

REFINEMENT MONOIDS WITH WEAK COMPARABILITY
AND APPLICATIONS TO REGULAR RINGS
AND C^* -ALGEBRAS

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ABSTRACT. We prove a cancellation theorem for simple refinement monoids satisfying the weak comparability condition, first introduced by K.C. O'Meara in the context of von Neumann regular rings. This result is then applied to von Neumann regular rings and C^* -algebras of real rank zero via the monoid of isomorphism classes of finitely generated projective modules.

INTRODUCTION

Let M be an (abelian) monoid. For $x, y \in M$ we will write $x \leq y$ if there exists $z \in M$ such that $y = x + z$. This preorder is sometimes called the *algebraic preorder* in M . Let M^* denote the set of nonzero elements of M . M is called *conical* if M^* is closed under addition. We will write $x \leq^* y$ if there exists $z \in M^*$ such that $y = x + z$. Note that the relation \leq^* is transitive if M is a conical monoid. An *order-unit* in M is a nonzero element $u \in M$ such that for each $x \in M$ there exists $n \geq 1$ such that $x \leq nu$. A monoid M is said to be *simple* if it is nonzero and every nonzero element of M is an order-unit, so that M has no nontrivial ideals (i.e. convex submonoids of M).

Definition. Let (M, u) be a monoid with order-unit. We say that (M, u) satisfies *weak comparability* provided that for all nonzero elements x in M such that $x \leq u$, there exists a positive integer $k = k(x)$ such that, if $y \in M$ and $ky \leq u$, then $y \leq x$. Note that, for conical M , if there is an element $x' \in M^*$ such that $x' \leq^* x$, then replacing $k(x)$ by $k(x')$ we obtain a positive integer k such that $ky \leq u$ implies $y \leq^* x$.

We say that a subset X of a monoid M is *cancellative* (respectively *strictly cancellative*) if, for $a, b, c \in X$, the relation $a + c = b + c$ (resp. $a + c \leq^* b + c$) implies $a = b$ (resp. $a \leq^* b$).

Our main results are the following:

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Theorem. *Let (M, u) be a simple refinement monoid satisfying weak comparability. Then M^* is cancellative.*

Corollary. *Let (M, u) be a simple refinement monoid with weak comparability. Assume that $u + a = u$ implies $a = 0$ for all $a \in M$. Then M is cancellative.*

We can apply these results to obtain cancellation results for rings R via the monoids of finitely generated projective right R -modules $V(R)$ provided we know that $V(R)$ is a refinement monoid. This is the case for von Neumann regular rings and for C^* -algebras with real rank zero. For von Neumann regular rings we recover O'Meara's Theorem [8, Theorem 1]. Also our methods prove that strict cancellation implies cancellation for C^* -algebras of real rank zero, giving a partial answer to [3, Question 3.2.3].

1. REFINEMENT MONOIDS

A monoid M is said to be a *refinement monoid* [11] if whenever $a + b = c + d$ in M , there exists $x, y, z, t \in M$ such that $a = x + y$, $b = z + t$ while $c = x + z$ and $d = y + t$. We say that M satisfies the *Riesz decomposition property* if whenever $p \leq q_1 + q_2$ in M , there exist $p_1, p_2 \in M$ such that $p = p_1 + p_2$ and $p_i \leq q_i$ for $i = 1, 2$. Clearly every refinement monoid satisfies the Riesz decomposition property, but the converse is not true in general. For example, the monoid $\{0, u, \infty\}$ (where $2u = \infty$) satisfies the Riesz decomposition property but it is not a refinement monoid.

An *atom* of a monoid M is a nonzero element a such that there is no b in M such that $0 \leq^* b \leq^* a$. M is an *atomic monoid* if each element of M can be written as a sum of atoms. Clearly a simple, nonatomic, refinement monoid has no atoms. The following lemma is known. In fact, much more is true; see [10, Theorem 6.2].

Lemma 1.1. *Let M be a simple nonatomic refinement monoid and let $x_1, \dots, x_n \in M^*$. Then for each $k \in \mathbb{N}$ there exists $x \in M^*$ such that $kx \leq x_i$ for all i . \square*

Lemma 1.2. *Let M be a simple, conical, nonatomic, refinement monoid and assume that $x \leq^* y + w$ for some $x, y, w \in M$ with $y \neq 0$. Then there exists $y' \in M^*$ such that $x \leq^* y' + w$ and $y' \leq^* y$.*

Proof. Write $x + t = y + w$ for some $t \in M^*$. Applying the refinement property, we have $x = x_1 + x_2$ and $t = t_1 + t_2$ such that $y = x_1 + t_1$ and $w = x_2 + t_2$. If $t_1 \neq 0$, write $t_1 = t'_1 + t''_1$ with $t'_1, t''_1 \in M^*$ and set $y' = x_1 + t'_1$. Then clearly $x \leq^* y' + w$. If $t_1 = 0$, then, by Lemma 1.1, we can write $t_2 = 2t'_2 + t''_2$ with $t'_2 \in M^*$ and $t'_2 \leq^* y$. Write $y = t'_2 + y'$ for some $y' \in M^*$. Then

$$x \leq^* x + t'_2 + t''_2 = y + t'_2 + t''_2 + x_2 = y' + 2t'_2 + t''_2 + x_2 = y' + w,$$

as desired. \square

Lemma 1.3. *If (M, u) is a simple, conical, nonatomic, refinement monoid satisfying weak comparability, then M^* is strictly cancellative.*

Proof. Assume x, y, z are elements of M^* such that $x + z \leq^* y + z$. There exists $m \in \mathbb{N}$ such that $z \leq mu$, whence $x + mu \leq^* y + mu$. Since M is conical, it suffices to prove the result for the case $z = u$.

So, assume that $x + u \leq^* y + u$ with $y \in M^*$. By Lemma 1.2, there exist $y', t \in M^*$ such that $y = y' + t$ and $x + u \leq^* y' + u$. By weak comparability, there

is $k \in \mathbb{N}$ such that, for $a \in M$, if $ka \leq u$, then $a \leq^* t$. Now, using [11, Lemma 1.11(i)] we get decompositions $x = x_1 + x_2$ and $y' = x_1 + y_1$ such that $kx_2 \leq u$. Therefore, we conclude that $x_2 \leq^* t$ and so $x = x_1 + x_2 \leq^* x_1 + y_1 + t = y' + t = y$. It follows that M^* satisfies strict cancellation. \square

Proposition 1.4. *Let (M, u) be a simple, conical, nonatomic, refinement monoid satisfying weak comparability.*

(a) *Assume that $x + u = u$ implies $x = 0$ for $x \in M$. Then M is strictly cancellative.*

(b) *Assume that $a + u = u$ for some $a \in M^*$. Then $x \leq^* y$ for all $x, y \in M^*$ and M^* is cancellative.*

Proof. (a) Since M^* is strictly cancellative by Lemma 1.3, it will suffice to prove that $x + z = z$ implies $x = 0$ for all $x, z \in M$.

Assume that $x + z = z$ for some $x \in M^*$. Since M is nonatomic, we then have $x' + z \leq^* z$ for some $x' \in M$. Now there exists a positive integer n such that $z \leq nu$ and so $x' + nu \leq^* nu$. By applying Lemma 1.3 we obtain $x' + u \leq^* u$, giving a contradiction.

(b) Assume that $a + u = u$ for some $a \in M^*$. Let x, y be elements in M^* . There exists $n \geq 1$ such that $x \leq na$ and thus $x + u \leq na + u = u$. We have $x + u \leq u \leq^* y + u$ and consequently Lemma 1.3 gives us $x \leq^* y$.

Now assume that $x + z = y + z$ for $x, y, z \in M^*$. Then $z \leq^* x$ and $z \leq^* y$ so that $x = z + x'$ and $y = z + y'$ for some $x', y' \in M^*$. Now $2z \leq^* z$ implies $z = 2z + t$ for some $t \in M^*$ and so

$$x = z + x' = 2z + t + x' = x + z + t = y + z + t = 2z + y' + t = z + y' = y,$$

showing cancellation in M^* . \square

Lemma 1.5. *Let M be a strictly cancellative, conical, atomless, refinement monoid. Assume that $2a + h = a + y$ for some $a, h, y \in M$. Then there exists $e \in M$ such that $2a = a + e$ and $y = h + e$.*

Proof. By applying the refinement property to the equality $a + a + h = a + y$, we get $a = a_1 + t_1 = a_2 + t_2$ and $h = a_3 + t_3$, with $a_1 + a_2 + a_3 = a$ and $t_1 + t_2 + t_3 = y$. If $a_3 = 0$, then we can take $e = t_1 + t_2$. So, assume that $a_3 \neq 0$. Since M has no atoms, there exists nonzero $x_1, x_2 \in M$ such that $a_3 = x_1 + x_2$.

For $i = 1, 2$, we have $a_i + x_i \leq^* a_1 + a_2 + x_1 + x_2 = a = a_i + t_i$. Since M is strictly cancellative, we get $x_i \leq^* t_i$ for $i = 1, 2$. Write $t_i = x_i + x'_i$ for some $x'_i \in M^*$, and set $e = x'_1 + x'_2$. Then we have

$$y = t_1 + t_2 + t_3 = x_1 + x'_1 + x_2 + x'_2 + t_3 = a_3 + t_3 + e = h + e$$

and also

$$\begin{aligned} a + e &= a_1 + a_2 + a_3 + x'_1 + x'_2 = a_1 + a_2 + x_1 + x_2 + x'_1 + x'_2 \\ &= a_1 + t_1 + a_2 + t_2 = 2a. \end{aligned}$$

This completes the proof. \square

We need a simple lemma:

Lemma 1.6. *The only conical atomic simple refinement monoid is the infinite cyclic monoid.*

Proof. Let M be a conical atomic simple refinement monoid and let a be an atom in M . By Riesz decomposition, every element b in M is of the form na for some nonnegative integer n . Assume there is a relation $na = ma$ for $0 \leq n < m$. Then $n > 0$ because M is conical. Now use the refinement property to get decompositions $a = \sum_{j=1}^m a_{ij}$ for $i = 1, \dots, n$ such that $a = \sum_{i=1}^n a_{ij}$ for $j = 1, \dots, m$. Since $n < m$, there exists some i such that the identity $a = \sum_{j=1}^m a_{ij}$ reads $a = ka$ for some $k > 1$. But then $a \leq^* a$, contradicting the fact that a is an atom. \square

Theorem 1.7. *Let (M, u) be a simple refinement monoid satisfying weak comparability. Then M^* is cancellative.*

Proof. If M is not conical, then it is easy to show, by using the simplicity, that M is a group. So, assume that M is conical (so that M^* is a semigroup). By Lemma 1.6 we can assume in addition that M is nonatomic. By Proposition 1.4, we only need to consider the case in which M is strictly cancellative.

So, let M be a strictly cancellative, simple, conical, nonatomic, refinement monoid. Let $x, y, z \in M^*$ such that $x + z = y + z$. By Lemma 1.1, there exists $w \in M^*$ such that $2w \leq x$. By simplicity there exists a positive integer k such that $z \leq kw$. Using Riesz decomposition we get $z = z_1 + \dots + z_k$ such that $z_i \leq w$ for all i . So we get

$$x + z_1 + \dots + z_k = y + z_1 + \dots + z_k$$

with $2z_i \leq x$ and we therefore can reduce the problem to the case in which $2z \leq x$. So, assuming this condition, write $x = 2z + t$ for some $t \in M$ and note that $2z + (t + z) = z + y$. By Lemma 1.5, there exists $e \in M$ such that $2z = z + e$ and $y = t + z + e$. Therefore

$$x = 2z + t = z + e + t = y,$$

showing cancellation in M^* . \square

Corollary 1.8. *Let (M, u) be a simple refinement monoid with weak comparability. Assume that $u + a = u$ implies $a = 0$ for all $a \in M$. Then M is cancellative.*

Proof. We can again assume that M is conical. Since M^* is cancellative by Theorem 1.7, it suffices to prove that $a + c = c$ implies $a = 0$ for all $a, c \in M$. Since M is simple, there exists $n \geq 1$ such that $c \leq nu$. So $a + nu = nu$ and Theorem 1.7 tells us that $a + u = u$. We conclude that $a = 0$. \square

Finally we note that the proof of Theorem 1.7 gives the following:

Corollary 1.9. *Let M be a simple strictly cancellative refinement monoid. Then M is cancellative.* \square

The hypothesis of Corollary 1.9 is weaker than those in Corollary 1.8, because there exist simple cancellative refinement monoids which do not satisfy weak comparability; see [9, Example 3.12].

2. APPLICATIONS TO CANCELLATION
RESULTS FOR SOME RINGS AND C^* -ALGEBRAS

Let R be a ring and let $V(R)$ be the monoid of isomorphism classes of finitely generated projective right R -modules. It is well-known that $V(R)$ is a refinement monoid whenever R is a von Neumann regular ring [6, Theorem 2.8]. Also, a von Neumann regular ring is unit-regular if and only if $V(R)$ is a cancellative monoid [6, Theorem 4.5]. So Corollary 1.8 gives immediately the following result, due to K.C. O'Meara.

Theorem 2.1 ([8, Theorem 1]). *Let R be a simple directly finite von Neumann regular ring satisfying weak comparability. Then R is unit-regular.* \square

Also we have the following consequence of Corollary 1.9:

Theorem 2.2. *Let R be a simple von Neumann regular ring such that $A \oplus C \prec B \oplus C$ implies $A \prec B$ for all finitely generated projectives A, B, C . Then R is unit-regular.* \square

Now we turn our attention to C^* -algebras. We will follow the background and notation of [2] and [3]. The monoid $V(A)$ can be identified with the monoid of Murray-von Neumann equivalence classes of projections in $M_\infty(A)$. For projections p, q in a C^* -algebra, we write $p \lesssim q$ if p is Murray-von Neumann equivalent to a subprojection of q . The notation $k \cdot p$ for a positive integer k and a projection p will mean the orthogonal sum of k copies of p .

Definition. We say that a unital C^* -algebra A satisfies weak comparability if for every nonzero projection p in A there exists a positive integer k such that, for any projection q in A , if $k \cdot q \lesssim 1$, then $q \lesssim p$.

It is clear that A satisfies weak comparability if and only if $(V(A), [1])$ satisfies weak comparability.

Lemma 2.3. *Let A be a C^* -algebra with real rank zero. Then $V(A)$ is a refinement monoid.*

Proof. By [5, Theorem 2.10], all the matrix algebras over A have real rank 0. So it suffices to consider two projections p, q in a unital C^* -algebra with real rank zero and show decompositions $q = q_{11} + q_{12}$ and $1 - q = q_{21} + q_{22}$ such that $q_{11} + q_{21} \sim p$ and $q_{12} + q_{22} \sim 1 - p$.

By [12, Theorem 3.2], q is homotopic to a projection of the form $p' + p''$ where $p' \leq p$ and $p'' \leq 1 - p$. So we can write $q = q_{11} + q_{12}$ where $q_{11} \sim p'$ and $q_{12} \sim p''$. Since $1 - q$ is homotopic to $1 - (p' + p'') = (p - p') + (1 - p - p'')$, we obtain a corresponding decomposition $1 - q = q_{21} + q_{22}$. Finally observe that $q_{11} + q_{21} \sim p' + (p - p') = p$ and $q_{12} + q_{22} \sim p'' + (1 - p - p'') = 1 - p$, as desired. \square

Theorem 2.4. *Let A be a finite simple unital C^* -algebra with real rank zero satisfying weak comparability. Then $V(A)$ has cancellation, the stable rank of A is one, and $K_0(A)$ is a weakly unperforated Riesz group.*

Proof. By Lemma 2.3, $V(A)$ is a refinement monoid. So, $V(A)$ is cancellative by Corollary 1.8. By [4, III.2.4], the stable rank of A is one. Finally, by [1, Theorem 4.2], $K_0(A)$ is weakly unperforated. \square

The following result answers [3, Question 3.2.3] for C^* -algebras with real rank zero.

Theorem 2.5. *Let A be a simple C^* -algebra with real rank zero such that $V(A)$ is strictly cancellative. Then the stable rank of A is one.*

Proof. It follows immediately from Lemma 2.3, Corollary 1.9 and [4, III.2.4]. \square

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