J. reine angew. Math. **588** (2005), 1—25

Journal für die reine und angewandte Mathematik

Semilattices of groups and inductive limits of Cuntz algebras

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Abstract. We characterize, in terms of elementary properties, the abelian monoids which are direct limits of finite direct sums of monoids of the form $(\mathbb{Z}/n\mathbb{Z}) \sqcup \{0\}$ (where 0 is a new zero element), for positive integers n. The key properties are the Riesz refinement property and the requirement that each element x has finite order, that is, $(n+1)x = x$ for some positive integer n. Such monoids are necessarily semilattices of abelian groups, and part of our approach yields a characterization of the Riesz refinement property among semilattices of abelian groups. Further, we describe the monoids in question as certain submonoids of direct products $\Lambda \times G$ for semilattices Λ and torsion abelian groups G. When applied to the monoids $V(A)$ appearing in the non-stable K-theory of C*-algebras, our results yield characterizations of the monoids $V(A)$ for C* inductive limits A of sequences of finite direct products of matrix algebras over Cuntz algebras $\mathcal{O}_n$. In particular, this completely solves the problem of determining the range of the invariant in the unital case of Rørdam's classification of inductive limits of the above type.

1. Introduction

As indicated in the abstract, the goal of this paper is to prove a semigroup-theoretic result motivated by, and with applications to, the classification theory of C*-algebras. The relevant C*-algebras, which we will call *Cuntz limits* for short, are the C* inductive limits of sequences of finite direct products of full matrix algebras over the Cuntz algebras $\mathcal{O}_n$. (We recall the definition of the latter for the information of non-C*-algebraic readers: for $2 \leqq n < \infty$, the Cuntz algebra $\mathcal{O}_n$, introduced in [4], is the unital C*-algebra generated by elements $s_1, \dots, s_n$ with relations $s_i^* s_j = \delta_{ij}$ and $\sum_{i=1}^{n} s_i s_i^* = 1$.) Our results will provide an analogue for Cuntz limits of the description of the range of the invariant for separable AF C*-algebras (namely, ordered K_0) by Elliott [8] and Effros, Handelman, and Shen [7]. We

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The research of the first author was partially supported by an NSF grant. The research of the second author was partially supported by DGESIC grant BFM2001-3141, by FQM-298 PAI III grant (Junta de Andalucía), and by PR2001-0276 grant from Secretaría de Estado de Educación y Universidades, M.E.C.D. (Spain).

begin by sketching the source of the problem and giving a precise formulation. Most of the remainder of the paper is purely semigroup-theoretic, except for the applications to C*-algebras in the final section.

In [20], Rørdam gave a K-theoretic classification of even Cuntz limits (i.e., C* inductive limits of sequences of finite direct products of matrix algebras over $\mathcal{O}_n$s with n even). The invariant which Rørdam used for his classification is equivalent, in the unital case, to the pair $(V(A), [1_A])$ where $V(A)$ denotes the (additive, commutative) monoid of Murray-von Neumann equivalence classes of projections (self-adjoint idempotents) in matrix algebras over a C*-algebra A, and $[1_A]$ is the class in $V(A)$ of the unit projection in A (cf. [1], Sections 4.6, 5.1, and 5.2). Thus, the unital case of the classification states that if A and B are unital even Cuntz limits, then $A \cong B$ if and only if $(V(A), [1_A]) \cong (V(B), [1_B])$, that is, there is a monoid isomorphism $V(A) \to V(B)$ sending $[1_A]$ to $[1_B]$ (cf. [20], Theorem 7.1). Rørdam has communicated to us [21] that his classification can be extended to all Cuntz limits by investing the work of Kirchberg [15] and Phillips [17].

As with any classification theorem, an accompanying problem is to describe the range of the invariant—that is, which pairs (M, u) (an abelian monoid M together with an element $u \in M$) appear as $(V(A), [1_A])$ for unital Cuntz limits A? This question reduces to an interesting problem in the theory of monoids which we shall describe shortly. The major aim of this paper is to solve this monoid problem, and then draw corresponding conclusions for Cuntz limits. For non-unital Cuntz limits A, Rørdam's classifying invariant amounts to a triple $(V(A), P(A), \tau)$ where $P(A)$ is a partial semigroup consisting of unitary equivalence classes of projections in A and $\tau : P(A) \to V(A)$ is a natural homomorphism. Thus, $V(A)$ is an important part of the classification in general, and pinning down its structure is of interest also in the non-unital case.

In trying to match a given pair (M, u) with a unital Cuntz limit, it is easy to eliminate u. First, one notes that u must be an *order-unit* in M, that is, for any $x \in M$, there exist $y \in M$ and $n \in \mathbb{N}$ such that $x + y = nu$. Second, if we can find a Cuntz limit B such that $V(B) \cong M$, then there is a projection p in some matrix algebra $M_n(B)$ whose class $[p]$ corresponds to u, and the C*-algebra $A = pM_n(B)p$ is a unital Cuntz limit satisfying $(V(A), [1_A]) \cong (M, u)$. Thus, we concentrate on the problem of describing those abelian monoids which appear as $V(A)$s. In the case of simple algebras, Rørdam's work provides the answer—the abelian monoids appearing as $V(A)$ for simple (unital) Cuntz limits A are the monoids $G \sqcup \{0\}$ for arbitrary countable torsion abelian groups G [20], Proposition 2.5 and Theorem 2.6, where $G \sqcup \{0\}$ is the monoid obtained from G by adjoining a new zero element. The answer is also known for the case of $\mathcal{O}_2$-limits (Cuntz limits involving only direct products of matrix algebras over $\mathcal{O}_2$), one of the basic ingredients of a class of C*-algebras classified by Lin in [16]. The monoids appearing as $V(A)$ for $\mathcal{O}_2$-limits are just the direct limits of sequences of *Boolean* monoids (finite direct sums of copies of the two-element monoid). These direct limits were shown by Bulman-Fleming and McDowell to be precisely the countable distributive upper semilattices, see [2], Theorem 3.1. While the result of [2] relies heavily on Shannon's categorical result [22], Theorem 2, a purely general algebraic proof has been given by the first and third authors [11], Theorem 6.6.

It is known that the functor $V(-)$ converts C* inductive limits to monoid inductive (direct) limits, that it converts finite direct products to direct sums, and that $V(M_m(A)) \cong V(A)$ for all A and m. Moreover, $V(\mathcal{O}_n) \cong (\mathbb{Z}/(n-1)\mathbb{Z}) \sqcup \{0\}$ (this follows from the computations in [5]; see also Section 7). Thus, the monoid problem boils down to the following task (where we have replaced $n-1$ by n for convenience):

Characterize those abelian monoids isomorphic to direct limits of sequences of finite direct sums of building blocks of the form $(\mathbb{Z}/n\mathbb{Z}) \sqcup \{0\}$.

In this paper, we solve the above problem, and thus characterize the monoids that appear as $V(A)$ for Cuntz limits A.

2. Background

Monoids. All monoids in this paper will be abelian, written additively, and so with additive identities denoted 0. The monoids that appear as $V(A)$ for Cuntz limits A enjoy several standard properties familiar from other classification results, such as conicality and refinement. Recall that a monoid M is *conical* if $x+y=0$ (for $x, y \in M$) always implies $x=y=0$, and that M satisfies the *Riesz refinement property* provided that for any $x_1, x_2, y_1, y_2 \in M$ satisfying $x_1+x_2=y_1+y_2$, there exist elements $z_{ij} \in M$ such that each $x_i = z_{i1}+z_{i2}$ and each $y_j = z_{1j}+z_{2j}$. It is convenient to record the latter four equations in the form of a *refinement matrix*:

	y_1	y_2
x_1	z_{11}	z_{12}
x_2	z_{21}	z_{22}

Following [6], a *refinement monoid* is any abelian monoid satisfying the Riesz refinement property.

Any abelian monoid M supports a translation-invariant pre-order $\leqq$ (often called the *algebraic pre-order*) defined by the existence of differences: $x \leqq y$ if and only if there exists $z \in M$ such that $x+z=y$. All inequalities in abelian monoids will be with respect to this pre-order. The monoid M satisfies the *Riesz decomposition property* provided that whenever $x \leqq y_1+y_2$ in M, there exist elements $x_i \in M$ such that $x=x_1+x_2$ and each $x_i \leqq y_i$. This property follows from the refinement property, but in general the two are not equivalent.

We can construct a monoid from any additive group G by adjoining a new additive identity, denoted 0 following our general convention. The new monoid can be expressed in the form $G \sqcup \{0\}$, which we sometimes abbreviate $G^{\sqcup 0}$. In case we need to refer to the zero of the group G, we write 0_G in order to distinguish this element from the zero of the monoid $G^{\sqcup 0}$.

Let M be an abelian monoid and $x \in M$. It is standard in the semigroup literature to say that x is *periodic* if the subsemigroup of M generated by x is finite. This does not, however, imply that this subsemigroup is a group. Thus, we shall say that x is *strongly periodic* provided the subsemigroup generated by x is a finite group; note that this occurs if and only if there is a positive integer m such that $(m+1)x=x$. The smallest such m is, of course, the order of the sub(semi)group generated by x; we will refer to it as the *order* of x. We say that M itself is *strongly periodic* provided every element of M is strongly periodic.

Semilattices. Recall that an *upper semilattice* (or $\vee$-*semilattice*) is a partially ordered set in which every pair of elements has a supremum. All semilattices in this paper will be upper semilattices, and they will also be assumed to have least elements, denoted 0. We will refer to them simply as *semilattices*, rather than using the precise but cumbersome term "$(\vee, 0)$-semilattice". If one takes $+ = \vee$, any semilattice becomes an abelian monoid in which $2x = x$ for all x; conversely, any abelian monoid with the latter property is a semilattice with respect to its algebraic pre-order. (It is an easy exercise to check that the pre-order is actually a partial order in this case.) Thus, for our purposes, it is convenient to take the name "semilattice" to mean any abelian monoid in which all elements satisfy the equation $2x = x$. Note that in a semilattice, $x \leqq y$ if and only if $x + y = y$. We shall generally write the operation in a semilattice as addition, except when it appears helpful to emphasize that an element $x \vee y$ is the supremum of elements x and y.

An *ideal* of a semilattice S is any nonempty, order-hereditary subset which is closed under finite suprema, that is, any submonoid of S which is hereditary with respect to the algebraic order. The collection of ideals of S is a complete lattice, denoted $\operatorname{Id} S$, in which infima are given by intersections. There is a canonical semilattice embedding of S into $\operatorname{Id} S$ given by $a \mapsto [0, a]$, where $[0, a]$ denotes the "closed interval" $\{x \in S \mid x \leqq a\}$.

A *distributive semilattice* is any semilattice which satisfies the Riesz decomposition (equivalently, refinement) property (cf. [11], Lemma 2.3). A semilattice S is distributive if and only if the ideal lattice $\operatorname{Id} S$ is distributive [12], Section II.5.

Semilattices of groups. Let M be an abelian monoid, and let $\Lambda(M)$ denote the set of idempotent (actually "idem-multiple") elements of M, that is, those $e \in M$ such that $2e = e$. Then $\Lambda(M)$ is a submonoid of M, and it is a semilattice. Note that the algebraic (pre-) order within $\Lambda(M)$ coincides with the restriction of the pre-order from M: if $e, f \in \Lambda(M)$ and $e \leqq f$ in M, then $e + x = f$ for some $x \in M$, whence $e + f = 2e + x = e + x = f$, and so $e \leqq f$ within $\Lambda(M)$. Consequently, we may use inequalities for idempotents with no danger of ambiguity.

The monoid M is a *semilattice of groups* provided M is a disjoint union of subgroups, that is, a disjoint union of subsemigroups each of which happens to be a group. (The collection of these subgroups is then a semilattice, where the supremum of subgroups G and G' is the unique subgroup containing $G + G'$.) The zero elements of these groups are then the idempotent elements of M, and so M will be a disjoint union of subgroups $G_M[e]$ indexed by the idempotents $e \in \Lambda(M)$. These subgroups may be described as follows:

$$G_M[e] = \{x \in M \mid e \leqq x \leqq e\}.$$

Note that whenever $e \leqq f$ in $\Lambda(M)$, the rule $x \mapsto x + f$ defines a group homomorphism $G_M[e] \to G_M[f]$.

If M is a semilattice of groups, then the homomorphisms above, together with the groups $G_M[e]$, define a functor from $\Lambda(M)$ (made into a category from its poset structure in the standard way) to the category of abelian groups. Conversely (e.g., [3], Theorem 4.11, or

[14], p. 89–90), given any functor $\mathscr{F}$ from a semilattice Λ to abelian groups, we can construct a corresponding semilattice of groups, say $M(\Lambda, \mathscr{F})$, whose underlying set is the disjoint union of the groups $\mathscr{F}(e)$ for $e \in \Lambda$. The addition operation in $M(\Lambda, \mathscr{F})$ is defined as follows: if $x, y \in M(\Lambda, \mathscr{F})$, there are unique $e, f \in \Lambda$ such that $x \in \mathscr{F}(e)$ and $y \in \mathscr{F}(f)$, and $x + y := \mathscr{F}(i)(x) + \mathscr{F}(j)(y)$ in $\mathscr{F}(e + f)$, where $i : e \to e + f$ and $j : f \to e + f$ are the unique morphisms in the category Λ corresponding to the relations $e \leqq e + f$ and $f \leqq e + f$.

Semilattices of groups are characterized by the standard semigroup-theoretic concept of regularity, which takes the following form in additive notation. An abelian monoid M is (*von Neumann*) *regular* provided that for each $x \in M$, there exists $y \in M$ such that $x + y + x = x$. Equivalently, M is regular if and only if $2x \leqq x$ for all $x \in M$. Observe that every strongly periodic monoid is regular.

It is well known that a semigroup S (not necessarily commutative) is a semilattice of groups if and only if S is regular and its idempotents are central [14], Theorem 2.1. We give a short proof of the commutative case below, for the reader's convenience.

Lemma 2.1. *An abelian monoid M is a semilattice of groups if and only if M is regular.*

Proof. ($\Rightarrow$) Any $x \in M$ lies in a group $G_M[e]$, for some $e \in \Lambda(M)$. Then $x + y = e$ for some $y \in G_M[e]$, whence $2x + y = x$.

($\Leftarrow$) For $e \in \Lambda(M)$, set $X(e) = \{x \in M \mid e \leqq x \leqq e\}$, and observe that $X(e)$ is a subsemigroup of M, containing e. If $x \in X(e)$, there exist $y, z \in M$ such that $e + y = x$ and $x + z = e$. Then $e + x = 2e + y = e + y = x$, which shows that e is an additive identity for $X(e)$. Since $z \leqq e$, we see that $z + e \in X(e)$, and then since $x + (z + e) = 2e = e$, we see that $z + e$ is an additive inverse for x within $X(e)$. Therefore $X(e)$ is a group.

It remains to prove that M is the disjoint union of the groups $X(e)$. Disjointness is clear, since if $x \in X(e) \cap X(f)$ for some $e, f \in \Lambda(M)$, then $e \leqq x \leqq f \leqq x \leqq e$, whence $e = f$. Given $x \in M$, we have $2x \leqq x$ by hypothesis, whence $2x + y = x$ for some $y \in M$. Set $e = x + y$, observing that $e \leqq x \leqq e$ and $2e = 2x + y + y = x + y = e$, that is, $e \in \Lambda(M)$ and $x \in X(e)$. Therefore M is the disjoint union of the subgroups $X(e)$, as desired. □

In view of Lemma 2.1, the terms "semilattice of abelian groups" and "regular abelian monoid" are equivalent; we shall use the latter from now on.

If M is a regular abelian monoid, then each element $a \in M$ lies in a group $G_M[\epsilon(a)]$ for a unique idempotent $\epsilon(a) \in \Lambda(M)$. Let a^- denote the additive inverse of a in the group $G_M[\epsilon(a)]$.

3. Regular refinement monoids

We begin by establishing some necessary conditions for the general type of direct limits that we are seeking to characterize, among which are the key properties of regularity and refinement. We also develop a new characterization of regular refinement monoids.

Proposition 3.1. *Let M be any direct limit of finite direct sums of monoids of the form $A^{\sqcup 0}$, for abelian groups A. Then the following statements hold*:

(a) *M is a regular conical refinement monoid.*

(b) *If all the groups A are torsion groups, then M is strongly periodic.*

(c) *For any idempotents $e \leqq f$ in M, the homomorphism $x \mapsto x + f$ from $G_M[e]$ to $G_M[f]$ is injective.*

(d) *For any idempotents $e \leqq f$ in M, the group $G_M[e] + f$ is a pure subgroup of $G_M[f]$.*

Proof. Statement (b) is clear. Note that (c) and (d) are equivalent to the following properties:

(c′) If $e \leqq f$ in $\Lambda(M)$ and $x \in G_M[e]$ such that $x + f = f$, then $x = e$.

(d′) If $e \leqq f$ in $\Lambda(M)$ and $x \in G_M[e]$, $y \in G_M[f]$ satisfy $x + f = my$ for some $m \in \mathbb{N}$, then there exists $z \in G_M[e]$ such that $x + f = m(z + f)$.

Thus, properties (a), (c), (d) can all be checked in terms of finite sets of equations involving finitely many elements. Therefore we need only verify them in the case when $M = A^{\sqcup 0}$.

(a) Obviously M is conical and regular. Suppose that $x_1 + x_2 = y_1 + y_2$ for some $x_i, y_j \in M$. If $x_1 = 0$, then there is a refinement matrix:

	y_1	y_2
x_1	0	0
x_2	y_1	y_2

Similar refinements exist if x_2, y_1, or y_2 is zero. Hence, we may assume that $x_i, y_j \in A$ for all i, j. In the group A, we have $x_2 = y_1 + x_1^- + y_2$, and so

	y_1	y_2
x_1	x_1	0
x_2	$y_1 + x_1^-$	y_2

is a refinement matrix.

(c′) Let $e \leqq f$ in $\Lambda(M)$ and $x \in G_M[e]$ such that $x + f = f$. If $e = 0$, then $x = 0 = e$. If $e \neq 0$, then $e = 0_A$, whence $f = 0_A$ and $x \in A$. Since A is a group, $x = 0_A = e$ in this case.

(d′) Let $e \leqq f$ in $\Lambda(M)$ and $x \in G_M[e]$, $y \in G_M[f]$ such that $x + f = my$ for some $m \in \mathbb{N}$. If $e = 0$, then $x = 0$, whence $x + f = f = m(e + f)$. If $e \neq 0$, then $e = 0_A$, whence $f = 0_A$ and $x, y \in A$. In this case, $y \in G_M[e]$, and $x + f = m(y + f)$. $\square$

Definition. We shall say that a regular abelian monoid M satisfies the *embedding condition*, abbreviated (emb), provided condition (c) of Proposition 3.1 holds. Further, M satisfies the *purity condition*, abbreviated (pur), provided M satisfies condition (d) of the proposition.

In view of the results above, any direct limit of finite direct sums of monoids of the form $(\mathbb{Z}/n\mathbb{Z})^{\sqcup 0}$ is a strongly periodic conical refinement monoid satisfying (emb) and (pur). Our main monoid-theoretic goal is to establish the converse statement (Theorem 6.4).

We next investigate the structure of regular abelian monoids M, for which some additional notation and terminology is helpful. Recall that $a \propto b$ (for some $a, b \in M$) means that $a \leqq mb$ for some $m \in \mathbb{N}$, and that $a \asymp b$ means that $a \propto b \propto a$. Since M is regular, $mb \leqq b$ for all $m \in \mathbb{N}$, and so $a \propto b$ if and only if $a \leqq b$. Thus, $a \asymp b$ if and only if $a \leqq b \leqq a$. Similarly, $a \propto b$ if and only if $\epsilon(a) \propto \epsilon(b)$, if and only if $\epsilon(a) \leqq \epsilon(b)$. Consequently, $a \asymp b$ if and only if a and b lie in the same group $G_M[e]$, for some idempotent $e \in \Lambda(M)$.

For any $a, b \in M$, the sum $\epsilon(a) + \epsilon(b)$ is an idempotent with $\epsilon(a) + \epsilon(b) \asymp a + b$, whence $\epsilon(a) + \epsilon(b) = \epsilon(a + b)$. In particular, this shows that $G_M[e] + G_M[f] \subseteq G_M[e + f]$ for all idempotents $e, f \in \Lambda(M)$. Now $e + f$ is the supremum of e and f in the semilattice $\Lambda(M)$, but there need not exist an infimum. We do, however, have a commutative diagram of abelian groups and group homomorphisms as follows:

$$\begin{array}{ccccc}
 & & G_M[e] & & \\
 & \overset{x \mapsto x+e}{\nearrow} & & \overset{y \mapsto y+f}{\searrow} & \\
\varinjlim_{g \leqq e, f} G_M[g] & & & & G_M[e+f] \\
 & \underset{x \mapsto x+f}{\searrow} & & \underset{z \mapsto z+e}{\nearrow} & \\
 & & G_M[f] & &
\end{array}$$

The resemblance of this diagram to a pullback behind a Mayer-Vietoris sequence in homological algebra provides a convenient name for the following monoid condition, which will be our key to the refinement property in regular abelian monoids.

Definition. Let M be a regular abelian monoid. We shall say that M satisfies the *Mayer-Vietoris property* (or MVP, for short) provided that, for all idempotents $e, f \in \Lambda(M)$:

(a) $G_M[e] + G_M[f] = G_M[e + f]$.

(b) Whenever $u \in G_M[e]$ and $v \in G_M[f]$ with $u + f = v + e$, there exists $w \in M$ such that $u = w + e$ and $v = w + f$. (Note that necessarily $w \in G_M[g]$ for some idempotent $g \leqq e, f$.)

The following result is in some sense a version of Proposition 1 and Corollary 4 of [6] with the finiteness assumption on the monoid removed.

Theorem 3.2. *A regular abelian monoid M is a refinement monoid if and only if $\Lambda(M)$ is a distributive semilattice and M satisfies the* MVP.

Proof. ($\Rightarrow$) Suppose that $e_1+e_2=f_1+f_2$ for some $e_i, f_j \in \Lambda(M)$. Refine this equation in M:

	f_1	f_2
e_1	x_{11}	x_{12}
e_2	x_{21}	x_{22}

Now if we set $g_{ij}=\epsilon(x_{ij})$ for all i, j, then $e_i=x_{i1}+x_{i2}\in G_M[g_{i1}+g_{i2}]$. Since e_i is idempotent, we obtain $g_{i1}+g_{i2}=e_i$ for $i=1,2$. Similarly, $g_{1j}+g_{2j}=f_j$ for $j=1,2$, which shows that $\Lambda(M)$ has refinement. Therefore $\Lambda(M)$ is a distributive semilattice.

Now let $e, f \in \Lambda(M)$. We have already observed that $G_M[e]+G_M[f]$ is contained in $G_M[e+f]$. To prove the reverse inclusion, consider an arbitrary element $a\in G_M[e+f]$. Note that $a=a+e+f$ and $a+a^-=e+f$. Take a refinement of the second equation:

	e	f
a	b	c
a^-	u	v

Now $a=a+e+f=(b+e)+(c+f)$. Since $b+u=e$, we have $b\leqq e$, whence $b+e\asymp e$ and so $b+e\in G_M[e]$. Similarly, $c+f\in G_M[f]$, and therefore $G_M[e+f]\subseteqq G_M[e]+G_M[f]$. This establishes the first half of the MVP.

Given $u\in G_M[e]$ and $v\in G_M[f]$ with $u+f=v+e$, take a refinement of this equation:

	v	e
u	a	b
f	c	d

Then $d\leqq e, f$. Put $w:=a+d^-$. Then $w+e=a+d^-+b+d=u+\epsilon(d)=u$ because $\epsilon(d)\leqq e\leqq u$, and $w+f=a+d^-+c+d=v+\epsilon(d)=v$ because $\epsilon(d)\leqq f\leqq v$. Therefore M satisfies the MVP.

($\Leftarrow$) Given $a_1+a_2=b_1+b_2$ in M, set $e_i=\epsilon(a_i)$ and $f_j=\epsilon(b_j)$ for $i,j=1,2$, so that $e_1+e_2=f_1+f_2$. Since $\Lambda(M)$ is distributive, it contains a refinement:

	f_1	f_2
e_1	g_{11}	g_{12}
e_2	g_{21}	g_{22}

By the MVP, each $G_M[e_i] = G_M[g_{i1}] + G_M[g_{i2}]$, and so each $a_i = c_{i1} + c_{i2}$ for some $c_{ij} \in G_M[g_{ij}]$. Note that $c_{1j} + c_{2j} \in G_M[g_{1j}] + G_M[g_{2j}] = G_M[f_j]$ for $j = 1, 2$, and that $(c_{11} + c_{21}) + (c_{12} + c_{22}) = a_1 + a_2 = b_1 + b_2$. Set $u := c_{11} + c_{21} + b_1^- \in G_M[f_1]$ and $v := b_2 + c_{12}^- + c_{22}^- \in G_M[f_2]$, and observe that

$$\begin{aligned} u + f_2 &= c_{11} + c_{21} + b_1^- + c_{12} + c_{12}^- + c_{22} + c_{22}^- \\ &= b_1 + b_2 + b_1^- + c_{12}^- + c_{22}^- = v + f_1. \end{aligned}$$

By the MVP, there exists an element $w \in M$ such that $u = w + f_1$ and $v = w + f_2$, and $w \in G_M[h]$ for some idempotent $h \leqq f_1, f_2$. Then

$$c_{11} + c_{21} + w^- = u + b_1 + w^- = w + f_1 + b_1 + w^- = b_1 + h + f_1 = b_1,$$

$$c_{12} + c_{22} + w = c_{12} + c_{22} + f_2 + w = c_{12} + c_{22} + v = b_2 + f_2 = b_2.$$

Since $h \leqq f_1 \leqq e_1 + e_2$, distributivity in $\Lambda(M)$ implies that $h = h_1 + h_2$ for some idempotents $h_i \leqq e_i$. Applying the MVP a final time, we obtain $w = w_1 + w_2$ for some $w_i \in G_M[h_i]$. We check that

$$(c_{11} + w_1^-) + (c_{21} + w_2^-) = c_{11} + c_{21} + w^- = b_1,$$

$$(c_{12} + w_1) + (c_{22} + w_2) = c_{12} + c_{22} + w = b_2,$$

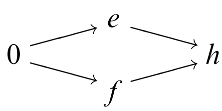
$$(c_{i1} + w_i^-) + (c_{i2} + w_i) = a_i + h_i = a_i \quad (i = 1, 2),$$

where the last equalities hold because $h_i \leqq e_i \leqq a_i$. Therefore we have a refinement:

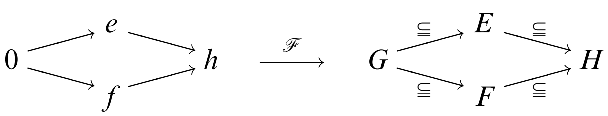
	b_1	b_2
a_1	$c_{11} + w_1^-$	$c_{12} + w_1$
a_2	$c_{21} + w_2^-$	$c_{22} + w_2$

□

In particular, Theorem 3.2 describes the conditions needed to obtain refinements in a regular abelian monoid $M(\Lambda, \mathscr{F})$ constructed from a semilattice Λ and a functor $\mathscr{F}$ from Λ to abelian groups as in Section 2. For example, take $\Lambda = \mathbf{2}^2$, the Boolean monoid of subsets of a 2-element set. Viewed as a category obtained from a poset, Λ looks like this:



Suppose that H is an abelian group with subgroups E, F, G such that $G \subseteq E \cap F$. Then we can define a functor $\mathscr{F}$ from Λ to the category of abelian groups as follows:



Form the monoid $M = M(\Lambda, \mathscr{F})$. Then Theorem 3.2 says that M has refinement if and only if $E \cap F = G$ and $E + F = H$.

Because the group homomorphisms in the diagram above are embeddings, the monoid M is isomorphic to a submonoid of $\Lambda \times H$, namely

$$(\{0\} \times G) \sqcup (\{e\} \times E) \sqcup (\{f\} \times F) \sqcup (\{h\} \times H).$$

In fact, arbitrary regular abelian monoids with (emb) can be put into a similar form, as follows.

Theorem 3.3. *Let M be a regular abelian monoid satisfying the embedding condition. Then there exist a semilattice Λ, an abelian group G, and subgroups $G_e \subseteq G$ for all $e \in \Lambda$ such that*:

(a) $G = \bigcup_{e \in \Lambda} G_e$.

(b) $G_e \subseteqq G_f$ *for all* $e \leqq f$ *in* Λ.

(c) M *is isomorphic to the submonoid* $\bigsqcup_{e \in \Lambda} (\{e\} \times G_e) \subseteqq \Lambda \times G$.

The monoid M is a refinement monoid if and only if:

(a′) Λ *is distributive*.

(b′) $G_e + G_f = G_{e+f}$ *for all* $e, f \in \Lambda$.

(c′) $G_e \cap G_f = \bigcup_{g \in \Lambda, g \leqq e, f} G_g$ *for all* $e, f \in \Lambda$.

Moreover, M is conical if and only if

(d′) $G_0 = \{0\}$,

and M satisfies the purity condition if and only if

(e′) G_e *is a pure subgroup of* G *for all* $e \in \Lambda$.

Proof. Set $\Lambda = \Lambda(M)$, and for $e \leqq f$ in Λ, let $\phi_{e,f} : G_M[e] \to G_M[f]$ denote the homomorphism $x \mapsto x + f$. The collection of groups $G_M[e]$ and transition maps $\phi_{e,f}$ forms a direct system in the category of abelian groups. Let G be the direct limit of this system, with limiting maps $\eta_e : G_M[e] \to G$ for $e \in \Lambda$, and set $G_e = \eta_e(G_M[e])$ for $e \in \Lambda$. Conditions (a) and (b) are clear, and the isomorphism required in (c) is given by the rule $a \mapsto \big(\epsilon(a), \eta_{\epsilon(a)}(a)\big)$.

It follows from Theorem 3.2 that M is a refinement monoid if and only if (a′), (b′), (c′) hold, and the remaining equivalences are clear. (Note that (e′) is equivalent to the statement that G_e is pure in G_f whenever $e \leqq f$ in Λ.) $\square$

For certain applications, it is useful to be able to restrict to strongly periodic monoids in which the orders of the elements are controlled, as follows.

Recall that a *generalized integer* or *supernatural number* is a formal product of non-negative powers of the positive prime integers, thus

$$\prod_p p^{t(p)} = 2^{t(2)}3^{t(3)}5^{t(5)}\cdots p^{t(p)}\cdots,$$

where each exponent $t(p) \in \{0\} \cup \mathbb{N} \cup \{\infty\}$. If $\mathfrak{m} = \prod_p p^{s(p)}$ and $\mathfrak{n} = \prod_p p^{t(p)}$ are generalized integers, the statement $\mathfrak{m}|\mathfrak{n}$ means that $s(p) \leqq t(p)$ for all primes p. Ordinary positive integers are treated as generalized integers in the obvious manner.

Definition. For any regular abelian monoid M and generalized integer $\mathfrak{m}$, we set

$$M[\mathfrak{m}] = \{x \in M \mid (m+1)x = x \text{ for some positive integer } m|\mathfrak{m}\}.$$

Note that $M[\mathfrak{m}]$ is a submonoid of M containing $\Lambda(M)$, and that it is also a semilattice of groups, since the sets

$$M[\mathfrak{m}] \cap G_M[e] = \{x \in M \mid mx = e \text{ for some positive integer } m|\mathfrak{m}\}$$

are subgroups of $G_M[e]$ for each $e \in \Lambda(M)$.

Proposition 3.4. *Let M be a regular refinement monoid satisfying the embedding and purity conditions, and let $\mathfrak{m}$ be a generalized integer. Then $M[\mathfrak{m}]$ is a regular refinement monoid satisfying the embedding and purity conditions.*

Proof. We have already observed that $M[\mathfrak{m}]$ is a semilattice of groups, and that $\Lambda(M[\mathfrak{m}]) = \Lambda(M)$, whence $\Lambda(M[\mathfrak{m}])$ is a distributive semilattice. It is clear that (emb) passes from M to $M[\mathfrak{m}]$.

Let e, f, g be idempotents in M with $e+f=g$. If $z \in G_{M[\mathfrak{m}]}[g]$, then $mz = g$ for some positive integer $m|\mathfrak{m}$. By the MVP, $z = b+c$ for some $b \in G_M[e]$ and $c \in G_M[f]$. Add mc^- to both sides of the equation $mb+mc=g$, to obtain $mb+f = mc^-+e$. The MVP now implies that there exists an element $w \in M$ such that $mb = w+e$ and $mc^- = w+f$; moreover, $w \in G_M[h]$ for some idempotent $h \leqq e, f$. Since $w+e = mb$, it follows from (pur) and (emb) that $w = mv$ for some $v \in G_M[h]$. Set $v' = (v+e)^- \in G_M[e]$. Since $mb = w+e = m(v+e)$, the element $b+v' \in G_M[e]$ satisfies $m(b+v') = e$. Moreover, $c+v+f \in G_M[f]$, and $mc^- = w+f = mv+f$ implies $m(c+v+f) = f$. Finally,

$$\begin{aligned}(b+v')+(c+v+f) &= b+c+(v+e)^-+v+f\\ &= z+(v+e)^-+(v+e)+f = z+e+f = z.\end{aligned}$$

Thus, $G_{M[\mathfrak{m}]}[g] = G_{M[\mathfrak{m}]}[e] + G_{M[\mathfrak{m}]}[f]$. Now suppose that $u \in G_{M[\mathfrak{m}]}[e]$ and $v \in G_{M[\mathfrak{m}]}[f]$ with $u+f = v+e$. By the MVP in M, there exists an element $w \in M$ such that $u = w+e$ and $v = w+f$. Put $h = \epsilon(w)$, and choose $m \in \mathbb{N}$, with $m|\mathfrak{m}$, such that $mu = e$ and $mv = f$. Since $mw+e = mu = e$, (emb) implies that $mw = h$, so that $w \in G_{M[\mathfrak{m}]}[h]$. This shows that $M[\mathfrak{m}]$ satisfies the MVP. Therefore, by Theorem 3.2, $M[\mathfrak{m}]$ is a refinement monoid.

Let $e \leqq f$ be idempotents in M, and consider elements $x \in G_{M[\mathfrak{m}]}[e]$ and $y \in G_{M[\mathfrak{m}]}[f]$ such that $x + f = ny$ for some $n \in \mathbb{N}$. Choose $m \in \mathbb{N}$, with $m|\mathfrak{m}$, such that $mx = e$ and $my = f$, and let $d = \mathrm{GCD}(m, n)$. Then $m = m'd$ and $n = n'd$ for some $m', n' \in \mathbb{N}$, and $\mathrm{GCD}(m', n') = 1$. Note that $m'x + f = m'ny = n'my = f$, whence $m'x = e$ by (emb). Now $x + f = d(n'y)$ with $n'y \in G_M[f]$. Using (pur) and (emb) in M, we obtain an element $z \in G_M[e]$ such that $x = dz$. Moreover, $mz = m'x = e$, and so $z \in M[\mathfrak{m}]$. Since n' and m' are relatively prime, there exists $n^* \in \mathbb{N}$ such that $n^*n' \equiv 1 \pmod{m'}$, whence $n^*n \equiv d \pmod{m}$, and so $n^*nz = dz$. Thus $x = dz = n(n^*z)$ with $n^*z \in G_{M[\mathfrak{m}]}[e]$, which establishes (pur) in $M[\mathfrak{m}]$. □

4. Direct limits

Since our aim is to express certain monoids as direct limits of appropriate building blocks, it is helpful to set down general conditions for such direct limits at the outset. We shall use the following version of [11], Lemma 3.4, which many readers will recognize as an analogue of a key step in other classification results. It is a monoid-theoretical version of Shannon's result [22], Theorem 2. For a map $\phi : X \to Y$, we put

$$\ker \phi = \{(x, y) \in X \times X \mid \phi(x) = \phi(y)\}.$$

Lemma 4.1. *Let $\mathscr{B}$ be a class of finite abelian monoids which is closed under finite direct sums and let M be an abelian monoid. Then M is a direct limit of monoids from $\mathscr{B}$ if and only if the following two conditions are satisfied*:

(1) *For each $x \in M$, there exist $B \in \mathscr{B}$ and a homomorphism $\phi : B \to M$ such that $x \in \phi(B)$.*

(2) *For any $B \in \mathscr{B}$ and any homomorphism $\phi : B \to M$, there exist $B' \in \mathscr{B}$ and homomorphisms $B \xrightarrow{\psi} B' \xrightarrow{\phi'} M$ such that $\phi'\psi = \phi$ and $\ker \phi = \ker \psi$.*

Proof. The given conditions clearly imply the two hypotheses of [11], Lemma 3.4, hence they imply that M is a direct limit of members of $\mathscr{B}$.

Conversely, suppose that $M = \varinjlim_{i \in I} B_i$, a direct limit with all B_i in $\mathscr{B}$, transition maps $f_{ij} : B_i \to B_j$, and limiting maps $f_i : B_i \to M$, for all $i \leqq j$ in the directed partially ordered set I. As $M = \bigcup_{i \in I} f_i(B_i)$, Condition (1) is satisfied. Now let $\phi : B \to M$ be a monoid homomorphism, with $B \in \mathscr{B}$. Since B is finite, $\phi(B) \subseteq f_i(B_i)$ for some $i \in I$. Choose elements $x_b \in B_i$ such that $f_i(x_b) = \phi(b)$ for all $b \in B$ and $x_0 = 0$. For all $c, d \in B$, we have $f_i(x_c + x_d) = \phi(c + d) = f_i(x_{c+d})$. By finiteness, there is some $j \in I$, with $j \geqq i$, such that $f_{ij}(x_c + x_d) = f_{ij}(x_{c+d})$ for all $c, d \in B$. Now replace i by j and each x_b by $f_{ij}(x_b)$. This allows us to assume, without loss of generality, that $x_c + x_d = x_{c+d}$ for all $c, d \in B$. Hence, there is a monoid homomorphism $\psi : B \to B_i$, given by $\psi(b) = x_b$, such that $f_i\psi = \phi$. For each $(x, y) \in \ker \phi$, we have $f_i\psi(x) = f_i\psi(y)$, and so there is some $k \in I$, with $k \geqq i$, such that $f_{ik}\psi(x) = f_{ik}\psi(y)$ for all $(x, y) \in \ker \phi$. Now replace i and ψ by k and $f_{ik}\psi$. This allows to assume that $\ker \phi \subseteq \ker \psi$. Since the reverse inclusion follows from $f_i\psi = \phi$, we conclude that (2) above is satisfied with $B' = B_i$ and $\phi' = f_i$. □

In an arbitrary category admitting all direct limits (in categorical language, directed colimits), the class of all direct limits of members from a given class is not necessarily closed under direct limits—even in case the category we are starting with is a partially ordered set! However, strengthening the assumptions leads to the following useful positive result.

Corollary 4.2. *Let $\mathscr{B}$ be a class of finite abelian monoids which is closed under finite direct sums. Then the class of all direct limits of monoids from $\mathscr{B}$ is closed under direct limits.*

Proof. Denote by $\mathscr{L}$ the class of all direct limits of monoids from $\mathscr{B}$. Let $M = \varinjlim_{i \in I} M_i$, a direct limit with all $M_i \in \mathscr{L}$, transition maps $f_{ij} : M_i \to M_j$ and limiting maps $f_i : M_i \to M$, for all $i \leqq j$ in the directed partially ordered set I. Since the M_i satisfy Condition (1) of Lemma 4.1 and $M = \bigcup_{i \in I} f_i(M_i)$, we see that M satisfies Condition (1) of Lemma 4.1. Now let $\phi : B \to M$ be a monoid homomorphism, with $B \in \mathscr{B}$. Since B is finite, we see as in the proof of Lemma 4.1 that there are $i \in I$ and a monoid homomorphism $\phi' : B \to M_i$ such that $\phi = f_i\phi'$ and $\ker\phi = \ker\phi'$. Since $M_i \in \mathscr{L}$, Lemma 4.1 shows that there exists $B' \in \mathscr{B}$ together with monoid homomorphisms $\psi : B \to B'$ and $\phi'' : B' \to M_i$ such that $\phi' = \phi''\psi$ and $\ker\phi' = \ker\psi$. Therefore, $\phi = (f_i\phi'')\psi$ with $f_i\phi'' : B' \to M$ and $\ker\phi = \ker\psi$. Using Lemma 4.1 again, we conclude that M belongs to $\mathscr{L}$. □

Remark 4.3. Both Lemma 4.1 and Corollary 4.2 can be extended to the case where all members of $\mathscr{B}$ are *finitely generated* monoids. To obtain this, we observe that in the proof of Lemma 4.1, the monoid $B/\ker\phi$ is finitely generated, thus, by Redei's Theorem (see [19], or [9] for a simple proof), finitely presented.

For the remainder of the paper, we restrict $\mathscr{B}$ to be the class of finite direct sums of monoids of the form $(\mathbb{Z}/n\mathbb{Z})^{\sqcup 0}$ for $n \in \mathbb{N}$, and we let $\mathscr{L}$ denote the class of all direct limits of monoids from $\mathscr{B}$. Further, write $\mathscr{R}_{\text{ep}}$ for the class of all strongly periodic conical refinement monoids satisfying the conditions (emb) and (pur). It follows from Proposition 3.1 that $\mathscr{L}$ is contained in $\mathscr{R}_{\text{ep}}$, and the main goal of Sections 5 and 6 is to prove the reverse inclusion.

Lemma 4.4. *The class $\mathscr{L}$ is closed under direct limits, finite direct sums, and retracts.*

Proof. Corollary 4.2 implies that $\mathscr{L}$ is closed under direct limits, and it is straightforward to verify that $\mathscr{L}$ is closed under finite direct sums.

Now consider a monoid M which is a retract of a monoid $M' \in \mathscr{L}$, that is, there are morphisms $\varepsilon : M \to M'$ and $\mu : M' \to M$ such that $\mu\varepsilon = \mathrm{id}_M$. Put $\rho = \varepsilon\mu$, and observe that $\rho^2 = \rho$ and $\mu\rho = \mu$. We claim that M is the direct limit of the sequence

$$M' \xrightarrow{\rho} M' \xrightarrow{\rho} M' \xrightarrow{\rho} \cdots,$$

with constant limiting morphism $\mu : M' \to M$. Suppose that we have a monoid C and morphisms $\varphi_n : M' \to C$ for $n \in \mathbb{N}$ such that $\varphi_n = \varphi_{n+1}\rho$ for all n. Since ρ is idempotent, $\varphi_n = \varphi_0$ for all n, and so $\varphi_0\varepsilon$ is the unique morphism $\psi : M \to C$ such that $\psi\mu = \varphi_0$. This establishes the claim, and since $\mathscr{L}$ is closed under direct limits, we conclude that $M \in \mathscr{L}$. □

Corollary 4.5. *For any finite abelian group A, the monoid $A^{\sqcup 0}$ belongs to $\mathscr{L}$.*

Proof. By the fundamental structure theorem of finite abelian groups, $A = \bigoplus_{i=1}^{n} A_i$ for some finite cyclic groups A_i. Now set $M = \bigoplus_{i=1}^{n} A_i^{\sqcup 0}$, and note that the inclusion map $A \hookrightarrow M$ extends to a unique monoid embedding $\varepsilon : A^{\sqcup 0} \hookrightarrow M$.

For $i = 1, \dots, n$, the canonical injection $A_i \hookrightarrow A$ extends to a unique monoid embedding $\mu_i : A_i^{\sqcup 0} \hookrightarrow A^{\sqcup 0}$. The maps μ_i induce a monoid homomorphism $\mu : M \to A^{\sqcup 0}$ given by the rule $\mu(a_1, \dots, a_n) = \sum_{i=1}^{n} \mu_i(a_i)$. It is clear that $\mu\varepsilon$ is the identity map on $A^{\sqcup 0}$, whence $A^{\sqcup 0}$ is a retract of M. Therefore, by Lemma 4.4, $A^{\sqcup 0} \in \mathscr{L}$. □

5. Finite monoids

The first major step towards our main result is to show that every finite monoid from $\mathscr{R}_{\mathrm{ep}}$ belongs to $\mathscr{L}$. We do this in the present section, after recalling some facts about join-irreducible elements in semilattices.

Every finite semilattice is, of course, a lattice, and it is distributive as a semilattice if and only if it is distributive as a lattice. A nonzero (i.e., non-minimum) element p in a semilattice S is *join-irreducible* if p is not the supremum of any pair of elements less than p, that is, if $p = x \vee y$ implies that $p \in \{x, y\}$, for any $x, y \in S$. We denote by $\mathrm{J}(S)$ the set of all join-irreducible elements of S, and, for each $a \in S$, we put $\mathrm{J}_S(a) = \{p \in \mathrm{J}(S) \mid p \leqq a\}$. It is well-known (see [12], Exercise I.6.13) that in case S is finite, every element of S is the supremum of the join-irreducible elements it dominates, that is, $a = \bigvee \mathrm{J}_S(a)$ for all $a \in S$. Furthermore, an element $p \in S$ is join-irreducible if and only if p has a unique *lower cover*, that is, an element $x < p$ in S such that no $y \in S$ satisfies $x < y < p$. In that case we shall denote by p_* the unique lower cover of p.

The following lemma is folklore.

Lemma 5.1. *For every join-irreducible element p in a finite distributive lattice D, there exists a unique largest $u \in D$ such that $p \nleqq u$.*

Proof. Since D is distributive and p is join-irreducible, $p \nleqq x$ and $p \nleqq y$ implies that $p \nleqq x \vee y$, for any $x, y \in D$. Set $u = \bigvee \{x \in D \mid p \nleqq x\}$. □

The element u of Lemma 5.1 is traditionally denoted by $p^{\dagger}$.

For an abelian group G, let us denote by $\operatorname{Sub} G$ the lattice of all subgroups of G. The following lemma is also folklore. It is valid in the much more general context of a homomorphism from a finite distributive lattice to a modular lattice with zero.

Lemma 5.2. *Let G be an abelian group, D a finite distributive lattice, $f : D \to \operatorname{Sub} G$ a lattice homomorphism, and $\big(H_p \mid p \in \mathrm{J}(D)\big)$ a family of subgroups of G such that $f(p) = f(p_*) \oplus H_p$ for all $p \in \mathrm{J}(D)$. Then*

$$f(a) = f(0) \oplus \bigoplus_{p \in \mathrm{J}_D(a)} H_p$$

for all $a \in D$.

Proof. We argue by induction on a. As the result is trivial for $a = 0$ (in which case $\mathrm{J}_D(a)$ is empty), we only deal with the induction step. Let b be a lower cover of a in D and let $p \leqq a$ be minimal with respect to the property $p \nleqq b$. Then p is join-irreducible, and, by the minimality statement, $p_* \leqq b$. Hence, $p \wedge b = p_*$ and $p \vee b = a$. For any $q \in \mathrm{J}(D)$ such that $q \leqq a$, it follows from the join-irreducibility of q and the distributivity of D that either $q \leqq b$ or $q \leqq p$. If $q \nleqq b$, then $q \leqq p$, and $q < p$ is ruled out because that would imply $q \leqq p_* \leqq b$, a contradiction. Hence, we have proved the statement

$$\mathrm{J}_D(a) = \mathrm{J}_D(b) \cup \{p\}. \tag{5.1}$$

Now we compute:

$$f(b) + H_p = f(b) + f(p_*) + H_p = f(b) + f(p) = f(b \vee p) = f(a)$$

because $f(p_*) \subseteqq f(b)$, while

$$f(b) \cap H_p = f(b) \cap f(p) \cap H_p = f(b \wedge p) \cap H_p = f(p_*) \cap H_p = \{0\}$$

because $H_p \subseteqq f(p)$. Therefore, $f(a) = f(b) \oplus H_p$, and thus, by (5.1) and the induction hypothesis, $f(a) = f(0) \oplus \bigoplus_{q \in \mathrm{J}_D(a)} H_q$. □

Proposition 5.3. *Any finite monoid in $\mathscr{R}_{\mathrm{ep}}$ belongs to $\mathscr{L}$.*

Proof. Let M be a finite monoid in $\mathscr{R}_{\mathrm{ep}}$. In view of Theorem 3.3, we may assume that

$$M = \bigsqcup_{e \in \Lambda} (\{e\} \times G_e) \subseteqq \Lambda \times G$$

for some finite semilattice Λ and some finite abelian group G with subgroups G_e (for $e \in \Lambda$) satisfying the conditions (a), (b), and (a′)–(e′) of the theorem. Finally, since Λ is finite, it is a distributive lattice, and condition (c′) implies that $G_e \cap G_f = G_{e \wedge f}$ for all $e, f \in \Lambda$. Note that the rule $e \mapsto G_e$ provides a lattice homomorphism $\Lambda \to \operatorname{Sub} G$.

For any $p \in \mathrm{J}(\Lambda)$, the group G_{p_*} is a finite, pure subgroup of G_p, and so, by Kulikov's Theorem (see [10], Theorem 27.5), $G_p = G_{p_*} \oplus H_p$ for some subgroup H_p of G_p. Lemma 5.2 thus yields that

$$G_e = \bigoplus_{p \in \mathrm{J}_\Lambda(e)} H_p \tag{5.2}$$

for all $e \in \Lambda$. In particular, taking $e = 1$ (the maximum element of Λ), we obtain $G = \bigoplus_{p \in \mathrm{J}(\Lambda)} H_p$. Let $\pi_q : G \to H_q$, for $q \in \mathrm{J}(\Lambda)$, denote the projections corresponding to this direct sum.

We next define maps $\varepsilon_p : M \to G^{\sqcup 0}$ and $\mu_p : G^{\sqcup 0} \to M$, for $p \in \mathrm{J}(\Lambda)$, by the rules

$$\varepsilon_p(e,x) = \begin{cases} \pi_p(x) & (p \leqq e), \\ 0 & (p \nleqq e), \end{cases} \quad \mu_p(y) = \begin{cases} (0,0) & (y = 0), \\ (p, \pi_p(y)) & (y \in G). \end{cases}$$

It is clear that μ_p is a monoid homomorphism, and we claim that ε_p is one as well. Hence, we must show that

$$\varepsilon_p(e \vee f, x+y) = \varepsilon_p(e,x) + \varepsilon_p(f,y) \tag{5.3}$$

for all $(e,x), (f,y) \in M$. If $p \leqq e$ and $p \leqq f$, then both sides of (5.3) equal $\pi_p(x+y)$, while if $p \nleqq e$ and $p \nleqq f$, both sides are zero. If $p \nleqq e$ but $p \leqq f$, then in view of (5.2), $\pi_p(x) = 0$ (because $p \notin \mathrm{J}_\Lambda(e)$), whence both sides of (5.3) equal $\pi_p(y)$. A symmetric observation covers the remaining situation, and thus (5.3) holds in all cases.

Finally, we define homomorphisms $\varepsilon : M \to (G^{\sqcup 0})^{\mathrm{J}(\Lambda)}$ and $\mu : (G^{\sqcup 0})^{\mathrm{J}(\Lambda)} \to M$ by the rules

$$\varepsilon(e,x) = \big(\varepsilon_p(e,x)\big)_{p \in \mathrm{J}(\Lambda)}, \quad \mu\big((y_p)_{p \in \mathrm{J}(\Lambda)}\big) = \sum_{p \in \mathrm{J}(\Lambda)} \mu_p(y_p).$$

For any nonzero $(e,x) \in M$, we compute that

$$\begin{aligned} \mu\varepsilon(e,x) &= \sum_{p \in \mathrm{J}(\Lambda)} \mu_p \varepsilon_p(e,x) = \sum_{p \in \mathrm{J}_\Lambda(e)} \mu_p \pi_p(x) \\ &= \sum_{p \in \mathrm{J}_\Lambda(e)} (p, \pi_p(x)) = \Big(e, \sum_{p \in \mathrm{J}_\Lambda(e)} \pi_p(x)\Big) = (e,x), \end{aligned}$$

where the final equality comes from (5.2). Thus, $\mu\varepsilon = \mathrm{id}_M$, and so M is a retract of $(G^{\sqcup 0})^{\mathrm{J}(\Lambda)}$. We conclude from Corollary 4.5 and Lemma 4.4 that $M \in \mathscr{L}$. □

Remark 5.4. The direct limits that exist by virtue of Proposition 5.3 necessarily involve systems of non-injective homomorphisms, even in the case of semilattices—while every distributive semilattice is a direct limit of finite Boolean semilattices [11], Theorem 6.6, most distributive semilattices are not directed unions of finite Boolean subsemilattices. This is just because finite distributive semilattices need not be Boolean, the three-element chain $\{0,1,2\}$ being the simplest example. This semilattice can be expressed as a direct limit of copies of $\mathbf{2}^2$; see [11], Example 6.8.

6. Characterization of the monoids in $\mathscr{R}_{\mathrm{ep}}$

Because of Proposition 5.3, we will be able to conclude that $\mathscr{R}_{\mathrm{ep}} = \mathscr{L}$ once we show that every monoid in $\mathscr{R}_{\mathrm{ep}}$ is a direct limit of finite members of $\mathscr{R}_{\mathrm{ep}}$. In fact, we will show that monoids in $\mathscr{R}_{\mathrm{ep}}$ are directed unions of finite submonoids from $\mathscr{R}_{\mathrm{ep}}$. This also provides a generalization of Pudlák's result, [18], Fact 4, p. 100, that every distributive semilattice is the directed union of its finite distributive subsemilattices.

Theorem 6.1. *Each monoid M in $\mathscr{R}_{\mathrm{ep}}$ is the directed union of those finite submonoids of M which belong to $\mathscr{R}_{\mathrm{ep}}$.*

Proof. We must show that any finite subset X of M is contained in some finite submonoid of M lying in $\mathscr{R}_{\mathrm{ep}}$. For convenience, assume that $0 \in X$. We first reduce to the case where there is a bound on the orders of the elements of M, by observing that M is the directed union of all $M[m]$, for $m \in \mathbb{N}$; thus, $X \subseteq M[m]$ for some m. By Proposition 3.4, $M[m] \in \mathscr{R}_{\mathrm{ep}}$, and so we may replace M by $M[m]$.

Hence, we may assume that $(m+1)x = x$ for all $x \in M$, where m is a fixed positive integer. We start as in the proof of Proposition 5.3. By Theorem 3.3, we may assume that

$$M = \bigsqcup_{e \in \Lambda} \big(\{e\} \times G_e\big) \subseteq \Lambda \times G$$

for some distributive semilattice Λ and some abelian group G with subgroups G_e satisfying all the conditions of the theorem.

Next, we set $G_A = \bigcup_{e \in A} G_e$ for every ideal A of Λ. Observe that the union defining G_A is directed, and that $G_{[0,e]} = G_e$ for all $e \in \Lambda$. Hence, if $A \subseteq B$ in $\mathrm{Id}\,\Lambda$, then G_A is a pure subgroup of G_B. Since $mG_A = \{0\}$, it follows from Kulikov's Theorem that G_A must be a direct summand of G_B. Notice also that $G_A + G_B = G_{A \vee B}$ and $G_A \cap G_B = G_{A \cap B}$ for arbitrary $A, B \in \mathrm{Id}\,\Lambda$. Thus, the rule $A \mapsto G_A$ defines a lattice homomorphism $\mathrm{Id}\,\Lambda \to \mathrm{Sub}\,G$.

Write the elements $x \in X$ in the form $x = (e_x, g_x) \in M$. Denote by $\boldsymbol{D}$ the sublattice of $\mathrm{Id}\,\Lambda$ generated by the principal ideals $[0, e_x]$ for $x \in X$. Since $\mathrm{Id}\,\Lambda$ is distributive, $\boldsymbol{D}$ is finite (in fact, $|\boldsymbol{D}| \leqq 2^{2^{|X|}}$). Moreover, the ideal $\{0\}$ belongs to $\boldsymbol{D}$ because $0 \in X$. For each $P \in \mathrm{J}(\boldsymbol{D})$, choose a subgroup H_P of G_P such that $G_P = G_{P_*} \oplus H_P$, where P_* denotes the unique lower cover of P in the lattice $\boldsymbol{D}$. Lemma 5.2 now implies that

$$G_A = \bigoplus_{P \in \mathrm{J}_{\boldsymbol{D}}(A)} H_P$$

for all $A \in \boldsymbol{D}$. In particular, taking A to be the largest element, say I, of $\boldsymbol{D}$, we obtain $G_I = \bigoplus_{P \in \mathrm{J}(\boldsymbol{D})} H_P$.

For each $x \in X$, we have

$$g_x \in G_{e_x} = G_{[0,e_x]} = \bigoplus_{P \in \mathrm{J}_{\boldsymbol{D}}([0,e_x])} H_P.$$

Since X is finite, there exist finitely generated subgroups $H'_P \subseteq H_P$ for $P \in \mathrm{J}(\boldsymbol{D})$ such that

$$g_x \in \bigoplus_{P \in \mathrm{J}_{\boldsymbol{D}}([0,e_x])} H'_P \tag{6.1}$$

for $x \in X$. Since each $mH_P = 0$, the groups H'_P are all finite. Define finite subgroups

$$G'_A = \bigoplus_{P \in \mathrm{J}_{\boldsymbol{D}}(A)} H'_P \subseteq G_A \tag{6.2}$$

for all $A \in \boldsymbol{D}$. Observe that

$$G'_A + G'_B = G'_{A+B} \quad \text{and} \quad G'_A \cap G'_B = G'_{A\cap B} \quad \text{for all } A, B \in \boldsymbol{D}, \tag{6.3}$$

and that

$$G'_A \text{ is a pure subgroup of } G'_B \quad \text{for all } A \subseteq B \text{ in } \boldsymbol{D}. \tag{6.4}$$

For each $x \in X$, since $[0, e_x]$ is the supremum of all join-irreducible elements of $\boldsymbol{D}$ below it, there are elements $u_P^x \in P$, for $P \in \mathrm{J}_{\boldsymbol{D}}([0, e_x])$, such that $e_x = \bigvee_{P \in \mathrm{J}_{\boldsymbol{D}}([0,e_x])} u_P^x$. Setting $u_P = \bigvee_{x \in X,\, [0,e_x] \supseteq P} u_P^x$ for $P \in \mathrm{J}(\boldsymbol{D})$, we obtain that $u_P \in P$ and

$$e_x = \bigvee_{P \in \mathrm{J}_{\boldsymbol{D}}([0,e_x])} u_P \tag{6.5}$$

for all $x \in X$. Since each G'_P is a finite subset of the directed union $G_P = \bigcup_{e \in P} G_e$, there exist elements $v_P \in P$ such that $G'_P \subseteq G_{v_P}$ for all $P \in \mathrm{J}(\boldsymbol{D})$. Finally, for each $P \in \mathrm{J}(\boldsymbol{D})$, recall the notation $P^\dagger$ for the unique largest element of $\boldsymbol{D}$ not containing P (see Lemma 5.1), choose $w_P \in P \setminus P^\dagger$, and put $\psi(P) = u_P \vee v_P \vee w_P$. We define a map $\varphi : \boldsymbol{D} \to \Lambda$ by the rule

$$\varphi(A) = \bigvee_{P \in \mathrm{J}_{\boldsymbol{D}}(A)} \psi(P),$$

and we claim that:

(1) φ is a semilattice embedding.

(2) $\varphi(\boldsymbol{D})$ is a finite distributive subsemilattice of Λ.

(3) $\varphi(A) \in A$ for all $A \in \boldsymbol{D}$.

(4) $\varphi([0, e_x]) = e_x$ for all $x \in X$.

The third statement is clear since $\psi(P) \in P$ for all $P \in \mathrm{J}(\boldsymbol{D})$. In particular, $\varphi(\{0\}) = 0$. It is also clear that φ is a semilattice homomorphism. To finish the proof of (1), consider $A, B \in \boldsymbol{D}$ such that $A \nsubseteq B$. There exists $P \in \mathrm{J}(\boldsymbol{D})$ such that $P \subseteq A$ but $P \nsubseteq B$, and then $B \subseteq P^\dagger$. From $P \subseteq A$ it follows that $w_P \leqq \varphi(A)$. On the other hand, from $w_P \notin P^\dagger$ it follows that $w_P \notin B$, and so $w_P \nleqq \varphi(B)$. Therefore, $\varphi(A) \nleqq \varphi(B)$, and (1) is proved. It now follows that $\varphi(\boldsymbol{D})$ is a finite subsemilattice of Λ, isomorphic to $\boldsymbol{D}$ and hence distributive, establishing (2). Finally, for $x \in X$, it follows from (3) that $\varphi([0, e_x]) \leqq e_x$. On the other hand,

$$\varphi([0, e_x]) = \bigvee_{P \in \mathrm{J}_{\boldsymbol{D}}([0,e_x])} \psi(P) \geqq \bigvee_{P \in \mathrm{J}_{\boldsymbol{D}}([0,e_x])} u_P = e_x$$

by (6.5), and (4) is proved.

Now we set $N = \bigsqcup_{A \in \boldsymbol{D}} \left(\{\varphi(A)\} \times G'_A\right) \subseteq \Lambda \times G$. In view of (6.3), N is a finite submonoid of $\Lambda \times G$. Since

$$G'_A = \sum_{P \in \mathrm{J}_{\boldsymbol{D}}(A)} H'_P \subseteq \sum_{P \in \mathrm{J}_{\boldsymbol{D}}(A)} G_{v_P} \subseteq \sum_{P \in \mathrm{J}_{\boldsymbol{D}}(A)} G_{\psi(P)} = G_{\varphi(A)}$$

for all $A \in \boldsymbol{D}$, we see that $N \subseteqq M$. By (2), $\Lambda(N) \cong \varphi(\boldsymbol{D})$ is a (finite) distributive semilattice. It now follows from (6.3) and Theorem 3.3 that N is a refinement monoid. It is clear that N is conical and satisfies (emb), and N satisfies (pur) by (6.4). Thus, N belongs to $\mathscr{R}_{\mathrm{ep}}$.

Finally, for every $x \in X$,

$$g_x \in \bigoplus_{P \in \mathrm{J}_{\boldsymbol{D}}([0, e_x])} H'_P = G'_{[0, e_x]}$$

by (6.1) and (6.2), whence $x = (e_x, g_x) \in N$. Therefore, X is contained in N. □

Remark 6.2. It is tempting to try to reduce the proof of Theorem 6.1 to the case where Λ is finite, by applying Pudlák's result. After putting M into the form given by Theorem 3.3, we can choose a finite set $E \subseteqq \Lambda$ such that $X \subseteqq \bigsqcup_{e \in E} (\{e\} \times G_e)$; then, by Pudlák's result, Λ has a finite distributive subsemilattice Λ' containing E, and X is contained in the submonoid $M' = \bigsqcup_{e \in \Lambda'} (\{e\} \times G_e)$ of M. The temptation is to replace M by M'. However, there is no guarantee that M' satisfies the second part of the MVP, and so we do not know whether M' is a refinement monoid.

Remark 6.3. The proof above yields an explicit upper bound for the cardinality of N (the desired finite submonoid containing X), as a function of m (fixed positive integer such that $X \subseteqq M[m]$) and $n = |X|$. Now $\boldsymbol{D}$ is the sublattice of $\mathrm{Id}\,\Lambda$ generated by $X \cup \{0\}$. For fixed $x \in X$, we pick elements $g_{P,x} \in H_P$, for $P \in \mathrm{J}_{\boldsymbol{D}}([0, e_x])$, such that $g_x = \sum_{P \in \mathrm{J}_{\boldsymbol{D}}([0, e_x])} g_{P,x}$; then put $U_P = \{g_{P,x} \mid x \in X,\ [0, e_x] \supseteqq P\}$ and we define H'_P as the subgroup of H_P generated by U_P, for all $P \in \mathrm{J}(\boldsymbol{D})$. By definition, the subgroups H'_P satisfy (6.1). Hence, the subset

$$Y = \bigcup_{P \in \mathrm{J}(\boldsymbol{D})} \big(\{\varphi(P)\} \times U_P\big)$$

is a generating subset of the submonoid N of the proof of Theorem 6.1, with $|Y| \leqq |\mathrm{J}(\boldsymbol{D})| \cdot n$. Since $\boldsymbol{D}$ is distributive, every element of $\boldsymbol{D}$ is a supremum of infima of elements of the form $[0, e_x]$, thus every join-irreducible element of $\boldsymbol{D}$ has the form $\bigwedge_{x \in I} [0, e_x]$, for some subset I of X. Therefore, $|\mathrm{J}(\boldsymbol{D})| \leqq 2^n$, and hence, since $N \subseteqq M[m]$, we obtain the estimates $|N| \leqq (m+1)^{|Y|} \leqq (m+1)^{2^n n}$.

We are now ready to establish the key result of the paper, namely that $\mathscr{R}_{\mathrm{ep}} = \mathscr{L}$.

Theorem 6.4. *An abelian monoid M is a direct limit of finite direct sums of monoids of the form $(\mathbb{Z}/n\mathbb{Z}) \sqcup \{0\}$ if and only if:*

(a) *M is a strongly periodic conical refinement monoid.*

(b) *For all idempotents $e \leqq f$ in M, the homomorphism $G_M[e] \to G_M[f]$ given by $x \mapsto x + f$ is injective, and $G_M[e] + f$ is a pure subgroup of $G_M[f]$.*

Proof. Proposition 3.1, Theorem 6.1, Proposition 5.3, and Lemma 4.4. □

Of course, in case M is countable, the direct limit of Theorem 6.4 may be taken indexed by the natural numbers.

It is easy to restrict the set of cyclic groups used as building blocks in the theorem, as follows.

Corollary 6.5. *Let* $\mathfrak{m}$ *be a generalized integer and* M *an abelian monoid. Then* M *is a direct limit of finite direct sums of monoids of the form* $(\mathbb{Z}/n\mathbb{Z}) \sqcup \{0\}$ *with* $n|\mathfrak{m}$ *if and only if* M *satisfies the conditions of Theorem* 6.4 *and*:

(c) *The order of each element of* M *divides* $\mathfrak{m}$.

Proof. We verify the nontrivial direction, $(\Leftarrow)$. By Theorem 6.4, M is the direct limit of a direct system of monoids M_i and transition maps $f_{ij} : M_i \to M_j$ where each M_i is a finite direct sum of monoids of the form $(\mathbb{Z}/n\mathbb{Z})^{\sqcup 0}$. It is routine to verify that each f_{ij} maps $M_i[\mathfrak{m}]$ to $M_j[\mathfrak{m}]$, and that $M[\mathfrak{m}]$ is the direct limit of the restricted system $(M_i[\mathfrak{m}], f_{ij}|_{M_i[\mathfrak{m}]})$. Assumption (c) says that $M = M[\mathfrak{m}]$, and it only remains to observe that each $M_i[\mathfrak{m}]$ is a finite direct sum of monoids $(\mathbb{Z}/n\mathbb{Z})^{\sqcup 0}$ with $n|\mathfrak{m}$. □

For the applications to C*-algebras, we need to incorporate order-units into our direct limits. Recall that an *order-unit* in an abelian monoid M is an element $u \in M$ such that each $x \in M$ satisfies $x \leqq nu$ for some $n \in \mathbb{N}$. (In case M is regular, the condition for u to be an order-unit becomes "$x \leqq u$ for all $x \in M$", because $2u \leqq u$.) We now work in the category whose objects are pairs (M, u) consisting of abelian monoids M paired with specified order-units u, and whose morphisms are normalized monoid homomorphisms, that is, a morphism from (M, u) to (M', u') is any monoid homomorphism from M to M' that sends u to u'. The existence and form of isomorphisms, direct limits, and direct products in this category are clear. We use the term "direct product" rather than "direct sum" here because the natural construction (via Cartesian products) produces categorical products which are not coproducts.

Given $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, let us write $\bar{m}$ for the coset $m + n\mathbb{Z}$, viewed as an element of the monoid $(\mathbb{Z}/n\mathbb{Z})^{\sqcup 0}$; we observe that $\bar{m}$ is an order-unit for this monoid.

Corollary 6.6. *Let* (M, u) *be an abelian monoid with order-unit. Then* (M, u) *is a direct limit of finite direct products of pairs of the form* $\big((\mathbb{Z}/n\mathbb{Z}) \sqcup \{0\}, \bar{m}\big)$ *if and only if* M *satisfies the conditions of Theorem* 6.4.

Proof. The implication $(\Rightarrow)$ is immediate from Theorem 6.4. Conversely, if M satisfies the conditions of the theorem, then M is the direct limit of a direct system of monoids M_i and transition maps f_{ij} where each M_i is a finite direct product of monoids of the form $(\mathbb{Z}/n\mathbb{Z})^{\sqcup 0}$. Let I denote the directed set indexing this direct system, and $g_i : M_i \to M$ the limiting maps. There exist $i_0 \in I$ and $u_{i_0} \in M_{i_0}$ such that $g_{i_0}(u_{i_0}) = u$. After replacing I by the cofinal subset $\{i \in I \mid i \geqq i_0\}$, we may assume that i_0 is the least element of I. Set $u_i = f_{i_0 i}(u_{i_0}) \in M_i$ for all i, so that $g_i(u_i) = u$.

Next, set $M_i' = \{x \in M_i \mid x \leqq u_i\}$ for all i, and observe that M_i' is a submonoid of M_i (remember that $2u_i \leqq u_i$). Moreover, u_i is an order-unit for M_i'. Now any $y \in M$ satisfies $y \leqq u$, whence $y = g_i(x)$ for some $i \in I$ and $x \in M_i$ satisfying $x \leqq u_i$, that is, $x \in M_i'$. Thus,

(M, u) is a direct limit of the pairs (M_i', u_i). It is straightforward to verify that each (M_i', u_i) is a finite direct product of pairs of the form $\big((\mathbb{Z}/n\mathbb{Z})^{\sqcup 0}, \overline{m}\big)$. $\square$

7. Cuntz limits

Recall that we are using the term *Cuntz limit* as an abbreviation for "C* inductive limit of a sequence of finite direct products of full matrix algebras over Cuntz algebras $\mathcal{O}_n$ for $n \in \mathbb{N}$". (In particular, we are not incorporating the algebra $\mathcal{O}_\infty$ into our scheme.) We summarize various standard facts about the monoids $V(A)$ that will be needed in applying our monoid-theoretic results to C*-algebras.

First, $V(-)$ is a functor from C*-algebras to abelian monoids that preserves finite direct products and inductive (direct) limits [1], (5.2.3)–(5.2.4). Further, $V\big(M_m(A)\big) \cong V(A)$ for any $m \in \mathbb{N}$ and any A, and $V(A)$ is countable if A is separable [1], p. 28. It is routine to check that for any unital C*-algebra A, the class $[1_A]$ is an order-unit in $V(A)$, and that the canonical isomorphism $V\big(M_m(A)\big) \to V(A)$ sends $[1_{M_m(A)}]$ to $m[1_A]$.

The basic K-theoretic information concerning the Cuntz algebras $\mathcal{O}_n$ is usually summarized in the statements $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ and $K_1(\mathcal{O}_n) = 0$ [5], Theorems 3.7–3.8. However, Cuntz also showed that the Murray-von Neumann equivalence classes of nonzero projections in $\mathcal{O}_n$ form a subgroup of $V(\mathcal{O}_n)$ which maps isomorphically onto $K_0(\mathcal{O}_n)$ under the natural map $V(\mathcal{O}_n) \to K_0(\mathcal{O}_n)$ [5], p. 188. In addition, the relation $n \cdot 1_{\mathcal{O}_n} \sim 1_{\mathcal{O}_n}$ (a direct consequence of the defining relations for $\mathcal{O}_n$) implies that every projection in a matrix algebra over $\mathcal{O}_n$ is equivalent to a projection in $\mathcal{O}_n$ itself. It follows that $V(\mathcal{O}_n)\backslash\{0\}$ is a group isomorphic to $K_0(\mathcal{O}_n)$, that is, $V(\mathcal{O}_n) \cong \big(\mathbb{Z}/(n-1)\mathbb{Z}\big) \sqcup \{0\}$. It is routine to check that this isomorphism sends $[1_{\mathcal{O}_n}]$ to the coset $\overline{1}$ in $\mathbb{Z}/(n-1)\mathbb{Z}$, and thus we have

$$(7.1) \qquad \big(V\big(M_m(\mathcal{O}_n)\big), [1_{M_m(\mathcal{O}_n)}]\big) \cong \big(\big(\mathbb{Z}/(n-1)\mathbb{Z}\big) \sqcup \{0\}, \overline{m}\big)$$

for all $m \geqq 1$ and $n \geqq 2$. The remaining basic fact that we shall need is the following lemma. It is essentially equivalent to [20], Lemma 6.1; we sketch a proof for the reader's convenience.

Lemma 7.1. *Let A be a finite direct product of full matrix algebras over Cuntz algebras, B a* C**-algebra, and $q \in B$ a projection. Then any normalized monoid homomorphism*

$$\alpha : \big(V(A), [1_A]\big) \to \big(V(B), [q]\big)$$

is induced by a C**-algebra map $\phi : A \to B$ that sends 1_A to q. That is, $V(\phi) = \alpha$.*

Proof. Write $A = \bigoplus_{j=1}^{r} M_{k_j}(\mathcal{O}_{n_j})$ for some $k_j, n_j \in \mathbb{N}$, and let $p_1, \dots, p_r$ be the corresponding orthogonal central projections in A summing to 1_A. Each p_j is an orthogonal sum of pairwise equivalent projections $e_1^{(j)}, \dots, e_{k_j}^{(j)}$ such that $e_1^{(j)} A e_1^{(j)} \cong \mathcal{O}_{n_j}$. In $V(A)$, we have $n_j[e_1^{(j)}] = [e_1^{(j)}]$ for all j and

$$\sum_{j=1}^{r} k_j[e_1^{(j)}] = \sum_{j=1}^{r} [p_j] = [1_A],$$

whence $n_j\alpha([e_1^{(j)}]) = \alpha([e_1^{(j)}])$ and $\sum_{j=1}^{r} k_j\alpha([e_1^{(j)}]) = [q]$ in $V(B)$. Consequently, q is an orthogonal sum of projections $q_1, \dots, q_r$ such that $k_j\alpha([e_1^{(j)}]) = [q_j]$, and each q_j is an orthogonal sum of pairwise equivalent projections $f_1^{(j)}, \dots, f_{k_j}^{(j)}$ such that $\alpha([e_1^{(j)}]) = [f_1^{(j)}]$.

Since $n_j[f_1^{(j)}] = [f_1^{(j)}]$, the projection $f_1^{(j)}$ is an orthogonal sum of n_j projections each equivalent to $f_1^{(j)}$, and so there exist $t_1^{(j)}, \dots, t_{n_j}^{(j)} \in f_1^{(j)}Bf_1^{(j)}$ such that $(t_l^{(j)})^* t_m^{(j)} = \delta_{lm} f_1^{(j)}$ and $\sum_{l=1}^{n_j} t_l^{(j)}(t_l^{(j)})^* = f_1^{(j)}$. Consequently, there exists a unital C*-algebra map $\phi_j : \mathcal{O}_{n_j} \to f_1^{(j)}Bf_1^{(j)}$. Define a C*-algebra map

$$\phi = \bigoplus_{j=1}^{r} M_{k_j}(\phi_j) : A \to \bigoplus_{j=1}^{r} M_{k_j}(f_1^{(j)}Bf_1^{(j)}) \cong \bigoplus_{j=1}^{r} q_jBq_j \subseteqq B.$$

It follows from the definition of ϕ that $\phi(1_A) = q$ and $[\phi(e_1^{(j)})] = [f_1^{(j)}]$ for all j. Since the classes $[e_1^{(1)}], \dots, [e_1^{(r)}]$ generate $V(A)$, we conclude that $V(\phi) = \alpha$. $\square$

Theorem 7.2. *An abelian monoid M is isomorphic to $V(A)$ for some Cuntz limit A if and only if*:

(a) *M is a countable, strongly periodic, conical refinement monoid.*

(b) *For all idempotents $e \leqq f$ in M, the homomorphism $G_M[e] \to G_M[f]$ given by $x \mapsto x + f$ is injective, and $G_M[e] + f$ is a pure subgroup of $G_M[f]$.*

Proof. ($\Rightarrow$) Recall (7.1). Since $V(-)$ preserves direct limits and finite direct products, the present implication follows from Theorem 6.4.

($\Leftarrow$) Since M is countable, Theorem 6.4 implies that M is the direct limit of a sequence of the form

$$M_1 \xrightarrow{\alpha_1} M_2 \xrightarrow{\alpha_2} M_3 \xrightarrow{\alpha_3} \cdots$$

where each M_i is a finite direct product of monoids $(\mathbb{Z}/n_{ij}\mathbb{Z})^{\sqcup 0}$ for some $n_{ij} \in \mathbb{N}$. Hence, if A_i is the direct product of the Cuntz algebras $\mathcal{O}_{n_{ij}+1}$ for the corresponding indices j, then there exists an isomorphism $h_i : V(A_i) \to M_i$. Each of the homomorphisms

$$h_{i+1}^{-1}\alpha_i h_i : V(A_i) \to V(A_{i+1})$$

sends $[1_{A_i}]$ to the class of a projection in A_{i+1}, and so, by Lemma 7.1, $h_{i+1}^{-1}\alpha_i h_i$ is induced by a C*-algebra map $\phi_i : A_i \to A_{i+1}$. Therefore $M \cong V(A)$ where A is the C* inductive limit of the sequence

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \cdots. \quad \square$$

A structural description of the monoids appearing in Theorem 7.2 is easily obtained with the help of Theorem 3.3, as follows.

Corollary 7.3. *Let M be an abelian monoid. Then $M \cong V(A)$ for some Cuntz limit A if and only if*

$$M \cong \bigsqcup_{e \in \Lambda} (\{e\} \times G_e) \subseteqq \Lambda \times G$$

where:

(a) *Λ is a countable distributive semilattice.*

(b) *G is a countable torsion abelian group.*

(c) *G_e is a pure subgroup of G for all $e \in \Lambda$.*

(d) $G_0 = \{0\}$ *and* $\bigcup_{e \in \Lambda} G_e = G$.

(e) $G_e + G_f = G_{e+f}$ *and* $G_e \cap G_f = \bigcup_{g \in \Lambda, g \leqq e, f} G_g$ *for all* $e, f \in \Lambda$.

We can also characterize the monoids $V(A)$ for Cuntz limits A with a restricted set of building blocks $\mathcal{O}_n$, as follows.

Corollary 7.4. *Let M be an abelian monoid and $\mathfrak{m}$ a generalized integer. Then $M \cong V(A)$ for some* C* *inductive limit of a sequence of finite direct products of full matrix algebras over Cuntz algebras $\mathcal{O}_n$ with $n - 1 \mid \mathfrak{m}$ if and only if M satisfies the conditions of Theorem* 7.2 *and the order of each element of M divides $\mathfrak{m}$.*

Proof. Theorem 7.2 and Corollary 6.5. □

Finally, we establish the unital cases of the above results.

Theorem 7.5. *Let (M, u) be an abelian monoid with order-unit. Then $(M, u) \cong (V(A), [1_A])$ for some unital Cuntz limit A if and only if*:

(a) *M is a countable, strongly periodic, conical refinement monoid.*

(b) *For all idempotents $e \leqq f$ in M, the homomorphism $G_M[e] \to G_M[f]$ given by $x \mapsto x + f$ is injective, and $G_M[e] + f$ is a pure subgroup of $G_M[f]$.*

Proof. ($\Rightarrow$) Theorem 7.2.

($\Leftarrow$) Corollary 6.6 implies that (M, u) is the direct limit of a sequence of the form

$$(M_1, u_1) \xrightarrow{\alpha_1} (M_2, u_2) \xrightarrow{\alpha_2} (M_3, u_3) \xrightarrow{\alpha_3} \cdots$$

where each (M_i, u_i) is a finite direct product of pairs $((\mathbb{Z}/n_{ij}\mathbb{Z})^{\sqcup 0}, \overline{m}_{ij})$ for some $n_{ij}, m_{ij} \in \mathbb{N}$. In view of (7.1), there exist isomorphisms $h_i : (V(A_i), [1_{A_i}]) \to (M_i, u_i)$ where A_i is the direct product of the matrix algebras $M_{m_{ij}}(\mathcal{O}_{n_{ij}+1})$. Each of the normalized homomorphisms

$$h_{i+1}^{-1} \alpha_i h_i : (V(A_i), [1_{A_i}]) \to (V(A_{i+1}), [1_{A_{i+1}}])$$

is induced by a unital C*-algebra map $\phi_i : A_i \to A_{i+1}$ (Lemma 7.1). Therefore $(M, u) \cong (V(A), [1_A])$ where A is the C* inductive limit of the sequence

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \cdots. \quad \square$$

Corollary 7.6. *Let (M, u) be an abelian monoid with order-unit. Then $(M, u) \cong (V(A), [1_A])$ for some unital Cuntz limit A if and only if*

$$(M, u) \cong \Big(\bigsqcup_{e \in \Lambda} (\{e\} \times G_e), (1, u_1) \Big) \subseteq (\Lambda \times G_1, (1, u_1))$$

where:

(a) Λ *is a countable distributive semilattice with maximum element* 1.

(b) G_1 *is a countable torsion abelian group.*

(c) G_e *is a pure subgroup of* G_1 *for all* $e \in \Lambda$, *and* $G_0 = \{0\}$.

(d) $G_e + G_f = G_{e+f}$ *and* $G_e \cap G_f = \bigcup_{g \in \Lambda, g \leqq e, f} G_g$ *for all* $e, f \in \Lambda$.

(e) $u_1 \in G_1$.

Proof. ($\Rightarrow$) By Corollary 7.3, M is isomorphic to a monoid of the form

$$M' = \bigsqcup_{e \in \Lambda} (\{e\} \times G_e) \subseteq \Lambda \times G$$

for some countable distributive semilattice Λ and some countable torsion abelian group G with subgroups G_e satisfying the conditions of that corollary. An isomorphism $M \to M'$ must carry u to an order-unit $u' = (\varepsilon, u_\varepsilon) \in M'$. For each $e \in \Lambda$, there exists $n \in \mathbb{N}$ such that $(e, 0) \leqq nu' = (\varepsilon, nu_\varepsilon)$, whence $e \leqq \varepsilon$. Thus, ε is the largest element of Λ, and we rename it in the standard way: $\varepsilon = 1$. Conditions (a)–(e) are now all satisfied.

($\Leftarrow$) With the help of Theorem 3.3, it is clear that M satisfies conditions (a) and (b) of Theorem 7.5. $\square$

Corollary 7.7. *Let (M, u) be an abelian monoid with order-unit, and $\mathfrak{m}$ a generalized integer. Then $(M, u) \cong (V(A), [1_A])$ for some unital* C* *inductive limit of a sequence of finite direct products of full matrix algebras over Cuntz algebras $\mathcal{O}_n$ with $n - 1 \mid \mathfrak{m}$ if and only if M satisfies the conditions of Theorem* 7.5 *and the order of each element of M divides* $\mathfrak{m}$.

Proof. Theorem 7.5 and Corollary 6.5. $\square$

Acknowledgments. Part of this work was done during visits of the second author to the Department of Mathematics of the University of California at Santa Barbara (USA) and the Département de Mathématiques de l'Université de Caen (France). The second author wants to thank both host centers for their warm hospitality.

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Eingegangen 16. August 2004