# GENERATING CLASSES OF REGULAR REFINEMENT MONOIDS

E. PARDO AND F. WEHRUNG

Abstract. TO BE FILLED IN.

## 1. Basic concepts

For abelian groups $A$ and $B$, let $A \leqslant B$ (resp., $A \leqslant_{\mathrm{ess}} B$) hold, if $A$ is a subgroup (resp., an essential subgroup) of $B$.

Every commutative monoid $M$ is endowed with its *algebraic* preordering, defined by $x \leq y$ iff there exists $z \in M$ such that $x+z=y$, for all $x, y \in M$. Then let $x \equiv y$ hold, if $x \leq y \leq x$. We denote by $\mathbf{\Lambda}(M)$ the $(\vee, 0)$-semilattice of all idempotent elements of $M$. For an element $x$ in $M$, we denote by $\epsilon(x)$ the unique $u \in \mathbf{\Lambda}(M)$, if it exists, such that $x \equiv u$. We put

$$G_M[u] = \{x \in M \mid x \equiv u\}\,, \qquad \text{for all } u \in \mathbf{\Lambda}(M).$$

A non-unit element $p$ of $M$ is *prime*, if $p \leq x+y$ implies that either $p \leq x$ or $p \leq y$, for all $x, y \in M$. We say that $M$ is *regular*, if $2x \leq x$ holds for all $x \in M$. Equivalently, $M$ is a disjoint union of groups (which turn out to be the $G_M[a]$, where $a$ ranges over $\mathbf{\Lambda}(M)$), see [14, Theorem 2.1] or [9, Lemma 2.1]. We say that $M$ is *conical*, if 0 is the only unit of $M$. An *o-ideal* of $M$ is a nonempty subset $I$ of $M$ such that $x+y \in I$ iff $x \in I$ and $y \in I$, for all $x, y \in M$, and we denote by $\operatorname{Id} M$ the lattice of all ideals of $M$, partially ordered by containment. We say that $M$ is a *refinement monoid*, if for any elements $a_0, a_1, b_0, b_1 \in M$ such that $a_0 + a_1 = b_0 + b_1$, there are elements $c_{i,j} \in M$, for $i, j < 2$, such that $a_i = c_{i,0} + c_{i,1}$ and $b_i = c_{0,i} + c_{1,i}$ for all $i < 2$. For regular commutative monoids, the refinement property can be conveniently characterized by the distributivity of the semilattice of idempotents together with the so-called *Mayer-Vietoris Property* (see [9, Theorem 3.2]), which consists of the conjunction of the two following properties:

(MVP$_\vee$) $G_M[a+b] = G_M[a] + G_M[b]$, for all $a, b \in \mathbf{\Lambda}(M)$.

(MVP$_\wedge$) For all $a, b \in \mathbf{\Lambda}(M)$ and all $(x, y) \in G_M[a] \times G_M[b]$, if $x+b = y+a$, then there exists $z \in M$ such that $x = z+a$ and $y = z+b$.

For a semigroup $S$, we set $S^{\sqcup 0} = S \sqcup \{0\}$, where $\sqcup$ stands for disjoint union and the new zero element is the new unit element.

We put $P \downarrow a = \{x \in P \mid x \leq a\}$, for any element $a$ in a partially ordered set $P$. A nonzero element $p$ in a $(\vee, 0)$-semilattice $S$ is *join-irreducible*, if $p = x \vee y$ implies that either $p = x$ or $p = y$, for all $x, y \in S$. We denote by $\mathrm{J}(S)$ the partially ordered set of join-irreducible elements of $S$. In case $S$ is finite, $\mathrm{J}(S)$ consists exactly of those $p \in S \setminus \{0\}$ such that $\{x \in S \mid x < p\}$ has a largest element, then denoted by $p_*$.

We put $\ker f = \{(x, y) \in X \times X \mid f(x) = f(y)\}$, for every function $f$ with domain $X$.

---

*Date*: March 7, 2014.

2000 *Mathematics Subject Classification.* Primary 20M17, 46L35; Secondary 06A12, 06F05, 46L80.

## 2. Partial orders of abelian groups and the monoids Mon $\mathcal{G}$

We recall some concepts used in [3]. A *partial order of abelian groups* is a poset-indexed direct system of abelian groups, that is, a system of the form

$$\mathcal{G} = \left(G_i, g_i^{i'} \mid i \leq i' \text{ in } I\right), \tag{2.1}$$

where $I$ is a partially ordered set, $(G_i \mid i \in I)$ is a family of abelian groups, and $\left(g_i^{i'} \mid i \leq i' \text{ in } I\right)$ is a family of group homomorphisms such that $g_i^i = \mathrm{id}_{G_i}$ and $g_i^k = g_j^k \circ g_i^j$, for all $i \leq j \leq k$ in $I$; we say that $\mathcal{G}$ is *based on* $I$. For partial orders of abelian groups $\mathcal{G} = \left(G_i, g_i^{i'} \mid i \leq i' \text{ in } I\right)$ and $\mathcal{H} = \left(H_j, h_j^{j'} \mid j \leq j' \text{ in } J\right)$, a *morphism* from $\mathcal{G}$ to $\mathcal{H}$ consists of an order-preserving map $\varphi\colon I \to J$ together with a family $(\psi_i \mid i \in I)$ of group homomorphisms $\psi_i\colon G_i \to H_{\varphi(i)}$ such that the equality $h_{\varphi(i)}^{\varphi(i')} \circ \psi_i = \psi_{i'} \circ g_i^{i'}$ holds for all $i \leq i'$ in $I$. This way the class of partial orders of abelian groups becomes a category, introduced in [3]. With a partial order of abelian groups $\mathcal{G}$ as in (2.1) we associate the commutative monoid $\mathrm{Mon}(\mathcal{G})$ defined by the generators $(i, x)$, where $i \in I$ and $x \in G_i$, and the relations

$$(i, x) + (j, y) = (j, g_i^j(x) + y), \quad \text{for all } i \leq j \text{ in } I \text{ and all } (x, y) \in G_i \times G_j. \tag{2.2}$$

An explicit description of $\mathrm{Mon}(\mathcal{G})$ is given in [3, p. 166–167]. For calculating in these monoids, it is important to observe that an equality of the form

$$(i, x) = \sum ((i_k, x_k) \mid k < n)$$

holds in Mon $\mathcal{G}$ iff $i = \max\{i_k \mid k < n\}$ and $x = \sum\left(g_{i_k}^i(x_k) \mid k < n\right)$ in $G_i$. It is also proved there [3, Proposition 1] that $\mathrm{Mon}(\mathcal{G})$ is a primely generated regular refinement monoid, and that every primely generated regular refinement monoid is isomorphic to $\mathrm{Mon}(\mathcal{G})$ for some partial order of abelian groups $\mathcal{G}$ (see [3, Theorem 2]). In fact, the latter result is given by an equivalence between the category of partial orders of abelian groups and the category of regular refinement monoids with suitably defined morphisms. In particular, the finitely generated, regular, conical refinement monoids are exactly the monoids of the form Mon $\mathcal{G}$, for partial orders $\mathcal{G}$ of abelian groups based on finite partially ordered sets. We apply these results in the following lemma.

**Lemma 2.1.** *Every regular refinement monoid $M$ with finite semilattice of idempotents is a direct limit of finitely generated regular refinement monoids with the same semilattice of idempotents as $M$.*

*Proof.* Put $\Lambda = \mathbf{\Lambda}(M)$, $I = \mathrm{J}(\Lambda)$, $G_i = G_M[i]$, and $g_i^j\colon G_i \to G_j$, $x \mapsto x + j$, for all $i \leq j$ in $I$. Define $\mathcal{G}$ as in (2.1). As $\Lambda$ is finite, every element of $\Lambda$ is a (finite) join of elements of $I$. As $G_a = \sum_{i \in \mathrm{J}(a)} G_i$ holds for all $a \in \Lambda$ and every element of $\bigcup_{i \in I} G_i$ is prime, $M$ is primely generated. It follows from [3, Theorem 2] that there exists a unique *isomorphism* from $\mathrm{Mon}(\mathcal{G})$ onto $M$ that sends $(i, x)$ to $x$, for $i \in I$ and $x \in G_i$.

Now let $J$ be the set of all families $\xi = (X_i \mid i \in I)$ such that

(i) $X_i$ is a finitely generated subgroup of $G_i$, for all $i \in I$;
(ii) $i \leq j$ implies that $g_i^j(X_i) \leqslant X_j$, for all $i \leq j$ in $I$,

and then put $\mathcal{G}_\xi = \left(X_i, g_i^j[\xi] \mid i \le j \text{ in } I\right)$ where $g_i^j[\xi]$ denotes the restriction of $g_i^j$ from $X_i$ to $X_j$, for $i \le j$. It is straightforward to verify that $J$ is an upwards directed partially ordered set and that $\mathcal{G}$ is the direct limit of $(\mathcal{G}_\xi \mid \xi \in J)$ with the obvious transition morphisms and limiting morphisms. Hence $\mathrm{Mon}(\mathcal{G})$ is the direct limit of $(\mathrm{Mon}(\mathcal{G}_\xi) \mid \xi \in J)$ with the obvious transition morphisms and limiting morphisms. Observe that each monoid $\mathrm{Mon}(\mathcal{G}_\xi)$ is finitely generated. $\square$

**Lemma 2.2.** *For any prime number $p$, there are an abelian group $G$ of exponent $p$ with infinite subgroups $A_0$, $A_1$, $A_2$, $A_3$ such that $G = A_0 \oplus A_3 = A_1 \oplus A_2$ but for any finitely generated $X \leqslant G$, $X = (X \cap A_0) + (X \cap A_3) = (X \cap A_1) + (X \cap A_2)$ implies that $X \cap A_0 = \{0\}$.*

*Proof.* Denote by $\mathbb{F}_p$ the $p$-element field and put $G = \mathbb{F}_p^{(\mathbb{Z})}$, the free $\mathbb{F}_p$-vector space on $\mathbb{Z}$. Denote the canonical basis of $G$ by $(\delta_n \mid n \in \mathbb{Z})$, and denote by $f$ the automorphism of $G$ defined by $f(\delta_n) = \delta_{n+1}$, for all $n \in \mathbb{Z}$. We put

$$\begin{aligned} A_0 &= \langle \delta_{2n} \mid n \in \mathbb{Z} \rangle\,, \\ A_1 &= f(A_0) = \langle \delta_{2n+1} \mid n \in \mathbb{Z} \rangle\,, \\ A_2 &= (\mathrm{id}_G - f)(A_0) = \langle \delta_{2n} - \delta_{2n+1} \mid n \in \mathbb{Z} \rangle\,, \\ A_3 &= (f - f^2)(A_0) = \langle \delta_{2n+1} - \delta_{2n+2} \mid n \in \mathbb{Z} \rangle\,. \end{aligned}$$

Of course, $G = A_0 \oplus A_3 = A_1 \oplus A_2$. Now let $X$ be a subgroup of $G$ such that $X = (X \cap A_i) \oplus (X \cap A_j)$ holds for all $(i,j) \in \{(0,3),(1,2)\}$; put $X_i = X \cap A_i$, for all $i \in \{0,1,2,3\}$. We claim that $f^2(X_0) \leqslant X_0$. Indeed, let $x \in X_0$. As $x \in X = X_1 \oplus X_2$ and $x = f(x) + (x - f(x))$ with $f(x) \in A_1$ and $x - f(x) \in A_2$, we get $f(x) \in X_1$ and $x - f(x) \in X_2$. As $f(x) \in X = X_0 \oplus X_3$ and $f(x) = f^2(x) + (f(x) - f^2(x))$ with $f^2(x) \in A_0$ and $f(x) - f^2(x) \in A_3$, we get $f^2(x) \in X_0$, thus establishing our claim.

In particular, if $X$ is finite-dimensional, then, as $G$ does not have any nonzero finitely generated subgroup which is closed under $f^2$, we obtain that $X_0 = \{0\}$. $\square$

The following result shows that one cannot replace "direct limit" by "directed union" in the statement of Lemma 2.1. Because of [17, Theorem 4.3], the situation is different with monoids satisfying the embedding condition (emb).

**Proposition 2.3.** *There exists a regular conical refinement monoid with finitely many idempotents which is not a directed union of finitely generated refinement submonoids.*

*Proof.* Let $G$, $A_0$, $A_1$, $A_2$, $A_3$ be abelian groups satisfying the conditions of Lemma 2.2, denote by $\Lambda^*$ the powerset of $\{0,1,2,3\}$, and set $\Lambda = \Lambda^* \cup \{\bot\}$ where $\bot$ is a new zero element. We put $G_\bot = \{0\}$ (*the element $\bot$ is put there only to ensure conicality of the monoid*), and

$$A_u = \sum_{i \in p} A_i \quad \text{and} \quad G_u = G/A_u, \qquad \text{for all } u \in \Lambda^*,$$

where we identify $G/A_\varnothing = G/\{0\}$ with $G$. Next, we define a group homomorphism $g_u^v \colon G_u \to G_v$, for all $u \le v$ in $\Lambda$. For $u = \bot$ there exists a unique homomorphism $g_\bot^v \colon \{0\} \to G_v$. For $u \le v$ in $\Lambda^*$, let $g_u^v$ be the canonical projection from $G/A_u$ onto $G/A_v$. The desired monoid is

$$M = \bigcup_{u \in \Lambda} (\{u\} \times G_u),$$

endowed with the addition given by $(u,x)+(v,y)=(u\vee v, g_u^{u\vee v}(x)+g_v^{u\vee v}(y))$, for all $(u,x),(v,y)\in M$. It is straightforward to verify, for example by using [9, Theorem 3.2], that $M$ is a regular conical refinement monoid.

Fix any element $a\in A_0\setminus\{0\}$, and let $N$ be a refinement submonoid of $M$ containing $\mathbf{\Lambda}(M)\cup\{(\varnothing,a)\}$. Suppose that $N$ is finitely generated. As $\mathbf{\Lambda}(M)\subseteq N$, there are submonoids $H_u\subseteq G_u$, for all $u\in\Lambda$, such that

$$N=\bigcup_{u\in\Lambda}(\{u\}\times H_u).$$

As all groups $G_u$ have finite exponent, $H_u$ is, in fact, a *subgroup* of $G_u$, for all $u\in\Lambda$, and hence $N$ is regular. As $N$ is finitely generated, all $H_u$, for $u\in\Lambda$, are finitely generated.

We claim that $H_\varnothing\cap(A_i+A_j)=(H_\varnothing\cap A_i)+(H_\varnothing\cap A_j)$, for all $(i,j)\in\{(0,3),(1,2)\}$. Indeed, let $x\in H_\varnothing\cap(A_i+A_j)$. As $(\{i\},x+A_i)=(\varnothing,x)+(\{i\},0)$ belongs to $N$, we obtain that $x+A_i\in H_{\{i\}}$. Similarly, $0+A_j$ belongs to $H_{\{j\}}$, and $g_{\{i\}}^{\{i,j\}}(x+A_i)=g_{\{j\}}^{\{i,j\}}(0+A_j)=0$ in $H_{\{i,j\}}$. Hence (*we use here the assumption that $N$ satisfies refinement*), there exists, by [9, Theorem 3.2], $y\in H_\varnothing$ such that $x+A_i=y+A_i$ and $0+A_j=y+A_j$, and so $x\in(H_\varnothing\cap A_i)+(H_\varnothing\cap A_j)$, therefore establishing our claim.

As $H_\varnothing$ is finitely generated and by the properties required from $G$ and the $A_i$s, it follows that $H_\varnothing\cap A_0=\{0\}$, a contradiction as $a\in H_\varnothing\cap A_0$. □

## 3. Approximating regular conical refinement monoids from below

The present section will be devoted to the proof of the following result.

**Theorem 3.1.** *Every regular conical refinement monoid is a direct limit of finitely generated regular conical refinement monoids.*

Let $M$ be a regular conical refinement monoid. In order to prove that $M$ is a direct limit of finitely generated regular conical refinement monoids, we apply Lemma 4.1 and Remark 4.3 of [9], with $\mathcal{B}$ defined as the class of all finitely generated regular conical refinement monoids. Observe that $\mathcal{B}$ is, indeed, closed under finite direct sums, so the abovecited results apply.

We first need to verify that every $a\in M$ belongs to some submonoid $B$ of $M$ belonging to $\mathcal{B}$. It suffices to put $B=G\cup\{0\}$, where $G$ is defined as the subgroup of $G_M[a]$ generated by $a$. Hence the main part of the proof of Theorem 3.1 consists of verifying the "Triangle Lemma", which is item (2) of [9, Lemma 4.1]. So let $B$ be a finitely generated regular conical refinement monoid and let $f\colon B\to M$ be a monoid homomorphism, we must prove that there are $C\in\mathcal{B}$ and monoid homomorphisms $\varphi\colon B\to C$ and $g\colon C\to M$ such that $f=g\circ\varphi$ and $\ker f=\ker\varphi$.

Put $\Lambda=\mathbf{\Lambda}(M)$ and $G_a=G_M[a]$, for all $a\in\Lambda$. We shall abbreviate $\downarrow a=\Lambda\downarrow a$, for all $a\in\Lambda$. As $B$ is finitely generated, $\epsilon\circ f(B)$ is a finite join-subsemilattice of $\Lambda$. Put $e_x=\epsilon\circ f(x)$, for all $x\in B$, and denote by $\mathbb{D}$ the sublattice of $\operatorname{Id}\Lambda$ generated by $\{\downarrow e_x\mid x\in B\}$. As $\operatorname{Id}\Lambda$ is a distributive lattice and $B$ is finitely generated, $\mathbb{D}$ is a finite distributive lattice. Define a *choice function* on $\mathbb{D}$ as a map $\gamma\colon\mathbb{D}\to\Lambda$ such that $\gamma(A)\in A$, for all $A\in\mathbb{D}$.

**Lemma 3.2.** *For any choice function $\gamma$ on $\mathbb{D}$, there exists a $(\vee,0)$-embedding $\eta\colon\mathbb{D}\hookrightarrow\Lambda$ such that the following conditions hold:*

(i) *$\eta$ is a choice function on $\mathbb{D}$.*

(ii) $\eta(\downarrow e_x) = e_x$, *for all* $x \in B$.
(iii) $\gamma \leq \eta$, *that is,* $\gamma(A) \leq \eta(A)$ *for all* $A \in \mathbb{D}$.

*Outline of proof.* As in the construction of $\varphi$ in the proof of [9, Theorem 6.1]. As, for all $x \in B$, the principal ideal $\downarrow e_x$ is the join of all join-irreducible elements of $\mathbb{D}$ below it, there are elements $u_P \in P$, for $P \in \mathrm{J}(\mathbb{D})$, such that

$$e_x = \bigvee (u_P \mid P \in \mathrm{J}_{\mathbb{D}}(\downarrow e_x)) \,, \quad \text{for all } x \in B.$$

Denote by $P^\dagger$ the largest element of $\mathbb{D}$ such that $P \nsubseteq P^\dagger$ (see [9, Lemma 5.1]). By possibly enlarging the elements $u_P$, we may assume that $u_P \in P \setminus P^\dagger$, for all $P \in \mathrm{J}(\mathbb{D})$. Finally, for all $A \in \mathbb{D}$, the element $\gamma(A)$ belongs to $A = \bigvee (P \mid P \in \mathrm{J}_{\mathbb{D}}(A))$, hence we may further enlarge the elements $u_P$ in such a way that

$$\gamma(A) \leq \bigvee (u_P \mid P \in \mathrm{J}_{\mathbb{D}}(A)) \,, \quad \text{for all } A \in \mathbb{D}.$$

The map $\eta \colon \mathbb{D} \to \Lambda$ defined by the rule

$$\eta(A) = \bigvee (u_P \mid P \in \mathrm{J}_{\mathbb{D}}(A)) \,, \quad \text{for all } A \in \mathbb{D},$$

is as required. □

For all $a \leq b$ in $\Lambda$, set $\tau_a^b \colon G_a \to G_b$, $x \mapsto x + b$, the canonical group homomorphism from $G_a$ to $G_b$. For any $A \in \operatorname{Id} \Lambda$, let

$$\left(G_A, \tau_a^A \mid a \in A\right) = \varinjlim \left(G_b, \tau_a^b \mid a \leq b \text{ in } A\right),$$

where the direct limit is evaluated in the category of abelian groups. We may assume that $G_{\downarrow a} = G_a$ and $\tau_a^{\downarrow a} = \mathrm{id}_{G_a}$, for all $a \in \Lambda$.

Let $A \subseteq B$ in $\operatorname{Id} \Lambda$. It follows from the universal property of the direct limit that there exists a unique group homomorphism $\tau_A^B \colon G_A \to G_B$ such that the equality $\tau_a^B = \tau_A^B \circ \tau_a^A$ holds for all $a \in A$. Hence $\tau_A^A = \mathrm{id}_{G_A}$ and $\tau_A^C = \tau_B^C \circ \tau_A^B$ holds for all $A \subseteq B \subseteq C$ in $\operatorname{Id} \Lambda$. We define a submonoid $\mathbb{M}$ of $\mathbb{D} \times G_\Lambda$ (where $\mathbb{D}$ is viewed as a join-semilattice) by

$$\mathbb{M} = \bigcup_{A \in \mathbb{D}} (\{A\} \times G_A),$$

endowed with the addition defined by the rule

$$(A, x) + (B, y) = \left(A \vee B, \tau_A^{A \vee B}(x) + \tau_B^{A \vee B}(y)\right), \quad \text{for all } (A, x), (B, y) \in \mathbb{M}.$$

**Lemma 3.3.** *The monoid* $\mathbb{M}$ *is a regular conical refinement monoid, with semilattice of idempotents isomorphic to* $\mathbb{D}$.

*Proof.* It is obvious that $\mathbb{M}$ is regular and that $\mathbf{\Lambda}(\mathbb{M}) = \mathbb{D} \times \{0\} \cong \mathbb{D}$. In order to verify that $\mathbb{M}$ is a refinement monoid, it suffices, by [9, Theorem 3.2], to verify the Mayer-Vietoris property.

**(MVP$_\vee$)** We must verify that $G_{A \vee B} = \tau_A^{A \vee B}(G_A) + \tau_B^{A \vee B}(G_B)$, for all $A, B \in \mathbb{D}$. Let $x \in G_{A \vee B}$. There are $c \in A \vee B$ and $y \in G_c$ such that $x = \tau_c^{A \vee B}(y)$. By possibly enlarging $c$, we may assume that $c = a \vee b$, for some $(a, b) \in A \times B$. As $y$ belongs to $G_c = \tau_a^{a \vee b}(G_a) + \tau_b^{a \vee b}(G_b)$, there exists $(u, v) \in G_a \times G_b$ such that $y = \tau_a^{a \vee b}(u) + \tau_b^{a \vee b}(v)$. Hence,

$$x = \tau_a^{A \vee B}(u) + \tau_b^{A \vee B}(v) = \tau_A^{A \vee B}(\tau_a^A(u)) + \tau_B^{A \vee B}(\tau_b^B(v)) \in \tau_A^{A \vee B}(G_A) + \tau_B^{A \vee B}(G_B).$$

**(MVP$_\wedge$)** We must verify that for all $A, B \in \mathbb{D}$ and all $(x, y) \in G_A \times G_B$ such that $\tau_A^{A\vee B}(x) = \tau_B^{A\vee B}(y)$, there exists $z \in G_{A\cap B}$ such that $x = \tau_{A\cap B}^A(z)$ and $y = \tau_{A\cap B}^B(z)$. There are $(a', b') \in A \times B$ and $(u', v') \in G_{a'} \times G_{b'}$ such that $x = \tau_{a'}^A(u')$ and $y = \tau_{b'}^B(v')$. As $\tau_{a'}^{A\vee B}(u') = \tau_{b'}^{A\vee B}(v')$, there exists $c \in A \vee B$ such that $\tau_{a'}^c(u') = \tau_{b'}^c(v')$. By possibly enlarging $c$, we may assume that $c = a \vee b$, for some $(a, b) \in A \times B$ such that $a \geq a'$ and $b \geq b'$. So $u = \tau_{a'}^a(u')$ belongs to $G_a$, $v = \tau_{b'}^b(v')$ belongs to $G_b$, $x = \tau_a^A(u)$, $y = \tau_b^B(v)$, and $\tau_a^{a\vee b}(u) = \tau_b^{a\vee b}(v)$. By (MVP$_\wedge$) for $M$, there are $d \leq a, b$ in $\Lambda$ and $w \in G_d$ such that $u = \tau_d^a(w)$ and $v = \tau_d^b(w)$. Hence, putting $z = \tau_d^{A\cap B}(w)$, we obtain that $x = \tau_a^A(u) = \tau_d^A(w) = \tau_{A\cap B}^A(z)$, and, similarly, $y = \tau_{A\cap B}^B(z)$. □

As $f(x) \in G_{\epsilon\circ f(x)} = G_{e_x}$ for all $x \in B$, we can define a map $\overline{f}\colon B \to \mathbb{M}$, $x \mapsto (\downarrow e_x, f(x))$. It is obvious that $\overline{f}$ is zero-preserving. For all $x, y \in B$, we compute

$$\begin{aligned}\overline{f}(x) + \overline{f}(y) &= \Big(\downarrow e_x \vee \downarrow e_y, \tau_{\downarrow e_x}^{\downarrow e_x \vee \downarrow e_y}(f(x)) + \tau_{\downarrow e_y}^{\downarrow e_x \vee \downarrow e_y}(f(y))\Big)\\ &= \Big(\downarrow e_{x+y}, \tau_{e_x}^{e_{x+y}}(f(x)) + \tau_{e_y}^{e_{x+y}}(f(y))\Big)\\ &= (\downarrow e_{x+y}, f(x+y) + e_{x+y})\\ &= (\downarrow e_{x+y}, f(x+y))\\ &= \overline{f}(x+y),\end{aligned}$$

and so $\overline{f}$ is a monoid homomorphism from $B$ to $\mathbb{M}$. Trivially, $\ker f = \ker \overline{f}$.

Now it follows from Lemmas 2.1 and 3.3 that $\mathbb{M}$ is a direct limit of finitely generated regular conical refinement monoids. In particular, by Lemma 4.1 and Remark 4.3 in [9], the Triangle Lemma holds for $\mathbb{M}$. By applying this to the homomorphism $\overline{f}\colon B \to \mathbb{M}$, we obtain a finitely generated regular conical refinement monoid $C$ and homomorphisms $\varphi\colon B \to C$ and $\overline{g}\colon C \to \mathbb{M}$ such that $\overline{f} = \overline{g} \circ \varphi$ and $\ker \overline{f} = \ker \varphi$. Observe that $\ker \varphi = \ker f$ as well. Let $\overline{g}(y) = (K_y, \tilde{g}(y))$, for all $y \in C$. In particular, the map $C \to \mathbb{D}$, $y \mapsto K_y$ is a monoid homomorphism.

By Redei's Theorem (see [19], or [7] for a simple proof), every finitely generated commutative monoid is finitely presented. In particular, $C$ is finitely presented. Thus, by possibly enlarging a given generating subset of $C$, we may assume that $C$ has a presentation of the form

$$y_k = y_i + y_j, \qquad \text{for all } (i, j, k) \in \Gamma, \tag{3.1}$$

where $\{y_i \mid i < m\}$ is a finite generating subset of $C$ and $\Gamma$ is a set of triples of elements of $\{0, 1, \dots, m-1\}$. For all $i < m$, as $\tilde{g}(y_i)$ belongs to $G_{K_{y_i}}$, there are $b_i \in K_{y_i}$ and $z_i \in G_{b_{y_i}}$ such that $\tilde{g}(y_i) = \tau_{b_i}^{K_{y_i}}(z_i)$. For each $(i, j, k) \in \Gamma$, it follows from the equality $\overline{g}(y_k) = \overline{g}(y_i) + \overline{g}(y_j)$ that

$$K_{y_k} = K_{y_i} \vee K_{y_j} \tag{3.2}$$

and $\tilde{g}(y_k)\tau_{K_{y_i}}^{K_{y_k}}(\tilde{g}(y_i)) + \tau_{K_{y_j}}^{K_{y_k}}(\tilde{g}(y_j))$. The latter equation can be written

$$\tau_{b_k}^{K_{y_k}}(z_k) = \tau_{b_i}^{K_{y_k}}(z_i) + \tau_{b_j}^{K_{y_k}}(z_j),$$

and thus there exists $b'_{i,j,k} \in K_k$ such that $b_i \vee b_j \vee b_k \leq b'_{i,j,k}$ and

$$\tau_{b_k}^{b'_{i,j,k}}(z_k) = \tau_{b_i}^{b'_{i,j,k}}(z_i) + \tau_{b_j}^{b'_{i,j,k}}(z_j).$$

For fixed $k$, we can replace $b'_{i,j,k}$ by the join $b'_k$ of all $b'_{i,j,k}$ such that $(i,j,k) \in \Gamma$, thus obtaining the equation

$$\tau_{b_k}^{b'_k}(z_k) = \tau_{b_i}^{b'_k}(z_i) + \tau_{b_j}^{b'_k}(z_j),$$

that is,

$$z_k + b'_k = z_i + z_i + b'_k, \quad \text{for all } (i,j,k) \in \Gamma. \tag{3.3}$$

An easy application of Lemma 3.2 yields a $(\vee, 0)$-embedding $\eta \colon \mathbb{D} \hookrightarrow \Lambda$ such that

(i) $\eta$ is a choice function on $\mathbb{D}$.
(ii) $\eta(\downarrow e_x) = e_x$, for all $x \in B$.
(iii) $b'_i \leq \eta(K_{y_i})$, for all $i < m$.

In particular, it follows from (3.2) and (3.3) that $\eta(K_{y_k}) = \eta(K_{y_i}) \vee \eta(K_{y_j})$ and $z_k + \eta(K_{y_k}) = z_i + z_j + \eta(K_{y_k})$, for all $(i,j,k) \in \Gamma$. Hence,

$$z_k + \eta(K_{y_k}) = (z_i + \eta(K_{y_i})) + (z_j + \eta(K_{y_j})).$$

Hence, as (3.1) is a presentation of $C$, there exists a unique monoid homomorphism $g \colon C \to M$ such that $g(y_i) = z_i + \eta(K_{y_i})$ for all $i < m$.

**Lemma 3.4.** *The equality* $\tilde{g}(y) = \tau_{\eta(K_y)}^{K_y}(g(y))$ *holds, for all* $y \in C$.

*Proof.* There are $I \subseteq \{0, 1, \dots, m-1\}$ and a family $(k_i \mid i \in I)$ of positive integers such that $y = \sum_{i \in I} k_i y_i$. We first observe that $g(y) = \sum_{i \in I} k_i g(y_i) = \sum_{i \in I} k_i z_i + \eta(K_y)$, and so

$$g(y) = \sum_{i \in I} k_i (z_i + \eta(K_y)). \tag{3.4}$$

Now we can compute

$$\begin{aligned}
\tilde{g}(y) &= \sum_{i \in I} k_i \cdot \tau_{K_{y_i}}^{K_y}(\tilde{g}(y_i)) && \text{(because } \overline{g} \text{ is a monoid homomorphism)}\\
&= \sum_{i \in I} k_i \cdot \left(\tau_{K_{y_i}}^{K_y} \circ \tau_{b_i}^{K_{y_i}}(z_i)\right) && \text{(by the definition of } b_i \text{ and } z_i\text{)}\\
&= \sum_{i \in I} k_i \cdot \tau_{b_i}^{K_y}(z_i) \\
&= \sum_{i \in I} k_i \cdot \tau_{\eta(K_{y_i})}^{K_y}(z_i + \eta(K_{y_i})) && \text{(because } \tau_{b_i}^{\eta(K_{y_i})}(z_i) = z_i + \eta(K_{y_i})\text{)}\\
&= \sum_{i \in I} k_i \cdot \tau_{\eta(K_{y_i})}^{K_y}(g(y_i)) && \text{(by the definition of } g\text{)}\\
&= \tau_{\eta(K_y)}^{K_y}\left(\sum_{i \in I} k_i \cdot \tau_{\eta(K_{y_i})}^{\eta(K_y)}(g(y_i))\right) && \text{(because } \eta(K_{y_i}) \leq \eta(K_y) \text{ for all } i \in I\text{)}\\
&= \tau_{\eta(K_y)}^{K_y}\left(\sum_{i \in I} k_i \cdot (z_i + \eta(K_y))\right) && \text{(by the definition of } \tau_{\eta(K_{y_i})}^{\eta(K_y)}\text{)}\\
&= \tau_{\eta(K_y)}^{K_y}(g(y)) && \text{(by (3.4)).}
\end{aligned}$$

□

Now for all $x \in B$, we obtain, using Lemma 3.4, that

$$(\downarrow e_x, f(x)) = \overline{f}(x) = \overline{g} \circ \varphi(x) = (K_{\varphi(x)}, \tau_{\eta(K_{\varphi(x)})}^{K_{\varphi(x)}}(g \circ \varphi(x)).$$

In particular, $K_{\varphi(x)} = {\downarrow} e_x$, thus $\eta(K_{\varphi(x)}) = e_x$, and thus $\tau_{\eta(K_{\varphi(x)})}^{K_{\varphi(x)}} = \mathrm{id}_{G_{e_x}}$, and so

$$f(x) = \tau_{\eta(K_{\varphi(x)})}^{K_{\varphi(x)}}(g \circ \varphi(x)) = g \circ \varphi(x).$$

Therefore, $f = g \circ \varphi$. As we have already observed that $\ker f = \ker \varphi$, this concludes the proof of Theorem 3.1.

## References


[1] G. Birkhoff, *Subgroups of abelian groups*, Proc. London Math. Soc. II, Ser. **38** (1934), 385–401.
[2] S. Bulman-Fleming and K. McDowell, *Flat semilattices*, Proc. Amer. Math. Soc. **72** (1978), 228–232.
[3] H. Dobbertin, *Primely generated regular refinement monoids*, J. Algebra **91** (1984), 166–175.
[4] E. G. Effros, D. E. Handelman, and C-L. Shen, *Dimension groups and their affine representations*, Amer. J. Math. **102**, no. 2 (1980), 385–407.
[5] G. A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), 29–44.
[6] Eršov, Ju. L. "Теория Нумераций". (Russian) [Theory of Numerations] Математическая Логика и Основания Математики. [Monographs in Mathematical Logic and Foundations of Mathematics] "Nauka", Moscow, 1977. 416 p.
[7] P. Freyd, *Redei's finiteness theorem for commutative semigroups*, Proc. Amer. Math. Soc. **19**, no. 4 (1968), 1003.
[8] L. Fuchs, "Infinite Abelian Groups. Vol. I". Pure and Applied Math. **36**. New York - London, Academic Press, 1970. xi+290 p.
[9] K. R. Goodearl, E. Pardo, and F. Wehrung, *Semilattices of groups and inductive limits of Cuntz algebras*, J. Reine Angew. Math. **588** (2005), 1–25.
[10] K. R. Goodearl and F. Wehrung, *Representations of distributive semilattices in ideal lattices of various algebraic structures*, Algebra Universalis **45**, no. 1 (2001), 71–102.
[11] G. Grätzer, "General Lattice Theory. Second edition", new appendices by the author with B. A. Davey, R. Freese, B. Ganter, M. Greferath, P. Jipsen, H. A. Priestley, H. Rose, E. T. Schmidt, S. E. Schmidt, F. Wehrung, and R. Wille. Birkhäuser Verlag, Basel, 1998. xx+663 p.
[12] P. A. Grillet, *Directed colimits of free commutative semigroups*, J. Pure Appl. Algebra **9**, no. 1 (1976), 73–87.
[13] T. J. Head, *Purity in compactly generated modular lattices*, Acta Math. Hungar. **17** (1966), 55–59.
[14] J. M. Howie, "An Introduction to Semigroup Theory". L.M.S. Monographs **7**. Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1976. x+272 p.
[15] T. Y. Lam, "A First Course in Noncommutative Rings". Graduate Texts in Mathematics **131**. Springer-Verlag, New York, 1991. xvi+397 p.
[16] R. N. McKenzie, G. F. McNulty, and W. F. Taylor, "Algebras, Lattices, Varieties. Vol. I", The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1987. xvi+361 p.
[17] E. Pardo and F. Wehrung, *Semilattices of groups and nonstable K-theory of extended Cuntz limits*, K-Theory **37** (2006), 1–23.
[18] P. Pudlák, *On congruence lattices of lattices*, Algebra Universalis **20** (1985), 96–114.
[19] L. Redei, "The Theory of Finitely Generated Commutative Semigroups". Translation edited by N. Reilly. Intl. Series of Monographs in Pure and Applied Math. **82**, Oxford - Edinburgh - New York, Pergamon Press, 1965. xiii+350 p.

Departamento de Matemáticas, Universidad de Cádiz, Apartado 40, 11510 Puerto Real (Cádiz), Spain
*E-mail address*, E. Pardo: enrique.pardo@uca.es
*URL*, E. Pardo: http://www2.uca.es/dept/matematicas/PPersonales/PardoEspino/EMAIN.HTML

CNRS, UMR 6139, Département de Mathématiques, Université de Caen, 14032 Caen Cedex, France
*E-mail address*, F. Wehrung: wehrung@math.unicaen.fr
*URL*, F. Wehrung: http://www.math.unicaen.fr/~wehrung