

TOWARDS A K-THEORETIC CHARACTERIZATION OF GRADED ISOMORPHISMS BETWEEN LEAVITT PATH ALGEBRAS

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ABSTRACT. In [16], Hazrat gave a K-theoretic invariant for Leavitt path algebras as graded algebras. Hazrat conjectured that this invariant classifies Leavitt path algebras up to graded isomorphism, and proved the conjecture in some cases. In this paper, we prove that a weak version of the conjecture holds for all finite essential graphs.

1. INTRODUCTION

Leavitt path algebras of row-finite graphs have been introduced in [1] and [7]. They have become a subject of significant interest, both for algebraists and for analysts working in C^* -algebras. The Cuntz-Krieger algebras $C^*(E)$ (the C^* -algebra counterpart of these Leavitt path algebras) are described in [27]. While sharing some striking similarities, the algebraic and analytic theories present some remarkable differences, as has been shown for instance in [30], [8], [5] (see also [10]).

For a field K , the algebras $L_K(E)$ are natural generalizations of the algebras investigated by Leavitt in [20], and are a specific type of path K -algebras associated to a graph E (modulo certain relations). The family of algebras which can be realized as the Leavitt path algebras of a graph includes matrix rings $M_n(K)$ for $n \in \mathbb{N} \cup \{\infty\}$ (where $M_\infty(K)$ denotes matrices of countable size with only a finite number of nonzero entries), the Toeplitz algebra, the Laurent polynomial ring $K[x, x^{-1}]$, and the classical Leavitt algebras $L(1, n)$ for $n \geq 2$. Constructions such as direct sums, direct limits, and matrices over the previous examples can be also realized in this setting.

A great deal of effort has been focused on trying to unveil the algebraic structure of $L_K(E)$ via the graph nature of E . Concretely, the literature on Leavitt path algebras includes necessary and sufficient conditions on a graph E so that the corresponding Leavitt path algebra $L_K(E)$ is simple [1], purely infinite simple [2] or exchange [9]. Another remarkable approach has been the research of their monoids of finitely generated projective modules $V(L_K(E))$ [7] and the computation of its algebraic K-theory [4]. The availability of these data, and the tight connection between properties of Leavitt path algebras and those of graph C^* -algebras suggested the convenience of studying classification results for (purely infinite simple) Leavitt path algebras using K-theoretical invariants, in a similar way to that

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of [19, 26] for the case of Kirchberg algebras, and in particular for purely infinite simple graph C^* -algebras. In this direction, important connections between Leavitt path algebras, graph C^* -algebras and symbolic dynamics have been recently investigated by various authors, see e.g. [3], [17], [22] and [23]. In particular, substantial advances on the above classification problem have been undertaken in [3], but at this moment there is no complete classification result available yet.

The graded structure of Leavitt path algebras has been an important tool for studying the ideal structure of Leavitt path algebras, as well as suitable versions of the Uniqueness Theorems for graph C^* -algebras in this context [30]. Also, it becomes relevant for the recent work of S. Paul Smith on noncommutative geometry [29]. Very recently, Hazrat [16, 17] proposed an approach to the classification problem when the equivalence relation considered is that of graded isomorphisms between algebras. After introducing a suitable notion of graded K_0 -group, he proved that for a strongly \mathbb{Z} -graded algebra A the groups $K_0^{gr}(A)$ and $K_0(A_0)$ have a natural structure of $\mathbb{Z}[x, x^{-1}]$ -module, and also that, thanks to Dade's Theorem, we have $K_0^{gr}(A) \cong K_0(A_0)$ as ordered $\mathbb{Z}[x, x^{-1}]$ -modules. Hazrat conjectures in [16, Conjecture 1] that, given row-finite graphs E and F , we have that $L(E) \cong_{gr} L(F)$ if and only if there is an order-preserving $\mathbb{Z}[x, x^{-1}]$ -module isomorphism $(K_0^{gr}(L(E)), [1_{L(E)}]) \cong (K_0^{gr}(L(F)), [1_{L(F)}])$. Moreover, he proves that the conjecture holds for finite graphs being acyclic, comet or, more generally, polycephalic [16, Theorem 9].

In this paper we prove that a weak version of Hazrat's conjecture holds for finite graphs with neither sources nor sinks. Namely we prove that, under the above hypothesis on the graded K_0 -groups, the algebra $L(F)$ is graded-isomorphic to a certain deformation $L^g(E)$ of the algebra $L(E)$. (See the beginning of Section 4 for the precise definition of $L^g(E)$.) We also obtain a description of the automorphisms of $L(E)$ which induce the identity on K_0^{gr} , for the same class of graphs (Theorem 6.5). This is closely related to the uniqueness part of [16, Conjecture 3].

The article is organized as follows. Section 2 includes the basic definitions and examples that will be used throughout. In Section 3 we will show that Hazrat's invariant is equivalent to other K-theoretic invariants, and also to an invariant used for classifying subshifts of finite type (see [31]). This section also contains lifting results of certain isomorphisms between the graded K_0 -groups to graded Morita equivalences and graded isomorphisms, respectively (see Theorem 3.12 and Theorem 3.15). These results confirm the existence part of [16, Conjecture 3] in the case where the isomorphism between the K_0^{gr} -groups is induced by a strong shift equivalence (see also [17, Proposition 15(2)]). In Section 4 we state the main result of the paper and fix the strategy to prove it. The hard part of the proof is contained in Section 5. Section 6 contains a study of the uniqueness of liftings of maps between the K-theoretic invariants.

2. PRELIMINARES

We briefly recall some graph-theoretic definitions and properties; more complete explanations and descriptions can be found in [1]. A *graph* $E = (E^0, E^1, r, s)$ consists of two sets E^0, E^1 and maps $r, s : E^1 \rightarrow E^0$. (Some authors use the phrase 'directed' graph for this structure.) The elements of E^0 are called *vertices* and the elements of E^1 *edges*. An edge $e \in E^1$ is said to point from $s(e)$ to $r(e)$. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the

graph is called *row-finite*. A vertex v for which $s^{-1}(v)$ is empty is called a *sink*; a vertex w for which $r^{-1}(w)$ is empty is called a *source*. If F is a subgraph of E , then F is called *complete* in case $s_F^{-1}(v) = s_E^{-1}(v)$ for every $v \in F^0$ having $s_F^{-1}(v) \neq \emptyset$.

A *path* μ in a graph E is a sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, $s(\mu) := s(e_1)$ is the *source* of μ , $r(\mu) := r(e_n)$ is the *range* of μ , and n is the *length* of μ . An edge e is an *exit* for a path $\mu = e_1 \dots e_n$ if there exists i such that $s(e) = s(e_i)$ and $e \neq e_i$. If μ is a path in E , and if $v = s(\mu) = r(\mu)$, then μ is called a *closed path based at v* . If $\mu = e_1 \dots e_n$ is a closed path based at $v = s(\mu)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then μ is called a *cycle*.

The following notation is standard. Let A be a $p \times p$ matrix having non-negative integer entries (i.e., $A = (a_{ij}) \in M_p(\mathbb{Z}^+)$). The graph E_A is defined by setting $(E_A)^0 = \{v_1, v_2, \dots, v_p\}$, and defining $(E_A)^1$ by inserting exactly a_{ij} edges in E_A having source vertex v_i and range vertex v_j . Conversely, if E is a finite graph with vertices $\{v_1, v_2, \dots, v_p\}$, then we define the *adjacency matrix* A_E of E by setting $(A_E)_{ij}$ as the number of edges in E having source vertex v_i and range vertex v_j .

Definition 2.1. Let E be any row-finite graph, and K any field. The *Leavitt path K -algebra* $L_K(E)$ of E with coefficients in K is the K -algebra generated by a set $\{v \mid v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of variables $\{e, e^* \mid e \in E^1\}$, which satisfy the following relations:

- (1) $s(e)e = er(e) = e$ for all $e \in E^1$.
- (2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$.
- (3) (The ‘‘CK1 relations’’) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$.
- (4) (The ‘‘CK2 relations’’) $v = \sum_{\{e \in E^1 \mid s(e)=v\}} ee^*$ for every vertex $v \in E^0$ for which $s^{-1}(v)$ is nonempty.

When the role of the coefficient field K is not central to the discussion, we will often denote $L_K(E)$ simply by $L(E)$. The set $\{e^* \mid e \in E^1\}$ will be denoted by $(E^1)^*$. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $\mu = e_1 \dots e_n$ is a path, then we denote by μ^* the element $e_n^* \dots e_1^*$ of $L_K(E)$.

An alternate description of $L_K(E)$ is given in [1], where it is described in terms of a free associative algebra modulo the appropriate relations indicated in Definition 2.1 above. As a consequence, if A is any K -algebra which contains a set of elements satisfying these same relations (we call such a set an *E -family*), then there is a (unique) K -algebra homomorphism from $L_K(E)$ to A mapping the generators of $L_K(E)$ to their appropriate counterparts in A .

If E is a finite graph then $L_K(E)$ is unital, with $\sum_{v \in E^0} v = 1_{L_K(E)}$. Conversely, if $L_K(E)$ is unital, then E^0 is finite. If E^0 is infinite then $L_K(E)$ is a ring with a set of local units; one such set of local units consists of sums of distinct elements of E^0 . There is a canonical \mathbb{Z} -grading on $L_K(E)$, which is given by $L_K(E) = \bigoplus_{n \in \mathbb{Z}} L_K(E)_n$, where, for each $n \in \mathbb{Z}$, the *degree n component* $L_K(E)_n$ is spanned by elements of the form $\{pq^* \mid \text{length}(p) - \text{length}(q) = n\}$. The set of *homogeneous elements* is $\bigcup_{n \in \mathbb{Z}} L_K(E)_n$, and an element of $L_K(E)_n$ is said to be *n -homogeneous* or *homogeneous of degree n* . The K -linear extension of the assignment $pq^* \mapsto qp^*$ (for p, q paths in E) yields an involution on $L_K(E)$, which we denote simply as $*$. Information regarding the ‘‘ C^* -algebra of a graph’’, also known as the ‘‘Cuntz-Krieger graph C^* -algebra’’, may be found in [27]. In particular, the graph C^* -algebra $C^*(E)$ of a graph E is the completion of $L_C(E)$ in a suitable norm [27].

Recall that for a ring R , we denote by $K_0(R)$ the Grothendieck group of R . This is the group F/S , where F is the free group generated by isomorphism classes of finitely generated projective left R -modules, and S is the subgroup of F generated by symbols of the form $[P \oplus Q] - [P] - [Q]$. As is standard, we denote the isomorphism class of R in $K_0(R)$ by $[1_R]$; we will call it the order-unit of the K_0 -group. The group $K_0(R)$ is the universal group of the monoid $V(R)$ of isomorphism classes of finitely generated projective left R -modules (with binary operation in $V(R)$ given by $[A] + [B] = [A \oplus B]$).

For a row-finite graph E , the *monoid of E* , denoted M_E , is the monoid generated by the set E^0 of vertices of E modulo appropriate relations, specifically,

$$M_E = \langle a_v, v \in E^0 \mid a_v = \sum_{\{e \in s^{-1}(v)\}} a_{r(e)}, \text{ for } v \in E^0 \text{ which is not a sink} \rangle.$$

It is shown in [7, Theorem 2.5] that $V(L(E)) \cong M_E$ for any row-finite graph E . This yields $K_0(L(E)) \cong \text{Grot}(M_E) := G$, where $\text{Grot}(M_E)$ denotes the universal group of the monoid M_E . If E is finite, then M_E is finitely generated, hence so is its universal group G . Thus G admits a presentation $\pi : \mathbb{Z}^n \rightarrow G$ (an epimorphism). Here $\ker(\pi)$ is the subgroup of relations, which, in case E does not have sinks, corresponds to the image of the group homomorphism $I - A_E^t : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$, where A_E^t is the transpose of the incidence matrix A_E of E . Hence we get

$$K_0(L(E)) \cong G \cong \mathbb{Z}^n / \ker(\pi) = \mathbb{Z}^n / \text{im}(I - A_E^t) = \text{coker}(I - A_E^t).$$

Moreover, under this isomorphism the element $[1_{L(E)}]$ is represented by $(1, 1, \dots, 1)^t + \text{im}(I - A_E^t)$ in $\text{coker}(I - A_E^t)$.

Now, we will recall the notion of fractional skew monoid ring [6] in the particular case of the monoid being \mathbb{Z}^+ . Let A be a unital ring, let $p^2 = p \in A$ be an idempotent, and let $\alpha : A \rightarrow pAp$ be an isomorphism. Then the fractional skew monoid ring of A over \mathbb{Z}^+ by α is the ring R generated by an isomorphic copy of A and two generators t_+, t_- satisfying the following relations:

- (1) $t_- t_+ = 1$ and $t_+ t_- = p$;
- (2) $t_+^n a = \alpha^n(a) t_+^n$ for all $a \in A, n \in \mathbb{Z}^+$;
- (3) $a t_-^n = t_-^n \alpha^n(a)$ for all $a \in A, n \in \mathbb{Z}^+$.

We denote $R := A[t_+, t_-; \alpha]$. By [6, 2.2] the elements $r \in R$ can all be written as ‘polynomials’ of the form

$$r = a_n t_+^n + \dots + a_1 t_+ + a_0 + t_- a_{-1} + \dots + t_-^m a_{-m},$$

with coefficients $a_i \in A$. By [6, Proposition 1.6], R is a \mathbb{Z} -graded ring $R = \bigoplus_{i \in \mathbb{Z}} R_i$, with $R_i = A t_+^i$ for $i > 0$ and $R_i = t_-^{-i} A$ for $i < 0$, while $A_0 = A$. This construction is an exact algebraic analog of the construction of the crossed product of a C^* -algebra by an endomorphism introduced by Paschke [25]. In fact, if A is a C^* -algebra and the corner isomorphism α is a $*$ -homomorphism, then Paschke’s C^* -crossed product, which he denotes by $A \rtimes_{\alpha} \mathbb{N}$, is just the completion of $A[t_+, t_-; \alpha]$ in a suitable norm. If E is a finite graph with no sources, then it is known that $L(E) = L(E)_0[t_+, t_-; \alpha]$ for suitable elements $t_+, t_- \in L(E)$, being α the corner isomorphism of $L(E)_0$ defined by the rule $\alpha(a) = t_+ a t_-$ for all $a \in L(E)_0$ [6].

3. EQUIVALENT FORMS OF HAZRAT'S INVARIANT

In this section, we will relate the invariant proposed by Hazrat for classifying Leavitt path algebras up to graded isomorphism to a couple of K-theoretic invariants, and to an invariant (the so-called dimension triple) which appears in the classification of subshifts of finite type.

Let Γ be a group and let A be a Γ -graded ring. We will denote by $\text{Gr-}A$ the category of graded A -modules and graded homomorphisms, see e.g. [18]. A graded right A -module is projective in the category $\text{Mod-}A$ of right A -modules if and only if it is projective in the category $\text{Gr-}A$ of graded right A -modules (see e.g. [18, Proposition 1.2.13]). Recall from [16, 2.5] that if Γ is a group and A is a Γ -graded ring, then the graded Grothendieck group $K_0^{gr}(A)$ is defined as the universal enveloping group of the abelian monoid of isomorphism classes of graded finitely generated projective modules over A . Also, for $g \in \Gamma$, the g -suspension functor $\mathcal{T}_g : \text{Gr-}A \rightarrow \text{Gr-}A$ (defined by the rule $(\mathcal{T}_g(M))_h = M_{hg}$) is an isomorphism with the property that $\mathcal{T}_g \mathcal{T}_h = \mathcal{T}_{gh}$ for $g, h \in \Gamma$. Moreover, \mathcal{T}_g restricts to the category of graded finitely generated projective A -modules, and hence it induces a structure of $\mathbb{Z}[\Gamma]$ -module on $K_0^{gr}(A)$ by the rule $g[P] := [\mathcal{T}_g(P)]$.

By Dade's Theorem [24, Theorem 3.1.1], if Γ is a group and A is a strongly Γ -graded ring, then the functor $(-)_0 : \text{Gr-}A \rightarrow \text{Mod-}A_0$ (defined by the rule $M \mapsto M_0$) is an additive functor with inverse $- \otimes_{A_0} A : \text{Mod-}A_0 \rightarrow \text{Gr-}A$, so that it induces an equivalence of categories. Furthermore, since $A_g \otimes_{A_0} A_h \cong A_{gh}$ for $g, h \in \Gamma$ as A_0 -bimodules, the suspension functor $\mathcal{T}_g : \text{Gr-}A \rightarrow \text{Gr-}A$ induces a functor $\mathcal{T}_g : \text{Mod-}A_0 \rightarrow \text{Mod-}A_0$, given by the rule $M \mapsto M \otimes_{A_0} A_g$, such that $\mathcal{T}_g \mathcal{T}_h = \mathcal{T}_{gh}$ for $g, h \in \Gamma$ making the following diagram commutative:

$$\begin{array}{ccc} \text{Gr-}A & \xrightarrow{\mathcal{T}_g} & \text{Gr-}A \\ \downarrow (-)_0 & & \downarrow (-)_0 \\ \text{Mod-}A_0 & \xrightarrow{\mathcal{T}_g} & \text{Mod-}A_0 \end{array}$$

Therefore $K_i(A_0)$ is also a $\mathbb{Z}[\Gamma]$ -module, and $K_i^{gr}(A) \cong K_i(A_0)$ as $\mathbb{Z}[\Gamma]$ -modules for $i \geq 0$.

In the case of Leavitt path algebras, we will specialize the above situation to the natural \mathbb{Z} -grading, so that $\mathbb{Z}[\Gamma]$ will be $\mathbb{Z}[x, x^{-1}]$, and in particular we only need to take care of \mathcal{T}_1 (which represents $x \cdot$) to have a complete picture of the action.

Recall that two Γ -graded rings A and B are said to be *graded Morita equivalent* in case there exists an equivalence of categories $F : \text{Gr-}A \rightarrow \text{Gr-}B$ such that $F \circ \mathcal{T}_g = \mathcal{T}_g \circ F$ for all $g \in \Gamma$. By [15, Theorem 5.4], A and B are graded Morita equivalent if and only if there is a graded right A -module P , which is a finitely generated projective generator in $\text{Mod-}A$, such that $\text{End}_A(P)$ and B are isomorphic as graded rings.

Given a graded equivalence $F : \text{Gr-}A \rightarrow \text{Gr-}B$, there is an induced ordered $\mathbb{Z}[\Gamma]$ -module homomorphism $K_0^{gr}(F) : K_0^{gr}(A) \rightarrow K_0^{gr}(B)$ defined by $K_0^{gr}(F)([X]) = [F(X)]$ for a finitely generated graded projective module X . Note that $K_0^{gr}(F) = K_0^{gr}(G)$ if $F \cong G$. If $\phi : A \rightarrow B$ is a graded isomorphism then ϕ induces an obvious graded Morita equivalence Φ such that $K_0^{gr}(\Phi) = K_0^{gr}(\phi)$.

Examples 3.1. (1) Let e be a homogeneous full idempotent in the Γ -graded ring A , then A is graded Morita equivalent to eAe , cf. [18, Example 2.3.1].

(2) Assume that A, B are Γ -graded rings and $F: \text{Gr-}A \rightarrow \text{Gr-}B$ is a graded equivalence. Set $P = F(A_A)$. Then P has a natural structure of graded A - B -bimodule, and $F \cong - \otimes_A P$, see e.g. [18, Theorem 2.3.5]. Now assume that $P_B \cong B_B$ as graded modules. We can give to B a structure of A - B -bimodule such that the above is a graded isomorphism of bimodules. Therefore $F \cong - \otimes_A B$, and in particular we obtain a graded ring isomorphism ϕ as the composition of the graded isomorphisms

$$A \rightarrow \text{End}_A(A) \rightarrow \text{End}_B(A \otimes_A B) \rightarrow \text{End}_B(B) \rightarrow B.$$

It is easily checked that the Morita equivalence induced by ϕ is exactly $- \otimes_A B$. Therefore

$$K_0^{gr}(F) = K_0^{gr}(- \otimes_A P) = K_0^{gr}(- \otimes_A B) = K_0^{gr}(\phi).$$

We now state Hazrat's main conjecture.

Conjecture 3.2. [16, Conjecture 1] Given finite graphs E and F , we have that $L(E) \cong_{gr} L(F)$ if and only if

$$(K_0^{gr}(L(E)), [1_{L(E)}]) \cong_{\mathbb{Z}[x, x^{-1}]} (K_0^{gr}(L(F)), [1_{L(F)}]).$$

Hazrat proves that the conjecture holds for acyclic, comet and polycephalic finite graphs [16, Theorem 9]. The isomorphism appearing in the Conjecture is assumed to preserve the order in both directions, to commute with the action of \mathcal{T}_* , and to take the canonical order-unit $[1_{L(E)}]$ of $K_0^{gr}(L(E))$ to the canonical order-unit $[1_{L(F)}]$ of $K_0^{gr}(L(F))$. These are the isomorphisms in the category of triples (G, G^+, u) , where G is a $\mathbb{Z}[x, x^{-1}]$ -module, G^+ is a positive cone in G , and u is a distinguished order-unit in G (see [18, 3.5]). This will be referred to later as the order-unit preserving category.

Because our tools require the use of the structure of strongly \mathbb{Z} -graded rings, we will need to ask our graphs to contain no **sinks** [16, Theorem 4]. Also, we will consider a different picture of the \mathbb{Z} -graded structure of a Leavitt path algebra $L(E)$ over a finite graph E , which links grading with the existence of special non necessarily unital endomorphisms of $L(E)_0$. To be concrete, assume that E is a finite graph that contains no sources. If $E^0 = \{v_1, \dots, v_n\}$, then for each $1 \leq i \leq n$ there exists at least one $e_i \in E^1$ such that $r(e_i) = v_i$. If we define $t_+ = \sum_{i=1}^n e_i \in L(E)_1$ and $t_- = \sum_{i=1}^n e_i^* \in L(E)_{-1}$, then it is easy to see that $t_- t_+ = 1$. Hence, by [6, Lemma 2.4],

$$L(E) = L(E)_0[t_+, t_-; \alpha].$$

Since this picture of $L(E)$ plays a central role in our arguments, from now on we will ask our graphs to have no **sources**. Moreover, if $p := t_+ t_-$, then $\alpha: L(E)_0 \rightarrow pL(E)_0 p$ is a corner isomorphism defined by the rule $\alpha(a) = t_+ a t_-$. Under this picture, $L(E)_n = L(E)_0 t_+^n = L(E)_0 p_n t_+^n$ and $L(E)_{-n} = t_-^n L(E)_0 = t_-^n p_n L(E)_0$ for $n > 0$, where $p_n = \alpha^n(1)$. Moreover, since $L(E)_0$ is a unital ultramatricial algebra (see e.g. [7, Proof of Theorem 5.3]), we can classify it via its Bratteli diagram X_A . This Bratteli diagram can be described fixing at each level of the diagram the same number of vertices, namely $|E^0|$ vertices at each level, while the edges between two different levels are codified by the entries of the adjacency matrix $A := A_E$ of the graph E .

The following standard definition will be useful in order to shorten our statements.

Definition 3.3. An *essential graph* is a graph with neither sources nor sinks.

We will show that the K-theoretic part of [16, Conjecture 1] can be rephrased in equivalent forms which are more useful to relate the K-theoretic isomorphisms with the behavior of the graded structure of the Leavitt path algebras involved.

Given any $M \in \text{Mod-}L(E)_0$, we will define a right $L(E)_0$ -module M^α as follows:

- (1) As abelian group, $M^\alpha := Mp$;
- (2) The action is defined by the rule

$$\begin{aligned} M^\alpha \times L(E)_0 &\longrightarrow M^\alpha \\ (m, a) &\longmapsto m * a := m\alpha(a) \end{aligned}$$

Notice that, whenever $m \in M^\alpha$, $m * 1 = m\alpha(1) = m \cdot p = m$, so that M^α is a unital module.

Lemma 3.4. *The right $L(E)_0$ -modules M^α and $M \otimes_{L(E)_0} L(E)_0 t_+$ are isomorphic.*

Proof. Since $pt_+ = t_+$, the module $M \otimes_{L(E)_0} L(E)_0 t_+$ is unital. Moreover,

$$M \otimes_{L(E)_0} L(E)_0 t_+ = M \otimes_{L(E)_0} L(E)_0 pt_+.$$

Hence, the map

$$\begin{aligned} M \otimes_{L(E)_0} L(E)_0 pt_+ &\xrightarrow{f} M^\alpha \\ m \otimes (xp)t_+ &\longmapsto mxp \end{aligned}$$

is a group isomorphism. Also, for any $r \in L(E)_0$,

$$f((m \otimes (xp)t_+)r) = f(m \otimes (xp)\alpha(r)t_+) = (mxp)\alpha(r) = (mxp) * r = f(m \otimes (xp)t_+) * r$$

which ends the proof. \square

Define $\mathcal{T} := \mathcal{T}_1 : \text{Mod-}L(E)_0 \rightarrow \text{Mod-}L(E)_0$ by

$$\mathcal{T}(M) = M \otimes_{L(E)_0} L(E)_1 = M \otimes_{L(E)_0} L(E)_0 t_+.$$

This corresponds to the translation functor when $L(E)$ is strongly graded, i.e., when E has no sinks ([16, Theorem 4]). By Lemma 3.4, the induced isomorphism

$$\mathcal{T}_* : K_0(L(E)_0) \rightarrow K_0(L(E)_0)$$

is given by the rule $\mathcal{T}_*([P]) = [P^\alpha]$.

We will make use of the following well-known (and easy to prove) lemma:

Lemma 3.5. *Let F be a finite graph. Then $p = t_+ t_-$ is a full idempotent in $L(F)_0$ if and only if F does not have sinks.*

Assume that E is an essential graph. It follows from Lemma 3.5 and the fact that $\alpha : L(E)_0 \rightarrow pL(E)_0 p$ is an isomorphism that the induced map

$$\alpha_* : K_0(L(E)_0) \rightarrow K_0(L(E)_0)$$

is a group isomorphism.

Lemma 3.6. *If E is a finite essential graph, then the maps α_* and \mathcal{T}_* are mutually inverse isomorphisms.*

Proof. Since both maps are group isomorphisms, it suffices to show that

$$\text{Id}_{K_0(L(E)_0)} = \mathcal{T}_* \circ \alpha_*.$$

Let $e^2 = e \in L(E)_0$. Then

$$(\mathcal{T}_* \circ \alpha_*)([eL(E)_0]) = \mathcal{T}_*([\alpha(e)L(E)_0]) = [(\alpha(e)L(E)_0)^\alpha].$$

The map

$$\begin{array}{ccc} eL(E)_0 & \xrightarrow{g} & (\alpha(e)L(E)_0)^\alpha \\ x & \longmapsto & \alpha(x) \end{array}$$

is well-defined and additive. As $\alpha : L(E)_0 \rightarrow pL(E)_0p$ is a corner isomorphism, g is injective. Now, let $y \in (\alpha(e)L(E)_0)^\alpha$. Thus, $y = \alpha(e)rp$ for some $r \in L(E)_0$. Then,

$$y = \alpha(e)rp = \alpha(e \cdot 1)rp = \alpha(e)prp$$

and since $prp \in pL(E)_0p = \alpha(L(E)_0)$, there exists a unique $s \in L(E)_0$ such that $\alpha(s) = prp$, whence

$$\alpha(e)prp = \alpha(e)\alpha(s) = \alpha(es)$$

and hence $y = g(es)$, so that g is onto.

Finally, for any $x \in eL(E)_0$ and any $r \in L(E)_0$,

$$g(xr) = \alpha(xr) = \alpha(x)\alpha(r) = \alpha(x) * r = g(x) * r,$$

Thus $[eL(E)_0] = [(\alpha(e)L(E)_0)^\alpha]$ in $K_0(L(E)_0)$, so that $(\mathcal{T}_* \circ \alpha_*)([eL(E)_0]) = [eL(E)_0]$ for all idempotents e in $L(E)_0$. Since $K_0(L(E)_0)$ is generated by the elements $[eL(E)_0]$, for $e = e^2 \in L(E)_0$, we conclude that $\mathcal{T}_* \circ \alpha_* = \text{Id}_{K_0(L(E)_0)}$. \square

As a consequence we have the following result.

Proposition 3.7. *Let E, F be finite essential graphs, and α, β the respective corner isomorphisms. Then, the following are equivalent:*

- (1) *There exists an isomorphism $\Psi : (K_0(L(E)_0), K_0^+(L(E)_0)) \rightarrow (K_0(L(F)_0), K_0^+(L(F)_0))$ such that $\Psi \circ (\mathcal{T}_E)_* = (\mathcal{T}_F)_* \circ \Psi$*
- (2) *There exists an isomorphism $\Phi : (K_0(L(E)_0), K_0^+(L(E)_0)) \rightarrow (K_0(L(F)_0), K_0^+(L(F)_0))$ such that $\Phi \circ \alpha_* = \beta_* \circ \Phi$*
- (3) *There exists an isomorphism of ordered $\mathbb{Z}[x, x^{-1}]$ -modules $K_0^{gr}(L(E)) \cong K_0^{gr}(L(F))$.*

Moreover, the result holds also in the order-unit preserving category. \square

Notation. We will denote the condition in Proposition 3.7(1) as

$$(K_0(L(E)_0), K_0^+(L(E)_0)) \cong_{\mathbb{Z}[x, x^{-1}]} (K_0(L(F)_0), K_0^+(L(F)_0)).$$

and that in Proposition 3.7(2) as

$$(K_0(L(E)_0), K_0^+(L(E)_0), \alpha_*) \cong (K_0(L(F)_0), K_0^+(L(F)_0), \beta_*).$$

In order to give the next equivalent condition, we need to recall some facts about conjugacy of subshifts of finite type [31]. Recall [21, Definition 7.3.1] that given A, B nonnegative integral square matrices, and $\ell \geq 1$, a shift equivalence of lag ℓ from A to B is a pair of rectangular nonnegative integral matrices (S, R) satisfying the four shift equivalence equations:

- (1) (i) $AR = RB$; (ii) $SA = BS$.
- (2) (i) $A^\ell = RS$; (ii) $B^\ell = SR$.

We will denote this equivalence by $A \sim_{SE} B$.

In order to relate this equivalence with our problem, recall that if E is a finite graph with no sinks, and $A := A_E^t$ denotes the transpose of the adjacency matrix, then the ultramatricial algebra $L(E)_0$ is a direct limit of matricial algebras $L(E)_{0,n}$ ($n \geq 0$) whose connecting maps are induced by the (CK2) relation (see e.g. [7, Proof of Theorem 5.3]). Moreover, by the picture of the Bratteli diagram associated to this ultramatricial algebra, $K_0(L(E)_0) \cong \varinjlim (K_0(L(E)_{0,n}), A)$, where $K_0(L(E)_{0,n}) \cong \mathbb{Z}^N$ for all $n \geq 0$, and $N = |E^0|$. We will denote $G_m := K_0(L(E)_{0,m})$ ($m \geq 0$) and $G := K_0(L(E)_0)$. Then, we have a commutative diagram

$$\begin{array}{ccccccc}
 G_0 & \xrightarrow{A} & G_1 & \xrightarrow{A} & G_2 & \xrightarrow{A} & G_3 \cdots \cdots \cdots \rightarrow G \\
 \downarrow A & & \downarrow A & & \downarrow A & & \downarrow A & & \downarrow \delta_A \\
 G_0 & \xrightarrow{A} & G_1 & \xrightarrow{A} & G_2 & \xrightarrow{A} & G_3 \cdots \cdots \cdots \rightarrow G
 \end{array}$$

that induces a group isomorphism $\delta_A : G \rightarrow G$ by multiplication by A on the direct limit.

With that in mind, we can show the following result

Lemma 3.8. *Let E be a finite essential graph, let A be the transpose of the incidence matrix of E , and let δ_A and α_* be the automorphisms of $K_0(L(E)_0)$ defined above. Then, α_* is the inverse of δ_A .*

Proof. Let $G = \varinjlim (G_n, A)$, where $G_n = \mathbb{Z}^{|E^0|}$ for all $n \geq 0$. The map corresponding to $\alpha : L(E)_0 \rightarrow L(E)_0$ at each step of the direct limit is the identity map $\text{id} : G_k \rightarrow G_{k+1}$. Hence, we have a commutative diagram

$$\begin{array}{ccccccc}
 G_0 & \xrightarrow{A} & G_1 & \xrightarrow{A} & G_2 & \xrightarrow{A} & G_3 \cdots \cdots \cdots \rightarrow G \\
 \downarrow A & \nearrow \text{id} & \downarrow A & \nearrow \text{id} & \downarrow A & \nearrow \text{id} & \downarrow A & & \downarrow \delta_A \\
 G_0 & \xrightarrow{A} & G_1 & \xrightarrow{A} & G_2 & \xrightarrow{A} & G_3 \cdots \cdots \cdots \rightarrow G \\
 & & & & & & & & \uparrow \alpha_*
 \end{array}$$

Since both maps are automorphisms of $K_0(L(E)_0)$, it suffices to show that $\delta_A \circ \alpha_* = \text{Id}_G$. Following the diagram,

$$(\delta_A \circ \alpha_*)(\iota_{n,\infty}(g)) = \delta_A(\alpha(\iota_{n,\infty}(g))) = \delta_A(\iota_{n+1,\infty}(g)) = \iota_{n+1,\infty}(Ag) = \iota_{n+1,\infty}\iota_{n,n+1}(g) = \iota_{n,\infty}(g),$$

where $\iota_{n,\infty} : G_n \rightarrow G$ is the canonical map. This concludes the proof. \square

Remark 3.9. It follows from Lemmas 3.6 and 3.8 that $\mathcal{T}_* = \delta_A$ when E is a finite essential graph. In fact, it is easy to prove directly that this equality holds for any finite graph without sinks: Setting $A_i = L(E)_i$, we have, for $v \in E^0$,

$$\mathcal{T}_*([vA_0]) = [vA_0 \otimes_{A_0} A_1] = [vA_1] = \sum_{e \in s^{-1}(v)} [eA_0] = \sum_{e \in s^{-1}(v)} [r(e)A_0].$$

Similarly, if $|\gamma| \geq 1$, we have

$$\mathcal{T}_*([\gamma\gamma^*A_0]) = [\gamma\gamma^*A_1] = \sum_{e \in s^{-1}(r(\gamma))} [\gamma ee^*\gamma^*A_1] = \sum_{e \in s^{-1}(r(\gamma))} [\gamma' ee^*(\gamma')^*A_0],$$

where $\gamma = f\gamma'$ for $f \in E^1$ and $|\gamma'| = |\gamma| - 1$. \square

It is important to notice that

$$(K_0(L(E)_0), K_0^+(L(E)_0), \delta_{A_E^t}) \cong (K_0(L(F)_0), K_0^+(L(F)_0), \delta_{A_F^t})$$

is equivalent to $A_E^t \sim_{SE} A_F^t$ with lag l for some $l \geq 1$ [21, Theorem 7.5.8]. Now, we can state the main result of this section

Theorem 3.10. *Let E, F be finite essential graphs, let α, β be the respective corner isomorphisms. Set $A = A_E^t$ and $B = A_F^t$, and let δ_A, δ_B be the automorphisms of $K_0(L(E)_0)$ and $K_0(L(F)_0)$ defined above. Then, the following are equivalent:*

- (1) $A \sim_{SE} B$.
- (2) $(K_0(L(E)_0), K_0^+(L(E)_0)) \cong_{\mathbb{Z}[x, x^{-1}]} (K_0(L(F)_0), K_0^+(L(F)_0))$.
- (3) $(K_0(L(E)_0), K_0^+(L(E)_0), \alpha_*) \cong (K_0(L(F)_0), K_0^+(L(F)_0), \beta_*)$.
- (4) $(K_0(L(E)_0), K_0^+(L(E)_0), \delta_A) \cong (K_0(L(F)_0), K_0^+(L(F)_0), \delta_B)$.
- (5) *There is an ordered $\mathbb{Z}[x, x^{-1}]$ -module isomorphism $K_0^{gr}(L(E)) \cong K_0^{gr}(L(F))$.*

Moreover, the equivalences also hold in the order-unit preserving category. \square

For a non-negative matrix $A \in M_N(\mathbb{Z}^+)$, we denote by $(\Delta_A, \Delta_A^+, \delta_A)$ the dimension triple of A . Here $\Delta_A = \varinjlim (G_n, A)$, with $G_n = \mathbb{Z}^N$ for all n , and δ_A is induced by multiplication by A at each level, so that $K_0(L(E)_0) \cong \Delta_A$ and the dimension triple is isomorphic to $(K_0(L(E)_0), K_0^+(L(E)_0), \delta_A)$, where E is the graph with incidence matrix A^t .

Note that every shift equivalence $(S, R): A \sim_{SE} B$ of lag l , where $A \in M_N(\mathbb{Z}^+)$ and $B \in M_M(\mathbb{Z}^+)$, induces an isomorphism Φ of dimension triples such that $\Phi(\iota_{m+n, \infty}(x)) = \iota_{m+n, \infty}(Sx)$ for $x \in G_n = \mathbb{Z}^N$, where m is a fixed non-negative integer. The inverse map $\Psi: \Delta_B \rightarrow \Delta_A$ is defined by $\Psi(\iota_{m+n, \infty}(x)) = \iota_{m+n, \infty}(Rx)$ for $x \in \mathbb{Z}^M$ and $n \geq 0$. (See the diagram on [21, page 234] for an illustrative picture in the case $l = 2$.) Conversely, given an isomorphism of dimension triples $\Phi: (\Delta_A, \Delta_A^+, \delta_A) \rightarrow (\Delta_B, \Delta_B^+, \delta_B)$, there is a shift equivalence $(S, R): A \sim_{SE} B$ of lag l for some $l \geq 1$ and $m \geq 0$ such that Φ is the map described above (see [21, Theorem 7.5.8]).

Now we consider the special case given by a strong shift equivalence.

Let $A \in M_N(\mathbb{Z}^+)$ and $B \in M_M(\mathbb{Z}^+)$ be two non-negative matrices. An *elementary shift equivalence* $(S, R): A \approx B$ from A to B is a pair of non-negative integral matrices R and S such that $A = RS$ and $B = SR$. Note that this is just a shift equivalence of lag $l = 1$. A *strong shift equivalence of lag l* from A to B is a sequence of l elementary shift equivalences

$$(3.1) \quad (S_1, R_1): A = A_0 \approx A_1, (S_2, R_2): A_1 \approx A_2, \dots, (S_l, R_l): A_{l-1} \approx A_l = B,$$

see [21, Definition 7.2.1]. Note that each strong shift equivalence as above induces a shift equivalence of lag l from A to B , and thus an isomorphism Φ from the dimension triple $(\Delta_A, \Delta_A^+, \delta_A)$ associated to A to the dimension triple $(\Delta_B, \Delta_B^+, \delta_B)$ associated to B [21, 7.5], such that

$$\Phi(\iota_{n,\infty}(x)) = \iota_{m+n,\infty}(S_l \cdots S_1 x)$$

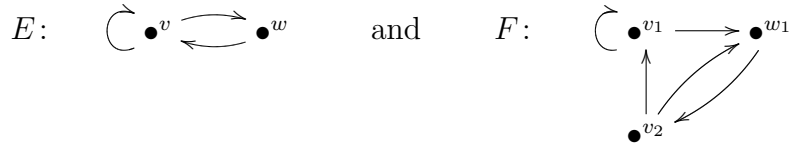
for $x \in \mathbb{Z}^N$, where m is a fixed non-negative integer. By Theorem 3.10, Φ induces an isomorphism, also denoted by Φ , from $K_0^{gr}(L(E))$ onto $K_0^{gr}(L(F))$. Letting $S = S_l \cdots S_1$, $R = R_l \cdots R_1$, we say that (S, R) is a strong shift equivalence from A to B . Two non-negative integral matrices A and B are said to be *strongly shift equivalent* if there is a strong shift equivalence from A to B .

In the situation above, we say that Φ is the morphism determined by $((S, R), m)$, or just by (S, m) . (Observe that Φ only depends on S and m .)

Some examples can be useful to clarify these concepts.

Examples 3.11. (1) Let E be an essential finite graph, and consider the corner isomorphism $\alpha: L(E) \rightarrow pL(E)p$ built before. Then α induces a graded Morita-equivalence from $L(E)$ to itself. This corresponds to the elementary shift equivalence (I_N, A) , with $m = 1 = l$, where $A = A_E^t$. The inverse is given by the elementary shift equivalence (A, I_N) with $m = 0$ and $l = 1$. It is realized by the graded Morita equivalence inverse to the one defined by α .

(2) Let

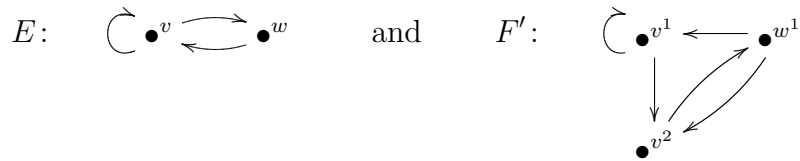


be the graphs considered in [3, Example 1.10], where F is obtained from E by an in-splitting. Then the corresponding graded Morita equivalence induces at the level

of K_0^{gr} a strong shift equivalence (S, R) , with $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$, $R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, with

$m = l = 1$. It is important to remark here that, since $m = 1$, the induced map $\Phi: K_0^{gr}(L(E)) \rightarrow K_0^{gr}(L(F))$ does not preserve the canonical order-units. This corresponds to the fact that the process described in [3] gives only a graded Morita equivalence in this case, but not an isomorphism. Considering the same pair (S, R) , but with $m = 0, l = 1$, one obtains another map $\Phi': K_0^{gr}(L(E)) \rightarrow K_0^{gr}(L(F))$ which does preserve the order-unit. We will show later that this map can be lifted to a graded isomorphism from $L(E)$ to $L(F)$.

(3) We consider now the graphs



arising in [3, Example 1.13]. The graph E is the same as the one considered in the previous example, but now the graph F' is obtained from E by an out-splitting instead

of an in-splitting. In this case the corresponding graded Morita equivalence is indeed a graded isomorphism. At the level of K_0^{gr} , it corresponds to the strong shift equivalence

$$(S, R), \text{ with } S = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \text{ with } m = 0, l = 1.$$

The first part of the following result has been proved by Hazrat [17, Proposition 15(2)].

Theorem 3.12. *Let E and F be finite essential graphs, and set $A = A_E^t$, $B = A_F^t$. If A and B are strongly shift equivalent then $L(E)$ and $L(F)$ are graded Morita equivalent. Moreover, if (S, R) is a strong shift equivalence from A to B and m is any non-negative integer, then there is a graded Morita equivalence $\Psi: \text{Gr-}L(E) \longrightarrow \text{Gr-}L(F)$ such that $K_0^{gr}(\Psi)$ is the isomorphism induced by $((S, R), m)$.*

Proof. Let (S, R) be a strong shift equivalence from A to B and let m be a non-negative integer. By [31], there is a finite set of finite essential graphs E_0, E_1, \dots, E_n , with $E_0 = E$ and $E_n = F$, such that each E_i is obtained from E_{i-1} by an out-splitting, or an in-splitting, or the inverse of one of these operations. (See also [21, Chapters 2 and 7]). Now there are matrices S_i, R_i corresponding to these operations, so that $A_{E_{i-1}}^t = R_i S_i$ and $A_{E_i}^t = S_i R_i$ for $i = 1, \dots, n$ and $S = S_n \cdots S_1$ and $R = R_n \cdots R_1$. By [3, Propositions 1.11 and 1.14] each of these operations induces a graded Morita equivalence

$$\Psi_i: \text{Gr-}L(E_{i-1}) \longrightarrow \text{Gr-}L(E_i)$$

such that the induced map $K_0^{gr}(\Psi_i)$ is the map corresponding to $((S_i, R_i), m_i)$ for some $m_i \geq 0$ (where we identify $(K_0^{gr}(L(E_i)), K_0^{gr}(L(E_i))^+, \mathcal{T}_*)$ with the corresponding dimension triple $(\Delta_{A_{E_i}^t}, \Delta_{A_{E_i}^t}^+, \delta_{A_{E_i}^t})$). Now $\Psi' := \Psi_n \circ \cdots \circ \Psi_1$ is a graded Morita equivalence from $\text{Gr-}L(E)$ to $\text{Gr-}L(F)$ and

$$K_0^{gr}(\Psi) = K_0^{gr}(\Psi_n) \cdots K_0^{gr}(\Psi_1) = (S_n \cdots S_1, m_1 + \cdots + m_n) = (S, m'),$$

where $m' = m_1 + \cdots + m_n$.

Now if $m' = m$ we have reached the desired map $((S, R), m)$. If $m' < m$, consider

$$((S_{n+1}, R_{n+1}), m - m') = ((I_M, B^{m-m'}), m - m'),$$

with lag $m - m'$, see Example 3.11(1). Set $m_{n+1} := m - m'$. Then the composition of all (S_i, m_i) , $i = 1, \dots, n+1$, is precisely the map from the dimension triple of A to the dimension triple of B given by (S, m) . Similarly, if $m' > m$, consider

$$((S_{n+1}, R_{n+1}), m_{n+1}) = ((B^{m'-m}, I_M), 0),$$

with lag $m' - m$, see Example 3.11(1). Then we again get that the composition of all (S_i, m_i) is precisely (S, m) .

In any case, by Example 3.11(1), there is a graded Morita equivalence $\Psi_{n+1}: \text{Gr-}L(F) \rightarrow \text{Gr-}L(F)$ such that $K_0^{gr}(\Psi_{n+1}) = (S_{n+1}, m_{n+1})$. Now set

$$\Psi := \Psi_{n+1} \circ \Psi': \text{Gr-}L(E) \longrightarrow \text{Gr-}L(F)$$

be the composition, which is a graded Morita equivalence. (Of course $\Psi_{n+1} = \text{Id}$ if $m = m'$.) Then clearly $K_0^{gr}(\Psi)$ is the map induced by $((S, R), m)$. \square

Now we can make the following definition.

Definition 3.13. Let E and F be finite essential graphs, and set $A = A_E^t$, $B = A_F^t$. We say that $L(E)$ and $L(F)$ are *strongly shift K_0^{gr} -isomorphic* if there exists a strong shift equivalence (S, R) from A to B and a non-negative integer m such that the induced isomorphism $\Phi: K_0^{gr}(L(E)) \rightarrow K_0^{gr}(L(F))$ satisfies that $\Phi([1_{L(E)}]) = [1_{L(F)}]$.

In this case we can deduce graded isomorphism using [31]. First, we need a lemma.

Lemma 3.14. *Let E be a finite graph without sinks. Assume that P, Q are graded finitely generated projective right $L(E)$ -modules such that $[P] = [Q]$ in $K_0^{gr}(L(E))$. Then P and Q are graded-isomorphic.*

Proof. Observe that $L(E)$ is strongly graded, so that $\text{Gr-}L(E)$ is equivalent to $\text{Mod-}L(E)_0$ by Dade's Theorem. It follows that there are finitely generated projective right $L(E)_0$ -modules P_0 and Q_0 such that $P \cong_{gr} P_0 \otimes_{L(E)_0} L(E)$ and $Q \cong_{gr} Q_0 \otimes_{L(E)_0} L(E)$, and $[P_0] = [Q_0]$ in $K_0(L(E)_0)$. Since $L(E)_0$ is an ultramatricial algebra, it has stable rank one, and so $P_0 \cong Q_0$. It follows that $P \cong_{gr} Q$. \square

Theorem 3.15. *Let E and F be finite essential graphs. If $L(E)$ and $L(F)$ are strongly shift K_0^{gr} -isomorphic then $L(E)$ and $L(F)$ are graded isomorphic. Moreover, let $\Phi: K_0^{gr}(L(E)) \rightarrow K_0^{gr}(L(F))$ be an isomorphism satisfying the conditions in Definition 3.13. Then there is a graded isomorphism $\phi: L(E) \rightarrow L(F)$ such that $K_0^{gr}(\phi) = \Phi$.*

Proof. By definition there is a strong shift equivalence (S, R) from $A := A_E^t$ to $B := A_F^t$ and a non-negative integer m such that the induced map $K_0^{gr}(L(E)) \rightarrow K_0^{gr}(L(F))$ is Φ .

By Theorem 3.12, there is a graded Morita equivalence

$$\Psi: \text{Gr-}L(E) \longrightarrow \text{Gr-}L(F)$$

such that $K_0^{gr}(\Psi) = \Phi$. We have

$$[1_{L(F)}] = \Phi([1_{L(E)}]) = K_0^{gr}(\Psi)([1_{L(E)}]).$$

Set $P := \Psi(L(E)_{L(E)})$. Then we get from the above that $[P] = [L(F)_{L(F)}]$ in $K_0^{gr}(L(F))$, so that, by Lemma 3.14, we get that $P \cong_{gr} L(F)_{L(F)}$. By Example 3.1(2), there is a graded isomorphism $\phi: L(E) \rightarrow L(F)$ such that $K_0^{gr}(\phi) = K_0^{gr}(\Psi)$. Since $K_0^{gr}(\Psi) = \Phi$ we obtain that $K_0^{gr}(\phi) = \Phi$, as desired. \square

Example 3.16. Let E and F be the graphs considered in Example 3.11(2). As stated there the map induced by $((S, R), 0)$ gives a strong shift equivalence Φ from the dimension triple of A to the dimension triple of B preserving the canonical order-units. By Theorem 3.15, there is a graded isomorphism $\phi: L(E) \rightarrow L(F)$ such that $K_0^{gr}(\phi) = \Phi$. Observe that, following the proof of this theorem, we need to compose the graded Morita equivalence given in the proof of [3, Proposition 1.11] with the inverse of the graded Morita equivalence induced by the corner isomorphism $\beta: L(F) \rightarrow pL(F)p$ corresponding to $L(F)$, in order to get a graded Morita equivalence taking $[1_{L(E)}]$ to $[1_{L(F)}]$.

Question 3.17. Let E and F be finite essential graphs. Assume there is an isomorphism

$$(K_0^{gr}(L(E)), K_0^{gr}(L(E))^+, [1_{L(E)}]) \cong (K_0^{gr}(L(F)), K_0^{gr}(L(F))^+, [1_{L(F)}]).$$

Write $A := A_E^t$ and $B = A_F^t$. Assume in addition that A and B are strongly shift equivalent. Is then $L(E)$ graded isomorphic to $L(F)$? A positive answer would follow from Theorem 3.15

if one knows that every automorphism of the dimension triple $(\Delta_B, \Delta_B^+, \delta_B)$ is induced by a strong shift equivalence from B to B .

4. TOWARDS A CHARACTERIZATION OF GRADED ISOMORPHISMS

The main goal of this section is to establish a strategy to prove a weak version of [16, Conjecture 1] in the case of finite essential graphs.

In order to state our main result, we need some definitions. Recall that an automorphism g of an algebra \mathcal{A} is said to be *locally inner* in case, given any finite number of elements x_1, \dots, x_n in \mathcal{A} there is an inner automorphism h of \mathcal{A} such that $h(x_i) = \varphi(x_i)$ for $i = 1, \dots, n$. It is well-known ([14, Lemma 15.23(b)]) that any automorphism φ of an ultramatricial algebra \mathcal{A} , such that $K_0(\varphi) = \text{id}$, is locally inner.

Let E be a finite essential graph, and write $L(E) \cong L(E)_0[t_+, t_-, \alpha]$ as in Section 3. Let g be a locally inner automorphism of $L(E)_0$. Then we define $L^g(E)$ by

$$L^g(E) = L(E)_0[t_+, t_-, g \circ \alpha].$$

Observe that $L^g(E)$ is a graded algebra with the same graded K -theory as $L(E)$.

Theorem 4.1. *If E, F are finite essential graphs, then the following are equivalent:*

- (1) $(K_0(L(E)_0), K_0^+(L(E)_0), [1_{L(E)_0}]) \cong_{\mathbb{Z}[x, x^{-1}]} (K_0(L(F)_0), K_0^+(L(F)_0), [1_{L(F)_0}])$.
- (2) *There exists a locally inner automorphism g of $L(E)_0$ such that $L^g(E) \cong_{gr} L(F)$.*

Proof. (2) \Rightarrow (1). If $L^g(E) \cong_{gr} L(F)$, there is an order-unit preserving ordered $\mathbb{Z}[x, x^{-1}]$ -module isomorphism $K_0^{gr}(L^g(E)) \cong K_0^{gr}(L(F))$. So the result follows from Theorem 3.10 and the fact that $K_0^{gr}(L^g(E)) = K_0^{gr}(L(E))$.

(1) \Rightarrow (2) will be proven in Section 5. □

The proof of (1) \Rightarrow (2) is more involved, and we will postpone it until next section. We will explain in the present section the strategy we will follow to prove it, and we will give the final argument of the proof, assuming the results obtained in the next section.

We do not know whether the existence of a graded isomorphism between $L^g(E)$ and $L(F)$ implies the existence of a graded isomorphism between $L(E)$ and $L(F)$.

Example 4.2. A particular case to apply this argument is the problem of isomorphism between L_2 and L_{2-} (see e.g. [3, Section 2]). If L_2 is graded isomorphic to L_{2-} , then by Theorem 4.1 (2) \Rightarrow (1) and Theorem 3.10 the corresponding matrices A and B are shift equivalent, and so flow equivalent. By [13] we then have $\det(I - A_2) = \det(I - A_{2-})$, but $\det(I - A_2) = 1$ and $\det(I - A_{2-}) = -1$ (by direct calculation). Thus, L_2 and L_{2-} are not graded isomorphic. It is still open to decide the existence of a (nongraded) isomorphism.

We now begin to explain our strategy to show Theorem 4.1.

4.3. Starting Data: Let E, F be finite essential graphs, such that $N = |E^0|, M = |F^0|$. We fix pictures $L(E) = L(E)_0[t_+, t_-; \alpha]$ and $L(F) = L(F)_0[s_+, s_-; \beta]$. Also, we define $A := A_E^t \in M_N(\mathbb{Z}^+)$ and $B := A_F^t \in M_M(\mathbb{Z}^+)$, and fix $\delta_A : K_0(L(E)_0) \rightarrow K_0(L(E)_0)$ and $\delta_B : K_0(L(F)_0) \rightarrow K_0(L(F)_0)$ to be the group isomorphisms given by multiplication by A (respectively B). If we write $\mathcal{M}(E) := L(E)_0 = \varinjlim (\mathcal{M}(E)_n, j_{n,m}^E)$, where $\mathcal{M}(E)_n = L(E)_{0,n}$, and similarly $\mathcal{M}(F) := L(F)_0 = \varinjlim (\mathcal{M}(F)_n, j_{n,m}^F)$ (whose connecting maps are induced

by the (CK2) relation on E and F respectively), then we denote $G_m := K_0(\mathcal{M}(E)_m)$, $H_m := K_0(\mathcal{M}(F)_m)$, $G := K_0(\mathcal{M}(E))$ and $H := K_0(\mathcal{M}(F))$. Notice that for every $m \geq 0$ we have $G_m \cong \mathbb{Z}^N$ and $H_m \cong \mathbb{Z}^M$, as well as that $G = \varinjlim(G_m, A)$ and $H = \varinjlim(H_m, B)$.

In order to attain our goal, we start by establishing some previous facts. The proof of the next two lemmas is trivial.

Lemma 4.4. *If there exists a nonnegative integer matrix $S \in M_{M,N}(\mathbb{Z}^+)$ such that $SA = BS$, then for every $l, k, m \geq 1$ there exists a commutative diagram*

$$\begin{array}{ccccc}
 & & G_{l+1} & \xrightarrow{A^k} & G_{k+l+1} \\
 & \nearrow & \downarrow & \nearrow & \downarrow \\
 G_l & \xrightarrow{id_{\mathbb{Z}^N} A^k} & G_{k+l} & \xrightarrow{id_{\mathbb{Z}^N}} & G_{k+l+1} \\
 \downarrow S & & \downarrow S & & \downarrow S \\
 & \nearrow & H_{m+1} & \xrightarrow{B^k} & H_{m+k+1} \\
 H_m & \xrightarrow{id_{\mathbb{Z}^M}} & H_{m+k} & \xrightarrow{id_{\mathbb{Z}^M}} & H_{m+k+1} \\
 & \searrow & \downarrow & \searrow & \downarrow \\
 & & H_{m+k} & & H_{m+k+1}
 \end{array}$$

Lemma 4.5. *If there exists a nonnegative integer matrix $R \in M_{N,M}(\mathbb{Z}^+)$ such that $AR = RB$, then for every $l, k, m \geq 1$ there exists a commutative diagram*

$$\begin{array}{ccccc}
 & & G_{l+1} & \xrightarrow{A^k} & G_{k+l+1} \\
 & \nearrow & \downarrow & \nearrow & \downarrow \\
 G_l & \xrightarrow{id_{\mathbb{Z}^N} A^k} & G_{k+l} & \xrightarrow{id_{\mathbb{Z}^N}} & G_{k+l+1} \\
 \downarrow R & & \downarrow R & & \downarrow R \\
 & \nearrow & H_{m+1} & \xrightarrow{B^k} & H_{m+k+1} \\
 H_m & \xrightarrow{id_{\mathbb{Z}^M}} & H_{m+k} & \xrightarrow{id_{\mathbb{Z}^M}} & H_{m+k+1} \\
 & \searrow & \downarrow & \searrow & \downarrow \\
 & & H_{m+k} & & H_{m+k+1}
 \end{array}$$

□

Given $t \geq 0$, we will denote $\Omega_t := id_{\mathbb{Z}^N} : G_t \rightarrow G_{t+1}$ and $\Omega'_t := id_{\mathbb{Z}^M} : H_t \rightarrow H_{t+1}$. If we have

$$(K_0(\mathcal{M}(E)), K_0^+(\mathcal{M}(E)), [1_{\mathcal{M}(E)}]) \cong_{\mathbb{Z}[x, x^{-1}]} (K_0(\mathcal{M}(F)), K_0^+(\mathcal{M}(F)), [1_{\mathcal{M}(F)}]),$$

then by Theorem 3.10

$$(K_0(\mathcal{M}(E)), K_0^+(\mathcal{M}(E)), [1_{\mathcal{M}(E)}], \delta_A) \cong (K_0(\mathcal{M}(F)), K_0^+(\mathcal{M}(F)), [1_{\mathcal{M}(F)}], \delta_B)$$

through an ordered group isomorphism $\Phi : K_0(\mathcal{M}(E)) \rightarrow K_0(\mathcal{M}(F))$ such that $\Phi([1_{\mathcal{M}(E)}]) = [1_{\mathcal{M}(F)}]$. Equivalently, by Theorem 3.10, $A \sim_{SE} B$ with lag l for some $l \geq 1$. Hence, there exists a pair of nonnegative integral matrices $R \in M_{N \times M}(\mathbb{Z}^+)$ and $S \in M_{M \times N}(\mathbb{Z}^+)$ satisfying the four shift equivalence equations:

- (1) (i) $AR = RB$; (ii) $SA = BS$.
- (2) (i) $A^l = RS$; (ii) $B^l = SR$.

Notice that, when $l = 1$, we have strong shift equivalence of A_E and A_F , therefore $L(E)$ and $L(F)$ are strongly shift K_0^{gr} -isomorphic, and so by Theorem 3.15, we have that $L(E) \cong_{gr} L(F)$.

Consequently, we shall assume in the sequel that $l > 1$. Indeed, (R, SA^t) implements a shift equivalence between A and B of lag $l + t$ for any $t \geq 1$ (see [21, [Proposition 7.3.2(2)]]).

Remark 4.6. By the proof of [21, Theorems 7.5.7 & 7.5.8], the maps S and R define explicitly the ordered group isomorphisms Φ and Φ^{-1} respectively. In particular, S and R preserve the order unit (whenever Φ does).

Now, because of Theorem 3.10 we have

$$(K_0(\mathcal{M}(E)), K_0^+(\mathcal{M}(E)), [1_{\mathcal{M}(E)}], \alpha_*) \cong (K_0(\mathcal{M}(F)), K_0^+(\mathcal{M}(F)), [1_{\mathcal{M}(F)}], \beta_*)$$

for the respective corner isomorphisms α and β , through the same isomorphism of ordered groups Φ . Thus, by Lemmas 4.4 and 4.5 we have, for a suitable $m \geq 1$, the following commutative diagram (tagged (\dagger))

$$\begin{array}{ccccccc}
 & & G_1 & \xrightarrow{A^l} & G_{l+1} & \xrightarrow{A^l} & G_{2l+1} & \cdots & G \\
 & \nearrow \Omega_0 & & & \nearrow \Omega_l & & \nearrow \Omega_{2l} & & \nearrow \alpha_* \\
 G_0 & \xrightarrow{A^l} & G_l & \xrightarrow{A^l} & G_{2l} & \cdots & G & & \\
 \downarrow S & & \downarrow S & & \downarrow S & & \downarrow S & & \downarrow \Phi \\
 & \nearrow R & & & \nearrow R & & \nearrow R & & \\
 & H_{m+1} & \xrightarrow{B^l} & H_{m+l+1} & \xrightarrow{B^l} & H_{m+2l+1} & \cdots & H \\
 \downarrow S & & \downarrow S & & \downarrow S & & \downarrow S & & \downarrow \Phi \\
 H_m & \xrightarrow{B^l} & H_{m+l} & \xrightarrow{B^l} & H_{m+2l} & \cdots & H & & \\
 & \nearrow \Omega'_m & & & \nearrow \Omega'_{m+l} & & \nearrow \Omega'_{m+2l} & & \nearrow \beta_*
 \end{array}$$

All the group morphisms appearing in this diagram are order preserving, and, by Remark 4.6, those defined by matrices R, S, A^l, B^l also preserve order-unit, while the maps Ω_r, Ω'_t are the restriction of α_* (respectively β_*) to the corresponding G_r (respectively H_t) by the argument in the proof of Lemma 3.8.

Our aim is to lift, in an inductive way, this diagram to a commutative diagram of algebras (tagged (\diamond))

$$\begin{array}{ccccccc}
 & & \mathcal{M}(E)_1 & \xrightarrow{j_{1,l+1}^E} & \mathcal{M}(E)_{l+1} & \xrightarrow{j_{l+1,2l+1}^E} & \mathcal{M}(E)_{2l+1} & \cdots & \mathcal{M}(E) \\
 & \nearrow \alpha_{|\mathcal{M}(E)_0} & & & \nearrow \alpha_{|\mathcal{M}(E)_l} & & \nearrow \alpha_{|\mathcal{M}(E)_{2l}} & & \nearrow \alpha \\
 \mathcal{M}(E)_0 & \xrightarrow{j_{0,l}^E} & \mathcal{M}(E)_l & \xrightarrow{j_{l,2l}^E} & \mathcal{M}(E)_{2l} & \cdots & \mathcal{M}(E) & & \\
 \downarrow \phi_0 & & \downarrow \phi_l & & \downarrow \phi_{2l} & & \downarrow \phi_{2l+1} & & \downarrow \psi_0 \\
 & \nearrow \psi_m & & & \nearrow \psi_{m+l} & & \nearrow \psi_{m+2l} & & \nearrow \psi_1 \\
 & \mathcal{M}(F)_{m+1} & \xrightarrow{j_{m+1,m+l+1}^F} & \mathcal{M}(F)_{m+l+1} & \xrightarrow{j_{m+l+1,m+2l+1}^F} & \mathcal{M}(F)_{m+2l+1} & \cdots & \mathcal{M}(F) \\
 \downarrow \beta_{|\mathcal{M}(F)_m} & & \downarrow \beta_{|\mathcal{M}(F)_{m+l}} & & \downarrow \beta_{|\mathcal{M}(F)_{m+2l}} & & \downarrow \beta & & \downarrow \beta \\
 \mathcal{M}(F)_m & \xrightarrow{j_{m,m+l}^F} & \mathcal{M}(F)_{m+l} & \xrightarrow{j_{m+l,m+2l}^F} & \mathcal{M}(F)_{m+2l} & \cdots & \mathcal{M}(F) & & \\
 & & & & & & & &
 \end{array}$$

We will show how to do this in Section 5. For now, we assume that we have established that diagram (\diamond) commutes. Using this, we are ready to finish the proof of Theorem 4.1. Observe that, using the commutativity of the front and the front-back diagrams, we have well-defined algebra isomorphisms

$$\varphi_0, \varphi_1: \mathcal{M}(E) \rightarrow \mathcal{M}(F),$$

defined by

$$\varphi_\varepsilon(j_{kl+\varepsilon, \infty}^E(x)) = j_{kl+m+\varepsilon, \infty}^F(\phi_{kl+\varepsilon}(x))$$

for $x \in \mathcal{M}(E)_{kl+\varepsilon}$, $\varepsilon = 0, 1$.

Similarly, we have well-defined algebra isomorphisms $\psi_\varepsilon: \mathcal{M}(F) \rightarrow \mathcal{M}(E)$, for $\varepsilon = 0, 1$. Observe that ψ_ε is the inverse of φ_ε .

Notice that we have $\beta \cdot \varphi_0 = \varphi_1 \cdot \alpha$.

Lemma 4.7. *Let $g = \psi_0\varphi_1 \in \text{Aut}(\mathcal{M}(E))$. Then g is a locally inner automorphism of $\mathcal{M}(E)$.*

Proof. By the previous comment, it suffices to observe that $K_0(g) = \text{id}$. For this it is enough to show that, for any $k \in \mathbb{N}$, we have

$$K_0(j_{kl+1, (k+2)l}^E) = K_0(\psi_{m+(k+1)l} \cdot j_{m+kl+1, m+(k+1)l}^F \cdot \phi_{kl+1}).$$

Note that

$$K_0(\psi_{m+(k+1)l} \cdot j_{m+kl+1, m+(k+1)l}^F \cdot \phi_{kl+1}) = RB^{l-1}S = A^{l-1}RS = A^{2l-1} = K_0(j_{kl+1, (k+2)l}^E),$$

where we have used that $RB = AR$ and that $RS = A^l$. This shows the result. \square

Proof of Theorem 4.1: Set $g = \psi_0\varphi_1$. By Lemma 4.7, g is a locally inner automorphism of $\mathcal{M}(E)_0$. Put $\alpha' = g\alpha$. In order to show that

$$L^g(E) = \mathcal{M}(E)[t_+, t_-, \alpha'] \cong \mathcal{M}(F)[s_+, s_-, \beta] = L(F),$$

we first observe that $\beta \cdot \varphi_0 = \varphi_1 \cdot \alpha'$. Indeed, we have

$$\varphi_1 \cdot \alpha' = \varphi_1 \cdot \varphi_0^{-1} \cdot \varphi_1 \cdot \alpha = \varphi_1 \cdot \alpha = \beta \cdot \varphi_0.$$

Since

$$s_+\varphi_0(a)s_- = \beta(\varphi_0(a)) = \varphi_1(\alpha'(a))$$

for all $a \in \mathcal{M}(E)$, the universal property of $\mathcal{M}(E)[t_+, t_-; \alpha']$ gives that there is a unique algebra homomorphism

$$\varphi: \mathcal{M}(E)[t_+, t_-; \alpha'] \rightarrow \mathcal{M}(F)[s_+, s_-; \beta]$$

such that $\varphi|_{\mathcal{M}(E)} = \varphi_0$, $\varphi(t_+) = s_+$ and $\varphi(t_-) = s_-$. Clearly φ is a graded homomorphism.

The inverse of φ is the unique algebra homomorphism

$$\psi: \mathcal{M}(F)[s_+, s_-; \beta] \rightarrow \mathcal{M}(E)[t_+, t_-; \alpha']$$

such that $\psi|_{\mathcal{M}(F)} = \psi_0$, $\psi(s_+) = t_+$ and $\psi(s_-) = t_-$. \square

Remark 4.8. Clearly, we have $L(E) \cong_{gr} L^g(E)$ when g is an inner automorphism of $\mathcal{M}(E)$. As we shall see now, one can easily show that there are locally inner automorphisms g of $\mathcal{M}(E)$ such that $L(E) \not\cong_{gr} L^g(E)$, and indeed such that $L^g(E) \not\cong L(F)$ for any finite graph F . However the automorphism g produced in the proof of Theorem 4.1 is of a special form, and we have not been able to determine whether $L(E) \cong L^g(E)$ in that case.

Now let us give an example of a locally inner g such that $L^g(E)$ is not finitely generated as a K -algebra. This of course implies that $L^g(E) \not\cong L(F)$ for any finite graph F , because $L(F)$ is a finitely generated K -algebra.

Let K be an infinite field.

Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and let E be the graph with adjacency matrix A^t . So $E^0 = \{v, w\}$, and $E^1 = \{e, f, g\}$, with $s(e) = r(e) = s(f) = v$ and $r(f) = s(g) = r(g) = w$. We consider the elements $t_+ = e + g$, $t_- = e^* + g^*$, and the corner isomorphism α given by $\alpha(x) = t_+ x t_-$ for $x \in \mathcal{M}(E)$.

Observe that $\mathcal{M}(E)_n = K \times M_{n+1}(K)$, with transition maps given by

$$(\lambda, Z) \mapsto \left(\lambda, \begin{pmatrix} \lambda & 0 \\ 0 & Z \end{pmatrix} \right).$$

With this picture, α is given by

$$(\lambda, Z) \mapsto \left(\lambda, \begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix} \right).$$

We can build a locally inner automorphism g of $\mathcal{M}(E)$ as follows. Let $(\xi_n : n \geq 1)$ be a sequence of elements of K^\times . We require that ξ_{n+1} does not belong to the multiplicative group generated by ξ_1, \dots, ξ_n for all $n \geq 1$. (Note that such a sequence exists, since K is an infinite field.) Then define $U_0 = (1, 1)$, and inductively $U_n = (\xi_n, X_n)$, with $X_n = \text{diag}(\xi_n, X_{n-1})$ for $n \geq 1$, where $X_0 = 1$. Set

$$g(x) = U_n x U_n^{-1}, \quad x \in \mathcal{M}(E)_n.$$

Then g is a locally inner automorphism of $\mathcal{M}(E)$. Observe that $\alpha \circ g = g \circ \alpha$.

Set $L^g(E) = \mathcal{M}(E)[s_+, s_-, g \circ \alpha]$. We show that $L^g(E)$ is not finitely generated as a K -algebra. Assume that $L^g(E)$ is finitely generated. Then there is $n_0 \geq 1$ such that $L^g(E)$ is generated by $\mathcal{M}(E)_{n_0}$ and s_+, s_- . Set $f^{(n_0)} = (1, 0) \in \mathcal{M}(E)_{n_0}$, and denote by $e_{ij}^{(n)}$, $1 \leq i, j \leq n+1$ the matrix units of the factor $M_{n+1}(K)$ of $\mathcal{M}(E)_n$. Set

$$\mathcal{B} = \{f^{(n_0)}\} \cup \{e_{ij}^{(n_0)} : 1 \leq i, j \leq n_0 + 1\},$$

which is a linear basis for $\mathcal{M}(E)_{n_0}$. Then every element x in $\mathcal{M}(E)$ can be written as a linear combination of the family

$$\mathcal{B}' = \{b_r s_+ b_{r-1} s_+ \cdots s_+ b_0 s_- b_{-1} s_- \cdots s_- b_{-r} : b_{-r}, \dots, b_0, \dots, b_r \in \mathcal{B}\}.$$

(This uses the fact that $s_- b s_+ = (g\alpha)^{-1}(s_+ s_- b s_+ s_-) \in \mathcal{M}(E)_{n_0}$ for all $b \in \mathcal{B}$.) Hence we can write

$$e_{1, n_0+2}^{(n_0+1)} = \sum_{b' \in \mathcal{F}} \lambda_{b'} b',$$

where \mathcal{F} is a finite linearly independent subset of \mathcal{B}' , and $\lambda_{b'} \in K^\times$. Observe that for

$$b' = b_r s_+ b_{r-1} s_+ \cdots s_+ b_0 s_- b_{-1} s_- \cdots s_- b_{-r} \in \mathcal{B}',$$

we have

$$\begin{aligned}
g(b') &= g(b_r s_+ b_{r-1} s_+ \cdots s_+ b_0 s_- b_{-1} s_- \cdots s_- b_{-r}) \\
&= g\left(b_r (s_+ b_{r-1} s_-) (s_+^2 b_{r-2} s_-^2) \cdots (s_+^r b_0 s_-^r) \cdots (s_+ b_{-r+1} s_-) b_{-r}\right) \\
&= g\left(b_r (g\alpha(b_{r-1})) (g\alpha)^2(b_{r-2}) \cdots (g\alpha)^r(b_0) \cdots (g\alpha)(b_{-r+1}) b_{-r}\right) \\
&= g(b_r) (g\alpha(g(b_{r-1}))) (g\alpha)^2(g(b_{r-2})) \cdots (g\alpha)^r(g(b_0)) \cdots (g\alpha)(g(b_{-r+1})) g(b_{-r}) \\
&= \zeta_{b'} b'
\end{aligned}$$

where $\zeta_{b'} \in \langle \xi_1, \dots, \xi_{n_0} \rangle$, the subgroup of K^\times generated by ξ_1, \dots, ξ_{n_0} . Therefore we obtain

$$\xi_{n_0+1} e_{1, n_0+2}^{(n_0+1)} = g(e_{1, n_0+2}^{(n_0+1)}) = \sum_{b' \in \mathcal{F}} \lambda_{b'} \zeta_{b'} b'.$$

Since \mathcal{F} is a linearly independent family, we get $\xi_{n_0+1} = \zeta_{b'}$ for all $b' \in \mathcal{F}$. Since $\zeta_{b'} \in \langle \xi_1, \dots, \xi_{n_0} \rangle$, we obtain a contradiction with the choice of the family (ξ_n) .

5. LIFTING THE DIAGRAM

In this section we will proceed to prove Theorem 4.1. According to the arguments in Section 4, it is enough to perform the correct lifting of the diagram (\dagger) to a diagram (\diamond) .

The goal is to define algebra isomorphisms $\varphi_0, \varphi_1: \mathcal{M}(E) \rightarrow \mathcal{M}(F)$ by

$$\varphi_\varepsilon(j_{kl+\varepsilon, \infty}^E(x)) = j_{kl+m+\varepsilon, \infty}^F(\phi_{kl+\varepsilon}(x))$$

for $x \in \mathcal{M}(E)_{kl+\varepsilon}$, $\varepsilon = 0, 1$ lifting the group diagram, so that $\beta \cdot \varphi_0 = \alpha \cdot \varphi_1$. In order to do that in a clear way, we will divide the proof in several intermediate steps.

Notation. We start by fixing some notation that will be useful in the sequel. We will denote $E^0 = \{v_1, \dots, v_N\}$ and $F^0 = \{w_1, \dots, w_M\}$. Also, for $d \geq 1$, we will denote $E^d v_i = \{\gamma \in E^d \mid r(\gamma) = v_i\}$ and $v_i E^d = \{\gamma \in E^d \mid s(\gamma) = v_i\}$ (analogously for the w_j 's of F^0); also, we will denote $v_j E_d v_i := v_j E^d \cap E^d v_i$. For a path γ in E of length t and $d \geq t$, we will denote by γE^{d-t} the set of paths $\lambda \in E^d$ such that the initial segment of λ of length t is equal to γ . Of course, similar notations apply to F .

In order to define correctly the maps α, β , we fix for each $1 \leq i \leq N$ an edge $e_i \in r^{-1}(v_i)$, and analogously we fix for each $1 \leq j \leq M$ an edge $f_j \in r^{-1}(w_j)$. (Note that we use our hypothesis that E and F do not have sources here.) Then, we define:

$$\begin{aligned}
(1) \quad t_+ &:= \sum_{i=1}^N e_i \quad \text{and} \quad t_- := \sum_{i=1}^N e_i^* \\
(2) \quad s_+ &:= \sum_{j=1}^M f_j \quad \text{and} \quad s_- := \sum_{j=1}^M f_j^*
\end{aligned}$$

So, the map α is defined by the rule $\alpha(a) = t_+ a t_-$, while the map β is defined by the rule $\beta(b) = s_+ b s_-$.

For a path γ in E^d we denote by $\widehat{\gamma}$ the path $e_i \gamma$ in E^{d+1} , where $v_i = s(\gamma)$. Similarly, for $\lambda \in w_j F^d$, we set $\widehat{\lambda} := f_j \lambda \in F^{d+1}$. Note that, for $\gamma_1, \gamma_2 \in E^d$, we have $\alpha(\gamma_1 \gamma_2^*) = \widehat{\gamma}_1 \widehat{\gamma}_2^*$, and similarly, for $\lambda_1, \lambda_2 \in F^d$, we have $\beta(\lambda_1 \lambda_2^*) = \widehat{\lambda}_1 \widehat{\lambda}_2^*$.

The proof consists of constructing certain partitions of the sets of paths of E and F , and using them to define the liftings of R and S along all the diagram. Once the first three commutative squares of the diagram have been lifted, the rest of the partitions and the corresponding liftings are completely determined by the choices made so far, so in a sense there is only a finite number of choices involved in the process. Accordingly, we will divide the section in four subsections, corresponding to the liftings of the first three commutative diagrams and to the final argument showing how to continue the process inductively to obtain the lifting of all the squares.

5.1. Lifting the first (“left end”) commutative square. We start with the commutative square

$$\begin{array}{ccc} \mathbb{Z}^N & \xrightarrow{\Omega_0} & \mathbb{Z}^N \\ \downarrow S & & \downarrow S \\ \mathbb{Z}^M & \xrightarrow{\Omega'_m} & \mathbb{Z}^M \end{array}$$

and we lift it to a commutative square

$$\begin{array}{ccc} \mathcal{M}(E)_0 & \xrightarrow{\alpha} & \mathcal{M}(E)_1 \\ \downarrow \phi_0 & & \downarrow \phi_1 \\ \mathcal{M}(F)_m & \xrightarrow{\beta} & \mathcal{M}(F)_{m+1} \end{array}$$

where ϕ_0, ϕ_1 are unital morphisms. Because of the previous data, $1 \leq m \leq l$ are fixed across the whole process. Here $A = (a_{ik})$ and $B = (b_{jt})$ are the transposes of the adjacency matrices for E and F respectively. The matrices S and R have components denoted by s_{ji} and r_{ij} respectively. We will consistently use i to denote an index in the range $\{1, \dots, N\}$, and j to denote an index in the range $\{1, \dots, M\}$.

Now, we will define ϕ_0 . Since S is a group homomorphism preserving the order-unit, we have that $|F^m w_j| = \sum_{i=1}^N s_{ji}$. So, we can consider a partition

$$F^m w_j = \bigsqcup_{i=1}^N \Lambda_j^{m, v_i}$$

where $|\Lambda_j^{m, v_i}| = s_{ji}$ for all j, i . We will write

$$\Lambda_j^{m, v_i} = \{\lambda_{j1}^0(v_i) \dots, \lambda_{js_{ji}}^0(v_i)\}.$$

We define a map $\phi_0: \mathcal{M}(E)_0 \rightarrow \mathcal{M}(F)_m$ by the rule

$$\phi_0(v_i) = \sum_{j=1}^M \sum_{\lambda \in \Lambda_j^{m, v_i}} \lambda \lambda^*.$$

Note that

$$[\phi_0(v_i)] = \sum_{j=1}^M s_{ji}[w_j] = S[v_i],$$

so that ϕ_0 lifts S .

Also, since $1 = \sum_{i=1}^N v_i$ and $F^m = \bigsqcup_{j=1}^M F^m w_j$, we have

$$\phi_0(1) = \sum_{i=1}^N \phi_0(v_i) = \sum_{i=1}^N \sum_{j=1}^M \sum_{\lambda \in \Lambda_j^{m,v_i}} \lambda \lambda^* = \sum_{j=1}^M \sum_{\lambda \in F^m w_j} \lambda \lambda^* = \sum_{\lambda \in F^m} \lambda \lambda^* = 1$$

where the last equality follows from (CK2) and the fact that F does not have sinks.

Now, we proceed to define ϕ_1 . For this, we consider $F^{m+1}w_j$. Observe that

$$|F^{m+1}w_j| = \sum_{k=1}^M b_{jk} |F^m w_k| = \sum_{k=1}^M \sum_{i=1}^N b_{jk} s_{ki} = \sum_{i=1}^N \left(\sum_{k=1}^M s_{jk} a_{ki} \right),$$

where we have used that $SA = BS$. Notice that a_{ki} is the number of edges in $v_i E^1 v_k$. It follows that we can consider a partition

$$(5.1) \quad F^{m+1}w_j = \bigsqcup_{i=1}^N \bigsqcup_{k=1}^M \bigsqcup_{\{e \in v_i E^1 v_k\}} \Lambda_j^{m+1,e},$$

with $|\Lambda_j^{m+1,e}| = s_{jk}$ for all $e \in v_i E^1 v_k$. We will write

$$\Lambda_j^{m+1,e} = \{\lambda_{j1}^1(e), \dots, \lambda_{j s_{jk}}^1(e)\}.$$

Moreover, we can choose the partition (5.1) in such a way that

$$(5.2) \quad \Lambda_j^{m+1, \widehat{v}_i} = \Lambda_j^{m+1, e_i} = \{\widehat{\lambda} \mid \lambda \in \Lambda_j^{m, v_i}\}.$$

This makes sense because the family on the right hand side is a family of s_{ji} paths of length $m+1$ ending in w_j . In fact, we can fix as a condition

$$\mathbf{(Ex 1)} \quad \lambda_{jt}^1(e_k) = \widehat{\lambda_{jt}^0(v_k)}$$

for all the possible values of j, k, t . This is coherent with the possible values of these variables, and clearly Condition (Ex 1) implies (5.2).

Now we define $\phi_1: \mathcal{M}(E)_1 \rightarrow \mathcal{M}(F)_{m+1}$ as follows. Given $e, f \in E^1$ such that $r(e) = v_k = r(f)$, we put:

$$\phi_1(e f^*) = \sum_{j=1}^M \sum_{p=1}^{s_{jk}} \lambda_{j,p}^1(e) \lambda_{j,p}^1(f)^*.$$

It is easy to show that ϕ_1 is a well-defined unital homomorphism, such that $K_0(\phi_1) = S$. Let us show that $\phi_1 \alpha = \beta \phi_0$:

$$\phi_1(\alpha(v_i)) = \phi_1(e_i e_i^*) = \sum_{j=1}^M \sum_{\lambda \in \Lambda_j^{m+1, e_i}} \lambda \lambda^* = \sum_{j=1}^M \sum_{\lambda \in \Lambda_j^{m, v_i}} \widehat{\lambda} \widehat{\lambda}^* = \beta(\phi_0(v_i)).$$

where we have used (5.2) for the third equality.

5.2. Lifting the second (“diagonal”) commutative square. We will take the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}^N & \xrightarrow{\Omega_l} & \mathbb{Z}^N \\ \uparrow R & & \uparrow R \\ \mathbb{Z}^M & \xrightarrow{\Omega'_m} & \mathbb{Z}^M \end{array}$$

and we will lift it to a commutative diagram

$$\begin{array}{ccc} \mathcal{M}(E)_l & \xrightarrow{\alpha} & \mathcal{M}(E)_{l+1} \\ \uparrow \psi_m & & \uparrow \psi_{m+1} \\ \mathcal{M}(F)_m & \xrightarrow{\beta} & \mathcal{M}(F)_{m+1} \end{array}$$

where ψ_m, ψ_{m+1} are unital morphisms satisfying that $\psi_m \cdot \phi_0 = j_{0,l}^E$ and $\psi_{m+1} \cdot \phi_1 = j_{1,l+1}^E$.

First, we will define ψ_m . Since $\sum_{j=1}^M r_{ij} s_{jk} = (A^l)_{ik}$ is the number of paths in $v_k E^l v_i$ and $|\Lambda_j^{m,v_k}| = s_{jk}$, we can consider a partition

$$(5.3) \quad v_k E^l v_i = \bigsqcup_{j=1}^M \bigsqcup_{\{\lambda \in \Lambda_j^{m,v_k}\}} \Gamma_i^{l,\lambda}$$

with $|\Gamma_i^{l,\lambda}| = r_{ij}$ for all $\lambda \in \Lambda_j^{m,v_k}$. Notice that (5.3) implies

$$(5.4) \quad E^l v_i = \bigsqcup_{k=1}^N \bigsqcup_{j=1}^M \bigsqcup_{\{\lambda \in \Lambda_j^{m,v_k}\}} \Gamma_i^{l,\lambda}$$

and

$$(5.5) \quad v_k E^l = \bigsqcup_{i=1}^N \bigsqcup_{j=1}^M \bigsqcup_{\{\lambda \in \Lambda_j^{m,v_k}\}} \Gamma_i^{l,\lambda}$$

If $\lambda = \lambda_{jt}^0(v_k) \in F^m w_j$, we will write

$$\Gamma_i^{l,\lambda} = \Gamma_i^{l,\lambda_{jt}^0(v_k)} = \{\gamma_{i1}^m(\lambda_{jt}^0(v_k)), \dots, \gamma_{ir_{ij}}^m(\lambda_{jt}^0(v_k))\}.$$

In particular, $s(\gamma_{ip}^m(\lambda_{jt}^0(v_k))) = v_k$ by (5.3).

We define a map $\psi_m : \mathcal{M}(F)_m \rightarrow \mathcal{M}(E)_l$ by the rule

$$\psi_m(\lambda_1 \lambda_2^*) = \sum_{i=1}^N \sum_{p=1}^{r_{ij}} \gamma_{ip}^m(\lambda_1) \gamma_{ip}^m(\lambda_2)^*,$$

where $\lambda_1, \lambda_2 \in F^m w_j$.

As in the previous cases, it is easy to show that ψ_m is a well-defined unital morphism satisfying that $K_0(\psi_m) = R$. Now,

$$(5.6) \quad \psi_m \cdot \phi_0(v_k) = \sum_{j=1}^M \sum_{\lambda \in \Lambda_j^{m, v_k}} \psi_m(\lambda \lambda^*) = \sum_{j=1}^M \sum_{i=1}^N \sum_{\lambda \in \Lambda_j^{m, v_k}} \sum_{\gamma \in \Gamma_i^{l, \lambda}} \gamma \gamma^* = \sum_{\gamma \in v_k E^l} \gamma \gamma^* = j_{0,l}^E(v_k),$$

where we have used (5.5) for the third equality.

Now, we will define ψ_{m+1} . Since, for $e \in E^1 v_k$, we have

$$|e E^l v_i| = |e(v_k E^l v_i)| = \sum_{j=1}^M r_{ij} s_{jk},$$

we may consider a partition

$$(5.7) \quad e E^l v_i = \bigsqcup_{j=1}^M \bigsqcup_{\{\lambda \in \Lambda_j^{m+1, e}\}} \Gamma_i^{l+1, \lambda},$$

with $|\Gamma_i^{l+1, \lambda}| = r_{ij}$ for $\lambda \in \Lambda_j^{m+1, e}$. Given any $\lambda \in F^{m+1} w_j$, we will write

$$\Gamma_i^{l+1, \lambda} = \{\gamma_{i1}^{m+1}(\lambda), \dots, \gamma_{ir_{ij}}^{m+1}(\lambda)\}.$$

For $e \in E^1 v_k$, left multiply (5.3) by e to get

$$(5.8) \quad e E^l v_i = e(v_k E^l v_i) = \bigsqcup_{j=1}^M \bigsqcup_{\{\lambda \in \Lambda_j^{m, v_k}\}} e \Gamma_i^{l, \lambda}.$$

Since there is a bijection $\lambda_{jt}^0(v_k) \longleftrightarrow \lambda_{jt}^1(e)$ between Λ_j^{m, v_k} and $\Lambda_j^{m+1, e}$, we may take as definition of the $\gamma_{ip}^{m+1}(\lambda_{jt}^1(e))$ the following:

$$(5.9) \quad \gamma_{ip}^{m+1}(\lambda_{jt}^1(e)) := e \gamma_{ip}^m(\lambda_{jt}^0(v_k)),$$

for $e \in E^1 v_k$, which is coherent with (5.7).

Now, we define a map $\psi_{m+1} : \mathcal{M}(F)_{m+1} \rightarrow \mathcal{M}(E)_{l+1}$ by the rule

$$\psi_{m+1}(\lambda_1 \lambda_2^*) = \sum_{i=1}^N \sum_{p=1}^{r_{ij}} \gamma_{ip}^{m+1}(\lambda_1) \gamma_{ip}^{m+1}(\lambda_2)^*.$$

for any $\lambda_1, \lambda_2 \in F^{m+1} w_j$. As in the previous cases, it is easy to show that ψ_{m+1} is a well-defined unital morphism satisfying that $K_0(\psi_{m+1}) = R$.

Given $e, f \in E^1 v_k$, we compute that

$$\psi_{m+1} \cdot \phi_1(e f^*) = \sum_{j=1}^M \sum_{p=1}^{s_{jk}} \psi_{m+1}(\lambda_{jp}^1(e) \lambda_{jp}^1(f)^*) = e j_{0,l}^E(v_k) f^* = j_{1,l+1}^E(e f^*),$$

where we have used (5.9) and (5.6).

Now, using (5.9) and (Ex 1) we compute

$$e_k \gamma_{ip}^m(\lambda_{jt}^0(v_k)) = \gamma_{ip}^{m+1}(\lambda_{jt}^1(e_k)) = \gamma_{ip}^{m+1}(\widehat{\lambda_{jt}^0(v_k)}).$$

Summarizing, for any $\lambda \in \Lambda_j^{m, v_k}$ we proved that

$$\text{(Ex 2)} \quad \widehat{\gamma_{ip}^m(\lambda)} = \gamma_{ip}^{m+1}(\widehat{\lambda}).$$

Finally, given $\lambda_1 \in \Lambda_j^{m, v_{k_1}}$, $\lambda_2 \in \Lambda_j^{m, v_{k_2}}$, we compute that

$$\alpha \cdot \psi_m(\lambda_1 \lambda_2^*) = \sum_{i=1}^N \sum_{p=1}^{r_{ij}} \widehat{\gamma_{ip}^m(\lambda_1)} \widehat{\gamma_{ip}^m(\lambda_2)}^* = \sum_{i=1}^N \sum_{p=1}^{r_{ij}} \gamma_{ip}^{m+1}(\widehat{\lambda_1}) \gamma_{ip}^{m+1}(\widehat{\lambda_2})^* = \psi_{m+1} \cdot \beta(\lambda_1 \lambda_2^*),$$

showing that $\alpha \cdot \psi_m = \psi_{m+1} \cdot \beta$, as desired.

5.3. Lifting the third (“second from the left end”) commutative square. Next, we will perform the lifting of the third commutative square of diagram (†) to that of diagram (◇). We will take the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}^N & \xrightarrow{\Omega_l} & \mathbb{Z}^N \\ \downarrow S & & \downarrow S \\ \mathbb{Z}^M & \xrightarrow{\Omega'_{m+l}} & \mathbb{Z}^M \end{array}$$

and we will lift it to a commutative diagram

$$\begin{array}{ccc} \mathcal{M}(E)_l & \xrightarrow{\alpha} & \mathcal{M}(E)_{l+1} \\ \downarrow \phi_l & & \downarrow \phi_{l+1} \\ \mathcal{M}(F)_{m+l} & \xrightarrow{\beta} & \mathcal{M}(F)_{m+l+1} \end{array}$$

where ϕ_l, ϕ_{l+1} are unital morphisms satisfying that $\phi_l \cdot \psi_m = j_{m, m+l}^F$ and $\phi_{l+1} \cdot \psi_{m+1} = j_{m+1, m+l+1}^F$.

As in the previous case, we start by defining a suitable partition for $F^{m+l} w_j$. For $\lambda \in F^m w_k$, we can define a partition

$$(5.10) \quad \lambda F^l w_j = \bigsqcup_{i=1}^N \bigsqcup_{\{\gamma \in \Gamma_i^{l, \lambda}\}} \Lambda_j^{m+l, \gamma}$$

where $|\Lambda_j^{m+l, \gamma}| = s_{ji}$ for $\gamma \in \Gamma_i^{l, \lambda}$. Given any $\gamma \in \Gamma_i^{l, \lambda}$, we will write

$$\Lambda_j^{m+l, \gamma} = \{\lambda_{jq}^l(\gamma) \mid 1 \leq q \leq s_{ji}\}.$$

Now, we will fix a condition analog to (5.9). We observe that for any $1 \leq j, k \leq M$ we have $\sum_{i=1}^N s_{ji} r_{ik} = (B^l)_{jk}$. Thus, we can consider an enumeration

$$(5.11) \quad w_k F^l w_j = \{\lambda_{ipq}^{jk} \mid 1 \leq i \leq N, 1 \leq p \leq r_{ik}, 1 \leq q \leq s_{ji}\}.$$

Since $r(\lambda_{kt}^0(v_d)) = w_k$ for all t, d , we can fix as definition

$$(5.12) \quad \lambda_{jq}^l(\gamma_{ip}^m(\lambda_{kt}^0(v_d))) := \lambda_{kt}^0(v_d) \cdot \lambda_{ipq}^{jk}.$$

Now, we define $\phi_l : \mathcal{M}(E)_l \rightarrow \mathcal{M}(F)_{m+l}$ as follows: for any $\gamma_1, \gamma_2 \in E^l v_i$,

$$\phi_l(\gamma_1 \gamma_2^*) = \sum_{j=1}^M \sum_{q=1}^{s_{ji}} \lambda_{jq}^l(\gamma_1) \lambda_{jq}^l(\gamma_2)^*.$$

It is easy to show that ϕ_l is a unital morphism satisfying $K_0(\phi_l) = S$.

Given $\lambda_1, \lambda_2 \in F^m w_k$, we compute

$$\phi_l \cdot \psi_m(\lambda_1 \lambda_2^*) = \sum_{i=1}^N \sum_{p=1}^{r_{ik}} \phi_l(\gamma_{ip}^m(\lambda_1) \gamma_{ip}^m(\lambda_2)^*) = \sum_{i=1}^N \sum_{j=1}^M \sum_{p=1}^{r_{ik}} \sum_{q=1}^{s_{ji}} \lambda_{jq}^l(\gamma_{ip}^m(\lambda_1)) \lambda_{jq}^l(\gamma_{ip}^m(\lambda_2))^*$$

that by (5.12) equals

$$\lambda_1 \left(\sum_{i=1}^N \sum_{j=1}^M \sum_{p=1}^{r_{ik}} \sum_{q=1}^{s_{ji}} \lambda_{ipq}^{jk} (\lambda_{ipq}^{jk})^* \right) \lambda_2^*$$

where the sums ranges through the set

$$\bigsqcup_{j=1}^M (w_k F^l w_j) = w_k F^l,$$

so that the above sum equals $j_{m,m+l}^F(\lambda_1 \lambda_2^*)$, as desired.

Now, we proceed to define a partition for $F^{m+l+1} w_j$. For $\lambda \in F^{m+1} w_k$, we can define a partition

$$(5.13) \quad \lambda F^l w_j = \bigsqcup_{i=1}^N \bigsqcup_{\{\gamma \in \Gamma_i^{l+1, \lambda}\}} \Lambda_j^{m+l+1, \gamma}$$

where $|\Lambda_j^{m+l+1, \gamma}| = s_{ji}$ for $\gamma \in \Gamma_i^{l+1, \lambda}$. Given any $\gamma = \gamma_{ip}^{m+1}(\lambda) \in \Gamma_i^{l+1, \lambda}$, we will write

$$\Lambda_j^{m+l+1, \gamma} = \{\lambda_{js}^{l+1}(\gamma) \mid 1 \leq s \leq s_{ji}\}.$$

In analogy with (5.12), for any $\lambda = \lambda_{kq}^1(f) \in F^{m+1} w_k$ we fix as definition

$$(5.14) \quad \lambda_{js}^{l+1}(\gamma_{ip}^{m+1}(\lambda)) := \lambda \cdot \lambda_{ips}^{jk}.$$

Now, we define $\phi_{l+1} : \mathcal{M}(E)_{l+1} \rightarrow \mathcal{M}(F)_{m+l+1}$ as follows: for any $\gamma_1, \gamma_2 \in E^{l+1} v_i$,

$$\phi_{l+1}(\gamma_1 \gamma_2^*) = \sum_{j=1}^M \sum_{s=1}^{s_{ji}} \lambda_{js}^{l+1}(\gamma_1) \lambda_{js}^{l+1}(\gamma_2)^*.$$

It is easy to show that ϕ_{l+1} is a unital morphisms satisfying $K_0(\phi_{l+1}) = S$. A similar computation to the one given above shows that $\phi_{l+1} \cdot \psi_{m+1} = j_{m+1, m+l+1}^F$.

Now, let $\lambda \in F^m w_k$. Using **(Ex 2)**, (5.14) and (5.12), we get

$$\lambda_{js}^{l+1}(\widehat{\gamma_{ip}^m(\lambda)}) = \lambda_{js}^{l+1}(\gamma_{ip}^{m+1}(\widehat{\lambda})) = \widehat{\lambda} \cdot \lambda_{ips}^{jk} = \widehat{\lambda} \cdot \lambda_{ips}^{jk} = \lambda_{js}^l(\widehat{\gamma_{ip}^m(\lambda)}).$$

Therefore, for $\gamma \in \Gamma_i^{l, \lambda}$, we have

$$\text{(Ex 3)} \quad \lambda_{js}^{l+1}(\widehat{\gamma}) = \widehat{\lambda_{js}^l(\gamma)}.$$

As before, this implies that $\phi_{l+1} \cdot \alpha = \beta \cdot \phi_l$.

5.4. The inductive argument. We are ready to state the induction hypothesis and show the inductive step. Note that everything is determined by the choices we have made in the previous subsections.

Let $n \in \mathbb{N}$, let $\varepsilon = 0, 1$, and suppose that for any $1 \leq k \leq n$ we have constructed:

(1) Partitions:

(a) Given $\gamma \in E^{(k-1)l+\varepsilon}$, $1 \leq i \leq N$, we have

$$\gamma E^l v_i = \bigsqcup_{j=1}^M \bigsqcup_{\{\lambda \in \Lambda_j^{m+(k-1)l+\varepsilon, \gamma}\}} \Gamma_i^{kl+\varepsilon, \lambda},$$

with $|\Gamma_i^{kl+\varepsilon, \lambda}| = r_{ij}$ for all $\lambda \in \Lambda_j^{m+(k-1)l+\varepsilon, \gamma}$. The elements of the subsets are denoted

$$\Gamma_i^{kl+\varepsilon, \lambda} = \{\gamma_{ir}^{(k-1)l+m+\varepsilon}(\lambda) \mid 1 \leq r \leq r_{ij}\},$$

where $\lambda = \lambda_{jq}^{(k-1)l+\varepsilon}(\gamma)$, and are defined by the rule

$$(5.15) \quad \gamma_{ir}^{(k-1)l+m+\varepsilon}(\lambda_{jq}^{(k-1)l+\varepsilon}(\gamma)) := \gamma \cdot \gamma_{ir}^m(\lambda_{jq}^0(r(\gamma))).$$

(b) Given $\lambda \in F^{(k-1)l+m+\varepsilon} w_i$, $1 \leq j \leq M$, we have

$$\lambda F^l w_j = \bigsqcup_{i=1}^N \bigsqcup_{\{\gamma \in \Gamma_i^{kl+\varepsilon, \lambda}\}} \Lambda_j^{kl+m+\varepsilon, \gamma},$$

with $|\Lambda_j^{kl+m+\varepsilon, \gamma}| = s_{ji}$ for all $\gamma \in \Gamma_i^{kl+\varepsilon, \lambda}$. The elements of the subsets are denoted

$$\Lambda_j^{kl+m+\varepsilon, \gamma} = \{\lambda_{jq}^{kl+\varepsilon}(\gamma) \mid 1 \leq q \leq s_{ji}\},$$

where $\gamma = \gamma_{ip}^{(k-1)l+m+\varepsilon}(\lambda)$, and are defined by the rule

$$(5.16) \quad \lambda_{jq}^{kl+\varepsilon}(\gamma_{ip}^{(k-1)l+m+\varepsilon}(\lambda)) := \lambda \cdot \lambda_{ipq}^{jt}.$$

(2) Exchange identities:

(a) Given $\lambda \in \Lambda_j^{m+(k-1)l, \gamma}$, we have

$$\text{(Ex 2k+1)} \quad \gamma_{ir}^{m+(k-1)l+1}(\widehat{\lambda}) = \gamma_{ir}^{m+(k-1)l}(\widehat{\lambda}).$$

(b) Given $\gamma \in \Gamma_i^{kl, \lambda}$, we have

$$\text{(Ex 2k)} \quad \lambda_{js}^{kl+1}(\widehat{\gamma}) = \widehat{\lambda_{js}^{kl}(\gamma)}.$$

(3) Unital homomorphisms:

(a) $\psi_{m+(k-1)l+\varepsilon} : \mathcal{M}(F)_{m+(k-1)l+\varepsilon} \rightarrow \mathcal{M}(E)_{kl+\varepsilon}$ defined by the rule

$$\psi_{m+(k-1)l+\varepsilon}(\lambda_1 \lambda_2^*) = \sum_{i=1}^N \sum_{r=1}^{r_{ij}} \gamma_{ir}^{m+(k-1)l+\varepsilon}(\lambda_1) \gamma_{ir}^{m+(k-1)l+\varepsilon}(\lambda_2)^*,$$

where $r(\lambda_1) = r(\lambda_2) = w_j$.

(b) $\phi_{kl+\varepsilon} : \mathcal{M}(E)_{kl+\varepsilon} \rightarrow \mathcal{M}(F)_{m+kl+\varepsilon}$ defined by the rule

$$\phi_{kl+\varepsilon}(\gamma_1 \gamma_2^*) = \sum_{j=1}^M \sum_{q=1}^{s_{ji}} \lambda_{jq}^{kl+\varepsilon}(\gamma_1) \lambda_{jq}^{kl+\varepsilon}(\gamma_2)^*,$$

where $r(\gamma_1) = r(\gamma_2) = v_i$.

satisfying:

(1) $K_0(\psi_{m+(k-1)l+\varepsilon}) = R$ and $K_0(\phi_{kl+\varepsilon}) = S$.

(2) $\psi_{m+(k-1)l+\varepsilon} \cdot \phi_{(k-1)l+\varepsilon} = j_{(k-1)l+\varepsilon, kl+\varepsilon}^E$ and $\phi_{kl+\varepsilon} \cdot \psi_{m+(k-1)l+\varepsilon} = j_{m+(k-1)l+\varepsilon, m+kl+\varepsilon}^F$.

(3) $\alpha \cdot \psi_{m+(k-1)l} = \psi_{m+(k-1)l+1} \cdot \beta$ and $\phi_{kl+1} \cdot \alpha = \beta \cdot \phi_{kl}$.

Now, we will show that the same holds for $k = n + 1$. First, we will take the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}^N & \xrightarrow{\Omega_{(n+1)l}} & \mathbb{Z}^N \\ \uparrow R & & \uparrow R \\ \mathbb{Z}^M & \xrightarrow{\Omega'_{m+nl}} & \mathbb{Z}^M \end{array}$$

and we will lift it to a commutative diagram

$$\begin{array}{ccc} \mathcal{M}(E)_{(n+1)l} & \xrightarrow{\alpha} & \mathcal{M}(E)_{(n+1)l+1} \\ \uparrow \psi_{m+nl} & & \uparrow \psi_{m+nl+1} \\ \mathcal{M}(F)_{m+nl} & \xrightarrow{\beta} & \mathcal{M}(F)_{m+nl+1} \end{array}$$

We now define a partition of $E^{(n+1)l+\varepsilon} v_i$. For $\gamma \in E^{nl+\varepsilon} v_k$, we can take a partition

$$(5.17) \quad \gamma E^l v_i = \bigsqcup_{j=1}^M \bigsqcup_{\{\lambda \in \Lambda_j^{m+nl+\varepsilon, \gamma}\}} \Gamma_i^{(n+1)l+\varepsilon, \lambda}$$

where $|\Gamma_i^{(n+1)l+\varepsilon, \lambda}| = r_{ij}$ for all $\lambda \in \Lambda_j^{m+nl+\varepsilon, \gamma}$. Given any $\lambda = \lambda_{jq}^{nl+\varepsilon}(\gamma) \in \Lambda_j^{m+nl+\varepsilon, \gamma}$, we will denote

$$\Gamma_i^{(n+1)l+\varepsilon, \lambda} = \{\gamma_{ir}^{m+nl+\varepsilon}(\lambda) \mid 1 \leq r \leq r_{ij}\}.$$

In analogy with (5.9), we can fix as definition

$$(5.18) \quad \gamma_{ir}^{m+nl+\varepsilon}(\lambda_{jq}^{nl+\varepsilon}(\gamma)) := \gamma \cdot \gamma_{ir}^m(\lambda_{jq}^0(r(\gamma))).$$

Now, we take $\gamma \in E^{nl}v_k$, and we compute, using (Ex 2n) and (5.18) that

$$\gamma_{ir}^{m+nl+1}(\widehat{\lambda_{jq}^{nl}(\gamma)}) = \gamma_{ir}^{m+nl+1}(\lambda_{jq}^{nl+1}(\widehat{\gamma})) = \widehat{\gamma} \cdot \gamma_{ir}^m(\lambda_{jq}^0(r(\gamma))) = \gamma_{ir}^{m+nl}(\widehat{\lambda_{jq}^{nl}(\gamma)}).$$

This gives that for all $\lambda \in \Lambda_j^{m+nl,\gamma}$,

$$\text{(Ex 2n+1)} \quad \gamma_{ir}^{m+nl+1}(\widehat{\lambda}) = \gamma_{ir}^{m+nl}(\widehat{\lambda}).$$

In the same way as in the previous cases, we define, for $\lambda_1, \lambda_2 \in F^{m+nl+\varepsilon}w_j$

$$\psi_{m+nl+\varepsilon}(\lambda_1\lambda_2^*) = \sum_{i=1}^N \sum_{r=1}^{r_{ij}} \gamma_{ir}^{m+nl+\varepsilon}(\lambda_1)\gamma_{ir}^{m+nl+\varepsilon}(\lambda_2)^*.$$

It is easy to show that $\psi_{m+nl+\varepsilon}$ is a unital morphism such that $K_0(\psi_{m+nl+\varepsilon}) = R$.

Given $\gamma_1, \gamma_2 \in E^{nl+\varepsilon}v_k$, one checks, using (5.18) and (5.5) that

$$\psi_{m+nl+\varepsilon} \cdot \phi_{nl+\varepsilon}(\gamma_1\gamma_2^*) = \gamma_1 \left(\sum_{\gamma \in v_k E^l} \gamma\gamma^* \right) \gamma_2^* = j_{nl+\varepsilon, (n+1)l+\varepsilon}^E(\gamma_1\gamma_2^*).$$

Finally, (Ex 2n+1) gives that $\psi_{m+nl+1} \cdot \beta = \alpha \cdot \psi_{m+nl}$.

The last step of the proof, consisting in defining the sets $\Lambda_j^{m+(n+1)l+\varepsilon,\gamma}$ and the maps $\phi_{(n+1)l+\varepsilon}$ satisfying the appropriate relations, is similar to the case $n = 0$ which we did before in detail. We leave the details to the reader. \square

6. UNIQUENESS OF LIFTINGS

Let us recall the strong classification conjecture, due to Hazrat:

Conjecture 6.1. [16, Conjecture 3] The graded Grothendieck group K_0^{gr} is a fully faithful functor from the category of unital Leavitt path algebras with graded homomorphisms modulo inner-automorphisms to the category of pre-ordered abelian groups with order-unit.

The conjecture has been proved by Hazrat for acyclic graphs ([16, Theorem 1]). The conjecture means that given any order-preserving $\mathbb{Z}[t, t^{-1}]$ -isomorphism $f: K_0^{\text{gr}}(L(E)) \rightarrow K_0^{\text{gr}}(L(F))$, such that $f([R]) = [S]$, there exists a graded isomorphism $\psi: R \rightarrow S$ such that $K_0(\psi) = f$. Moreover, given any two such isomorphisms φ and ψ , with $K_0(\psi) = K_0(\varphi)$, there is an inner automorphism τ of S such that $\psi = \tau \circ \varphi$.

In this section, we will study the uniqueness part of this conjecture, obtaining, for any finite graph without sources E , the form of the automorphisms θ of $L(E)$ such that $K_0^{\text{gr}}(\theta) = \text{id}$. Their form suggests that they are not necessarily inner automorphisms, and indeed this class of automorphisms always includes the ones given by the torus action (or gauge action, in the language of C^* -algebras), which are not inner in general. In order to get a complete description of this group of automorphisms, it remains to elucidate when certain injective algebra endomorphisms are surjective.

Let \mathfrak{G} be the group of graded automorphisms φ of $L(E)$ such that the restriction of φ to $\mathcal{M}(E)$ is locally inner. We first observe that this is the kernel of the map $\text{gr-Aut}(L(E)) \rightarrow \text{Aut}_{\mathbb{Z}[t,t^{-1}]}(K_0^{\text{gr}}(L(E)))$.

We need a lemma:

Lemma 6.2. *Let F be a row-finite graph having a tail v_0, v_1, v_2, \dots , that is, for each $i \geq 0$, there is exactly one arrow e_i such that $s(e_i) = v_i$, and in addition we have $r(e_i) = v_{i+1}$. Then $v_{i+1}L(F) \cong \mathcal{T}(v_iL(F))$, where \mathcal{T} denotes the translation functor.*

Proof. Consider the graded isomorphism $\gamma: v_{i+1}L(F) \rightarrow \mathcal{T}(v_iL(F))$ given by $\gamma(v_{i+1}x) = e_ix$. \square

Proposition 6.3. *Let E be a finite graph. Then the group \mathfrak{G} of graded K -algebra automorphisms of $L(E)$ whose restriction to $\mathcal{M}(E)$ is locally inner is precisely the group of graded K -algebra automorphisms φ of $L(E)$ such that $K_0^{\text{gr}}(\varphi) = \text{id}$.*

Proof. If φ is a graded automorphism of $L(E)$ such that $K_0^{\text{gr}}(\varphi) = \text{id}$, then the restriction φ_0 of φ to $\mathcal{M}(E)$ must satisfy $K_0(\varphi_0) = \text{id}$, and so φ_0 must be locally inner.

Conversely, let $\varphi \in \mathfrak{G}$. It suffices to show that the graded finitely generated projective $L(E)$ -modules are direct sums of translates of projective modules of the form $P \otimes_{\mathcal{M}(E)} L(E)$, where P is a finitely generated projective $\mathcal{M}(E)$ -module. This is clear for finite graphs without sinks, because then, by Dade's Theorem and [16, Theorem 4], every graded finitely generated projective $L(E)$ -module is induced by a finitely generated projective $\mathcal{M}(E)$ -module. If E does have sinks then we apply the desingularization process to obtain a new graph F without sinks such that $L(E)$ and $L(F)$ are Morita-equivalent. Indeed there is a homogeneous idempotent e in $L(F)$ such that $L(E)$ is graded-isomorphic to $eL(F)e$, and such that $L(F)eL(F) = L(F)$. It follows from [17, Example 2] that there is a graded equivalence between $gr - L(E)$ and $gr - L(F)$. Now $L(F)$ is strongly graded (by [16, Theorem 4]), so by Dade's Theorem its graded finitely generated projective modules come from the finitely generated projective $\mathcal{M}(F)$ -modules. The finitely generated projective $\mathcal{M}(F)$ -modules are generated by the finitely generated projective modules of the form $P \otimes_{\mathcal{M}(E)} e\mathcal{M}(F)$, for P a finitely generated projective $\mathcal{M}(E)$ -module, and the projective modules of the form $v_i\mathcal{M}(F)$, where v_0 is a sink in E and v_1, v_2, \dots , is the tail added to v_0 in F . Now observe that, by Lemma 6.2, we have $v_iL(F) \cong \mathcal{T}^i(v_0L(F))$, where \mathcal{T} denotes the translation functor. Therefore the graded finitely generated projective $L(F)$ -modules are direct sums of translates of graded finitely generated projective modules of the form $P \otimes_{\mathcal{M}(E)} eL(F)$, where P is a finitely generated projective $\mathcal{M}(E)$ -module, and therefore the same is true for the graded finitely generated projective $L(E)$ -modules. \square

Let E be a finite graph without sources, and let u, z be invertible elements in $\mathcal{M}(E)$ such that $z(uvu^{-1}) = (uvu^{-1})z$ for all $v \in E^0$. We define the algebra endomorphism $\theta_{u,z}$ of $L(E)$ by the rules $\theta_{u,z}(v) = uvu^{-1}$ for $v \in E^0$, and $\theta_{u,z}(e) = (ueu^{-1})z$, $\theta_{u,z}(e^*) = z^{-1}(ue^*u^{-1})$ for $e \in E^1$. Then it is easily checked that $\theta_{u,z}$ induces a graded algebra endomorphism of $L(E)$. It is clearly injective, because it is graded and $\theta_{u,z}(v) \neq 0$ for all $v \in E^0$ (see [30, Theorem 4.8]). It is surjective if and only if u^\pm and z^\pm belong to the image of $\theta_{u,z}$. More concretely, if $s, t \in \mathcal{M}(E)$ are elements such that $\theta_{u,z}(s) = u^{-1}$ and $\theta_{u,z}(t) = u^{-1}z^{-1}u$, then s, t are invertible, satisfy $(sxs^{-1})t = t(sxs^{-1})$ for every $x \in E^0$, and $\theta_{s,t} = \theta_{u,z}^{-1}$.

Lemma 6.4. *With the above notation, we have $K_0^{\text{gr}}(\theta_{u,z}) = \text{id}$.*

Proof. By Proposition 6.3, it suffices to check that the restriction of $\theta_{u,z}$ to $\mathcal{M}(E)$ is a locally inner endomorphism of $\mathcal{M}(E)$.

Let Q_n be the set of paths γ such that either $|\gamma| = n$ or $|\gamma| < n$ and $r(\gamma)$ is a sink. We will use the filtration $\mathcal{M}(E) = \bigcup_{n=0}^{\infty} \mathcal{M}(E)_n$ of $\mathcal{M}(E)$, where $\mathcal{M}(E)_n$ is the linear span of the elements $\gamma\mu^*$, where $\gamma, \mu \in Q_n$, $|\gamma| = |\mu|$ and $r(\gamma) = r(\mu)$.

For $v \in E^0$, define $u^v = uv \in \mathcal{M}(E)$ and, for $e \in E^1$, define $u^e = uee^* \in \mathcal{M}(E)$. Furthermore, for $\gamma = e_1e_2 \cdots e_n \in E^n$ with $n \geq 2$, define

$$u^\gamma = u^{e_1e_2 \cdots e_n} = (ue_1u^{-1})z(ue_2u^{-1})z \cdots (ue_{n-1}u^{-1})z(ue_n)\gamma^* \in \mathcal{M}(E).$$

Finally define, for each $n \geq 0$,

$$u_n = \sum_{\gamma \in Q_n} u^\gamma.$$

It is easy to check that u_n is an invertible element in $\mathcal{M}(E)$, with inverse given by

$$u_n^{-1} = \sum_{\gamma \in Q_n} (u^\gamma)^\dagger,$$

where $(u^v)^\dagger = vu^{-1}$ for $v \in E^0$, $(u^e)^\dagger = ee^*u^{-1}$ for $e \in E^1$, and

$$(u^\gamma)^\dagger = \gamma(e_n^*u^{-1})z^{-1}(ue_{n-1}^*u^{-1}) \cdots z^{-1}(ue_1^*u^{-1})$$

for $\gamma = e_1 \cdots e_n \in E^n$ with $n \geq 2$.

One can check that, for $x \in \mathcal{M}(E)_n$, we have

$$\theta_{u,z}(x) = u_n x u_n^{-1},$$

which shows that $\theta_{u,z}$ is a locally inner endomorphism of $\mathcal{M}(E)$. It follows that $K_0^{\text{gr}}(\theta_{u,z}) = \text{id}$, as desired. \square

Theorem 6.5. *Let E and F be finite graphs without sources. Let φ and ψ be two graded algebra isomorphisms from $L_K(E)$ onto $L_K(F)$ such that $K_0^{\text{gr}}(\varphi) = K_0^{\text{gr}}(\psi)$. Then there exist invertible elements u, z in $\mathcal{M}(F)_0$, with $z(uvu^{-1}) = (uvu^{-1})z$ for all $v \in F^0$, such that*

$$\psi = \theta_{u,z} \circ \varphi.$$

Proof. It is enough to determine the structure of the graded algebra automorphisms φ of $L(E)$ such that $K_0^{\text{gr}}(\varphi) = \text{id}$. So, let φ be such an automorphism, and denote by φ_0 the restriction of φ to $\mathcal{M}(E)$, which is an algebra automorphism of the ultramatricial algebra $\mathcal{M}(E)$ such that $K_0(\varphi_0) = \text{id}$. By [14, Lemma 15.23(b)], there exists $u \in \mathcal{M}(E)$ such that $\varphi(x) = uxu^{-1}$ for all $x \in \mathcal{M}(E)_1$.

Set $A = \mathcal{M}(E)$, set $t_+ = \sum_{i=1}^n e_i$, $t_- = \sum_{i=1}^n e_i^*$, where $e_i \in E^1$ satisfy that $r(e_i) = v_i$ for all i , and $E^0 = \{v_1, \dots, v_n\}$. As observed before, we have $L(E) = A[t_+, t_-, \alpha]$, where $\alpha: A \rightarrow pAp$ is the corner-isomorphism given by $\alpha(x) = t_+xt_-$ for $x \in A$.

It is easy to show that φ is determined by φ_0 and $\varphi(t_+) = a_0t_+$, $\varphi(t_-) = t_-b_0$, where $a_0 \in Ap$ and $b_0 \in pA$ must satisfy the relations $b_0a_0 = p$ and

$$\varphi_0(\alpha(x)) = a_0\alpha(\varphi_0(x))b_0$$

for all $x \in A$. In particular we have $a_0b_0 = \varphi_0(p) = upu^{-1}$.

Set $\zeta = \alpha(u)u^{-1}a_0 \in pAp$, and $\varpi = b_0u\alpha(u^{-1})$. Then, we have

$$\zeta\varpi = \alpha(u)u^{-1}a_0b_0u\alpha(u^{-1}) = \alpha(u)u^{-1}upu^{-1}u\alpha(u^{-1}) = \alpha(1) = p$$

and

$$\varpi\zeta = b_0u\alpha(u^{-1})\alpha(u)u^{-1}a_0 = b_0u\alpha(1)u^{-1}a_0 = b_0\varphi_0(\alpha(1))a_0 = \alpha(\varphi_0(1)) = p.$$

So ζ is invertible in pAp , with inverse ϖ . Set $z = \alpha^{-1}(\zeta)$, an invertible element of A . Then, for $v \in E^0$ we have

$$\begin{aligned} z(uvu^{-1})z^{-1} &= \alpha^{-1}(\zeta\alpha(\varphi_0(v))\varpi) = \alpha^{-1}(\alpha(u)u^{-1}a_0\alpha(\varphi_0(v))b_0u\alpha(u^{-1})) \\ &= \alpha^{-1}(\alpha(u)u^{-1}\varphi_0(\alpha(v))u\alpha(u^{-1})) = \alpha^{-1}(\alpha(u)\alpha(v)\alpha(u^{-1})) = uvu^{-1}, \end{aligned}$$

showing that $z(uvu^{-1}) = (uvu^{-1})z$ for all $v \in E^0$. Moreover, if $e \in E^1$, then

$$\begin{aligned} \varphi(e) &= \varphi_0(et_-)a_0t_+ = uet_-u^{-1}a_0t_+ = uet_-(pu^{-1}a_0)t_+ \\ &= uet_-\alpha(u^{-1})\zeta t_+ = ueu^{-1}\alpha^{-1}(\zeta) = ueu^{-1}z. \end{aligned}$$

Similarly $\varphi(e^*) = z^{-1}ue^*u^{-1}$. Since φ and $\theta_{u,z}$ agree on the generators of $L(E)$, we conclude that $\varphi = \theta_{u,z}$, as desired. \square

Let u, z be invertible elements in $\mathcal{M}(E)_0$, with $z(uvu^{-1}) = (uvu^{-1})z$ for all $v \in F^0$. It is not clear whether all the endomorphisms $\theta_{u,z}$ are surjective. As we are going to show, this question has a positive answer when, in addition, we have that $z \in Z(\mathcal{M}(E))$.

We denote by $U(R)$ the group of units of a unital ring R .

Proposition 6.6. *Let E be a finite graph without sources. Let \mathfrak{G} be the group of graded automorphisms φ of $L(E)$ such that the restriction of φ to $\mathcal{M}(E)$ is locally inner. Then there is a group homomorphism*

$$\zeta: U(\mathcal{M}(E)) \times U(Z(\mathcal{M}(E))) \rightarrow \mathfrak{G}.$$

The kernel of ζ is the set of elements (u, z) in $U(\mathcal{M}(E)) \times U(Z(\mathcal{M}(E)))$ such that $u \in Z(\mathcal{M}(E))$ and $ueu^{-1} = ez^{-1}$ for all $e \in E^1$.

Proof. Define $\zeta: U(\mathcal{M}(E)) \times U(Z(\mathcal{M}(E))) \rightarrow \mathfrak{G}$ by $\zeta(u, z) = \theta_{u,z}$. It is easy to check that for $(u, z) \in U(\mathcal{M}(E)) \times U(Z(\mathcal{M}(E)))$ we have $\theta_{u,z}(x) = uxu^{-1}$ for all $x \in \mathcal{M}(E)$. Using this it is straightforward to check that $\theta_{u_1, z_1} \circ \theta_{u_2, z_2} = \theta_{u_1u_2, z_1z_2}$ for $(u_1, z_1), (u_2, z_2) \in U(\mathcal{M}(E)) \times U(Z(\mathcal{M}(E)))$. This shows that ζ is a group homomorphism. If (u, z) belongs to the kernel of ζ , then u must belong to the center of $\mathcal{M}(E)$ by the previous observation, and $ueu^{-1} = ez^{-1}$. Conversely if these conditions are satisfied then for $e \in E^1$ we have

$$ze^* = (ze^*)(ueu^{-1}(ue^*u^{-1})) = z(e^*e)z^{-1}(ue^*u^{-1}) = e^*eue^*u^{-1} = ue^*u^{-1},$$

so that $ue^*u^{-1} = ze^*$ and we conclude that $\theta_{u,z} = \text{id}$. \square

We now compute the group \mathfrak{G} in a particular example. This gives a counterexample to the uniqueness part of [16, Conjecture 3], in the sense that we obtain a non-inner graded automorphism φ of a Leavitt path algebra such that $K_0^{gr}(\varphi) = \text{Id}$, due to the presence of the torus action. Indeed, the same obstruction to innerness appears already for the Leavitt path algebra of the single loop $K[x, x^{-1}]$.

Example 6.7. Let

$$\mathfrak{T} := K\langle x, y \mid yx = 1 \rangle$$

be the (algebraic) Toeplitz algebra, which can be realized as the Leavitt path algebra associated to the graph E with $E^0 = \{v, w\}$ and $E^1 = \{\alpha, \beta\}$, with $s(\alpha) = r(\alpha) = v = s(\beta)$, and $r(\beta) = w$. The algebra $L(E)$ is isomorphic to \mathfrak{T} via an isomorphism that sends $\alpha + \beta$ to x and $\alpha^* + \beta^*$ to y .

Define $e_{ij} = x^i(1 - xy)y^j$ for $i, j \geq 0$. This is a set of matrix units in \mathfrak{T} , the ideal $I = \bigoplus_{i,j} e_{ij}K \cong M_\infty(K)$ is the only non-trivial ideal of \mathfrak{T} and $\mathfrak{T}/I \cong K[x, x^{-1}]$. The 0-component of \mathfrak{T} is precisely

$$\mathfrak{T}_0 = \left\{ \sum_{i=0}^n a_i e_{ii} + a_{n+1} x^{n+1} y^{n+1} \mid a_0, \dots, a_{n+1} \in K, n \geq 0 \right\} \cong K \cdot 1 \oplus \left(\bigoplus_{\mathbb{Z}^+} K \right).$$

In particular, \mathfrak{T}_0 is a commutative algebra, and so it follows from Theorem 6.5 that $\mathfrak{G} \cong \zeta(U(\mathfrak{T}_0) \times U(\mathfrak{T}_0))$. Now, it is shown in [12, Theorem 4.3] that

$$\text{Aut}_{K\text{-alg}}(\mathfrak{T}) \cong \mathbb{T} \rtimes \text{Inn}(\mathfrak{T}) \cong \mathbb{T} \rtimes GL_\infty(K).$$

Here \mathbb{T} denotes the ‘‘torus action’’ of K^\times on \mathfrak{T} , which, for $a \in K^\times$, sends x to ax and y to $a^{-1}y$. Note that this is precisely $\zeta((1, a \cdot 1)) \in \mathfrak{G}$. The group $GL_\infty(K)$ is the group of units of the multiplicative monoid $1 + M_\infty(K)$, and it acts by conjugation on \mathfrak{T} . Denote by $D_\infty(K)$ the group of units of the monoid $1 + \left(\bigoplus_{i=0}^\infty e_{ii}K \right)$. For $u = \sum_{i=0}^n a_i e_{ii} + x^{n+1} y^{n+1} \in D_\infty(K)$, we have the formula

$$uxu^{-1} = x \left(\sum_{i=0}^n a_i^{-1} a_{i+1} e_{ii} + x^{n+1} y^{n+1} \right),$$

(where $a_{n+1} = 1$). It follows from this that

$$\zeta(1, D_\infty(K)) = \zeta(U(\mathfrak{T}_0), 1) = D_\infty(K) \subset GL_\infty(K) \subseteq \text{Inn}(\mathfrak{T}),$$

which shows that

$$\mathfrak{G} = \mathbb{T} \times D_\infty(K).$$

An element in \mathfrak{G} is an inner automorphism of \mathfrak{T} if and only its \mathbb{T} -component is 1.

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REFERENCES

- [1] G. ABRAMS, G. ARANDA PINO, The Leavitt path algebra of a graph, *J. Algebra* **293** (2005), 319–334.
- [2] G. ABRAMS, G. ARANDA PINO, Purely infinite simple Leavitt path algebras, *J. Pure Appl. Algebra*, **207** (2006), 553–563.
- [3] G. ABRAMS, P. N. ÁNH, A. LOULY, E. PARDO, Flow invariants in the classification of Leavitt algebras, *J. Algebra* **333** (2011), 202–231.
- [4] P. ARA, M. BRUSTENGA, G. CORTIÑAS, K-theory of Leavitt path algebras, *Münster J. Math.* **2** (2009), 5–33.
- [5] P. ARA, G. CORTIÑAS, Tensor products of Leavitt path algebras, *Proc. Amer. Math. Soc.* **141** (2013), 2629–2639.
- [6] P. ARA, M.A. GONZÁLEZ-BARROSO, K.R. GOODEARL, E. PARDO, Fractional skew monoid rings. *J. Algebra* **278** (2004), no. 1, 104–126.
- [7] P. ARA, M.A. MORENO, E. PARDO, Nonstable K-Theory for graph algebras, *Algebra Rep. Th.* **10** (2007), 157–178.
- [8] P. ARA, E. PARDO, Stable rank of Leavitt path algebras, *Proc. Amer. Math. Soc.* **136** (2008), 2375–2386.
- [9] G. ARANDA PINO, E. PARDO, M. SILES MOLINA, Exchange Leavitt path algebras and stable rank, *J. Algebra* **305** (2006), 912–936.
- [10] G. ARANDA PINO, F. PERERA, M. SILES MOLINA, EDs., *Graph algebras: bridging the gap between analysis and algebra*, ISBN: 978-84-9747-177-0, University of Málaga Press, Málaga, Spain (2007).
- [11] T. BATES, Applications of the gauge-invariant uniqueness theorem for the Cuntz-Krieger algebras of directed graphs, *Bull. Austral. Math. Soc.* **65** (2002), 57–67.
- [12] V. V. BAVULA, The group of automorphisms of the algebra of polynomial integro-differential operators, *J. Algebra* **348** (2011), 233–263.
- [13] J. FRANKS, Flow equivalence of subshifts of finite type, *Ergod. Th. & Dynam. Sys.* **4** (1984), 53–66.
- [14] K. R. GOODEARL “Von Neumann Regular Rings”, Second Edition, Krieger, Malabar 1991.
- [15] R. GORDON, E. L. GREEN, Graded Artin algebras. *J. Algebra* **76** (1982), 111–137.
- [16] R. HAZRAT, The graded Grothendieck group and the classification of Leavitt path algebras, *Math. Annalen* **355** (2013), 273–325.
- [17] R. HAZRAT, The dynamics of Leavitt path algebras, *J. Algebra* **384** (2013), 242–266.
- [18] R. HAZRAT, Graded Rings and graded Grothendieck groups. Monograph, rhazrat.wordpress.com/preprint.
- [19] E. KIRCHBERG, The classification of purely infinite C*-algebras using Kasparov theory, preprint.
- [20] W.G. LEAVITT, The module type of a ring, *Trans. Amer. Math. Soc.* **103** (1962), 113–130.
- [21] D. LIND, B. MARCUS, “An Introduction to Symbolic Dynamics and Coding”, Cambridge Univ. Press, Cambridge, 1995, Reprinted 1999 (with corrections). ISBN 0-521-55900-6.
- [22] K. MATSUMOTO, Classification of Cuntz-Krieger algebras by orbit equivalence of topological Markov shifts, *Proc. Amer. Math. Soc.* **141** (2013), 2329–2342.
- [23] K. MATSUMOTO, K-groups of the full group actions on one-sided topological Markov shifts, *Discrete Contin. Dyn. Syst.* **33** (2013), 3753–3765.
- [24] C. NĂSTĂSESCU, F. VAN OYSTAEYEN, “Methods of graded rings”, Lecture Notes in Mathematics, 1836, Springer Verlag, Berlin, 2004.
- [25] W.L. PASCHKE, The crossed product of a C*-algebra by an endomorphism, *Proc. Amer. Math. Soc.* **80** (1980), 113–118.
- [26] N.C. PHILLIPS, A classification theorem for nuclear purely infinite simple C*-algebras, *Doc. Math.* **5** (2000), 49–114.
- [27] I. RAEBURN, “Graph Algebras”, CBMS Reg. Conf. Ser. Math., vol. 103, Amer. Math. Soc., Providence, RI, 2005.
- [28] M. RØRDAM, Classification of Cuntz-Krieger algebras, *K-Theory* **9** (1995), 31–58.
- [29] S. PAUL SMITH, Category equivalences involving graded modules over path algebras of quivers, *Adv. Math.* **230** (2012), 1780–1810.
- [30] M. TOMFORDE, Uniqueness theorems and ideal structure for Leavitt path algebras, *J. Algebra* **318** (2007), no. 1, 270–299.

- [31] R.F. WILLIAMS, Classification of subshifts of finite type, *Ann. Math.* **98(2)** (1973), 120–153.

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