PURELY INFINITE SIMPLE SKEW GROUP RINGS

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Abstract. In this note we prove that, if R is a purely infinite simple unital ring, G is a group, and $\alpha : G \to \mathrm{Aut}(R)$ is an outer action on R, then the skew group ring $R *_\alpha G$ is a purely infinite simple ring.

Introduction

In 1981, Cuntz [4] introduced the concept of a purely infinite simple C*-algebra. This notion has played a central role in the development of the theory of C*-algebras in the last two decades. A suitable notion of purely infinite simple in the algebraic context was introduced by Ara, Goodearl and Pardo [3]. Recall that a unital simple ring R is *purely infinite* if every nonzero right ideal of R contains an infinite idempotent. This concept is left-right symmetric, and coincides with the notion of introduced by Cuntz in case of C*-algebras (see [3]). Moreover, as as shown in [3, Theorem 1.6], it is equivalent to the following: (1) R is not a division ring; (2) For every nonzero element $a \in R$, there exist elements $x, y \in R$ such that $xay = 1$.

In the context of C*-algebras, Jeong [5] and Jeong, Kodaka and Osaka [6] showed that the reduced C*-crossed product $A \times_\alpha G$ of a purely infinite unital C*-algebra A by an outer action α of a countable abelian group G is always purely infinite. The aim is to extend this result to arbitrary groups acting on C*-algebras, and to the purely algebraic context.

In this note, we show that the skew group ring $R *_\alpha G$ associated to an outer action of a group G on a purely infinite ring R is always purely infinite. We recall the basic definitions we will need in the sequel. Let R be a unital ring, let G be a group, and let $\alpha : G \to \mathrm{Aut}(R)$ be an action of G on R. If the identity is the only element of G that maps to an inner automorphism, then the action is said to be *outer*. The *skew group ring* $R *_\alpha G$ (also denoted RG) is the free left R-module with basis G. Thus, the elements of RG are finite sums of the form $\Sigma a_g g$, where $a_g \in R$ and $g \in G$. Multiplication is defined according to the rule $(ag)(bh) = a\alpha(g)(b)(gh)$ for $a, b \in R$ and $g, h \in G$. The *support* of an element is $\mathrm{supp}(\Sigma a_g g) = \{g \in G \mid a_g \neq 0\}$. The *length* of $\Sigma a_g g$ is the cardinality of $\mathrm{supp}(\Sigma a_g g)$, and is denoted $\mathrm{len}(\Sigma a_g g)$ (see [7], [9]).

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1. The main result

The present we present here is similar to that of [2, Theorem 5.3]. The key point is [2, Lemma 5.2], that we quote her for the sake of completeness.

Lemma 1.1. *([2, Lemma 5.2]) If R is an simple ring containing an idempotent $e \neq 0, 1$, then R is generated (as ring) by its idempotents.*

Theorem 1.2. *Let R be a unital purely infinite simple ring, let G be a group, and let α be an outer action of G on R. Then the skew group ring $R *_\alpha G$ is purely infinite.*

Proof. Since R is purely infinite, it contains an infinite idempotent, and so there exists an idempotent $e \neq 0, 1$. Hence, $R *_\alpha G$ cannot be a division ring, because it contains the ring R.

Let γ be an arbitrary nonzero element of $R *_\alpha G$. Choose $\alpha, \beta \in R *_\alpha G$ such that $\alpha\gamma\beta$ is a nonzero element whose length is minimal for such nonzero products. Suppose that $\operatorname{len}(\alpha\gamma\beta) = n$. Now write

$$\alpha\gamma\beta = \sum_{i=1}^{n} a_i g_i,$$

where the g_i are distinct elements of G, and each a_i is a nonzero element of R. After replacing $\alpha\gamma\beta$ by $\alpha\gamma\beta g_1^{-1}$, we can assume that

$$\alpha\gamma\beta = a_1 + \sum_{i=2}^{n} a_i g_i.$$

Since R is purely infinite and simple, there exists $a, b \in R$ such that $aa_1b = 1$. Hence, after replacing $\alpha\gamma\beta$ by $a\alpha\gamma\beta b$, we can assume that $a_1 = 1$. So, if $n = 1$, we are done.

Suppose that $n \geq 2$. Thus,

$$\alpha\gamma\beta = 1 + a_2 g_2 + \sum_{i=3}^{n} a_i g_i,$$

where $a_2 \neq 0$ and $g_2 \neq 1$. For any idempotent $e \in R$, we have

$$e\alpha\gamma\beta(1-e) = ea_2\alpha(g_2)(1-e)g_2 + \sum_{i=3}^{n} ea_i\alpha(g_i)(1-e)g_i.$$

Since $\operatorname{len}(e\alpha\gamma\beta(1-e)) < n$, we have $e\alpha\gamma\beta(1-e) = 0$. Then, $ea_2\alpha(g_2)(e) = ea_2$. A symmetric argument, involving $(1-e)\alpha\gamma\beta e$, shows that $a_2\alpha(g_2)(e) = ea_2\alpha(g_2)(e)$, and so $ea_2 = a_2\alpha(g_2)(e)$. Thus, by Lemma 1.1, $a_2\alpha(g_2)(x) = xa_2$ for all $x \in R$, and so $Ra_2 = a_2R$. Since R is simple, thus $Ra_2 = Ra_2R = a_2R = R$, and so a_2 is an invertible element of R. But then $\alpha(g_2)$ is inner, which contradicts our assumptions. Therefore $n = 1$, and the proof is complete. ☐

In the particular case of G being a finite group, the action is G-Galois (see [1], [8]), and then RG is Morita equivalent to the ring

$$R^G = \{a \in R \mid \alpha(g)(a) = a \text{ for all } g \in G\}.$$

Then, we have the following result:

Corollary 1.3. *Let R be a unital purely infinite simple ring, let G be a finite group, and let α be an outer action of G on R. Then the fixed subring of R under G, R^G, is purely infinite.*

Proof. It is a direct consequence of the above remark, Theorem 1.2 and [3, Corollary 1.7]. □

References

[1] R. Alfaro, P. Ara, A. del Rio, Regular skew group rings, *J. Austral. Math. Soc.(Series A)* **58** (1995), 167-182.

[2] P. Ara, M.A. González-Barroso, K.R. Goodearl, E. Pardo, Fractional skew monoid rings, *J. Algebra* (submitted).

[3] P. Ara, K.R. Goodearl, E. Pardo, K_0 of purely infinite simple regular rings, *K-Theory* **26** (2000), 69-100.

[4] J. Cuntz, K-Theory for certain C*-algebras, *Ann. Math.* **113** (1981), 181-197.

[5] J.A. Jeong, Purely infinite simple C*-crossed products, *Proc. Amer. Math. Soc.* **123(10)** (1995), 3075-3078.

[6] J.A. Jeong, K. Kodaka, H. Osaka, Purely infinite simple C*-crossed products II, *Canad, Math. Bull.* **39(2)** (1996), 203-210.

[7] S. Montgomery, "Fixed Rings of Finite Automorphism Groups of Associative Rings", *Lecture Notes in Math.* **818**, Springer-Verlag, Berlin, 1980.

[8] J. Osterburg, Smash products and *G*-Galois actions, *Proc. Amer. Math. Soc.* **98** (1986), 217-221.

[9] D.S. Passman, "Infinite crossed products", *Pure and Applied Mathematics* **135**, Academic Press, London, 1989.

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