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METRIC COMPLETIONS OF ORDERED GROUPS AND K_0 OF EXCHANGE RINGS

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Abstract. We give a description of the closure of the natural affine continuous function representation of $K_0(R)$ for any exchange ring R. This goal is achieved by extending the results of Goodearl and Handelman, about metric completions of dimension groups, to a more general class of pre-ordered groups, which includes K_0 of exchange rings. As a consequence, the results about K_0^+ of regular rings, which the author gave in an earlier paper, can be extended to a wider class of rings, which includes C^*-algebras of real rank zero, among others. Also, the framework of pre-ordered groups developed here allows other potential applications.

1. Introduction

Goodearl and Handelman [22] studied the representation $\Phi : G \to \mathrm{Aff}(S(G, u))$ of a dimension group with order-unit (G, u) into the space $\mathrm{Aff}(S(G, u))$ of affine and continuous functions on its state space, as well as its completion in order-unit norm. In particular, they proved that the completion of a dimension group with order-unit in this norm is an archimedean, norm-complete dimension group with order-unit, and in fact that it is isomorphic (as ordered group) to the closure of the image of G into $\mathrm{Aff}(S(G, u))$. These results are applied successfully to describe K_0 in the case of unperforated, unit-regular rings, as well as to determine properties of some subclasses of this class of rings. Lately, Ara and Goodearl, in [2], used the representation of $K_0(R)$ into the space of affine and continuous functions on $S(K_0(R), [R]))$, in order to determine how far a non-artinian, simple regular ring R is from being a matrix ring of every dimension, and how far $K_0(R)$ is from being a dimension group. The key aspect of their proof is the property which they called *condition (D)*: $\Phi(K_0(R)^+)$ is dense in $\mathrm{Aff}(S(K_0(R), [R]))^+$. They showed that any non-artinian, strictly unperforated, simple unit-regular ring satisfies this condition. Subsequently, the author, in [29], proved that these results generalize to all non-artinian simple regular rings, and characterized the regular rings that satisfy condition (D) as those that have no nontrivial artinian homomorphic images; in fact, a somewhat more technical version of the result is obtained, which applies even if there are artinian homomorphic images, which gives a description of $\Phi(K_0(R)^+)$

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for any regular ring R. We prove this by considering the N^*-metric completion of R – which is an unperforated, unit-regular ring, as shown by Burgess and Handelman ([10]) and Torrens ([34]) – and then using the results about representations of dimension groups given by Goodearl and Handelman in [22]. We can do this because there exists an affine homeomorphism between $S(K_0(R),[R])$ and $\mathbb{P}(R)$ – the space of pseudo-rank functions over a regular ring R (see [16, Proposition 17.12]).

Our goal is to prove the analogous result for exchange rings. Exchange rings, a class of rings with the property that direct sums of projective modules have common refinements, have been widely studied (see [36], [27], [23], [33], [13], [3], [1]). In fact, it is a large class, which includes regular rings ([27] or [23]), strongly π-regular rings ([33]), and, as recently shown by Ara, Goodearl, O'Meara and Pardo, unital C^*-algebras of real rank zero ([3, Theorem 7.2]), so that our results generalize those of [29] to a more general class of rings containing C^*-algebras of real rank zero, among others.

There are some problems to solve in order to achieve our goal. On the one hand, in contrast with the case of regular rings or C^*-algebras of real rank zero, no theory of pseudo-rank functions has been developed for exchange rings. In fact, it seems a difficult task to extend that theory to this larger class of rings. Even in the case of C^*-algebras of real rank zero, where this theory has been developed ([9]), analogs of the results about N^*-metric completions for regular rings do not exist. Thus, the technique used in [29] does not work in this general context. On the other hand, K_0 of an exchange ring can fail to be a dimension group, since Moncasi, in [26] constructed a stably finite regular ring R whose $K_0(R)$ fails to be a Riesz group; moreover, $\mathrm{tor}(K_0(R)) \neq 0$. Goodearl, in [20], constructed an analogous example for the case of C^*-algebras of real rank zero. Thus, it is not possible to prove the desired results within the framework of Riesz groups.

Nevertheless, we are able to avoid these difficulties by working with asymptotic versions (in a metric sense) of the Riesz refinement and interpolation properties. (Notice that, by dealing with ordered groups, we implicitly avoid the problem of defining pseudo-rank functions on exchange rings.) The point is that the monoid of isomorphism classes of finitely generated projective right modules for an exchange ring R – denoted by $V(R)$ – is a *refinement monoid*, as shown by Ara, Goodearl, O'Meara and Pardo in [3, Proposition 1.1] (As finitely generated projective right modules over an exchange ring satisfy the finite exchange property, this result is, in fact, a restatement in monoid language of the exchange property for modules, introduced by Crawley and Jónsson in [14]) . Thus, the asymptotic properties are available in the applications of interest to us, since we will prove that the Grothendieck group of a refinement monoid has these properties. In this note, we prove that the results about representations and metric completions of dimension groups are also true in the context of these weaker "asymptotic" properties. Some additional applications are developed using the main result. Namely, the framework of pre-ordered groups we develop here gives an affirmative answer to Open Problems 1, 7 and 8 of [17]. Moreover, in [30], we obtain, as an application of the main result of this paper, a version of the theorem of representation of dimension groups ([15]) for simple Riesz groups, as well as an affirmative answer to Open Problem 2 of [17] for the case of simple groups.

Here is a brief outline of the paper. In Section 2, we introduce the notion of *asymptotic interpolation group*, and we show that the Grothendieck group of a refinement monoid lies in this class. Also, we show some "asymptotic" versions of the results of [22] that we need to prove the main result. This is contained in Section 3, where we show that the completion of an asymptotic refinement group G in its order-unit norm is an archimedean, norm-complete dimension group with order-unit, and in fact that it is isomorphic (as ordered group) to the closure of the image of G in $\mathrm{Aff}(S(G,u))$ (that is, the analog of Goodearl and Handelman's result). Section 4 is devoted to applying these results to give a description of the closure of the natural affine continuous function representation of $K_0(R)$ for any exchange ring R, as well as some other applications, as to obtain a result of comparability of finitely generated projective modules on exchange rings in terms of comparability of elements of $\Phi(K_0(R))$. In Section 5 we give a generalization of the result of Zhang about "halving projections" in simple C^*-algebras of real rank zero ([41, Theorem I(i)]), as well as those of Ara and Goodearl about approximation of simple regular rings by matrix rings ([2, Corollary 2.8]) in the context of refinement monoids, that allows us to avoid the specific techniques of both classes of rings.

Throughout this note we will refer to [17] for the background on ordered abelian groups, to [36], [23], [27] and [3] for the background on exchange rings, and to [16] and [6] for the concrete applications to the field of regular rings and C^*-algebras respectively.

2. Asymptotic properties for ordered groups

In this section we will define a property somewhat weaker than interpolation for ordered groups, and we will prove some results on groups satisfying this property that will be necessary to show that the results about representations of dimension groups, given by Goodearl and Handelman in [22], also hold for groups with fewer restrictions. As we shall see, important key examples of pre-ordered groups satisfying this property are K_0 of exchange rings. We will start by showing a useful result, that is analogous to [17, Proposition 2.1], and that allows us to define this new class.

Given a pre-ordered abelian group with order-unit (G,u), we denote by $S(G,u)$ the space of (normalized) states on (G,u). This is a convex compact subset of a locally compact real vector space ([17, Chapter 4]). Also, we denote by $\mathrm{Aff}(S(G,u))$ the space of affine and continuous real-valued functions on $S(G,u)$, which has a natural structure of partially ordered abelian group with order-unit ([17, Chapter 7]). We denote by Φ the natural evaluation map from (G,u) to $\mathrm{Aff}(S(G,u))$. Recall that the order-unit norm on (G,u) can be defined as $\|x\|_u = \|\Phi(x)\|_\infty$ ([17, Proposition 7.12]).

Proposition 2.1. *Let (G,u) be a pre-ordered group with order-unit, and let $\|.\| = \|.\|_u$ be its order-unit norm. Then, the following are equivalent:*

(a) For all $x_1, x_2, y_1, y_2 \in G$ be such that $\{x_1,x_2\} \leq \{y_1,y_2\}$ and for all $\varepsilon > 0$ there exist $z \in G$ and $d \in G^+$ such that $\{x_1,x_2\} \leq z \leq \{y_1+d, y_2+d\}$ and $\|d\| < \varepsilon$.

(b) For all $x, y_1, y_2 \in G^+$ such that $x \leq y_1 + y_2$ and for all $\varepsilon > 0$ there exist $x_1, x_2, d \in G^+$ such that $x + d = x_1 + x_2$ and $x_i \leq y_i + d$ $\forall i$, while $\|d\| < \varepsilon$.

Proof. First, assume that G satisfies (b), and let $x_1, x_2, y_1, y_2 \in G$ be such that $\{x_1,x_2\} \leq \{y_1,y_2\}$. Then $y_j - x_i \in G^+$ $\forall i,j$, and $y_2 - x_1 \leq (y_2 - x_1) + (y_1 - x_2) = (y_1 - x_1) + (y_2 - x_2)$. So, given $\varepsilon > 0$, there exist $z_1, z_2, d \in G^+$ such that $(y_2 - x_1) + d = z_1 + z_2$ and $z_j \leq (y_j - x_j) + d$ with $\|d\| < \epsilon$. Set $z = x_1 + z_1$. Hence, $x_1 \leq z$, and as $z_1 \leq (y_1 - x_1) + d$ we have that $z \leq y_1 + d$. Since $(y_2 - x_1) + d = z_1 + z_2$, we have that $z = (y_2 - z_2) + d \leq y_2 + d$. As $z_2 \leq (y_2 - x_2) + d$, we obtain $x_2 \leq z$. Thus, we conclude that $\{x_1, x_2\} \leq z \leq \{y_1 + d, y_2 + d\}$.

Now assume that G satisfies (a), and let $x, y_1, y_2 \in G^+$ be such that $x \leq y_1 + y_2$. Thus it is easy to see that $\{0, x - y_2\} \leq \{x, y_1\}$, and hence, given $\varepsilon > 0$, $\exists x_1 \in G$, $\exists d \in G^+$ such that $\{0, x - y_2\} \leq x_1 \leq \{x + d,\ y_1 + d\}$ and $\|d\| < \epsilon$. So, $x + d = x_1 + x_2$ for some $x_2 \in G^+$, $x_1 \leq y_1 + d$, and $x - y_2 \leq x_1 = x + d - x_2$, whence $x_2 \leq x + d - x + y_2 = y_2 + d$. ■

Observe that if the conditions in Proposition 2.1 hold, then by induction, the corresponding conditions for larger numbers of elements also hold.

Definition. A pre-ordered abelian group with order-unit (G, u) is said to satisfy the **asymptotic interpolation property** (and so we say that G is an asymptotic interpolation group) provided G satisfies the condition given in Proposition 2.1(a), and it is said to satisfy the **asymptotic refinement property** (and so we say that G is an asymptotic refinement group) provided G satisfies the condition given in Proposition 2.1(b).

Observe that, as all order-unit norms are equivalent, the properties we have defined above do not depend of the choice of the order-unit.

Clearly, the interpolation groups lies in this new class. Another family of pre-ordered groups which satisfy asymptotic interpolation is that of *finite tensor products of interpolation groups with order-unit*, as pointed out by Wehrung in [39] (in that paper, Wehrung constructed some examples of this kind – see [39, Examples 1.4 and 1.5] – that fail to be interpolation groups). We will show that the class of Grothendieck groups of refinement monoids with order-unit also lies in this new class. Recall that an abelian monoid M is said to be a *refinement monoid* provided that whenever $x_1, x_2, y_1, y_2 \in M$ are such that $x_1 + x_2 = y_1 + y_2$, then $\exists z_{ij} \in M$ such that $\sum_j z_{ij} = x_i$ and $\sum_i z_{ij} = y_j$ for $i, j = 1, 2$. We will define the following pre-ordering $\leq$ for any abelian monoid M: for any $x, y \in M$, we will say that $x \leq y$ if and only if there exists $z \in M$ such that $x + z = y$. It is sometimes called the *algebraic ordering*, but since it is the only ordering we will use, we do not use any special notation. An element $u \in M$ is said to be an *order-unit* if, whenever $x \in M$, there exists $n \in \mathbb{N}$ such that $x \leq nu$. By using this order-unit, the state space $S(M, u)$ and the natural evaluation map $\Phi : M \longrightarrow \mathrm{Aff}(S(M, u))$ of a monoid M with order-unit u are defined in the same way as for a pre-ordered abelian group with order-unit. Given any abelian monoid M, we will denote by $G(M)$ the *universal group* – or *Grothendieck group* – of M, and we will use $[x]$ to denote the class of $x \in M$ inside $G(M)$ (Recall that the equivalence relation on M is given by the following rule: for any two elements $x, y \in M$, $[x] = [y]$ if there exists $z \in M$ such that $x + z = y + z$.) $G(M)$ will be viewed as a pre-ordered group with positive cone $G(M)^+ = \{[x] \mid x \in M\}$.

Lemma 2.2. *Let (M, u) be a refinement monoid with order-unit, and let $a_0, a_1, b_0, b_1 \in M$ be such that $\{a_0, a_1\} \leq \{b_0, b_1\}$. Then, $\forall \varepsilon > 0$ $\exists c_\varepsilon, d_\varepsilon \in M$ with $\|d_\varepsilon\| < \varepsilon$, such that $\{a_0, a_1\} \leq c_\varepsilon \leq \{b_0 + d_\varepsilon, b_1 + d_\varepsilon\}$.*

Proof. Assume that we have $\{a_0, a_1\} \leq \{b_0, b_1\}$. Thus, for all $n \in \mathbb{N}$ we have, by [38, Lemma 2.8], that there exist $\overline{c}, \overline{d}, d_0, d_1 \in M$ and $\overline{m}, m_0, m_1 \in \mathbb{N}$ such that $a_0 \leq \overline{c}$, $a_1 \leq \overline{c} + \overline{d}$, $\overline{c} \leq b_i + d_i$ and $2^{\overline{m}+n+1}\overline{d} \leq 2^{\overline{m}}\overline{c}$, $2^{n+m_i} d_i \leq 2^{m_i} b_i$. Set $c = \overline{c} + \overline{d}$ and $d = \overline{d} + d_0 + d_1$. Hence, $\{a_0, a_1\} \leq c \leq \{b_0 + d, b_1 + d\}$.

Observe that $\|\overline{d}\| \leq \frac{\|\overline{c}\|}{2^{n+1}}$ and that $\|d_i\| \leq \frac{\|b_i\|}{2^n}$. Thus, if $k = \max\{\|b_0\|, \|b_1\|\}$, then $\|c\| \leq \|\overline{c}\| + \|\overline{d}\| \leq \|\overline{c}\| + \frac{1}{2^{n+1}}\|\overline{c}\| = (1 + \frac{1}{2^{n+1}})\|\overline{c}\| \leq (1 + \frac{1}{2^{n+1}})(\|b_i\| + \frac{1}{2^n}\|b_i\|) = (\frac{2^{n+1}+1}{2^{n+1}}\frac{2^n+1}{2^n})\|b_i\| \leq 2k$, whence $\|d\| \leq \|\overline{d}\| + \|d_0\| + \|d_1\| \leq \frac{3k}{2^n} < \frac{k}{2^{n-2}}$, and taking $n > 2 + \log_2(\frac{k}{\varepsilon})$ we obtain the desired result. ■

Our next result constitutes the main motivation for our definition of asymptotic refinement groups.

Proposition 2.3. *Let (M, u) be a refinement monoid with order-unit, and let G be its Grothendieck group. Then:*

(a) If $x, y_1, y_2 \in G^+$ and $x \leq y_1 + y_2$, then $\forall \varepsilon > 0$ $\exists d \in M$ with $\|[d]\| < \varepsilon$ and $x_1, x_2 \in G^+$ such that $x = x_1 + x_2$ and $x_1 \leq y_1$, $x_2 \leq y_2 + [d]$.

(b) G is an asymptotic refinement group.

Proof. First of all, notice that if $x, y \in M$ and $[x] \leq [y]$, then $\forall \varepsilon > 0$ $\exists d \in M$ with $\|[d]\| < \varepsilon$ such that $x \leq y + d$. To see this, observe that there exists $z \in M$ such that $x + z \leq y + z$. Applying [37, Lemma 1.11], $\exists d \in M$ with $\|[d]\| < \varepsilon$ such that $x \leq y + d$, which ends the proof of the claim. Now,

(a) holds by the claim and the fact that M is a refinement monoid.

(b) is easy to see by noticing that $\exists m \in \mathbb{N}$ such that $x_i + mu \geq 0$, $y_j + mu \geq 0$ for $i, j \in \{1, 2\}$, and using the claim and Lemma 2.2. ■

In fact, as we will see now, the asymptotic refinement groups are, in some sense, asymptotic dimension groups.

Lemma 2.4. *Let (G, u) be an asymptotic refinement group. If $x \leq my$ for some $x, y \in G^+$ and some $m \in \mathbb{N}$, then $\forall \varepsilon > 0$ there exists a decomposition $x + d = x_1 + \cdots + x_m$ with $x_i \in G^+$ and $d \in G^+$ such that $x_1 \leq x_2 \leq \cdots \leq x_m \leq y + d$, and $\|d\| < \varepsilon$.*

Proof. We will prove it by induction on m. For $m = 1$ it is trivial; so we assume that it holds for $m - 1$, and we will check it for m. If $x \leq my$, then we have $x + d_1 = z_1 + \cdots + z_m$, $z_i \leq y + d_1$ for $z_i \in G^+$ and $d_1 \in G^+$ with $\|d_1\| < \frac{\varepsilon}{5}$. So $z_i \leq \{x+d_1, y+d_1\}$ $\forall i$, and thus $\exists x' \in G$ such that $z_i \leq x' \leq \{x+d_1+d_2, y+d_1+d_2\}$ $\forall i$, $d_2 \in G^+$ with $\|d_2\| < \frac{\varepsilon}{5}$. Now set $z = x+(d_1+d_2)-x'$. Then, $z = x+(d_1+d_2)-x' \leq x + (d_1 + d_2) - z_m = z_1 + \cdots + z_{m-1} + d_2 \leq (m-1)x' + d_2 \leq (m-1)(x' + d_2)$. By the induction hypothesis, $z + d_3 = x_1 + \cdots + x_{m-1}$ with $x_1 \leq \cdots \leq x_{m-1} \leq x' + d_2 + d_3$ for $d_3 \in G^+$ with $\|d_3\| < \frac{\varepsilon}{5}$. Set $x_m = x' + d_2 + d_3$, and observe that $x + (d_1 + 2d_2 + 2d_3) = x_1 + \cdots + x_m$ and $x_1 \leq \cdots \leq x_m \leq y + (d_1 + 2d_2 + 2d_3)$. Set $d = d_1 + 2d_2 + 2d_3$. Then the induction step works. ■

Lemma 2.5. *Let (G, u) be an asymptotic refinement group, let $a, b \in G^+$ and let $m, n \in \mathbb{N}$. If $ma \leq mb$, then $\forall \varepsilon > 0$ $\exists d_n, d_n{}' \in G^+$ with $\{\|d_n\|, \|d_n{}'\|\} < \varepsilon$ such that $a + d_n = a_n + w_n$ for some $a_n, w_n \in G^+$ satisfying $na_n \leq (m-1)w_n$ and $w_n \leq b + d_n{}'$.*

Proof. We will prove it by induction on n. If $n = 1$, by Lemma 2.4, $a + d = x_1 + \cdots + x_m$ for some $x_i \in G^+$ with $x_1 \leq x_2 \leq \cdots \leq x_m \leq b + d$ with $\|d\| < \varepsilon$. Set $a_1 = x_1 + \cdots + x_{m-1}$ and $w_1 = x_m$; then $a + d = a_1 + w_1$ with $w_1 \leq b + d$ and $a_1 \leq (m-1)w_1$.

Now assume that for some $n \in \mathbb{N}$ we have $a + d_1 = a_n + w_n$ and $b + d_2 = b_n + w_n$ for some $a_n, b_n, w_n, d_1, d_2 \in G^+$ with $na_n \leq (m-1)w_n$ and $\{\|d_1\|, \|d_2\|\} < \frac{\varepsilon}{4}$.

Then, $ma_n \leq m(b_n + d_1)$ and so, by the result of the previous paragraph, we get $a_n + d_3 = a_{n+1} + z_{n+1}$ and $b_n + d_1 + d_3 = b_{n+1} + z_{n+1}$ for some $a_{n+1}, b_{n+1}, z_{n+1}, d_3 \in G^+$ with $a_{n+1} \leq (m-1)z_{n+1}$ and $\|d_3\| < \frac{\varepsilon}{2(n+1)}$. Setting $w_{n+1} = z_{n+1} + w_n + nd_3$, we obtain $a + (d_1 + (n+1)d_3) = a_{n+1} + w_{n+1}$ with $w_{n+1} \leq b + (d_1 + d_2 + (n+1)d_3)$ and

$$
\begin{aligned}
(n+1)a_{n+1} &\leq a_{n+1} + na_{n+1} \leq (m-1)z_{n+1} + (m-1)w_n + nd_3 \\
&\leq (m-1)(z_{n+1} + w_n + nd_3) = (m-1)w_{n+1};
\end{aligned}
$$

thus, setting $d_n = d_1 + (n+1)d_3$, ${d_n}' = d_1 + d_2 + (n+1)d_3$, we end the proof. ■

In particular, we have the following result:

Corollary 2.6. *Let (G, u) be an asymptotic refinement group, and let $a, b \in G$ and $m \in \mathbb{N}$ be such that $ma \leq mb$. Then, $\forall \varepsilon > 0$ $\exists d \in G^+$ with $\|d\| < \varepsilon$ such that $a \leq b + d$.*

Proof. First, notice that taking a suitable $n \in \mathbb{N}$, we have $\{a + nu, b + nu\} \in G^+$, whence we can assume that $a, b \in G^+$. Thus, by Lemma 2.5, $a \leq a + d_n = a_n + w_n \leq b + d'_n + a_n$, where $\|d'_n\|$ could be chosen arbitrarily small and $\|a_n\| < \frac{m-1}{n}(\|b\| + \|d'_n\|)$ for arbitrarily big $n \in \mathbb{N}$. Thus the result holds by taking $\|d'_n\| < \frac{\varepsilon}{2}$ and $n > \frac{2(m-1)(\|b\| + \frac{\varepsilon}{2})}{\varepsilon}$. ■

Corollary 2.6 means that any asymptotic refinement group is "asymptotically unperforated", so that, as we observed above, there are asymptotic dimension groups. Now, in order to show the main result, we will prove a number of results that are "asymptotic" versions of those that Goodearl and Handelman showed in [22] (and we will give, for each one, the reference which appears in [17]). To prove it, we use essentially the same techniques that they used in that paper, except that various norm-estimates need to be computed since the results only hold asymptotically. Remember that this difficulty cannot be avoided to give an answer to the question of describing K_0 of exchange rings, as we noticed in the Introduction.

Lemma 2.7 (cf. [17, Lemma 2.18])**.** *Let (G, u) be an asymptotic refinement group. Let $p \in G$, $z \in G^+$. If $p \leq z$ and $2p \leq z$, then $\forall \varepsilon > 0$ $\exists q, s \in G^+$ such that $p \leq q$, $2q \leq z + s$ and $\|s\| < \varepsilon$.*

Proof. We have $\{0, p\} \leq \{z, z - p\}$, and hence, by definition, there exist $r \in G, s_1 \in G^+$ such that $\|s_1\| < \frac{\varepsilon}{2}$ and $\{0, p\} \leq r \leq \{z + s_1, z + s_1 - p\}$. It follows that $\{0, p\} \leq \{r, z + s_1 - r\}$, and again by definition there exist $q \in G, s_2 \in G^+$ such that $\|s_2\| < \frac{\varepsilon}{4}$ and $\{0, p\} \leq q \leq \{r + s_2, z + (s_1 + s_2) - r\}$. Then, $q \in G^+$, $p \leq q$ and $2q \leq z + (s_1 + s_2) - r + r + s_2 = z + (s_1 + 2s_2)$. Set $s = s_1 + 2s_2$, and the proof is complete. ■

Proposition 2.8 (cf. [17, Proposition 2.19])**.** *Let (G, u) be an asymptotic refinement group. Let $x, y, z \in G^+$ and let $n \in \mathbb{N}$. If $2^n x \leq 2^n y + z$, then for all $\varepsilon > 0$ there exist $v, w, d, d' \in G^+$ such that $d \leq d'$, $\|d'\| < \varepsilon$, $x + d = v + w$, $v \leq y + d$ and $2^n w \leq z + d'$.*

Proof. First assume that $n = 1$, so that $2x \leq 2y + z$. Set $p = x - y$, whence $2p \leq z$. As $2p \leq z \leq 2z$, we have that $\exists d_1 \in G^+$ with $\|d_1\| < \frac{\varepsilon}{4}$ and $p \leq z + d_1$ (because of Corollary 2.6). Now, by Lemma 2.7, $\exists q, d_2 \in G^+$ with $\|d_2\| < \frac{\varepsilon}{4}$ such that $p \leq q$ and $2q \leq z + (d_1 + d_2)$. Then $x \leq y + q$, and hence $\exists v, w, d \in G^+$ with $\|d\| < \frac{\varepsilon}{4}$ such that $x+d = v+w$, $v \leq y+d$ and $w \leq q+d$. Also, $2w \leq 2q+2d \leq z+(d_1+d_2+2d)$. Take $d' = d_1 + d_2 + 2d$.

Now let $n > 1$, and assume that the result holds for lower powers of 2. Since $2^{n-1}(2x) \leq 2^{n-1}(2y) + z$, the induction hypothesis implies that $2x + d_1 = p + q$ for some $p, q, d_1, {d_1}' \in G^+$ with $d_1 \leq {d_1}'$ and $\|{d_1}'\| < \frac{\varepsilon}{2}$ such that $p \leq 2y + d_1$ and $2^{n-1}q \leq z + {d_1}'$. Then, $2x + d_1 \leq 2y + d_1 + q$, i.e., $2x \leq 2y + q$. By the first case proved, $x + d_2 = v + w$ for some $v, w, d_2, {d_2}' \in G^+$ with $d_2 \leq {d_2}'$ and $\|{d_2}'\| < \frac{\varepsilon}{2^n}$ such that $v \leq y+d_2$, $2w \leq q+{d_2}'$. As $2^n w \leq 2^{n-1}q+2^{n-1}{d_2}' \leq z+({d_1}'+2^{n-1}{d_2}')$, by taking $d = d_2, d' = {d_1}' + 2^{n-1}{d_2}'$, we complete the induction step. ■

Corollary 2.9 (cf. [17, Corollary 2.20]). *Let (G, u) be an asymptotic refinement group. Let $x, y \in G$ and $z \in G^+$, and let $n \in \mathbb{N}$. If $2^n x \leq 2^n y + z$, then for all $\varepsilon > 0$ there exist $v \in G$, $w, d, d' \in G^+$ such that $d \leq d'$, $\|d'\| < \varepsilon$, $x + d = v + w$, $v \leq y + d$ and $2^n w \leq z + d'$.*

Proof. Since G is directed, $\exists t \in G$ such that $t \leq x$, $t \leq y$. Set $x' = x - t$, $y' = y - t$. Then $x', y' \in G^+$ and $2^n x' \leq 2^n y' + z$. By Proposition 2.8, $x' + d = v' + w$, $v', w \in G^+$ such that $v' \leq y' + d$ and $2^n w \leq z + d'$, where $0 \leq d \leq d'$ with $\|d'\| < \varepsilon$. Set $v = v' + t$. Then $x + d = v + w$ and $v \leq y + d$. ■

Proposition 2.10 (cf. [17, Proposition 2.21]). *Let (G, u) be an asymptotic refinement group. Let $z \in G^+$ and let $n \in \mathbb{N}$. Set $X = \{x \in G^+ \mid 2^n x \leq z\}$. Then, for all $\varepsilon > 0$ and for all $x_1, x_2, \ldots, x_k \in X$ there exist elements $\overline{x}, d \in G^+$ such that $\|d\| < \varepsilon$, $\{x_1, \ldots, x_k\} \leq \overline{x}$ and $2^n \overline{x} \leq (z + d)$.*

Proof. It is enough to consider the case $k = 2$. Given $x_1, x_2 \in X$, set $z' = z - 2^n x_2$. Then $z' \in G^+$ and $2^n x_1 \leq 2^n x_2 + z'$. By Proposition 2.8, $x_1 + d = v + w$, where $v, w \in G^+$ are such that $v \leq x_2 + d$ and $2^n w \leq z' + d'$ for some $0 \leq d \leq d'$ with $\|d'\| < \frac{\varepsilon}{2^{n+1}}$. Set $\overline{x} = x_2 + w + d$, and observe that each $x_i \leq \overline{x}$. Since $2^n w \leq z' + d' = z + d' - 2^n x_2$, we have $2^n \overline{x} \leq z + (2^n d + d')$. ■

Lemma 2.11 (cf. [17, Lemma 7.14]). *Let (G, u) be an asymptotic refinement group. Let $x \in G$, and assume that $\|x\| < \frac{k}{2^n}$ for some positive integers k, n. Then, for all $\varepsilon > 0$ there exist $\overline{u}, y, z \in G^+$ such that $u \leq \overline{u}$, $\|\overline{u} - u\| < \varepsilon$, $x = y - z$ and $2^n y \leq k\overline{u}$, $2^n z \leq k\overline{u}$.*

Proof. Without loss of generality, we can assume that G is nonzero. By [17, Lemma 7.13] and Corollary 2.6, $\exists d_1 \in G^+$ with $\|d_1\| < \frac{\varepsilon}{3}$ such that

$$-k(u + d_1) \leq 2^n x \leq k(u + d_1).$$

Set $u_1 = u + d_1$, and notice that, in particular, $2^n x \leq 2^n 0 + k u_1$. According to Corollary 2.9, $x + d_2 = v + w$, where $v \in G$, $w \in G^+$ are such that $v \leq 0 + d_2$ and $2^n w \leq k u_1 + {d_2}'$, where $0 \leq d_2 \leq {d_2}'$ and $\|{d_2}'\| < \frac{\varepsilon}{6}$. Set $u_2 = u_1 + d_2 + {d_2}'$. Now, $-k u_2 \leq 2^n x$, whence $-k u_2 + 2^n d_2 \leq 2^n (x + d_2) = 2^n v + 2^n w$, and hence $2^n(-v + d_2) \leq 2^n w + k u_2$. As $-v + d_2, w, k u_2 \in G^+$, Proposition 2.8 says that $-v + d_2 + d_3 = a + z$, where $a, z \in G^+$ are such that $a \leq w + d_3$ and $2^n z \leq k u_2 + {d_3}'$ for $0 \leq d_3 \leq {d_3}'$ and $\|{d_3}'\| < \frac{\varepsilon}{3 \cdot (2^n + 2)}$. Set $u_3 = u_2 + d_3 + {d_3}'$, $y = w + d_3 - a$, so that $y \in G^+$ and $x = v + w - d_2 = w + d_3 + v - d_2 - d_3 = w + d_3 - a - z = y - z$. As $a \geq 0$, we also have $2^n y \leq 2^n w + 2^n d_3 \leq k(u_3 + 2^n d_3)$. Set $\overline{u} = u_3 + 2^n d_3$. ■

An immediate application of these results allows us to adapt the proof of [17, Proposition 7.15] to the case of asymptotic refinement groups. This result, which we enunciate without proof, gives an affirmative answer to [17, Open Problem 7]. In order to state this result, recall that a subgroup H of a preordered abelian group G is an *ideal* if is a convex, directed subgroup.

Proposition 2.12. *Let (G, u) be an asymptotic refinement group, and let H be an ideal of G. Then*

$$\|x + H\| = \inf\{\|y\| \mid y \in x + H\}$$

for any $x \in G$.

3. The main result

In this section, we will use the results of the above section to obtain results about representation of asymptotic refinement groups analogous to those that Goodearl and Handelman obtained in [22]. Our results, in fact, follow from these of Goodearl and Handelman, once we are able to show that the completion of an asymptotic refinement group with respect to its order-unit norm is an archimedean, norm-complete dimension group with order-unit. We will use a functional-like method to obtain this result.

Lemma 3.1. *Let (G, u) be an asymptotic refinement group, let $S := S(G, u)$ be its state space, and let $\Phi : G \to \mathrm{Aff}(S)$ be the natural map. Then, $\overline{\Phi(G^+)} = \overline{\Phi(G)} \cap \mathrm{Aff}(S)^+$.*

Proof. Clearly, $\overline{\Phi(G^+)} \subseteq \overline{\Phi(G)} \cap \mathrm{Aff}(S)^+$. Conversely, take any element $f \in \overline{\Phi(G)} \cap \mathrm{Aff}(S)^+$. Now, given $n \in \mathbb{N}$, there exists an element $x \in G$ such that $\|\Phi(x) - f\|_\infty < \frac{1}{2^n}$. In particular, since $f \geq 0$, we have that

$$\frac{-1}{2^n} \ll \Phi(x),$$

and so $0 \ll \Phi(2^n x + u)$. We know, by [17, Theorem 4.12], that there exists $m \in \mathbb{N}$ such that $0 \leq m(2^n x + u)$. Set $x = y - z$, where $y, z \in G^+$. Thus,

$$m2^n z \leq m(2^n y + u).$$

By Corollary 2.6 there is a $d_1 \in G^+$ with $\|d_1\| < \frac{1}{2^{n+1}}$ such that

$$2^n z \leq 2^n y + (u + d_1),$$

and by Proposition 2.8 there exist elements $v, w, d_2, d_2' \in G^+$ such that $d_2 \leq d_2'$ with $\|d_2'\| < \frac{1}{2^{n+1}}$ such that $z + d_2 = v + w$, $v \leq y + d_2$ and $2^n w \leq u + d_1 + d_2'$. Hence, if we define $t = y + d_2 - v$ (and notice that $t \in G^+$), we have that

$$x = y - z = (y + d_2) - (z + d_2) = (y + d_2 - v) - w = t - w$$

with $t, w \in G^+$ and $\|x - t\| = \|w\| \leq \frac{1}{2^n}\|u + d_1 + d_2'\| \leq \frac{1}{2^n}(1 + \frac{1}{2^n})$. Thus,

$$\|f - \Phi(t)\|_\infty < \|f - \Phi(x)\|_\infty + \|\Phi(x) - \Phi(t)\|_\infty$$

$$< \frac{1}{2^n} + \frac{1}{2^n}(1 + \frac{1}{2^n}) = \frac{1}{2^n}(2 + \frac{1}{2^n}) = \frac{2^{n+1} + 1}{2^{2n}} < \frac{1}{2^{n-2}}. \quad \blacksquare$$

Notation. Let (G, u) be an asymptotic refinement group, let $\overline{G}$ be the completion of G in the order-unit norm, and let $\phi : G \to \overline{G}$ be the natural morphism. $\overline{G}$ is an abelian group, endowed with the ordering induced by the positive cone $\overline{G}^+ := \overline{\phi(G^+)}$. We will endow $\overline{G}$ with a norm $\|.\|_*$ defined by the rule

$$\|x\|_* = \lim_{n\to\infty} \|x_n\|,$$

where $(x_n)_{n\geq 1} \subseteq G$ is a Cauchy sequence (in the order-unit norm) such that $\phi(x_n) \to x$. With this norm, the morphism ϕ is an isometry.

Let $S =: S(G, u)$ be the space of states on G, and let $\Phi : G \to \mathrm{Aff}(S)$ be the natural affine representation of G. If $\overline{\Phi(G)}$ is the closure of $\Phi(G) \subseteq \mathrm{Aff}(S)$ in the supremum norm, then we will endow it with an ordering compatible with the inclusion, defining

$$\overline{\Phi(G)}^+ = \overline{\Phi(G)} \cap \mathrm{Aff}(S)^+.$$

Proposition 3.2. *Let (G, u) be an asymptotic refinement group, and let $\overline{G}$ be its completion in the order-unit norm. Then, $\overline{G} \cong \overline{\Phi(G)}$ as pre-ordered groups.*

Proof. Let $S =: S(G, u)$ be the space of states on G. Since $\Phi : G \to \mathrm{Aff}(S)$ and $\phi : G \to \overline{G}$ are both isometries, and $\mathrm{Aff}(S)$ is complete in its own norm, there exists a unique isometry $f : \overline{G} \to \mathrm{Aff}(S)$ such that the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[dr, "\Phi"'] & \overline{G} \arrow[d, "f"] \\ & \mathrm{Aff}(S) \end{tikzcd}$$

Clearly, f is a group morphism by definition. Suppose that we have an element $\overline{x} \in \overline{G}$ such that $f(\overline{x}) = 0$. Then, take a Cauchy sequence $(x_n)_{n\geq 1} \subseteq G$ such that $\phi(x_n) \to \overline{x}$. As f is an isometry (and thus a continuous map with respect to the metric of $\overline{G}$ and $\mathrm{Aff}(S)$), we have $\Phi(x_n) = f(\phi(x_n)) \to f(\overline{x}) = 0$. Since Φ is an isometry, we have that $(x_n)_{n\geq 1}$ is a nullsequence, and hence $\overline{x} = 0$. Then, f is one-to-one. Moreover, as the three maps are isometries, we have that

$$\overline{\Phi(G)} = \overline{f(\phi(G))} = f(\overline{\phi(G)}) = f(\overline{G}),$$

and thus f is onto $\overline{\Phi(G)}$. Similarly,

$$f(\overline{G}^+) = f(\overline{\phi(G^+)}) = \overline{f(\phi(G^+))} = \overline{\Phi(G^+)}.$$

Since $\overline{\Phi(G^+)} = \overline{\Phi(G)} \cap \mathrm{Aff}(S)^+$ by Lemma 3.1, we have $f(\overline{G}^+) = \overline{\Phi(G)}^+$. ■

Corollary 3.3. *Let (G, u) be an asymptotic refinement group, let $\overline{G}$ be its completion in the order-unit norm, let $\phi : G \to \overline{G}$ be the natural morphism, and let $\overline{u} := \phi(u)$. Then, $(\overline{G}, \overline{u})$ is a partially ordered abelian group with order-unit, unperforated and archimedean. Moreover, $\|.\|_* = \|.\|_{\overline{u}}$, and the map $S(\phi) : S(\overline{G}, \overline{u}) \to S(G, u)$ is an affine homeomorphism.*

Proof. Let $S =: S(G, u)$ be the space of states on G. By Proposition 3.2, the natural ordering of $\overline{G}$ coincides with the ordering induced by the map $\Phi : G \to \mathrm{Aff}(S)$ on the closure of $\Phi(G)$ (that is, the map f defined in Proposition 3.2 is an order-embedding). Thus, $\overline{G}$ is isomorphic to a closed subgroup of $\mathrm{Aff}(S)$, whence it is partially ordered, archimedean, unperforated, and $\overline{u}$ is an order-unit, showing the first part of the result.

Now, given $x \in \overline{G}$, we have $\|x\|_{\overline{u}} = \inf\{\frac{k}{n} \mid k, n \in \mathbb{N} \text{ and } -n\overline{u} \leq kx \leq n\overline{u}\}$. As $f : \overline{G} \to \mathrm{Aff}(S)$ is an order-embedding, the last expression equals

$$\inf\{\frac{k}{n} \mid k, n \in \mathbb{N} \text{ and } -n \cdot 1 \leq kf(x) \leq n \cdot 1\} = \|f(x)\|_\infty = \|x\|_*,$$

where the last equality derives from the fact that f is an isometry with the norm $\|\cdot\|_*$.

Finally, consider the affine continuous map $S(\phi) : S(\overline{G}, \overline{u}) \to S(G, u)$. Clearly, $S(\phi)$ is onto, because each $s \in S(G, u)$ extends by continuity to $\overline{s} \in S(\overline{G}, \overline{u})$. Also, as we observed in the last paragraph, $\|\overline{x}\|_{\overline{u}} = \lim_{n\to\infty} \|x_n\|_u$ for any $\overline{x} = \lim_{n\to\infty} \phi(x_n)$, and thus every $s \in S(\overline{G}, \overline{u})$ is continuous with respect to the metric inherited from G by the completion process. Hence, $s = \overline{s_{|G}}$, showing that $S(\phi)$ is one-to-one. Since $S(\overline{G}, \overline{u})$ is compact, $S(\phi)$ is a homeomorphism. ■

Theorem 3.4. *Let (G, u) be an asymptotic refinement group, let $\overline{G}$ be its completion in the order-unit norm, let $\phi : G \to \overline{G}$ be the natural morphism, and let $\overline{u} =: \phi(u)$. Then, $(\overline{G}, \overline{u})$ is an archimedean, norm-complete dimension group with order-unit.*

Proof. According to Lemma 3.3, we only need to prove that $\overline{G}$ is an interpolation group. We will give a proof of this analogous to that of Goodearl and Handelman ([17, Theorem 15.3]).

To see that $\overline{G}$ is an interpolation group, let f_1, f_2, g_1, g_2 be any elements in $\overline{G}$ satisfying $f_i \leq g_j$ for all i, j. Choose sequences $\{x_{11}, x_{12}, \dots\}, \{x_{21}, x_{22}, \dots\}, \{y_{11}, y_{12}, \dots\}, \{y_{21}, y_{22}, \dots\}$ in G such that $\|\phi(x_{in}) - f_i\| < \frac{1}{2^{n+2}}$ and $\|\phi(y_{jn}) - g_j\| < \frac{1}{2^{n+2}}$ $\forall i, j, n$. For all i, n, observe that

$$\begin{aligned}\|x_{i,n+1} - x_{in}\| = \|\phi(x_{i,n+1}) - \phi(x_{in})\| &\leq \|\phi(x_{i,n+1}) - f_i\| + \|\phi(x_{in}) - f_i\| \\ &< \frac{1}{2^{n+3}} + \frac{1}{2^{n+2}} < \frac{1}{2^{n+1}}.\end{aligned}$$

Similarly, $\|y_{j,n+1} - y_{jn}\| < \frac{1}{2^{n+1}}$ $\forall j, n$. We shall construct Cauchy sequences $\{e_n\}$ and $\{z_n\}$ in G such that $e_n \to 0$ and $x_{in} \leq z_n \leq y_{jn} + e_n$ $\forall i, j, n$. The limit of the sequence $\{\phi(z_n)\}$ will then provide an element of $\overline{G}$ to interpolate between f_1, f_2, g_1, g_2.

We first construct elements $a_1, a_2, \dots$ in G^+ such that $\|a_n\| < \frac{1}{2^n}$ $\forall n$, while also $x_{in} - a_n \leq x_{i,n+1} \leq x_{in} + a_n$ and $y_{jn} - a_n \leq y_{j,n+1} \leq y_{jn} + a_n$ $\forall i, j, n$. For each i, n, we have $\|x_{i,n+1} - x_{in}\| < \frac{1}{2^{n+1}}$, whence by Lemma 2.11 $\exists \overline{u} \geq u$ with $\|\overline{u} - u\| < \frac{1}{4}$ and $x_{i,n+1} - x_{in} = p_{in} - q_{in}$ for some $p_{in}, q_{in} \in G^+$ satisfying $2^{n+1}p_{in} \leq \overline{u}$ and $2^{n+1}q_{in} \leq \overline{u}$. Similarly, each $y_{j,n+1} - y_{jn} = r_{jn} - s_{jn}$ for some $r_{jn}, s_{jn} \in G^+$ satisfying $2^{n+1}r_{jn} \leq \widetilde{u}$ and $2^{n+1}s_{jn} \leq \widetilde{u}$ for some $\widetilde{u} \geq u \geq 0$ satisfying $\|\widetilde{u} - u\| < \frac{1}{4}$. As $\overline{u} = u + v$, $\widetilde{u} = u + w$, taking $u' = u + v + w$ we can change $\overline{u}$ and $\widetilde{u}$ to u' in these expressions while preserving the desired properties, and notice that $\|u' - u\| \leq \|\overline{u} - u\| + \|\widetilde{u} - u\| < \frac{1}{2}$. By Proposition 2.10, $\exists \widehat{u} \geq u'$ with $\|\widehat{u} - u'\| < \frac{1}{2}$ and $a_n \in G^+$ such that $\{p_{in}, q_{in}, r_{jn}, s_{jn}\} \leq a_n$ $\forall i, j$ and $2^{n+1}a_n \leq \widehat{u}$. So, as $\|\widehat{u}\| \leq \|\widehat{u} - u'\| + \|u' - u\| + \|u\| < 2$, we have that $\|a_n\| < \frac{1}{2^n}$. The required properties of a_n are clear.

Next, we construct elements $b_1, b_2, \dots$ in G^+ such that $\|b_n\| < \frac{1}{2^n}$ $\forall n$, while also $x_{in} \leq y_{jn} + b_n$ $\forall i, j, n$. Fix n for a while, and let $f : \overline{G} \to \mathrm{Aff}(S)$ be as in the proof of Proposition 3.2. For each i, j we have

$$(f\phi)(x_{in}) - \frac{1}{2^{n+2}} \ll f(f_i) \le f(g_j) \ll (f\phi)(y_{jn}) + \frac{1}{2^{n+2}},$$

whence $\Phi(x_{in}) \ll \Phi(y_{jn}) + \frac{1}{2^{n+1}}$, and so $\Phi(2^{n+1}x_{in}) \ll \Phi(2^{n+1}y_{jn} + u)$. So, by [17, Theorem 7.8] and Corollary 2.6 we have that $\exists d \in G^+$ with $\|d\| < \frac{1}{6}$ and $2^{n+1}x_{in} < 2^{n+1}y_{jn} + (u + d)$. According to Proposition 2.8, $x_{in} + d_{ijn} = p_{ij} + q_{ij}$ for some $p_{ij}, q_{ij} \in G^+$ satisfying $p_{ij} \le y_{jn} + d_{ijn}$ and $2^{n+1}q_{ij} \le u + d + d'_{ijn}$ for some $0 \le d_{ijn} \le d'_{ijn}$ with $\|d_{ijn}{}'\| < \frac{1}{6}$. Using Proposition 2.10 we obtain an element $b_n \in G^+$ such that $2^{n+1}b_n \le u + (d + \overline{d} + d'_{11n} + d'_{12n} + d'_{21n} + d'_{22n})$, where $\overline{d} \in G^+$ with $\|\overline{d}\| < \frac{1}{6}$, and $b_n \ge q_{ij}$ $\forall i, j$. Then $\|b_n\| < \frac{1}{2^n}$ and $x_{in} + d_{ijn} = p_{ij} + q_{ij} \le y_{jn} + b_n + d_{ijn}$, i.e., $x_{in} \le y_{jn} + b_n$ for all i, j.

Finally, we construct elements $z_1, z_2 \cdots \in G$ such that $x_{in} \le z_n \le y_{jn} + b_n + d_n$ $\forall i, j, n$, where $d_n \in G^+$, $\|d_n\| < \frac{1}{2^n}$, while also $\|z_{n+1} - z_n\| < \frac{3}{2^n}$ $\forall n$. As $x_{i1} \le y_{j1} + b_1$ $\forall i, j$, by asymptotic interpolation there exist elements $z_1 \in G$, $d_1 \in G^+$ with $\|d_1\| < \frac{1}{2}$, such that $x_{i1} \le z_1 \le y_{j1} + b_1 + d_1$. Now suppose that $z_1, d_1, \ldots, z_n, d_n$ have been constructed for some n. Then, $x_{i,n+1} \le y_{j,n+1} + b_{n+1}$, $x_{i,n+1} \le x_{in} + a_n \le z_n + a_n$ and $z_n - (b_n + d_n) - a_n \le y_{jn} - a_n \le y_{j,n+1} \le y_{j,n+1} + b_{n+1}$ $\forall i, j$. Hence $\exists z_{n+1} \in G$, $\exists d_{n+1} \in G^+$ with $\|d_{n+1}\| < \frac{1}{2^{n+1}}$ such that

$$\begin{matrix} x_{1,n+1} & & & & y_{1,n+1} + (b_{n+1} + d_{n+1}) \\ x_{2,n+1} & \le & z_{n+1} & \le & y_{2,n+1} + (b_{n+1} + d_{n+1}) \\ z_n - (b_n + d_n) - a_n & & & & z_n + (a_n + d_{n+1}) \end{matrix}.$$

Since $-(a_n + b_n + d_n) \le z_{n+1} - z_n \le a_n + d_{n+1}$, we conclude that

$$\|z_{n+1} - z_n\| \le \max\{\|a_n + b_n + d_n\|, \|a_n + d_{n+1}\|\} < \frac{3}{2^n}.$$

This completes the induction step.

The z_n form a Cauchy sequence in G, and hence there exists $h \in \overline{G}$ such that $\phi(z_n) \to h$. Note also that, if we define $e_n =: b_n + d_n$, then $\phi(e_n) \to 0$. Since $\phi(x_{in}) \le \phi(z_n) \le \phi(y_{jn}) + \phi(e_n)$ $\forall i, j, n$, we conclude that $f_i \le h \le g_j$ $\forall i, j$. Therefore $\overline{G}$ has interpolation, as desired. ∎

In particular, Theorem 3.4 gives an affirmative (partial) answer to [17, Open Problem 8], in the case that the group G satisfies the asymptotic refinement property. Now, using [17, Corollary 13.6], we can compute the closure of $\Phi(G)$ in $\mathrm{Aff}(S(G, u))$.

Theorem 3.5. *Let (G, u) be an asymptotic refinement group. Let Φ denote the natural representation map of G, and consider*

$$A = \{p \in \mathrm{Aff}\,(S(G, u)) \mid p(s) \in s(G) \textit{ for every discrete state } s \in \partial_e S(G, u)\}.$$

Then, $\Phi(G^+)$ is dense in A^+, and $\Phi(G)$ is dense in A.

Proof. Let $\overline{G}$ be the completion of G, let $\phi : G \to \overline{G}$ be the natural map, and let $S = S(G, u)$, $\overline{S} = S(\overline{G}, \overline{u})$. If $\Phi : G \to \mathrm{Aff}\,(S)$ and $\overline{\Phi} : \overline{G} \to \mathrm{Aff}\,(\overline{S})$ denote the natural representation maps of G and $\overline{G}$ respectively, and $\overline{A}$ denotes the analog of the set A corresponding to the group $\overline{G}$, then $\Phi(G^+) \subseteq A^+$ and $\overline{\Phi}(\overline{G}^+) \subseteq \overline{A}^+$. In fact, $\Phi(\overline{G}^+) = \overline{A}^+$ by [17, Theorem 15.7], because $\overline{G}$ is an archimedean norm-complete dimension group by Theorem 3.4.

Now, given the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Aff}\,(S) & \xrightarrow{\mathrm{Aff}(S(\phi))} & \mathrm{Aff}\,(\overline{S}) \\ \Phi\uparrow & & \uparrow\overline{\Phi} \\ G & \xrightarrow{\phi} & \overline{G}, \end{array}$$

observe that $\mathrm{Aff}(S(\phi))$ is an isometric isomorphism of partially ordered Banach spaces with order-unit. Notice that, by definition of the morphism $S(\phi) : S(\overline{G}, \overline{u}) \to S(G, u)$ (explicitly, for all $\overline{s} \in S(\overline{G}, \overline{u})$ we have $S(\phi)(\overline{s}) = \overline{s}\phi$), $\overline{s}$ discrete implies $S(\phi)(\overline{s})$ discrete. Conversely, if $s = S(\phi)(\overline{s})$, then $\overline{s}(\lim_{n\to\infty} \phi(x_n)) = \lim_{n\to\infty} s(x_n)$ for any Cauchy sequence $(x_n)_{n\geq 1}$. Hence, if $s(G)$ is discrete, then by uniform continuity, $\overline{s}(\overline{G}) = s(G)$; thus, $S(\phi)$ preserves discrete states and their ranges, whence $\mathrm{Aff}\,(S(\phi))(A^+) = \overline{A}^+$. As a result, $\overline{A}^+ = \overline{\Phi}(\overline{G}^+) = \mathrm{Aff}(S(\phi))(A^+)$. Since $\phi(G^+)$ is dense in $\overline{G}^+$, then $\Phi(G^+) = \mathrm{Aff}(S(\phi))^{-1} \circ \overline{\Phi} \circ \phi(G^+)$ is dense in $\mathrm{Aff}(S(\phi))^{-1} \circ \overline{\Phi}(\overline{G}^+) = A^+$. ∎

In particular we give an affirmative answer to [17, Open Problem 1]. Also, an argument similar to that in the proof of Theorem 3.4 allows us to adapt the proof of [17, Theorem 12.7] to obtain a description of what kinds of properties are satisfied by the completion of an asymptotic refinement group with respect to the norm associated to any state on the group. We enunciate the result without proof.

Theorem 3.6. *Let (G, u) be an asymptotic refinement group. Let $s \in S(G, u)$ be any state on G, and set $\overline{G}$ equal to the s-metric completion of G. Then, $\overline{G}$ is a Dedekind-complete lattice-ordered abelian group.*

4. A description of K_0 for exchange rings

This section is mainly devoted to obtain a description of the closure of the natural affine continuous function representation of $K_0(R)$ for any exchange ring R. This work was done for the concrete case of C^*-algebras of real rank zero and stable rank one with K_0 unperforated by Blackadar and Handelman in [9], and for any regular ring by Pardo in [29]. The line of the proof for the second one is, essentially, the use of the results and techniques of [22] on the N^*-metric completion of regular rings. Nevertheless, as we noted in the Introduction, the results of [22] cannot be applied directly in the general case of exchange rings, because K_0 usually fails to be a dimension group, and also because there does not exist an analog of the metric completion of regular rings in the general case of exchange rings. Namely, in the regular case, Moncasi, in [26], gave an example of a stably finite regular ring R of stable rank 2 such that $K_0(R)$ does not satisfy the interpolation property. In fact, for each $n \in \mathbb{N}$, he constructed such an example whose K_0 is isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$, whence $K_0(R)$ also fails to be unperforated. Lately, Goodearl, in [20], inspired by Moncasi, obtained an analogous example in the case of C^*-algebras of real rank zero.

So, we will deal with asymptotic refinement groups, in order to obtain a background framework that allows us to describe K_0 in this case. The key point is the fact that $V(R)$ – the monoid of isomorphism classes of finitely generated projective right R-modules – for an exchange ring R is a *refinement monoid*. The particular case of regular rings was proved by Goodearl and Handelman ([21, Lemma 3.8]), and the case of C^*-algebras of real rank zero by Zhang ([40]). The exchange case (which covers these cases) was proved by Ara, Goodearl, O'Meara and Pardo ([3, Proposition 1.1]), by doing a restatement in monoid language of the exchange property for modules, introduced by Crawley and Jónsson in [14]. So, Proposition 2.3 can be applied in this context, and thus we obtain the following result.

Lemma 4.1. *If R is an exchange ring, then $(K_0(R), [R])$ is an asymptotic refinement group.*

Notice that this result applies, in fact, to any ring R whose monoid $V(R)$ turns out to be a refinement monoid. Nevertheless, as it is observed in [3], in the case of exchange rings there is a faithful connexion between the properties of such a ring and those of its associated monoid. Now, by using the results of the last section, we can obtain a description of K_0 for exchange rings. Namely, we have the following result.

Theorem 4.2. *Let R be an exchange ring, let $K_0(R)$ be its Grothendieck group, let $S := S(K_0(R), [R])$ be the state space on $K_0(R)$, and let $\Phi : K_0(R) \to \mathrm{Aff}(S)$, the natural map. For each $s \in S$, set $B_s = s(K_0(R))$ if s is discrete, and $B_s = \mathbb{R}$ otherwise. Set $A = \{p \in \mathrm{Aff}(S) \mid p(s) \in B_s \,\forall s \in \partial_e S\}$. Then, $\Phi(K_0(R)^+)$ is dense in A^+, and $\Phi(K_0(R))$ is dense in A.*

Proof. The result is a direct consequence of Lemma 4.1 and Theorem 3.5. ■

Theorem 4.2 allows us to characterize when an exchange ring satisfies condition (D) of [2]: $\Phi(K_0(R)^+)$ is dense in $\mathrm{Aff}(S(K_0(R), [R]))^+$.

Corollary 4.3. *An exchange ring R satisfies condition (D) if and only if the space of states on $K_0(R)$ contains no discrete extremal states.*

In fact, Corollary 4.3 can be stated in terms of algebraic properties of exchange rings. This was shown by the author in the particular case of regular rings ([29, Corollary 2.5]), and also it is easy to show this result in the case of unital C^*-algebras of real rank zero, by using some results from [9, Section III]. In both cases, we use the good relationship that exists between states on the Grothendieck group and pseudo-rank functions in the regular case (see [16, Chapter 17]), or with lower semi-continuous dimension functions in the case of C^*-algebras of real rank zero (see [9]). Unfortunately, in the general case of exchange rings, it is not known if such a good relation exists, so this technique cannot be applied. Nevertheless, when we were finishing this paper, Ken Goodearl communicated to us that it is possible to avoid this difficulty. We thank Ken Goodearl, who allows us to include his result here.

In order to state this result, recall that, if R is an exchange ring, $L(R)$ is the lattice of two-sided ideals of R, and $L(V(R))$ is the lattice of order-ideals – convex submonoids – of $V(R)$, then by [3, Proposition 1.4], the map $\phi : L(R) \to L(V(R))$ given by $\phi(I) = V(I)$ is a surjective lattice morphism. In fact, for any $S \in L(V(R))$, the fibre ϕ^{-1} equals the interval $[I_0(S), I_1(S)]$, where $I_0(S)$ is the ideal generated by $\{e = e^2 \in R \mid \langle eR\rangle \in S\}$, and $I_1(S)$ is the ideal of R containing $I_0(S)$ such that $I_1(S)/I_0(S) = J(R/I_0(S))$ (here $J(-)$ denotes the Jacobson radical). Also, notice that $S(K_0(R), [R]) \cong S(V(R), \langle R\rangle)$, whence we can identify the states of the group and those of the monoid. Moreover, if $s \in S(V(R), \langle R\rangle)$, then $\ker s \in L(V(R))$.

Lemma 4.4. *Let R be an exchange ring, and let $s \in S(V(R), \langle R\rangle)$. Then s is a discrete state (on the Grothendieck group of R) if and only if s is a rational convex* *combination of discrete extremal states and* $R/I_1(\ker s)$ *is a semisimple artinian ring.*

Proof. Necessity is clear. To see sufficiency, suppose that s is a discrete state on $K_0(R)$. Then, s is a rational convex combination of extremal discrete states, because of Corollary 3.3, Theorem 3.4 and [17, Proposition 6.22]. Now, if we look at s as a state on $V(R)$, we have that $\ker s \in L(V(R))$. Set $K = I_1(\ker s)$, and observe that $K \neq R$, as otherwise $s(\langle R\rangle) = 0$, which is impossible. We claim that $\overline{R} := R/K$ is semisimple artinian.

First notice that $J(R/K) = 0$, whence $\overline{R}$ is semiprimitive. Moreover, by [3, Proposition 1.4], $V(\overline{R}) \cong V(R)/\ker s$. Thus, the state $\overline{s} \in S(V(\overline{R}), \langle\overline{R}\rangle)$, given by $\overline{s}(\langle\overline{e}\overline{R}\rangle) = s(\langle eR\rangle)$ for any idempotent $e \in R$, is a discrete state on $K_0(\overline{R})$ such that $\ker \overline{s} = 0$ in $V(\overline{R})$. So, without lose of generality, we can assume that R is semiprimitive and $\ker s = 0$. Notice that, according to [28, Proposition 2.1], any idempotent e in an exchange ring R is primitive if and only if e is local. As $eJ(R)e = J(eRe)$, if R is semiprimitive and $e^2 = e \in R$ is local, then eRe is a division ring, whence eR is a minimal right ideal of R. Hence, to prove the claim it suffices to show that there exists a finite sequence of nonzero orthogonal primitive idempotents $e_1, \dots, e_n$ such that $e_1 + \cdots + e_n = 1$.

Observe that, as s is a discrete state, there exists an idempotent $e_1 \in R$ such that $s(\langle e_1R\rangle)$ takes the minimum nonzero value in the range of s. Suppose that there exist orthogonal idempotents $f, g \in R$ such that $e_1 = f + g$. Since $s(\langle e_1R\rangle) = s(\langle fR\rangle) + s(\langle gR\rangle)$, we have either $s(\langle fR\rangle) = 0$ or $s(\langle gR\rangle) = 0$; that is, $f = 0$ or $g = 0$. Thus, $e_1 \in R$ is a primitive idempotent.

Now suppose that $e_1 \neq 1$, and take $\widetilde{R} := (1 - e_1)R(1 - e_1)$. $\widetilde{R}$ is an exchange ring, and if f is an idempotent in $\widetilde{R}$, then the map $\widetilde{s} : V(\widetilde{R}) \to \mathbb{R}^+$ given by $\widetilde{s}(\langle f\widetilde{R}\rangle) = s(\langle fR\rangle)/s(\langle(1 - e_1)R\rangle)$ is a discrete state on $V(\widetilde{R})$ with $\ker \widetilde{s} = 0$. The same argument as above shows that there exists an idempotent $e_2 \in \widetilde{R}$ such that $\widetilde{s}(\langle e_2\widetilde{R}\rangle)$ takes the minimum nonzero value in the range of $\widetilde{s}$. Hence, e_2 is a primitive idempotent in $\widetilde{R}$, and so in R. Also, $s(\langle e_1R\rangle) \leq s(\langle e_2R\rangle)$. By recurrence, we can construct primitive orthogonal idempotents $e_1, \dots, e_k$ such that $s(\langle e_1R\rangle) \leq s(\langle e_iR\rangle)$ for all $i \leq k$. Then, as

$$1 = s(\langle R\rangle) \geq s(\langle(\sum_{i=1}^{k} e_i)R\rangle) = \sum_{i=1}^{k} s(\langle e_iR\rangle) \geq ks(\langle e_1R\rangle) > 0,$$

this process must terminate in a finite number of steps, and so there exist primitive orthogonal idempotents $e_1, \dots, e_n$ such that

$$1 = s(\langle R\rangle) = s(\langle(\sum_{i=1}^{n} e_i)R\rangle) = \sum_{i=1}^{n} s(\langle e_iR\rangle).$$

Thus, $s(\langle(1 - \sum_{i=1}^{n} e_i)R\rangle) = 0$, whence $e_1 + \cdots + e_n = 1$, which ends the proof. ∎

As a consequence we obtain

Corollary 4.5. *If R is an exchange ring, then R has no nontrivial artinian homomorphic images if and only if $S(K_0(R), [R])$ has no discrete states.*

Thus, by joining Corollary 4.3 and Corollary 4.5, we obtain the following result.

Corollary 4.6. *Let R be an exchange ring. Then R satisfies condition (D) if and only if R has no nontrivial artinian homomorphic images.*

In the particular case of any non-artinian simple exchange ring R, we have that $V(R)$ – and so $K_0(R)$ – is simple (that is, is a nonzero monoid such that every nonzero element is an order-unit) by [21, Proposition 1.2], and atomless because of simplicity and non-artinianity (where an atom of a monoid M is a nonzero element $a \in M$ such that there cannot exist any other element b satisfying $0 < b < a$). Thus $K_0(R)$ has no discrete states, and so Corollary 4.6 has the following consequence.

Corollary 4.7. *If R is a non-artinian simple exchange ring, then it satisfies condition (D).*

Corollary 4.7, together with Corollary 2.6 and [2, Lemma 2.3], implies that

Given any stably finite, non-artinian, simple exchange ring R and any nonzero finitely generated projective right R-module A, $n \in \mathbb{N}$, $\varepsilon > 0$, there exist finitely generated projective right R-modules B, C such that $nC \lesssim A \lesssim nB$, while

$$\|n[B] - [A]\|_{[R]} < \varepsilon \quad \textit{and} \quad \|n[C] - [A]\|_{[R]} < \varepsilon.$$

So, we obtain a general version of [2, Proposition 2.4]. This result presents some similarities with [41, Theorem I(i)], where Zhang shows that nonelementary simple C^*-algebras of real rank zero have approximate halving projections. In the last section of this paper we will give a result that generalizes those of Zhang ([41]) and Ara and Goodearl ([2]), in a more general context.

On the other side, Corollary 4.7 and [17, Proposition 14.15] imply that the group $(\Phi(K_0(R)), \underline{\ll})$ is a simple dimension group. This means that, in the case of simple groups, there exists a close connexion between unperforation and interpolation properties (this is false in general, as we can see in [17, Example 2.7]).

Our next result shows that, if (G, u) is strictly unperforated (that is, for any $x \in G$, if $nx \geq 0$ and $nx \neq 0$ for some $n \in \mathbb{N}$, then $x \geq 0$), then G is an interpolation group under mild (but unavoidable) hypothesis.

Proposition 4.8 ([29, Proposition 2.7])**.** *Let (G, u) be a simple, strictly unperforated, pre-ordered abelian group with order-unit such that $S(G, u) \neq \emptyset$, let $\Phi : G \to \mathrm{Aff}(S(G, u))$ be the natural homomorphism, and suppose that G satisfies condition (D). Assume also that $S(G, u)$ is a Choquet simplex. Then G is an interpolation group.*

In the case of abstract pre-ordered groups, the author, in [30], applied Proposition 4.8 to show that, given a torsionfree simple Riesz group (G, u), if we add to its positive cone all the elements $x \in G$ such that $nx \in G^+$ for some $n \in \mathbb{N}$, then G with this new cone is a simple dimension group. In particular, this result gives an affirmative answer to [17, Open Problem 2] in the simple case, and also allows us to obtain a version of the theorem of representation of dimension groups ([15]; see [17, Theorem 3.19]) in the case of torsionfree simple Riesz groups. In the case of simple exchange rings, Proposition 4.8 applies in the following form:

Corollary 4.9. *Let R a stably finite, non-artinian, simple exchange ring such that $K_0(R)$ is strictly unperforated. Then, $K_0(R)$ is a simple Riesz group.*

Proof. By hypothesis, $(K_0(R), [R])$ is a partially ordered abelian group with order-unit, simple and strictly unperforated. By Corollary 4.7, $K_0(R)$ satisfies condition (D), and since by Theorem 3.4 the norm-completion is a dimension group with affinely homeomorphic state space, $S(K_0(R),[R])$ is a nonempty Choquet simplex. Thus the result holds because of Proposition 4.8. ■

A particular case of this situation appears with the following construction: Given any field F, set $T_F = \varinjlim M_n(F)$, where the ordering in $\mathbb{N}$ is given by "$m \leq n$ when m divides n". Thus, if A is a F-algebra, then $K_0(A \otimes T_F) \cong K_0(A) \otimes \mathbb{Q}$. In particular, if $K_0(A)$ is simple, so is $K_0(A \otimes T_F)$; but the latter is also unperforated. Thus we have the following result:

Corollary 4.10. *Let R be a simple ring, let $F = Z(R)$ be its center, and let $S = R \otimes T_F$. Suppose that $S(K_0(R),[R]) \neq \emptyset$. Then:*

(a) If R satisfies condition (D), then $K_0(S)$ is a dimension group if and only if $S(K_0(R),[R])$ is a Choquet simplex.

(b) If R is an exchange ring, then S is an exchange ring, and $K_0(S)$ is a simple dimension group.

Proof. (a) Necessity is clear by [17, Theorem 10.17]. To prove sufficiency, simply notice that, as $S(K_0(R),[R]) \cong S(K_0(S),[S])$, $K_0(S)$ is a simple unperforated group satisfying condition (D) with $S(K_0(S),[S])$ being a Choquet simplex, so that Proposition 4.8 applies in this case.

(b) Since $S = R \otimes T_F = \varinjlim M_n(R)$ and R is an exchange ring, so are $M_n(R)$ for all $n \in \mathbb{N}$, and thus the direct limit is an exchange ring. As S is non-artinian, it satisfies condition (D) by Corollary 4.7, so that the result holds by part (a). ■

To end this section, we will show that elements of a refinement monoid which are comparable by the order into the space of affine and continuous functions on the space of states of its Grothendieck group are "almost" comparables. First we will state a result analogous to Proposition 2.8 for Grothendieck groups of refinement monoids.

Proposition 4.11. *Let (M,u) be a refinement monoid with order-unit, and $(G,[u])$ its Grothendieck group. Let $x,y,z \in M$ and $n \in \mathbb{N}$. If $2^n[x] \leq 2^n[y]+[z]$, then for all $\varepsilon > 0$ there exist $v,w,d \in M$ such that $\|[d]\| < \varepsilon$, $x = v+w$, $v \leq y$ and $2^n w \leq z+d$.*

Proof. Suppose that we have $x,y,z \in M$ and $n \in \mathbb{N}$ such that $2^n[x] \leq 2^n[y]+[z]$. Since $M \to G^+$ is a monoid epimorphism, by Proposition 2.8 there exist $r,s,d_1,d_2 \in M$ with $[d_1] \leq [d_2]$, $\|d_2\| < \frac{\varepsilon}{5\cdot 2^n}$ such that $[x]+[d_1] = [r]+[s]$, $[r] \leq [y]+[d_1]$ and $2^n[s] \leq [z]+[d_2]$. Now, by definition of G there exist $d_3,d_4,d_5 \in M$ such that $x+d_1+d_3 = r+s+d_3$, $r+d_4 \leq y+d_1+d_4$ and $2^n s+d_5 \leq z+d_2+d_5$. So, by [37, Lemma 1.11], we can assume that $x \leq r+s+d_3$, $r \leq y+d_1+d_4$ and $2^n s \leq z+d_2+d_5$ with $\|d_i\| < \frac{\varepsilon}{5\cdot 2^n}$ for $i = 3,4,5$. Thus, as $r \leq y+(d_1+d_4)$, by refinement there exist $r_1,r_2 \in M$ such that $r = r_1+r_2$, $r_1 \leq y$ and $r_2 \leq d_1+d_4$. Now,

$$x \leq r+s+d_3 = r_1+(r_2+s+d_3),$$

and hence there exist $x_1,x_2 \in M$ such that

$$x = x_1+x_2, \qquad x_1 \leq r_1 \leq y$$

and

$$2^n x_2 \leq 2^n r_2 + 2^n s + 2^n d_3$$

$$\leq z + (d_2 + d_5 + 2^n d_1 + 2^n d_4 + 2^n d_3).$$

Then, taking $v = x_1$, $w = x_2$ and $d = d_2 + d_5 + 2^n(d_1 + d_3 + d_4)$, we obtain the desired result. ■

Theorem 4.12. *Let (M, u) be a refinement monoid with order-unit, let $(G(M), [u])$ be its Grothendieck group, and let $\Phi : G(M) \to \mathrm{Aff}(S(G(M), [u]))$ be the natural map. If $x, y \in M$ and $\Phi(x) \leq \Phi(y)$, then $\forall \varepsilon > 0$ there exist decompositions $x = x_1 + x_2$, $y = x_1 + y_2$ such that $\|[x_2]\|_{[u]} < \varepsilon$.*

Proof. By Proposition 2.3, $(G(M), [u])$ is an asymptotic refinement group. As $\Phi(x) \leq \Phi(y)$, we have that $\Phi([y] - [x]) \geq 0$, that is,

$$\Phi([y] - [x]) \in \mathrm{Aff}(S(G(M), [u]))^+.$$

Thus, according to Lemma 3.1, for any $n \in \mathbb{N}$ there exists an element $t_n \in M$ such that

$$\|\Phi([x + t_n]) - \Phi([y])\|_\infty = \|\Phi([t_n]) - \Phi([y] - [x])\|_\infty < \frac{1}{2^n},$$

that is,

$$\Phi([y]) - \frac{1}{2^n} \ll \Phi([x + t_n]) \ll \Phi([y]) + \frac{1}{2^n}.$$

In particular, $\forall n \geq 1$,

$$\Phi(2^n[x]) \leq \Phi(2^n[x + t_n]) \ll \Phi(2^n[y] + [u]).$$

Now, by [17, Theorem 4.12], for each $n \geq 1$ there exists $m \in \mathbb{N}$ such that $m2^n[x] < m(2^n[y] + [u])$, and using Corollary 2.6 we have, given $\delta > 0$, an element $d_1 \in M$ with $\|[d_1]\|_{[u]} < \delta$ such that $2^n[x] \leq 2^n[y] + ([u + d_1])$. By Proposition 4.11 there exist $x_1, x_2, d_2 \in M$ with $\|[d_2]\|_{[u]} < \delta$ such that $x = x_1 + x_2$, $x_1 \leq y$ and $2^n x_2 \leq u + d_1 + d_2$, whence $\|[x_2]\|_{[u]} < \frac{1+2\delta}{2^n}$. Taking $\delta < \frac{1}{2}$, for all $n > 1 + \log_2(\frac{1}{\varepsilon})$ we have the desired decomposition. ■

As a consequence of Theorem 4.12, we obtain the following result for exchange rings.

Proposition 4.13. *Let R be an exchange ring, and let P, Q be finitely generated projective right R-modules. Let $\Phi : K_0(R) \to \mathrm{Aff}(S(K_0(R), [R]))$ be the natural map. If $\Phi([P]) \leq \Phi([Q])$, then for all $\varepsilon > 0$ there exist decompositions $P \cong P_1 \oplus P_2$, $Q \cong Q_1 \oplus Q_2$ such that $P_1 \cong Q_1$ and $\|[P_2]\|_{[R]} < \varepsilon$.*

Proof. As R is an exchange ring, we have that $V(R)$ is a refinement monoid. Also, $K_0(R) = G(V(R))$, whence the result is a consequence of Theorem 4.12. ■

Proposition 4.13, in the case of regular rings, gives [4, Lemma 3.2], which turns out to be the key for characterizing when K-theoretically simple regular rings are simple. In the case of unital C^*-algebras of real rank zero, Proposition 4.13 can be rewritten in the following form:

Corollary 4.14. *Let A be a unital C^*-algebra of real rank zero, and let $p, q \in A$ be projections. If $\tau(p) \leq \tau(q)$ for every quasi-trace $\tau \in \mathrm{QT}(A)$, then $\forall \varepsilon > 0$ there exist projections $p_1, p_2, q_1, q_2 \in A$ such that $p \sim p_1 \oplus p_2$, $q \sim q_1 \oplus q_2$, while $p_1 \sim q_1$ and $\sup\{\tau(p_2) \mid \tau \in \mathrm{QT}(A)\} < \varepsilon$.*

Proof. Given the affine homeomorphism $\mathrm{QT}(A) \cong S(K_0(A), [A])$ ([9, Section III]), we see that $\tau(p) \leq \tau(q)$ for every quasi-trace $\tau \in \mathrm{QT}(A)$ if and only if $\Phi([p]) \leq \Phi([q])$. Thus, the result follows from Proposition 4.13. ■

Corollary 4.14 could be viewed, in the simple case, as an approximation to an affirmative answer to the Fundamental Comparability Question 2, proposed by Blackadar in [8]. This question has an affirmative answer in the case of simple unital AF C^*-algebras ([7]), and in the case of irrational rotation algebras ([31], [32]), among others. Recently, Villadsen ([35]) constructed a counterexample to this question. This example, however, has real rank one, and thus remains out of the domain of application of Corollary 4.14.

5. Halving idempotents

In this section we will deal with the following problem: *can a non-artinian simple ring be approached by matrix rings of every dimension?* The key point is to show that every idempotent can be approximately "cut" into n equivalent idempotents for an arbitrary $n \in \mathbb{N}$. To be more explicit, given an idempotent $e \in R$ and $n \in \mathbb{N}$, we will show that there exist idempotents $f, g \in R$ such that

$$eR \cong n(fR) \oplus gR,$$

where g is "small" in some sense. The remark after Corollary 4.7 shows that this result holds for non-artinian simple exchange rings wherever we take as a measure of gR the value of $\|[gR]\|_{[R]}$. Zhang, in [41], shows that the result holds for nonelementary simple C^*-algebras of real rank zero wherever $n = 2^k$ for arbitrary $k \in \mathbb{N}$, and the measure of gR is fixed by the fact that it turns out to be a direct summand of both fR and an arbitrary nonzero principal right ideal generated by an idempotent.

Obviously, this kind of behavior fails in general. An easy example of this is the Weyl algebra $K\langle x, y \mid xy - yx = 1\rangle$, which is a noetherian, non-artinian simple *domain*. We will show, by using monoid-theoretic techniques, that the result holds for any ring R (not necessarily unital) whose monoid $V(R)$ is simple, atomless, and satisfies the Riesz decomposition property (that is, whenever $x, y_1, y_2 \in V(R)$ are such that $x \leq y_1 + y_2$, there exist $x_1, x_2 \in M$ such that $x = x_1 + x_2$, while $x_i \leq y_i$ for all i). Notice that refinement implies Riesz decomposition, but the converse is false (e.g., $M = \langle a, \infty \mid a + a = \infty\rangle$ is a Riesz monoid, but not a refinement monoid). Thus, the result holds for a wide class of simple rings that includes the non-artinian simple exchange rings. In order to state the following result, recall that an abelian monoid M is *conical* provided that $\forall x, y \in M$, $x + y = 0$ implies $x = y = 0$.

Lemma 5.1. *Let M be a simple, conical, atomless Riesz monoid. Then:*

(a) Given nonzero elements $x_1, \ldots, x_k \in M$ and $n \in \mathbb{N}$, there exists a nonzero element $y \in M$ such that $ny < x_i$ for all i.

(b) Given nonzero elements $p, r \in M$, $m \in \mathbb{N}$, there exist $q, s \in M$ such that $p = mq + s$ and $s \leq (m-1)r$.

Proof. (a) We will show this result in two steps:

Claim 1. *Given nonzero elements $x_1, \dots, x_k \in M$, there exists a nonzero element $y \in M$ such that $y \leq x_i$ for all i.* To see this, it is enough to consider the case $k = 2$. Let $x_1, x_2 \in M$ be any nonzero elements. As M is simple, there exists $m \in \mathbb{N}$ such that $x_1 \leq m x_2$. By Riesz decomposition there exist $a_1, \dots, a_m \in M$ such that $x_1 = a_1 + \cdots + a_m$ and $a_i \leq x_2$ for all i (and obviously $a_i \leq x_1$ for all i). Since $x_1 \neq 0$, there exists at least one $j \in \{1, \dots, m\}$ such that $a_j \neq 0$. Thus, take $y = a_j$.

Claim 2. *Given a nonzero element $x \in M$, and $n \in \mathbb{N}$, there exists a nonzero element $y \in M$ such that $ny < x$.* To see this, observe that, as M is atomless, there exists a strictly decreasing chain $x = x_0 > x_1 > \cdots > x_n > 0$ of nonzero elements of M. Let $t_1, \dots, t_n \in M$ be nonzero elements such that $x_i + t_i = x_{i-1}$ for $i \in \{1, \dots, n\}$. Then, by claim 1 there exists a nonzero element $y \in M$ such that $y \leq t_i$ for all i. Now,

$$ny \leq t_1 + \cdots + t_n < t_1 + \cdots + t_n + x_n$$

$$= t_1 + \cdots + t_{n-1} + x_{n-1} = \cdots = t_1 + x_1 = x_0 = x.$$

(b) Since M is simple, there exists $k \in \mathbb{N}$ such that $p \leq kr$. We will do the proof by induction on k. For $k \leq (m-1)$, we have $p = m0 + p$, and obviously $p \leq (m-1)r$. For $k = m$ we have, by [25, Lemma 3.1], a decomposition $p = p_1 + \cdots + p_m$ with $p_1 \leq \cdots \leq p_m \leq r$. Thus there exist $q_1, \dots, q_{m-1} \in M$ such that $p_1 + q_i = p_{i+1}$ for $i = 1, \dots, m-1$. Take $q = p_1$ and $s = q_1 + \cdots + q_{m-1}$. Then $p = mq + s$ with $s = q_1 + \cdots + q_{m-1} \leq p_2 + \cdots + p_m \leq (m-1)r$.

Now, assume that the result holds for k, and suppose that $p \leq (k+1)r$. First, notice that $p \leq (k+1)r = kr + r$. So, applying the Riesz property, we obtain a decomposition $p = p_1 + p_2$ with $p_1 \leq kr$ and $p_2 \leq r$. Then, applying induction, we have that $p_1 = mq_1 + s_1$ with $s_1 \leq (m-1)r$. Hence, $p_2 + s_1 \leq mr$. Again by induction $p_2 + s_1 = mq_2 + s_2$, with $s_2 \leq (m-1)r$. Thus, take $q = q_1 + q_2$ and $s = s_2$, and notice that

$$p = p_1 + p_2 = mq_1 + s_1 + p_2 = m(q_1 + q_2) + s_2 = mq + s$$

with $s \leq (m-1)r$. Thus, the induction step works. ∎

Theorem 5.2. *Let M be a simple, atomless, conical Riesz monoid. Let $p, r \in M$ be nonzero elements, and let $m \in \mathbb{N}$. Then, $p = mp_m + s$ for some $p_m, s \in M$ such that $s \leq p_m$ and $s \leq r$.*

Proof. First, we will see that, in order to show the desired result, it is enough to prove that there exist $q, t \in M$ such that $p = mq + t$ with $t \leq r$. Now, suppose that the claim holds. Then, given $p, r \in M$ (nonzero elements) and $m \in \mathbb{N}$ we have $q', t \in M$ such that $p = mq' + t$ and $t \leq r$. Now apply the same to $p = t$ and $r = q'$. Thus there exist $l, s \in M$ such that $t = ml + s$, with $s \leq q'$, and also $s \leq t \leq r$ by construction. Then, taking $q = q' + l$, we have $p = mq' + t = m(q' + l) + s = mq + s$ with $s \leq \{q, r\}$.

Now we will show that the claim of the last paragraph is true. To do this, take $p, r \in M$ (nonzero elements) and $m \in \mathbb{N}$. By Lemma 5.1(a) there exists a nonzero element $r' \in M$ such that $(m-1)r' < r$. Now apply Lemma 5.1(b) to p and r'. Then there exist $q, s \in M$ such that $p = mq + s$ with $s \leq (m-1)r' < r$. Thus, the claim of the first paragraph is true, and so the result holds. ∎

As a consequence, we obtain the following result:

Proposition 5.3. *Let R be a ring (not necessarily unital) such that $V(R)$ is simple, atomless and Riesz. Let A, B be nonzero finitely generated projective right R-modules, and $n \in \mathbb{N}$. Then there exist finitely generated projective right R-modules C, D, E, F such that $A \cong nC \oplus D$, $D \oplus E \cong C$, $D \oplus F \cong B$.*

Corollary 5.4. *Let R be a unital ring such that $V(R)$ is simple, atomless and Riesz. Then, for all $n \in \mathbb{N}$ there exists a finitely generated projective right R-module C such that*

$$M_n(\mathrm{End}_R(C)) \subseteq R \subseteq M_{n+1}(\mathrm{End}_R(C)),$$

where the inclusions are (non-unital) ring embeddings.

Note that any non-artinian, simple exchange ring satisfies the hypothesis of Proposition 5.3 and Corollary 5.4. Thus, these results apply for non-artinian simple exchange rings, and so [41, Theorem I(i)] is true for arbitrary $n \in \mathbb{N}$, and also the analogous result for regular rings, which generalizes [2, Corollary 2.8].

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References

1. P.Ara, *Strongly π-regular rings have stable range one*, Proc.AMS **124** (1996), pp. 3293–3298. MR **97a:**16024
2. P.Ara, K.R.Goodearl, *The almost isomorphism relation for simple regular rings*, Publ. Mat. Univ. Autònoma Barcelona **36(2)** (1992), pp. 369–388. MR **94f:**16024
3. P.Ara, K.R.Goodearl, K.C.O'Meara, E.Pardo, *Separative cancellation for projective modules over exchange rings*, Israel J. Math. (to appear).
4. P.Ara, K.R.Goodearl, E.Pardo, D.V.Tyukavkin, *K-theoretically simple von Neumann regular rings*, J.Algebra, **174** (1995), pp.659–677. MR **96g:**16012
5. P.Ara, E.Pardo, *Refinement monoids with weak comparability and applications to regular rings and C^*-algebras*, Proc.AMS **124** (1996), pp. 715-720. MR **96f:**46124
6. B.Blackadar, *"K-Theory for Operator Algebras"*, M.S.R.I.Publ.5, Springer Verlag, New York, 1986. MR **88g:**46082
7. B.Blackadar, *Traces on simple AF C^*-algebras*, J.Func.Anal., **38** (1980), pp.156–168. MR **82a:**46062
8. B.Blackadar, *Comparison theory for simple C^*-algebras*, "Operator algebras and application", D.E. Evans and M. Takesaki (eds.), LMS Lecture Notes Series **135**, Cambridge Univ. Press, 1988, pp. 21–54. MR **90g:**46078
9. B.Blackadar, D.E.Handelman, *Dimension functions and traces on C^*-algebras*, J.Func.Anal., **45** (1982), pp.297–340. MR **83g:**46050
10. W.D.Burgess, D.E.Handelman, *The N^*-metric completion of regular rings,* Math.Ann, **261** (1982), pp. 235–254. MR **84a:**16023
11. L.G.Brown, G.K.Pedersen, *C^*-algebras of real rank zero*, J.Func.Anal., **99** (1991), pp. 131–149. MR **92m:**46086
12. V.P.Camillo, H.-P.Yu, *Stable range one for rings with many idempotents*, Proc.AMS **347** (1995), 3141–3147. MR **95j:**16006
13. V.P.Camillo, H.-P.Yu, *Exchange rings, units and idempotents*, Comm.Algebra, **22** (1994), pp.4737–4749. MR **95d:**16013
14. P.Crawley, B.Jónsson, *Refinement for infinite direct decompositions of algebraic systems*, Pacific J.Math., **14** (1964), pp.797–855. MR **30:**49
15. E.G.Effros, D.E.Handelman, C-L.Shen, *Dimension groups and their affine representations*, American J.Math., **102** (1980), pp. 385–407. MR **83g:**46061

16. K.R.Goodearl, *Von Neumann regular rings*, Pitman, London, 1979; second ed., Krieger, Malabar, Fl., 1991. MR **80e:**16011; MR **93m:**16006
17. K.R.Goodearl, *Partially Ordered Abelian Groups with Interpolation*, Math.Surveys and Monographs, **20**, A.M.S., Providence, 1986. MR **88f:**06013
18. K.R.Goodearl, *Simple regular rings and rank functions*, Math.Annalen, **214** (1975), pp.267–287. MR **53:**13295
19. K.R.Goodearl, *Metrically complete regular rings*, Trans. Amer. Math. Soc., **272** (1982), pp.275–310. MR **83j:**16014
20. K.R.Goodearl, *C^*-algebra's of real rank zero whose K_0 are not Riesz groups*, Canad. Math. Bull. **39** (1996), 429–437. CMP 97:06
21. K.R.Goodearl, D.Handelman, *Simple self-injective rings*, Comm.Algebra, **3** (1975), pp.797–834. MR **52:**498
22. K.R.Goodearl, D.Handelman, *Metric completions of partially ordered abelian groups*, Indiana Univ.Math.J., **29** (1980), pp.861–895. MR **82b:**06020
23. K.R.Goodearl, R.B.Warfield, *Algebras over zero-dimensional rings*, Math.Ann., **223** (1976), pp.157–168. MR **54:**357
24. I.Halperin, *Regular rank rings*, Canadian J.Math., **17** (1965), pp.709–719. MR **33:**153
25. J.Hannah, K.C.O'Meara, *Depth of idempotent-generated subsemigroups of regular rings*, Proc.London Math.Soc., **59** (1989), pp. 464–482. MR **90h:**16021
26. J.Moncasi, *A regular ring whose K_0 is not a Riesz group*, Comm.Algebra, **13** (1985), pp.125–133. MR **86b:**16023
27. W.K.Nicholson, *Lifting idempotents and exchange rings*, Trans.AMS, **229** (1977), pp.269–278. MR **55:**12757
28. W.K.Nicholson, *I-rings*, Trans.AMS, **207** (1975), pp.361–377. MR **52:**481
29. E.Pardo, *On a density condition for ${K_0}^+$ of von Neumann regular rings*, Comm. Algebra, **22(2)** (1994), pp 707–719. MR **95a:**16014
30. E.Pardo, *On the representation of simple Riesz groups*, Comm. Algebra (to appear).
31. M.Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras*, Proc.London Math.Soc., **46** (1983), pp.301-333. MR **84g:**46085
32. M.Rieffel, *The cancellation theorem for projective modules over irrational rotation algebras*, Proc.London Math.Soc., **47** (1983), pp.301–333. MR **85g:**46085
33. J.Stock, *On rings whose projective modules have the exchange property*, J.Algebra, **103** (1986), pp.437–453. MR **88e:**16038
34. J.Torrens, *On the N^*-metric completion of regular rings*, Arch.Math., **47** (1986), pp. 529–534. MR **88b:**16025
35. J.Villadsen, *Simple C^*-algebras with perforation*, Preprint, 1995.
36. R.B.Warfield, *Exchange rings and decompositions of modules*, Math. Ann., **199** (1972), pp.31–36. MR **48:**11218
37. F.Wehrung, *Injective positively ordered monoids I*, J.Pure Appl.Algebra, **83** (1992), pp. 43–82. MR **93k:**06023
38. F.Wehrung, *Injective positively ordered monoids II*, J.Pure Appl.Algebra, **83** (1992), pp. 83–100. MR **93k:**06023
39. F.Wehrung, *Tensor products of structures with interpolation*, Pacific J.Math. **176** (1996), 267–285. CMP 97:08
40. S.Zhang, *Matricial structure and homotopy type of simple C^*-algebras with real rank zero*, J.Operator Theory, **26** (1991), pp.283–312, MR **94f:**46075
41. S.Zhang, *Diagonalizing projections in multiplier algebras and in matrices over a C^*-algebra*, Pacif.J.Math., **145** (1990), pp.181–200. MR **92h:**46088

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