$ for every $C(\mathcal{I}_{\mathcal{D}_\alpha})$. Thus, θ_α is surjective if and only if φ_α is injective. But if $\widehat{\theta}_\alpha$ is surjective then, given any $\chi_B \in C(\mathcal{I}_{\mathcal{R}_\alpha})$, there exists $A \in \mathcal{D}_\alpha$ such that $\widehat{\theta}_\alpha(\chi_A) = \chi_B$, and hence $B = \theta_\alpha(A)$, as desired. $\square$

Lemma 5.7. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system, and let E be the topological graph defined in Proposition 5.3. Then, given $e \in E^1_\alpha$, the following statements are equivalent:*

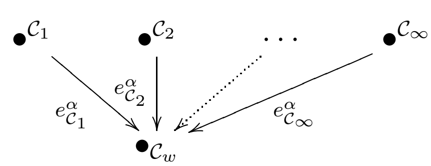
(1) $r(e) \in \mathcal{N}_A$.
(2) $d(e) \in \mathcal{N}_{\theta_\alpha(A)}$.
(3) $e \in \mathcal{M}^\alpha_{\theta_\alpha(A)}$.

Proof. (2) $\Leftrightarrow$ (3) is clear by definition. Now, let $e = e^\alpha_{\mathcal{C}}$ for some $\alpha \in \mathcal{L}$ and $\mathcal{C} \in \mathcal{D}(\mathcal{I}_{\mathcal{R}_\alpha})$. Suppose that $v_{\mathcal{C}'} = r(e^\alpha_{\mathcal{C}}) \in \mathcal{N}_A$, where $\mathcal{C}' = \{B \in \mathcal{I}_{\mathcal{D}_\alpha} : \theta_\alpha(B) \in \mathcal{C}\}$, whence $v_{\mathcal{C}} \in \mathcal{N}_{\theta_\alpha(B)}$ for every $B \in \mathcal{C}'$. Since $A \in \mathcal{C}'$, it follows that $v_{\mathcal{C}} \in \mathcal{N}_{\theta_\alpha(A)}$, as desired. Now, let us suppose that $d(e^\alpha_{\mathcal{C}}) = v_{\mathcal{C}} \in \mathcal{N}_{\theta_\alpha(A)}$, so that $\theta_\alpha(A) \in \mathcal{C}$. Since $r(e^\alpha_{\mathcal{C}}) = v_{\mathcal{C}'}$, where $\mathcal{C}' = \{B \in \mathcal{I}_{\mathcal{D}_\alpha} : \theta_\alpha(B) \in \mathcal{C}\}$, it follows that $A \in \mathcal{C}'$. Thus, $v_{\mathcal{C}'} \in \mathcal{N}_A$, as desired. $\square$

Example 5.8. Let $X = \mathbb{N} \cup \{w\}$, and let $\mathcal{B}$ be the minimal Boolean space generated by the subsets $\{F \subseteq \mathbb{N} : F \text{ finite }\} \cup \{\mathbb{N} \setminus F : F \text{ finite }\} \cup \{w\}$. We have that $\mathcal{D}(\mathcal{B})$ is the compact space $\{v_{C_i} : i = 1, 2, \dots, \infty\} \cup \{v_{\mathcal{C}_w}\}$, where $\mathcal{C}_w = \{A \in \mathcal{B} : w \in A\}$. Let $\mathcal{L} = \{\alpha\}$, and define

$$\theta_\alpha(A) = \begin{cases} \mathbb{N} & \text{if } A = \{w\} \\ \emptyset & \text{otherwise} \end{cases},$$

that is an action on the Boolean space $\mathcal{B}$. Therefore, $(\mathcal{B}, \mathcal{L}, \theta)$ is a Boolean dynamical system, and let E be its associated topological graph. Thus, $E^0 = \{v_{C_i} : i = 1, 2, \dots, \infty\} \cup \{v_{\mathcal{C}_w}\}$ and $E^1 = \{e^\alpha_{\mathcal{C}_i} : i = 1, \dots, \infty\}$. Then, $d(e^\alpha_{\mathcal{C}_i}) = v_{\mathcal{C}_i}$ and $r(e^\alpha_{\mathcal{C}_i}) = v_{\mathcal{C}_w}$ for every $i = 1, 2, \dots, \infty$. A picture of this topological graph will be as follows:



Example 5.9. Let $\mathcal{B}$ be the minimal Boolean algebra generated by

$$\{F : F \subseteq \mathbb{Z} \text{ finite }\} \cup \{\mathbb{Z} \setminus F : F \subseteq \mathbb{Z} \text{ finite}\}.$$

Let θ_a, θ_b and θ_c be actions on $\mathcal{B}$ given by the following graph

$$\begin{tikzcd} \cdots \arrow[r, shift left, "b"] & \bullet_{-2} \arrow[l, shift left, "c"] \arrow[r, shift left, "b"] & \bullet_{-1} \arrow[l, shift left, "c"] \arrow[r, shift left, "b"] & \bullet_0 \arrow[l, shift left, "c"] \arrow[loop above, "a"] \arrow[r, shift left, "b"] & \bullet_1 \arrow[l, shift left, "c"] \arrow[r, shift left, "b"] & \bullet_2 \arrow[l, shift left, "c"] \arrow[r, shift left, "b"] & \cdots \arrow[l, shift left, "c"] \end{tikzcd}$$

We have that $\mathcal{D}(\mathcal{B}) = \{\mathcal{C}_n : n \in \mathbb{Z}\} \cup \{\mathcal{C}_\infty\}$ where $\mathcal{C}_n = \{A \in \mathcal{B} : n \in A\}$ and $\mathcal{C}_\infty = \{\mathbb{Z} \setminus F : F \subseteq \mathbb{Z} \text{ finite}\}$.

Let us consider its associated topological graph E, where $E^0 = \{v_{\mathcal{C}_n} : n \in \mathbb{Z}\} \cup \{v_{\mathcal{C}_\infty}\}$ is the one point compactification of $\mathbb{Z}$, $E^1_a = \{e^a_{\mathcal{C}_0}\}$, $E^1_b = \{e^b_{\mathcal{C}_n} : n \in \mathbb{Z}\} \cup \{e^b_{\mathcal{C}_\infty}\}$ and $E^1_c = \{e^c_{\mathcal{C}_n} : n \in \mathbb{Z}\} \cup \{e^c_{\mathcal{C}_\infty}\}$. Hence,

$$E^1 = E^1_a \sqcup E^1_b \sqcup E^1_c$$

is a compact space because E^1_a, E^1_b and E^1_c are compact by Corollary 2.17. Then, we have that $d(e^a_{\mathcal{C}_0}) = v_{\mathcal{C}_0}$ and $r(e^a_{\mathcal{C}_0}) = v_{\mathcal{C}_0}$. Given $n \in \mathbb{Z}$, we have that $d(e^b_{\mathcal{C}_n}) = v_{\mathcal{C}_n}$ and $r(e^b_{\mathcal{C}_n}) = v_{\mathcal{C}_{n-1}}$, and $d(e^c_{\mathcal{C}_n}) = v_{\mathcal{C}_n}$ and $r(e^c_{\mathcal{C}_n}) = v_{\mathcal{C}_{n-1}}$. Finally, $d(e^b_{\mathcal{C}_\infty}) = d(e^c_{\mathcal{C}_\infty}) = v_{\mathcal{C}_\infty}$ and $r(e^b_{\mathcal{C}_\infty}) = r(e^c_{\mathcal{C}_\infty}) = v_{\mathcal{C}_\infty}$.

Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system and let E be the associated topological graph defined in Proposition 5.3. We will describe it in terms of $(\mathcal{B}, \mathcal{L}, \theta)$. To this end, we must first determine the source vertices $E^0_{sce} = E^0 \setminus \overline{r(E^1)}$. Recall that $r(E^1) = \bigcup_{\alpha\in\mathcal{L}} r(E^1_\alpha) = \bigcup_{\alpha\in\mathcal{L}} \mathcal{N}_{\mathcal{D}_\alpha}$ by Lemma 5.7.

Then we have that

$$\overline{r(E^1)} = \{v_\mathcal{C} \in E^0 : \forall A \in \mathcal{C},\, \exists \alpha \in \mathcal{L} \text{ such that } \mathcal{D}_\alpha \cap A \neq \emptyset\},$$

and hence

$$E^0_{sce} = \{v_\mathcal{C} \in E^0 : \exists A \in \mathcal{C} \text{ such that } \mathcal{D}_\alpha \cap A = \emptyset \ \ \forall \alpha \in \mathcal{L}\},$$

with closure

$$\overline{E^0_{sce}} = \{v_\mathcal{C} \in E^0 : \forall A \in \mathcal{C},\, \exists B \subseteq A \text{ such that } D_\alpha \cap A = \emptyset \ \ \forall \alpha \in \mathcal{L}\}.$$

Now, let us define the *compact vertices* as

$$E^0_{fin} = \{v_\mathcal{C} : \exists A \in \mathcal{C} \text{ such that } r^{-1}(\mathcal{N}_A) \subseteq E^1 \text{ is compact } \}.$$

By Lemma 5.7 we have that $r^{-1}(\mathcal{N}_A) = \bigsqcup_{\alpha\in\mathcal{L}} \mathcal{M}^\alpha_{\theta_\alpha(A)}$. So,

$$E^0_{fin} = \{v_\mathcal{C} : \exists A \in \mathcal{C} \text{ such that } \lambda_A < \infty\}.$$

Observe that $v_\mathcal{C} \in E^0_{sce}$ if and only if there exists $A \in \mathcal{C}$ such that $\lambda_A = 0$. Now, we define the *regular vertices* as the open set

$$\begin{aligned} E^0_{rg} := E^0_{fin} \setminus \overline{E^0_{sce}} = \{v_\mathcal{C} : {} & \exists A \in \mathcal{C} \text{ such that } \lambda_A < \infty \text{ and} \\ & \forall B \in \mathcal{B} \text{ with } B \subseteq A \ \exists \alpha \in \mathcal{L} \text{ such that } \mathcal{D}_\alpha \cap B \neq \emptyset\,\} \\ = \{v_\mathcal{C} : {} & \exists A \in \mathcal{C} \text{ such that } \forall B \in \mathcal{B} \text{ with } B \subseteq A \text{ then } 0 < \lambda_B < \infty\}, \end{aligned}$$

and the *singular vertices*

$$E^0_{sg} := E^0 \setminus E^0_{rg} = \{v_\mathcal{C} : \forall A \in \mathcal{C} \ \exists B \in \mathcal{B} \text{ with } B \subseteq A \text{ such that } \lambda_B \in \{0, \infty\}\}.$$

Theorem 5.10. *Let $(\mathcal{B},\mathcal{L},\theta)$ be a Boolean dynamical system, and let E be the associated topological graph defined in Proposition 5.3. Then, the faithful representation constructed in Corollary 5.4 is universal. Therefore, $C^*(\mathcal{B},\mathcal{L},\theta)\cong\mathcal{O}(E)$.*

Proof. Our strategy will be to prove that any representation $\{P_A,S_\alpha\}$ of $(\mathcal{B},\mathcal{L},\theta)$ induces a representation (T^0,T^1) of the associated topological graph E constructed in Proposition 5.3, such that $T^0(\chi_{\mathcal{N}_A})=P_A$ and $T^1(\chi_{E^1_\alpha})=S_\alpha$. Then the universality of (t^0,t^1) will induce the map $\eta:\mathcal{O}(E)\to C^*(P_A,S_\alpha)$ with $p_A=t^0(\chi_{\mathcal{N}_A})\mapsto T^0(\chi_{\mathcal{N}_A})=P_A$ and $s_\alpha=t^1(\chi_{E^1_\alpha})\mapsto T^1(\chi_{E^1_\alpha})=S_\alpha$.

First, we claim that the families $\{\chi_{\mathcal{N}_A}:A\in\mathcal{B}\}$ and $\{\chi_{\mathcal{M}^\alpha_A}:\alpha\in\mathcal{L},\ A\in\mathcal{I}_{\mathcal{R}_\alpha}\}$ generate $C_0(E^0)$ and $C_d(E^1)$ respectively. Recall that the definition for $\xi\in C_0(E^1)$ to be in $C_d(E^1)$ is that

$$\sum_{e\in d^{-1}(v)}|\xi(e)|^2<\infty$$

for all $v\in E^0$. Since d is injective on a given E^1_α, we just show that $\{\chi_{\mathcal{M}^\alpha_A}:A\in\mathcal{I}_{\mathcal{R}_\alpha}\}$ generates $C(E^1_\alpha)$ for each $\alpha\in\mathcal{L}$. Then, Proposition 2.16 proves the claim.

Therefore, we define $T^0:C_0(E^0)\longrightarrow\mathcal{A}$ by $\chi_{\mathcal{N}_A}\longmapsto P_A$ for every $A\in\mathcal{B}$, and $T^1: C_d(E^1)\longrightarrow\mathcal{A}$ by $\chi_{\mathcal{M}^\alpha_A}\longmapsto S_\alpha P_A$ for every $A\in\mathcal{I}_{\mathcal{R}_\alpha}$ and $\alpha\in\mathcal{L}$. T^0 is an $*$-homomorphism by [8, Lemma B.1], and T^1 is a well-defined linear map since it decreases the norm. Given $\alpha,\beta\in\mathcal{L}$, $A\in\mathcal{I}_{\mathcal{R}_\alpha}$ and $B\in\mathcal{I}_{\mathcal{R}_\beta}$,

$$T^1(\chi_{\mathcal{M}^\alpha_A})^*T^1(\chi_{\mathcal{M}^\beta_B})=(S_\alpha P_A)^*S_\beta P_B=\delta_{\alpha,\beta}P_{A\cap B}.$$

Observe that, given $e\neq e'$ with $d(e)=d(e')=v$, if $e\in E^1_\alpha$ for some $\alpha\in\mathcal{L}$ then $e'\notin E^1_\alpha$. Indeed, let $e=e^\alpha_{\mathcal{C}}$ and $e'=e^\beta_{\mathcal{C}'}$ for some $\alpha,\beta\in\mathcal{L}$, $\mathcal{C}\in\mathcal{D}(\mathcal{I}_{\mathcal{R}_\alpha})$ and $\mathcal{C}'\in\mathcal{D}(\mathcal{I}_{\mathcal{R}_\beta})$. By hypothesis $v_{\mathcal{C}}=d(e^\alpha_{\mathcal{C}})=d(e^\beta_{\mathcal{C}'})=v_{\mathcal{C}'}$, so $\mathcal{C}=\mathcal{C}'$. But since $e^\alpha_{\mathcal{C}}\neq e^\beta_{\mathcal{C}}$, it implies that $\alpha\neq\beta$.

Therefore,

$$\begin{aligned}\langle\chi_{\mathcal{M}^\alpha_A}|\chi_{\mathcal{M}^\beta_B}\rangle(v)&=\sum_{d(e)=v}\overline{\chi_{\mathcal{M}^\alpha_A}(e)}\chi_{\mathcal{M}^\beta_B}(e)\\&=\delta_{\alpha,\beta}\chi_{\mathcal{N}_A}\chi_{\mathcal{N}_B}(v)=\delta_{\alpha,\beta}\chi_{\mathcal{N}_A\cap\mathcal{N}_B}(v)\,,\end{aligned}$$

and hence

$$T^0(\langle\chi_{\mathcal{M}^\alpha_A}|\chi_{\mathcal{M}^\beta_B}\rangle)=\delta_{\alpha,\beta}P_{A\cap B}=T^1(\chi_{\mathcal{M}^\alpha_A})^*T^1(\chi_{\mathcal{M}^\beta_B})\,,$$

as desired. Now let $\alpha\in\mathcal{L}$, $A\in\mathcal{B}$ and $B\in\mathcal{I}_{\mathcal{R}_\alpha}$. Then,

$$T^0(\chi_{\mathcal{N}_A})T^1(\chi_{\mathcal{M}^\alpha_B})=P_AS_\alpha P_B=S_\alpha P_{\theta_\alpha(A)}P_B=S_\alpha P_{\theta_\alpha(A)\cap B}\,.$$

Thus, given $e\in E^1$, and Lemma 5.7, we have that

$$\begin{aligned}\pi_r(\chi_{\mathcal{N}_A})(\chi_{\mathcal{M}^\alpha_B})(e)&=\chi_{\mathcal{N}_A}(r(e))\chi_{\mathcal{M}^\alpha_B}(e)\\&=\chi_{\mathcal{M}^\alpha_{\theta_\alpha(A)}}(e)\chi_{\mathcal{M}^\alpha_B}(e)\\&=\chi_{\mathcal{M}^\alpha_{\theta_\alpha(A)}\cap\mathcal{N}_B}(e)\,.\end{aligned}$$

Hence,

$$T^0(\chi_{\mathcal{N}_A})T^1(\chi_{\mathcal{M}^\alpha_B})=S_\alpha P_{\theta_\alpha(A)\cap B}=T^1(\pi_r(\chi_{\mathcal{N}_A})(\chi_{\mathcal{M}^\alpha_B}))\,,$$

whence (T^0,T^1) is a Toeplitz E-pair.

Finally, let $f \in C_0(E^0_{rg})$. We need to prove that $T^0(f) = \Phi(\pi_r(f))$, where $\Phi : \mathcal{K}(C_d(E^1)) \longrightarrow B$ is the associated $*$-homomorphism associated to (T^0, T^1). Given $\varepsilon > 0$, we will construct $f' \in C_0(E^0_{rg})$ such that $\|f - f'\| < \varepsilon$ and such that $\Phi(\pi_r(f')) = T^0(f')$. Let K be a compact subset of E^0_{rg} such that $\|f_{E^0_{rg}\setminus K}\| < \varepsilon$. Given $v \in K$, we define the open subset $Z_v := \{w \in E^0_{rg} : \|f(v) - f(w)\| < \varepsilon\}$. Then, we can find $A_v \in \mathcal{B}_{reg}$ such that $v \in \mathcal{N}_{A_v} \subseteq Z_v$. Therefore, we have that $K \subseteq \bigcup\limits_{v\in K} \mathcal{N}_{A_v} \subseteq E^0_{rg}$, but since K is compact, there exist $v_1, \dots, v_n \in K$ such that $K \subseteq \bigcup\limits_{i=1}^{n} \mathcal{N}_{A_{v_i}}$. Observe that we also can assume that $\mathcal{N}_{A_{v_i}} \cap \mathcal{N}_{A_{v_j}} = \emptyset$ for $i \neq j$, and that $K = \bigcup\limits_{i=1}^{n} \mathcal{N}_{A_{v_i}}$. Then, we define $f' := \sum_{i=1}^n f(v_i)\chi_{\mathcal{N}_{A_{v_i}}} \in C_0(E^0_{rg})$. Clearly, $\|f - f'\| < \varepsilon$. We claim that

$$\pi_r(f') = \sum_{i=1}^{n} f(v_i) \left(\sum_{\alpha\in\Delta_{A_{v_i}}} \theta_{\chi_{\mathcal{M}^{\alpha}_{\theta_\alpha(A_{v_i})}}, \chi_{\mathcal{M}^{\alpha}_{\theta_\alpha(A_{v_i})}}} \right).$$

Indeed, let $\xi \in C_d(E^1)$ and $e \in E^1_\alpha$. Observe that $0 < |\Delta_{A_{v_i}}| = \lambda_{A_{v_i}} < \infty$ for every $i = 1, \dots, n$. Then we have that

$$\left(\sum_{i=1}^{n} f(v_i) \left(\sum_{\beta\in\Delta_{A_{v_i}}} \theta_{\chi_{\mathcal{M}^{\beta}_{\theta_\beta(A_{v_i})}}, \chi_{\mathcal{M}^{\beta}_{\theta_\beta(A_{v_i})}}} \right) \right)(\xi)(e) =$$

$$\sum_{i=1}^{n} f(v_i) \left(\sum_{\beta\in\Delta_{A_{v_i}}} \chi_{\mathcal{M}^{\beta}_{\theta_\beta(A_{v_i})}}(e) \langle \chi_{\mathcal{M}^{\beta}_{\theta_\beta(A_{v_i})}} | \xi\rangle (d(e)) \right) =$$

$$\sum_{i=1}^{n} f(v_i) \left(\sum_{\beta\in\Delta_{A_{v_i}}} \chi_{\mathcal{M}^{\beta}_{\theta_\beta(A_{v_i})}}(e) \left(\sum_{d(e')=d(e)} \overline{\chi_{\mathcal{M}^{\beta}_{\theta_\beta(A_{v_i})}}(e')} \xi(e') \right) \right) =$$

$$\sum_{i=1}^{n} f(v_i) \left(\sum_{\beta\in\Delta_{A_{v_i}}} \chi_{\mathcal{M}^{\beta}_{\theta_\beta(A_{v_i})}}(e) \xi(e) \right).$$

Observe that, by Lemma 5.7 and the fact that $\mathcal{N}_{A_{v_i}} \cap \mathcal{N}_{A_{v_j}} = \emptyset$ for $i \neq j$, we have that $r(e) \in K$ if and only if there exists a unique $1 \leq k \leq n$ such that $e \in \mathcal{M}^{\alpha}_{\theta_\alpha(A_{v_k})}$. Then,

$$\sum_{i=1}^{n} f(v_i) \left(\sum_{\beta\in\Delta_{A_{v_i}}} \chi_{\mathcal{M}^{\beta}_{\theta_\beta(A_{v_i})}}(e) \xi(e) \right) = \sum_{i=1}^{n} f(v_i) \chi_{\mathcal{N}_{A_{v_i}}}(r(e)) \xi(e) = \pi_r(f')\xi(e),$$

as desired.

Finally, since $\{P_A, S_\alpha\}$ is a representation of $(\mathcal{B}, \mathcal{L}, \theta)$, we have that

$$T^0(f') = \sum_{i=1}^{n} f(v_i) T^0\left(\chi_{\mathcal{N}_{A_{v_i}}}\right) = \sum_{i=1}^{n} f(v_i) P_{A_{v_i}} = \sum_{i=1}^{n} f(v_i) \sum_{\alpha\in\Delta_{A_{v_i}}} S_\alpha P_{\theta_\alpha(A_{v_i})} S^*_\alpha,$$

because $A_{v_i} \in \mathcal{B}_{reg}$. But

$$\Phi(\pi_r(f')) = \Phi\left(\sum_{i=1}^{n} f(v_i)\left(\sum_{\alpha\in\Delta_{A_{v_i}}} \theta_{\chi_{\mathcal{M}^{\alpha}_{\theta_\alpha(A_{v_i})}},\chi_{\mathcal{M}^{\alpha}_{\theta_\alpha(A_{v_i})}}}\right)\right) =$$

$$\sum_{i=1}^{n} f(v_i)\left(\sum_{\alpha\in\Delta_{A_{v_i}}} T^1(\chi_{\mathcal{M}^{\alpha}_{\theta_\alpha(A_{v_i})}})T^1(\chi_{\mathcal{M}^{\alpha}_{\theta_\alpha(A_{v_i})}})^*\right) = \sum_{i=1}^{n} f(v_i)\sum_{\alpha\in\Delta_{A_{v_i}}} S_\alpha P_{\theta_\alpha(A_{v_i})}(S_\alpha P_{\theta_\alpha(A_{v_i})})^* =$$

$$\sum_{i=1}^{n} f(v_i)\sum_{\alpha\in\Delta_{A_{v_i}}} S_\alpha P_{\theta_\alpha(A_{v_i})} S_\alpha^* = T^0(f')\,.$$

Thus, (T^0, T^1) is a Cuntz-Krieger E-pair, as desired. □

We can use the characterization of $C^*(\mathcal{B},\mathcal{L},\theta)$ as a topological graph to deduce the following results:

Corollary 5.11. [21, Section 6] *Let $(\mathcal{B},\mathcal{L},\theta)$ be a Boolean dynamical system.*

(1) $C^*(\mathcal{B},\mathcal{L},\theta)$ is nuclear,
(2) if $\mathcal{B}$ is a unital Boolean algebra then $C^*(\mathcal{B},\mathcal{L},\theta)$ is unital,
(3) if $\mathcal{B}$ and $\mathcal{L}$ are countable then $C^*(\mathcal{B},\mathcal{L},\theta)$ satisfies the Universal Coefficients Theorem.

Our intention now is to state a gauge invariant theorem for $C^*(\mathcal{B},\mathcal{L},\theta)$. By the universality of $\mathcal{O}(E)$, there exists a gauge action $\beta' : \mathbb{T} \curvearrowright \text{Aut}\,(\mathcal{O}(E))$ defined by $\beta'_z(t^0(f)) = t^0(f)$ and $\beta'_z(t^1(\xi)) = zt^1(\xi)$ for $f \in C_0(E^0)$, $\xi \in C_d(E^1)$ and $z \in \mathbb{T}$. Moreover, the map Φ : $C^*(\mathcal{B},\mathcal{L},\theta) \longrightarrow \mathcal{O}(E)$, defined by $p_A \longmapsto t^0(\chi_{\mathcal{N}_A})$ and $s_\alpha \longmapsto t^1(\chi_{\mathcal{M}_\alpha})$ for $A \in \mathcal{B}$ and $\alpha \in \mathcal{L}$, is an isomorphism. Then, it is clear that $\beta'_z \circ \Psi = \Psi \circ \beta_z$ for $z \in \mathbb{T}$, where β is the gauge action of $C^*(\mathcal{B},\mathcal{L},\theta)$ defined in Section 3. Therefore, using the above isomorphism Ψ, we will not make distinction between $C^*(\mathcal{B},\mathcal{L},\theta)$ and $\mathcal{O}(E)$, and between their respective gauge actions β and β'.

Theorem 5.12. *Let $(\mathcal{B},\mathcal{L},\theta)$ be a Boolean dynamical system, and let $\{P_A, S_\alpha\}$ be a Cuntz-Krieger representation of $(\mathcal{B},\mathcal{L},\theta)$ in $\mathcal{A}$. Suppose that $P_A \neq 0$ whenever $A \neq \emptyset$, and that there is a strongly continuous action γ of $\mathbb{T}$ on $C^*(P_A, S_\alpha) \subseteq \mathcal{A}$, such that for all $z \in \mathbb{T}$ we have that $\gamma_z \circ \pi_{S,P} = \pi_{S,P} \circ \beta_z$. Then, $\pi_{S,P}$ is injective.*

Proof. The result follows by Theorem 5.10, the above comment and [21, Theorem 4.5]. □

Finally we will compute the K-Theory of Cuntz-Krieger Boolean algebras. To do that, we will use the above characterization as topological graph C^*-algebra, and then we will use the results of Katsura [21, Section 6] to give a 6-term exact sequence that allows to compute the K-Theory of the Cuntz-Krieger Boolean algebra. The peculiarity of the space, that is 0-dimensional, implies that this computation reduces to computing the kernel and cokernel of a map between the K-groups of certain subspaces of the vertex spaces.

First recall that, given a topological graph E, there is a 6-term exact sequence

$$\begin{tikzcd} K_0(C_0(E^0_{rg})) \arrow[r, "\iota_*-[\pi_r]"] & K_0(C_0(E^0)) \arrow[r] & K_0(\mathcal{O}(E)) \arrow[d] \\ K_1(\mathcal{O}(E)) \arrow[u] & K_1(C_0(E_{rg})) \arrow[l] & K_1(C_0(E^0_{rg})) \arrow[l, "\iota_*-[\pi_r]"'] \end{tikzcd}$$

where $\iota : C_0(E^0_{rg}) \to C_0(E^0)$ is the natural map, and $\pi_r : C_0(E^0_{rg}) \to \mathcal{K}(C_d(E^1))$.

Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system, and let E be the associated topological graph. Recall that $E^0 = \mathcal{D}(\mathcal{B})$, that $E^0_{rg} = \mathcal{D}(\mathcal{B}_{reg})$, and that by the Stone's Representation Theorem we have that

$$\begin{tikzcd} K_0(C^*(\mathcal{B}_{reg})) \arrow[r, "\iota_*-[\pi_r]"] & K_0(C^*(\mathcal{B})) \arrow[r] & K_0(\mathcal{O}(E)) \arrow[d] \\ K_1(\mathcal{O}(E)) \arrow[u] & 0 \arrow[l] & 0 \arrow[l] \end{tikzcd} .$$

Observe that, since E^0 is a 0-dimensional space, we have that

$$K_0(C^*(\mathcal{B})) = K_0(C_0(E^0)) = C_0(E^0, \mathbb{Z}) = C(\mathcal{B}, \mathbb{Z}) ,$$

where $C(\mathcal{B}, \mathbb{Z})$ is the $\mathbb{Z}$-linear span of the functions defined on $\mathcal{B}$ by

$$\chi_A(B) = \begin{cases} 1 & \text{if } A \cap B \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for $A, B \in \mathcal{B}$.

Now, given $A \in \mathcal{B}_{reg}$, we have that the characteristic function $\chi_{\mathcal{N}_A} \in C_0(E^0_{rg})$, and hence $\pi_r(\chi_{\mathcal{N}_A}) = \sum_{\alpha\in\Delta_A} \Theta_{\chi_{\mathcal{M}^\alpha_{\theta_\alpha(A)}}, \chi_{\mathcal{M}^\alpha_{\theta_\alpha(A)}}}$. Therefore, the map $[\pi_r] : C(\mathcal{B}_{reg}, \mathbb{Z}) \to C(\mathcal{B}, \mathbb{Z})$ is given by $\chi_A \mapsto \sum_{\alpha\in\Delta_A} \chi_{\theta_\alpha(A)}$ for every $A \in \mathcal{B}_{reg}$.

Proposition 5.13 (cf. [21, Proposition 6.9])**.** *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system. Then, $K_0(C^*(\mathcal{B}, \mathcal{L}, \theta)) \cong$ Ker $(Id - [\pi_r])$ and $K_1(C^*(\mathcal{B}, \mathcal{L}, \theta)) \cong$ Coker $(Id - [\pi_r])$, where $Id - [\pi_r] : C(\mathcal{B}_{reg}, \mathbb{Z}) \to C(\mathcal{B}, \mathbb{Z})$ is given by $\chi_A \mapsto \chi_A - \sum_{\alpha\in\Delta_A} \chi_{\theta_\alpha(A)}$ for $A \in \mathcal{B}_{reg}$.*

Remark 5.14. We would like to remark that Corollary 5.11 is a generalization of [3, Corollary 3.11], that Theorem 5.12 is a generalization of [3, Corollary 3.10], and that Proposition 5.13 is a generalization of [3, Theorem 4.4].

6. An $*$-inverse semigroup

In this section we will associate to $C^*(\mathcal{B}, \mathcal{L}, \theta)$ an $*$-inverse semigroup, which will help us to construct the groupoid used to represent the above algebra as a groupoid C^*-algebra. In order to attain our goal we will first associate to $C^*(\mathcal{B}, \mathcal{L}, \theta)$ a suitable $*$-inverse semigroup.

Definition 6.1.

$$T = T_{(\mathcal{B},\mathcal{L},\theta)} := \{s_\alpha p_A s^*_\beta : \alpha, \beta \in \mathcal{L}^* , A \in \mathcal{B} , A \subseteq \mathcal{R}_\alpha \cap \mathcal{R}_\beta \neq \emptyset\} \cup \{0\} .$$

Recall that given $A \in \mathcal{B}$ we have that $s_A := p_A$.

Clearly, $T \subseteq C^*(\mathcal{B}, \mathcal{L}, \theta)$. Now,

Proposition 6.2. *T is an $*$-inverse semigroup.*

Proof. First notice that, given $\alpha, \beta \in \mathcal{L}^*$ and $A \in \mathcal{B}$,

$$s_\alpha p_A s_\beta^* = s_\alpha p_{A\cap\mathcal{R}_\alpha\cap\mathcal{R}_\beta} s_\beta^* \,,$$

so the assumption implies that $s_\alpha p_A s_\beta^* \neq 0$.

Now given $s_\alpha p_A s_\beta^*$, $s_\gamma p_B s_\delta^*$, we have that

$$s_\alpha p_A s_\beta^* \cdot s_\gamma p_B s_\delta^* = \begin{cases} s_{\alpha\gamma'} p_{\theta_{\gamma'}(A)\cap B} s_\delta^* & \text{if } \gamma = \beta\gamma' \text{and } \mathcal{R}_{\alpha\gamma'} \cap \mathcal{R}_\delta \neq \emptyset \\ s_\alpha p_{A\cap\theta_{\beta'}(B)} s_{\delta\beta'}^* & \text{if } \beta = \gamma\beta' \text{ and } \mathcal{R}_\alpha \cap \mathcal{R}_{\delta\beta'} \neq \emptyset \\ s_\alpha p_{A\cap B} s_\delta & \text{if } \gamma = \beta \text{ and } \mathcal{R}_\alpha \cap \mathcal{R}_\gamma \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

So T is closed under multiplication. Moreover,

$$(s_\alpha p_A s_\beta^*)^* = s_\beta p_A s_\alpha^*$$

for every $\alpha, \beta \in \mathcal{L}^*$ $A \in \mathcal{B}$ with $A \subseteq \mathcal{R}_\alpha \cap \mathcal{R}_\beta \neq \emptyset$. Thus, T is an $*$-semigroup with 0.

Next, notice that for any $s = s_\alpha p_A s_\beta^* \in T$, we have that $s = ss^*s$:

$$ss^*s = (s_\alpha p_A s_\beta^* \cdot s_\beta p_A s_\alpha^*) \cdot s_\alpha p_A s_\beta^* = s_\alpha p_A s_\alpha^* \cdot s_\alpha p_A s_\beta^* = s_\alpha p_A s_\beta^* = s \,.$$

Thus, every $s \in T$ is a partial isometry.

Finally, notice that the idempotents ss^*, for $s \in T$, have the form $s_\alpha p_A s_\alpha^*$. Hence,

$$s_\alpha p_A s_\alpha^* \cdot s_\beta p_B s_\beta^* = \begin{cases} s_\beta p_{\theta_{\beta'}(A)\cap B} s_\beta^* & \text{if } \beta = \alpha\beta' \\ s_\alpha p_{A\cap\theta_{\alpha'}(B)} s_\alpha^* & \text{if } \alpha = \beta\alpha' \\ s_\alpha p_{A\cap B} s_\alpha^* & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases}$$

and it is straightforward to check that these projections pairwise commute. Thus, T is an $*$-inverse semigroup by [26, Theorem 1.1.3]. $\square$

Corollary 6.3. $C^*(\mathcal{B}, \mathcal{L}, \theta) = \overline{span}\{x : x \in T\}$

Definition 6.4. We will define $\mathcal{E}(T)$ to be the set of idempotents of T.

In order to go forward, we want to keep control of the natural ordering of $\mathcal{E}(T)$.

Lemma 6.5. *Let $\alpha, \beta \in \mathcal{L}^*$, $A \in \mathcal{B}$. Then:*

(1) If either $\alpha \neq \emptyset$ or $\alpha = \beta = \emptyset$, then $s_\alpha p_A s_\alpha^* \leq s_\beta p_B s_\beta^*$ if and only if $\alpha = \beta\alpha'$ and $A \subseteq \theta_{\alpha'}(B)$.

(2) If $\alpha = \emptyset$ and $\beta \neq \emptyset$, then $s_\alpha p_A s_\alpha^* \leq s_\beta p_B s_\beta^*$ if and only if: (i) $\Delta_A = \{\beta\}$ and (ii) $\theta_\beta(A) \subseteq B$.

Proof. (1) $s_\alpha p_A s_\alpha^* \leq s_\beta p_B s_\beta^*$ if and only if $s_\alpha p_A s_\alpha^* = s_\alpha p_A s_\alpha^* \cdot s_\beta p_B s_\beta^*$ if and only if $\alpha = \beta\alpha'$ and $A \subseteq \theta_{\alpha'}(B)$ by Proposition 6.2.

(2) If $\alpha = \emptyset$, then $s_\alpha p_A s_\alpha^* = p_A$. Hence, if $p_A \leq s_\beta p_B s_\beta^*$, then $p_A = p_A \cdot s_\beta p_B s_\beta^* = s_\beta p_{\theta_\beta(A)\cap B} s_\beta^*$. Multiplying on the right side by s_β we have that $p_A s_\beta = s_\beta p_{\theta_\beta(A)\cap B}$, and multiplying on the left side by s_β^* we have that $s_\beta^* p_A s_\beta = p_{\theta_\beta(A)\cap B}$. Since $s_\beta^* p_A s_\beta = p_{\theta_\beta(A)}$, we have $\theta_\beta(A) \subseteq B$. Moreover, $p_A = s_\beta p_{\theta_\beta(A)} s_\beta^*$ means that $\Delta_A = \{\beta\}$.

Conversely, if $\Delta_A = \{\beta\}$ and $\theta_\beta(A) \subseteq B$, then $p_A = s_\alpha p_{\theta_\beta(A)} s_\beta^* = s_\beta p_{\theta_\beta(A)\cap B} s_\beta^* = p_A \cdot s_\beta p_B s_\beta^*$, whence $p_A \leq s_\beta p_B s_\beta^*$. $\square$

In order to prove the next property of T, we need a technical result.

Lemma 6.6. *If $\emptyset \neq \alpha \in \mathcal{L}^*$ and $A \in \mathcal{B}$ with $A \subseteq \mathcal{R}_{\alpha\alpha}$, then $p_A \neq p_A s_\alpha^*$.*

Proof. Suppose that $p_A = p_A s_\alpha^*$. Since p_A is a projection, we have that

$$p_A = p_A s_\alpha^* = (p_A s_\alpha^*)^* = s_\alpha p_A\,,$$

whence $p_A = s_\alpha p_A s_\alpha^*$, which only occurs if $\Delta_A = \{\alpha\}$ and $\theta_\alpha(A) = A$. Now, given any $\emptyset \neq B \subseteq A$, it also follows that $0 \neq p_B = p_B p_A = p_B p_A s_\alpha^* = p_B s_\alpha^*$, so $\theta_\alpha(B) = B$ by the above argument.

Now, consider the Boolean system

$$\tilde{A} = \{B \in \mathcal{B} : B \subseteq A\}\,,$$

with unique action θ_α. Then $(\theta_\alpha)_{|\tilde{A}} = \mathrm{id}$, whence $C^*(\tilde{A}, \alpha, \theta_\alpha) \cong C(\widehat{A}, \mathbb{T})$. Since $C^*(\tilde{A}, \alpha, \theta_\alpha)$ has a faithful representation and any representation of $(\tilde{A}, \alpha, \theta_\alpha)$ induces a representation of $(\mathcal{B}, \mathcal{L}, \theta)$, we get a contradiction. $\square$

Definition 6.7. A $*$-inverse semigroup S is E^*-unitary if for every $s \in S$, $e \in \mathcal{E}(S)$, if $e \leq s$ then $s \in \mathcal{E}(S)$.

Proposition 6.8. *T is a E^*-unitary inverse semigroup.*

Proof. We need to check the 6 possible cases:

(1) $s_\gamma p_B s_\gamma^* \leq s_\alpha p_A s_\beta^*$ if and only if

$$s_\gamma p_B s_\gamma^* = s_\alpha p_A s_\beta^* \cdot s_\gamma p_B s_\gamma^* = s_\alpha s_\beta^* s_\beta s_\delta p_B s_\gamma^* = \qquad (\gamma = \beta\delta)$$
$$= s_\alpha p_A p_{\mathcal{R}_{\alpha\beta}} s_\delta p_B s_\gamma^* = s_\alpha p_A s_\delta p_B s_\gamma^* = s_{\alpha\delta} p_{\theta_\delta(A)\cap B} s_\gamma^*$$

if and only if $\alpha\delta = \gamma = \beta\delta$, whence $\alpha = \beta$ and then $s_\alpha p_A s_\alpha^* \in \mathcal{E}(T)$.

(2) $s_\gamma p_B s_\gamma^* \leq s_\alpha p_A$ if and only if

$$s_\gamma p_B s_\gamma^* = s_\alpha p_A \cdot s_\gamma p_B s_\gamma^* = s_{\alpha\gamma} p_{\theta_\gamma(A)\cap B} s_\gamma^*$$

if and only if $\alpha\gamma = \gamma$, i.e., $\alpha = \emptyset$, whence $s_\alpha p_A = p_A \in \mathcal{E}(T)$.

(3) $s_\gamma p_B s_\gamma^* \leq p_A s_\alpha^*$, this case is analog to (2).

(4) $p_B \leq s_\alpha p_A s_\beta^*$ if and only if

$$p_B = s_\alpha p_A s_\beta^* \cdot p_B = s_\alpha p_{A\cap\theta_\beta(B)} s_\beta^* = s_\beta p_{A\cap\theta_\beta(B)} s_\alpha^*\,.$$

Thus,

$$p_{A\cap\theta_\beta(B)} = s_\alpha^* p_B s_\beta = s_\alpha^* s_\beta p_{\theta_\beta(B)}\,.$$

By Lemma 6.6 , the only possibility is that $\alpha = \beta$, whence $s_\alpha p_A s_\alpha^* \in \mathcal{E}(T)$.

(5) $p_B \leq s_\alpha p_A$ if and only if $p_B = s_\alpha p_A \cdot p_B = s_\alpha p_{A\cap B}$. Thus, by Lemma 6.6 $\alpha = \emptyset$, whence $s_\alpha p_A \in \mathcal{E}(T)$.

(6) $p_B \leq p_A s_\alpha^*$, this case is analog to case (5).

$\square$

Proposition 6.8 will play an important role in the sequel. We also need to determine the orthogonality of idempotents.

Lemma 6.9. *$s_\alpha p_A s_\alpha^* \cdot s_\beta p_B s_\beta^* = 0$ if and only if either*

(1) $\alpha \not\leq \beta$ *and* $\beta \not\leq \alpha$*, or*
(2) $\beta = \alpha\beta'$ *and* $\theta_{\beta'}(A) \cap B = \emptyset$*, or*
(3) $\alpha = \beta\alpha'$ *and* $\theta_{\alpha'}(B) \cap A = \emptyset$*.*

Proof. It is a simple computation, according Proposition 6.2. □

It is possible to use an *abstract* version of T, defined as follows:

Definition 6.10. Given $(\mathcal{B}, \mathcal{L}, \theta)$, define the set

$$\mathcal{S} = \mathcal{S}_{(\mathcal{B},\mathcal{L},\theta)} := \{(\alpha, A, \beta) : \alpha, \beta \in \mathcal{L}^*\, , A \in \mathcal{B}\, , \emptyset \neq A \subseteq \mathcal{R}_\alpha \cap \mathcal{R}_\beta\} \cup \{0\}$$

If we endow $\mathcal{S}$ with the natural involution $(\alpha, A, \beta)^* := (\beta, A, \alpha)$ and the operation induced by that of T (see Proposition 6.2), we conclude:

Proposition 6.11. $\mathcal{S}$ *is an* $*$*-inverse semigroup.*

The first difference between both semigroups arises when looking at the order relation defined on them. The reason is that, whenever $\emptyset \neq \beta \in \mathcal{L}^*$, the inequality $(\emptyset, A, \emptyset) \leq (\beta, B, \beta)$ cannot hold, so that there is no analog of Lemma 6.5(2) for $\mathcal{S}$. The ordering on $\mathcal{S}$ is described as follows.

Lemma 6.12. *Let* $\alpha, \beta \in \mathcal{L}^*$*,* $A \in \mathcal{B}$*. Then,* $(\alpha, A, \alpha) \leq (\beta, B, \beta)$ *if and only if* $\alpha = \beta\alpha'$ *and* $A \subseteq \theta_{\alpha'}(B)$*.*

Definition 6.13. The map

$$\begin{array}{cccc} \pi : & \mathcal{S} & \longrightarrow & T \\ & (\alpha, A, \beta) & \mapsto & s_\alpha p_A s_\beta^* \end{array}$$

is an onto $*$-semigroup homomorphism.

Notice that if $\Delta_A = \{\alpha\}$, then $(\alpha, \theta_\alpha(A), \alpha) < (\emptyset, A, \emptyset)$ in $\mathcal{S}$, but $\pi(\alpha, \theta_\alpha(A), \alpha) = \pi(\emptyset, A, \emptyset)$ in T, whence π is not injective in general. In this case notice that, if $0 \neq (\beta, B, \beta) \leq (\emptyset, A, \emptyset)$, then $0 \neq (\beta, B, \beta) \cdot (\alpha, \theta_\alpha(A), \alpha)$. So,

Lemma 6.14. *If* $\Delta_A = \{\alpha\}$*, then* $(\alpha, \theta_\alpha(A), \alpha)$ *is dense in* $(\emptyset, A, \emptyset)$ [12, Definition 2.9].

This will play a role in the sequel.

6.15. Let us fix the exact situations in which $\pi : \mathcal{S} \to T$ fails to be injective. For this end, suppose that

$$0 \neq s_\alpha p_A s_\beta^* = s_\gamma p_B s_\eta^*.$$

Then,

$$0 \neq s_\gamma^* s_\alpha p_A s_\beta^* s_\eta = p_B,$$

whence α and γ are comparable, as well as so does β and η.

Suppose that $\gamma \leq \alpha$, i.e. $\gamma = \alpha\gamma'$. We have two possibilities:

(1) If $\beta \leq \eta$, i.e. $\beta = \eta\beta'$, then

$$0 \neq p_B = s_\gamma^* s_\alpha p_A s_\beta^* s_\eta = s_{\gamma'}^* p_A s_{\beta'}^* = p_{\theta_{\gamma'}(A)} s_{\beta'\gamma'}^*.$$

Thus, by Lemma 6.6, $\beta' = \gamma' = \emptyset$, whence $\beta = \eta$, $\alpha = \gamma$ and $A = B$. So, $(\alpha, A, \beta) = (\gamma, B, \eta)$ in $\mathcal{S}$.

(2) If $\eta \leq \beta$, i.e. $\eta = \beta\eta'$, then, as above,

$$0 \neq p_B = p_{\theta_{\gamma'}(A)} s^*_{\gamma'} s_{\eta'} .$$

Again by by Lemma 6.6, $\gamma' = \eta'$, so that $s_\gamma p_B s^*_\eta = s_\alpha s_{\gamma'} p_B s^*_{\gamma'} s^*_\beta$. Thus, $p_A = s_{\gamma'} p_B s^*_{\gamma'}$, whence $\Delta_A = \{\gamma'\}$ and $B = \theta_{\gamma'}(A)$.

The case $\alpha \leq \gamma$ is proved in a similar way. Summarizing, the failure of injectivity for the map $\pi : \mathcal{S} \to T$ is directly connected to the existence of dense pairs of idempotents as in Lemma 6.14.

7. Tight representations of T and $\mathcal{S}$.

This intermediate step will help us to connect $C^*(\mathcal{B}, \mathcal{L}, \theta)$ with a universal C^*-algebra for a suitable family of representations of both T and $\mathcal{S}$. Concretely, the goal of this section is to prove that the maps

$$\iota : T \longrightarrow C^*(\mathcal{B}, \mathcal{L}, \theta) \qquad \text{and} \qquad \iota \circ \pi : \mathcal{S} \longrightarrow C^*(\mathcal{B}, \mathcal{L}, \theta)$$

are universal tight representations of $\mathcal{S}$ and T, respectively.

First we recall some definitions from [11].

Definition 7.1. Set $\mathcal{E} = \mathcal{E}(T)$ or $\mathcal{E}(S)$. Then:

(1) Given $X, Y \subseteq \mathcal{E}$ finite subsets,

$$\mathcal{E}^{X,Y} := \{z \in \mathcal{E} : z \leq x \text{ for all } x \in X \text{ and } z \perp y \text{ for all } y \in Y\} .$$

(2) Given any $F \subseteq \mathcal{E}$, we say that $Z \subseteq F$ is a cover for F if for every $0 \neq x \in F$ there exists $z \in Z$ such that $zx \neq 0$. Z is cover for $y \in \mathcal{E}$ if it is a cover for $F = \{x \in \mathcal{E} : x \leq y\}$.

(3) A representation φ of $\mathcal{E}$ is tight if for every $X, Y \subseteq \mathcal{E}$ finite subsets, and for every finite cover $Z \subseteq \mathcal{E}^{X,Y}$,

$$\bigvee_{z \in Z} \varphi(z) \geq \bigwedge_{x \in X} \varphi(x) \wedge \bigwedge_{y \in Y} \neg\varphi(y) .$$

Proposition 7.2 ([11, Prop. 11.8])**.** *If φ is a representation of $\mathcal{E}$ which satisfies :*

(1) $\mathcal{E}$ contains $X \subseteq \mathcal{E}$ finite such that $\bigvee_{x \in X} \varphi(x) = 1$, or

(2) $\mathcal{E}$ admits no finite cover,

then φ is tight if and only if for every $x \in \mathcal{E}$ and for every finite cover $Z \subseteq \mathcal{E}$ for x,

$$\bigvee_{z \in Z} \varphi(z) \geq \varphi(x) .$$

First observe that $C^*(\mathcal{B}, \mathcal{L}, \theta)$ is unital if and only if $\mathcal{B}$ is a unital Boolean algebra, with suprema $\mathbf{1}$, and in this case $p_{\mathbf{1}}$ will be a finite cover for T. If $C^*(\mathcal{B}, \mathcal{L}, \theta)$ is not unital, then we have that $\{p_A\}_{A \in \mathcal{B}}$ is an approximate unit of projections. In particular, given a finite set Y of elements of $\mathcal{E}$, there exists A such that $p_B e p_B = e$ for every $e \in Y$ and $B \in \mathcal{B}$ with $A \subseteq B$.

Now, let $X \subseteq \mathcal{E}$ be a finite cover. Then, X is of the form

$$\{p_A\} \cup \{s_{\alpha_i} p_{B_i} s^*_{\alpha_i}\}_{i=1}^n .$$

Let us define $C := A \cup \bigcup_{i=1}^{n} \mathcal{D}_{\alpha\alpha_i} \in \mathcal{B}$. Since $C^*(\mathcal{B}, \mathcal{L}, \theta)$ is not unital, and hence $\mathcal{B}$ has not suprema, there exists $\emptyset \neq D \in \mathcal{B}$ with $C \cap D = \emptyset$. Therefore

$$p_D \cap p_A = \emptyset \qquad \text{and} \qquad p_D \cdot s_{\alpha_i} p_{B_i} s_{\alpha_i}^* = 0 \qquad \forall i \in \{1, \ldots, n\}.$$

Then,

Corollary 7.3. *Proposition 7.2 apply to $\mathcal{E}(T)$ for every $(\mathcal{B}, \mathcal{L}, \theta)$.*

Next step is to identify finite covers for $\Sigma_x = \{y \in \mathcal{E}(T) : y \leq x\}$, $x \in \mathcal{E}(T)$. But first a (probably well known) result.

Lemma 7.4. *Let S be any $*$-inverse semigroup, and let $\mathcal{E}(S)$ its semilattice of idempotents. Let $x \in E$ and $s \in S$ such that $x \leq s^* s$. Then $\{e_1 \ldots, e_n\}$ is a finite cover for Σ_x if and only if $\{s e_1 s^*, \ldots, s e_n s^*\}$ is a finite cover for Σ_{sxs^*}.*

Now, we need to fix a concept.

Definition 7.5. Given $\emptyset \neq A \in \mathcal{B}$, we define an expansion of A to be a finite set $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathcal{L}^*$ such that $\theta_{\alpha_i}(A) \neq \emptyset$ for every $1 \leq i \leq n$. Moreover, we say that an expansion of A is complete if $\alpha_i \nleq \alpha_j$ and $\alpha_j \nleq \alpha_i$ whenever $i \neq j$, and for every $\beta \in \mathcal{L}^*$ with $\theta_\beta(A) \neq \emptyset$ there exists i such that either $\alpha_i \leq \beta$ or $\beta \leq \alpha_i$. Equivalently, $\{\alpha_1, \ldots, \alpha_n\}$ is a complete expansion for A if $p_A = \sum_{i=1}^{n} s_{\alpha_i} p_{\theta_{\alpha_i}(A)} s_{\alpha_i}^*$.

Definition 7.6. Given $\emptyset \neq A \in \mathcal{B}$, and $n \in \mathbb{N}$, we define

$$\Delta_A^n := \{\alpha \in \mathcal{L}^n : \theta_\alpha(A) \neq \emptyset\},$$

and $\Delta_A^{\leq n} = \bigcup_{k=1}^{n} \Delta_A^k$.

Definition 7.7. Given a cover Z of Σ, we say that $\hat{Z}$ is a refinement of Z if $\hat{Z}$ is a cover of Σ, and for every element $x \in \hat{Z}$ there exists $y \in Z$ with $x \leq y$.

7.8. Now we will analyse how look like the finite covers of Σ_x for $x = p_A$ and $x = s_\alpha p_A s_\alpha^*$. By Lemma 7.4 it will be enough to look at $x = p_A$. Then a finite cover for Σ_x has the form

$$Z = \{p_{B_i}\}_{i=1}^{n} \cup \{s_{\gamma_j} p_{C_j} s_{\gamma_j}^*\}_{j=1}^{m} \subseteq \Sigma_x .$$

Observe that we can joint all the idempotents $\{p_{B_i}\}_{i=1}^{n}$ in a single idempotent p_B where $B := \bigcup_{i=1}^{n} B_i$, so

$$Z = \{p_B\} \cup \{s_{\gamma_j} p_{C_j} s_{\gamma_j}^*\}_{j=1}^{m} \subseteq \Sigma_x .$$

Now, if $A \setminus B = A \setminus (A \cap B) \notin \mathcal{B}_{reg}$, it means that there exists $C \subseteq A \setminus B$ with either $\lambda_C = 0$ or $\lambda_C = \infty$. If $\lambda_C = 0$ then we have that

$$p_C \cdot s_{\gamma_j} p_{C_j} s_{\gamma_j}^* = s_{\gamma_j} p_{\theta_{\gamma_j}(C) \cap C_j} s_{\gamma_j}^* = s_{\gamma_j} p_\emptyset s_{\gamma_j}^* = 0 \qquad \forall j \in \{1, \ldots, m\},$$

contradicting the fact that Z is a cover of p_A. If $\lambda_C = \infty$, there exists $\beta \in \mathcal{L}$ such that $\beta \nleq \gamma_i$ for $1 \leq i \leq m$. Thus, if we consider the element $s_\beta p_{\theta_\beta(C)} s_\beta^*$, then

$$s_\beta p_{\theta_\beta(C)} s_\beta^* \cdot s_{\gamma_j} p_{C_j} s_{\gamma_j}^* = 0 \qquad \forall j \in \{1, \ldots, m\},$$

and moreover, since

$$p_A \cdot s_\beta p_{\theta_\beta(C)} s_\beta^* = p_A \cdot p_C s_\beta s_\beta^* = 0,$$

this contradicts that Z is a cover for Σ_x. Therefore, $A \setminus B$ must be in $\mathcal{B}_{reg}$ for Z to be a cover.

Notice that p_B covers all the elements of Σ_x that are dominated by $p_{A\cap B}$. Thus, without loss of generality, we can assume that

$$Z = \{s_{\gamma_i} p_{C_i} s_{\gamma_i}^*\}_{i=1}^n ,$$

since $Z \subseteq \Sigma_x$ with $\theta_{\gamma_i}(A) \neq \emptyset$ for every $1 \leq i \leq n$, where $x = p_A$ with $A \in \mathcal{B}_{reg}$, and that $\gamma_i \neq \gamma_j$ whenever $i \neq j$.

Next, we see that $\{\gamma_i\}_{i=1}^n$ must contain a complete expansion for A. Otherwise, there exists $\beta \in \mathcal{L}^*$ with $\theta_\beta(A) \neq \emptyset$ with $\alpha_i \nleq \beta$ and $\beta \nleq \alpha_i$ for every $1 \leq i \leq n$, and then $s_\beta p_{\theta_\beta(A)} s_\beta^* \leq p_A$ and $s_\beta p_{\theta_\beta(A)} s_\beta^* \leq p_A \cdot s_{\gamma_i} p_{C_i} s_{\gamma_i}^* = 0$ for every $1 \leq i \leq n$, contradicting that Z is a cover for p_A. We relabel the complete expansion as $\gamma_1, \dots, \gamma_l$ for some $1 \leq l \leq n$. We can also take it minimal, so for every $k \geq l$ there exists $1 \leq i \leq l$ with $\gamma_i \leq \gamma_k$.

Another important observation is that $D_i := \theta_{\gamma_i}(A) \setminus C_i \in \mathcal{B}_{reg}$ whenever $1 \leq i \leq l$. Indeed, let us first suppose that $\lambda_{D_i} = 0$. Then, $0 \neq s_{\gamma_i} p_{D_i} s_{\gamma_i}^*$ is the element that leads to contradiction with Z being a cover of p_A. Now suppose that there exists $E_i \subseteq D_i$ with $\lambda_{E_i} = \infty$. Then, there exists $\beta \in \Delta_{E_i}$ such that $\gamma_i\beta \nleq \gamma_i$ for every γ_j with $l+1 \leq j \leq n$. Thus, the element $s_{\gamma_i\beta} p_{\theta_\beta(E_i)} s_{\gamma_i\beta}^*$ is the element that leads to contradiction with Z being a cover of p_A.

We also have that, given γ_i with $1 \leq i \leq l$ such that $\gamma_i \nleq \gamma_j$ for every $j \geq l+1$, it must be $\theta_{\gamma_i}(A) \subseteq C_i$. Otherwise, the element $s_{\gamma_i} p_{\theta_{\gamma_i}(A)\setminus C_i} s_{\gamma_i}^*$ is the element that leads to contradiction with Z being a cover of p_A.

Now, we define $A_i := \theta_{\gamma_i}(A) \setminus C_i$ for those $i \leq l$ such that $A_i \neq \emptyset$. So, there exist $\gamma_{i_1}, \dots, \gamma_{i_{k(i)}}$ with $\gamma_i \lneq \gamma_{i_j}$ for $1 \leq j \leq k(i)$, and we define $E_{i,j} := C_{i_j}$ for $1 \leq j \leq k(i)$. We can relabel the A_is as $A_1, \dots, A_m$, and if we define $\beta_{i,j} := \gamma_{i_j} \setminus \gamma_i$ for $1 \leq j \leq k(i)$, then the sets $Z_i := \{s_{\beta_{i,j}} p_{E_{i,j}} s_{\beta_{i,j}}^*\}$ are finite covers of p_{A_i} for $1 \leq i \leq m$.

Now, must proceed as above with this new covers as many time as we need, and since they are finite covers, each step will have less elements than the previous. So, in a finite number of steps, there will be a refinement of the cover that will contain a complete expansion $\{\gamma_i\}$ of A with $C_i = \theta_{\gamma_i}(A)$.

Summarizing

Lemma 7.9. *If $Z \subseteq \Sigma_x$ is a finite cover for $x \in \mathcal{E}(T)$, there exists a refinement of $\hat{Z}$ of Z such that:*

(1) $\hat{Z} \subseteq \Sigma_x$ is a finite cover,
(2) The elements in $\hat{Z}$ are pairwise orthogonal,
(3) $\bigvee_{z\in Z} \rho(z) = \sum_{\hat{z}\in\hat{Z}} \rho(\hat{z})$ for every representation ρ of $\mathcal{E}(T)$.

We are ready to prove the main result of this section

Theorem 7.10. *The representation $\iota : T \longrightarrow C^*(\mathcal{B}, \mathcal{L}, \theta)$ is the universal tight representation of T.*

Proof. First notice that, because of Corollary 7.3 and Lemma 7.9, the representation $\iota : T \to C^*(\mathcal{B}, \mathcal{L}, \theta)$ is tight.

Now, let A be any C^*-algebra, and suppose that $\rho : T \to A$ is a tight representation. Consider $\hat{s}_a := \rho(s_a)$ for every $a \in \mathcal{L}$, and $\hat{p}_A := \rho(p_A)$ for every $A \in \mathcal{B}$. Then, $\{\hat{s}_a : a \in \mathcal{L}\} \cup \{\hat{p}_A : A \in \mathcal{B}\} \subset A$, and clearly:

(1) $\{\hat{p}_A : A \in \mathcal{B}\}$ is a set of projections in A.
(2) $\{\hat{s}_a : a \in \mathcal{L}\}$ is a set of partial isometries in A.

Since ρ is a $*$-homomorphism of semigroups, we clearly have that:

(1) $\hat{p}_A \hat{p}_B = \hat{p}_{A\cap B}$ for every $A, B \in \mathcal{B}$.
(2) $\hat{p}_A \hat{s}_a = \hat{s}_a \hat{p}_{\theta_a(A)}$ for every $a \in \mathcal{L}$ and $A \in \mathcal{B}$.
(3) $\hat{s}_a^* \hat{s}_b = \delta_{a,b} \hat{p}_{\mathcal{R}_a}$ for every $a, b \in \mathcal{L}$.

In order to prove the two remaining identities, we will use the fact that ρ is tight:

(1) Take $A, B \in \mathcal{B}$. Then, it is clear that $\{p_{A\setminus B}, p_{A\cap B}\}$ is a finite orthogonal cover of p_A, and so does $\{p_{B\setminus A}, p_{A\cap B}\}$ of p_B. Hence, $\hat{p}_A = \hat{p}_{A\setminus B} + \hat{p}_{A\cap B}$ and $\hat{p}_B = \hat{p}_{B\setminus A} + \hat{p}_{A\cap B}$, whence $\hat{p}_A + \hat{p}_B - \hat{p}_{A\cap B} = \hat{p}_{A\setminus B} + \hat{p}_{B\setminus A} + \hat{p}_{A\cap B}$. Since $\{p_{A\setminus B}, p_{B\setminus A}, p_{A\cap B}\}$ is an orthogonal finite cover of $p_{A\cup B}$, we conclude that $\hat{p}_{A\cup B} = \hat{p}_{A\setminus B} + \hat{p}_{B\setminus A} + \hat{p}_{A\cap B} = \hat{p}_A + \hat{p}_B - \hat{p}_{A\cap B}$, as desired.
(2) If $A \in \mathcal{B}_{reg}$, then $\{s_a p_{\theta_a(A)} s_a^* : a \in \Delta_A\}$ is an orthogonal finite cover of p_A. Hence,
$$\hat{p}_A = \rho(p_A) = \bigvee_{a\in\Delta_A} \rho(s_a p_{\theta_a(A)} s_a^*) = \bigvee_{a\in\Delta_A} \hat{s}_a \hat{p}_{\theta_a(A)} \hat{s}_a^* = \sum_{a\in\Delta_A} \hat{s}_a \hat{p}_{\theta_a(A)} \hat{s}_a^*,$$
so we are done.

Thus, by the Universal Property of $C^*(\mathcal{B}, \mathcal{L}, \theta)$, there exists a unique $*$-homomorphism
$$\begin{array}{rccc} \psi : & C^*(\mathcal{B}, \mathcal{L}, \theta) & \to & A \\ & s_a & \mapsto & \hat{s}_a \\ & p_A & \mapsto & \hat{p}_A \end{array}.$$
Since $\psi \circ \iota = \rho$, the universality of ι is proved. $\square$

Recall $\pi : S \to T$ is an onto $*$-semigroup homomorphism. By Lemma 6.14 and (6.15), the lack of injectivity of π is linked to the existence of dense pairs of idempotents in $\mathcal{S}$. By [12, Proposition 2.11], it is then immediate to conclude

Corollary 7.11. *The representation $\iota \circ \pi : S \longrightarrow C^*(\mathcal{B}, \mathcal{L}, \theta)$ is the universal tight representation of S. Therefore, $C^*(\mathcal{B}, \mathcal{L}, \theta) \cong C^*_{tight}(T) \cong C^*_{tight}(S)$.*

8. The tight groupoid of T

In this section we will benefit of the previous work to construct a groupoid $\mathcal{G}$ such that $C^*(\mathcal{B}, \mathcal{L}, \theta) \cong C^*(\mathcal{G})$. Now, we proceed to recall the construction of $\mathcal{G}_{\text{tight}}(T)$. Let us recall the construction in a generic form (see e.g. [15]):

- If S is an inverse semigroup, then $\mathcal{E} = \mathcal{E}(S) = \{\text{idempotents of } S\}$ is a semilattice with ordering $e \leq f$ if and only if $ef = e$, and $e \wedge f = ef$. It extends to an order in S, $s \leq t$ if and only if $s = ts^*s = ss^*t$. We denote by $e \perp f$ if and only if $ef = 0$, and $e \Cap f$ if and only if $ef \neq 0$.

- A character on $\mathcal{E}$ is a nonzero map $\phi : \mathcal{E} \to \{0,1\}$ with $\phi(0) = 0$, and $\phi(ef) = \phi(e)\phi(f)$ for every $e, f \in \mathcal{E}$. We denote the set of characters by $\widehat{\mathcal{E}}_0$. This is a topological space when equipped with the product topology inherited from $\{0,1\}^{\mathcal{E}}$. Since the zero map does not belong to $\widehat{\mathcal{E}}_0$, it is a locally compact space and totally disconnected Hausdorff space.
- A filter in $\mathcal{E}$ is a nonempty subset $\eta \subseteq \mathcal{E}$ such that:
  (1) $0 \notin \eta$,
  (2) closed under $\wedge$,
  (3) $f \geq e \in \eta$ implies $f \in \eta$.
- Given a filter η,
$$\begin{aligned} \phi_\eta : \mathcal{E} &\longrightarrow \{0,1\} \\ e &\longrightarrow [e \in \eta] \end{aligned}$$
  is a character. Conversely, if $\phi \in \widehat{\mathcal{E}}_0$, then $\eta_\phi = \{e \in \mathcal{E} | \phi(e) = 1\}$ is a filter. These correspondences are mutually inverses.
- A filter η is a ultrafilter if it is not properly contained in another filter. We denote $\widehat{\mathcal{E}}_\infty \subseteq \widehat{\mathcal{E}}_0$ the space of ultrafilters.
- Tight filters are defined in analogy with tight representations. The set of tight filters (tight spectrum) is a closed subspace $\widehat{\mathcal{E}}_{\text{tight}}$ of $\widehat{\mathcal{E}}_0$, containing $\widehat{\mathcal{E}}_\infty$ as a dense subspace.
- We can define a standard action of S on $\widehat{\mathcal{E}}_0$ as follows:
  (1) For each $e \in \mathcal{E}$, $D^\beta_e = \{\phi \in \widehat{\mathcal{E}}_0 : \phi(e) = 1\}$,
  (2) given $s \in S$,
$$\begin{aligned} \beta_s : D^\beta_{s^*s} &\longrightarrow D^\beta_{ss^*} \\ \phi &\longrightarrow \beta_s(\phi)(e) = \phi(s^*es) \end{aligned}$$
  When working with filters, $D^\beta_e = \{\eta \in \widehat{\mathcal{E}}_0 | e \in \eta\}$ while $\beta_s(\eta) = \{f \in \mathcal{E} : f \geq ses^* \text{ for every } e \in \eta\}$.
- β restricts to an action of S on ultrafilters and on tight filters.

Definition 8.1. Consider the set $\Omega = \{(s,x) \in S \times \widehat{\mathcal{E}}_{\text{tight}} : x \in D^\beta_{s^*s}\}$ and define $(s,x) \sim (t,y)$ if and only if $x = y$ and exists $e \in \mathcal{E}$ such that $x \in D^\beta_e$ and $se = te$.

Define $\mathcal{G}_{\text{tight}}(S) = \Omega/\sim$, with:

(1) $d([s,x]) = x$ and $r([s,x]) = \beta_s(x)$,
(2) $[s,z] \cdot [t,x] = [st,x]$ if and only if $z = \beta_t(x)$,
(3) $[s,x]^{-1} = [s^*, \beta_s(x)]$,
(4) $\mathcal{G}^{(0)}_{\text{tight}} = \{[e,x] : e \in \mathcal{E}\} \cong \widehat{\mathcal{E}}_{\text{tight}}$

$\mathcal{G}_{\text{tight}}(S)$ is the tight groupoid of the inverse semigroup S.

Then, we have

Theorem 8.2. $C^*(\mathcal{B}, \mathcal{L}, \theta) \cong C^*(\mathcal{G}_{tight}(T))$

Proof. This holds by Corollary 7.11 and [11, Theorem 13.3]. □

Moreover

Lemma 8.3. *$\mathcal{G}_{tight}(T)$ is Hausdorff*

Proof. By Proposition 6.2 and [15, Corollary 3.17] □

Also, we have the following

Lemma 8.4. *$\mathcal{G}_{tight}(T)$ is amenable.*

Proof. Since $C^*(\mathcal{G}_{\text{tight}}(T)) \cong C^*(\mathcal{B}, \mathcal{L}, \theta)$ is nuclear, then $C^*_{red}(\mathcal{G}_{\text{tight}}(T)) = C^*(\mathcal{G}_{\text{tight}}(T))$, and thus $C^*_{red}(\mathcal{G}_{\text{tight}}(T))$ is nuclear. Hence, the result holds by [6, Theorem 5.6.18]. □

9. Simplicity of $C^*(\mathcal{B}, \mathcal{L}, \theta)$

In this section we will characterise when $C^*(\mathcal{B}, \mathcal{L}, \theta)$ is simple, using information from $\mathcal{G}_{\text{tight}}(T)$. To this end, we use a result of [5].

Theorem 9.1 ([5, Theorem 5.1])**.** *Let $\mathcal{G}$ be an étale, Hausdorff, second countable, topological groupoid. If $\mathcal{G}$ is (elementary) amenable, then the following are equivalent:*

(1) $\mathcal{G}$ is minimal and essentially principal,
(2) $C^*(\mathcal{G})$ is simple.

Since $\mathcal{G}_{\text{tight}}(T)$ is the tight groupoid of an $*$-inverse semigroup, $\mathcal{G}_{\text{tight}}(T)$ is an étale, second countable, topological groupoid [11]. We know that $\mathcal{G}_{\text{tight}}(T)$ is Hausdorff and amenable. Hence, we need only to take care of $\mathcal{G}_{\text{tight}}(T)$ being essentially principal and minimal. As $\mathcal{G}_{\text{tight}}(T)$ is the tight groupoid of an inverse semigroup, we can benefit of the results of [15] for this task.

9.1. Essentially principal groupoids.

In this subsection we take care of the essential principal property. For this and related properties we refer to [15, Section 4]. In particular, we skip the definitions.

Recall the following facts.

Theorem 9.2 ([15, Theorem 4.7])**.** *$\mathcal{G}_{tight}(T)$ is essentially principal if and only if $\beta : T \curvearrowright \widehat{\mathcal{E}}_{tight}$ is topologically free.*

Definition 9.3 ([15, Definition 4.8])**.** Let $s \in T$, $e \in \mathcal{E}(T)$ such that $e \leq ss^*$. Then, we say that:

(1) e is fixed under s if $se = e$.
(2) e is weakly fixed under s, if $sfs^* \Cap f$ for every $f \in \mathcal{E}(T) \setminus \{0\}$ and $f \leq e$.

Theorem 9.4 ([15, Theorem 4.10])**.** *Since $\mathcal{G}_{tight}(T)$ is Hausdorff, the following statements are equivalent:*

(1) $\beta : T \curvearrowright \widehat{\mathcal{E}}_{tight}$ is topologically free.
(2) for every $s \in T$ and every $e \in \mathcal{E}(T)$ weakly fixed under s, there exists $F \subseteq \Sigma_e$ finite cover consisting of fixed elements.

Definition 9.5.

(1) We say that $\alpha = \alpha_1 \cdots \alpha_n \in \mathcal{L}^*$ is a cycle without exits if for any $\emptyset \neq A \in \mathcal{B}$ such that $\alpha_1 \in \Delta_A$ we have that $\Delta_{\theta_{\alpha_1 \cdots \alpha_t}(A)} = \{\alpha_{t+1}\}$ for $t < n$ and $\Delta_{\theta_\alpha(A)} = \{\alpha_1\}$.
(2) We say that $(\mathcal{B}, \mathcal{L}, \theta)$ satisfies Condition $(L_{\mathcal{B}})$ if given $\alpha \in \mathcal{L}^*$ a cycle without exits, there exists $\emptyset \neq B \in \mathcal{B}$ with $\theta_\alpha(B) \neq \emptyset$ such that $B \cap \theta_\alpha(B) = \emptyset$.

Then

Theorem 9.6. *The following are equivalent:*

(1) $(\mathcal{B}, \mathcal{L}, \theta)$ *satisfies condition* $(L_{\mathcal{B}})$*,*
(2) $\beta : T \curvearrowright \widehat{\mathcal{E}}_{tight}$ *is topologically free,*
(3) $\mathcal{G}_{tight}(T)$ *is essentially principal.*

Proof. (2) $\Leftrightarrow$ (3) by Theorem 9.2.

For (1) $\Leftrightarrow$ (2) notice that, since T is a E^*-unitary by Proposition 6.8, condition (2) in Theorem 9.4 is equivalent to the statement:

$\forall s \in S \backslash \mathcal{E}(S)$ and $\forall 0 \neq e \in \mathcal{E}(S)$ with $e \leq ss^*$, there exists $0 \neq f \leq e$ such that $sfs^* \cdot f = 0$.

We will separate 3 cases:

(1) Case $s = s_\alpha$: Then $e \leq s_\alpha s_\alpha^* = p_{\mathcal{R}_\alpha}$. Thus, without loss of generality, we can assume $e = p_A$ and $f \leq p_A$ (A being $\mathcal{R}_\alpha$). By Lemma 6.5, $f = s_\beta p_B s_\beta^*$ with $B \subseteq \theta_\beta(A) = \mathcal{R}_{\alpha\beta}$. Without loss of generality, we can assume that $|\alpha| < |\beta|$. Then

$$0 \neq sfs^* \cdot f = s_{\alpha\beta} p_B s_{\alpha\beta}^* s_\beta p_B s_\beta^*$$

implies $\beta = \alpha\hat{\beta}$. Assuming $|\alpha| < |\hat{\beta}|$, we have that $\hat{\beta} = \alpha\beta'$ and by recurence

$$\beta = \alpha\beta_1 = \alpha^2\beta_2 = \cdots = \alpha^n\beta_n = \cdots$$

Since $|\beta| < \infty$, β must be α^k for some $k \in \mathbb{N}$, and thus $0 \neq sfs^* \cdot f = s_{\alpha^{k+1}} p_{B \cap \theta_\alpha(B)} s_{\alpha^{k+1}}^*$ is equivalent to $\beta = \alpha^k$ and $B \cap \theta_\alpha(B) \neq \emptyset$. So, $sfs^* \cdot f = 0$ for a suitable nonzero idempotent f occurs exactly when one of the following two situations hold:

(a) There exists $\beta \in \mathcal{L}^*$ such that $\theta_\beta(\mathcal{R}_\alpha) \neq \emptyset$, $\beta \not\leq \alpha$ and $\alpha \not\leq \beta$; in particular, this is the case if α is a cycle with an exit.

(b) The path α is a cycle without an exit, and $\beta = \alpha^k$ for some $k \in \mathbb{N}$. Then, $sfs^* \cdot f = s_{\alpha^{k+1}} p_{B \cap \theta_\alpha(B)} s_{\alpha^{k+1}}^* = 0$ if and only if there exists $\emptyset \neq B \in \mathcal{B}$ such that $\theta_\alpha(B) \neq \emptyset$ and $B \cap \theta_\alpha(B) = \emptyset$.

This prove the equivalence for this case.

(2) Case $s = s_\alpha^*$: Then

$$e \leq s_\alpha s_\alpha^* = s_\alpha p_{\mathcal{R}_\alpha} s_\alpha^* = s_\alpha p_{\theta_\alpha(\mathcal{D}_\alpha)} s_\alpha^* \leq p_{\mathcal{D}_\alpha}.$$

Then, the argument is analog to the case (1).

(3) Case $s = s_\alpha p_B s_\beta^*$: Without loss of generality we can assume $|\alpha| > |\beta|$. Then, $e \leq s^*s = s_\beta p_B s_\beta^* \leq p_{\mathcal{R}_\beta}$ and $f = s_\gamma p_C s_\gamma^* \leq s_\beta p_B s_\beta^*$ if and only if $\gamma = \beta\hat{\gamma}$ and $C \subseteq \theta_{\hat{\gamma}}(B)$.

Thus, we are in the situation $s_{\hat{\gamma}} p_C s_{\hat{\gamma}}^* \leq p_B$, whence case (1) applies.

□

This picture allows to prove an analog of the Cuntz-Krieger Uniqueness Theorem for labelled graph C^*-algebras [2, Theorem 5.5] in our context. In order to prove such a theorem, we need to recall some facts:

Remark 9.7.

(1) By [15, Proposition 2.5], the set $\{D_e : e \in \mathcal{E}(T)\}$ is a basis of $\widehat{\mathcal{E}}_{\text{tight}}(T)$ by clopen compact sets.

(2) For any $s \in T$, the set $\Theta(s, D_{s^*s}) := \{[s, \eta] : \eta \in D_{s^*s}\}$ is a open bisection of $\mathcal{G}_{\text{tight}}(T)$ [11, Proposition 4.18]. Moreover, the isomorphism $C^*(\mathcal{B}, \mathcal{L}, \theta) \cong C^*(\mathcal{G}_{\text{tight}}(T))$ sends each $s \in T \subset C^*(\mathcal{B}, \mathcal{L}, \theta)$ to the characteristic function $1_{\Theta(s,D_{s^*s})} \in C^*(\mathcal{G}_{\text{tight}}(T))$.
(3) By [11, Proposition 4.15] and point (1) above, $\Theta(s, D_{s^*s})$ is open and compact for every $s \in T$.
(4) By point (1) above and [15, Proposition 3.8], the set $\{\Theta(s, D_{s^*s}) : s \in T\}$ is a basis of the topology of $\mathcal{G}_{\text{tight}}(T)$. In particular, since $\mathcal{G}_{\text{tight}}^{(0)} = \{[e, x] : e \in \mathcal{E}\} \cong \widehat{\mathcal{E}}_{\text{tight}}$, the set $\{\Theta(e, D_e) : e \in \mathcal{E}(T)\}$ is a basis of the topology of $\mathcal{G}_{\text{tight}}(T)^{(0)}$.

Now, we are ready to prove our theorem.

Theorem 9.8 (Cuntz-Krieger Uniqueness Theorem for $C^*(\mathcal{B}, \mathcal{L}, \theta)$)**.** *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system satisfying condition $(L_{\mathcal{B}})$, and let $C^*(\mathcal{B}, \mathcal{L}, \theta)$ be its associated C^*-algebra. Then, for any $*$-homomorphism $\pi : C^*(\mathcal{B}, \mathcal{L}, \theta) \to B$, the following are equivalent:*
(1) $\pi(s_\alpha P_A s_\alpha^*) \neq 0$ for every $\emptyset \neq A \in \mathcal{B}$ with $A \subseteq \mathcal{R}_\alpha$.
(2) π is injective.

Proof. By Lemma 8.3, Lemma 8.4 and Theorem 9.6, be can apply [13, Theorem 4.4] to $C^*(\mathcal{G}_{\text{tight}}(T))$. Thus, in order to conclude our result, it is enough to prove that $\pi|_{C_0(\mathcal{G}_{\text{tight}}(T)^{(0)})}$ is injective if and only if $\pi(s_\alpha P_A s_\alpha^*) \neq 0$ for every $\emptyset \neq A \in \mathcal{B}$ with $A \subseteq \mathcal{R}_\alpha$.

By Remark 9.7(2), if $\pi|_{C_0(\mathcal{G}_{\text{tight}}(T)^{(0)})}$ is injective then $\pi(s_\alpha P_A s_\alpha^*) \neq 0$ for every $\emptyset \neq A \in \mathcal{B}$ with $A \subseteq \mathcal{R}_\alpha$.

Conversely, suppose that $\pi(s_\alpha P_A s_\alpha^*) \neq 0$ for every $\emptyset \neq A \in \mathcal{B}$ with $A \subseteq \mathcal{R}_\alpha$. If there exists $0 \neq f \in C_0(\mathcal{G}_{\text{tight}}(T)^{(0)})$ such that $\pi(f) = 0$, then by Remark 9.7(4) there exists $e \in \mathcal{E}(T)$ such that $\Theta(e, D_e) \subseteq \text{supp}(f)$, whence $\pi(e) = 0$, contradicting the assumption. So we are done. □

9.2. **Minimal groupoids.** In this subsection we deal with the question of minimality of the groupoid. As in the previous subsection, we refer [15, Section 5] for definitions and results. We will use the following

Theorem 9.9 ([15, Theorem 5.5])**.** *The following statements are equivalent:*
(1) $\beta : T \curvearrowright \widehat{\mathcal{E}}_{tight}$ is irreducible,
(2) $\mathcal{G}_{tight}(T)$ is minimal,
(3) for every $0 \neq e, f \in \mathcal{E}(T)$ there exists $s_1, \ldots, s_n \in S$ such that $\{s_i f s_i^*\}_{i=1}^n$ is an outer cover for e.

By analogy with the case of graph C^*-algebras, we propose the following definition:

Definition 9.10. We say that $(\mathcal{B}, \mathcal{L}, \theta)$ is cofinal if for every $\emptyset \neq A \in \mathcal{B}$ and for every $\zeta \in \widehat{\mathcal{E}}_{\text{tight}}$ there exist $\alpha, \beta \in \mathcal{L}^*$ such that $s_\alpha p_{\theta_\beta(A)} s_\alpha^* \in \zeta$.

Recall that given $e \in \mathcal{E}$, we define the cylinder set of e in $\widehat{\mathcal{E}}_{\text{tight}}$ as

$$Z(e) := \{\zeta \in \widehat{\mathcal{E}}_{\text{tight}} : e \in \zeta\} .$$

For every $e \in \mathcal{E}$, $Z(e)$ is a compact open subset of $\widehat{\mathcal{E}}_{\text{tight}}$.

Then, we have

Proposition 9.11. *The following statements are equivalent:*

(1) $(\mathcal{B}, \mathcal{L}, \theta)$ *is cofinal.*
(2) $\mathcal{G}_{tight}(T)$ *is minimal.*

Proof. First, we will prove that cofinality implies condition (3) in Theorem 9.9. For this end, suppose that $e = s_\alpha p_A s_\alpha^*$and $f = s_\beta p_B s_\beta^*$. Since $A \subseteq \mathcal{R}_\alpha$ we have

$$s_\alpha p_A s_\alpha^* \leq s_\alpha p_{\mathcal{R}_\alpha} s_\alpha^* = s_\alpha s_\alpha^* \leq p_{\mathcal{D}_\alpha}.$$

As every cover of $p_{\mathcal{D}_\alpha}$ is a cover of $s_\alpha p_A s_\alpha^*$, we can assume without loss of generality that $e = p_A$ for some $A \in \mathcal{B}$. Since $p_B = s_\beta^* f s_\beta$, we can assume without loss of generality that $f = p_B$ for some $B \in \mathcal{B}$.

Given $\xi \in Z(p_A)$, cofinality implies that there exist $\alpha_\xi, \beta_\xi \in \mathcal{L}^*$ such that

$$s_{\alpha_\xi} p_{\theta_{\beta_\xi}(B)} s_{\alpha_\xi}^* \in \xi.$$

Hence,

$$Z(p_A) \subseteq \bigcup_{\xi \in Z(p_A)} Z(s_{\alpha_\xi} p_{\theta_{\beta_\xi}(B)} s_{\alpha_\xi}^*).$$

Since $Z(p_A)$ is compact, there exist $\alpha_{\xi_1}, \ldots, \alpha_{\xi_n}, \beta_{\xi_1}, \ldots, \beta_{\xi_n}$ such that

$$Z(p_A) \subseteq \bigcup_{i=1}^{n} Z(s_{\alpha_{\xi_i}} p_{\theta_{\beta_{\xi_i}}(B)} s_{\alpha_{\xi_i}}^*).$$

By [15, Proposition 3.7], this is equivalent to say that $\{s_{\alpha_{\xi_i}} p_{\theta_{\beta_{\xi_i}}(B)} s_{\alpha_{\xi_i}}^*\}_{i=1}^n$ is an outer cover for p_A. Notice that $s_{\alpha_{\xi_i}} p_{\theta_{\beta_{\xi_i}}(B)} s_{\alpha_{\xi_i}}^* = (s_{\alpha_{\xi_i}} s_{\beta_{\xi_i}}^*) p_B (s_{\alpha_{\xi_i}} s_{\beta_{\xi_i}}^*)^*$. Thus, the result holds for $s_i := (s_{\alpha_{\xi_i}} s_{\beta_{\xi_i}}^*)$.

Now, we will prove that condition (3) in Theorem 9.9 implies cofinality. For this end, take any $\emptyset \neq A \in \mathcal{B}$ and any $\xi \in \widehat{\mathcal{E}}_{\text{tight}}$. By the argument at the start of this proof, there exists $\emptyset \neq B \in \mathcal{B}$ such that $p_B \in \xi$. By condition (3) in Theorem 9.9, there exists $s_i := s_{\alpha_i} p_{C_i} s_{\beta_i}^*$ for $1 \leq i \leq n$ such that $\{s_i p_A s_i^*\}_{i=1}^n$ is an outer cover for p_B. Without loss of generality, we can assume that $\theta_{\beta_i}(A) \subseteq C_i$ for every $1 \leq i \leq n$, so that $s_i p_A s_i^* = s_{\alpha_i} p_{\theta_{\beta_i}(A)} s_{\alpha_i}^*$ for every $1 \leq i \leq n$. By multiplying by p_B, we conclude that $\{s_{\alpha_i} p_{(\theta_{\alpha_i}(B) \cap \theta_{\beta_i}(A))} s_{\alpha_i}^*\}_{i=1}^n$ is a finite cover for p_B. Since ξ is tight and $p_B \in \xi$, then there exists $1 \leq j \leq n$ such that

$$p_B \cdot s_{\alpha_j} p_{\theta_{\beta_j}(A)} s_{\alpha_j}^* = s_{\alpha_j} p_{(\theta_{\alpha_j}(B) \cap \theta_{\beta_j}(A))} s_{\alpha_j}^* \in \xi$$

by [15, (2.10)]. As ξ is a filter and $p_B \cdot s_{\alpha_j} p_{\theta_{\beta_j}(A)} s_{\alpha_j}^* \leq s_{\alpha_j} p_{\theta_{\beta_j}(A)} s_{\alpha_j}^*$, we conclude that $s_{\alpha_j} p_{\theta_{\beta_j}(A)} s_{\alpha_j}^* \in \xi$, as desired. ☐

Our next goal is to give a characterization of the cofinality of $(\mathcal{B}, \mathcal{L}, \theta)$ in terms of the elements in $\mathcal{B}$ and the actions θ. First we need the following definitions.

Definition 9.12. We say that an ideal $\mathcal{I}$ of $\mathcal{B}$ is hereditary if given $A \in \mathcal{I}$ and $\alpha \in \mathcal{L}$ then $\theta_\alpha(A) \in \mathcal{I}$. We also say that $\mathcal{I}$ is saturated if given $A \in \mathcal{B}_{reg}$ with $\theta_\alpha(A) \in \mathcal{I}$ for every $\alpha \in \Delta_A$ then $A \in \mathcal{I}$.

Given a collection $\mathcal{I}$ of elements of $\mathcal{B}$ we define the hereditary expansion of $\mathcal{I}$ as

$$\mathcal{H}(\mathcal{I}) := \{B \in \mathcal{B} : B \subseteq \bigcup_{i=1}^{n} \theta_{\alpha_i}(A_i) \text{ where } A_i \in \mathcal{I} \text{ and } \alpha_i \in \mathcal{L}^*\}\,.$$

Clearly, $\mathcal{H}(\mathcal{I})$ is the minimal hereditary ideal of $\mathcal{B}$ containing $\mathcal{I}$. Also, we define the saturation of $\mathcal{I}$, denoted by $\mathcal{S}(\mathcal{I})$, to be the minimal ideal of $\mathcal{B}$ generated by the set

$$\bigcup_{n=0}^{\infty} \mathcal{S}^{[n]}(\mathcal{I}),$$

defined by recurrence on $n \in \mathbb{Z}^+$ as follows:

(1) $\mathcal{S}^{[0]}(\mathcal{I}) := \mathcal{I}$

(2) For every $n \in \mathbb{N}$, $\mathcal{S}^{[n]}(\mathcal{I}) := \{B \in \mathcal{B}_{reg} : \theta_\alpha(B) \in \mathcal{S}^{[n-1]}(\mathcal{I}) \text{ for every } \alpha \in \Delta_B\}\,.$

Observe that if $\mathcal{I}$ is hereditary, then $\mathcal{S}(\mathcal{I})$ is also hereditary. Therefore, given a collection $\mathcal{I}$ of elements of $\mathcal{B}$, $\mathcal{S}(\mathcal{H}(\mathcal{I}))$ is the minimal hereditary and saturated ideal of $\mathcal{B}$ containing $\mathcal{I}$.

We set $\mathcal{L}^\infty := \prod_{n=1}^\infty \mathcal{L}$. Given $\alpha \in \mathcal{L}^\infty$ and $k \in \mathbb{N}$, we define $\alpha_{[1,k]} = \alpha_1 \cdots \alpha_k \in \mathcal{L}^k$.

Theorem 9.13. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ a Boolean dynamical system. Then the following statements are equivalent:*

(1) The only hereditary and saturated ideals of $\mathcal{B}$ are $\emptyset$ and $\mathcal{B}$,

(2) Given $A, B \in \mathcal{B}$, there exists $C \in \mathcal{B}_{reg} \cup \{\emptyset\}$ such that

(a) $B \setminus C \in \mathcal{H}(A)$, and

(b) For every $\alpha \in \mathcal{L}^\infty$ there exists $k \in \mathbb{N}$ such that $\theta_{\alpha_{[1,k]}}(C) \in \mathcal{H}(A)$.

(3) For every $0 \neq e, f \in \mathcal{E}(T)$, there exist $s_1, \ldots, s_n \in S$ such that $\{s_i f s_i^*\}_{i=1}^n$ is an outer cover for e.

(4) $(\mathcal{B}, \mathcal{L}, \theta)$ is cofinal.

(5) $\mathcal{G}_{tight}(T)$ is minimal.

Proof. First observe that (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) follows from Theorem 9.9 and Proposition 9.11.

(1) $\Rightarrow$ (2). Suppose that the only hereditary and saturated are $\emptyset$ and $\mathcal{B}$. Then, given $A \neq \emptyset$ we have that $\mathcal{S}(\mathcal{H}(A)) = \mathcal{B}$. By definition,

$$\mathcal{H}(A) = \{C \in \mathcal{B} : \exists \beta_1, \ldots, \beta_m \in \mathcal{L}^* \text{ and } n \in \mathbb{N} \text{ such that } C \subseteq \bigcup_{i=1}^{m} \theta_{\beta_i}(A)\}\,.$$

Since $\mathcal{S}(\mathcal{H}(A)) = \mathcal{B}$, by definition of saturation we have that $\mathcal{B} = \{C \cup D : C \in \mathcal{H}(A) \text{ and } D \in \mathcal{B}_{reg}\}$. Thus, given any $B \in \mathcal{B}$, there exists $D \in \mathcal{H}(A)$ such that $C := B \setminus D \in \mathcal{B}_{reg}$, and there exists $n \in \mathbb{N}$ such that $C \in \mathcal{S}^{[n]}(\mathcal{H}(A))$. Therefore, for every $\alpha \in \mathcal{L}^\infty$, we have that $\theta_{\alpha_{[1,n]}}(C) \in \mathcal{H}(A)$.

(2) $\Rightarrow$ (3). Without loss of generality, we can assume that $f = p_A$ and $e = p_B$ for some $\emptyset \neq A, B \in \mathcal{B}$. By hypothesis, there exists $C \in \mathcal{B}_{reg} \cup \{\emptyset\}$ such that $B \setminus C \in \mathcal{H}(A)$. So, there exist $\beta_1, \ldots, \beta_m \in \mathcal{L}^*$ such that $B \setminus C \subseteq \bigcup_{i=1}^{m} \theta_{\beta_i}(A)$. Thus, if we define $s_i := s_{\beta_i}^*$ for $1 \leq i \leq m$, then $s_i f s_i^* = p_{\theta_{\beta_i}(A)}$. Hence, since $\bigvee_{i=1}^m p_{\theta_{\beta_i}(A)} = p_{\bigcup_{i=1}^{m} \theta_{\beta_i}(A)}$, we can reduce the proof to the case that $e = p_C$. Now, if $\theta_\gamma(C) \in \mathcal{H}(A)$ for every $\gamma \in \Delta_C$ and $C \in \mathcal{B}_{reg}$, we have that

$C \in \mathcal{S}^{[1]}(\mathcal{H}(A))$, whence we can find a finite cover for p_C. Otherwise, there exists $\gamma_1 \in \Delta_C$ such that $\theta_{\gamma_1}(C) \notin \mathcal{H}(A)$. Now, we repeat the argument to find a finite cover for $p_{\theta_{\gamma_1}(C)}$. By recurrence, we either construct a finite path $\gamma = \gamma_1 \cdots \gamma_m$ such that $\theta_\gamma(C) \in \mathcal{H}(A)$, or we construct an infinite path $\alpha \in \mathcal{L}^\infty$ such that $\alpha_{[1,k]}(C) \notin \mathcal{H}(A)$ for every $k \in \mathbb{N}$. In the first case we obtain a finite cover for p_C. In the second case we get an infinite path, contradicting the hypothesis. So we are done.

(3) $\Rightarrow$ (1). Let $\emptyset \neq A \in \mathcal{B}$. We want to prove that $\mathcal{S}(\mathcal{H}(A)) = \mathcal{B}$. If we take $\emptyset \neq B \in \mathcal{B}$ then, by hypothesis, there exist $s_1, \dots, s_n$ such that $\{s_i f s_i^*\}_{i=1}^n = \{s_{\alpha_i} p_{\theta_{\beta_i}(A)} s_{\alpha_i}^*\}_{i=1}^n$ is an outer cover for p_B. So,

$$p_B \leq \bigvee_{i=1}^n s_{\alpha_i} p_{\theta_{\beta_i}(A)} s_{\alpha_i}^* \,.$$

We set $N_1 := \max \{|\alpha_i| : i = 1, \dots, n\}$. Since only regular sets can have finite covers, it must exists $C \in \mathcal{B}_{reg}$ such that

$$B \setminus C \subseteq \bigcup_{\alpha_i = \emptyset} \theta_{\beta_i}(A) \in \mathcal{H}(A) \,.$$

So we have that

$$p_C \leq \bigvee_{i=1, \alpha_i \neq \emptyset}^n s_{\alpha_i} p_{\theta_{\beta_i}(A)} s_{\alpha_i}^* \,,$$

and $C \in \mathcal{B}_{reg}$. Thus, we can assume that $B \in \mathcal{B}_{reg}$ and $\alpha_i \neq \emptyset$ with

$$p_B \leq \bigvee_{i=1}^n s_{\alpha_i} p_{\theta_{\beta_i}(A)} s_{\alpha_i}^* \,.$$

Now, we label $\Delta_B = \{\gamma_1, \dots, \gamma_m\}$, and relabel $\{\alpha_i\}$ so that there exist $0 = j_0 < j_1 < j_2 < \cdots < j_m = n$ with $\gamma_k \leq \alpha_i$ for every $j_{i-1} < k \leq j_i$ and $\gamma_k \not\leq \alpha_i$ otherwise. Then, we have that

$$s_{\gamma_i} p_{\theta_{\gamma_i}(B)} s_{\gamma_i}^* \leq \bigvee_{k=j_{i-1}+1}^{j_i} s_{\alpha_k} p_{\theta_{\beta_k}(A)} s_{\alpha_k}^* \qquad \text{for every } i = 1, \dots, m \,,$$

or equivalently

$$p_{\theta_{\gamma_i}(B)} \leq \bigvee_{k=j_{i-1}+1}^{j_i} s_{\alpha_k \setminus \gamma_i} p_{\theta_{\beta_k}(A)} s_{\alpha_k \setminus \gamma_i}^* \qquad \text{for every } i = 1, \dots, m \,.$$

Observe that we have $|\alpha_k \setminus \gamma_i| < |\alpha_k|$. Thus, we can assume that

$$p_{\theta_\gamma(B)} \leq \bigvee_{i=1}^n s_{\alpha_i} p_{\theta_{\beta_i}(A)} s_{\alpha_i}^* \qquad \text{for every } \gamma \in \Delta_B$$

with $N_2 := \max \{|\alpha_i| : i = 1, \dots, n\} = N_1 - 1 < N_1$. By hypothesis, we can also assume that $\theta_\gamma(A) \in \mathcal{B}_{reg}$ for every $\gamma \in \Delta_B$.

Therefore, after repeating this process N_1 times, we prove that $p_{\theta_\gamma(B)} \in \mathcal{B}_{reg}$ for every $\gamma \in \Delta_B^{\leq N_1 - 1}$, and $\theta_\gamma(B) \in \mathcal{H}(A)$ for every $\gamma \in \Delta_B^{N_1}$. Thus, $B \in \mathcal{S}^{[N_1]}(\mathcal{H}(A))$, and hence $B \in \mathcal{S}(\mathcal{H}(A))$.

□

9.3. **The simplicity result.** Now, we are ready to state a result, giving a characterization of simplicity for $C^*(\mathcal{B},\mathcal{L},\theta)$ in terms of properties enjoyed by $(\mathcal{B},\mathcal{L},\theta)$.

Theorem 9.14. *Let $(\mathcal{B},\mathcal{L},\theta)$ be a Boolean dynamical system, and let $C^*(\mathcal{B},\mathcal{L},\theta)$ be its associated C^*-algebra. Then, the following statements are equivalent:*

(1) $C^*(\mathcal{B},\mathcal{L},\theta)$ is simple.

(2) The following properties hold:

(a) $(\mathcal{B},\mathcal{L},\theta)$ satisfies condition $(L_{\mathcal{B}})$, and

(b) The only hereditary and saturated ideals of $\mathcal{B}$ are $\emptyset$ and $\mathcal{B}$.

Proof. By Theorem 9.1, $C^*(\mathcal{B},\mathcal{L},\theta) \cong C^*(\mathcal{G}_{\text{tight}}(T))$. By Lemma 8.3 and Lemma 8.4, $\mathcal{G}_{\text{tight}}(T)$ is Hausdorff and amenable. Then, the result holds by Theorem 9.6, Theorem 9.13 and Theorem 9.1. □

Theorem 9.14 generalizes [2, Theorem 6.4] (where only sufficient conditions are given) and [19, Theorem 3.8, 3.14 & 3.16] (which provided an equivalence, and solved a problem in Bates and Pask's result) in our context, the point being the use of a completely different approach to fix the conditions equivalent to simplicity, that are stated in terms of both the groupoid properties and the Boolean dynamical system.

10. Gauge invariant ideals

Now, using the characterization of the Cuntz-Krieger Boolean C^*-algebras as topological graph C^*-algebras explained in Section 5, we will use the work of Katsura [22] to determine the gauge invariant ideals of the Cuntz-Krieger Boolean C^*-algebras. We will restrict for simplicity, to the class of locally finite Boolean dynamical systems (see definition 3.6).

Given a Boolean dynamical system $(\mathcal{B},\mathcal{L},\theta)$, we will denote by $E_{(\mathcal{B},\mathcal{L},\theta)}$ the associated topological graph defined in Proposition 5.3. If there is no confusion, we will just write E.

Definition 10.1. Let $E = (E^0, E^1, d, r)$ be a topological graph. A subset X^0 of E^0 is said to be *positively invariant* if $d(e) \in X^0$ implies $r(e) \in X^0$ for each $e \in E^1$, and to be *negatively invariant* if for every $v \in X^0 \cap E^0_{rg}$ there exists $e \in E^1$ with $r(e) = v$ and $d(e) \in X^0$. A subset X^0 of E^0 is called *invariant* if X^0 is both positively and negatively invariant.

We define the *singular vertices* as $E^0_{sg} = E^0 \setminus E^0_{rg}$.

Definition 10.2. Let $E = (E^0, E^1, d, r)$ be a topological graph. A subset Y of E^0 is said to be *hereditary* if $r(e) \in Y$ implies $d(e) \in Y$, and *saturated* if $v \in E^0_{rg}$ with $d(r^{-1}(v)) \subseteq Y$ implies $v \in Y$.

Observe that a subset X^0 of E^0 is positively invariant if and only if $E^0 \setminus X^0$ is hereditary, and it is negatively invariant if and only if $E^0 \setminus X^0$ is saturated.

Lemma 10.3. *Let $(\mathcal{B},\mathcal{L},\theta)$ be a Boolean dynamical system, and let E be the associated topological graph. If $\mathcal{H}$ is an ideal of $\mathcal{B}$, then $\mathcal{H}$ is hereditary (definition 9.12) if and only if $Y := \bigcup_{A\in\mathcal{H}} \mathcal{N}_A$ is a hereditary subset of E^0.*

Proof. Suppose that $\mathcal{H}$ is a hereditary ideal of $\mathcal{B}$. Let $v_{\mathcal{C}} \in Y$, so there exists $A \in \mathcal{H}$ such that $v_{\mathcal{C}} \in \mathcal{N}_A$, and suppose that there exists $\alpha \in \mathcal{L}$ such that $v_{\mathcal{C}} \in r(E^1_\alpha)$. Let $\mathcal{C}' \in \mathcal{D}(\mathcal{I}_{\mathcal{R}_\alpha})$ such that $r(e^\alpha_{\mathcal{C}'}) = v_{\mathcal{C}}$, so that $\mathcal{C} = \{B \in \mathcal{I}_{\mathcal{D}_\alpha} : \theta_\alpha(B) \in \mathcal{C}'\}$. Since $A \in \mathcal{C}$, we have that $\theta_\alpha(A) \in \mathcal{C}'$, so $v_{\mathcal{C}'} \in \mathcal{N}_{\theta_\alpha(A)}$. As $A \in \mathcal{H}$, by hypothesis $\theta_\alpha(A) \in \mathcal{H}$, and therefore $v_{\mathcal{C}'} \in \mathcal{N}_{\theta_\alpha(A)} \subseteq Y$. Thus, $d(e^\alpha_{\mathcal{C}}) = v_{\mathcal{C}'} \in Y$, as desired.

Conversely, suppose that $Y := \bigcup_{A\in\mathcal{H}} \mathcal{N}_A$ is a hereditary subset of E^0, and suppose that there exists $A \in \mathcal{H}$ such that $\theta_\alpha(A) \notin \mathcal{H}$. We claim that there exists an ultrafilter $\mathcal{C}$ of $\mathcal{B}$ such that $A \in \mathcal{C}$ and $\theta_\alpha(B) \notin \mathcal{H}$ for every $B \in \mathcal{C}$. Indeed, let us consider the set Γ of all the filters $\mathcal{C}$ of $\mathcal{B}$ such that $A \in \mathcal{C}$ and $\theta_\alpha(B) \notin \mathcal{H}$ for every $B \in \mathcal{C}$. Γ is a partially ordered set with the inclusion.

First observe that $\Gamma \neq \emptyset$, because the minimal filter containing A belongs to Γ. Now, let $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ be an ascending sequence of filters of Γ. $\mathcal{C} = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$ is clearly a filter from Γ with $\mathcal{C}_n \subseteq \mathcal{C}$ for every $n \in \mathbb{N}$. Then, by Zorn's Lemma, there exist maximal elements in Γ. If $\mathcal{C}$ is a maximal element of Γ, we claim that $\mathcal{C}$ is an ultrafilter of $\mathcal{B}$. Indeed, we only have to check condition **F3**. Let $B \in \mathcal{C}$, and let $C, C' \in \mathcal{B} \setminus \{\emptyset\}$ such that $B = C \cup C'$ and $C \cap C' = \emptyset$. Suppose that $C, C' \notin \mathcal{C}$. Then, $C \cap D, C' \cap D \neq \emptyset$ for every $D \in \mathcal{C}$; otherwise, if there exists $D \in \mathcal{C}$ such that $C \cap D = \emptyset$, then $\mathcal{C} \ni (B \cap D) \subseteq C'$ by condition **F2**. Thus, $C' \in \mathcal{C}$ by condition **F1**, a contradiction, whence $C \cap D \neq \emptyset$ for every $D \in \mathcal{C}$. By the same argument $C' \cap D \neq \emptyset$ for every $D \in \mathcal{C}$. Now, suppose that there exists $D \in \mathcal{C}$ such that $\theta_\alpha(C \cap D) \in \mathcal{H}$. Then, for every $D' \in \mathcal{C}$ with $D' \subseteq D$, we have that $\theta_\alpha(C \cap D') \subseteq \theta_\alpha(C \cap D)$. So, $\theta_\alpha(C \cap D') \in \mathcal{H}$ too, since $\mathcal{H}$ is an ideal. Now, suppose that $\theta_\alpha(C \cap G) \in \mathcal{H}$ for some $G \in \mathcal{C}$. By the same argument as above, $\theta_\alpha(C \cap G') \in \mathcal{H}$ for every $G' \in \mathcal{C}$ with $G' \subseteq G$. Thus, $B \cap D \cap G \in \mathcal{C}$ and

$$\theta_\alpha(B \cap D \cap G \cap C) \cup \theta_\alpha(B \cap D \cap G \cap C') = \theta_\alpha(B \cap D \cap G) \notin \mathcal{H}\,.$$

But by the above arguments, we have that $\theta_\alpha(B \cap D \cap G) \in \mathcal{H}$ because $\mathcal{H}$ is an ideal, a contradiction. Therefore, we can assume that $\theta_\alpha(C \cap D) \notin \mathcal{H}$ for every $D \in \mathcal{C}$. Now, we construct the filter $\mathcal{C}' = \{B \in \mathcal{B} : C \cap D \subseteq B \text{ for some } D \in \mathcal{C}\}$. We clearly have that $\mathcal{C}' \in \Gamma$ with $\mathcal{C} \subsetneq \mathcal{C}'$, contradicting with the maximality of $\mathcal{C}$. Thus, $\mathcal{C}$ is an ultrafilter of $\mathcal{B}$, as desired.

Now, we claim that there exists an ultrafilter $\mathcal{C}'$ of $\mathcal{B}$ such that $\theta_\alpha(B) \in \mathcal{C}'$ for every $B \in \mathcal{C}$ and $C \notin \mathcal{H}$ for every $C \in \mathcal{C}'$, where $\mathcal{C}$ is the ultrafilter constructed above. Let Γ' be the set of all filters of $\mathcal{B}$ satisfying the above requirements. We have that $\Gamma' \neq \emptyset$ since the filter $\mathcal{D} = \{C :\in \mathcal{B} : \theta_\alpha(B) \subseteq C \text{ for some } B \in \mathcal{C}\}$ belongs to Γ'. Also, Γ' is a partially ordered set with the inclusion, and clearly every ascending sequence of filters of Γ' has an upper-bound. By the Zorn's Lemma, Γ' has maximal elements. Let $\mathcal{C}'$ be a maximal element. We claim that $\mathcal{C}'$ is an ultrafilter of $\mathcal{B}$. Indeed, we only have to check condition **F3**. Let $C \in \mathcal{C}'$ and let $D, D' \in \mathcal{B} \setminus \{\emptyset\}$ with $C = D \cap D'$ and $D \cap D' = \emptyset$ and $D, D' \notin \mathcal{C}'$. We have that $D \cap G, D' \cap G \neq \emptyset$ for every $G \in \mathcal{C}'$; otherwise, if there exists $G \in \mathcal{C}'$ such that $D \cap G = \emptyset$, then we have that $(C \cap G) \subseteq D'$. So, $D' \in \mathcal{C}$ by condition **F1**, a contradiction. Thus, $D \cap G \neq \emptyset$ for every $G \in \mathcal{C}'$. By the same argument we have that $D' \cap G \neq \emptyset$ for every $G \in \mathcal{C}'$. Finally suppose that there exists $G, G' \in \mathcal{C}'$ such that $D \cap G, D' \cap G' \in \mathcal{H}$. Then,

$$(C \cap G \cap G' \cap D) \cup (C \cap G \cap G' \cap D') = C \cap G \cap G' \notin \mathcal{H}\,,$$

but since $\mathcal{H}$ is an ideal, we have that $C \cap G \cap G' \in \mathcal{H}$, a contradiction. Therefore, suppose that $\emptyset \neq D \cap G \notin \mathcal{H}$ for every $G \in \mathcal{C}'$. Then, we can define the filter $\mathcal{C}'' = \{C \in \mathcal{B} : D \cap G \subseteq C \text{ for some } G \in \mathcal{C}'\}$. We have that $\mathcal{C}'' \in \Gamma'$ and $\mathcal{C}' \subsetneq \mathcal{C}''$, contradicting the maximality of $\mathcal{C}'$. Thus, $\mathcal{C}'$ is an ultrafilter, as desired.

Finally, since $\mathcal{C}' \in \mathcal{I}_{\mathcal{R}\alpha}$, we can define $e^{\alpha}_{\mathcal{C}'} \in E^1_{\alpha}$. But $v_{\mathcal{C}'} \notin Y$, since $B \notin \mathcal{H}$ for every $B \in \mathcal{C}'$. Observe that by Lemma 5.5 we have that $r(e^{\alpha}_{\mathcal{C}'}) = v_{\mathcal{C}}$. Moreover, $v_{\mathcal{C}} \in \mathcal{N}_A \subseteq Y$, since $A \in \mathcal{C}$. But this contradicts that Y is a hereditary set of E^0. Thus, $\theta_\alpha(A) \in \mathcal{H}$, as desired, whence $\mathcal{H}$ is a hereditary ideal of $\mathcal{B}$. □

Observe that, if $A \in \mathcal{B}_{reg}$, then given any $\mathcal{C} \in \mathcal{D}(\mathcal{I}_A)$ we have that $v_{\mathcal{C}} \in E^0_{rg}$.

Lemma 10.4. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system, and let E be the associated topological graph. If $\mathcal{H}$ is an ideal of $\mathcal{B}$, then $\mathcal{H}$ is saturated (definition 9.12) if and only if $Y := \bigcup_{A\in\mathcal{H}} \mathcal{N}_A$ is a saturated subset of E^0.*

Proof. First, suppose that $\mathcal{H}$ is a saturated subset of $\mathcal{B}$, and let $\mathcal{C} \in \mathcal{D}(\mathcal{B})$ such that $v_{\mathcal{C}} \in E^0_{rg}$. Recall that

$$r^{-1}(v_{\mathcal{C}}) = \{e^{\alpha}_{\mathcal{C}'} : \mathcal{C}' \in \mathcal{D}(\mathcal{B}) \text{ such that } \exists \alpha \in \mathcal{L} \text{ with } \mathcal{C} = \{A \in \mathcal{B} : \theta_\alpha(A) \in \mathcal{C}'\}\}\,.$$

Suppose that $d(e^{\alpha}_{\mathcal{C}'}) = v_{\mathcal{C}'} \in Y$ for every $e^{\alpha}_{\mathcal{C}'} \in r^{-1}(v_{\mathcal{C}})$. Hence, there exists $B_{\mathcal{C}'} \in \mathcal{C}'$ such that $B_{\mathcal{C}'} \in \mathcal{H}$. We claim that, for every $\alpha \in \mathcal{L}$ such that $\theta_\alpha(A) \neq \emptyset$ for every $A \in \mathcal{C}$, there exists $A \in \mathcal{C}$ such that $\theta_\alpha(A) \in \mathcal{H}$. Indeed, suppose that there exists $\alpha \in \mathcal{L}$ such that $\theta_\alpha(A) \notin \mathcal{H}$ for every $A \in \mathcal{C}$. Let Γ the set of all filters $\mathcal{F}$ of $\mathcal{B}$ such that $\theta_\alpha(A) \in \mathcal{F}$ and $\theta_\alpha(A) \notin \mathcal{H}$ for every $A \in \mathcal{C}$. Then, $\mathcal{F} = \{B \in \mathcal{B} : \theta_\alpha(A) \subseteq B \text{ for some } A \in \mathcal{C}\}$ is a filter in Γ, whence $\Gamma \neq \emptyset$. We have that Γ is a partially ordered set with the inclusion, and it is clear that Γ contains an upper-bound for every ascending chain. Therefore, by the Zorn's Lemma, Γ has maximal elements. Given any maximal element $\mathcal{C}' \in \Gamma$, we have that $\mathcal{C}'$ is an ultrafilter. Therefore, we have that $\mathcal{C}' \notin \mathcal{D}(\mathcal{I}_B)$ for every $B \in \mathcal{H}$, and hence $v_{\mathcal{C}'} \notin Y$. Moreover, by Lemma 5.5 we have that $r(e^{\alpha}_{\mathcal{C}'}) = v_{\mathcal{C}}$. But this contradicts the hypothesis that $d(r^{-1}(v_{\mathcal{C}})) \subseteq Y$. Thus, there exists $A \in \mathcal{C}$ such that $\theta_\alpha(A) \in \mathcal{H}$. Then, given any $\alpha \in \mathcal{L}$ such that $\theta_\alpha(A) \neq \emptyset$ for every $A \in \mathcal{C}$, there exists $A_\alpha \in \mathcal{C}$ such that $\theta_\alpha(A_\alpha) \in \mathcal{H}$.

Now, since $v_{\mathcal{C}} \in E^0_{rg}$, there exists $A \in \mathcal{C}$ such that $\lambda_A < \infty$, and given any $B \in \mathcal{B}$ with $B \subseteq A$ then $\lambda_B \neq 0$. So, A is a regular set of $\mathcal{B}$. If replace A by $A \cap \left(\bigcap_{\alpha\in\Delta_A} A_\alpha\right) \in \mathcal{C}$, we can suppose that $\theta_\alpha(A) \in \mathcal{H}$ for every $\alpha \in \Delta_A$. Then, since $\mathcal{H}$ is saturated, we have that $A \in \mathcal{H}$, and hence $v_{\mathcal{C}} \in \mathcal{N}_A \subseteq Y$. Thus, Y is a saturated subset of E^0.

Conversely, suppose that Y is a saturated subset of E^0, and let $\mathcal{H}$ be an ideal of $\mathcal{B}$. Let $A \in \mathcal{H}$ and regular such that $\{\theta_\alpha(A) : \alpha \in \mathcal{L}\} \subseteq \mathcal{H}$. We claim that for every ultrafilter $\mathcal{C} \in \mathcal{D}(\mathcal{I}_A)$ there exists $B_{\mathcal{C}} \in \mathcal{H}$ with $\mathcal{C} \in \mathcal{D}(\mathcal{I}_{B_{\mathcal{C}}})$. Indeed, since A is regular, we have that $v_{\mathcal{C}} \in E^0_{rg}$. Moreover, since $\{\theta_\alpha(A) : \alpha \in \mathcal{L}\} \subseteq \mathcal{H}$, we have that $d(r^{-1}(v_{\mathcal{C}})) \subseteq Y$. Therefore, since Y is saturated, it follows that $v_{\mathcal{C}} \in Y$, so $B_{\mathcal{C}} \in \mathcal{C}$ for some $B_{\mathcal{C}} \in \mathcal{H}$, as desired.

Let $\mathcal{C} \in \mathcal{D}(\mathcal{I}_A)$. By the above claim, there exists $B_{\mathcal{C}} \in \mathcal{H}$ with $\mathcal{C} \in \mathcal{D}(\mathcal{I}_{B_{\mathcal{C}}})$, and then $A \cap B_{\mathcal{C}} \in \mathcal{C} \cap \mathcal{H}$ and $\mathcal{N}_A \cap \mathcal{I}_{B_{\mathcal{C}}} = \mathcal{N}_{A\cap B_{\mathcal{C}}}$. Therefore, $\mathcal{N}_A = \bigcup_{\mathcal{C}\in\mathcal{D}(\mathcal{I}_A)} \mathcal{N}_{A\cap B_{\mathcal{C}}}$. But since $\mathcal{D}(\mathcal{I}_A)$ is compact by Corollary 2.17, we have that $\mathcal{N}_A = \mathcal{N}_{A\cap B_{\mathcal{C}_1}} \cup \cdots \cup \mathcal{N}_{A\cap B_{\mathcal{C}_n}}$ for some $n \in \mathbb{N}$. Hence, it is easy to check that $A = \bigcup_{i=1}^{n} (A \cap B_{\mathcal{C}_i})$. As $A \cap B_{\mathcal{C}_i} \in \mathcal{H}$ for every $i = 1, \ldots, n$, and $\mathcal{H}$ is an ideal, it follows that $A \in \mathcal{H}$, as desired. □

We have proved in the previous lemmas that, given a hereditary and saturated ideal $\mathcal{H}$ of $\mathcal{B}$, then $Y = \bigcup_{A\in\mathcal{H}} \mathcal{N}_A$ is a hereditary and saturated subset of E^0. The converse is also true. Indeed, let Y be a hereditary and saturated subset of E^0. Given $v \in Y$, pick $A_v \in \mathcal{B}$ such that $v \in \mathcal{N}_{A_v}$ and $\mathcal{N}_{A_v} \subseteq Y$. We define $\mathcal{H}$ to be the minimum ideal of $\mathcal{B}$ containing the A_v's. Observe that since every $\mathcal{N}_{A_v}$ is compact by Corollary 2.17, and since $\mathcal{H}$ is an ideal, $\mathcal{H}$ is independent of the choice of the A_v's. Now, following the proof of Lemmas 10.3 & 10.4, one can check that $\mathcal{H}$ is a hereditary and saturated ideal of $\mathcal{B}$. Thus, the following results follows:

Proposition 10.5. *Let $(\mathcal{B},\mathcal{L},\theta)$ be a Boolean dynamical system, and let E be the associated topological graph. Then, there is a bijection between the hereditary and saturated subsets of $\mathcal{B}$ and the invariant subsets of E.*

Example 10.6. Let $(\mathcal{B},\mathcal{L},\theta)$ be the Boolean dynamical system of Example 5.9. Then, the only hereditary and saturated subset of $\mathcal{B}$ is the set $\mathcal{H} = \{F : F \subseteq E^0 \text{ finite}\}$, the associated open hereditary and saturated subspace $Y = \bigcup_{A\in\mathcal{H}} \mathcal{N}_A$ of E^0 is $\{v_{\mathcal{C}_n} : n \in \mathbb{Z}\}$, and let $X = E^0 \setminus Y = \{\mathcal{C}_\infty\}$ is the associated invariant space.

Proposition 10.7. *Let $(\mathcal{B},\mathcal{L},\theta)$ be the Boolean dynamical system, and let $\mathcal{H}$ be a hereditary ideal of $\mathcal{B}$. If for any $\alpha \in \mathcal{L}$ and any $[A] \in \mathcal{B}/\mathcal{H}$ we define $\theta_\alpha([A]) = [\theta_\alpha(A)]$, then $(\mathcal{B}/\mathcal{H},\mathcal{L},\theta)$ is a Boolean dynamical system.*

Proof. We only need to prove that, given $\alpha \in \mathcal{L}$, the map $\theta_\alpha : \mathcal{B}/\mathcal{H} \longrightarrow \mathcal{B}/\mathcal{H}$ is a well-defined map. But this clear because $\mathcal{H}$ is a hereditary ideal of $\mathcal{B}$. Also, the range and domain of θ_α are $[\mathcal{R}_\alpha]$ and $[\mathcal{D}_\alpha]$ respectively. □

Let X^0 be an invariant space of E^0. If we define $X^1 = \{e \in E^1 : d(e) \in X^0\}$, then (X^0, X^1, d, r) is also a topological graph.

Proposition 10.8. *Let $(\mathcal{B},\mathcal{L},\theta)$ be a Boolean dynamical system, and let E be the associated topological graph. Given a hereditary and saturated ideal $\mathcal{H}$ of $\mathcal{B}$, define $X^0 := E^0 \setminus \bigcup_{A\in\mathcal{H}} \mathcal{N}_A$. Then, $E_{\mathcal{H}} := E_{(\mathcal{B}/\mathcal{H},\mathcal{L},\theta)} = (X^0, X^1, d, r)$.*

Proof. Since $E^0 = \mathcal{D}(\mathcal{B})$ and $\bigcup_{A\in\mathcal{H}} \mathcal{N}_A = \mathcal{D}(\mathcal{H})$, using Lemma 2.11 we can identify X^0 with $\mathcal{D}(\mathcal{B}/\mathcal{H}) = E^0_{\mathcal{H}}$ by $v_{\mathcal{C}} \mapsto v_{[\mathcal{C}]}$. By definition, $X^1 = \bigsqcup_{\alpha\in\mathcal{L}} \{e^\alpha_{\mathcal{C}} : \mathcal{C} \in \mathcal{D}(\mathcal{I}_{\mathcal{R}_\alpha}) \text{ and } [\mathcal{C}] \in \mathcal{D}(\mathcal{B}/\mathcal{H})\}$. So, we can identify it with $E^1_{\mathcal{H}} = \bigsqcup_{\alpha\in\mathcal{L}} \mathcal{D}(\mathcal{I}_{[\mathcal{R}_\alpha]})$ by $e^\alpha_{\mathcal{C}} \mapsto e^\alpha_{[\mathcal{C}]}$. With these identifications, it is clear that the maps d and r are the corresponding ones.

□

A topological graph $E = (E^0, E^1, d, r)$ is called row-finite if $r(E^1) = E^0_{rg}$.

Lemma 10.9. *Let $(\mathcal{B},\mathcal{L},\theta)$ be a locally finite Boolean dynamical system, then the associated topological graph E is row-finite.*

Proof. Recall that
$$r(E^1) = \{v_{\mathcal{C}} \in E^0 | \exists \alpha \in \mathcal{L}\,, \theta_\alpha(A) \neq \emptyset \ \forall A \in \mathcal{C}\}$$
and
$$E^0_{rg} = \{v_{\mathcal{C}} \in E^0 | \exists A \in \mathcal{C}\,,\ \forall B \subseteq A \text{ we have that } 0 < \lambda_B < \infty\}\,.$$

The inclusion $E^0_{rg} \subseteq r(E^1)$ is always valid, and the converse is obvious by locally finiteness of the Boolean dynamical system. □

Given a Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$ and a hereditary and saturated set $\mathcal{H}$ of $\mathcal{B}$, we define $I_{\mathcal{H}}$ as the ideal of $C^*(\mathcal{B}, \mathcal{L}, \theta)$ generated by

$$\{p_A : A \in \mathcal{H}\}.$$

Conversely, given an ideal I of $C^*(\mathcal{B}, \mathcal{L}, \theta)$ let us define $\rho_I : C^*(\mathcal{B}, \mathcal{L}, \theta) \longrightarrow C^*(\mathcal{B}, \mathcal{L}, \theta)/I$ to be the quotient map, and $\mathcal{H}_I := \{A \in \mathcal{B} : \rho_I(p_A) = 0\}$. Clearly $\mathcal{H}_I$ is a hereditary and saturated set of $\mathcal{B}$.

Then using Proposition 10.8 it follows:

Proposition 10.10 (cf. [22, Proposition 3.15]). *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system. If I is an ideal of $C^*(\mathcal{B}, \mathcal{L}, \theta)$, then there exists a natural surjection $C^*(\mathcal{B}/\mathcal{H}_I, \mathcal{L}, \theta) \to C^*(\mathcal{B}, \mathcal{L}, \theta)/I$ which is injective in $C^*(\mathcal{B}/\mathcal{H}_I)$.*

Proposition 10.11 (cf. [22, Proposition 3.16]). *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a locally finite Boolean dynamical system. For an ideal I of $C^*(\mathcal{B}, \mathcal{L}, \theta)$, the following statements are equivalent:*

(1) I is a gauge-invariant ideal,
(2) The natural surjection $C^*(\mathcal{B}/\mathcal{H}_I, \mathcal{L}, \theta) \to C^*(\mathcal{B}, \mathcal{L}, \theta)/I$ is an isomorphism,
(3) $I = I_{\mathcal{H}_I}$.

Theorem 10.12 (cf. [22, Corollary 3.25]). *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a locally Boolean dynamical system and let E the associated topological graph. Then the maps $I \to \mathcal{H}_I$ and $\mathcal{H} \to I_{\mathcal{H}}$ define a one-to-one correspondence between the set of all gauge invariant ideals of $C^*(\mathcal{B}, \mathcal{L}, \theta)$ and the set of all hereditary and saturated sets of $(\mathcal{B}, \mathcal{L}, \theta)$.*

Example 10.13. Let $(\mathcal{B}, \mathcal{L}, \theta)$ be the Boolean dynamical system from Example 5.9. By Example 10.6 there exists only one non-trivial hereditary and saturated subset $\mathcal{H}$. Then, the only gauge invariant ideal of $C^*(\mathcal{B}, \mathcal{L}, \theta)$ is the ideal $I_{\mathcal{H}}$ generated by the projections $\{p_F : F \subseteq E^0 \text{ finite}\}$. Then the quotients $C^*(\mathcal{B}, \mathcal{L}, \theta)/I_{\mathcal{H}}$ is isomorphic to $C^*(\mathcal{B}/\mathcal{H}, \mathcal{L}, \theta)$. Observe that $\mathcal{B}/\mathcal{H}$ has only one non-empty element $\{\infty\}$, and $\theta_a(\infty) = \emptyset$ and $\theta_b(\infty) = \theta_c(\infty) = \infty$, thus $C^*(\mathcal{B}/\mathcal{H}, \mathcal{L}, \theta)$ is isomorphic to the Cuntz algebra $\mathcal{O}_2$.

11. Examples

Our motivation to define the Boolean Cuntz-Krieger algebras was to study the labelled graph C^*-algebras [3, 2] from a more general point of view, and this is actually what we achieved here. However, at this point, it is not clear to us if the class of Boolean Cuntz-Krieger algebras is strictly bigger than this of labelled graph C^*-algebras, but our approach to these C^*-algebras clearly allows to extract largely more information than the usual one. Besides of that, as we showed that the Boolean Cuntz-Krieger algebras are compactly supported 0-dimensional topological graphs, the C^*-algebras that we can construct as Boolean Cuntz-Krieger algebras includes homeomorphism C^*-algebras over 0-dimensional compact spaces, and graph C^*-algebras, among others [21].

Example 11.1. (Weakly left-resolving labelled graphs) Let $(E, \mathcal{L}, \mathcal{B})$ be a labelled graph, where E is a directed graph, $\mathcal{L} : E^1 \to \mathcal{A}$ is a labelling map over an alphabet $\mathcal{A}$, and $\mathcal{B}$ is an accommodating set of vertices E^0 [3, Section 2] that contains $\{\{v\} : v \in E^0_{sink}\}$. We will suppose that $(E,\mathcal{L},\mathcal{B})$ is weakly left-resolving and that $\mathcal{B}$ is a Boolean algebra. Then, given $A,B\in\mathcal{B}$ and $\alpha\in\mathcal{L}(E^1)$, we have that $r(A\cup B,\alpha)=r(A,\alpha)\cup r(B,\alpha)$ by definition, and $r(A\cap B,\alpha)=r(A,\alpha)\cap r(B,\alpha)$ since $(E,\mathcal{L},\mathcal{B})$ is weakly left-resolving. We claim that $r(A\setminus B,\alpha)=r(A,\alpha)\setminus r(B,\alpha)$. Indeed, observe that

$$r(A\setminus B,\alpha)\cap r(B,\alpha)=r((A\setminus B)\cap B,\alpha)=r(\emptyset,\alpha)=\emptyset$$

since $(E,\mathcal{L},\mathcal{B})$ is weakly left-resolving and $A\setminus B\in\mathcal{B}$. Thus, since

$$r(A\setminus B,\alpha)\cup r(B,\alpha)=r(A\cup B,\alpha)=r(A,\alpha)\cup r(B,\alpha)=(r(A,\alpha)\setminus r(B,\alpha))\cup r(B,\alpha)\,,$$

it follows that $r(A\setminus B,\alpha)=r(A,\alpha)\setminus r(B,\alpha)$, as desired.

We can define $\bar{\mathcal{B}}$ as the Boolean subalgebra of $C^*(E,\mathcal{L},\mathcal{B})$ generated by $\{s_\alpha p_A s_\alpha^* : A\in\mathcal{B},\ \alpha\in\mathcal{L}(E^1)\cup\{\emptyset\}\}$, and given $\alpha\in\mathcal{L}(E^1)$ we define the action

$$\theta_\alpha(s_\beta p_A s_\beta^*)=\begin{cases} p_A & \beta=\alpha\\ p_{r(A,\alpha)} & \beta=\emptyset\\ 0 & \text{otherwise}\end{cases}$$

for $A\in\mathcal{B}$ and $\beta\in\mathcal{L}(E^1)\cup\{\emptyset\}$. Then $(\bar{\mathcal{B}},\mathcal{L}(E^1),\theta)$ is a Boolean dynamical system, and $C^*(E,\mathcal{L},\mathcal{B})\cong C^*(\bar{\mathcal{B}},\mathcal{L}(E^1),\theta)$.

Example 11.2. Now, we will construct a unital Boolean Cuntz-Krieger algebra that is not a graph C^*-algebra. Let us define the Boolean algebra

$$\mathcal{B}:=\{F\subseteq\mathbb{Z} : F \text{ finite}\}\cup\{\mathbb{Z}\setminus F : F\text{ finite}\}\,,$$

and let $\mathcal{L}:=\{\alpha_i\}_{i\in\mathbb{Z}}\cup\{\beta\}$. Then, given $A\in\mathcal{B}$, we define the actions

$$\theta_{\alpha_i}(A)=A+i=\{x+i : x\in A\}\qquad\text{for every } i\in\mathbb{Z}$$

$$\theta_\beta(A)=\begin{cases}\mathbb{N} & \text{if } 0\in A\\ \emptyset & \text{otherwise,}\end{cases}$$

and then $\mathcal{R}_{\alpha_i}=\mathcal{R}_\beta=\mathcal{D}_{\alpha_i}=\mathcal{D}_\beta=\mathbb{Z}\in\mathcal{B}$ for every $i\in\mathbb{Z}$. Thus, $(\mathcal{B},\mathcal{L},\theta)$ is a Boolean dynamical system. Then $C^*(\mathcal{B},\mathcal{L},\theta)$ is a unital C^*-algebra, and since $(\mathcal{B},\mathcal{L},\theta)$ satisfies condition $(L_{\mathcal{B}})$ and there are non-trivial hereditary and saturated ideals $C^*(\mathcal{B},\mathcal{L},\theta)$ is simple by Theorem 9.14. Since $\mathcal{B}_{reg}=\emptyset$, it follows from Theorem 5.13 that

$$K_0(C^*(\mathcal{B},\mathcal{L},\theta))=\left(\bigoplus_{i\in\mathbb{Z}}\mathbb{Z}\right)^1\quad\text{and } K_1(C^*(\mathcal{B},\mathcal{L},\theta))=0\,.$$

Therefore, since $C^*(\mathcal{B},\mathcal{L},\theta)$ is unital and has non-finitely generated K-theory, it can not be a graph C^*-algebra. There exists Labelled graph C^*-algebras that are not Morita equivalent to graph C^*-algebras [23].

Example 11.3. Let X be a Cantor set, and let $Y,Z\subseteq X$ be compact clopen subsets, and let $\varphi : Y\to Z$ be an isomorphism. Let $\bar{\varphi} : C(Z)\to C(Y)$ the induced isomorphism. We define $\mathcal{B}$ as the Boolean algebra of the compact and clopens of X, and $\mathcal{L}=\{\alpha\}$ with the single action $\theta_\alpha : \mathcal{B}\to\mathcal{B}$ defined as $\theta_\alpha(A):=\varphi^{-1}(A)$ for every $A\in\mathcal{B}$. Whence θ_α has compact range, with $\mathcal{R}_\alpha=Y$, and compact domain because $\theta_\alpha(Z)=Y$. Then $C^*(\mathcal{B},\mathcal{L},\theta)$ is generated by projections $\{p_A\}_{A\in\mathcal{B}}$ and a partial isometry s_α such that

$$p_A s_\alpha=s_\alpha p_{\varphi^{-1}(A)},\qquad s_\alpha^* s_\alpha=p_Y\qquad\text{and}\qquad s_\alpha s_\alpha^*=p_Z\,.$$

since $Z \in \mathcal{B}_{reg}$. Then $C^*(\mathcal{B}, \mathcal{L}, \theta)$ is isomorphic to the partial automorphism crossed product $C^*(C(X), \bar{\varphi})$ (see [10]).

Observe, that in this situation all the cycles are of the form α^n for $n \in \mathbb{N}$, and hence given $A \in \mathcal{B}$ we have that $\theta_{\alpha^n}(A) = \varphi^{-n}(A)$. Hence the Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$ satisfies condition $(L_{\mathcal{B}})$ if and only if for every $n \in \mathbb{N}$ there exists a clopen subset such that $\varphi^{-n}(A) \neq \emptyset$ and $\varphi^{-n}(A) \cap A = \emptyset$, and it is cofinal if given $A, B \in \mathcal{B} \setminus \emptyset$ there exist $n_1, \ldots, n_k \in \mathbb{Z}$ such that $A \subseteq \bigcup_{i=1}^k \varphi^{n_i}(B)$.

In particular, if $\varphi : X \to X$ is a homeomorphism then $C^*(\mathcal{B}, \mathcal{L}, \theta) \cong C(X) \times_{\bar{\varphi}} \mathbb{Z}$. Moreover, the associated Boolean system will be minimal if and only if φ is minimal if and only if the associated Boolean system satisfies condition $(L_{\mathcal{B}})$.

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